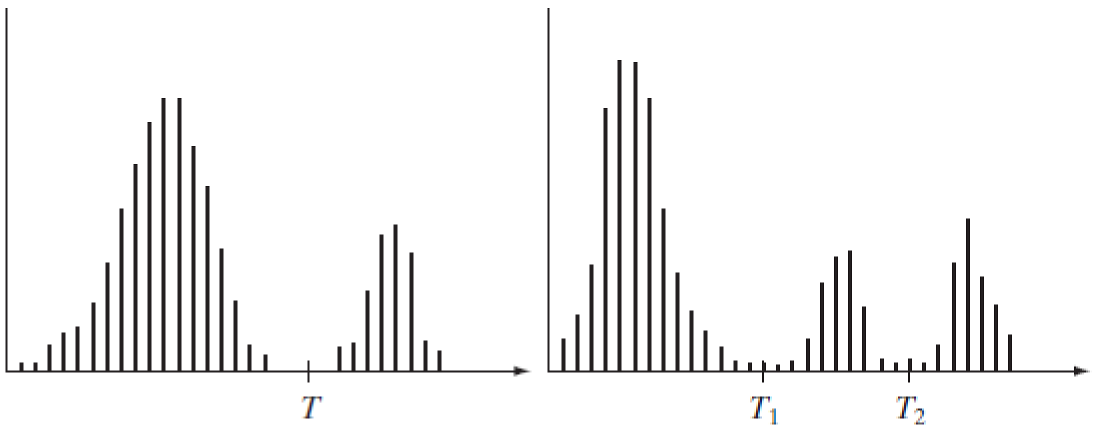
Image Segmentation

Lecture 3

Thresholding

- Image partitioning into regions directly from their intensity values.
- Consider an image corresponds to dark background and light foreground



a b

FIGURE 10.35 Intensity histograms that can be partitioned (a) by a single threshold, and (b) by dual thresholds.

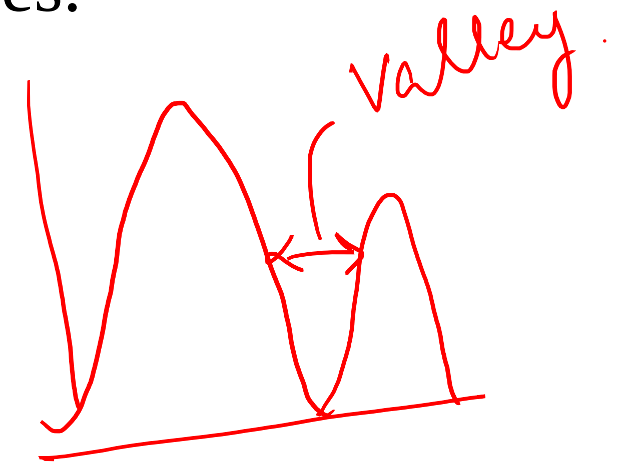
$$g(x, y) = \begin{cases} 1 & \text{if } f(x, y) > T \\ 0 & \text{if } f(x, y) \leq T \end{cases}$$

$$g(x, y) = \begin{cases} a & \text{if } f(x, y) > T_2 \\ b & \text{if } T_1 < f(x, y) \leq T_2 \\ c & \text{if } f(x, y) \leq T_1 \end{cases}$$

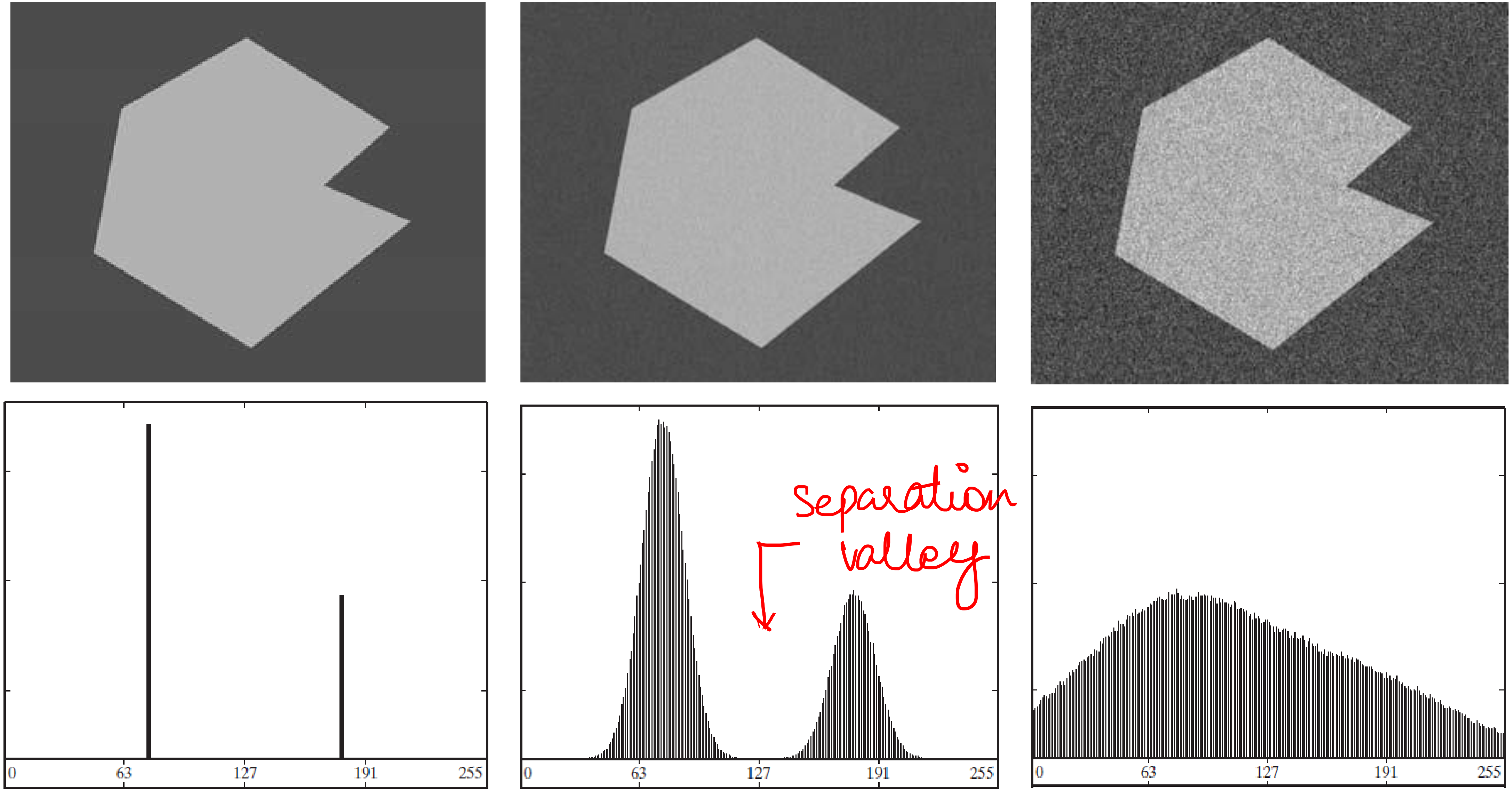
- Global thresholding- The value T is applicable over entire image.
- Variable thresholding - T changes over an image
- Local or regional thresholding -T depends on properties of neighborhood
- Dynamic or Adaptive thresholding- T depends on spatial coordinates themselves.

- The success of intensity thresholding is directly related to the width and depth of the valleys separating the histogram modes.
- Key factor affecting properties of the valleys are

1. Separation between peaks
2. The noise content in the image
3. The relative size of the background and objects .
4. The uniformity of the illumination source.
5. The uniformity of the reflectance properties of the image.



Noise in Thresholding

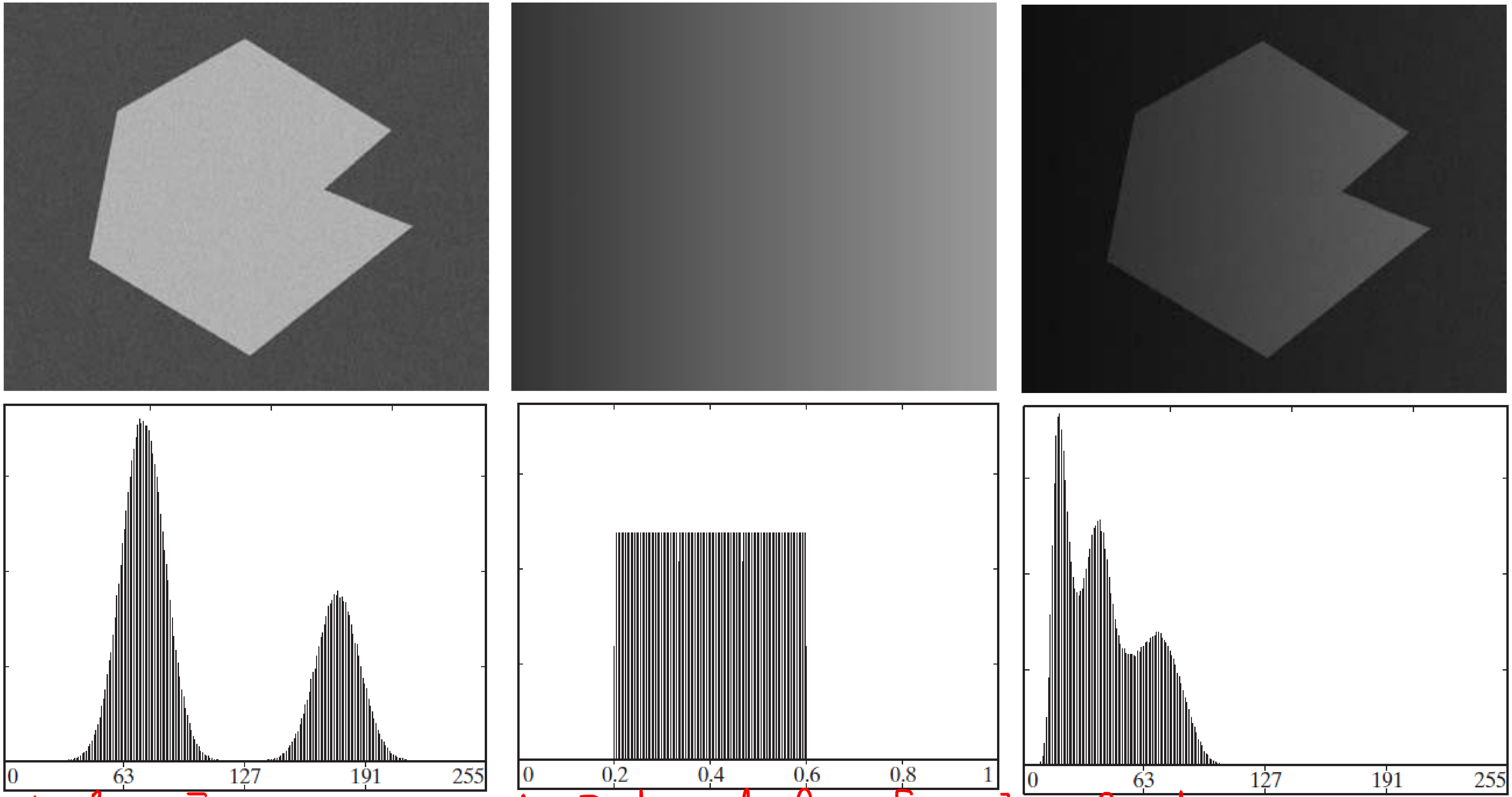


Noiseless image

Image + AGWN $\mu = 0$ $\sigma = 10$

Image + AGWN $\mu = 0$ $\sigma = 50$

Illumination in Thresholding



a) Noisy Image b) Intensity Ramp[0.2,0.6] a x b

Basic Global Thresholding

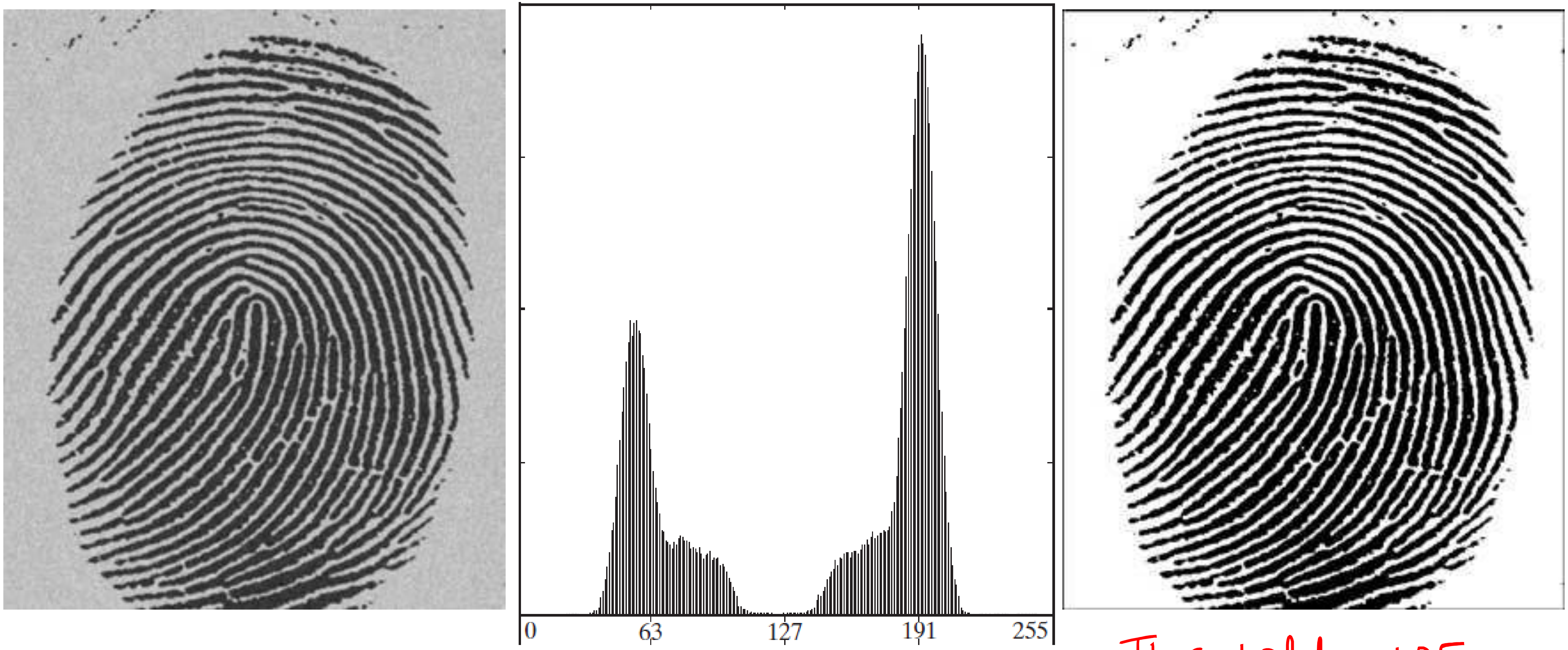
Algorithm

1. Select initial threshold estimate for the global threshold T.
2. Segment the image using T

   Region $G1$ (values > T) and region $G2$ (values < T).
3. Compute the average intensities $m1$ and $m2$ of regions $G1$ and $G2$ respectively.
4. Set $$T=(m1+m2)/2$$
5. Repeat steps 2 through 4 until the difference between the values of T in successive iterations is less than ΔT.

Global thresholding:

- The simple algorithm works well where there is a reasonably clear valley in histogram
- ΔT controls the speed of operation → number of iteration
- Larger the value → less iterations
- Smaller value → more iterations
- Initial threshold should be greater than minimum value of image intensity and lesser than maximum value of image intensity
- Avg intensity can be a good choice for initial threshold



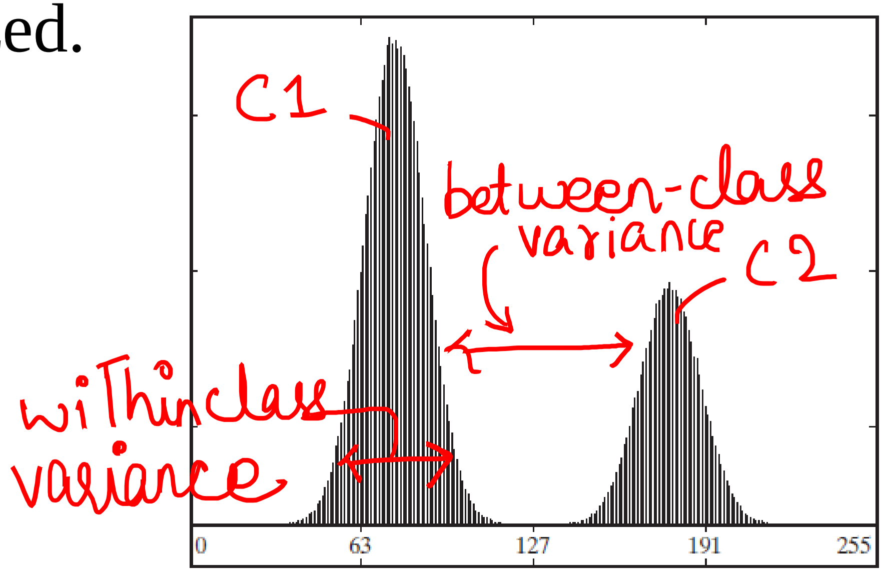
a b c

FIGURE 10.38 (a) Noisy fingerprint. (b) Histogram. (c) Segmented result using a global threshold (the border was added for clarity). (Original courtesy of the National Institute of Standards and Technology.)

Otsu’s Method

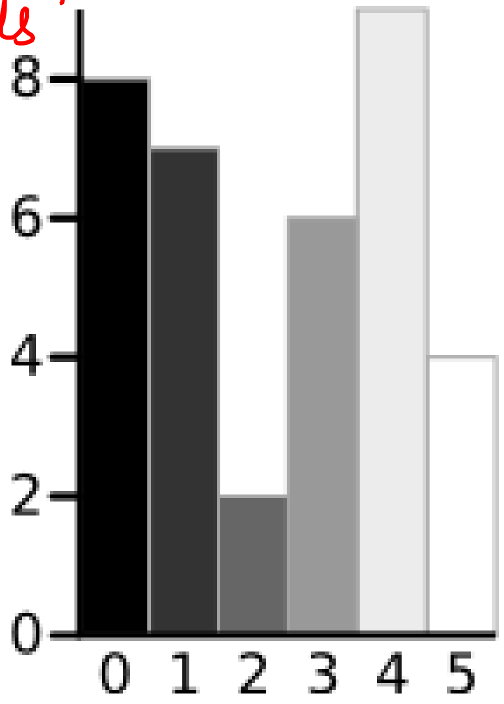
- Optimum Global Thresholding using Otsu’s Method.
- Otsu’s method is an adaptive thresholding way for binarization in image processing.
- It can find the optimal threshold value of the input image by going through all possible threshold values.(eg. 0 to 255)
- Basic idea is to obtain well separated classes.
- Threshold should be optimum such that *between the class variance* is maximized and *within the class variance* is minimized.

1. Histogram & Probability
2. Mean & Variance
3. within class variance
4. between-class variance
5. Otsu's thresholding



HISTOGRAM

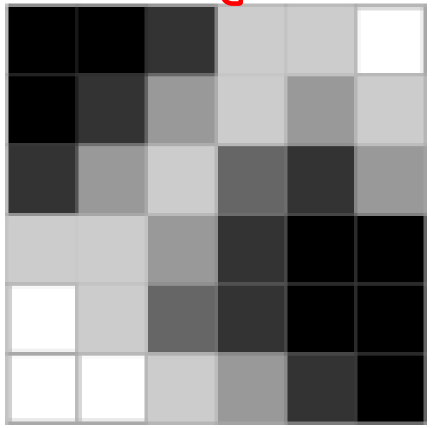
no of pixels



intensity

0 – 2

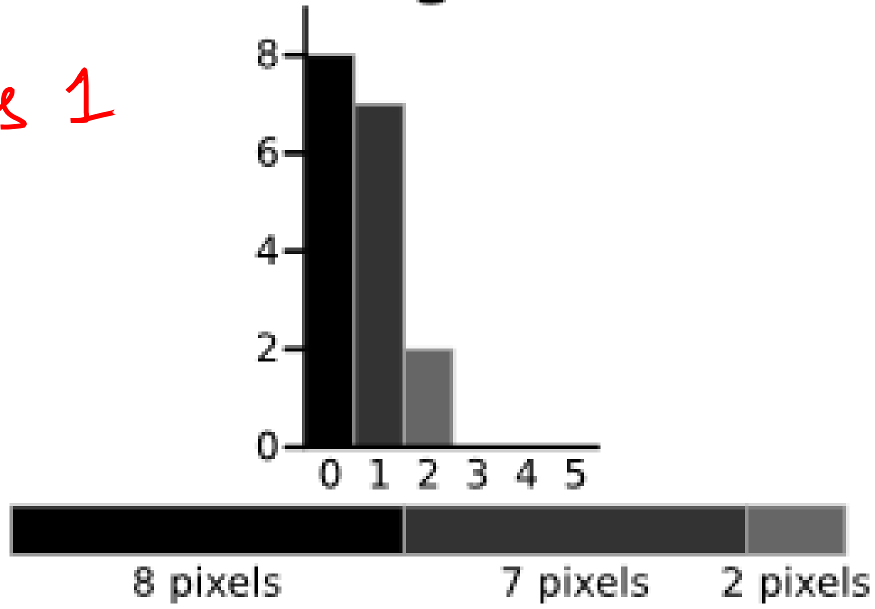
image



3 – 5

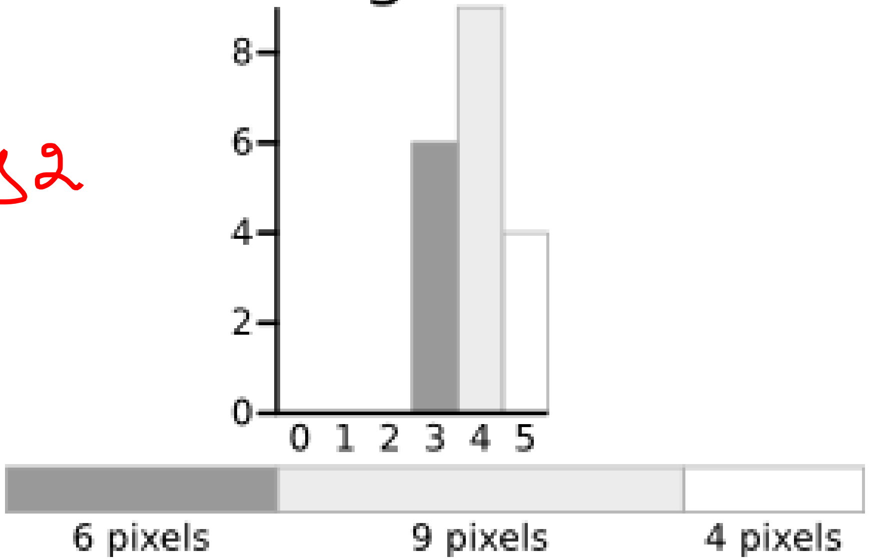
class 1

Background



class2

Foreground

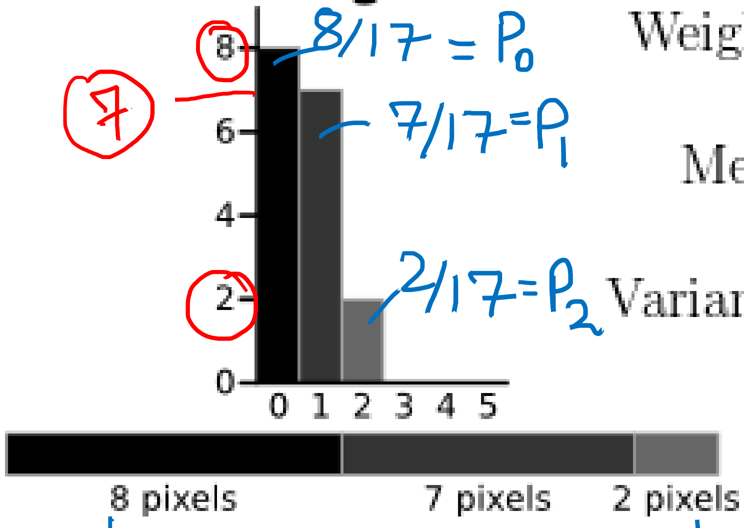


http://www.labbookpages.co.uk/software/imgProc/otsuThreshold.html

$P_i = n_i / MN$

$W_b \leftarrow$ Probability that a pixel is assigned to background class

Background



$$\text{Weight} \quad W_b = \frac{8+7+2}{36} = 0.4722$$

$$\text{Mean} \quad \mu_b = \frac{(0\times 8)+(1\times 7)+(2\times 2)}{17} = 0.6471$$

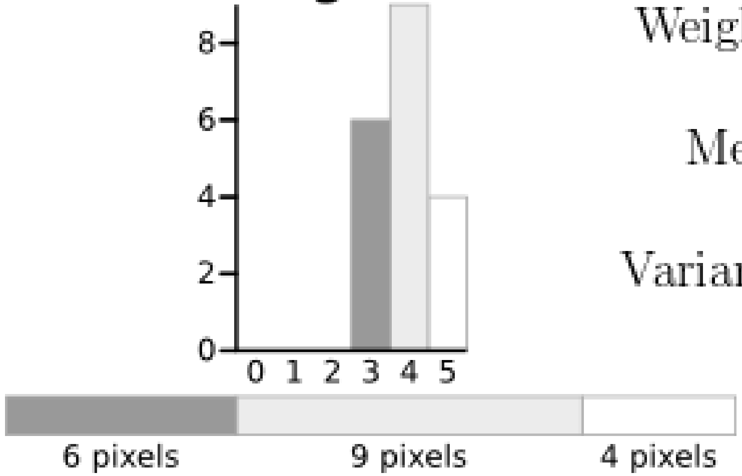
$$\text{Variance} \quad \sigma_b^2 = \frac{((0-0.6471)^2\times 8)+((1-0.6471)^2\times 7)+((2-0.6471)^2\times 2)}{17}$$

$$= \frac{(0.4187\times 8)+(0.1246\times 7)+(1.8304\times 2)}{17}$$

$$= 0.4637$$

$$\text{Mean} = \sum_{i=0}^{L-1} i \times P_i$$

Foreground



$$\text{Weight} \quad W_f = \frac{6+9+4}{36} = 0.5278$$

W_f = Probability that a pixel $\in$ foreground

$$\text{Mean} \quad \mu_f = \frac{(3\times 6)+(4\times 9)+(5\times 4)}{19} = 3.8947$$

$$\text{Variance} \quad \sigma_f^2 = \frac{((3-3.8947)^2\times 6)+((4-3.8947)^2\times 9)+((5-3.8947)^2\times 4)}{19}$$

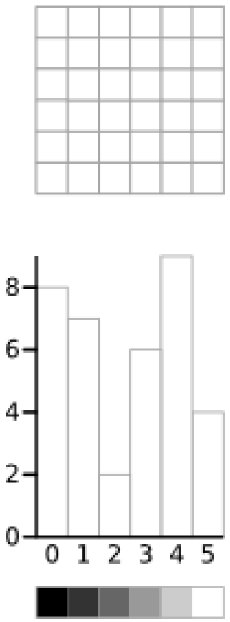
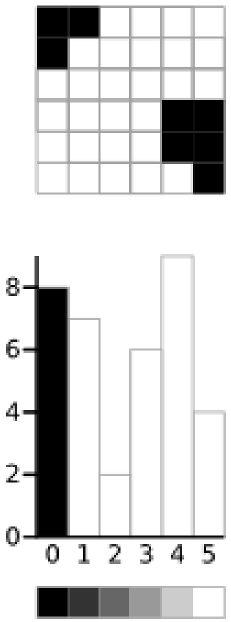
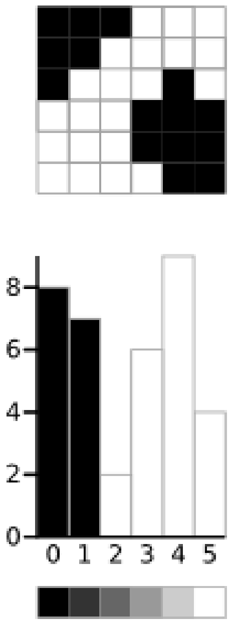
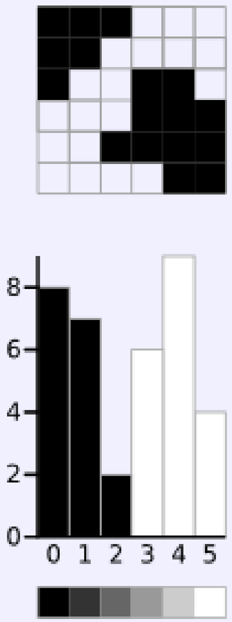
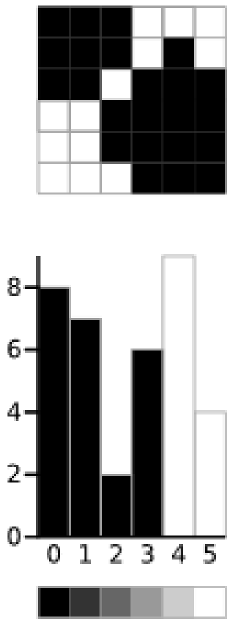
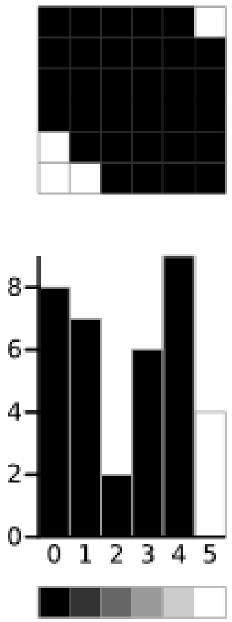
$$= \frac{(4.8033\times 6)+(0.0997\times 9)+(4.8864\times 4)}{19}$$

$$= 0.5152$$

http://www.labbookpages.co.uk/software/imgProc/otsuThreshold.html

$$\sigma_w^2 = W_b \sigma_b^2 + W_f \sigma_f^2$$

$$\text{Within Class Variance} \quad \sigma_W^2 = W_b\,\sigma_b^2 + W_f\,\sigma_f^2 = 0.4722 * 0.4637 + 0.5278 * 0.5152$$
$$= 0.4909$$

Threshold	T=0	T=1	T=2	T=3	T=4	T=5
						
Weight, Background	$W_b = 0$	$W_b = 0.222$	$W_b = 0.4167$	$W_b = 0.4722$	$W_b = 0.6389$	$W_b = 0.8889$
Mean, Background	$M_b = 0$	$M_b = 0$	$M_b = 0.4667$	$M_b = 0.6471$	$M_b = 1.2609$	$M_b = 2.0313$
Variance, Background	$\sigma^2_b = 0$	$\sigma^2_b = 0$	$\sigma^2_b = 0.2489$	$\sigma^2_b = 0.4637$	$\sigma^2_b = 1.4102$	$\sigma^2_b = 2.5303$
Weight, Foreground	$W_f = 1$	$W_f = 0.7778$	$W_f = 0.5833$	$W_f = 0.5278$	$W_f = 0.3611$	$W_f = 0.1111$
Mean, Foreground	$M_f = 2.3611$	$M_f = 3.0357$	$M_f = 3.7143$	$M_f = 3.8947$	$M_f = 4.3077$	$M_f = 5.000$
Variance, Foreground	$\sigma^2_f = 3.1196$	$\sigma^2_f = 1.9639$	$\sigma^2_f = 0.7755$	$\sigma^2_f = 0.5152$	$\sigma^2_f = 0.2130$	$\sigma^2_f = 0$
Within Class Variance	$\sigma^2_W = 3.1196$	$\sigma^2_W = 1.5268$	$\sigma^2_W = 0.5561$	$\boxed{\sigma^2_W = 0.4909}$	$\sigma^2_W = 0.9779$	$\sigma^2_W = 2.2491$

Min within class variance

$\sigma_w^2 = 4.909$

Variance of histogram

Within Class Variance $\sigma_W^2 = W_b\,\sigma_b^2 + W_f\,\sigma_f^2$ (as seen above)

Between Class Variance $\sigma_B^2 = \sigma^2 - \sigma_W^2$

$\sigma_B^2 = \sigma^2 - \sigma_W^2$

$= W_b(\mu_b - \mu)^2 + W_f(\mu_f - \mu)^2$ (where $\underbrace{\mu = W_b\,\mu_b + W_f\,\mu_f}$)

$= W_b\,W_f\,(\mu_b - \mu_f)^2$

$\boxed{\sigma_B^2 = W_b W_f (\mu_b - \mu_f)^2}$

Global mean $\rightarrow$ $\mu = W_b \mu_b + W_f \mu_f$

Threshold	T=0	T=1	T=2	T=3	T=4	T=5
Within Class Variance	σ^2_W = 3.1196	σ^2_W = 1.5268	σ^2_W = 0.5561	**σ^2_W = 0.4909**	σ^2_W = 0.9779	σ^2_W = 2.2491
Between Class Variance	σ^2_B = 0	σ^2_B = 1.5928	σ^2_B = 2.5635	**σ^2_B = 2.6287**	σ^2_B = 2.1417	σ^2_B = 0.8705

σ_B^2 should be max

σ_W^2 should be min

http://www.labbookpages.co.uk/software/imgProc/otsuThreshold.html

Otsu's Method

- Intensity levels: {0,1,2,…,L-1}
- Size of the image: M x N
- n_i : the number of pixels with intensity I
- Total pixels = $MN = n_0 + n_1 + n_2 + \cdots + n_{L-1}$
- The normalized histogram has components $p_i = n_i/MN$

$$\sum_{i=0}^{L-1} p_i = 1, \qquad p_i \geq 0$$

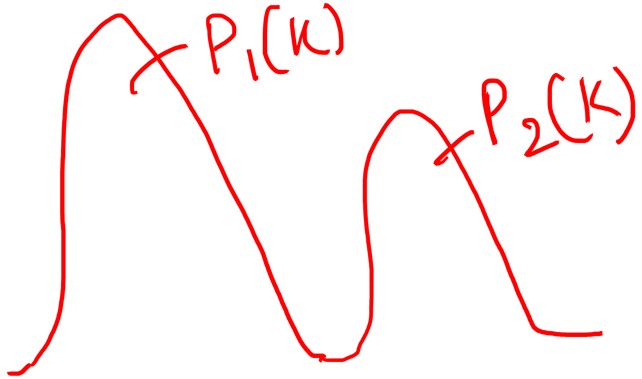
p_i is probability of a pixel having intensity i

- A Threshold $T(k) = k$, $0 < k < L\text{-}1$ is selected to threshold the input image into two classes C_1 and C_2.
- $C_1 \rightarrow$ all the pixels with intensity $[0, k]$
- $C_2 \rightarrow$ all the intensities $[k+1, L\text{-}1]$
- $P_1(k)$ and $P_2(k)$ are the probability that pixel is assigned to class C_1 and C_2 respectively or its probability of occurrence of class C_1 and C_2

$$P_1(k) = \sum_{i=0}^{k} p_i \qquad P_2(k) = \sum_{i=k+1}^{L-1} p_i = 1 - P_1(k)$$

K is threshold selected

Same as weight in example



- The mean intensity value of the pixels assigned to class C_1 is

$$m_1(k) = \sum_{i=0}^{k} iP(i/C_1)$$

$P(i/C_1)$ is the probability of intensity i, given that i comes from class C_1

$P(A/B) = \frac{P(B/A)\,P(A)}{P(B)}$

$$= \sum_{i=0}^{k} iP(C_1/i)P(i)/P(C_1)$$

$P(C_1/i) = 1$

$P(i) = p_i$ [from histogram]

$P(C_1) = P_1(k)$

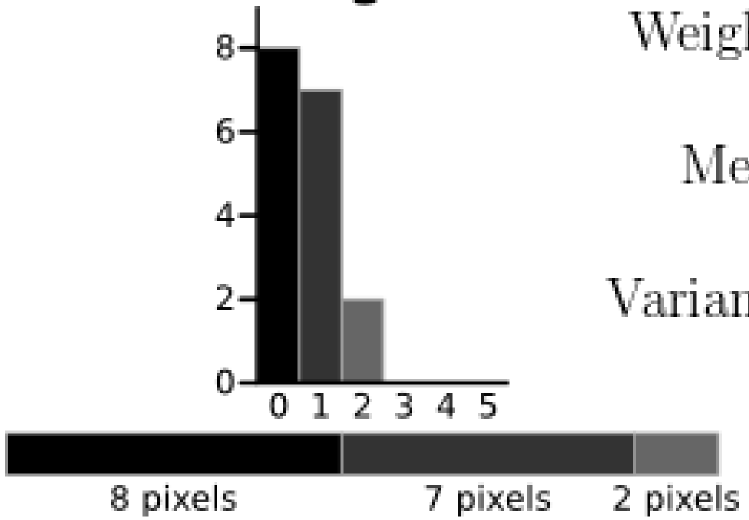
$$= \frac{1}{P_1(k)} \sum_{i=0}^{k} ip_i$$

- Bayes' Formula:

$$P(A/B) = P(B/A)P(A)/P(B)$$

$$\text{mean}_1 = \frac{1}{P_1(k)} \sum_{i=0}^{k} i P_i$$

Background



$$\text{Weight} \quad W_b = \frac{8+7+2}{36} = 0.4722$$

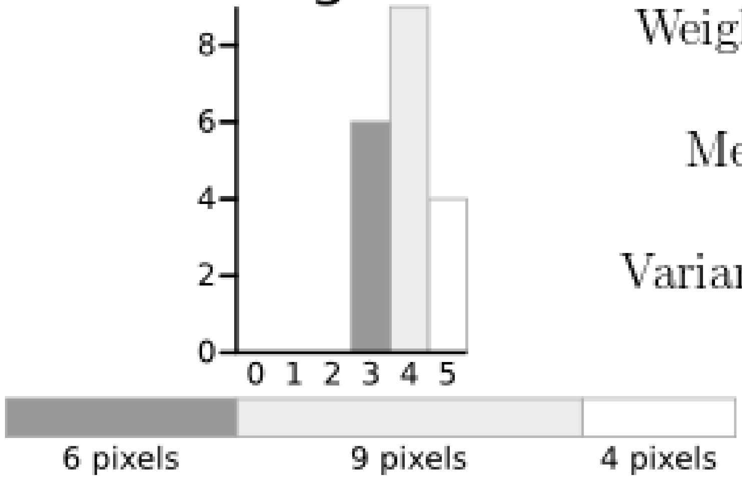
$$\text{Mean} \quad \mu_b = \frac{(0\times 8)+(1\times 7)+(2\times 2)}{17} = 0.6471 \leftarrow \frac{1}{0.4722}\left[\frac{0\times 8}{36} + \frac{1\times 7}{36} + \frac{4}{36}\right]$$

$$\text{Variance} \quad \sigma_b^2 = \frac{((0-0.6471)^2\times 8)+((1-0.6471)^2\times 7)+((2-0.6471)^2\times 2)}{17}$$

$$= \frac{(0.4187\times 8)+(0.1246\times 7)+(1.8304\times 2)}{17}$$

$$= 0.4637$$

Foreground



$$\text{Weight} \quad W_f = \frac{6+9+4}{36} = 0.5278$$

$$\text{Mean} \quad \mu_f = \frac{(3\times 6)+(4\times 9)+(5\times 4)}{19} = 3.8947$$

$$\text{Variance} \quad \sigma_f^2 = \frac{((3-3.8947)^2\times 6)+((4-3.8947)^2\times 9)+((5-3.8947)^2\times 4)}{19}$$

$$= \frac{(4.8033\times 6)+(0.0997\times 9)+(4.8864\times 4)}{19}$$

$$= 0.5152$$

- Similarly, the mean intensity value of the pixels assigned to class C_2 is

$$m_2(k) = \sum_{i=k+1}^{L-1} iP(i/C_2)$$

$$= \frac{1}{P_2(k)} \sum_{i=k+1}^{L-1} ip_i$$

- The cumulative mean (average intensity) upto level k is given by

$$m(k) = \sum_{i=0}^{k} ip_i$$

- The average intensity of the entire image (i.e., the global mean) is given by

$$m_G = \sum_{i=0}^{L-1} ip_i$$

- We can verify the things up till now using the two equations

$$P_1 + P_2 = 1$$

$$P_1 m_1 + P_2 m_2 = m_G$$

$$P_1 m_1 + P_2 m_2$$

$$= P_1(k) \times \frac{1}{P_1(k)} \sum_{i=0}^{k} i p_i + P_2(k) \times \frac{1}{P_2(k)} \sum_{i=k+1}^{L-1} i p_i$$

$$= \sum_{i=0}^{L-1} i p_i$$ $= m_G$ (Global mean)

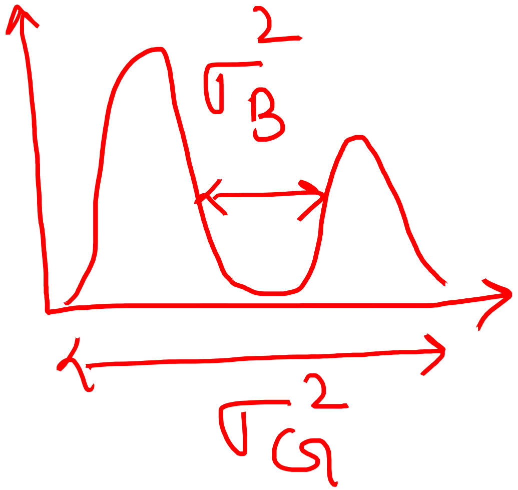
- In order to evaluate the "goodness" of the threshold at level k we use the normalized, dimensionless metric

$$\text{separability measure} \rightarrow \eta = \frac{\sigma_B^2}{\sigma_G^2} \begin{array}{l} \leftarrow \text{between-class variance} \\ \leftarrow \text{Global variance} \end{array}$$

where σ_G^2 is the *global variance*

$$p_i = \frac{n_i}{MN}$$

$$\sigma_G^2 = \sum_{i=0}^{L-1} (i - m_G)^2 p_i$$



and σ_B^2 is the *between-class variance*, defined as

$$\sigma_B^2 = P_1(m_1 - m_G)^2 + P_2(m_2 - m_G)^2$$

- This previous expression can also be written as

$$\sigma_B^2 = P_1 P_2 (m_1 - m_2)^2$$

$$= \frac{(m_G P_1 - m)^2}{P_1(1 - P_1)}$$

$$\eta = \frac{\sigma_B^2}{\sigma_G^2}$$

- σ_B^2 and η are measures of separability between classes.
- Larger σ_B^2 → larger between the class variance →more separability →higher η
- Generally $\sigma_G^2 > 0$
- If $\sigma_G^2 = 0$ → constant intensity in image → only one class possible → η = 0 → indicating no separation between classes.

- Reintroducing *k*, we have the final results

$$\eta(k) = \frac{\sigma_B^2(k)}{\sigma_G^2}$$

$$\sigma_B^2(k) = \frac{\left[m_G P_1(k) - m(k)\right]^2}{P_1(k)\left[1 - P_1(k)\right]}$$

- Ultimate goal is to find k for which σ_B^2 is maximum

- Then the optimum threshold is the value, k^*, that maximizes $\sigma_B^2(k)$:

$$\sigma_B^2(k^*) = \max_{0 \le k \le L-1} \sigma_B^2(k)$$

- In other words, to find k^*, we simply evaluate Eq for all integer values of k, select that value of k that yields maximum $\sigma_B^2(k)$.
- If the maximum exists for more than one value of k, it is customary to average the various values of k for which $\sigma_B^2(k)$ is maximum
- Once k^* has been obtained, the input image $f(x, y)$ is segmented as before:

$$g(x, y) = \begin{cases} 1 & \text{if } f(x, y) > k^* \\ 0 & \text{if } f(x, y) \le k^* \end{cases}$$

- The normalized metric evaluated at the optimum threshold value, can be used to obtain a quantitative estimate of the separability of classes, which in turn gives an idea of the ease of thresholding a given image.
- This measure has values in the range

$$0 \leq \eta(k^*) \leq 1$$

- Otsu's algorithm may be summarized as follows:

1. Compute the normalized histogram of the input image. Denote the components of the histogram by p_i , $i = 0, 1, 2, \ldots, L-1$.
2. Compute the cumulative sums, $P_1(k)$ for $k = 0, 1, 2, \ldots, L-1$
3. Compute the cumulative means, $m(k)$, for $k = 0, 1, 2, \ldots, L-1$
4. Compute the global intensity mean, m_G
5. Compute the between – class variance, $\sigma_B^2(k)$, for $k = 0, 1, 2, \ldots, L-1$
6. Obtain the Otsu threshold, k^*, as the value of for which $\sigma_B^2(k)$ is maximum. If the maximum is not unique, obtain k^* by averaging the values of k corresponding to the various maxima detected.
7. Obtain the separability measure, η^* at $k = k^*$

$$\sigma_B^2 = \frac{(m_G P_1 - m)^2}{P_1(1 - P_1)}$$

FIGURE 10.39
(a) Original image.
(b) Histogram (high peaks were clipped to highlight details in the lower values).
(c) Segmentation result using the basic global algorithm from Section 10.3.2.
(d) Result obtained using Otsu's method.

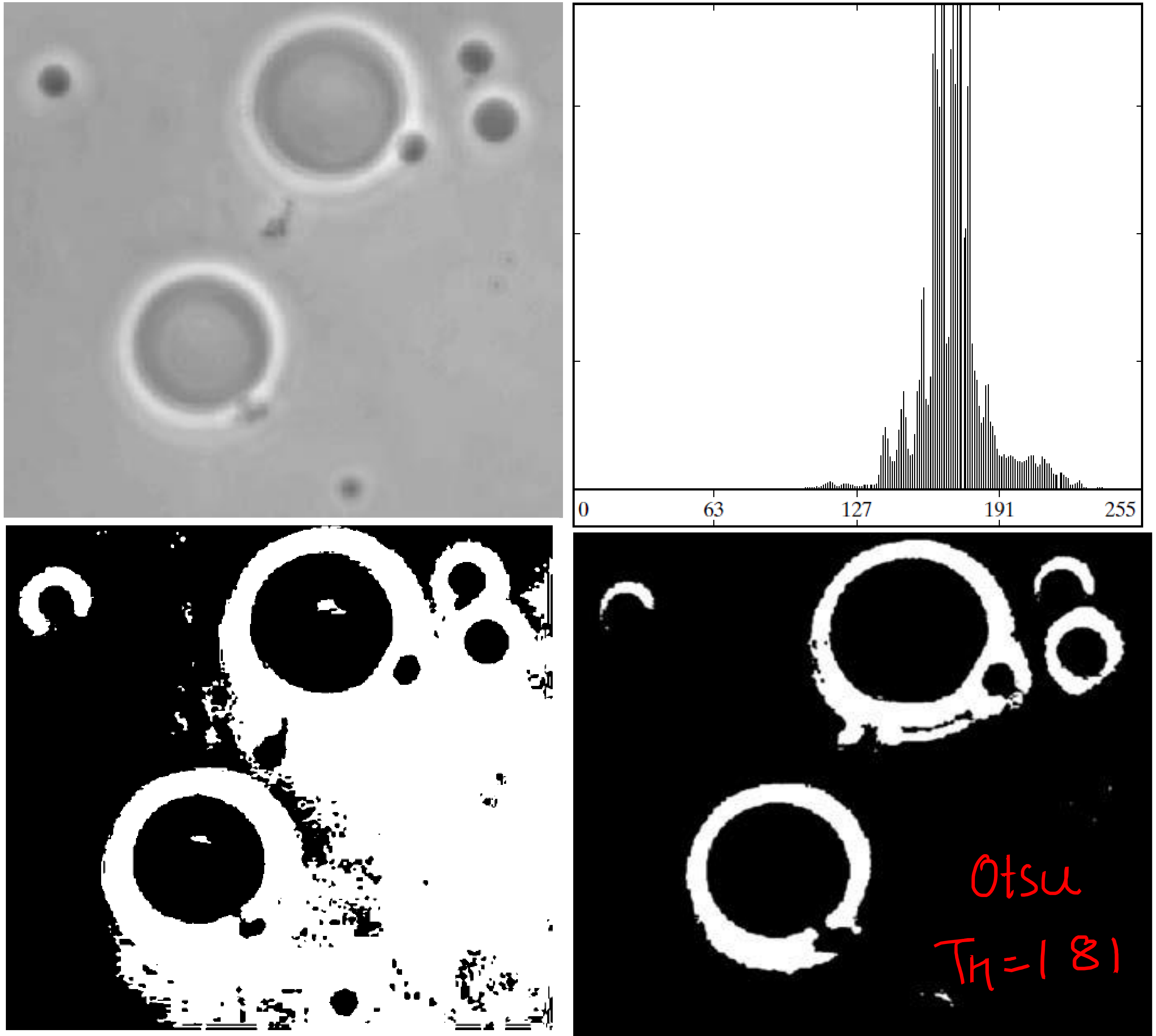
$T_H = 181$

$\eta = 0.467$

optical microscope image of polymersome cells

basic global thresholding →

$T_H = 169$



Otsu

$T_H = 181$

We know that

$$\rightarrow \sigma_B{}^2 = P_1(m_1 - m_G)^2 + P_2(m_2 - m_G)^2$$

$$\sigma_B{}^2 = P_1(m_1{}^2 - 2m_1m_G + m_G{}^2) + P_2(m_2{}^2 - 2m_2m_G + m_G{}^2)$$

$$\sigma_B{}^2 = P_1m_1{}^2 - 2(P_1m_1 + P_2m_2)m_G + (P_1 + P_2)m_G{}^2 + P_2m_2{}^2$$

$$\sigma_B{}^2 = P_1m_1{}^2 - 2m_G{}^2 + m_G{}^2 + P_2m_2{}^2$$

$$\sigma_B{}^2 = P_1m_1{}^2 - m_G{}^2 + P_2m_2{}^2$$

$$P_1m_1 + P_2m_2 = m_G \qquad\qquad P_1 + P_2 = 1$$

$$\sigma_B{}^2 = P_1m_1{}^2 - (P_1{}^2m_1{}^2 + 2P_1P_2m_1m_2 + P_2{}^2m_2{}^2) + P_2m_2{}^2$$

$$\sigma_B{}^2 = P_1(1 - P_1)m_1{}^2 - 2P_1P_2m_1m_2 + P_2(1 - P_2)m_2{}^2$$

$$\sigma_B{}^2 = P_1P_2m_1{}^2 - 2P_1P_2m_1m_2 + P_1P_2m_2{}^2$$

$$\boxed{\sigma_B{}^2 = P_1P_2(m_1 - m_2)^2}$$

$$\sigma_B^2 = P_1P_2(m_1 - m_2)^2$$

$$m_1 = \frac{1}{P_1(K)}\sum_{i=0}^{K} iP_i \qquad = P_1P_2\Big(\frac{1}{P_1}\underbrace{\sum_{i=0}^{k} ip_i} - \frac{1}{P_2}\underbrace{\sum_{i=k+1}^{L-1} ip_i}\Big)^2$$

To prove

$$\sigma_B^2 = \frac{(m_G P_1 - m)^2}{P_1(1-P_1)}$$

Cummulative mean

$$m = \sum_{i=0}^{k} ip_i \qquad m_G = \sum_{i=0}^{L-1} ip_i$$

$$= P_1P_2\Big(\frac{1}{P_1}\underbrace{m} - \frac{1}{P_2}\underbrace{(m_G - m)}\Big)^2$$

$$= \frac{P_1P_2}{P_1^2P_2^2}\big(P_2 m - P_1(m_G - m)\big)^2$$

$$= \frac{1}{P_1P_2}\big((1 - P_1)m - P_1(m_G - m)\big)^2 = \frac{(m_GP_1 - m)^2}{P_1(1 - P_1)}$$

Multiple Thresholds

- In the case of K classes $C_1, C_2, \ldots, C_K$ between the class variance is defined by :

$$\sigma_B^2 = \sum_{k=1}^{K} P_k (m_k - m_G)^2$$

- Where,

$$P_k = \sum_{i \in C_k} p_i \qquad m_k = \frac{1}{P_k} \sum_{i \in C_k} i p_i$$

- The K classes are separated by $K - 1$ thresholds, whose values k_1^*, $k_2^*, \ldots, k_{K-1}^*$ are given by:

$$\sigma_B^2(k_1^*, k_2^*, \ldots, k_{K-1}^*) = \max_{0<k_1<k_2<\ldots k_{n-1}<L-1} \sigma_B^2(k_1, k_2, \ldots k_{K-1})$$

- For three classes consisting of three intensity intervals (which are separated by three thresholds) the between-class variance is given by:

$$\sigma_B^2 = P_1(m_1 - m_G)^2 + P_2(m_2 - m_G)^2 + P_3(m_3 - m_G)^2$$

- Where,

$$P_1 = \sum_{i=0}^{k_1} p_i \qquad P_2 = \sum_{i=k_1+1}^{k_2} p_i \qquad P_3 = \sum_{i=k_2+1}^{L-1} p_i$$

$$m_1 = \frac{1}{P_1}\sum_{i=0}^{k_1} ip_i \qquad m_2 = \frac{1}{P_2}\sum_{i=k_1+1}^{k_2} ip_i \qquad m_3 = \frac{1}{P_3}\sum_{i=k_2+1}^{L-1} ip_i$$

- Following relationship holds

$$P_1m_1 + P_2m_2 + P_3m_3 = m_G \qquad P_1 + P_2 + P_3 = 1$$

$$\sigma_B^2(k_1^*, k_2^*) = \max_{0<k_1<k_2<L-1} \sigma_B^2(k_1, k_2)$$

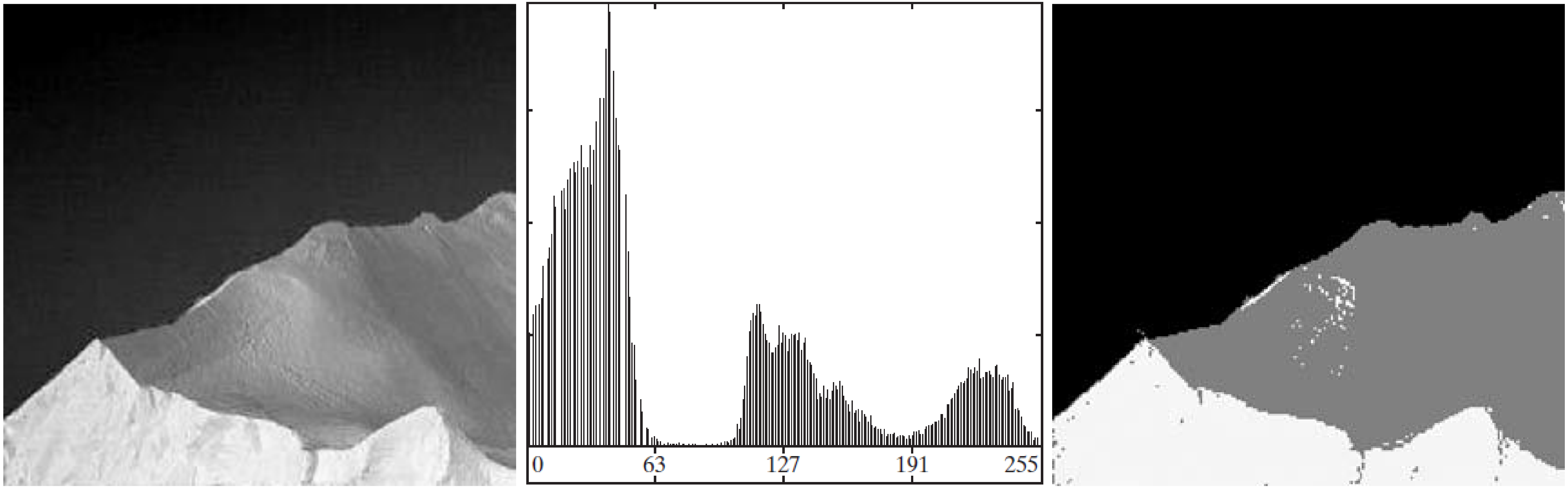
1. Select the value of k_1
2. Select the value of k_2 such that it is incremented through all possible values which are greater than k_1 but less than $L-1$
3. Compute between-class variance for each pair of k_1 and k_2
4. Increment the value of k_1 repeat the procedure.
5. Select maximum value of between-class variance and corresponding k_1 and k_2 are required thresholds.
6. In case of several maxima, corresponding values of k_1 and k_2 are averaged to obtain final results.

- The thresholded image is then given by

$$g(x, y) = \begin{cases} a & \text{if } f(x, y) \le k_1^* \\ b & \text{if } k_1^* < f(x, y) \le k_2^* \\ c & \text{if } f(x, y) > k_2^* \end{cases}$$

- where *a*, *b* and *c* are any three valid intensity values.
- Separability Measure

$$\eta(k_1^*, k_2^*) = \frac{\sigma_B^2(k_1^*, k_2^*)}{\sigma_G^2}$$



a b c

FIGURE 10.45 (a) Image of iceberg. (b) Histogram. (c) Image segmented into three regions using dual Otsu thresholds. (Original image courtesy of NOAA.)