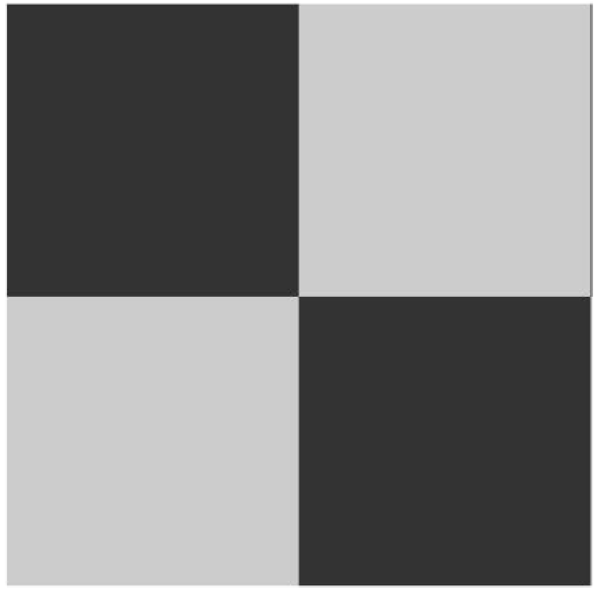
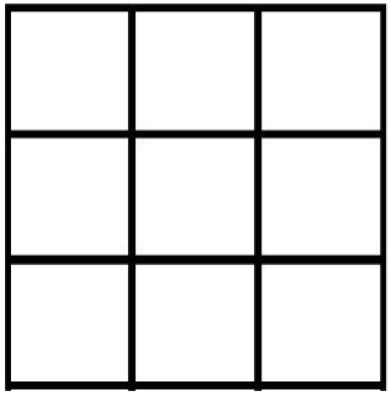
Spatial Aliasing

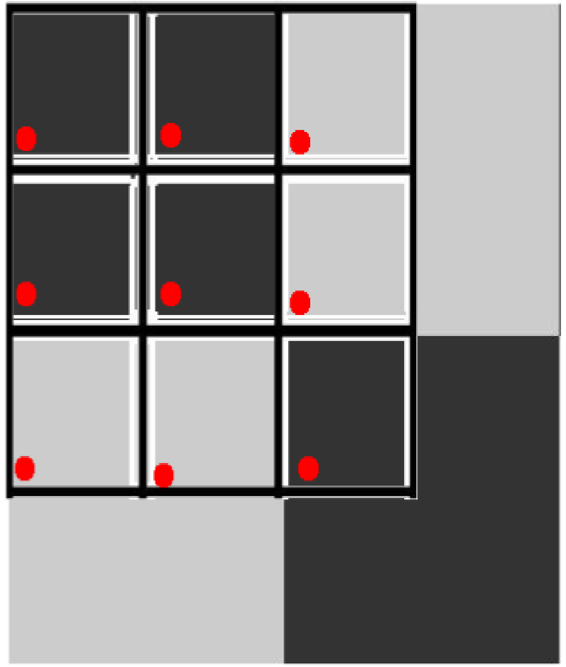
Case : 1

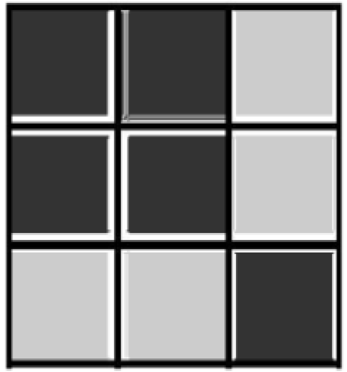


Original scene



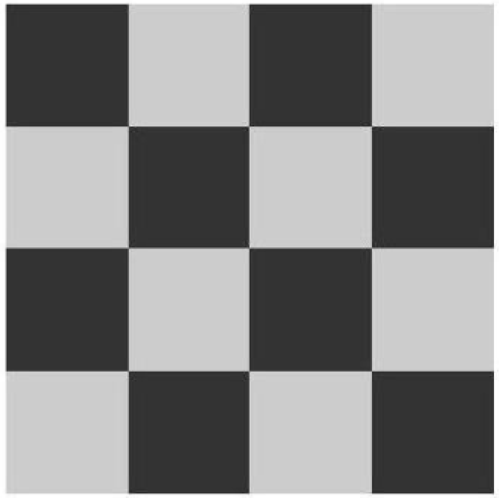
Pixel grid



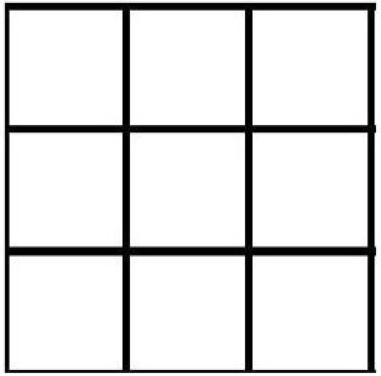


Perceived scene

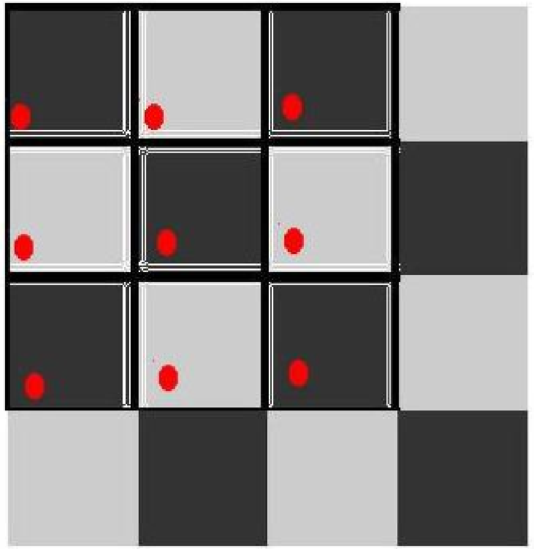
Case: 2

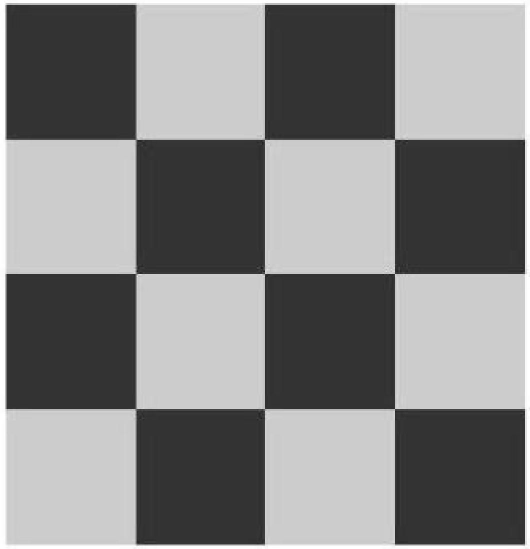


Original scene



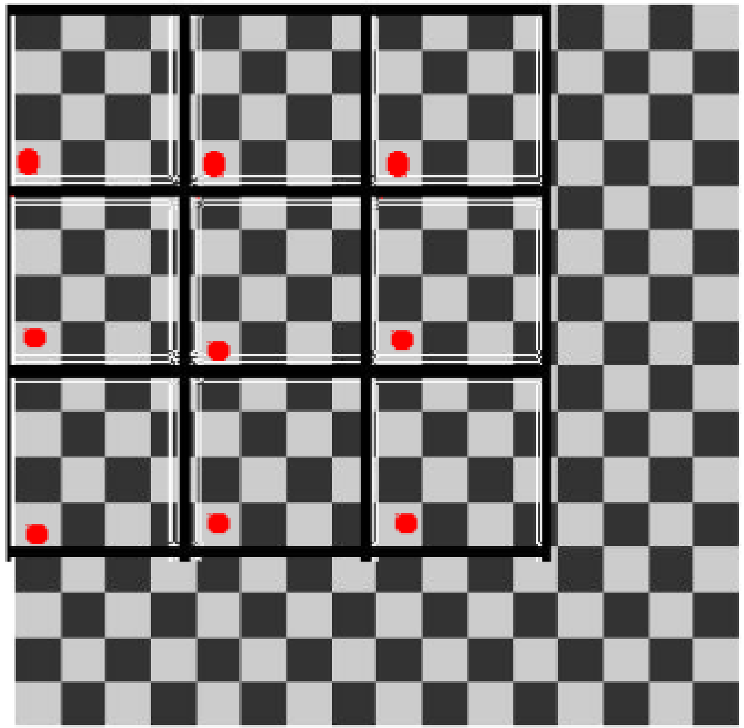
Pixel grid



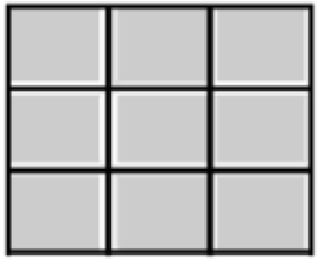


Perceived scene

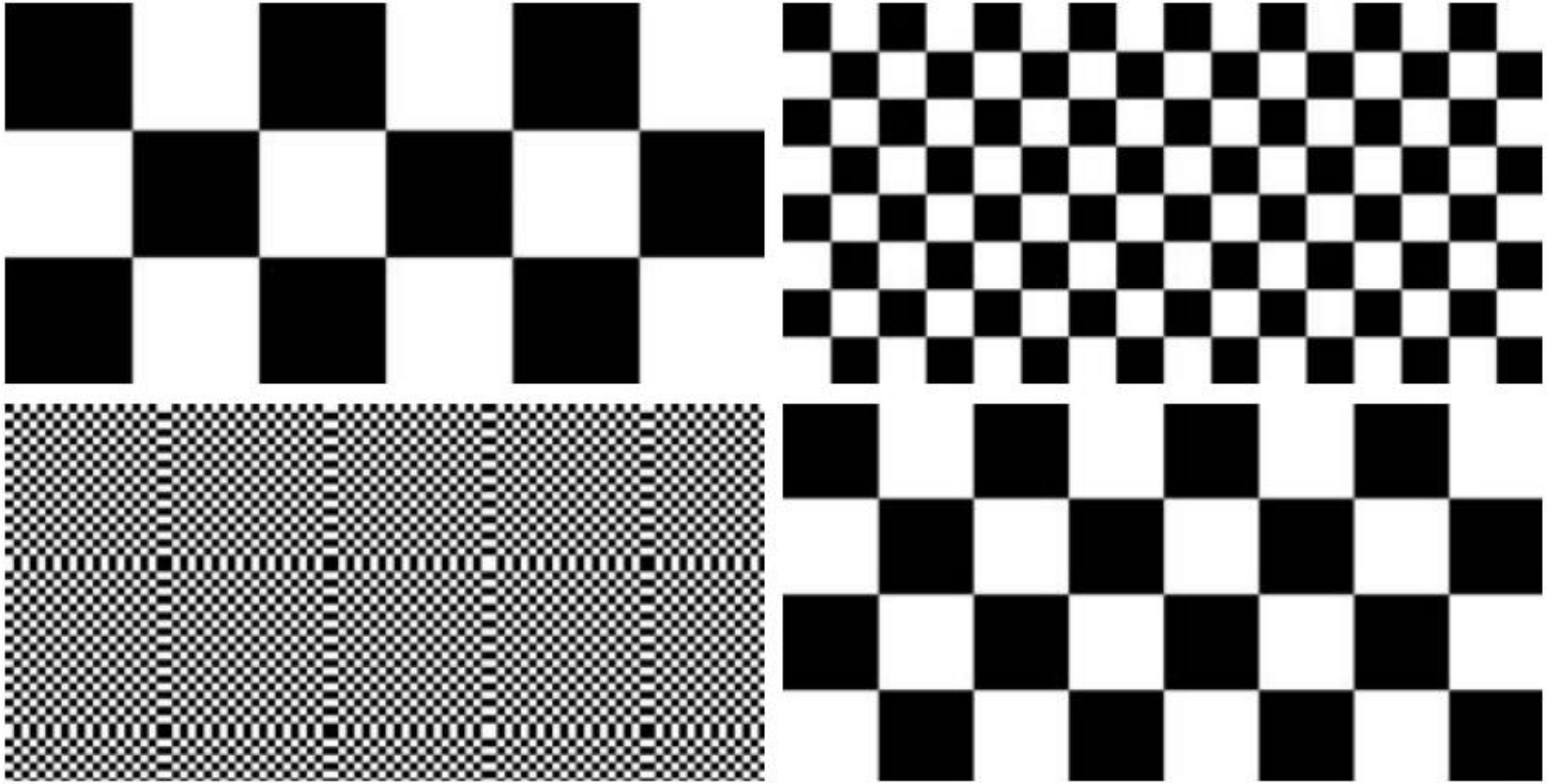
Case : 3



Original scene



Perceived scene



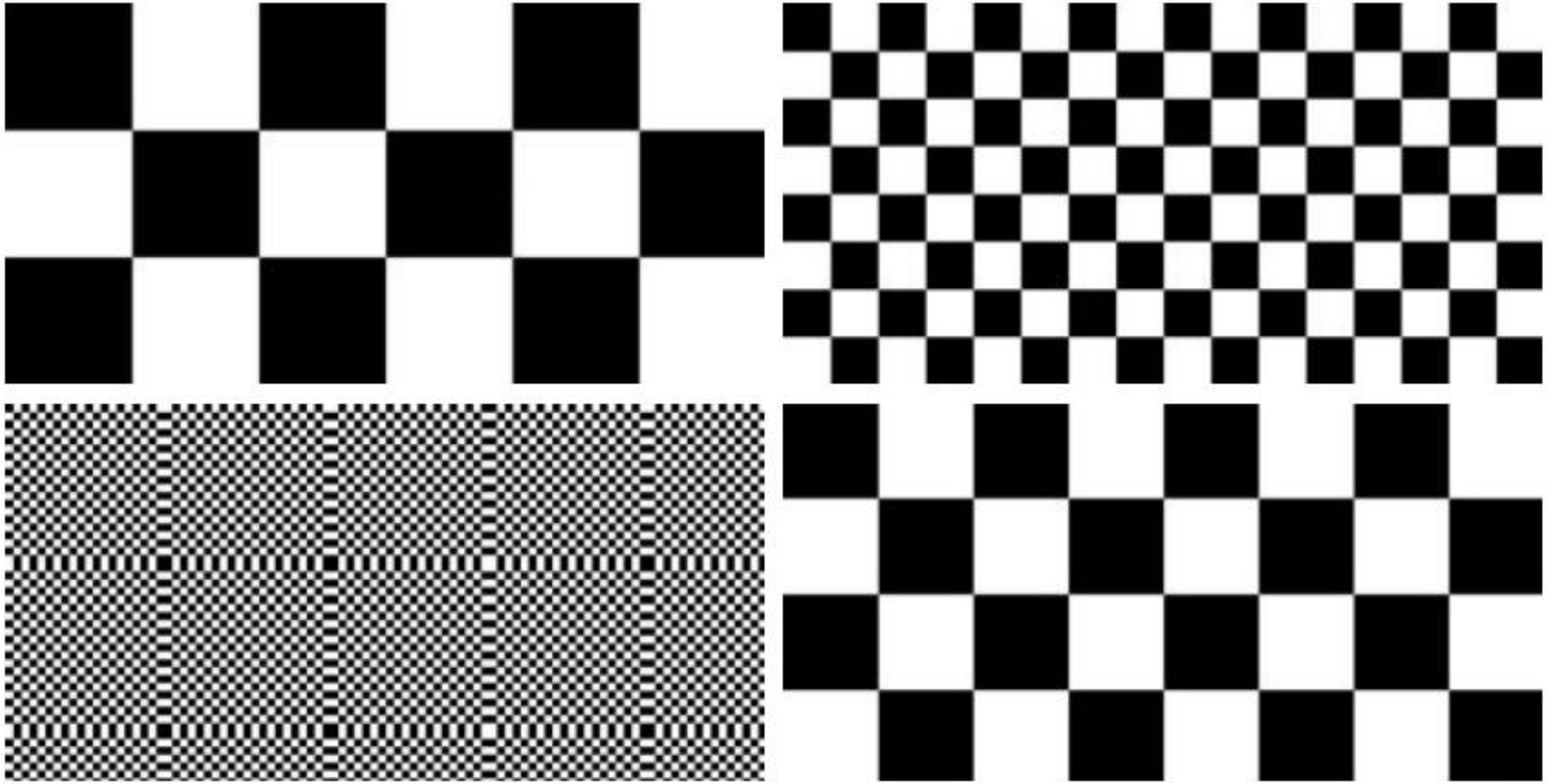
a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

16

96x96 pixel grid

96x96 squares with each square of size 1x1



a b
c d

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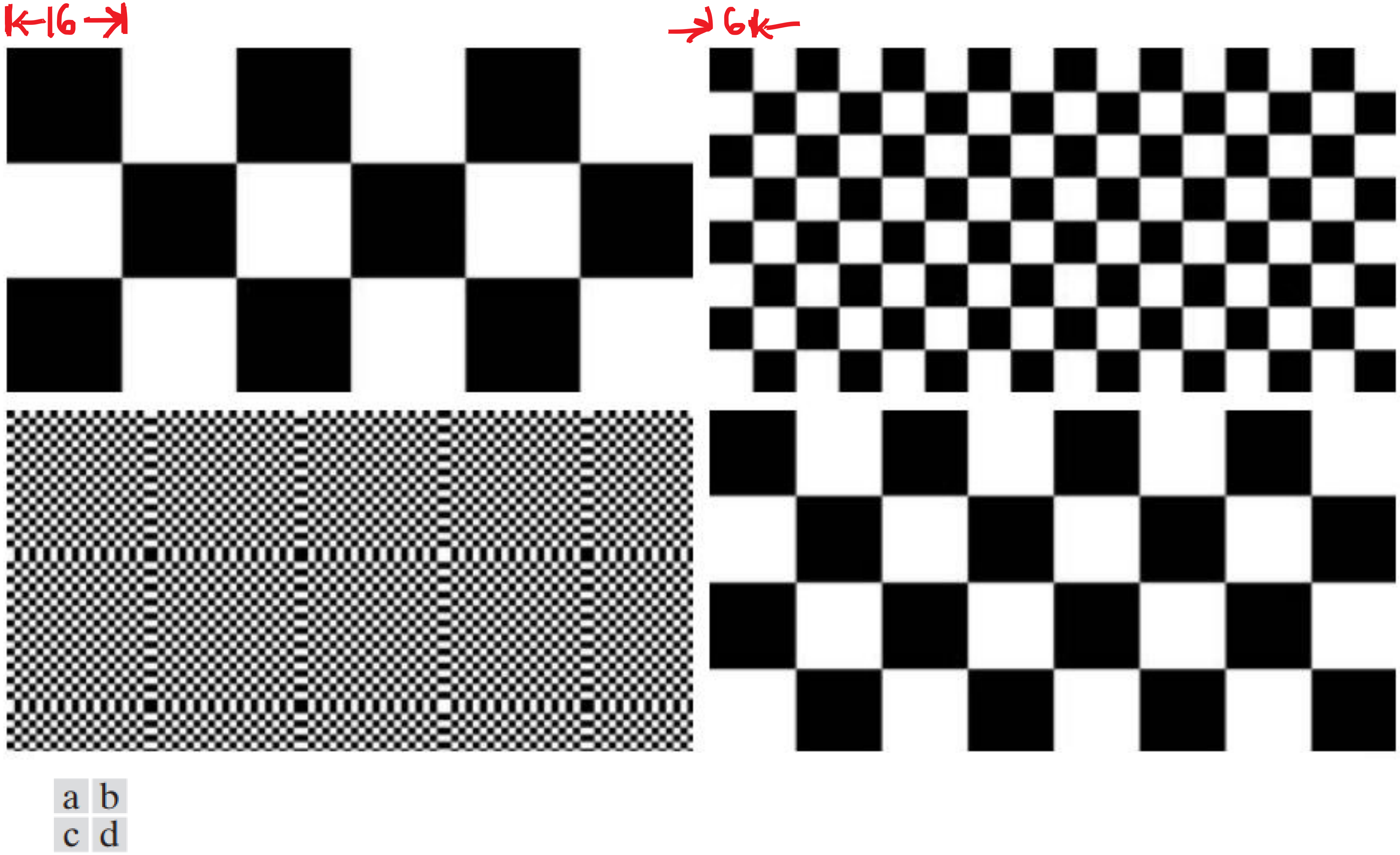
96x96 pixel grid

96x96 squares with each square of size 1x1



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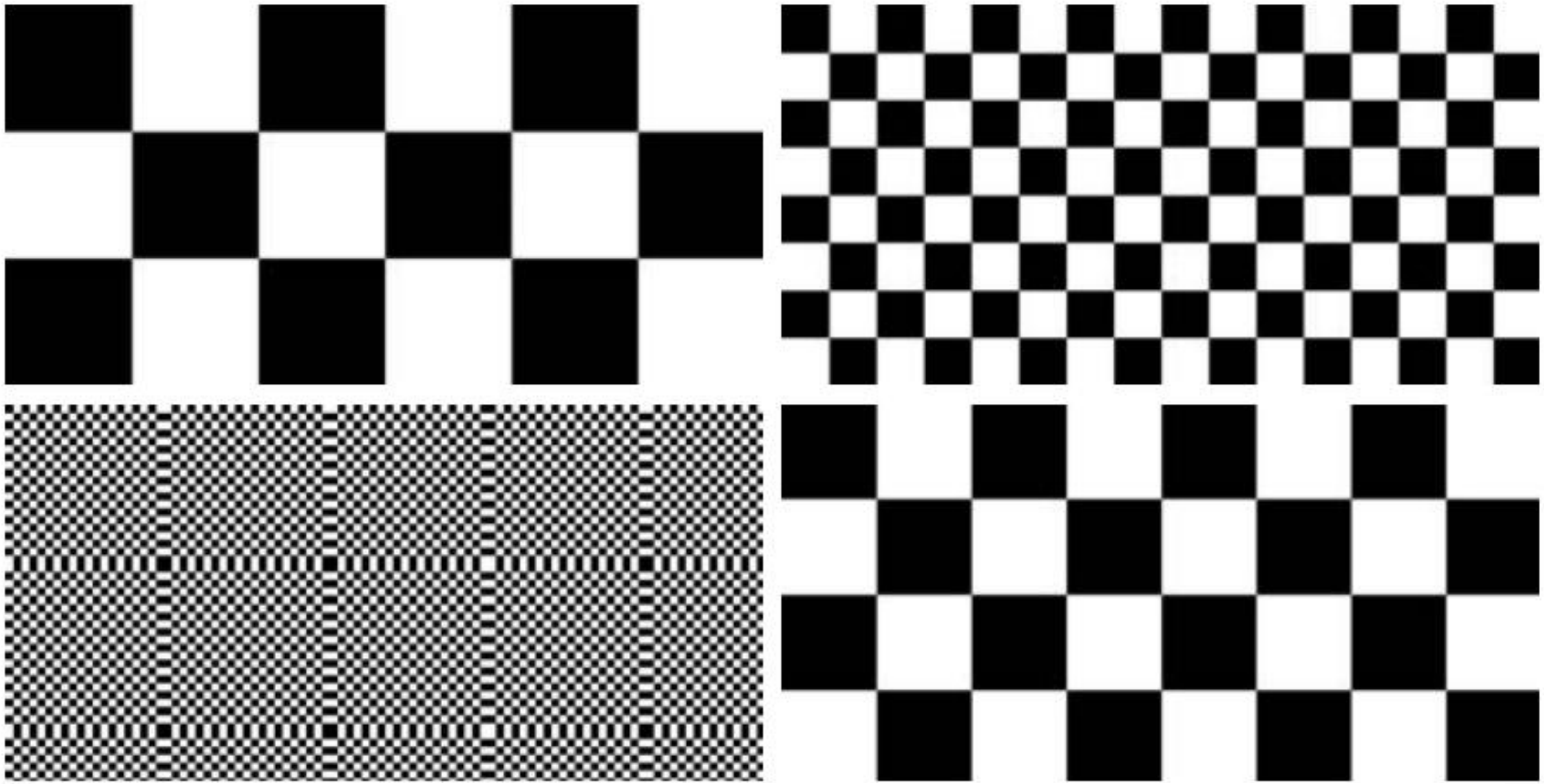
16

6

96x96 pixel grid

96x96 squares with each square of size 1x1

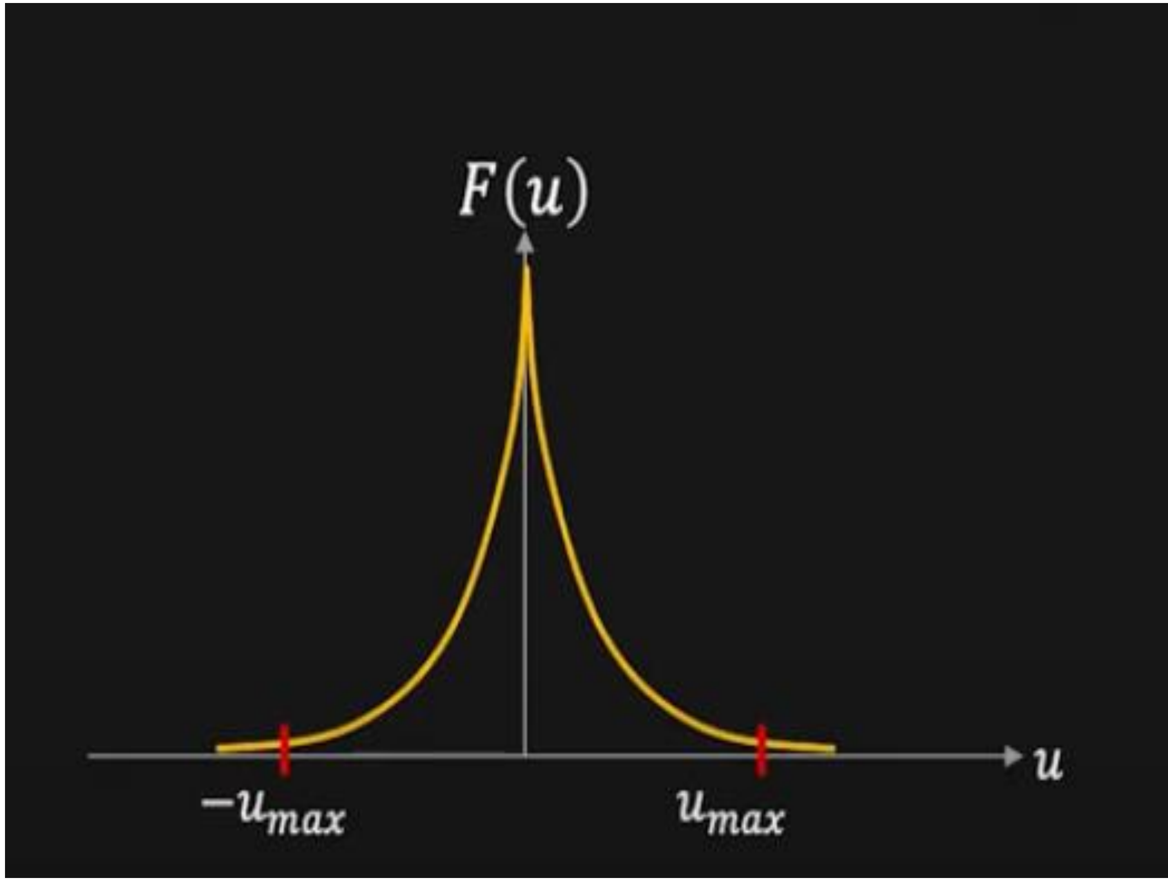


a b
c d

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Anti-aliasing in Images

- The effects of aliasing can be *reduced* by slightly defocusing the scene to be digitized so that high frequencies are attenuated.
- It is to be done at the “front-end,” *before* the image is sampled.
- Digital Cameras have anti-aliasing filtering built in , either in the lens or on the surface of the sensor itself.
- Difficult to illustrate aliasing using images obtained with such cameras.
- For re-sampling reduce artifacts (aliasing) – blurring.

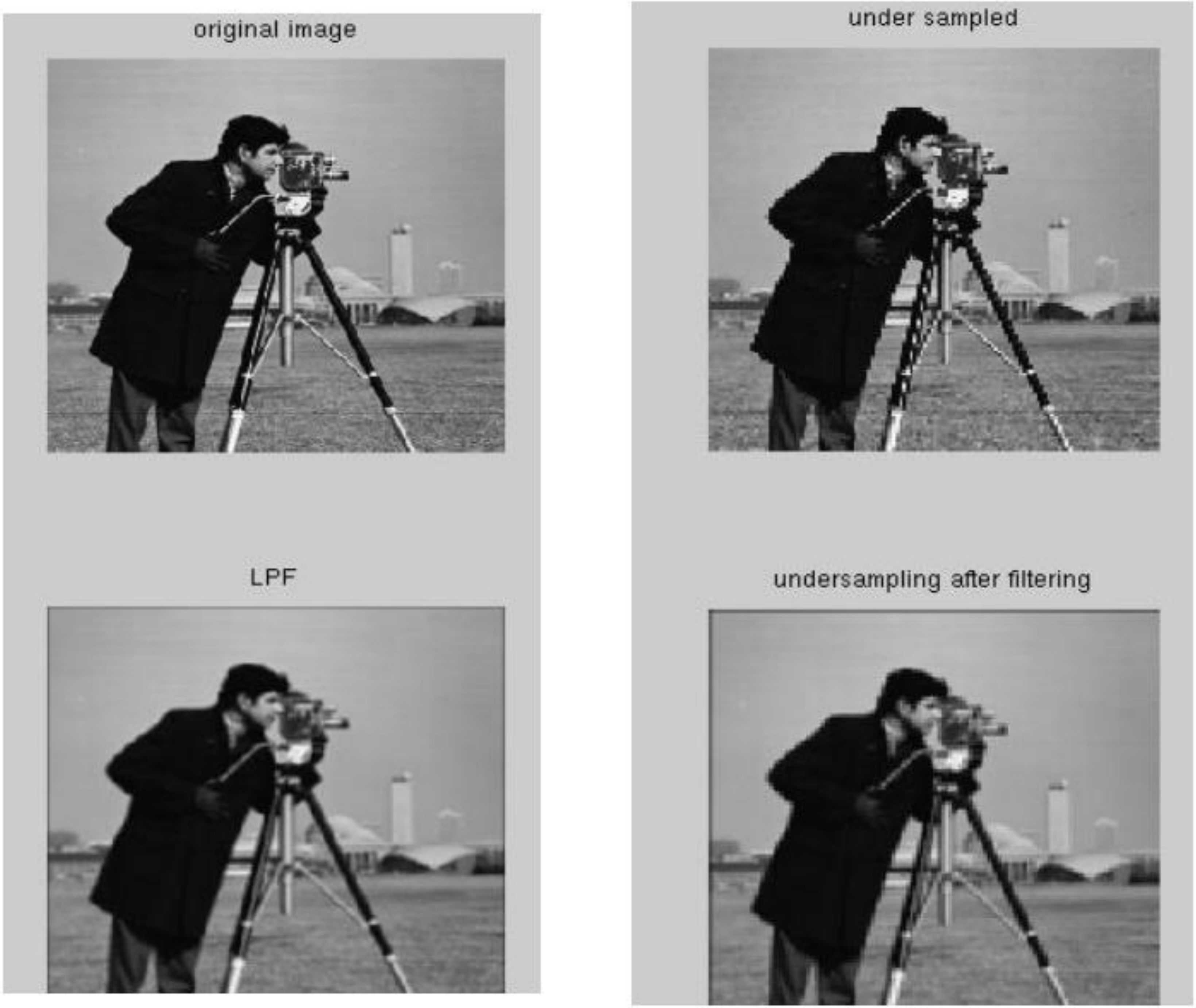


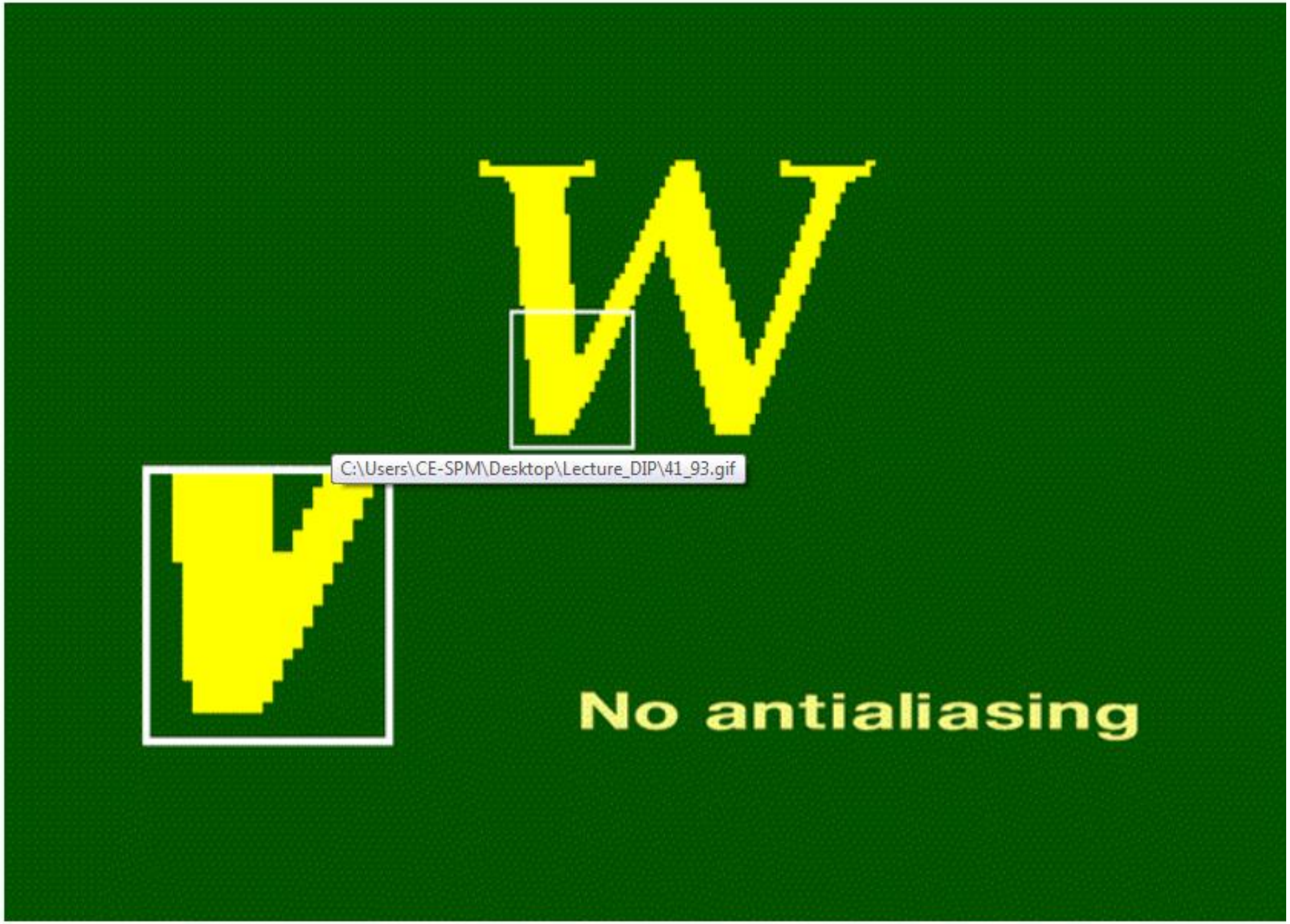
Typical Fourier Spectrum of natural Scenes

Aliasing artifacts “Moire Patterns”

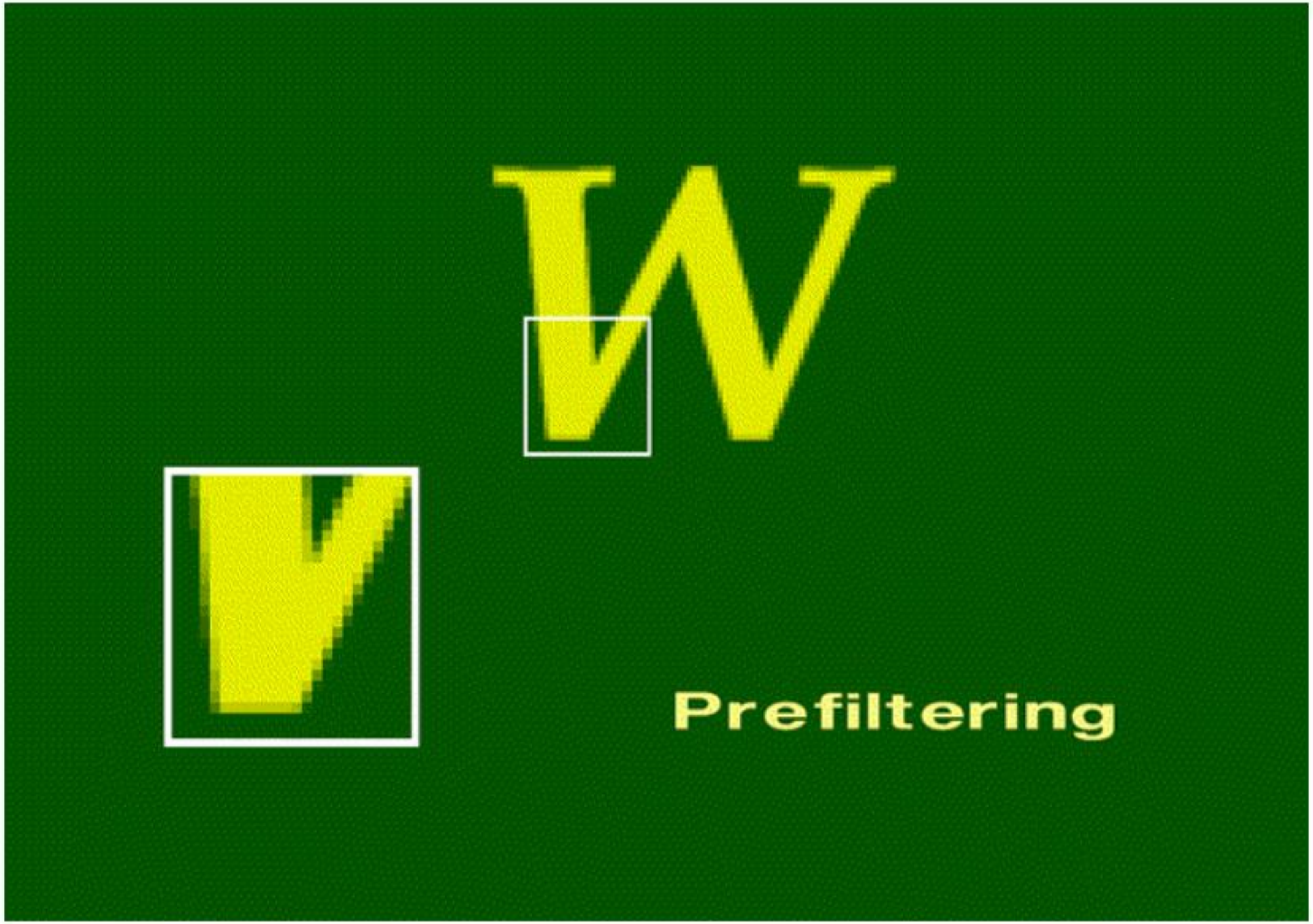
Interpolation and Re-Sampling

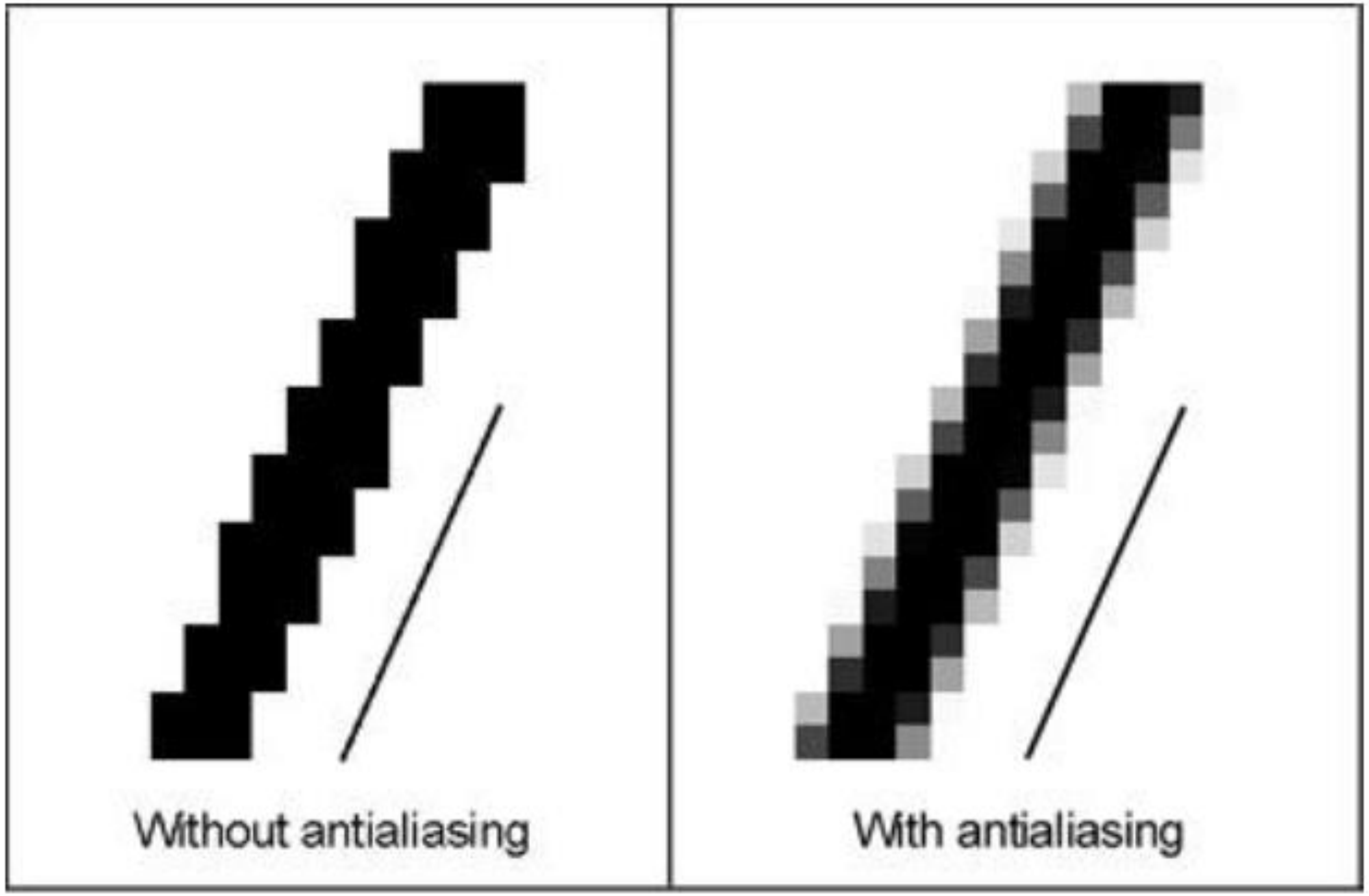
- Zooming-over sampling and shrinking-under sampling
- Nearest neighbor, bilinear, bicubic
- Zoom by factor -2 using pixel replication nearest neighbor
- Image shrinking- row and column deletion
- Blurring before resampling


original image
under sampled
LPF
undersampling after filtering



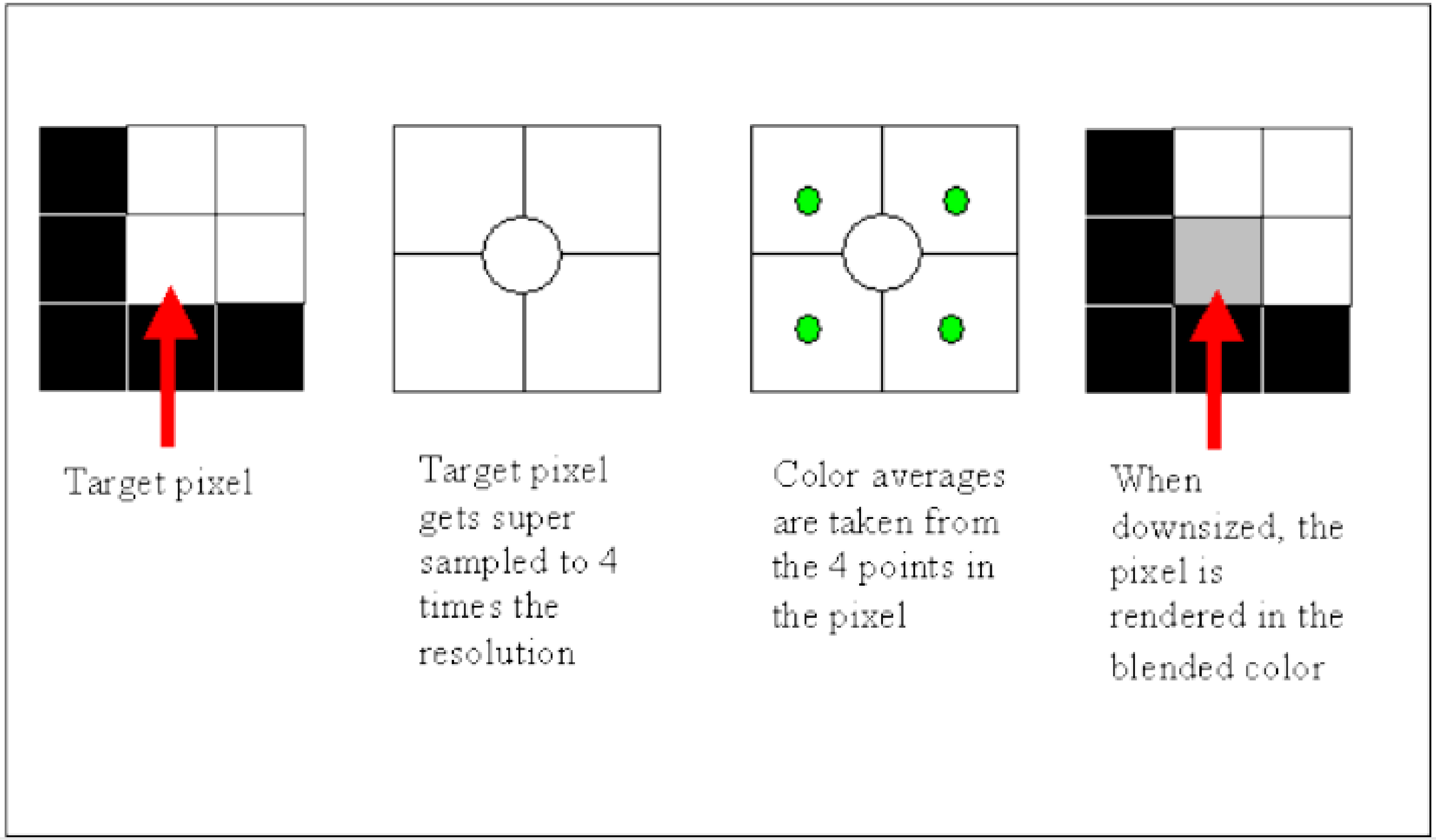
C:\Users\CE-SPM\Desktop\Lecture_DIP\41_93.gif
No antialiasing


W
Prefiltering


Without antialiasing
With antialiasing

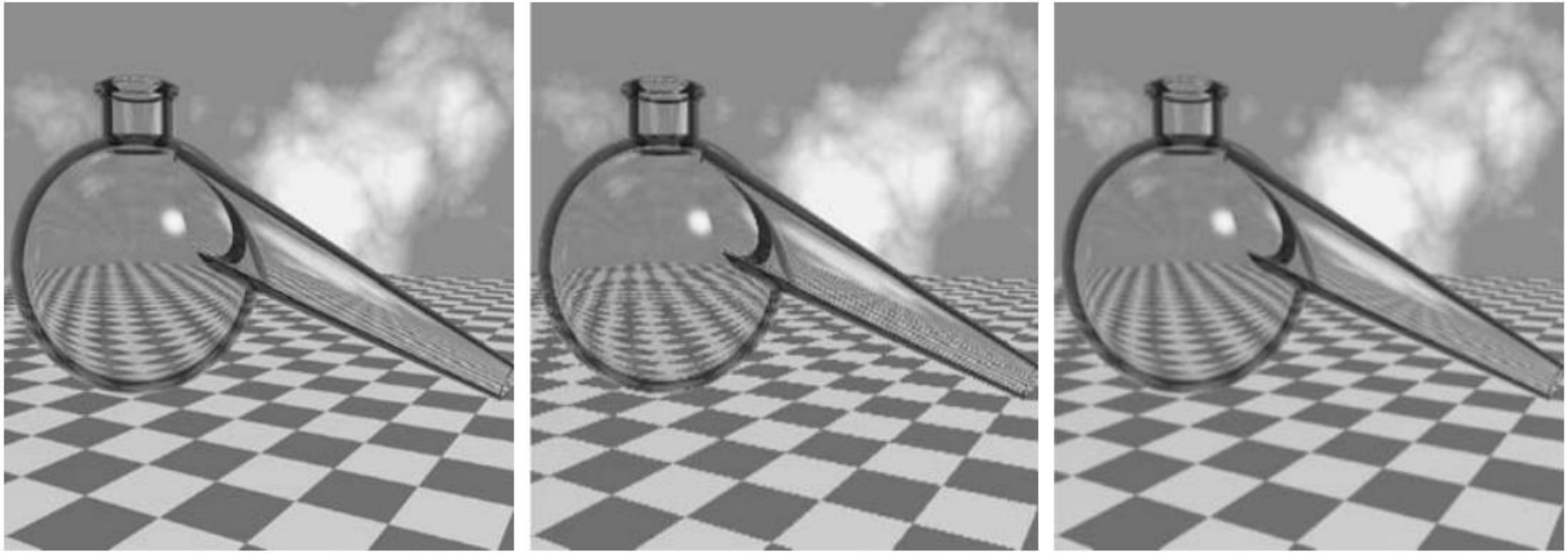
Super sample:

- Alternate method to reduce aliasing



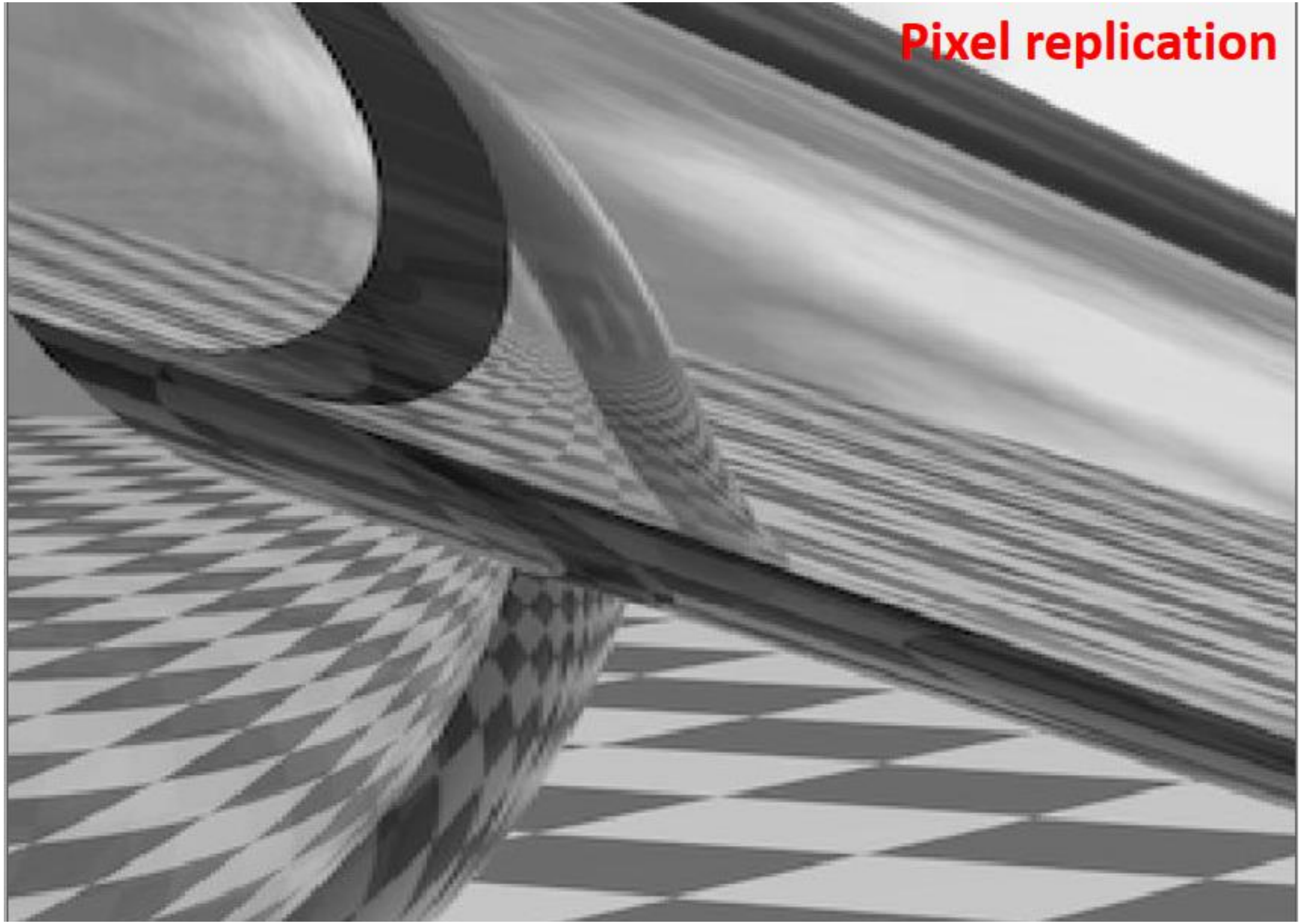
a b c

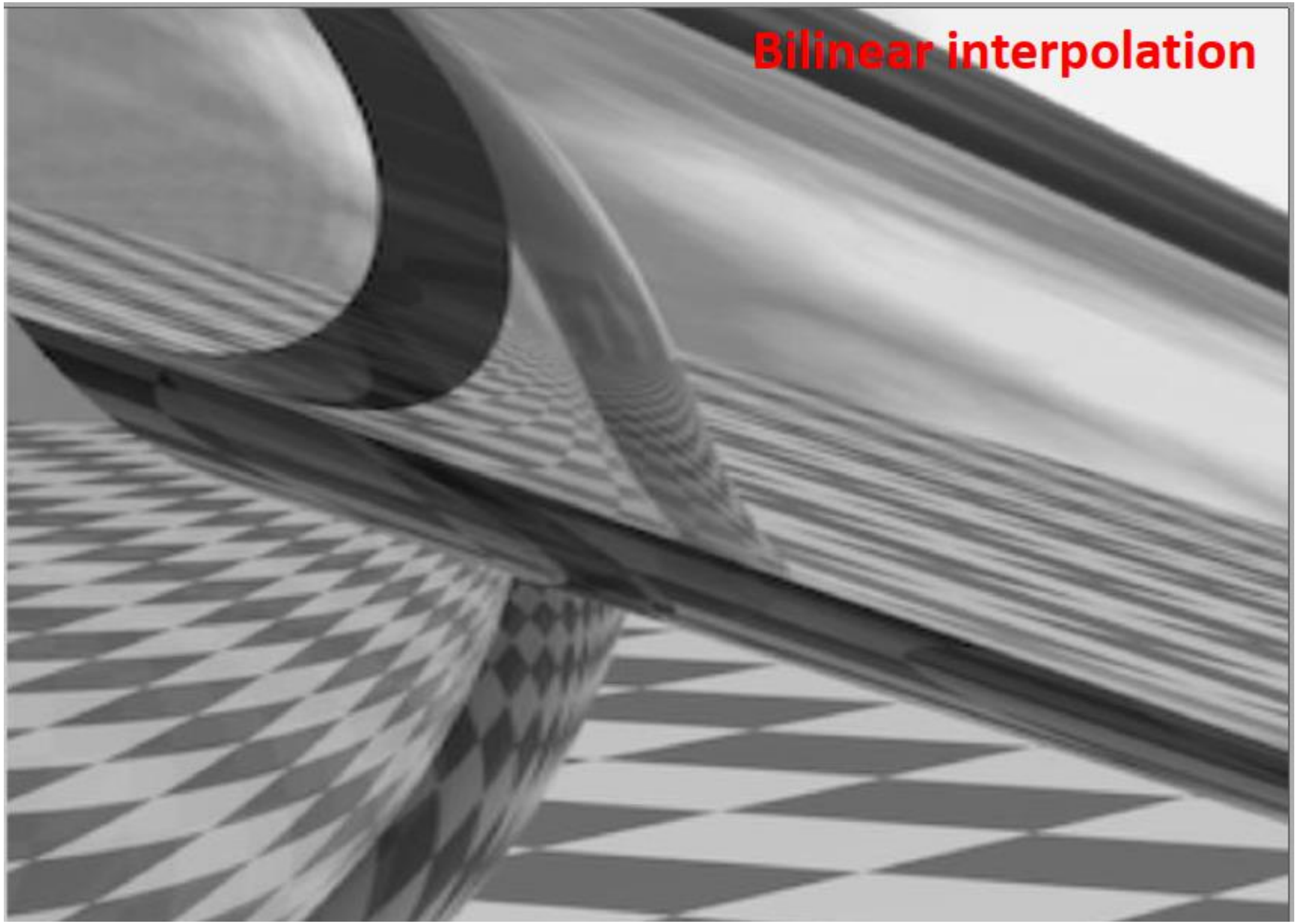
FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)



a b c

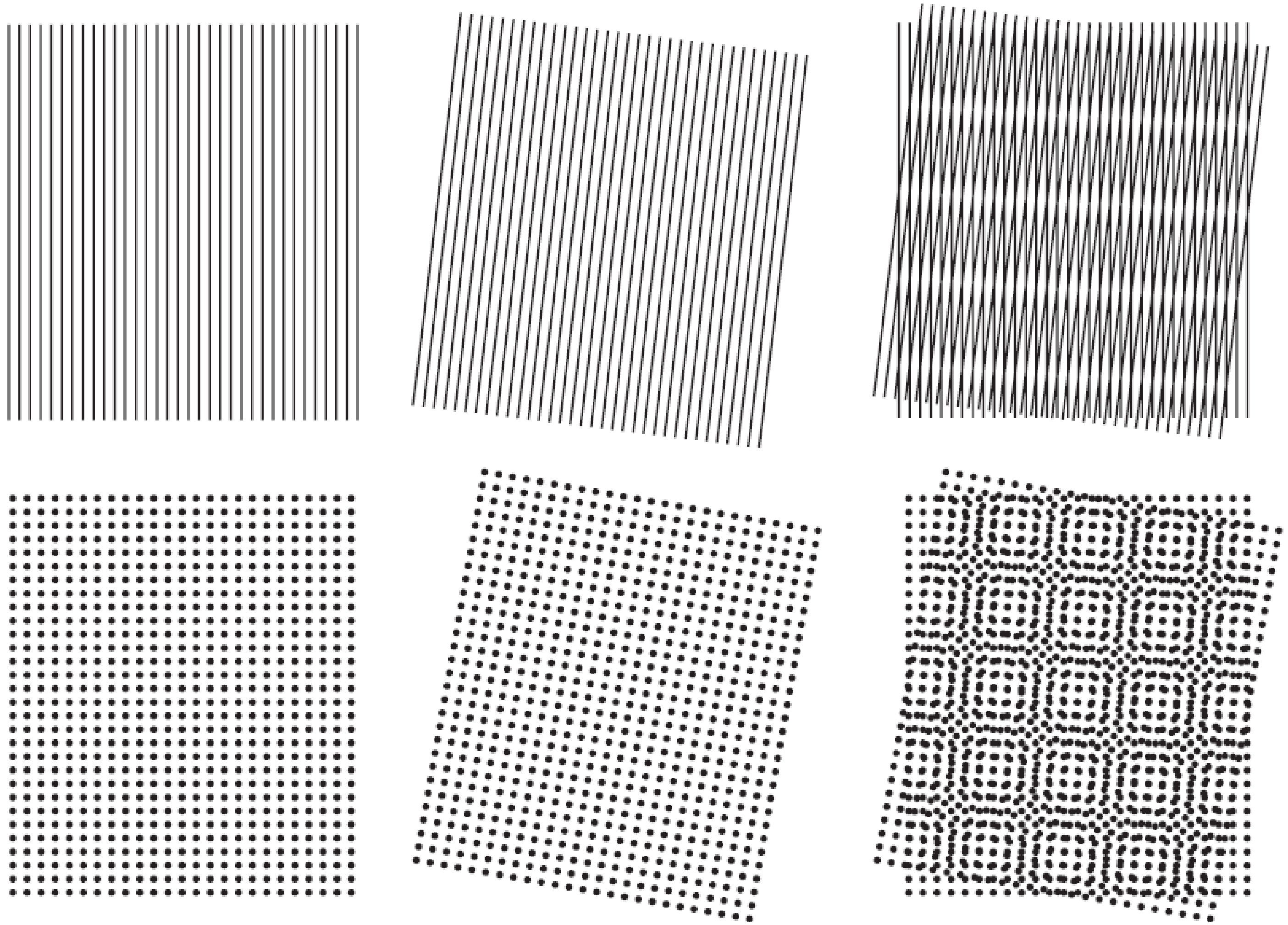
FIGURE 4.18 Illustration of jaggies. (a) A 1024×1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5×5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)


Pixel replication


Bilinear interpolation

Moirè Patterns

- In optics, moiré patterns refer to beat patterns produced between two gratings of approximately equal spacing.



a b c
d e f

FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

Moirè Patterns

- Moirè Patterns are generated from sampling scenes with periodic or nearly periodic components whose spacing is comparable to spacing between samples.
- Half Tone Images

FIGURE 4.21 A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

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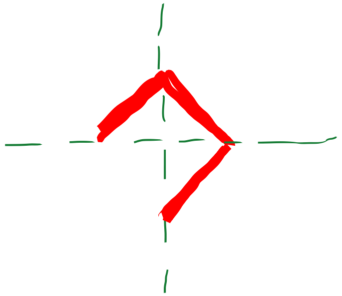

FIGURE 4.21 A newspaper image of size 246 $\times$ 168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

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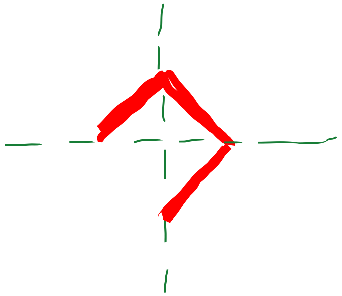

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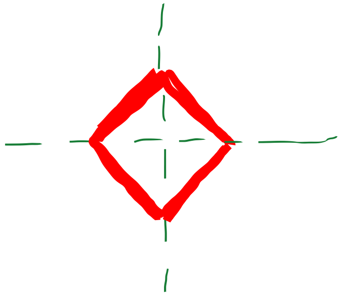

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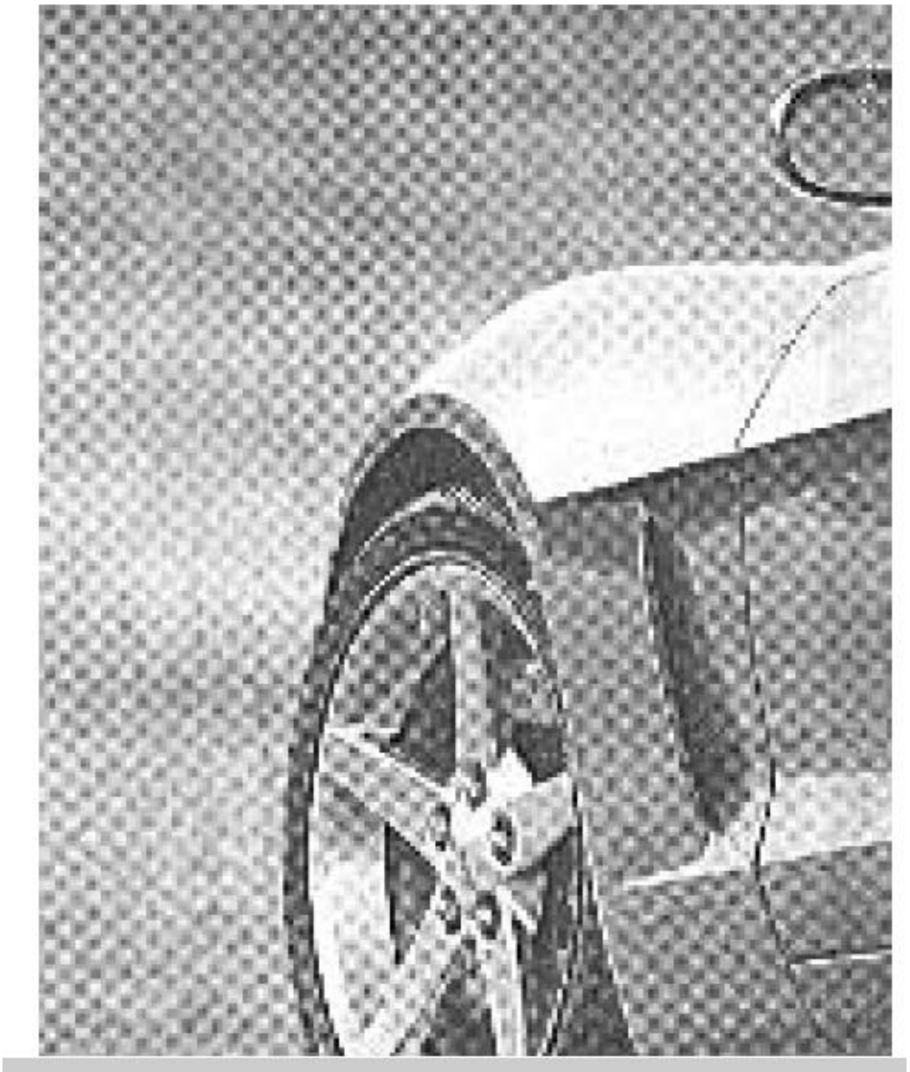

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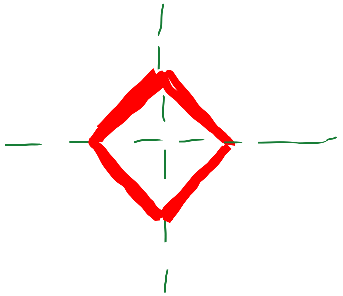


FIGURE 4.21 A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

Figure shows a newspaper image sampled at 400 dpi to avoid moirè effect

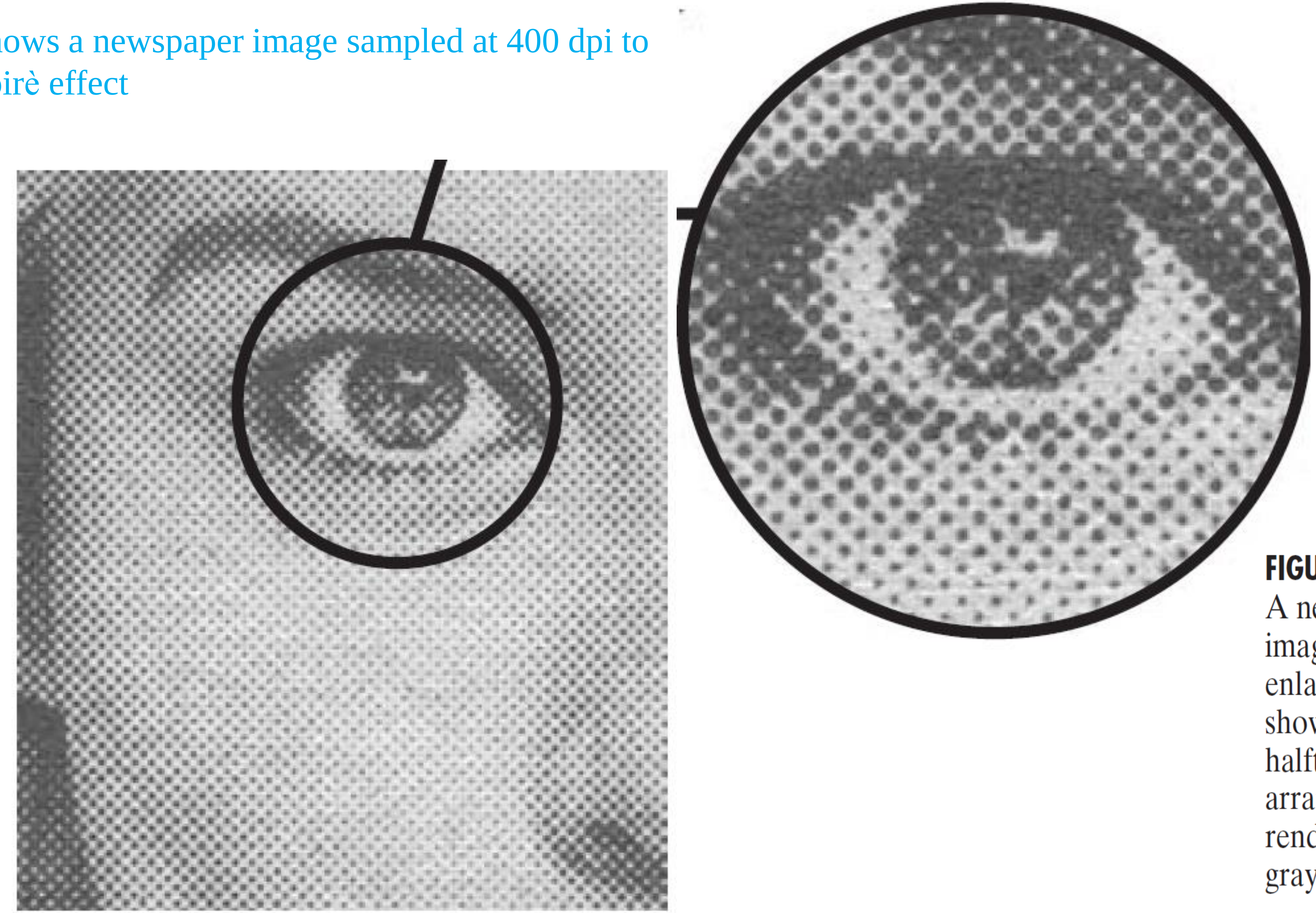
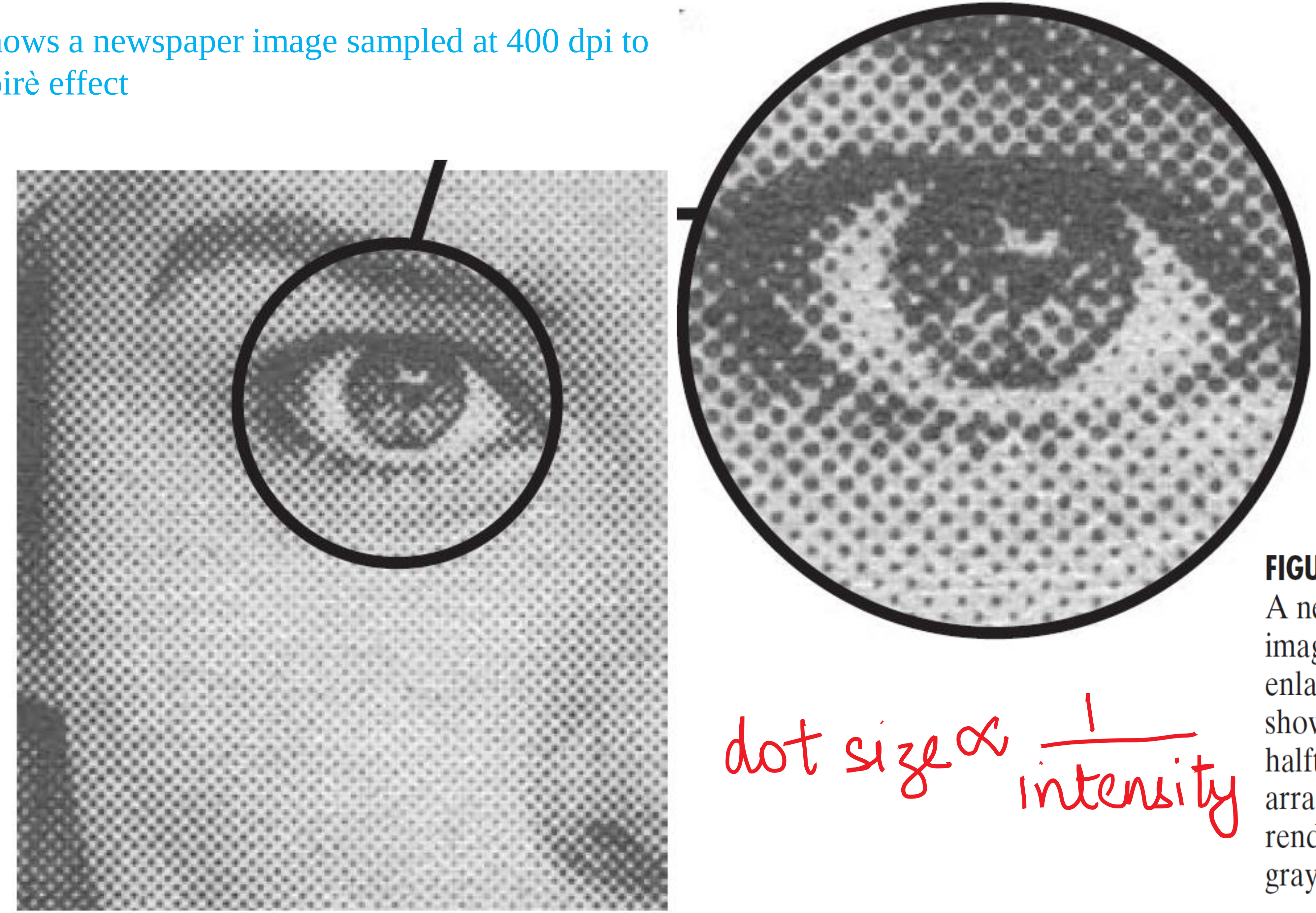


FIGURE 4.22 A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.

Figure shows a newspaper image sampled at 400 dpi to avoid moirè effect



dot size $\propto \frac{1}{\text{intensity}}$

FIGURE 4.22 A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.

Discrete Fourier Transform

- The Fourier transform of a sampled, band-limited function extending from –α to α is a *continuous*, *periodic* function that also extends from –α to α.
- In practice, we work with a finite number of samples, and the objective of this section is to derive the DFT corresponding to such sample sets.
- Discrete Fourier transform pair

DFT

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

IDFT

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{\,j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

where f_n is a set of M samples of $f(t)$

where F_m is a set of M discrete values corresponding to the discrete Fourier transform of the input sample set of $f(t)$

$$\widetilde{F}(\mu) = \int_{-\infty}^{\infty} \widetilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt$$

$$\tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)$$

$$
\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)e^{-j2\pi\mu t}\, dt
\end{aligned}
$$

$$
\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)
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&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} f_n\, e^{-j2\pi\mu n\Delta T}
\end{aligned}
$$

$$
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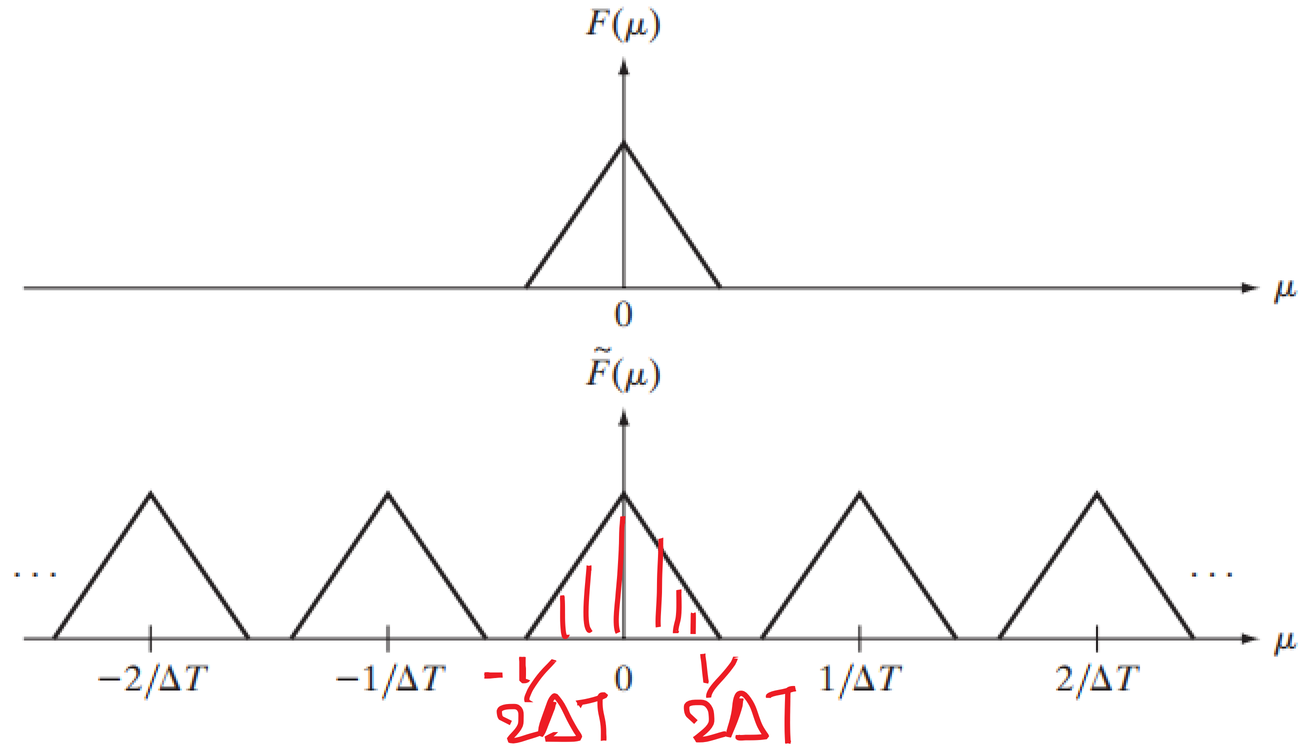
\end{aligned}
$$

$$\widetilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) \qquad f_n = f(n\Delta T)$$

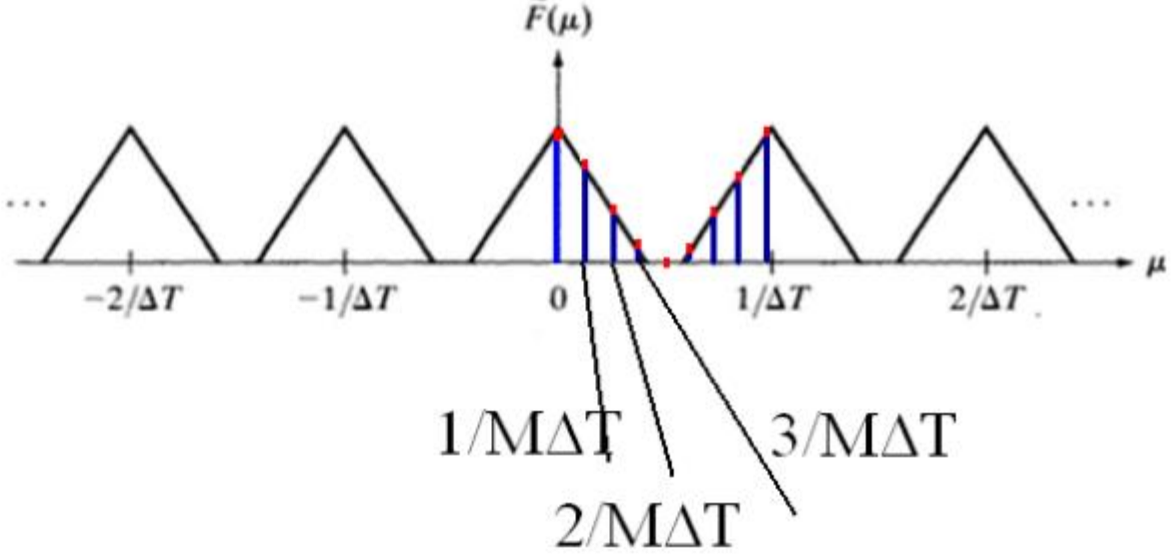
- Fourier Transform is a periodic function with period $1/\Delta T$



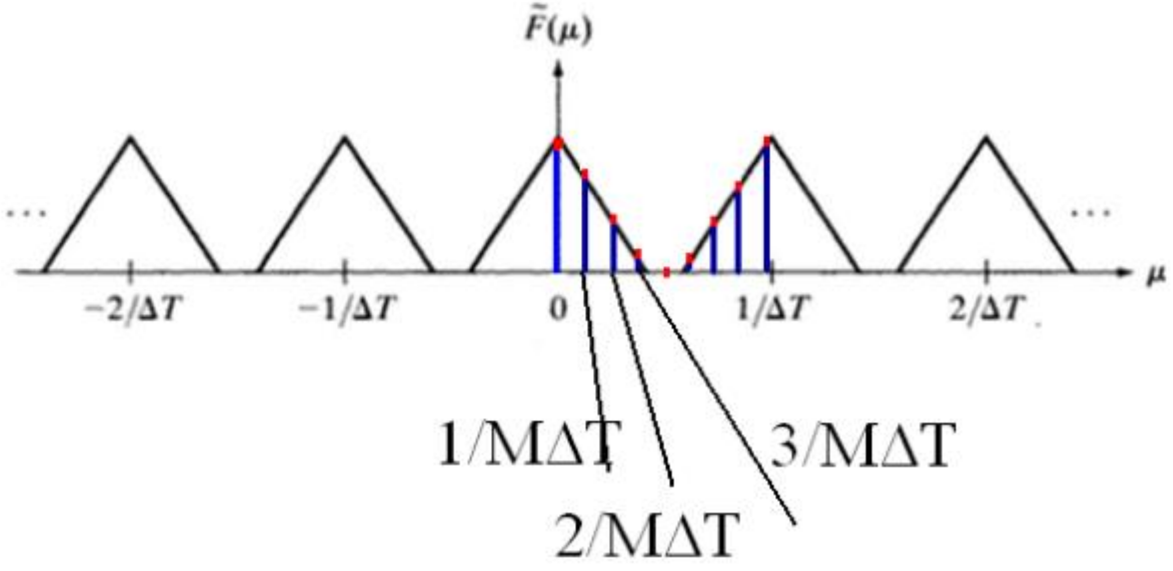
- If Fourier transform is sampled to M equally spaced samples in period $1/\Delta T$
- What is the separation between samples?

Samples	Space
M	$1/\Delta T$
1	(?)

space between two samples $=1/M\Delta T$



- Samples should be taken at frequencies



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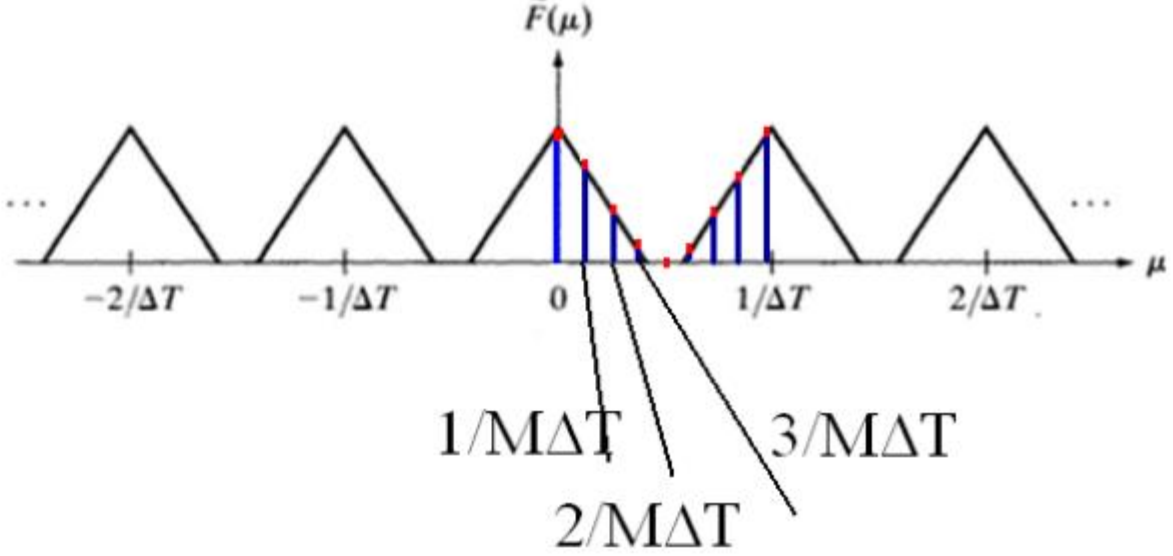
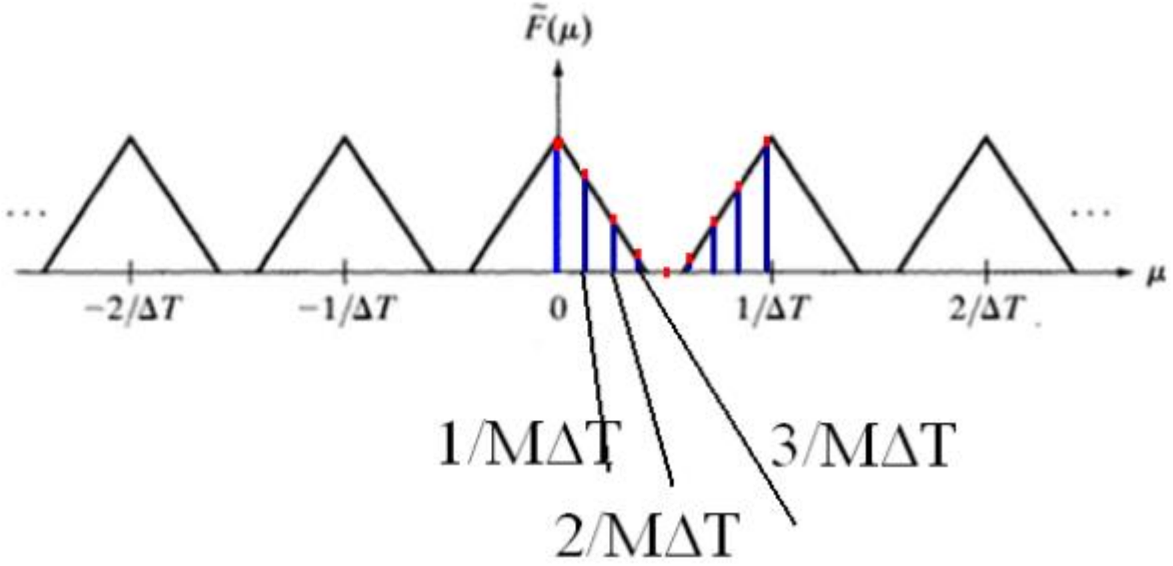


- Samples should be taken at frequencies

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$



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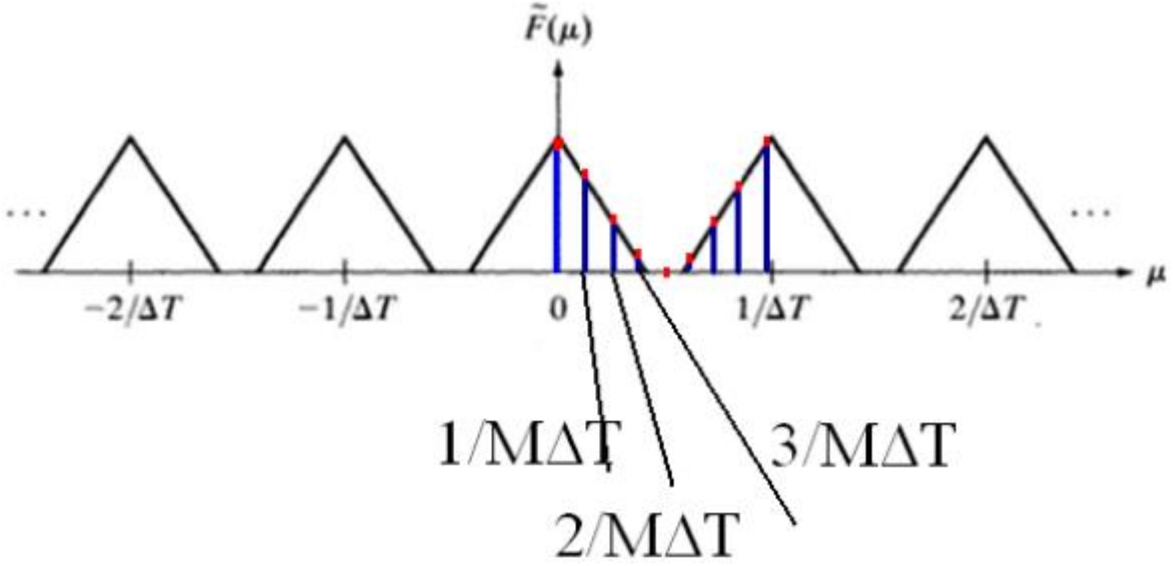
$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$



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$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

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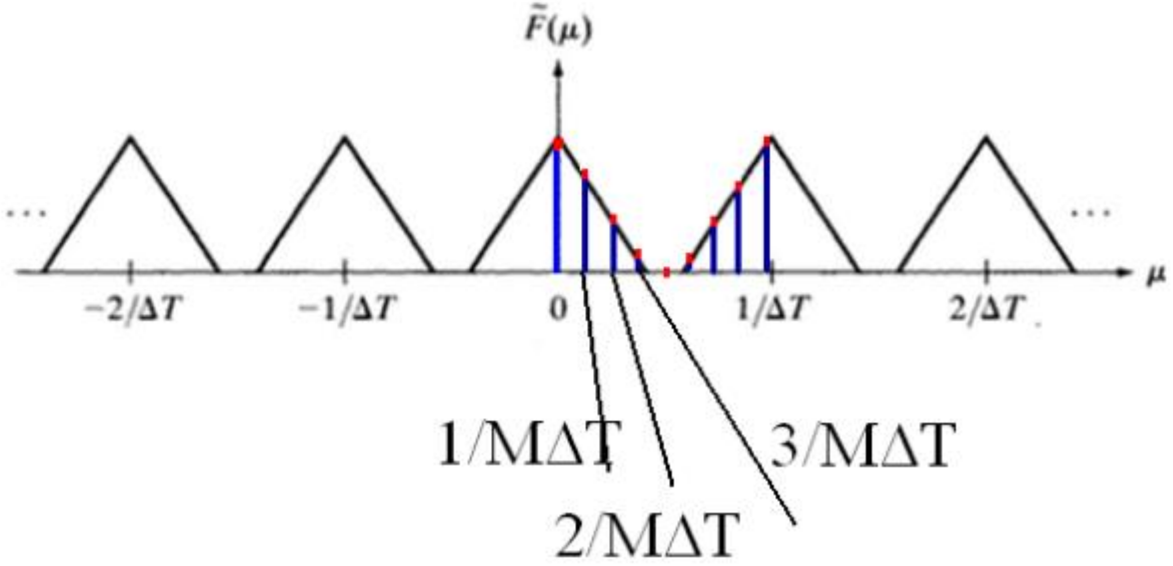

- Samples should be taken at frequencies

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1 \qquad \tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- The first expression is the discrete Fourier transform.
- Given a set $\{ f_n \}$ consisting of M samples of $f(t)$, DFT yields a sample set $\{ F_m \}$of M complex discrete values corresponding to the discrete Fourier transform of the input sample set.
- Conversely, given we can recover the sample set by using the *inverse discrete Fourier transform* (IDFT).

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{\,j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

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- Using different notations

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

Sample $f(t)$

$t = n\Delta T$

$$F(m) = \sum_{n=0}^{M-1} f(n)\, e^{-j2\pi mn/M}$$

$$\mu = \frac{m}{M\Delta T}$$

$t = n\Delta T$

$$F(u) = \sum_{x=0}^{M-1} f(x)\, e^{-j2\pi ux/M}$$

Both DFT and IDFT are infinitely periodic with period M

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

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For this reason, the process inherent in this equation often is referred to as circular convolution, and is a direct result of the periodicity of the DFT and its inverse.

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Relationship between the Sampling and Frequency Intervals

- If $f(x)$ consists of M samples of a function $f(t)$ taken ΔT units apart, the duration of the record comprising the set is $\{ f(x) \}$, $x = 0, 1, 2, \ldots, M-1$, is

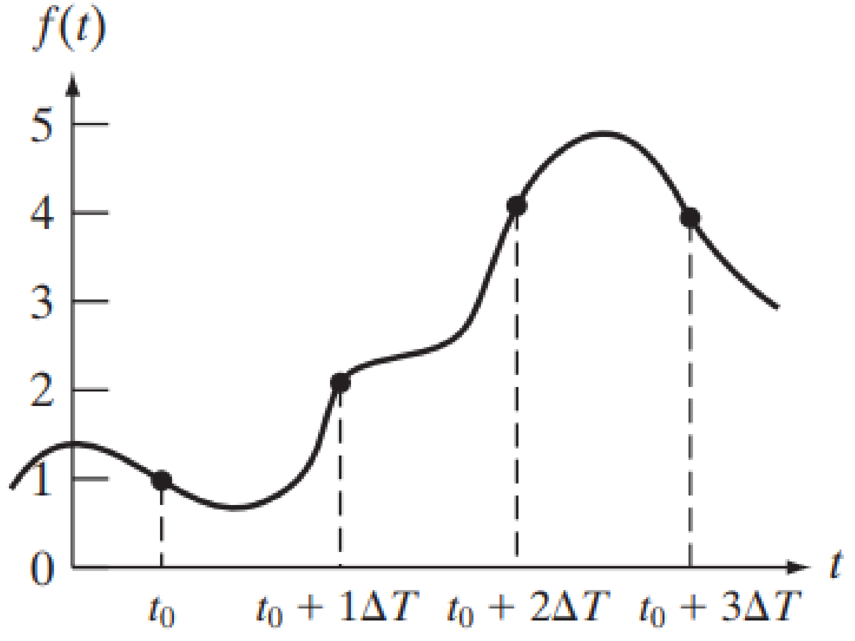
$$T = M\Delta T$$

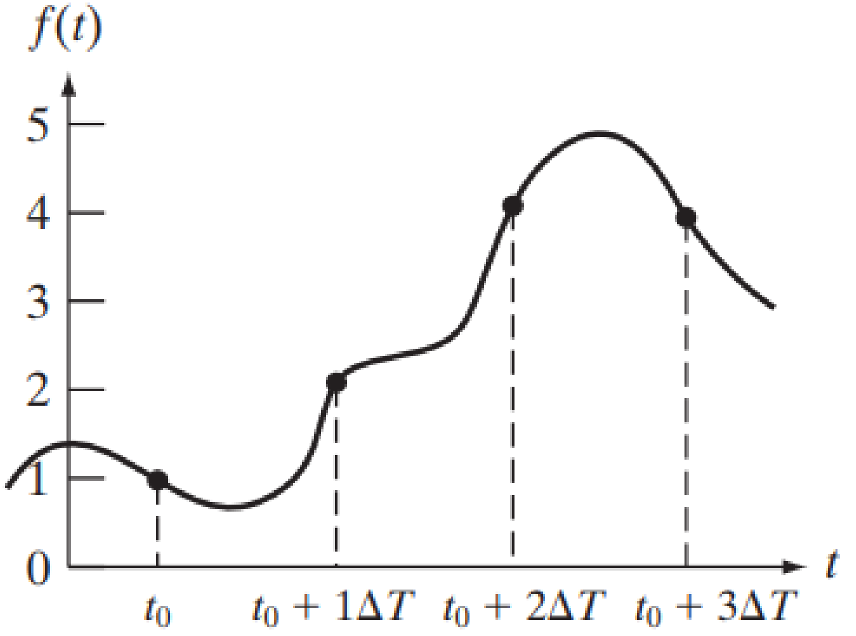
- The corresponding spacing, in the discrete frequency domain follows from

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

- The entire frequency range spanned by the M components of the DFT is

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$


f(t)
5
4
3
2
1
0
t
t0
t0 + 1ΔT
t0 + 2ΔT
t0 + 3ΔT



f(t)
5
4
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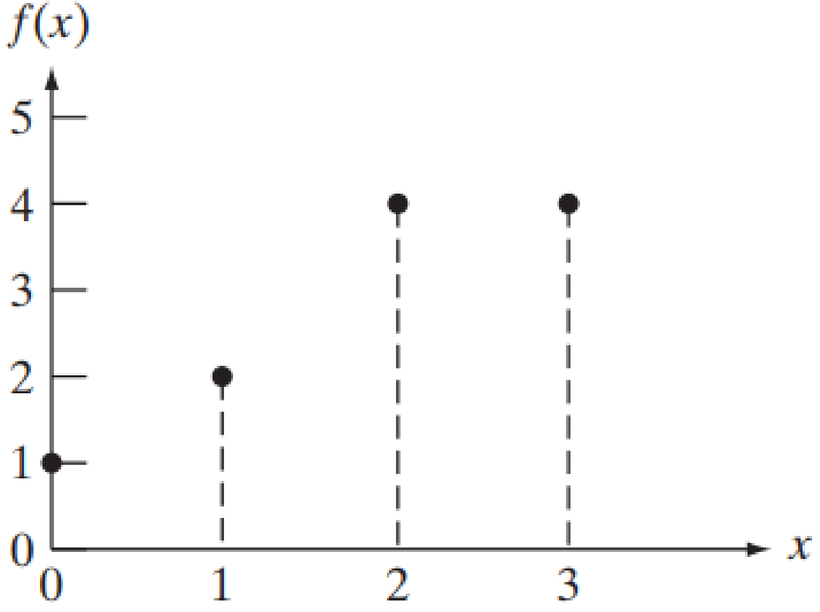

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5
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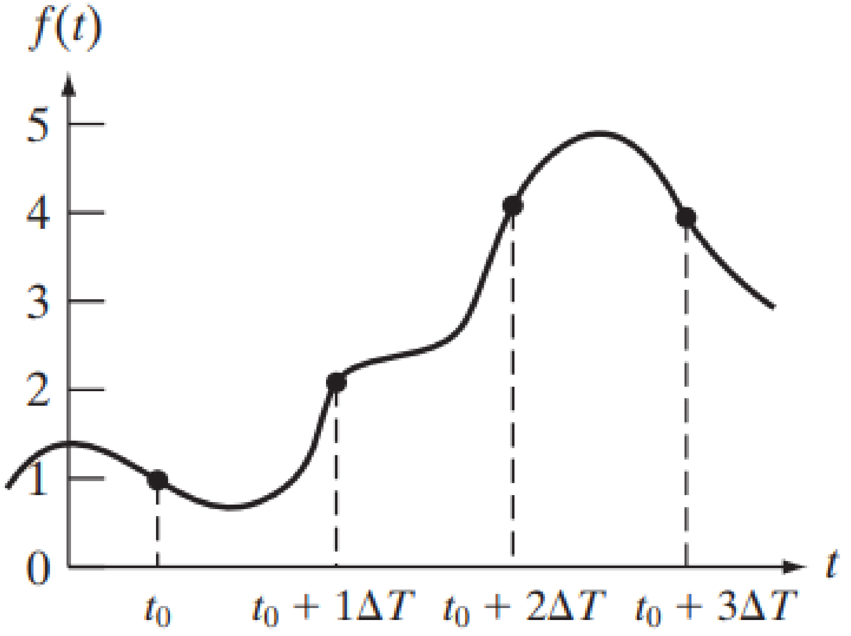

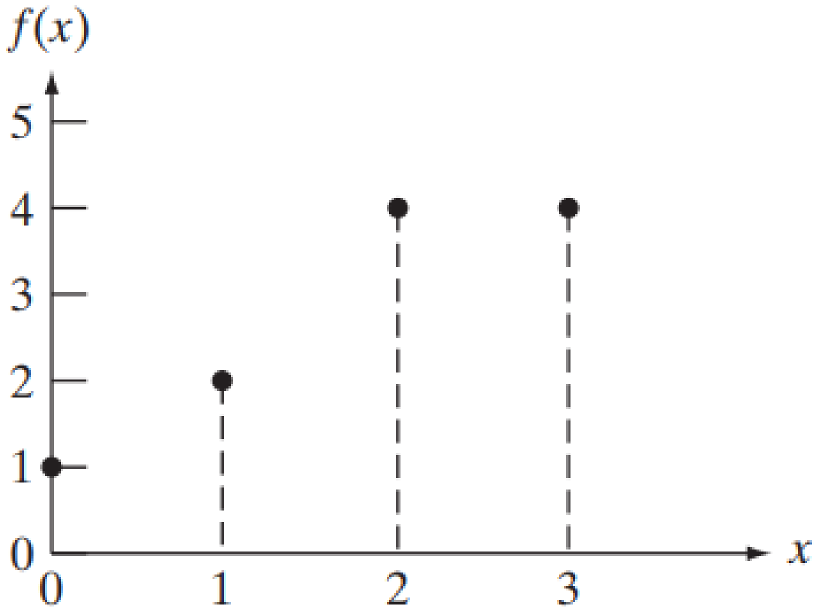

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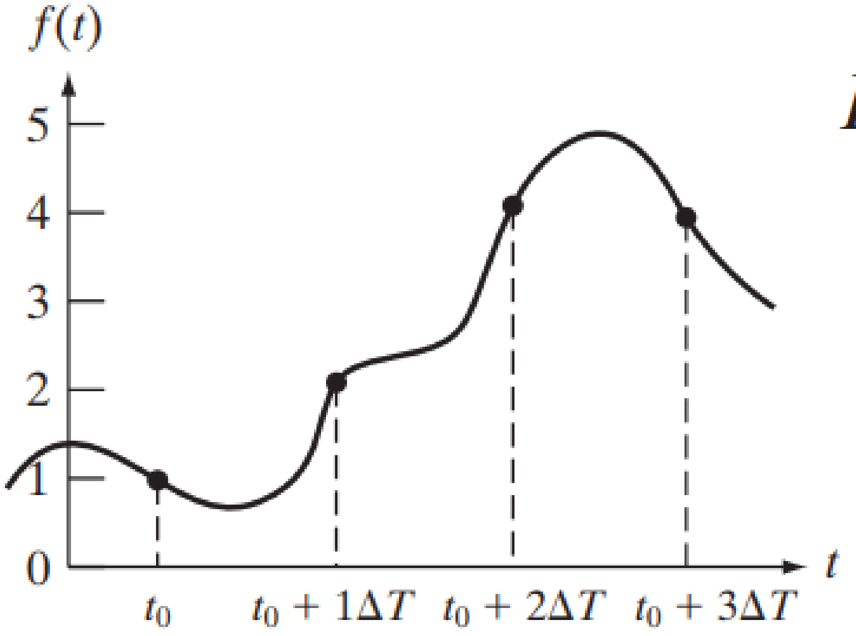

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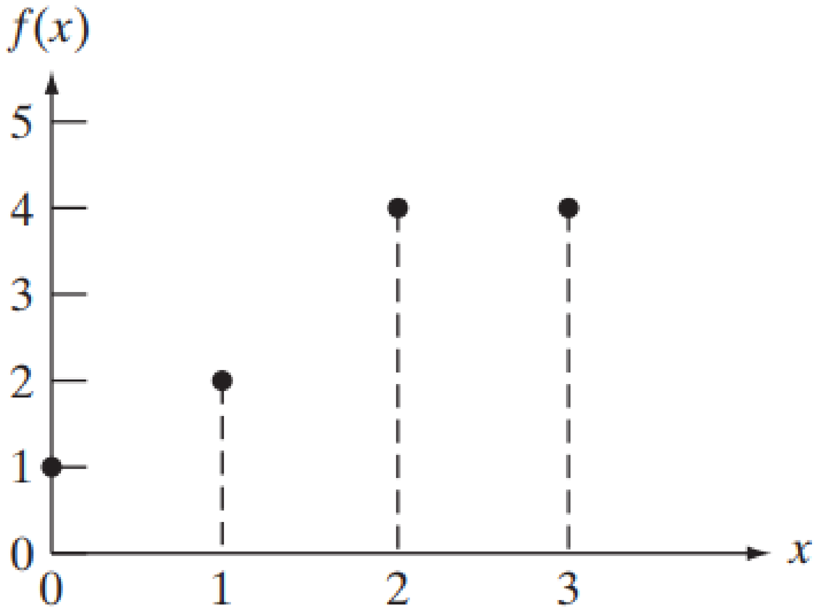

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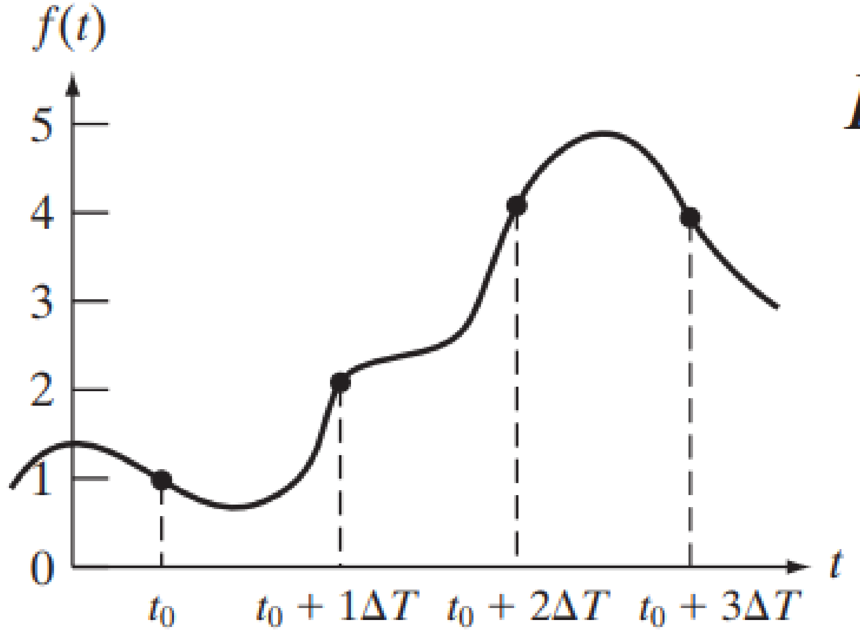

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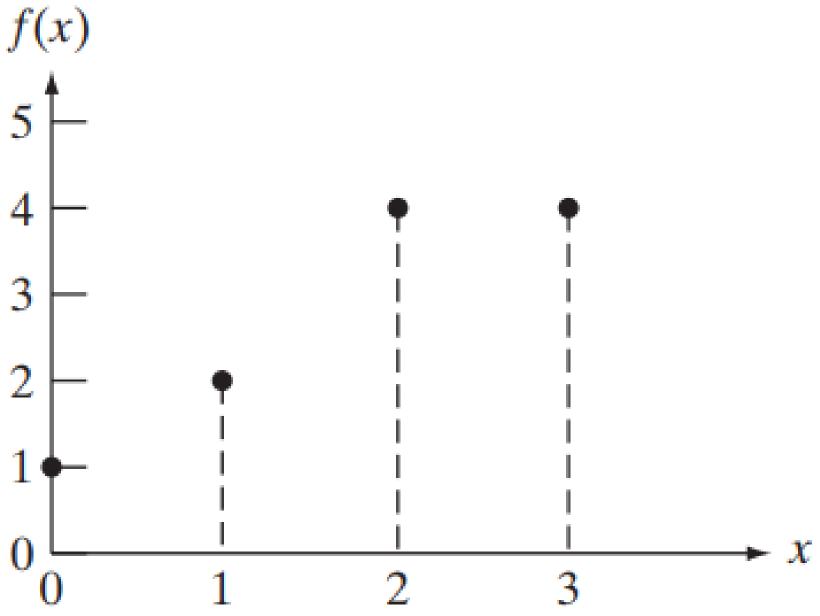

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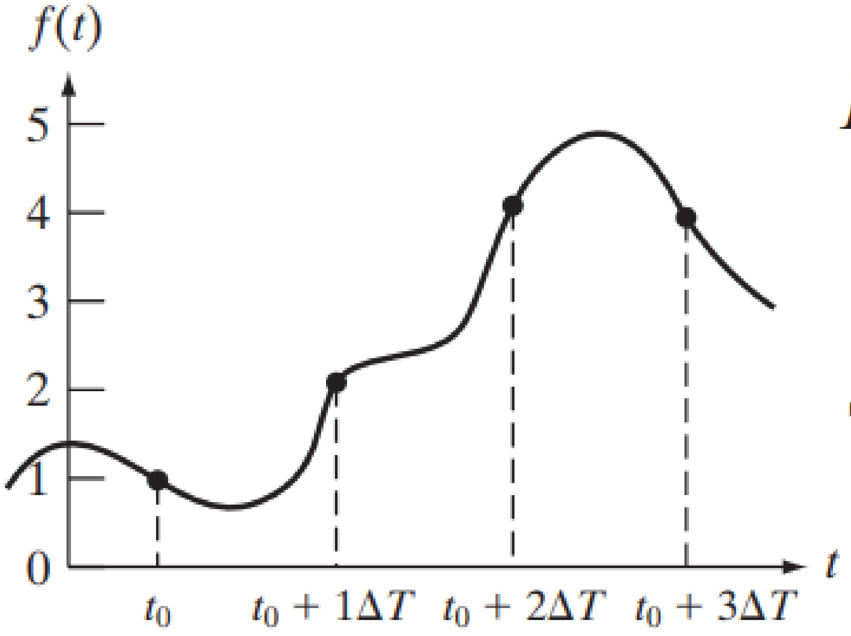

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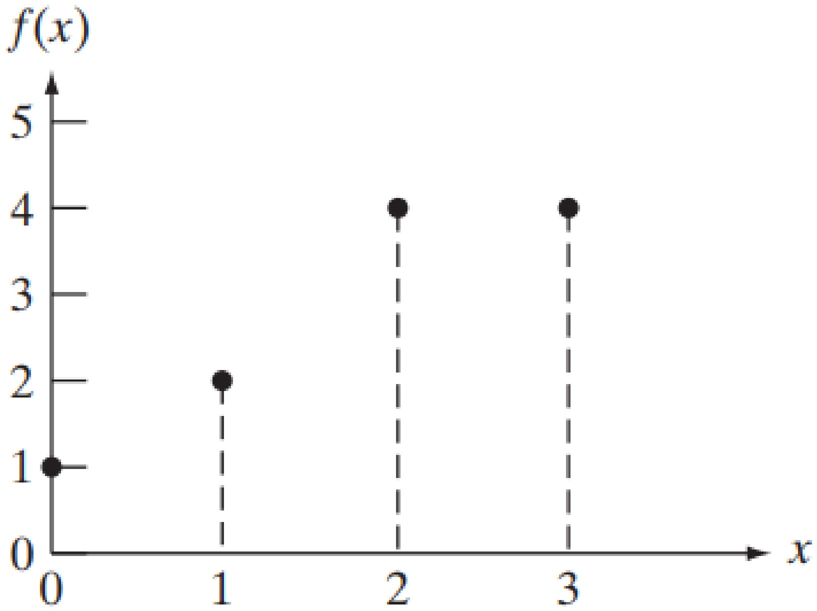

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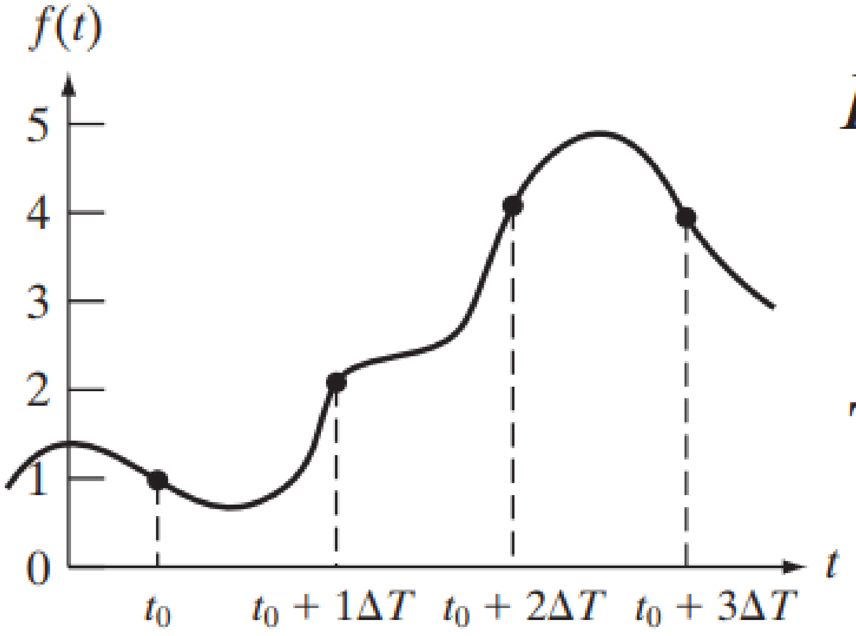

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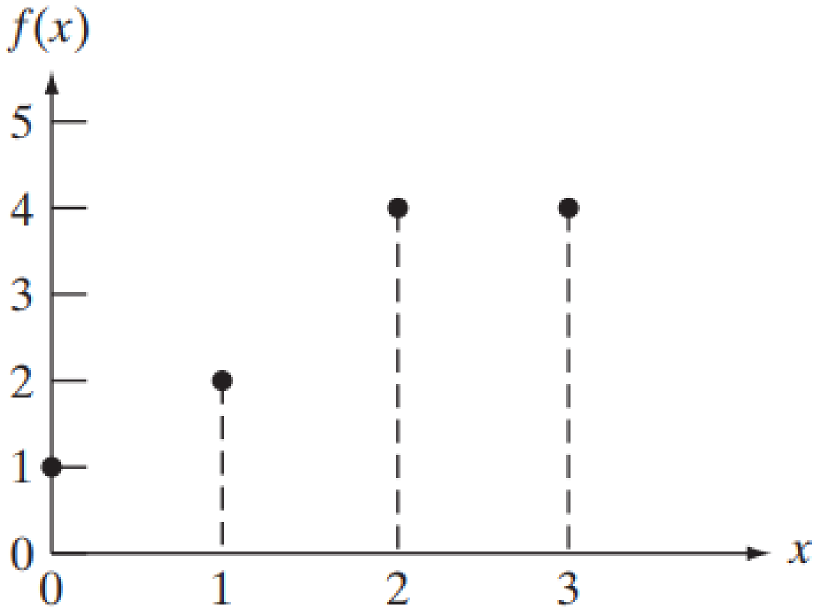

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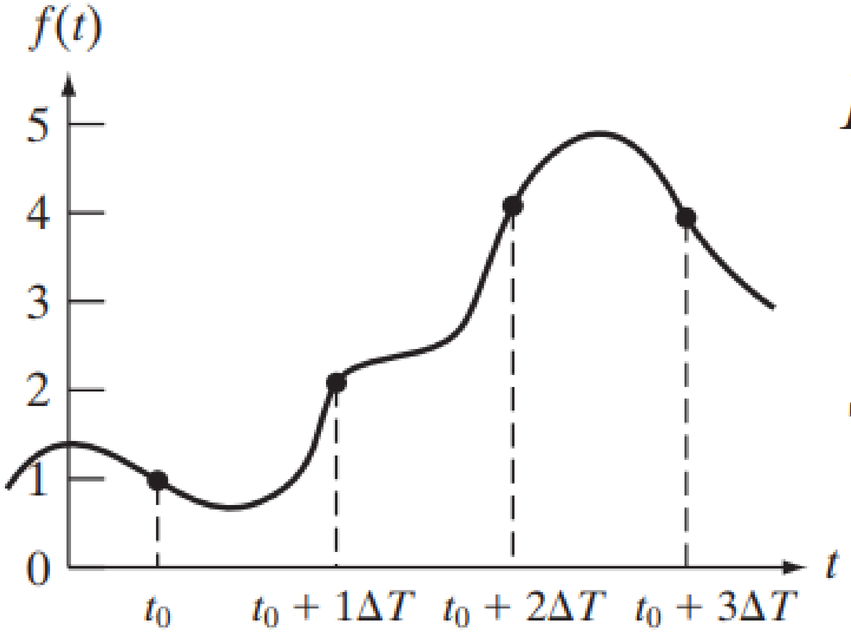

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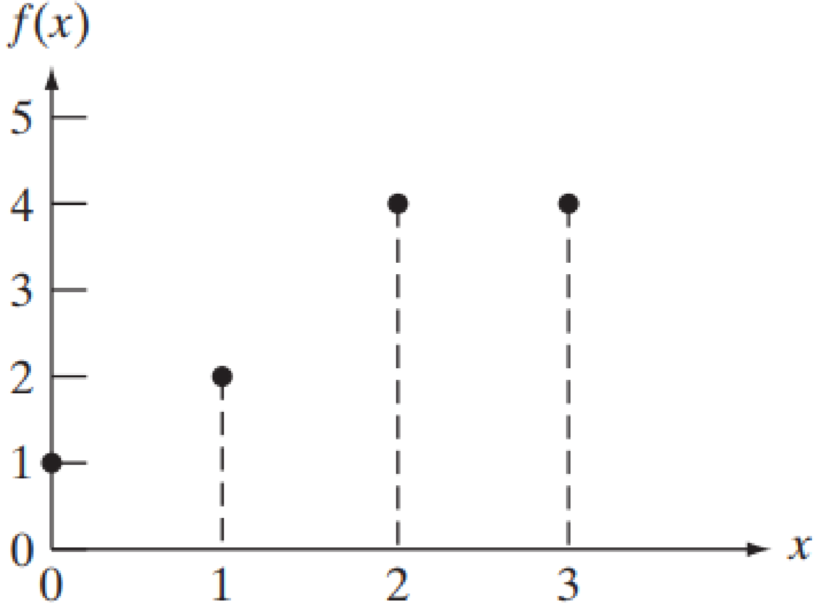

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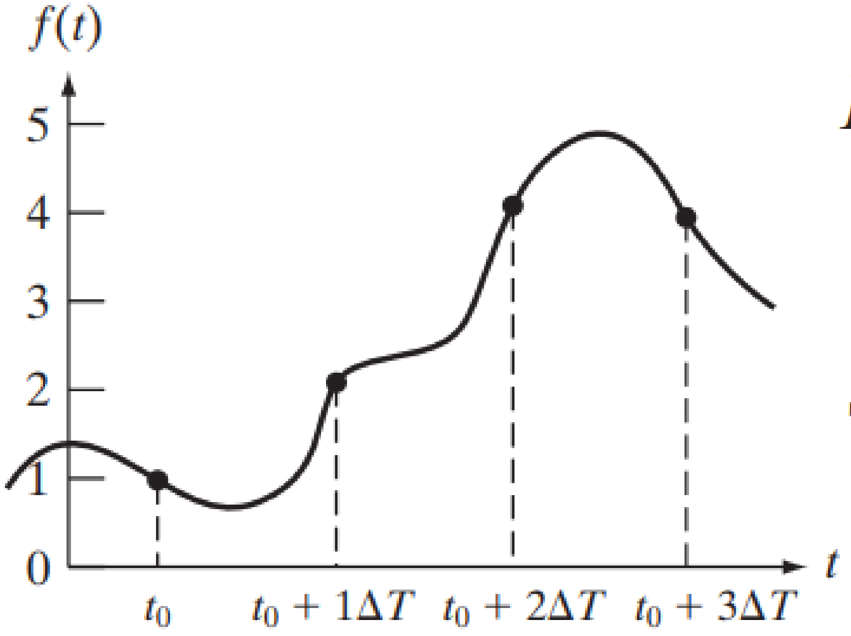

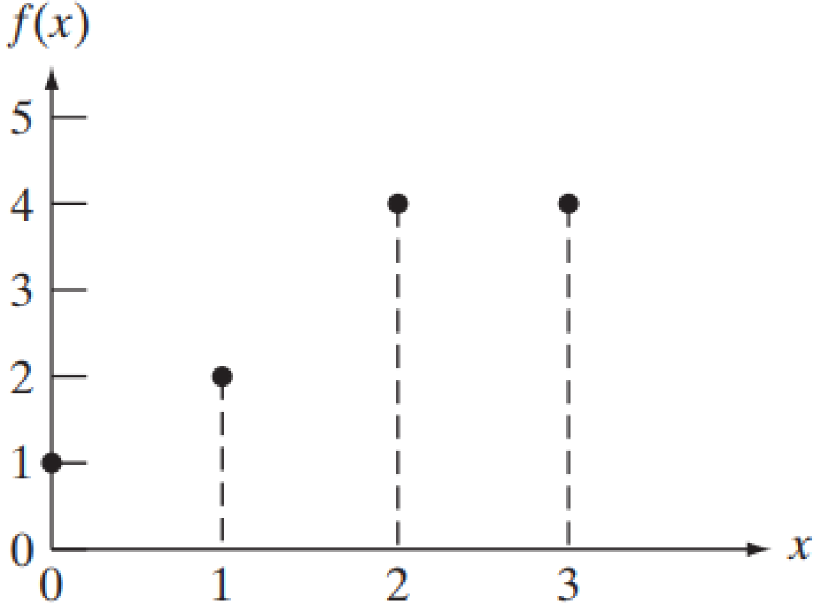

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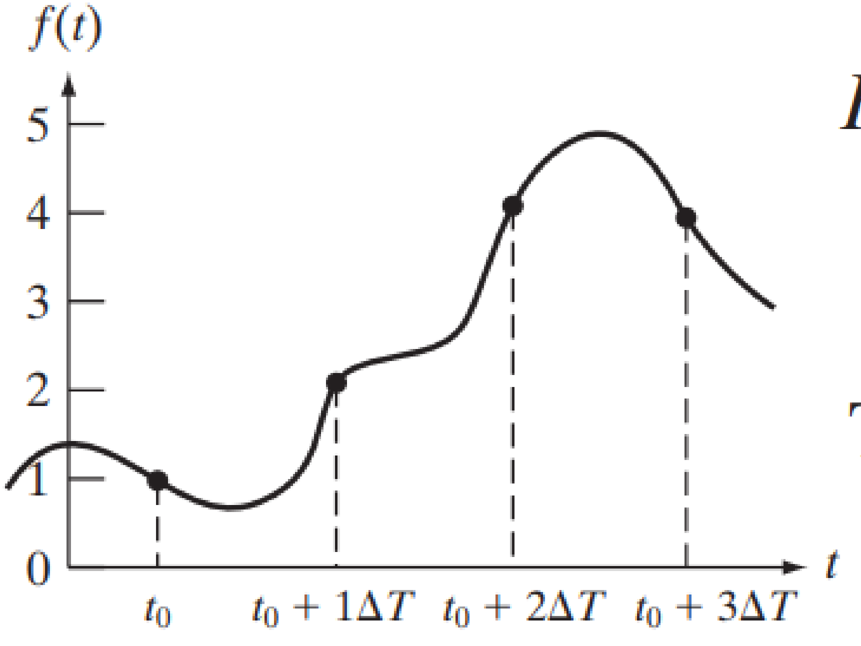

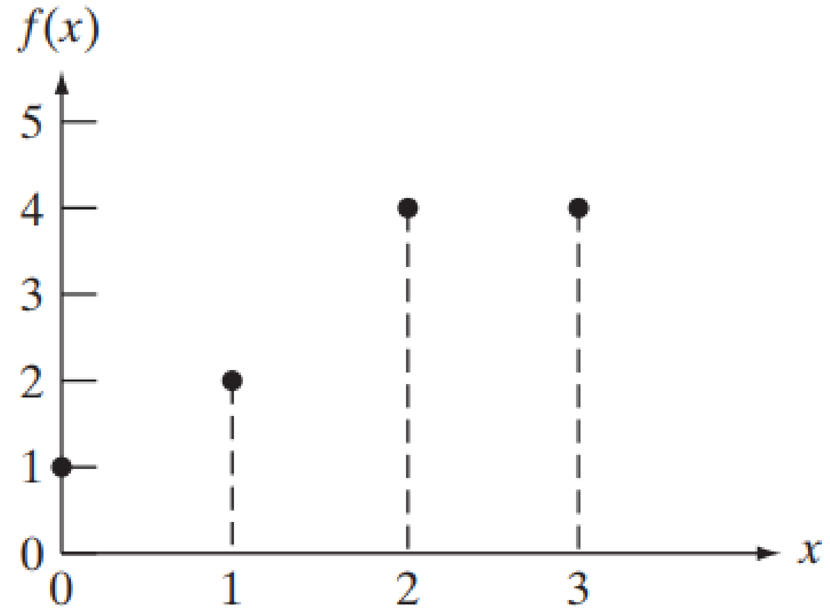

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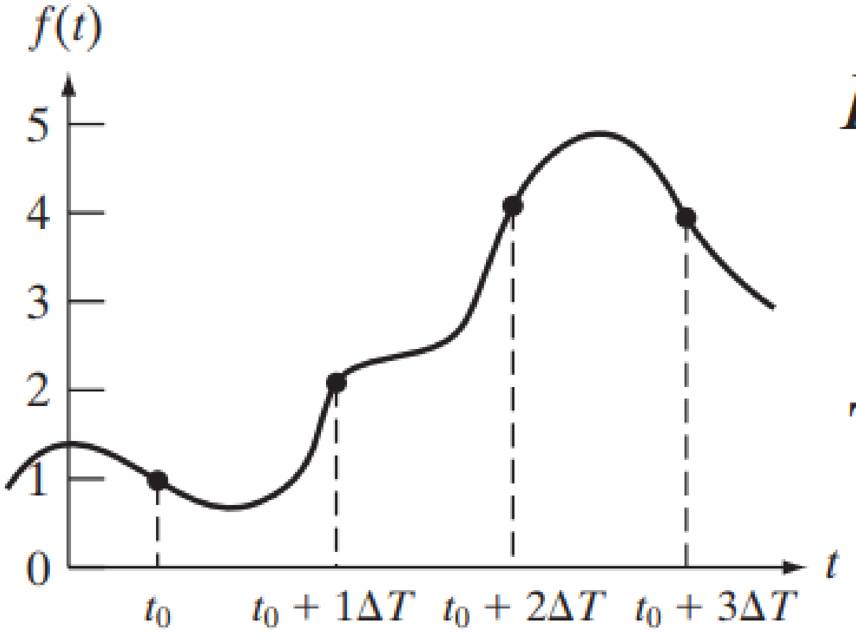

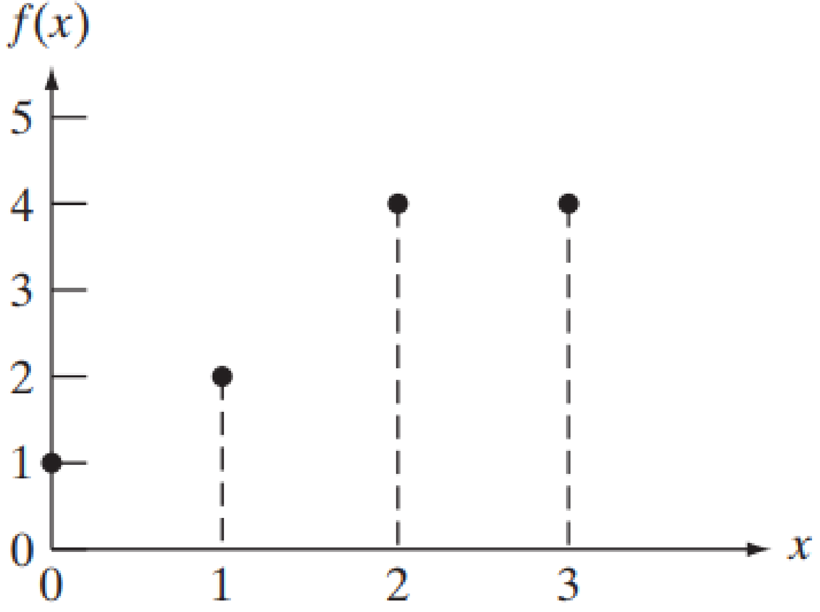

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- If instead we were given $F(u)$ and were asked to compute its inverse, we would proceed in the same manner, but using the inverse transform.

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&= \frac{1}{4}[4] = 1
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The 2-D Discrete Fourier Transform and its Inverse

- 2-D discrete Fourier transform (DFT)

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

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$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

Sample $f(t)$

$t = n\Delta T$

$$F(m) = \sum_{n=0}^{M-1} f(n)\, e^{-j2\pi mn/M}$$

$\mu = \frac{m}{M\Delta T}$

$t = n\Delta T$

$$F(u) = \sum_{x=0}^{M-1} f(x)\, e^{-j2\pi ux/M}$$

Some properties of the 2-D discrete Fourier Transform

- Suppose that a continuous function $f\,(t,\ z)$ is sampled to form a digital image, $f\,(x, y)$, consisting of M x N samples taken in the t- and z-directions, respectively. Let and ΔT and ΔZ denote the separations between samples.
- Then, the separations between the corresponding discrete, frequency domain variables are given by

$$\Delta u = \frac{1}{M\Delta T}$$

$$\Delta v = \frac{1}{N\Delta Z}$$

Translation and Rotation

- FT pair satisfies the following translation properties

$$f(x, y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

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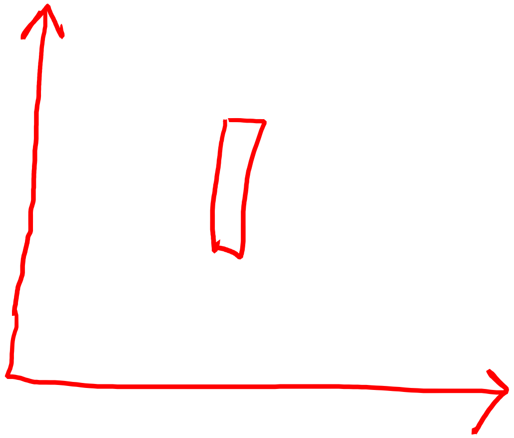

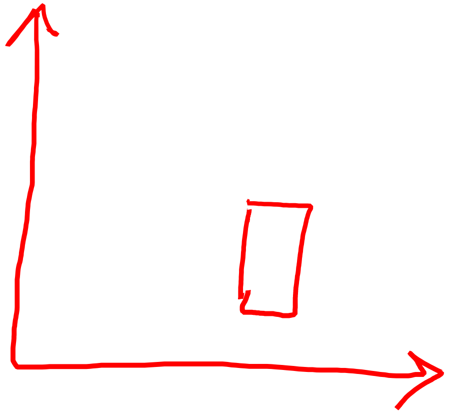

Proof of Translation Property

Proof of Translation Property

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[f(x,y)e^{j2\pi(u_0x+v_0y)}\right]e^{-j2\pi(ux/M+vy/N)}$$

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,e^{-j2\pi(ux/M+vy/N)}$$

Proof of Translation Property

$$\Im\left[f(x,y)e^{j2\pi(u_0x+v_0y)}\right]$$

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$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi[(u-u_0)x/M+(v-v_0)y/N]}$$

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DFT : A Complex no.

- Fourier transform theory
  - The Fourier transform F(u) is in general complex and we use polar coordinates:

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Image Transforms

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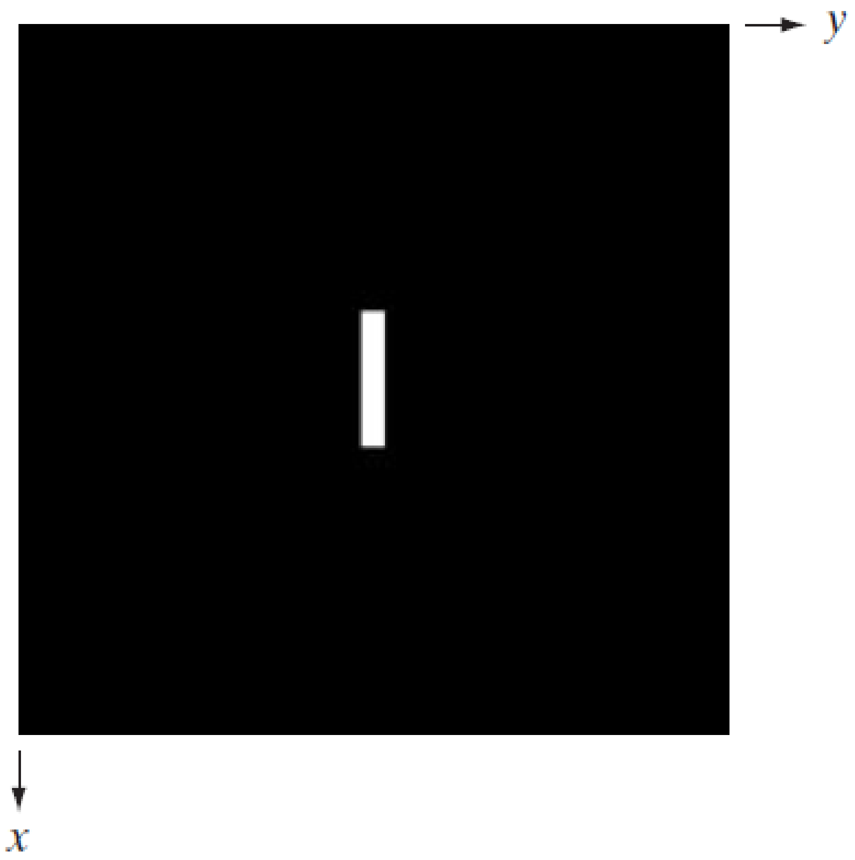
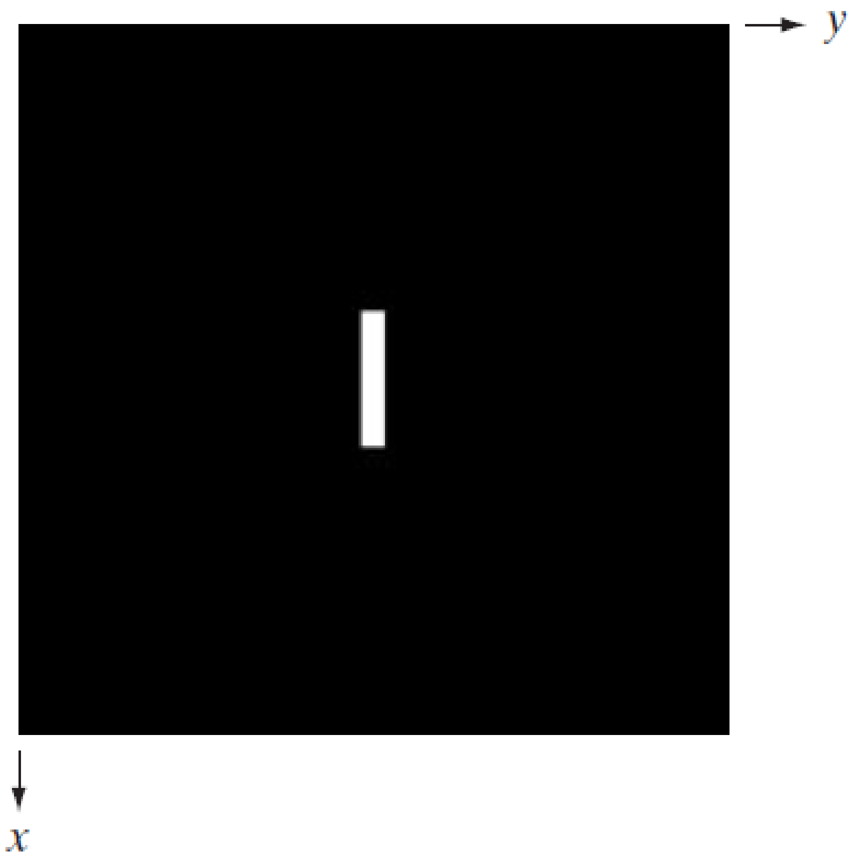

y
x



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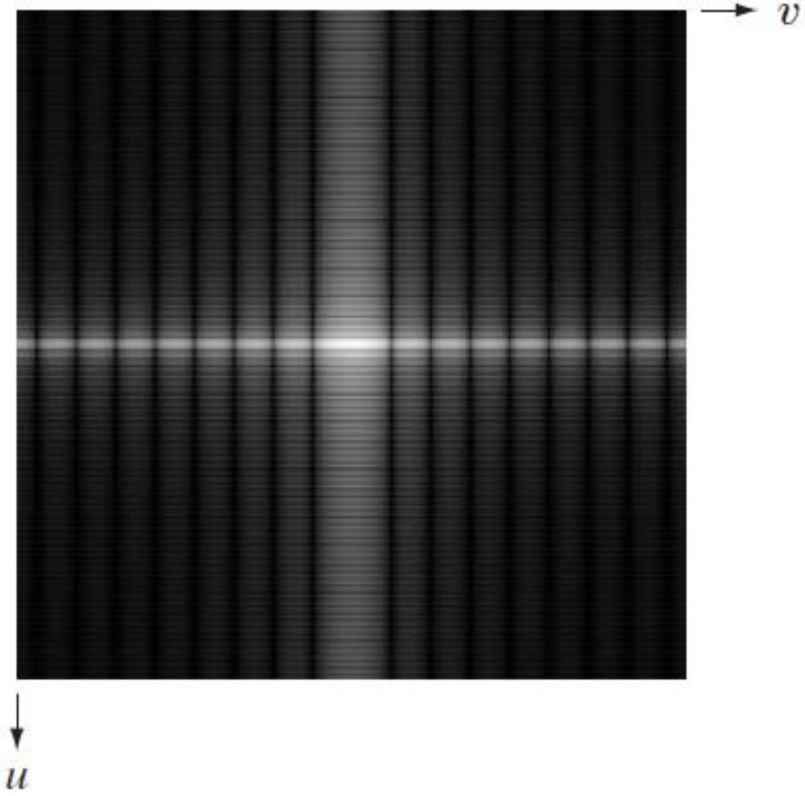

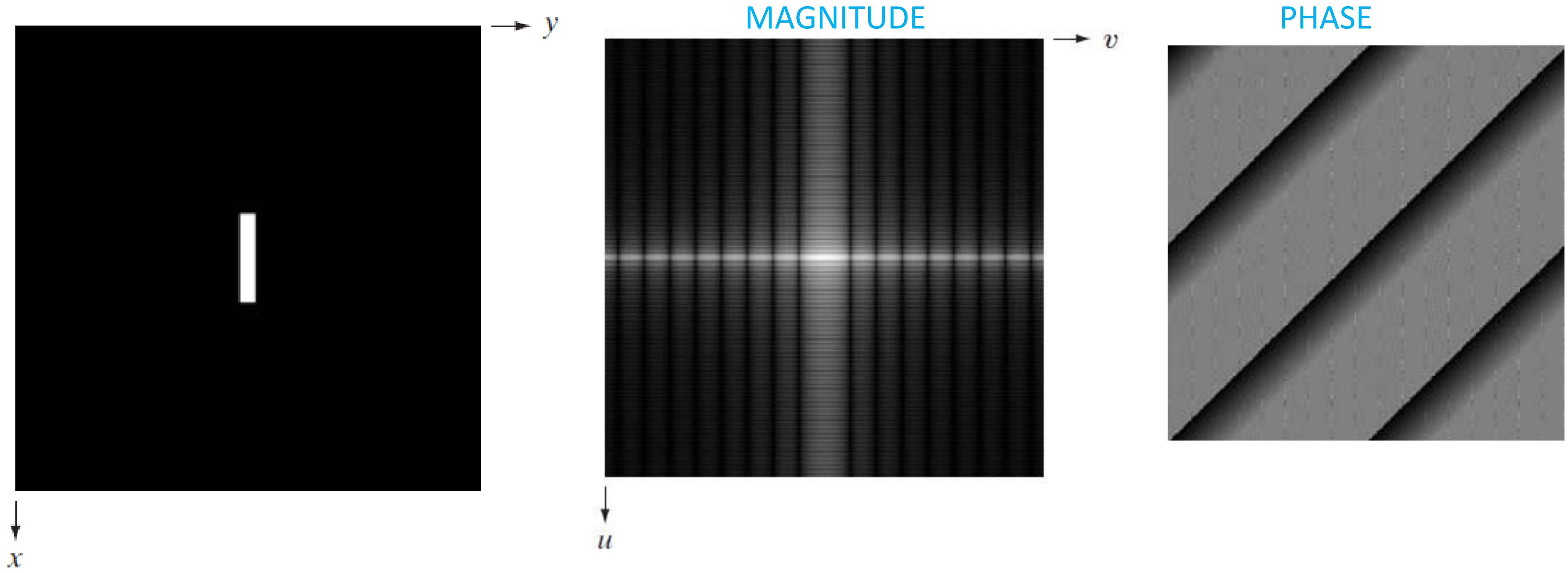
MAGNITUDE
PHASE
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v
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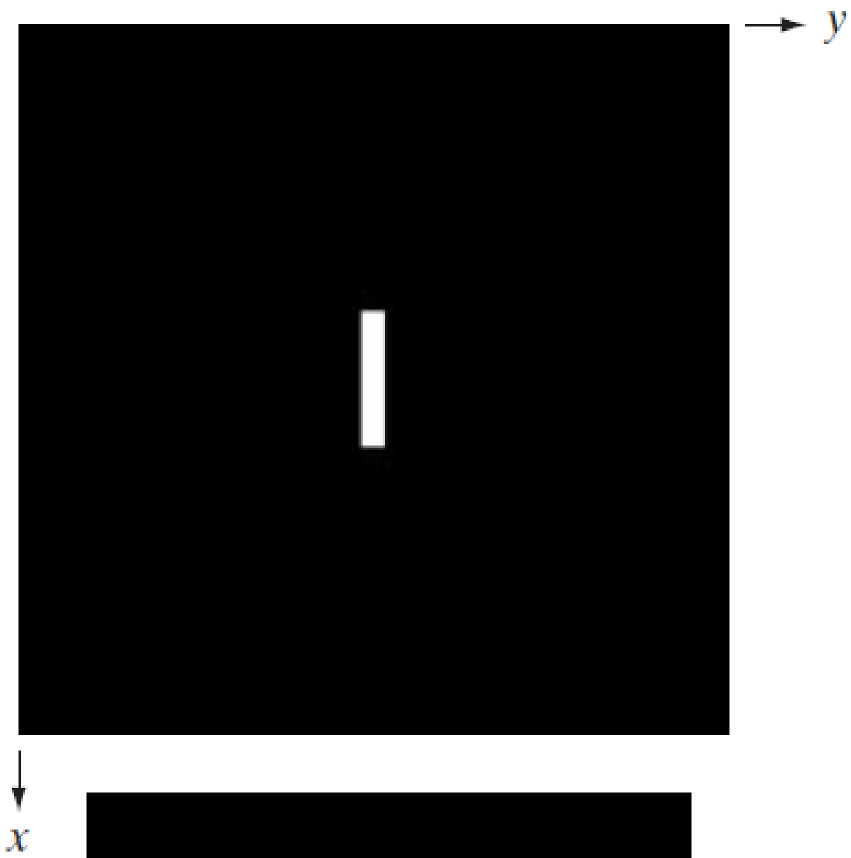

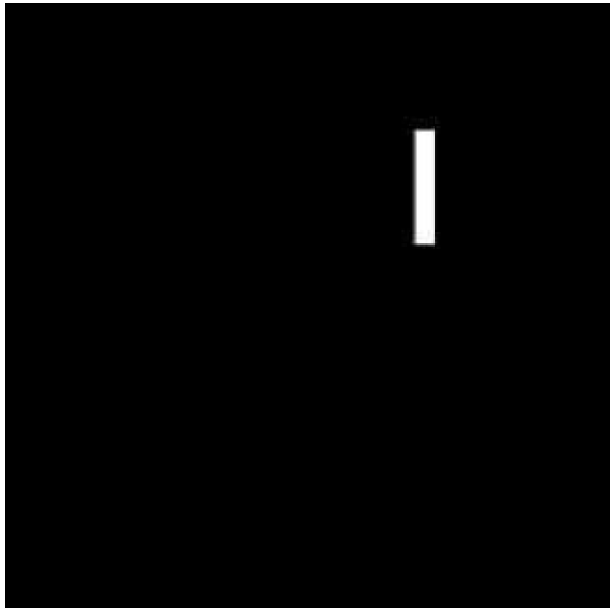

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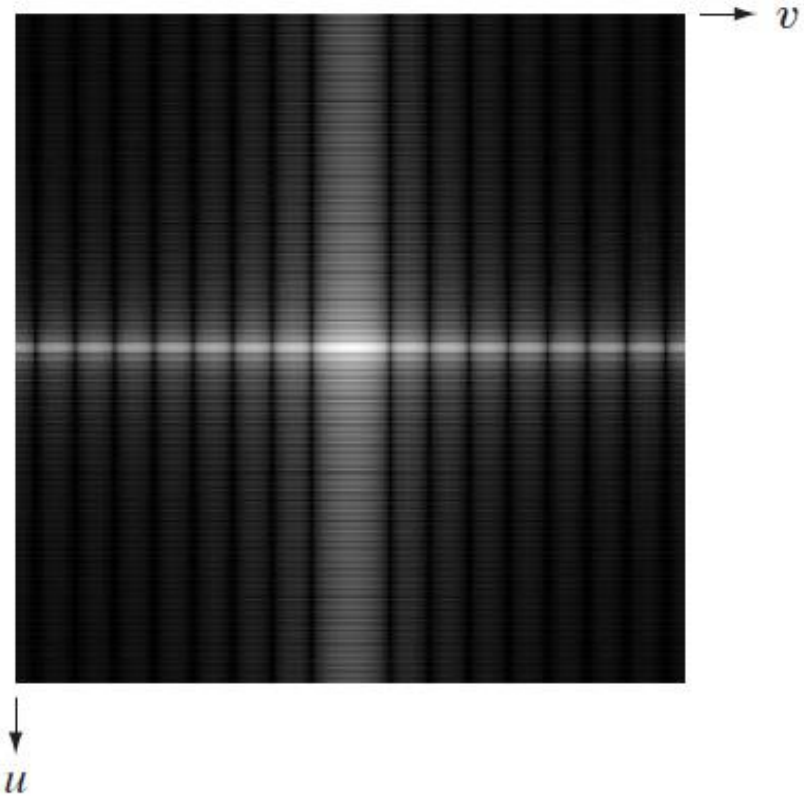

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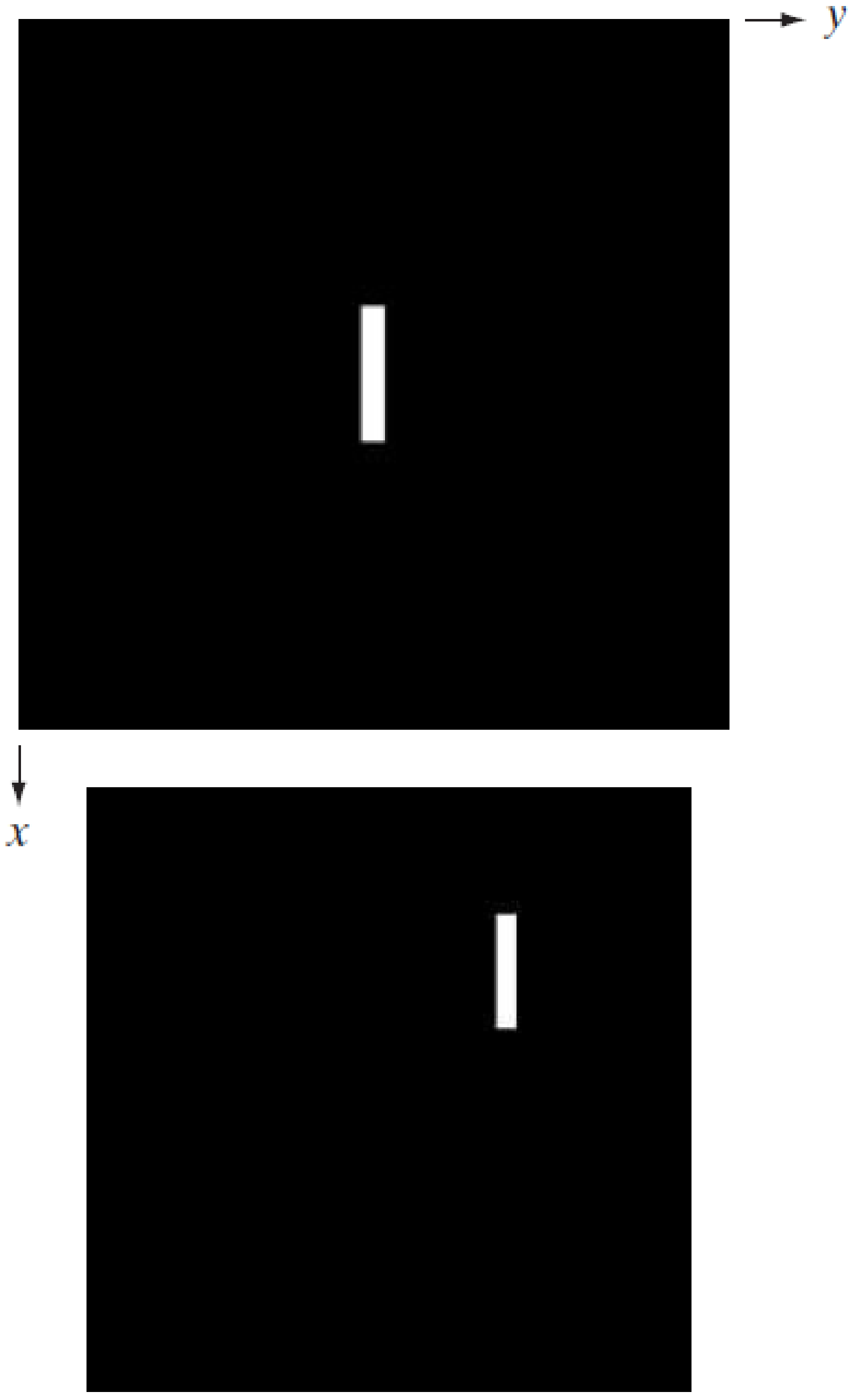

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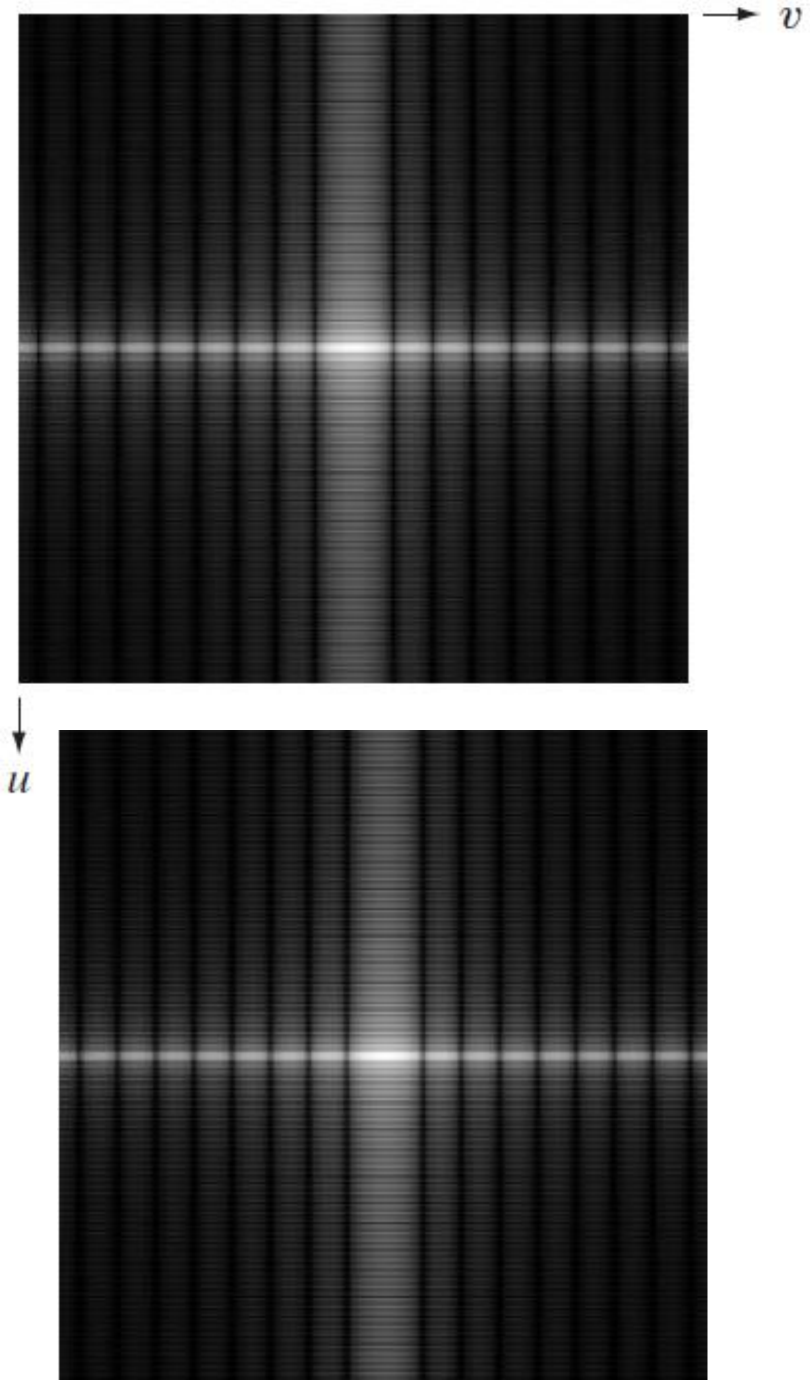
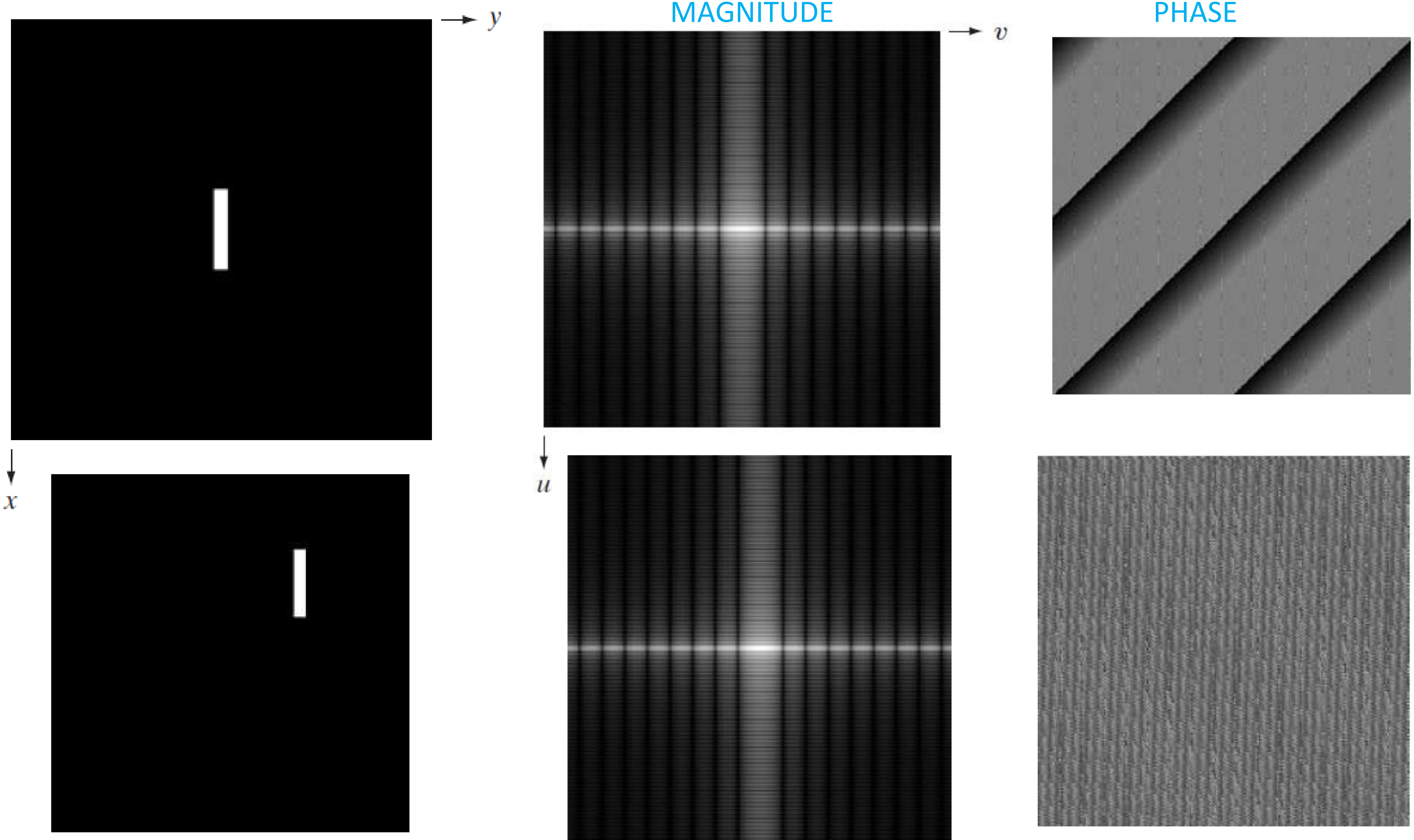
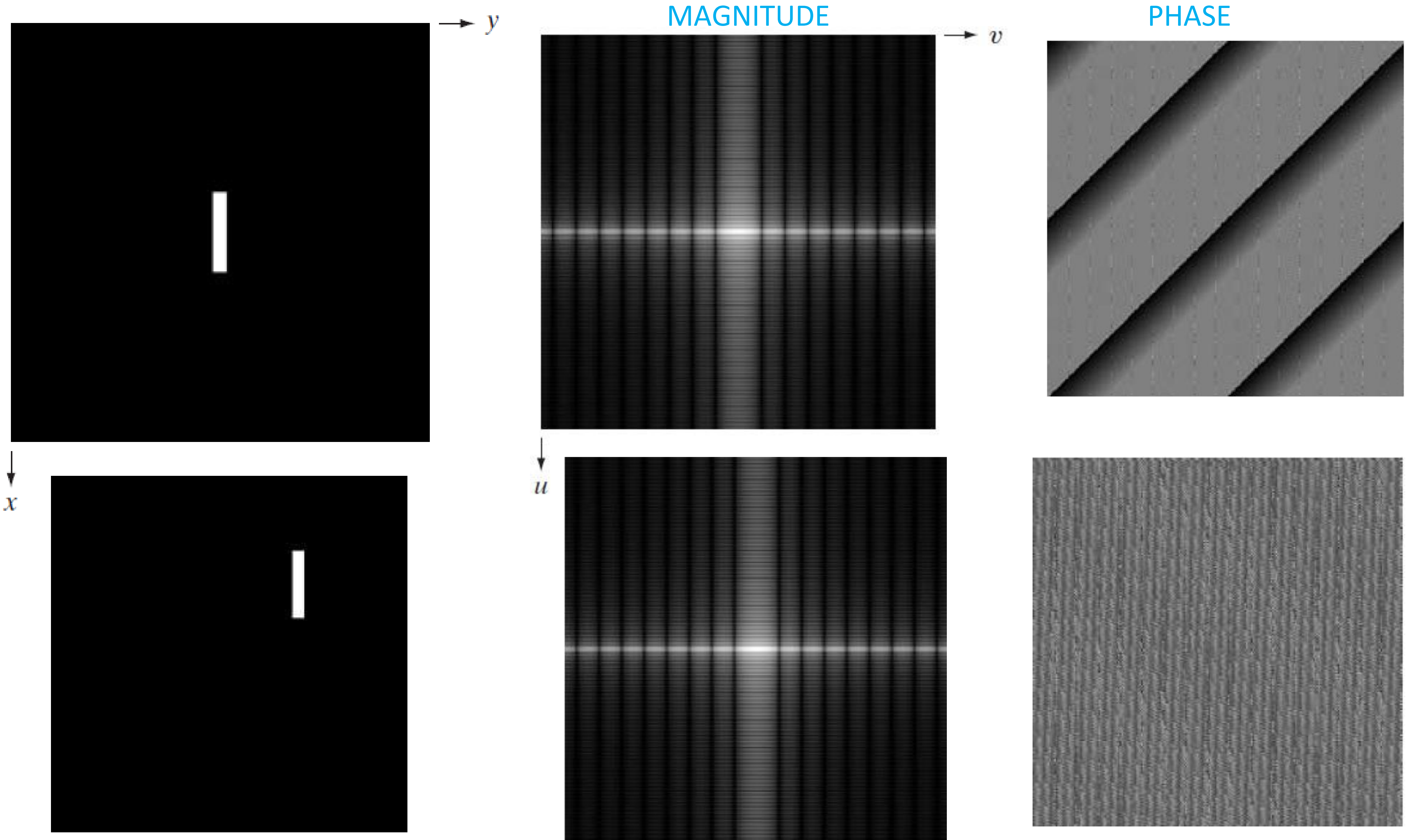


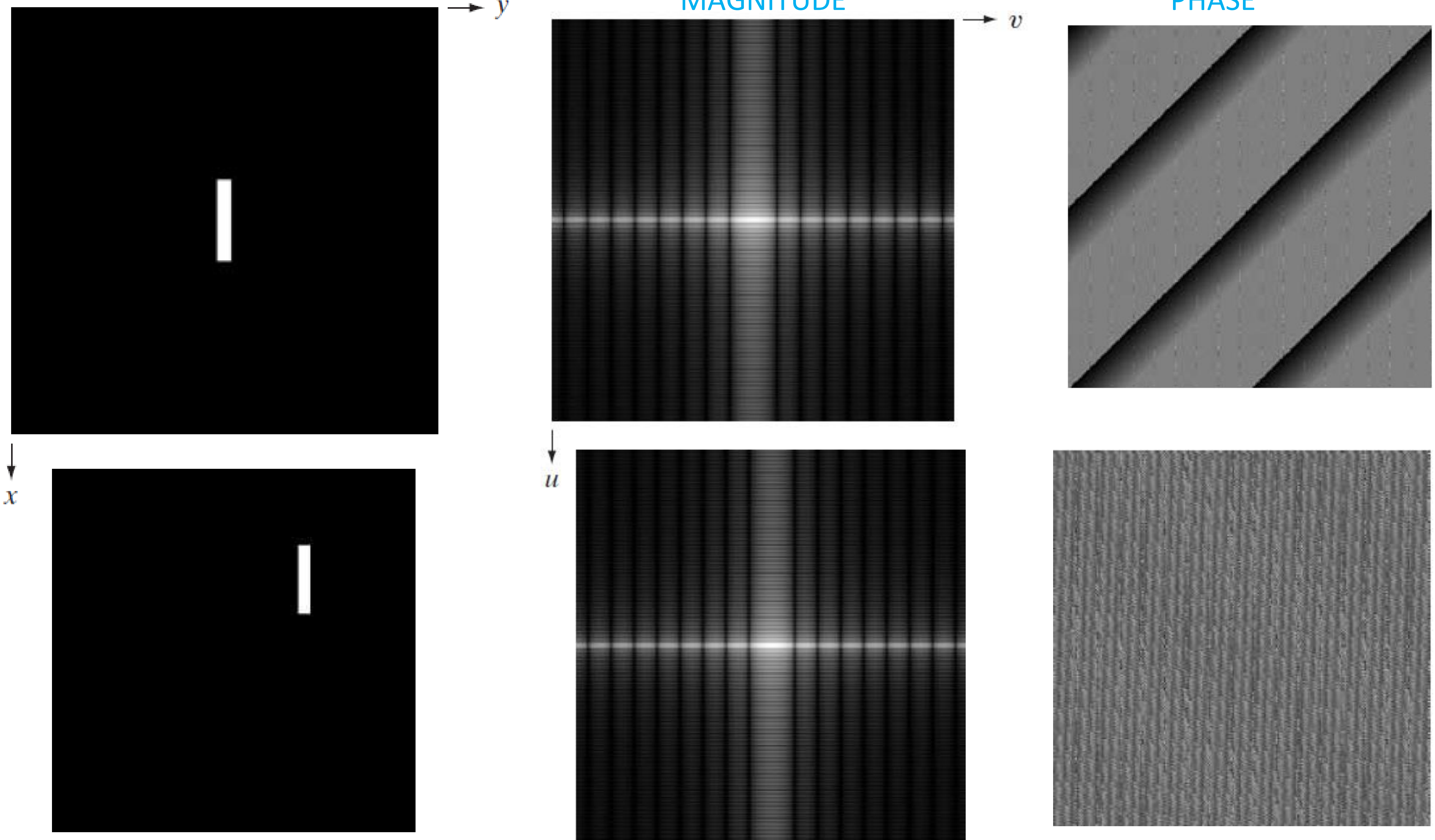
PHASE

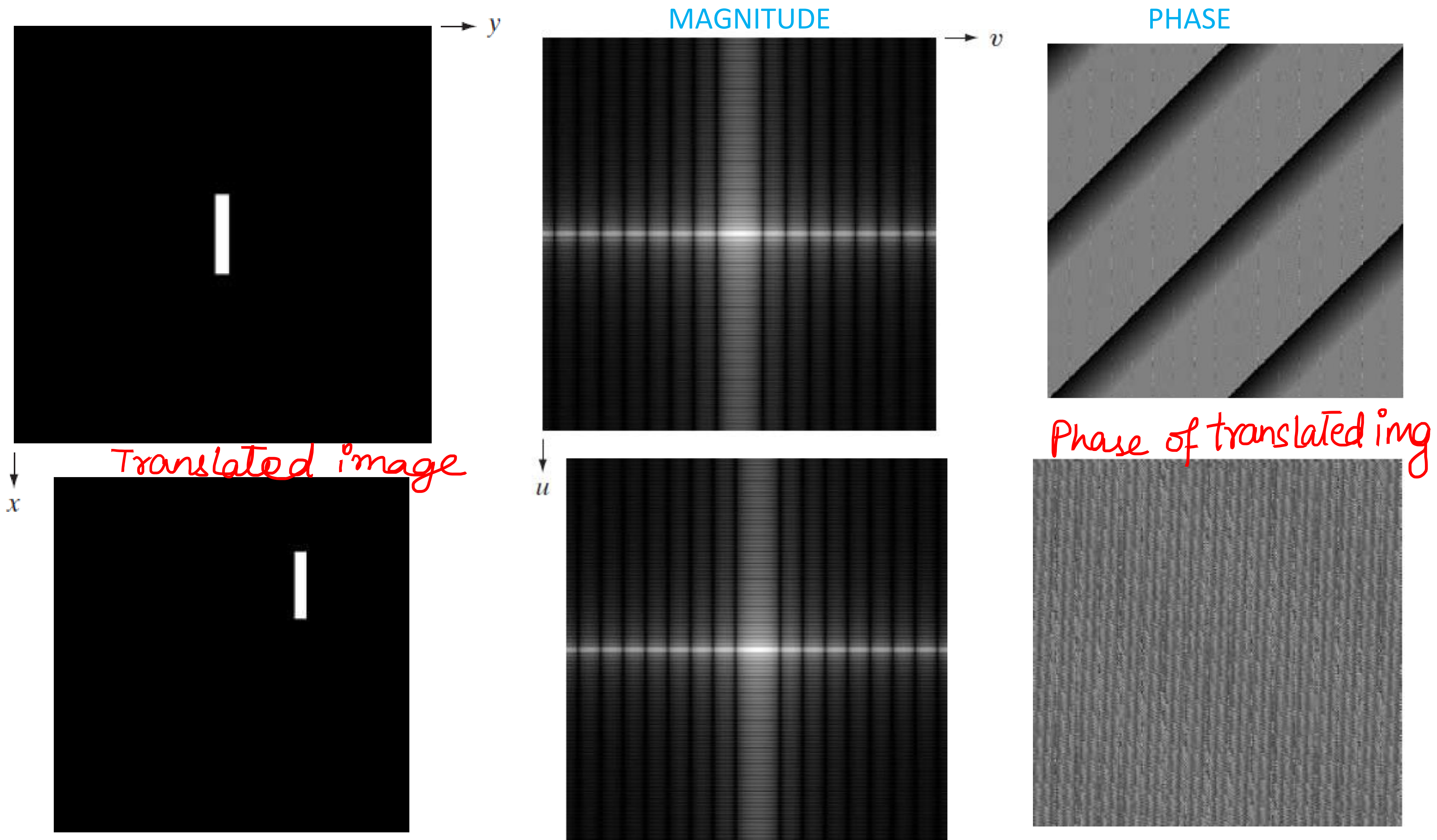
MAGNITUDE PHASE




MAGNITUDE
PHASE
y
v
x
u

MAGNITUDE PHASE




MAGNITUDE
PHASE
y
v
x
u
Translated image
Phase of translated img

Rotational Property

- Using the polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$$

- Results in the following transform pair:

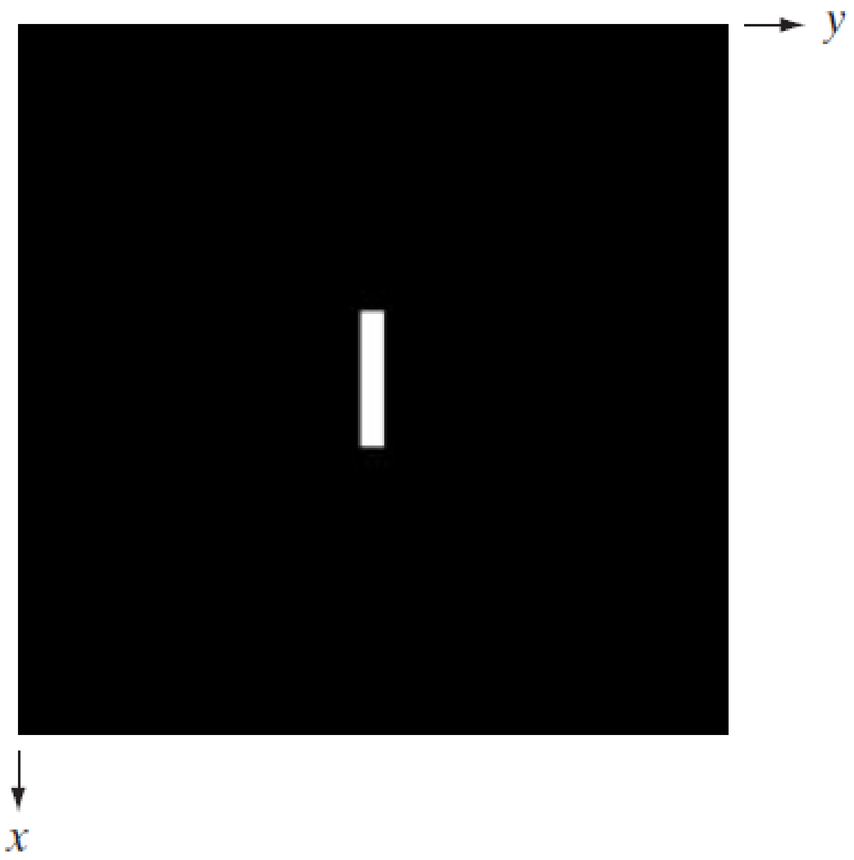
$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

- which indicates that rotating $f(x, y)$ by an angle θ_0 rotates $F(u, v)$ by the same angle. Conversely, rotating $F(u, v)$ rotates $f(x, y)$ by the same angle.

ORIGINAL

ROTATED

ORIGINAL



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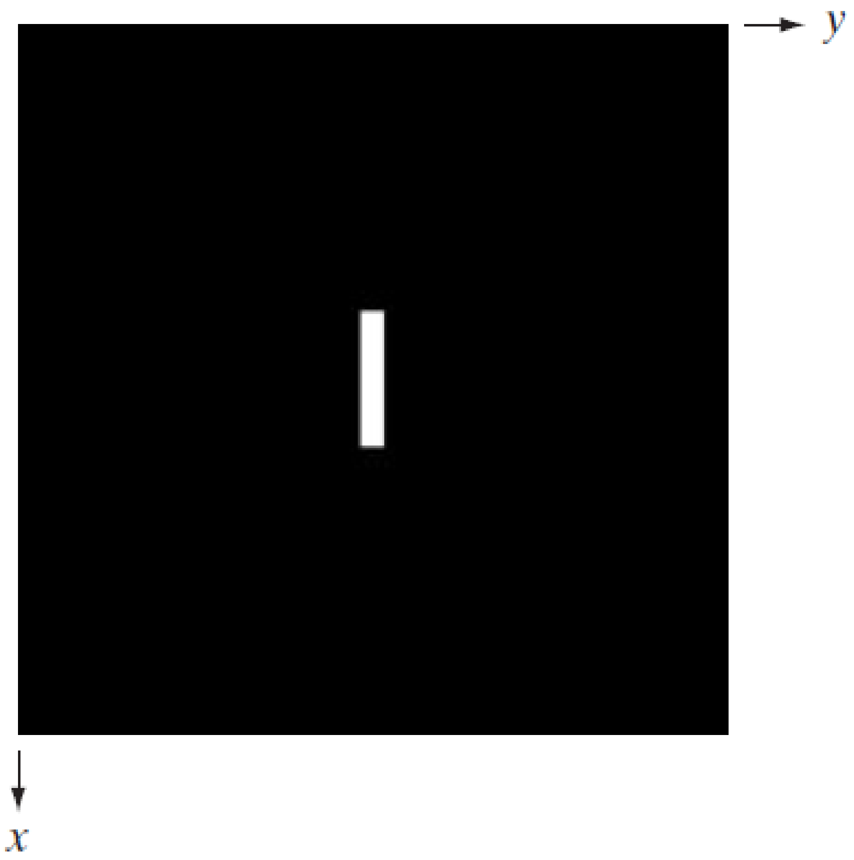

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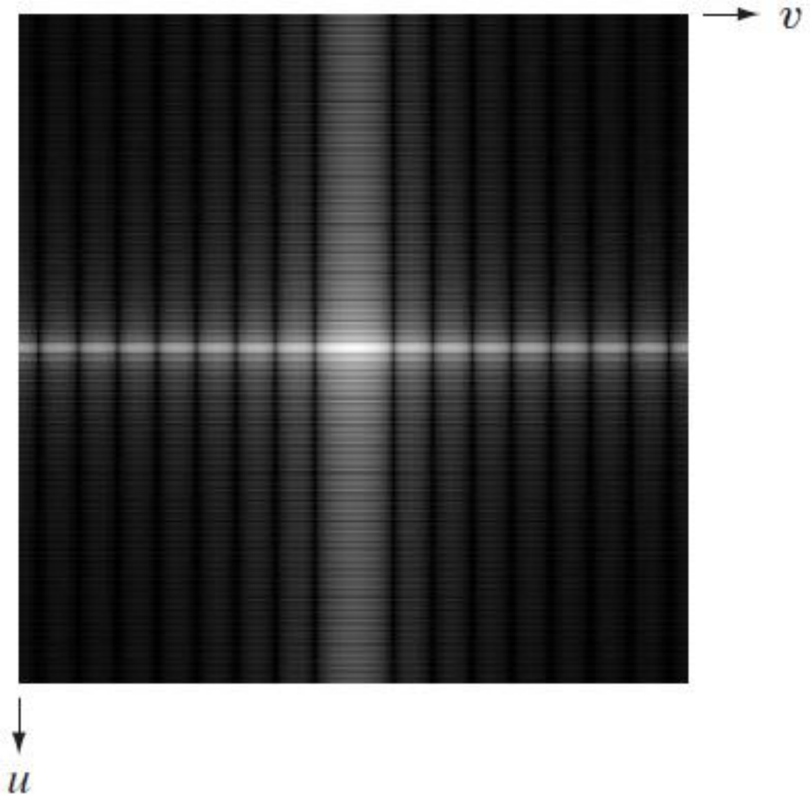

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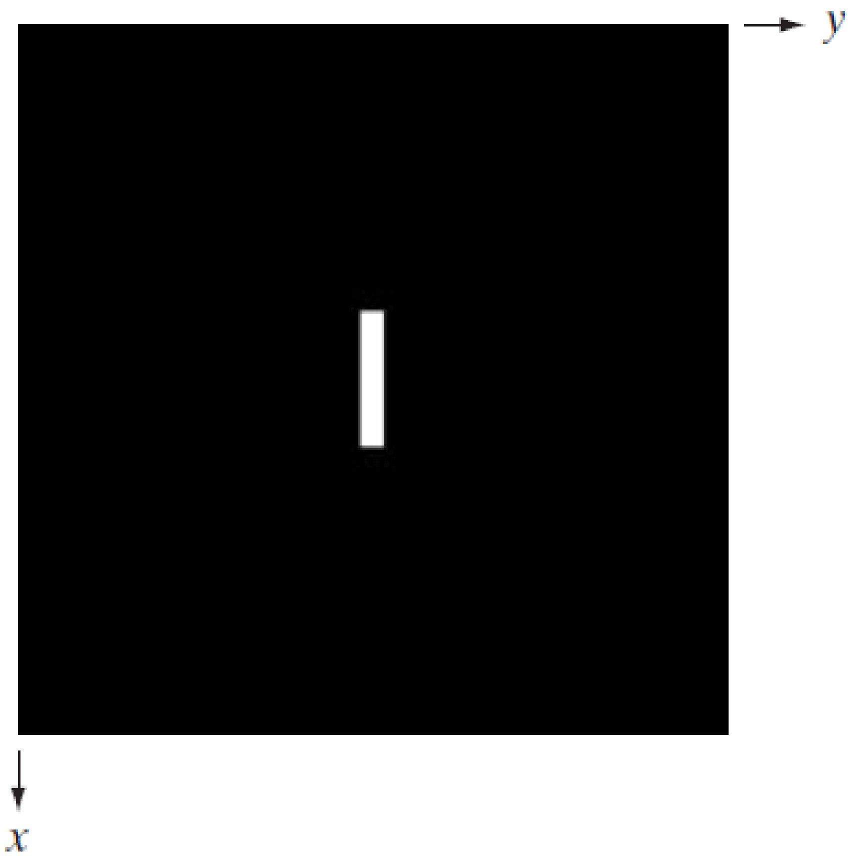

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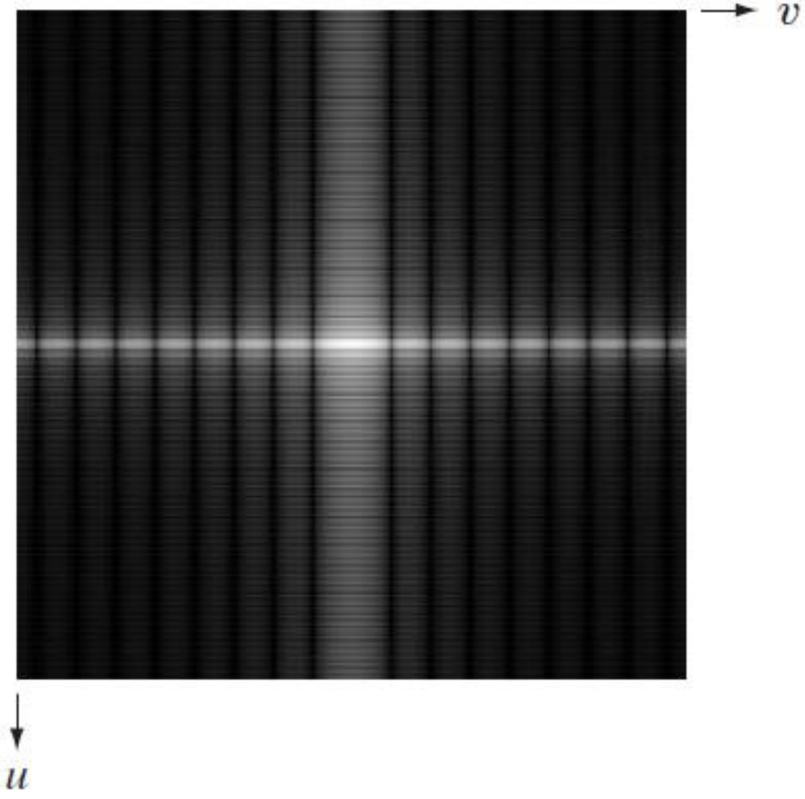

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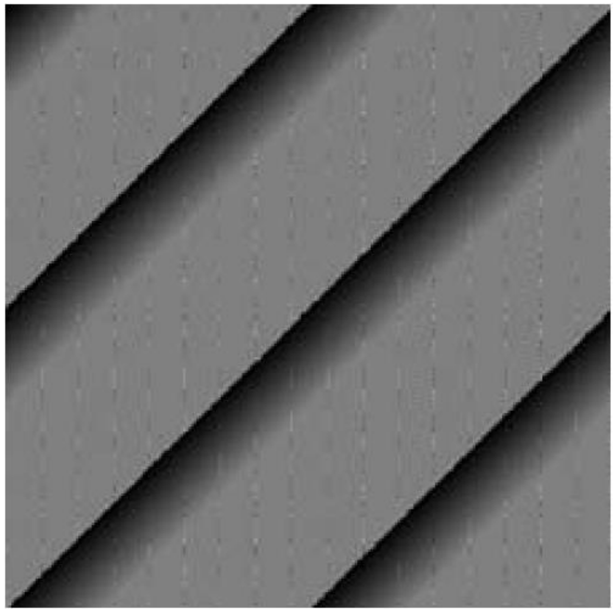

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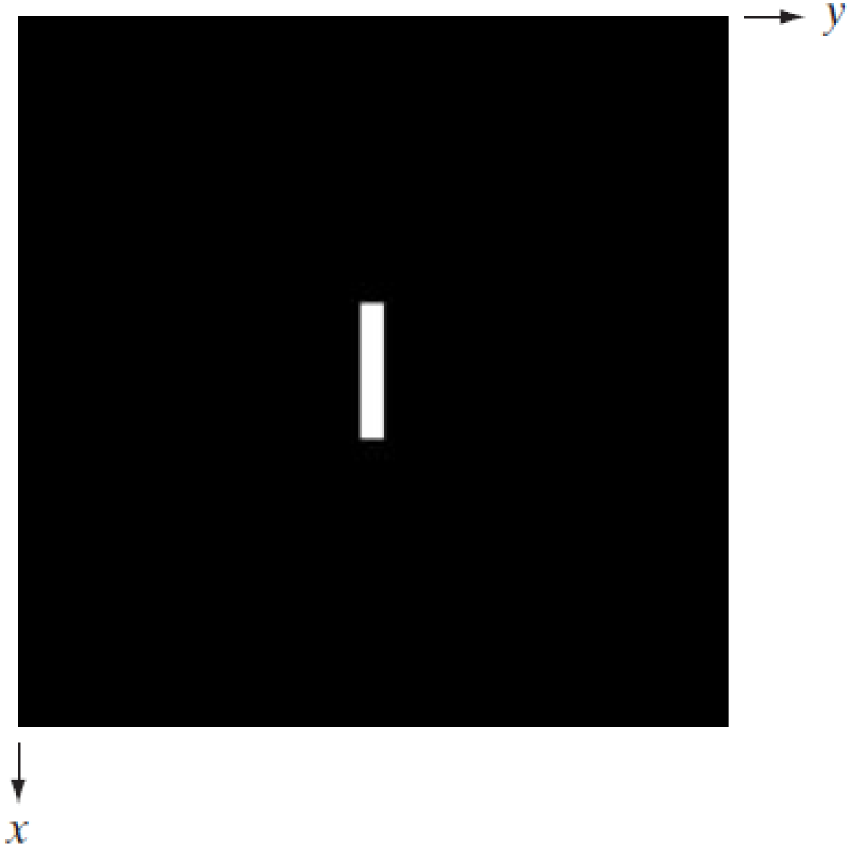

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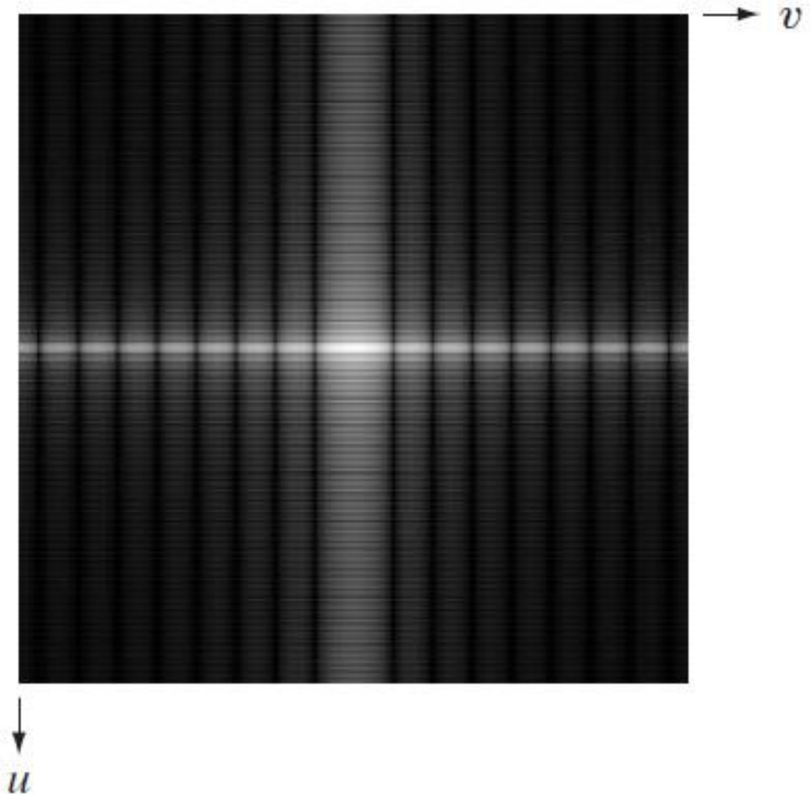

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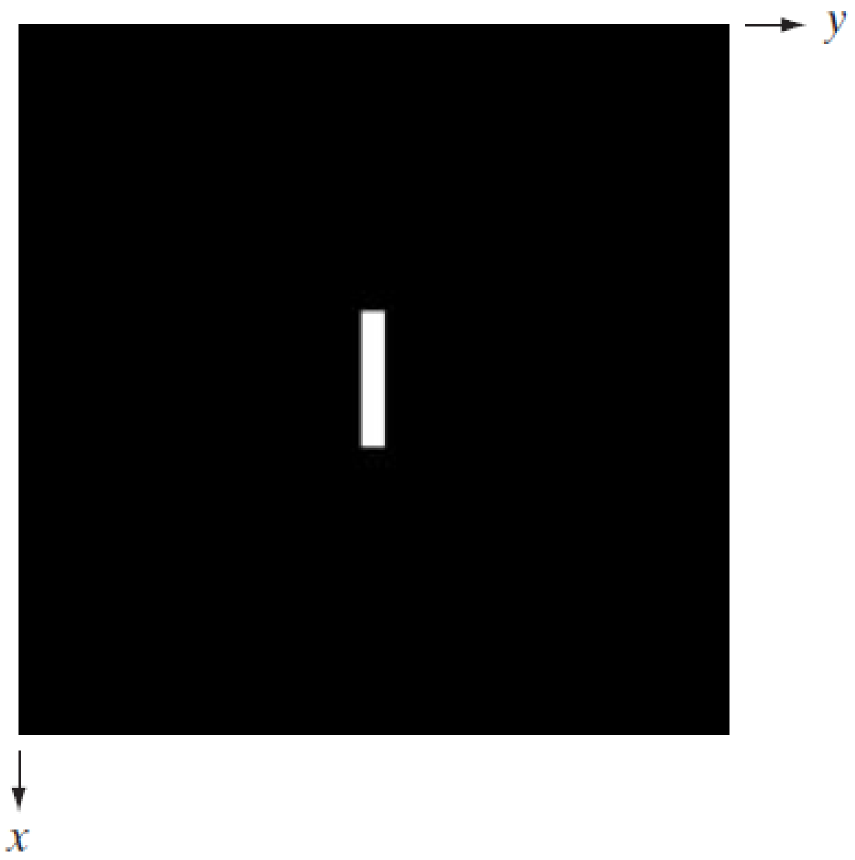

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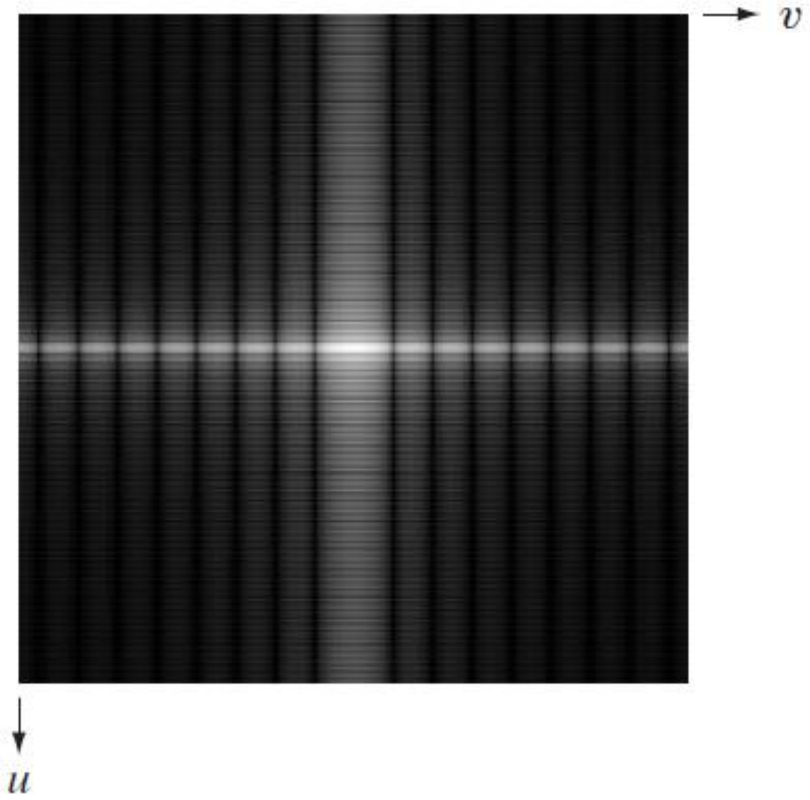

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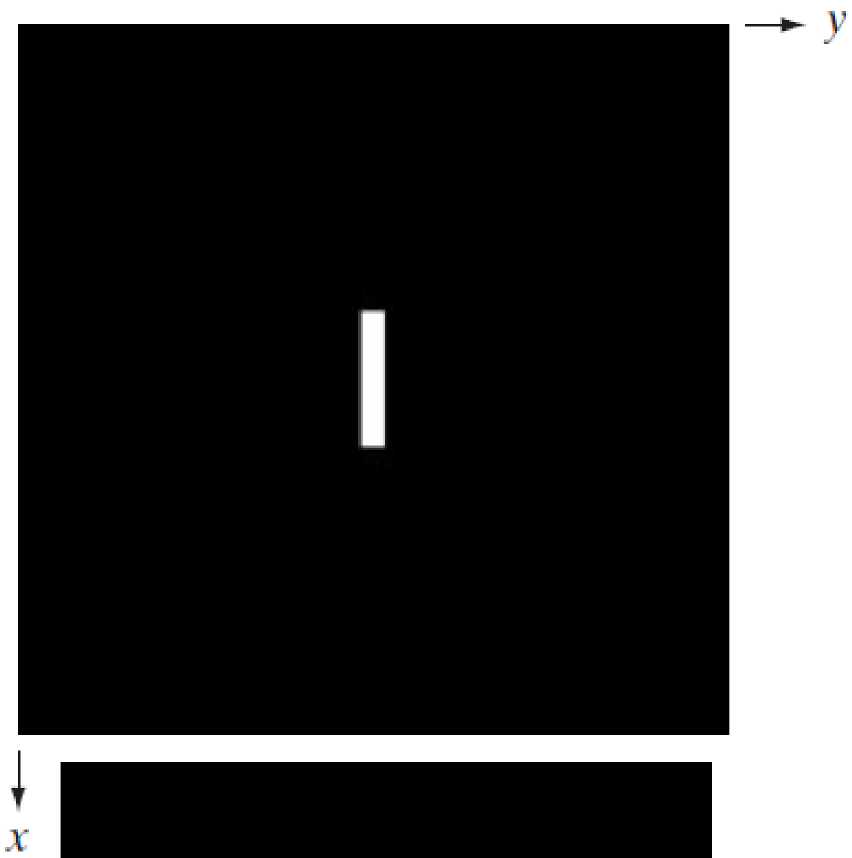

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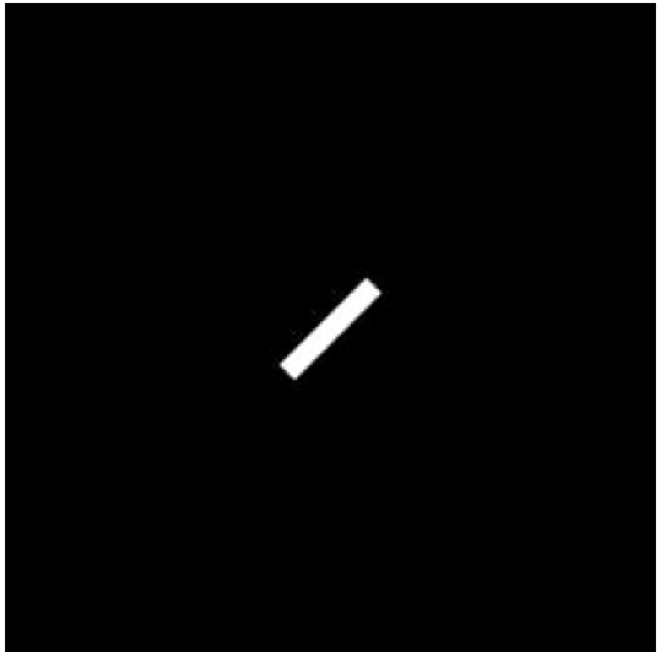

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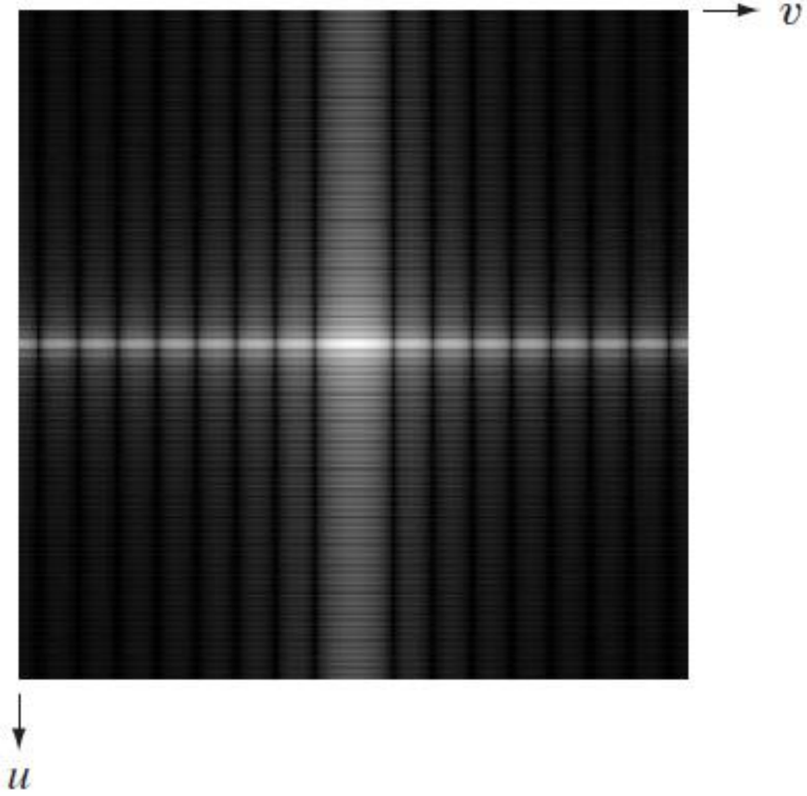

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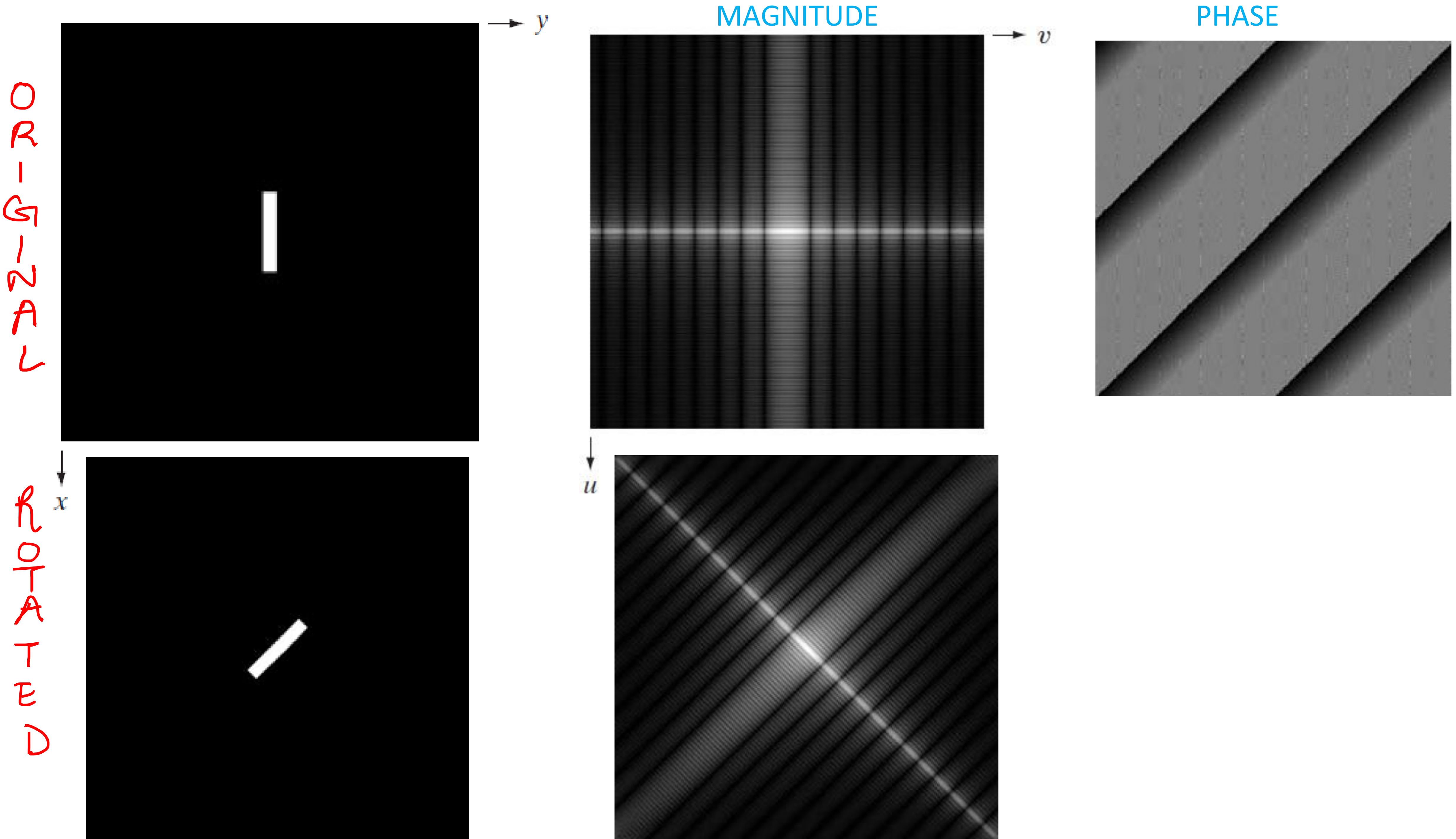
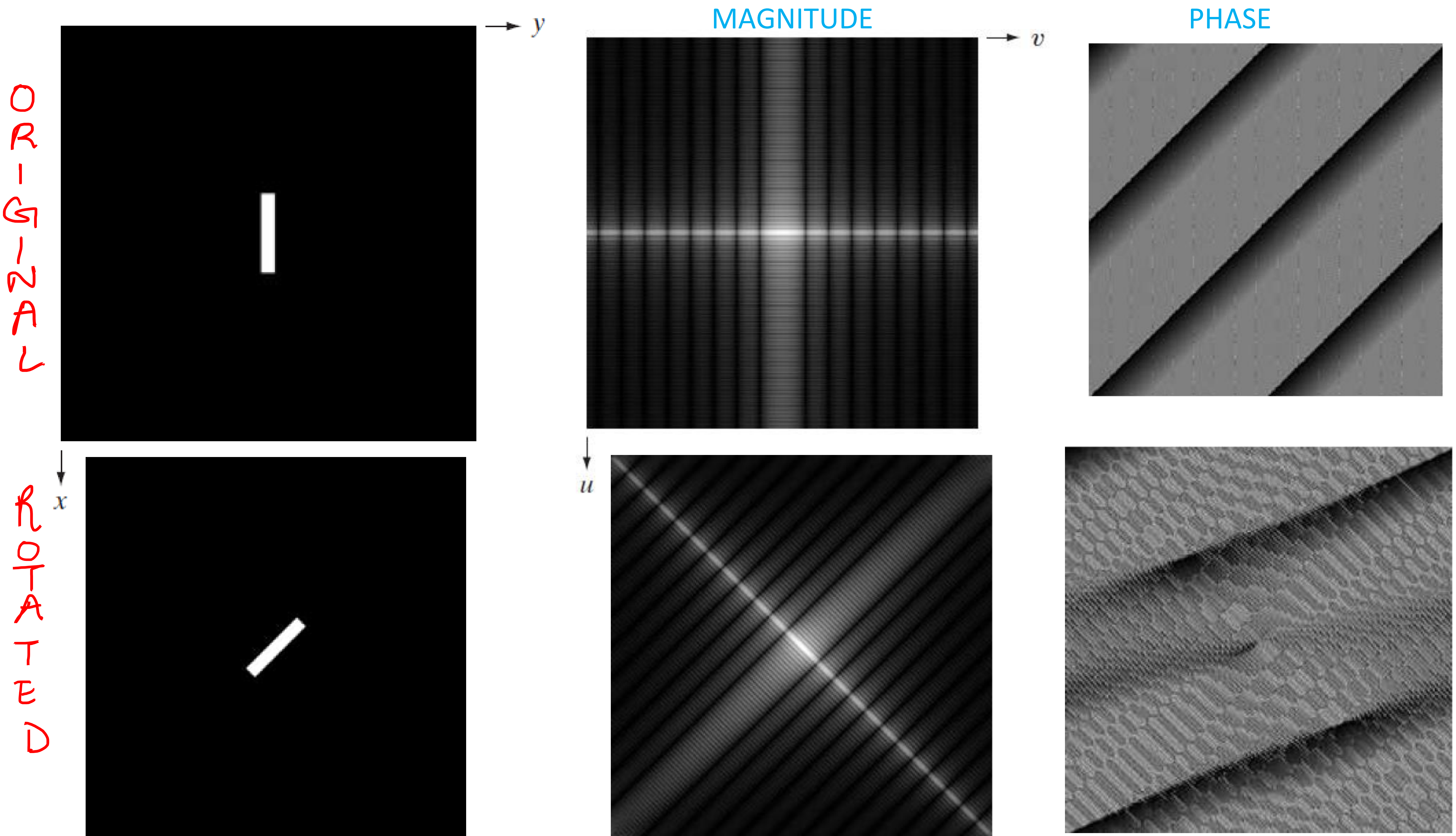

MAGNITUDE
PHASE
ORIGINAL
ROTATED
y
x
v
u


MAGNITUDE
PHASE
ORIGINAL
ROTATED
y
v
x
u

Periodicity

- As in the 1-D case, the 2-D Fourier transform and its inverse are infinitely periodic in the *u* and v directions; that is

$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

and

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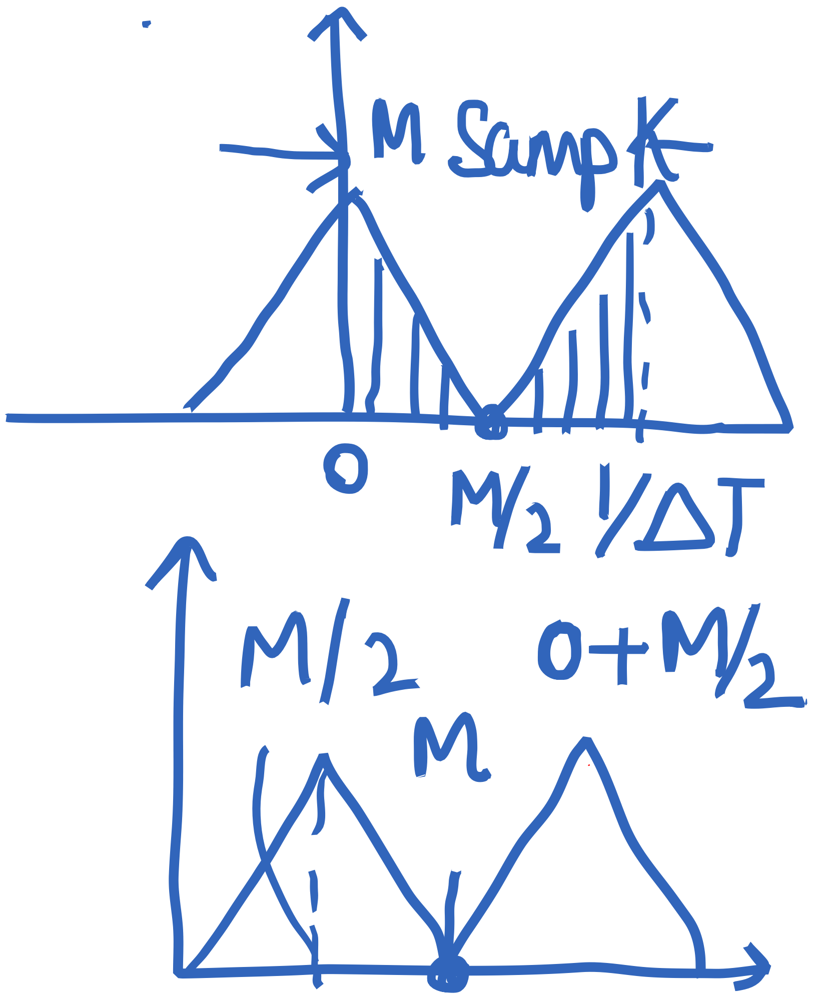
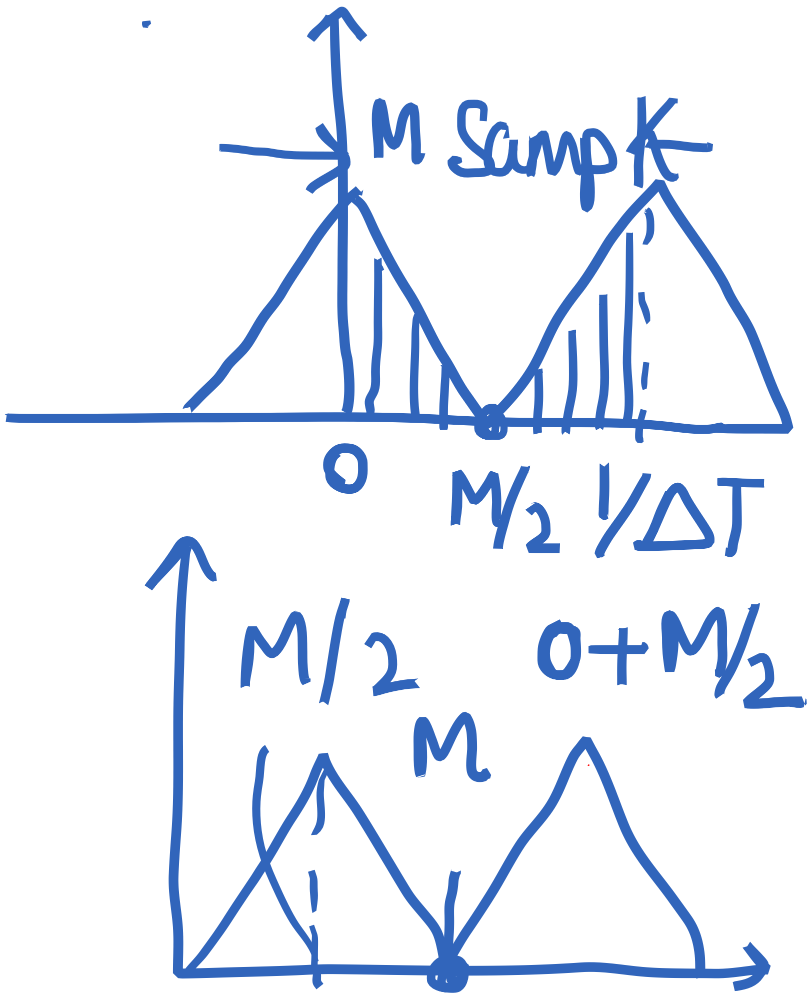
$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$


M Sampl
0
M/2
1/ΔT
M/2
0+M/2
M

- In 1-D DFT consider 0 to $M - 1$ samples
- We need to shift in frequency



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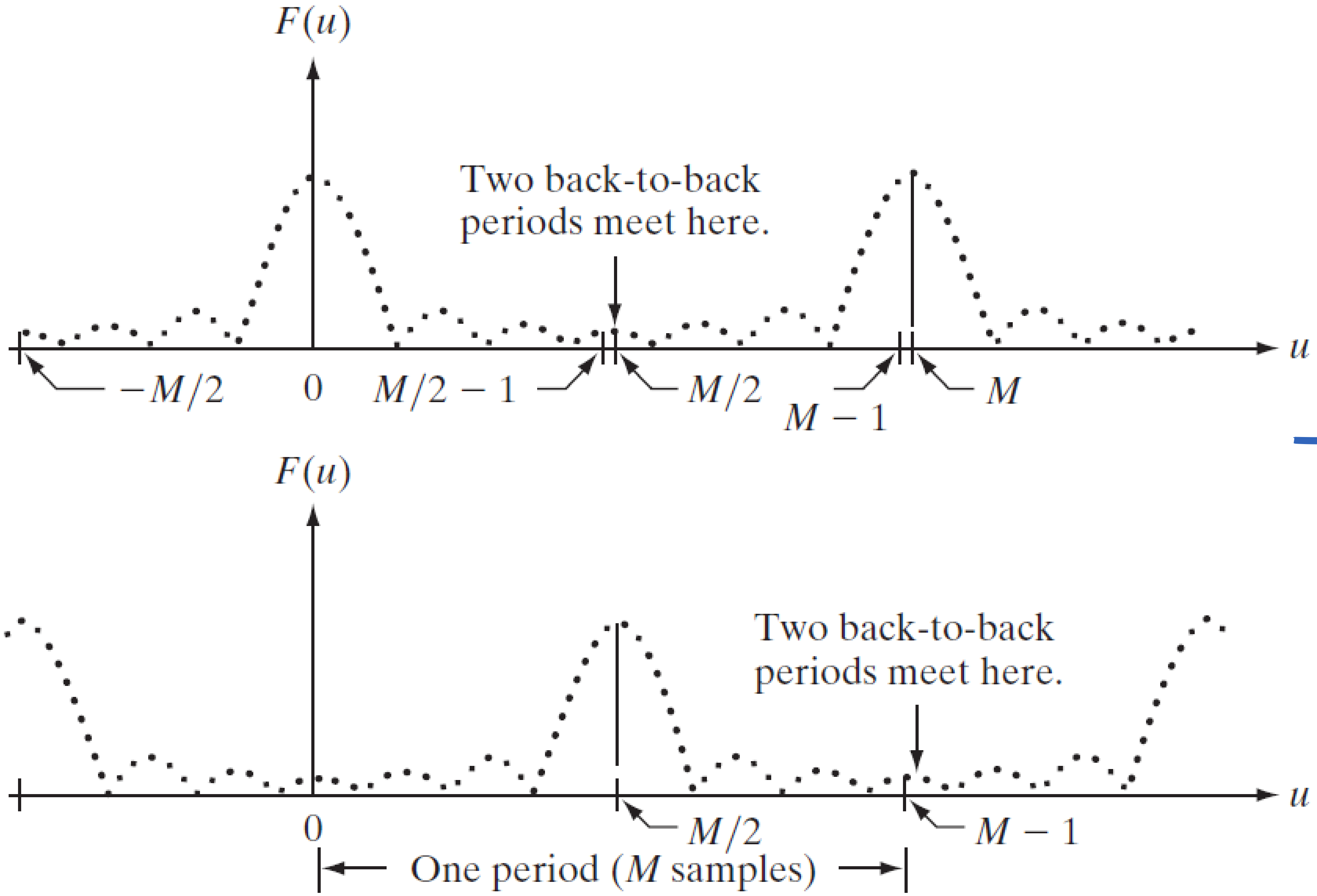

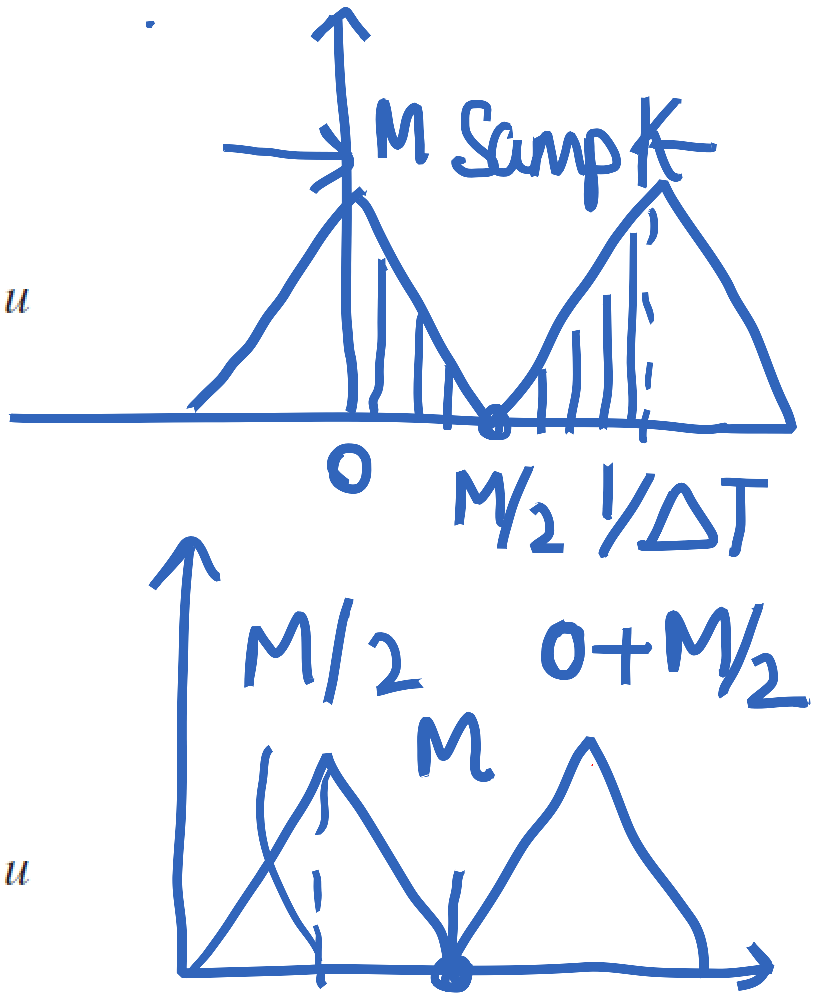

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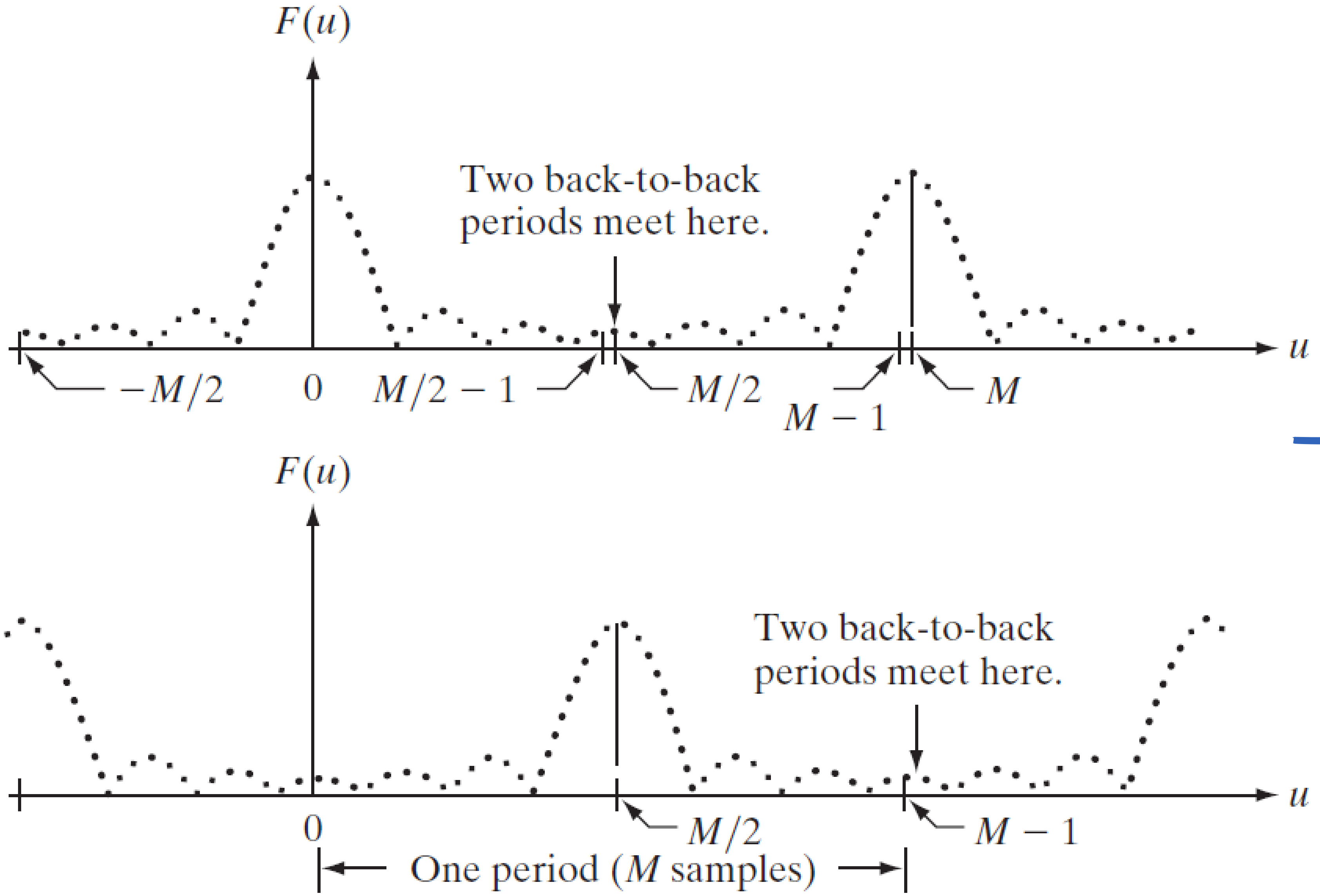

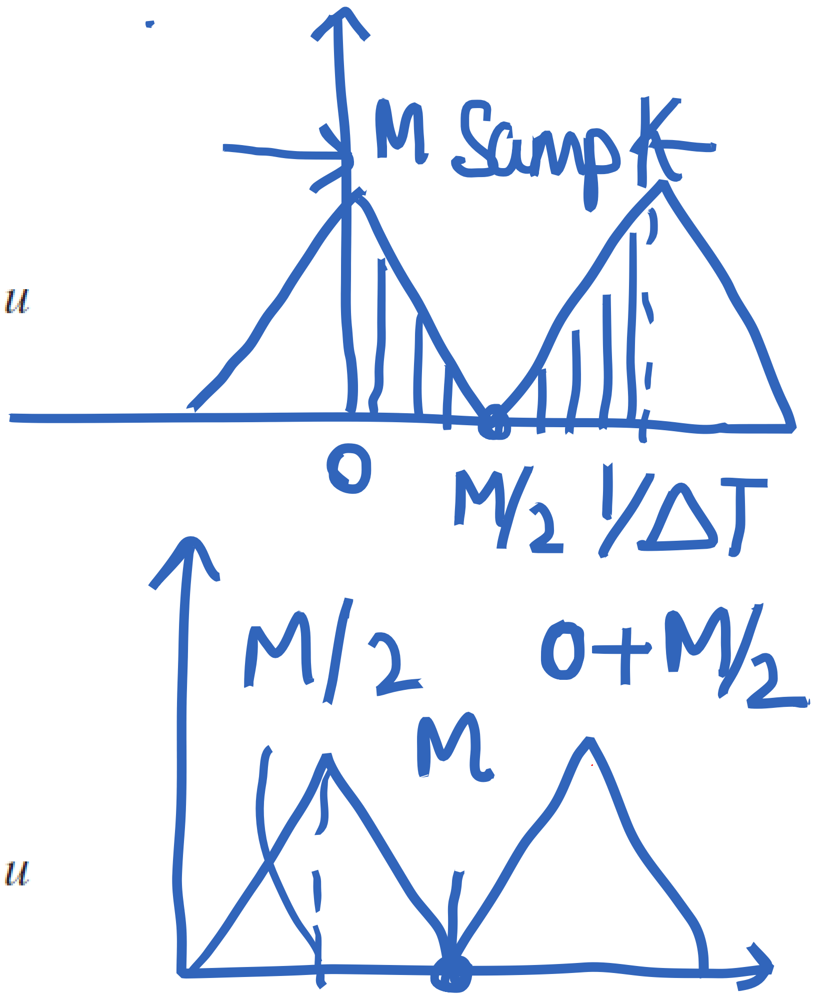

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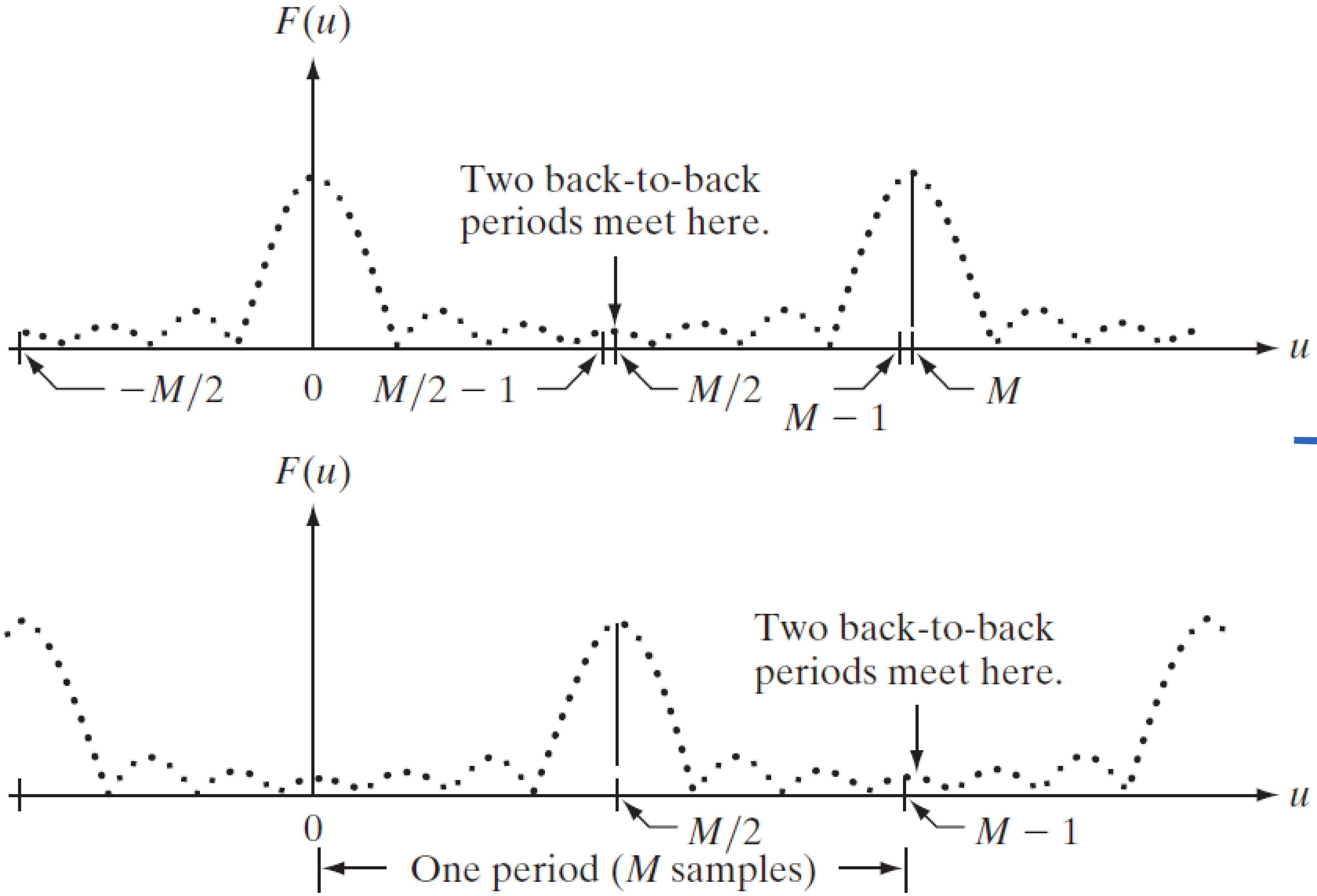

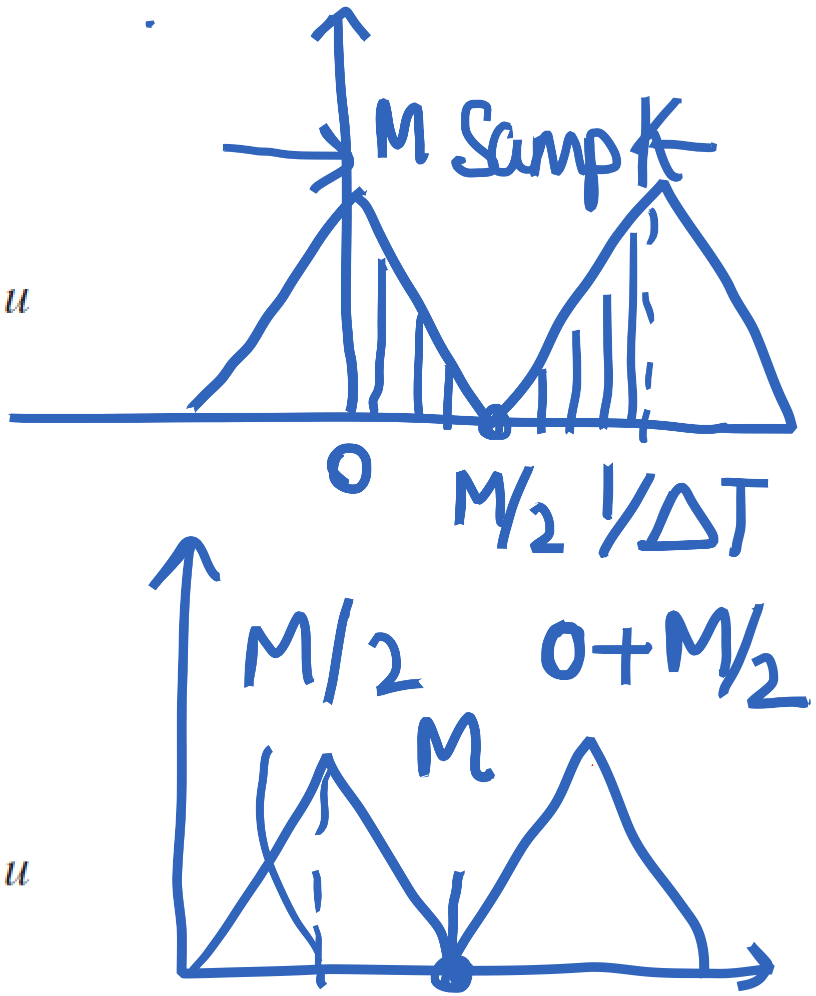

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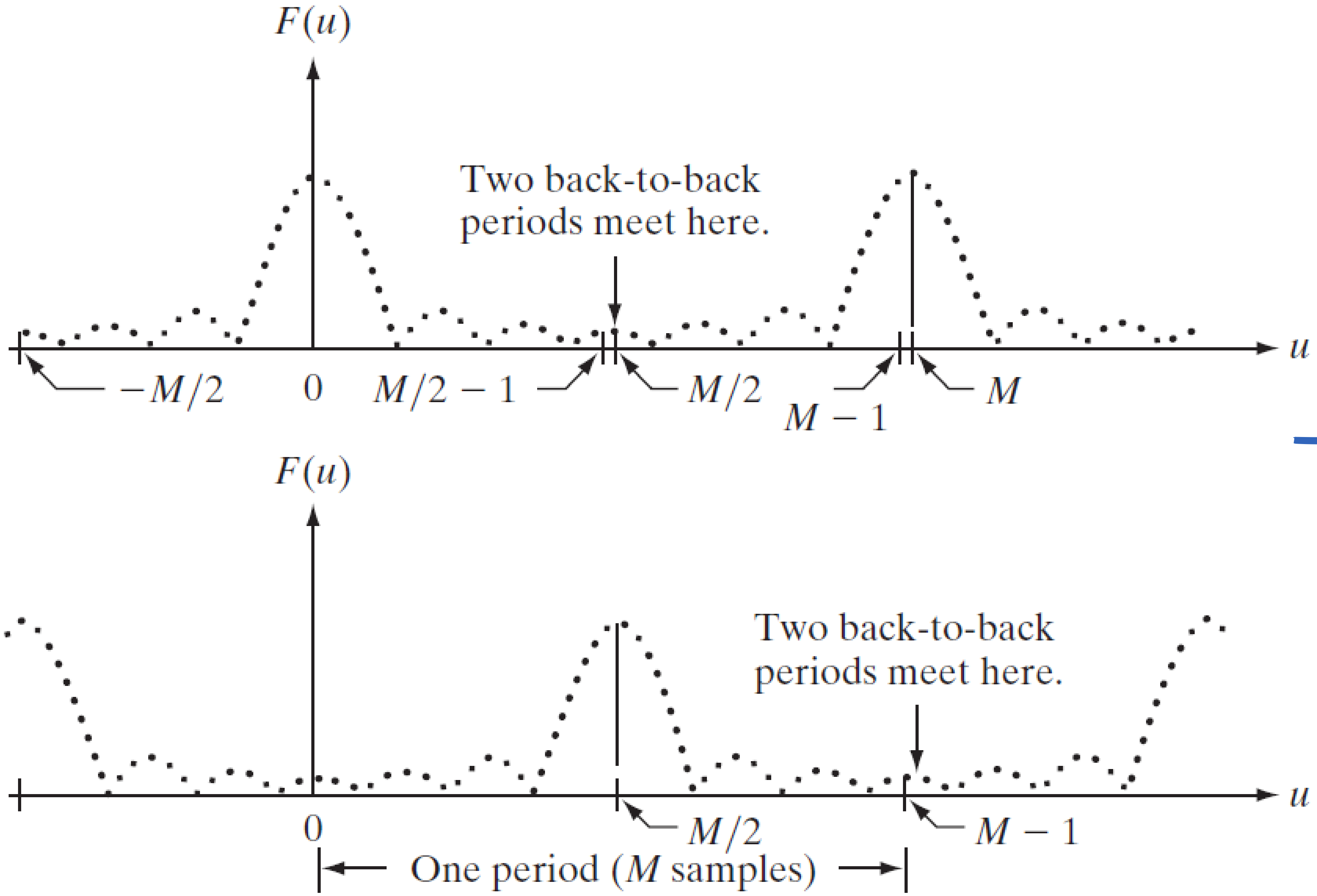

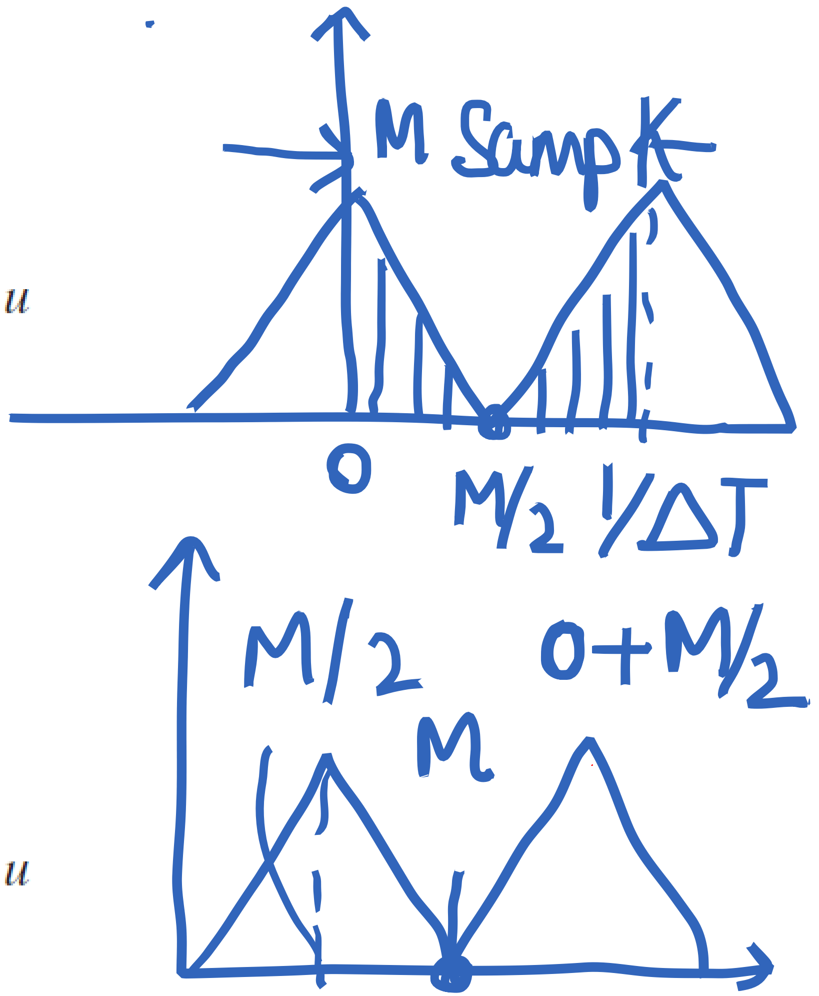

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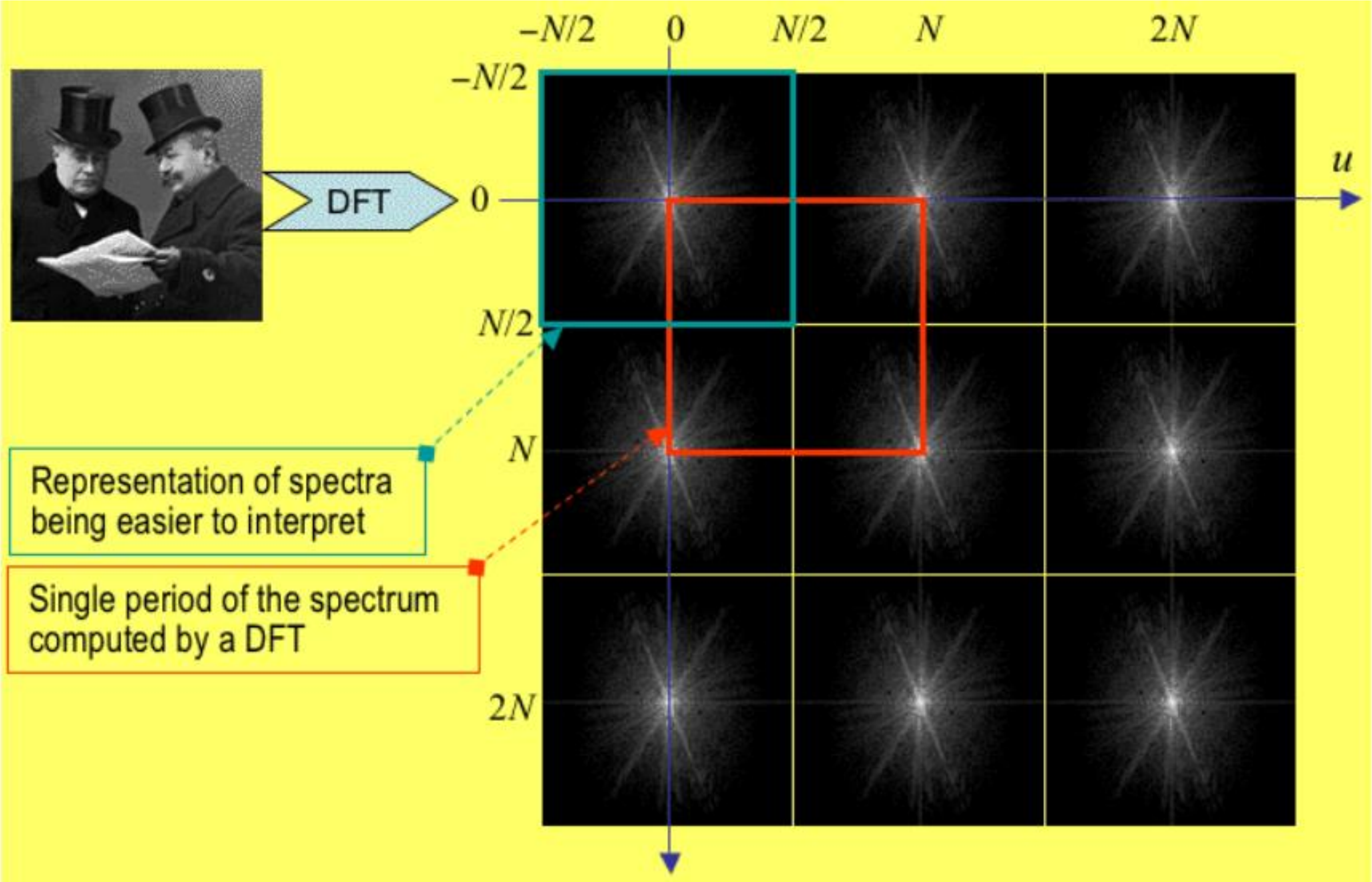

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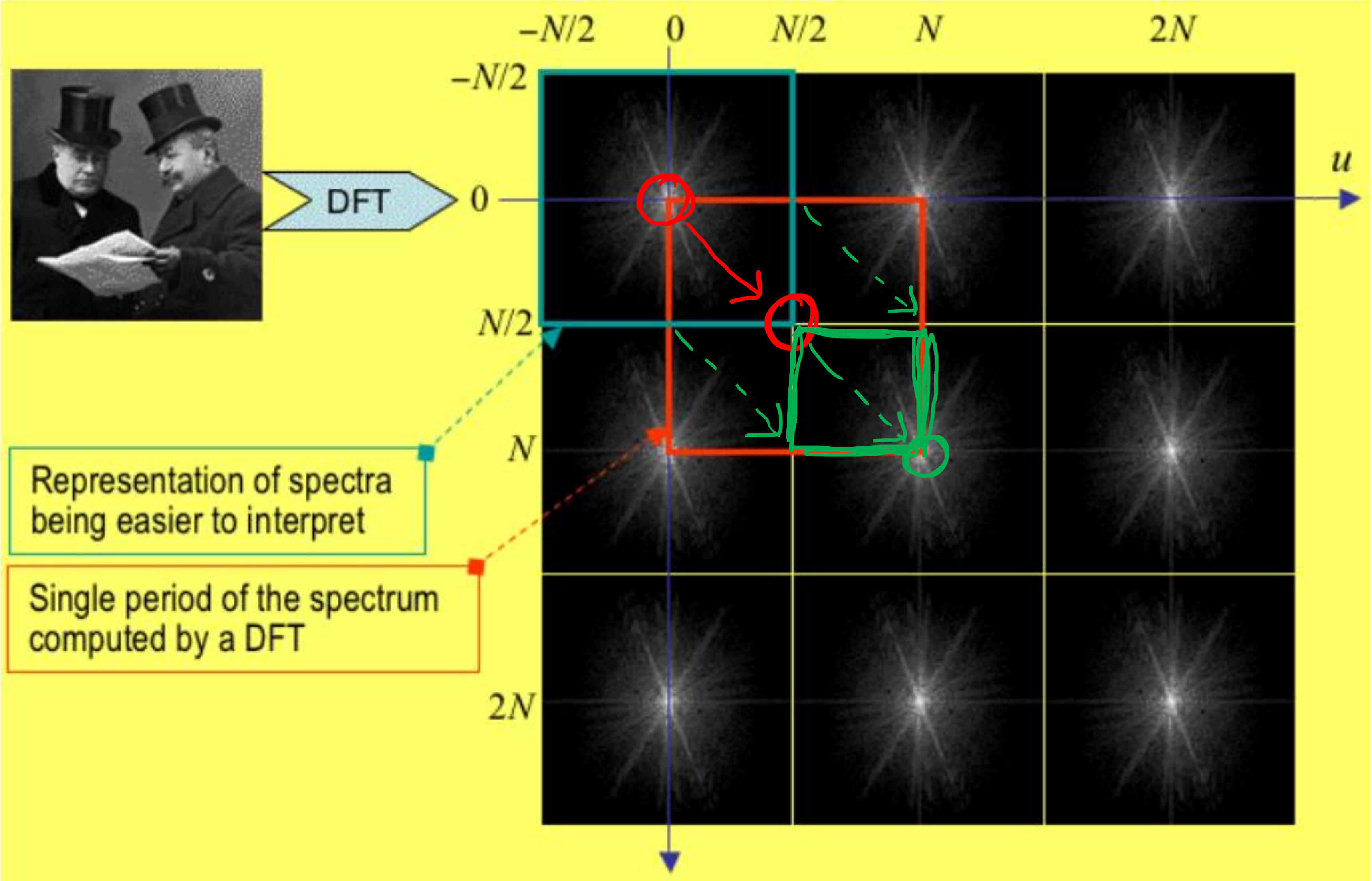
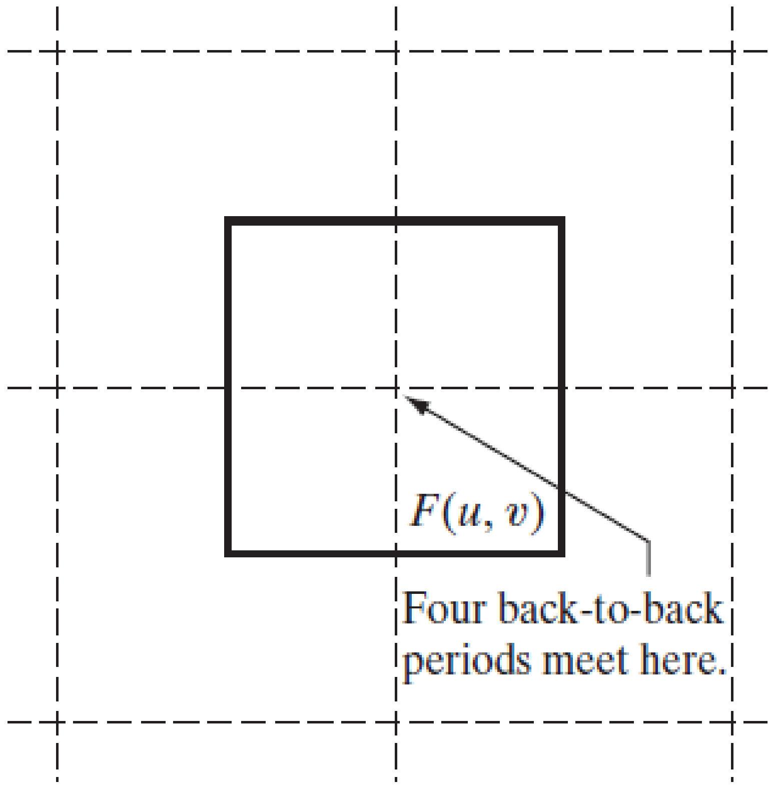
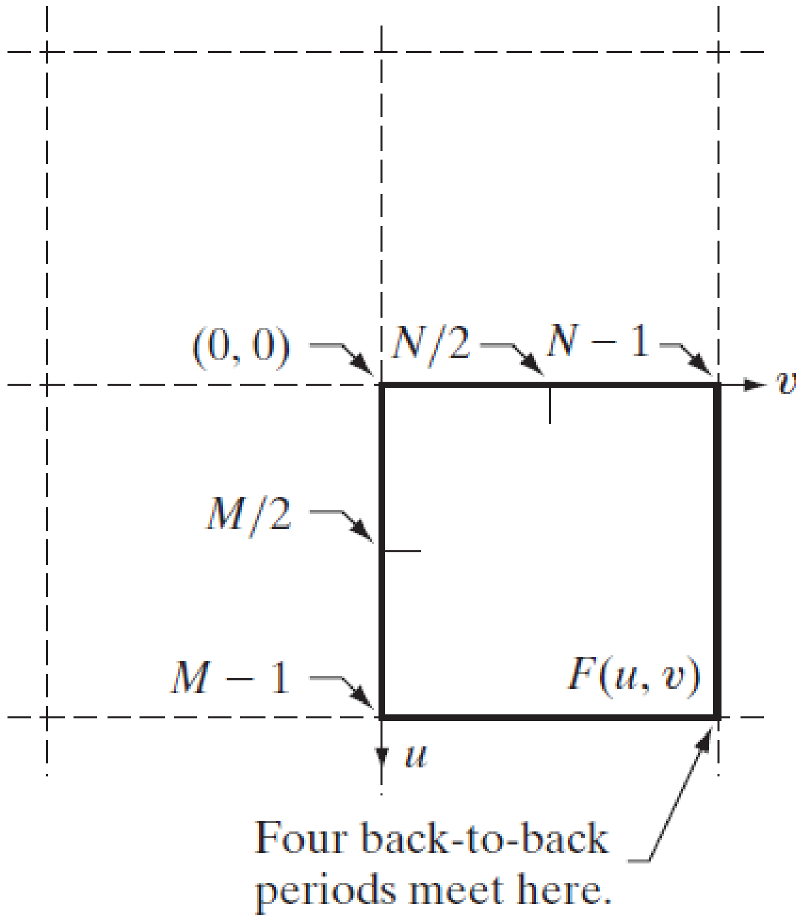




F(u, v)
Four back-to-back
periods meet here.



(0, 0)
N/2
N − 1
v
M/2
M − 1
F(u, v)
u
Four back-to-back
periods meet here.

Shift in Frequency

- How to get shift in frequency domain?
- By multiplying exponential in spatial domain.

- What is shift amount here?

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$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0)$$

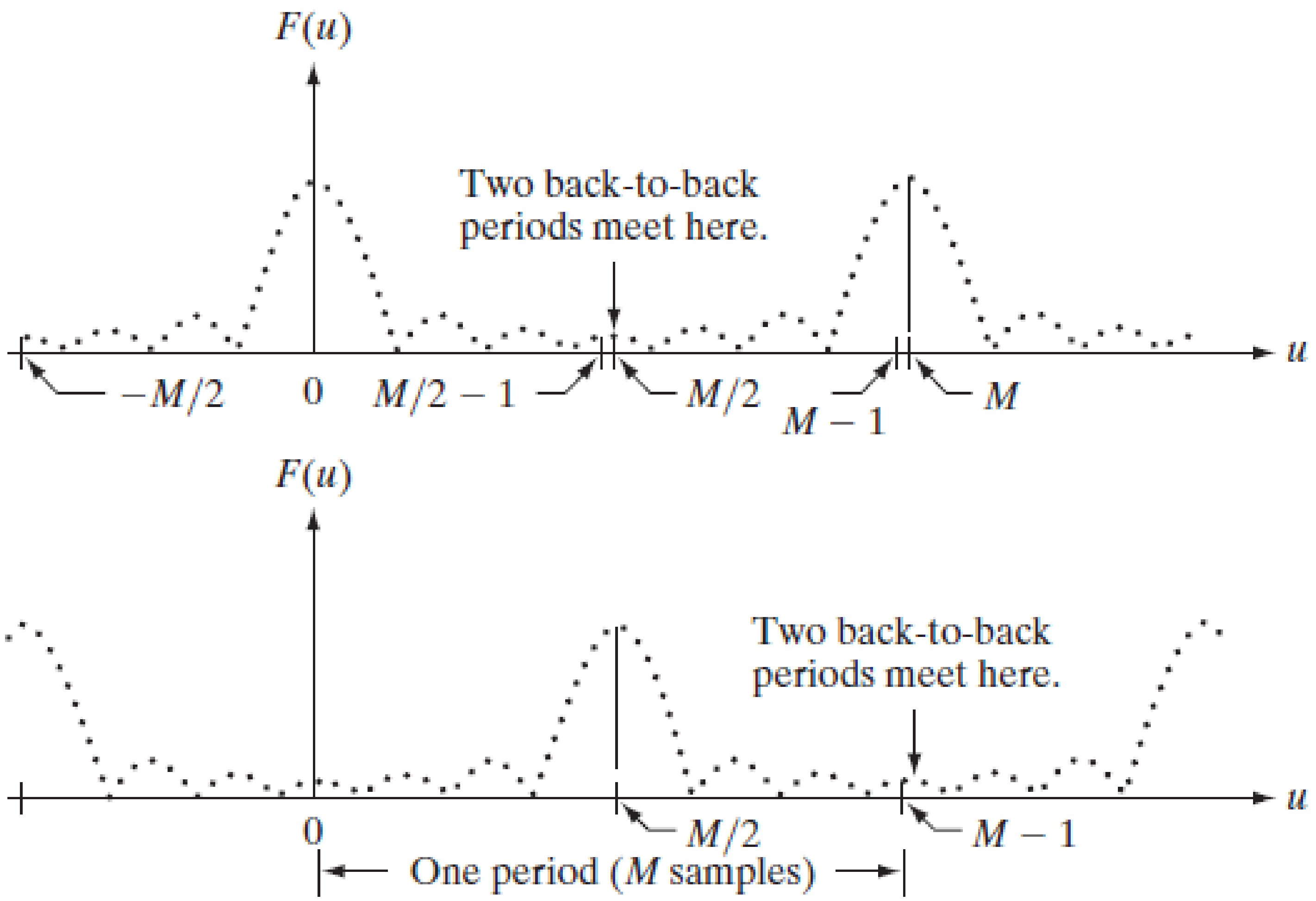
- What is shift amount here?

- Consider the 1-D spectrum in fig.(a) the transform data in the interval from 0 to $M - 1$ consists of two back-to-back half periods meeting at point $M / 2$. For display and filtering purposes, it is more convenient to have in this interval a complete period of the transform in which the data are contiguous, as in Fig.(b). It follows from Eq. that

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0)$$

- In other words, multiplying $f(x)$ by the exponential term shown shifts the data so that the origin, F(0), is located at $u0$ If we let $u0 = M / 2$, the exponential term becomes $e^{j\pi x}$which is equal to (-1)x because x is an integer. In this case,

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$



For 2-D

- u0=M/2
- v0=N/2

$$f(x,y)e^{j2\pi\left(\frac{x}{M}\left(\frac{M}{2}\right)+\frac{y}{N}\left(\frac{N}{2}\right)\right)} \leftrightarrow F(u-\frac{M}{2}, v-N/2)$$

$$f(x,y)e^{j2\pi\left(\frac{x}{M}\left(\frac{M}{2}\right)+\frac{y}{N}\left(\frac{N}{2}\right)\right)} = f(x,y)e^{j2\pi\left(\frac{x}{2}+\frac{y}{2}\right)}$$

$$= f(x,y)e^{j\pi(x+y)}$$

$x + y$ *is an integer so* $e^{j\pi(x+y)}$ *is* $(-1)^{x+y}$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(ux/M + vy/N)}$$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(ux/M + vy/N)}$$

$$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$F(0, 0) = MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M + vy/N)}$$

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$\begin{aligned} F(0,0) &= MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \\ &= MN\overline{f}(x,y) \end{aligned}$$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$F(0, 0) = MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$= MN\overline{f}(x, y)$$

$$|F(0, 0)| = MN|\overline{f}(x, y)|$$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$F(0, 0) = MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$= MN\bar{f}(x, y)$$

$$|F(0, 0)| = MN|\bar{f}(x, y)|$$

- Because the proportionality constant *MN* usually is large,
- |F(0,0)| typically is the largest component of the spectrum by a factor that can be several orders of magnitude larger than other terms.
- Because frequency components u and v are zero at the origin, (0, 0) sometimes is called the *dc component* of the transform.
- This terminology is from electrical engineering, where “dc” signifies direct current (i.e., current of zero frequency).

Scaled [0 -255]

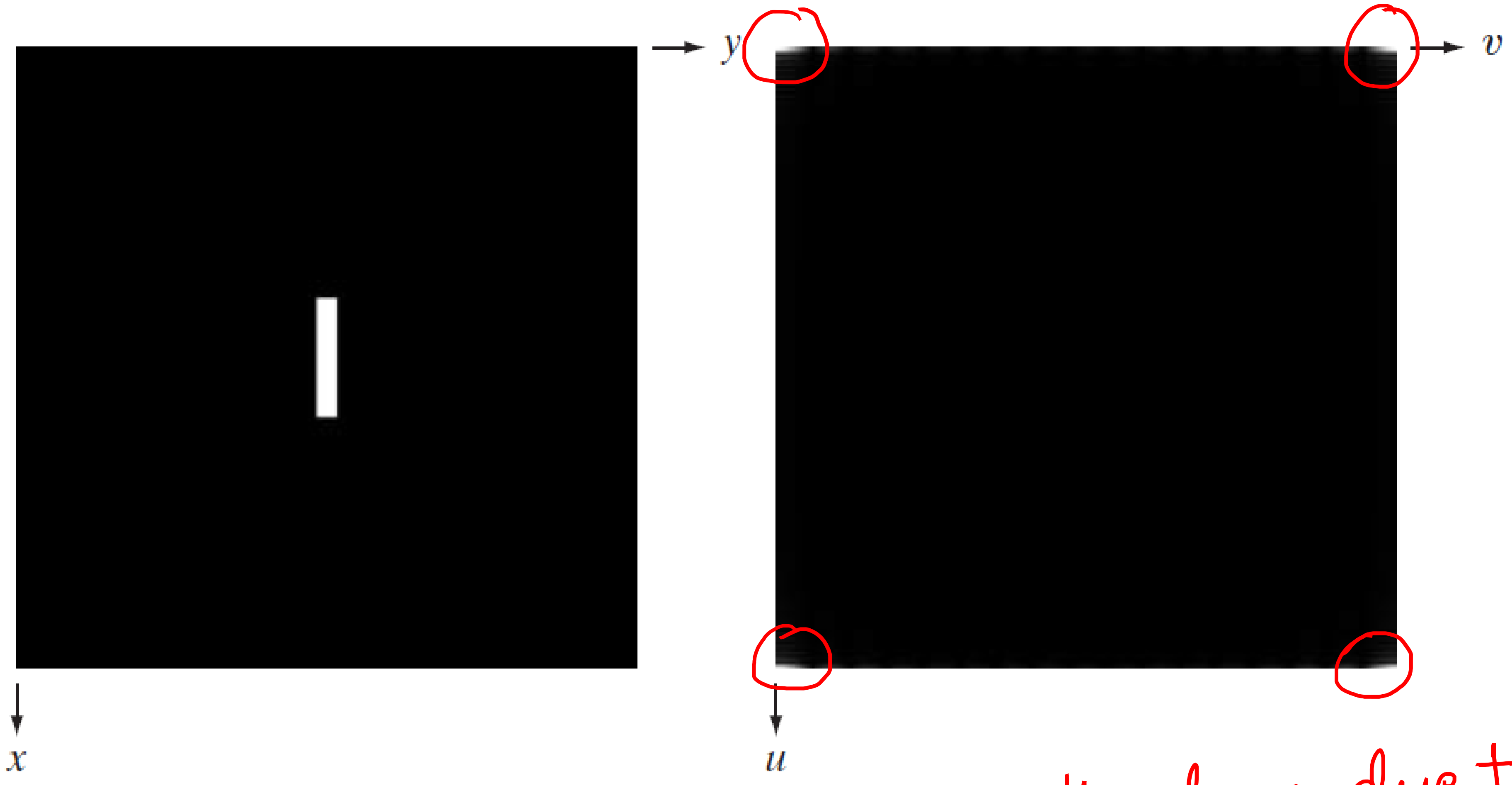
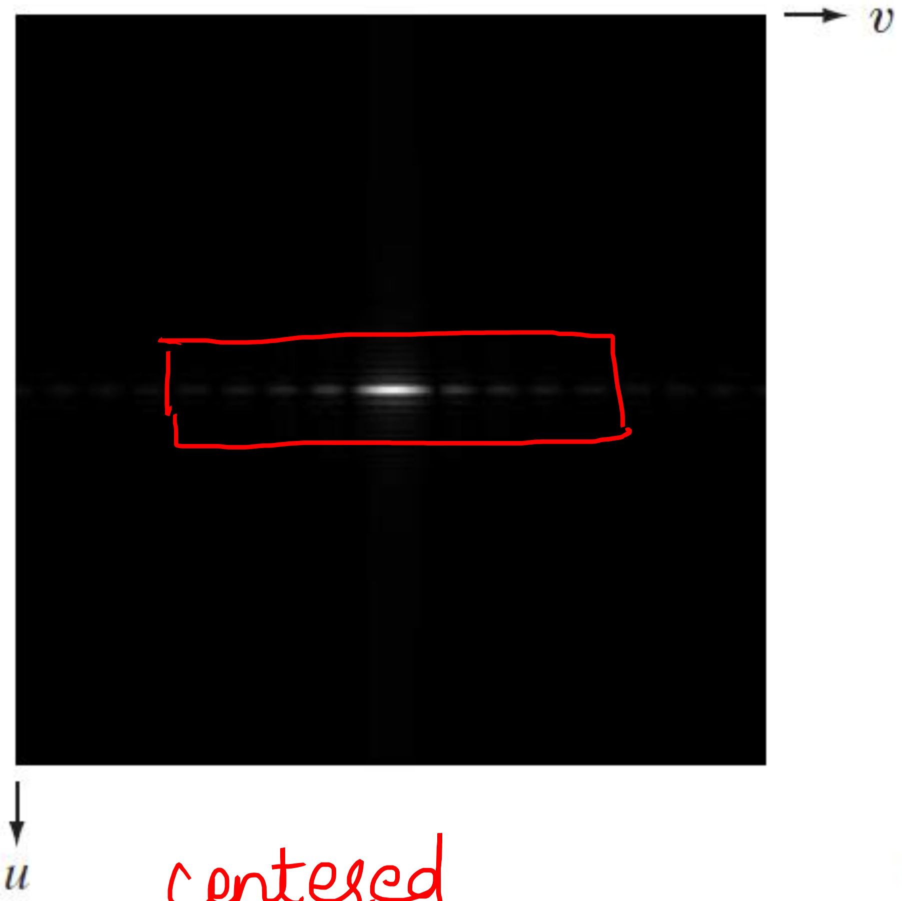


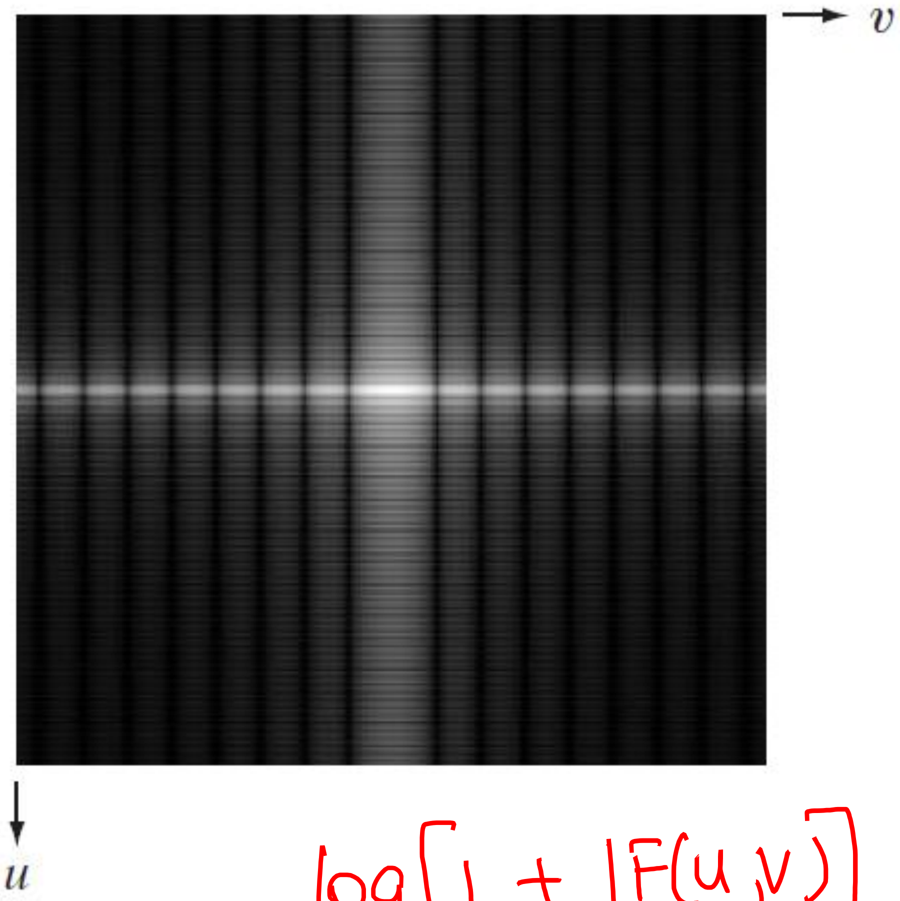
FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.

4 values due to periodicity

(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



centered spectrum



$\log[1 + |F(u,v)|]$

Significance of Spectrum and Phase Example 4.14 (From Book)

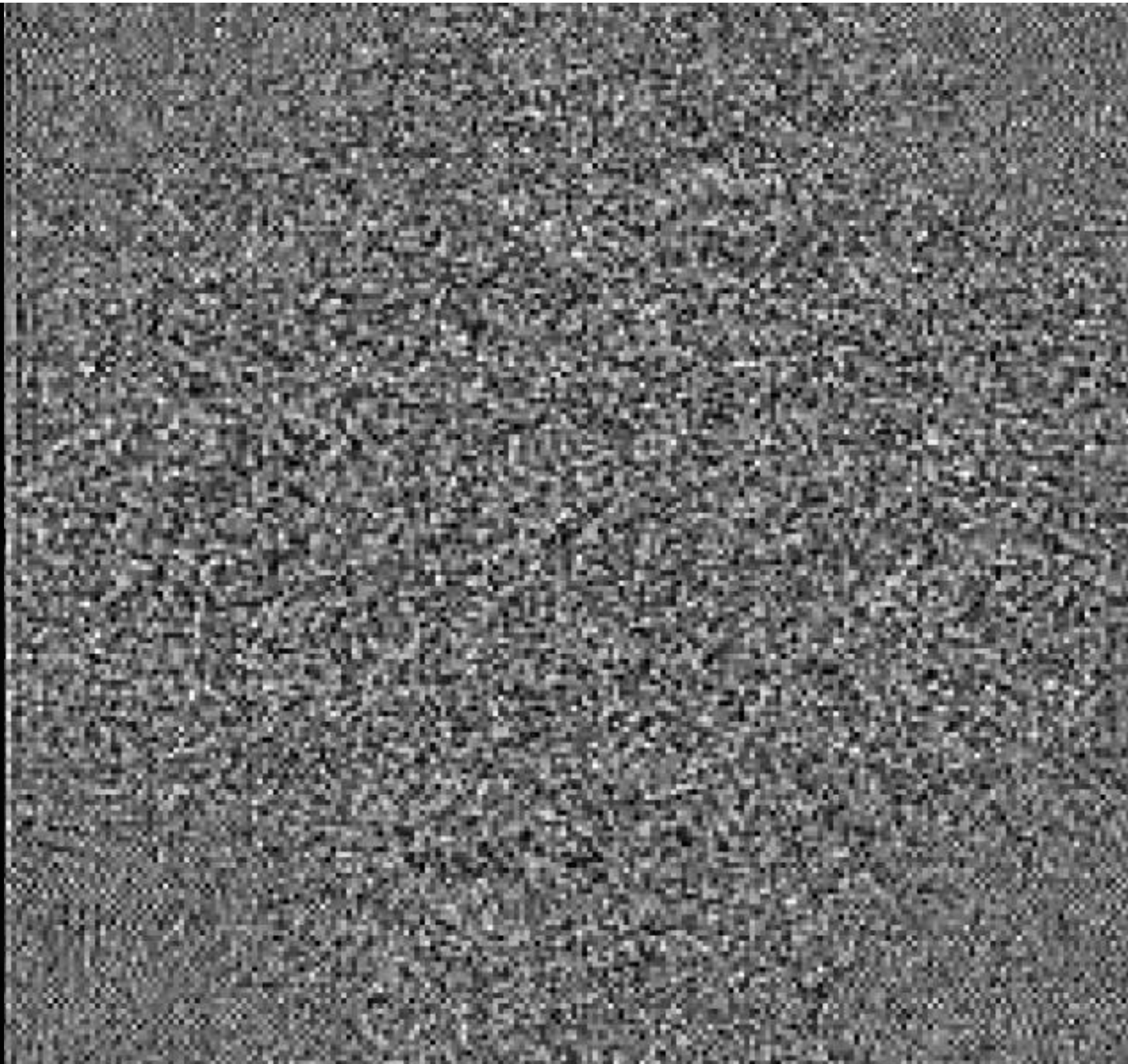
Significance of Spectrum and Phase Example 4.14 (From Book)

Significance of Spectrum and Phase Example 4.14 (From Book)

Significance of Spectrum and Phase Example 4.14 (From Book)

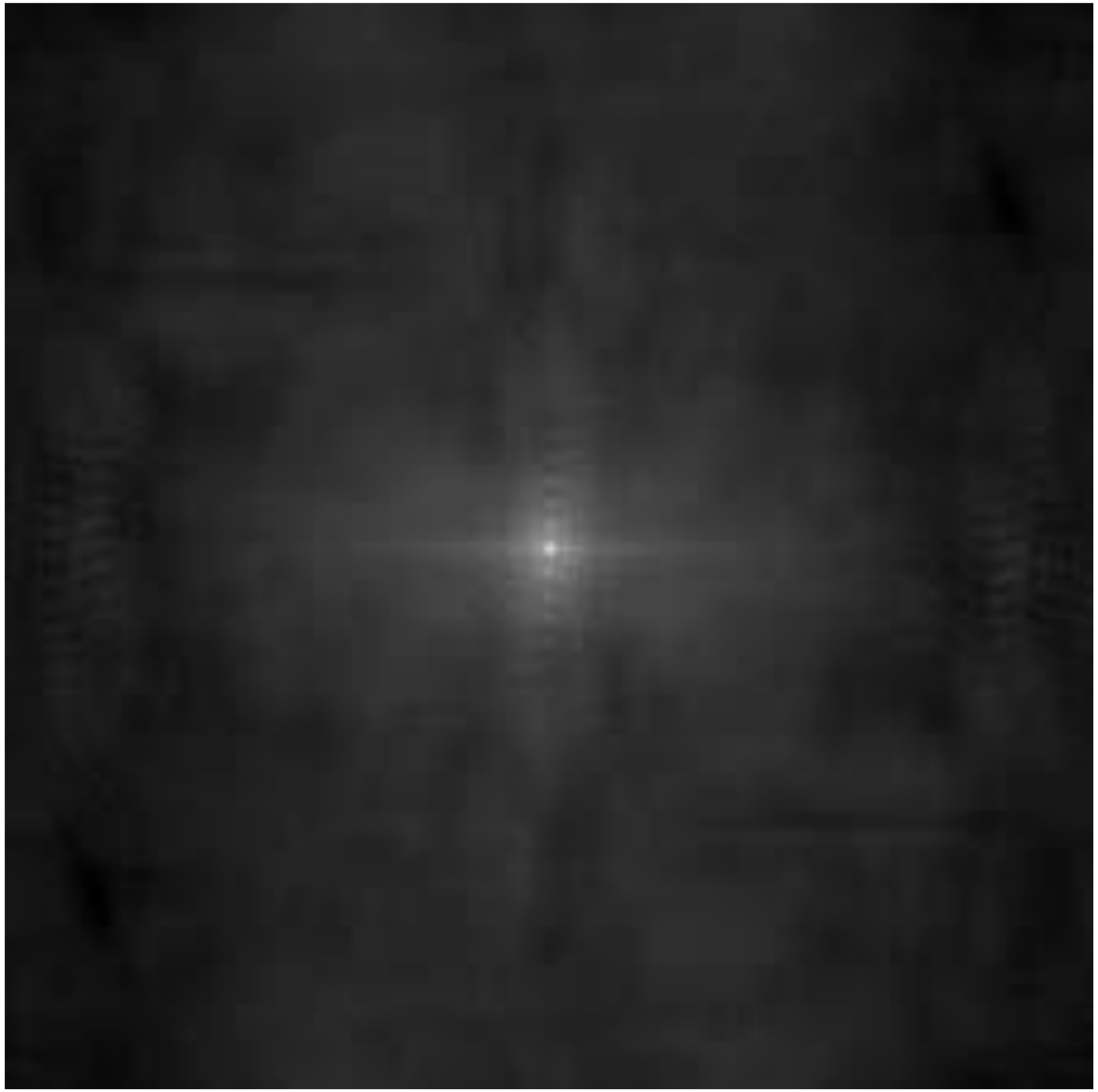
Original

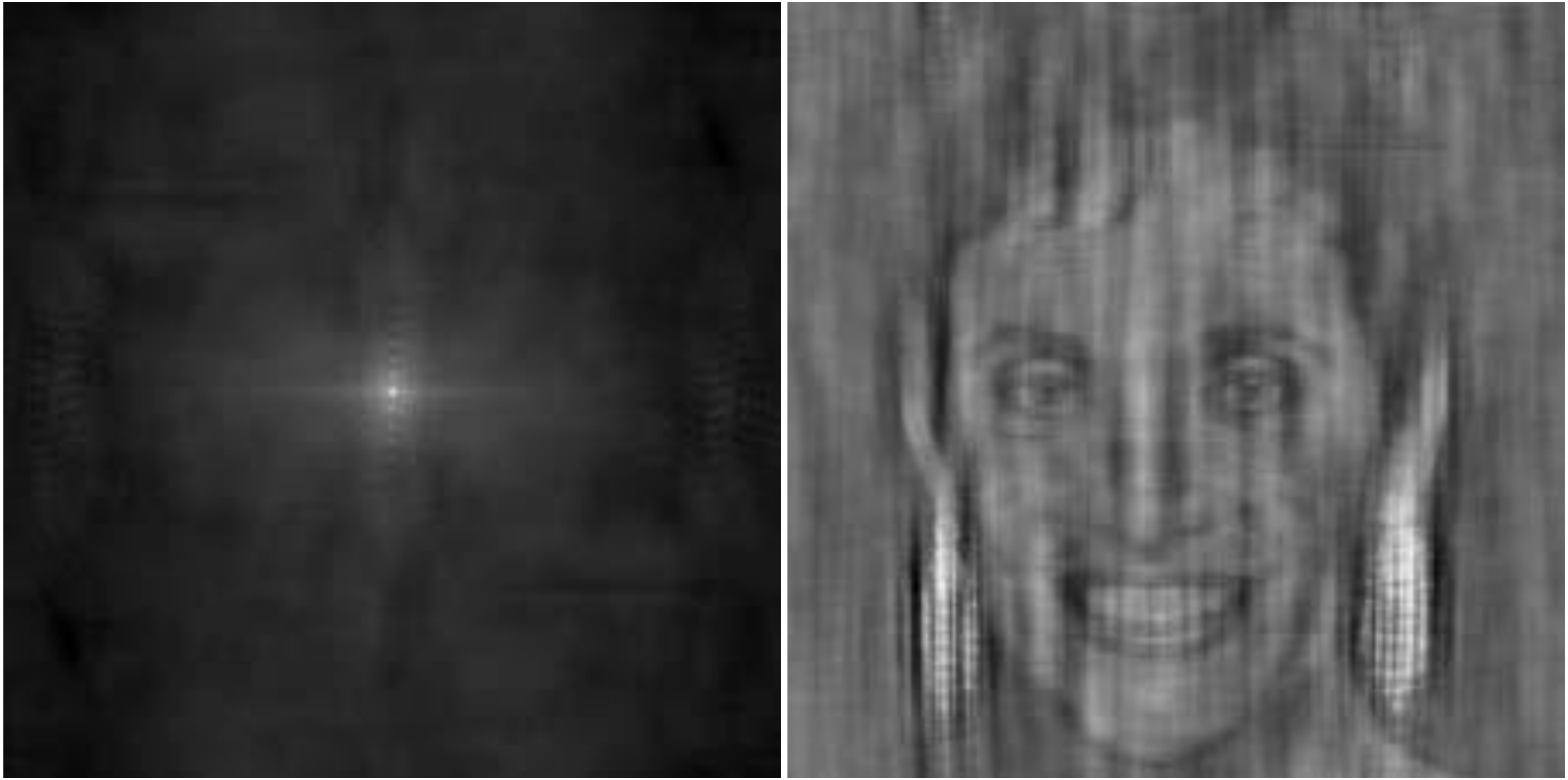
phase angle of DFT



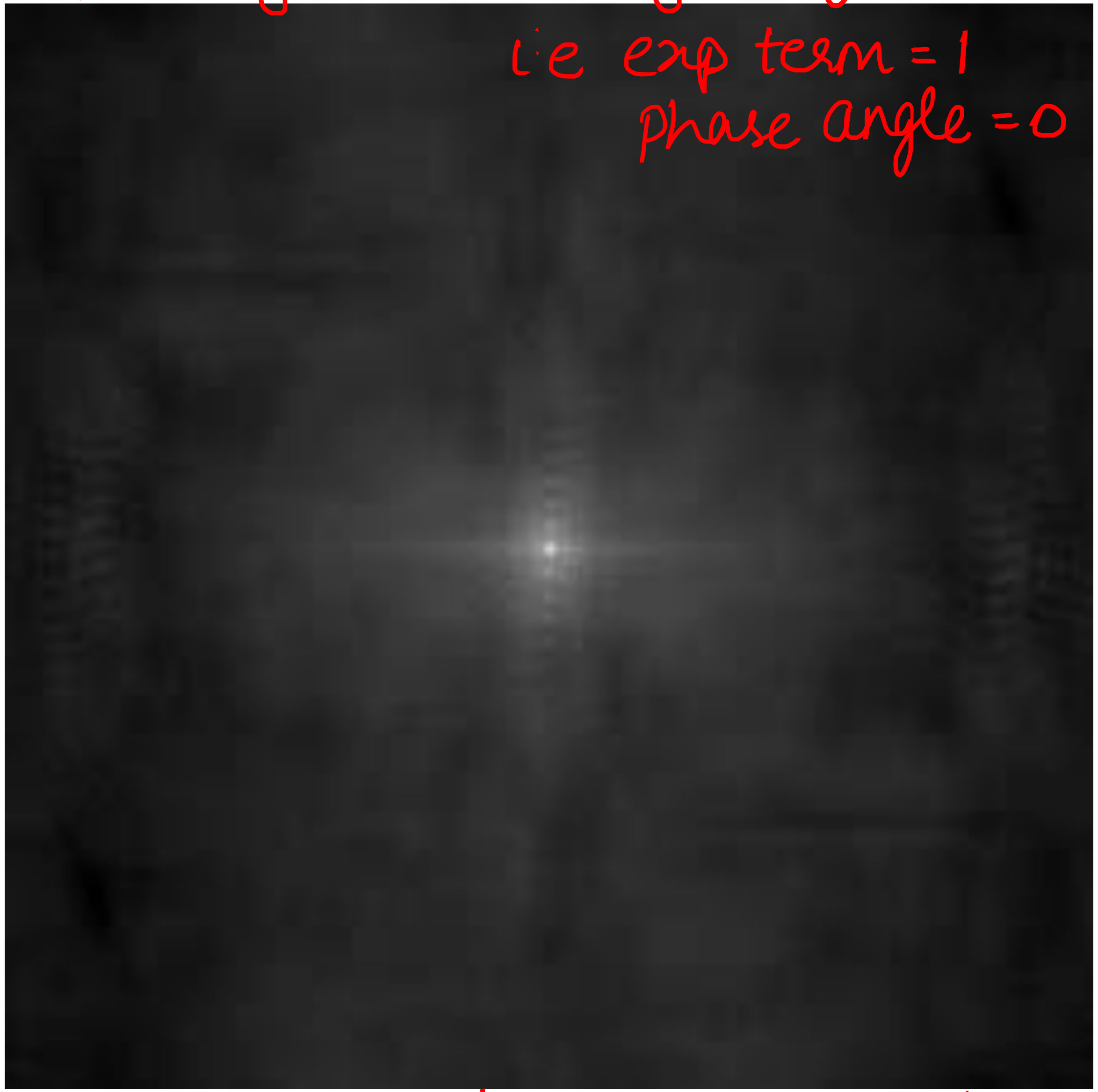
Original

Computing IDFT using only phase





computing IDFT using only spectrum

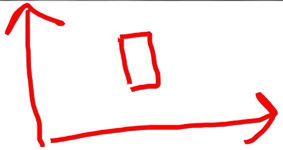


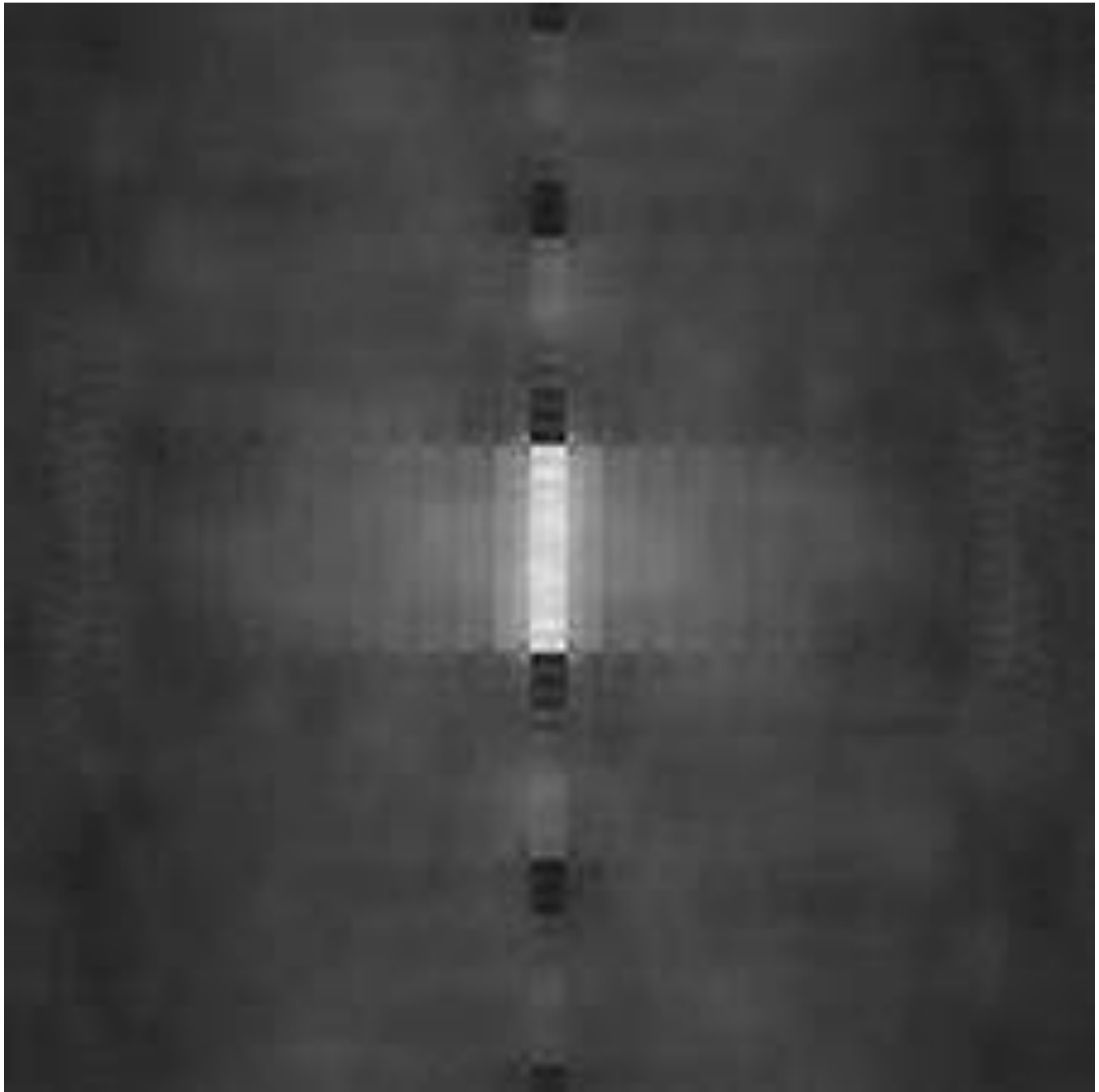
i.e exp term = 1
phase angle = 0

using phase ∠ of original

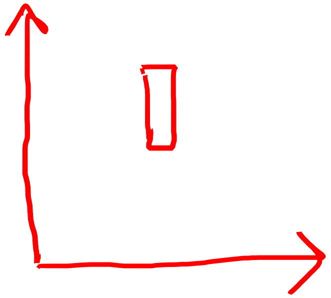
dc term max at center
no shape info

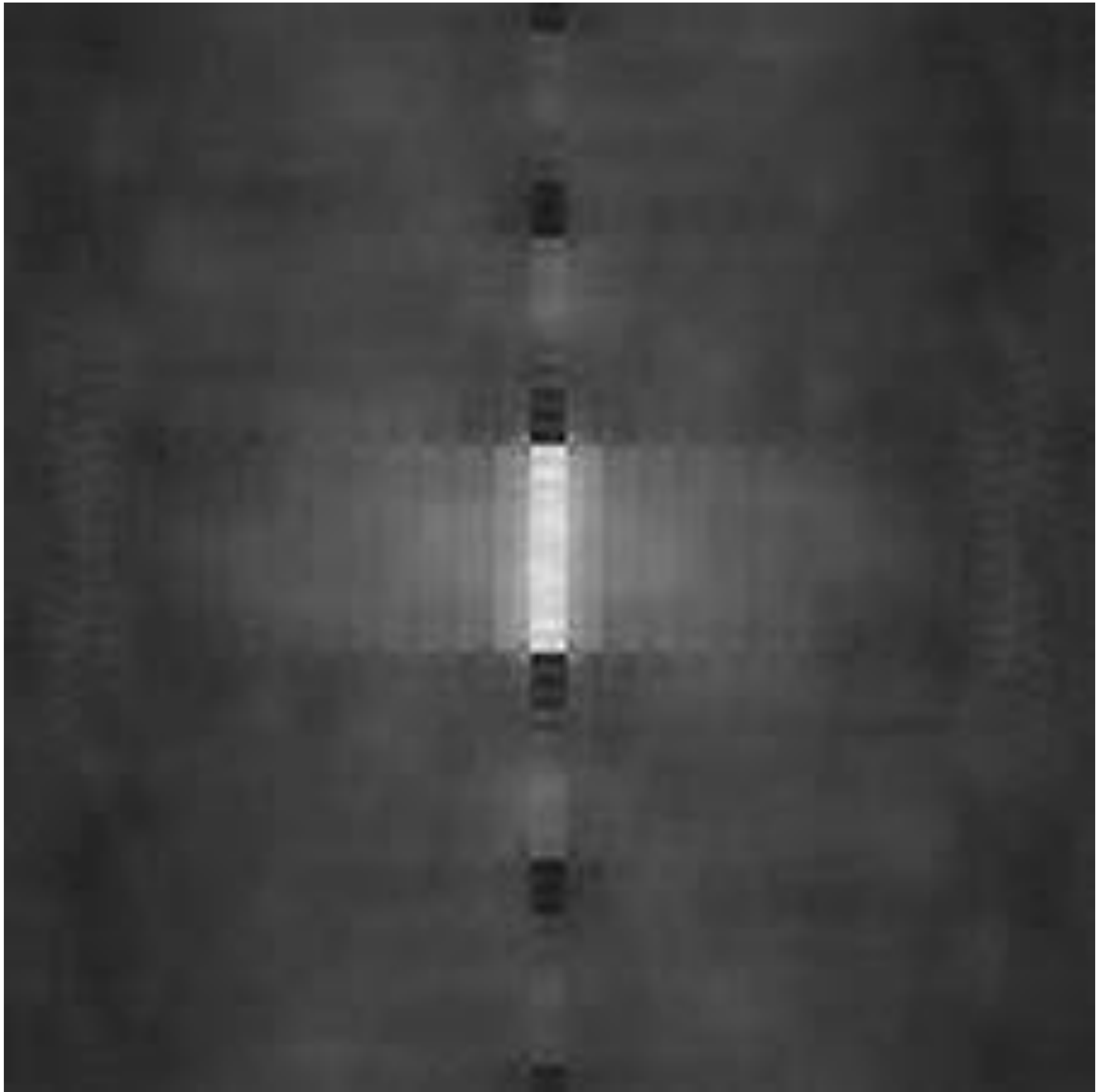
spectrum of



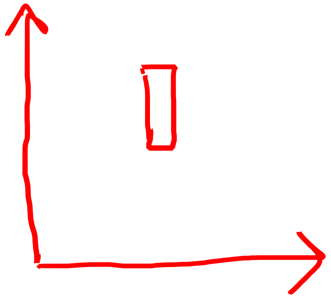


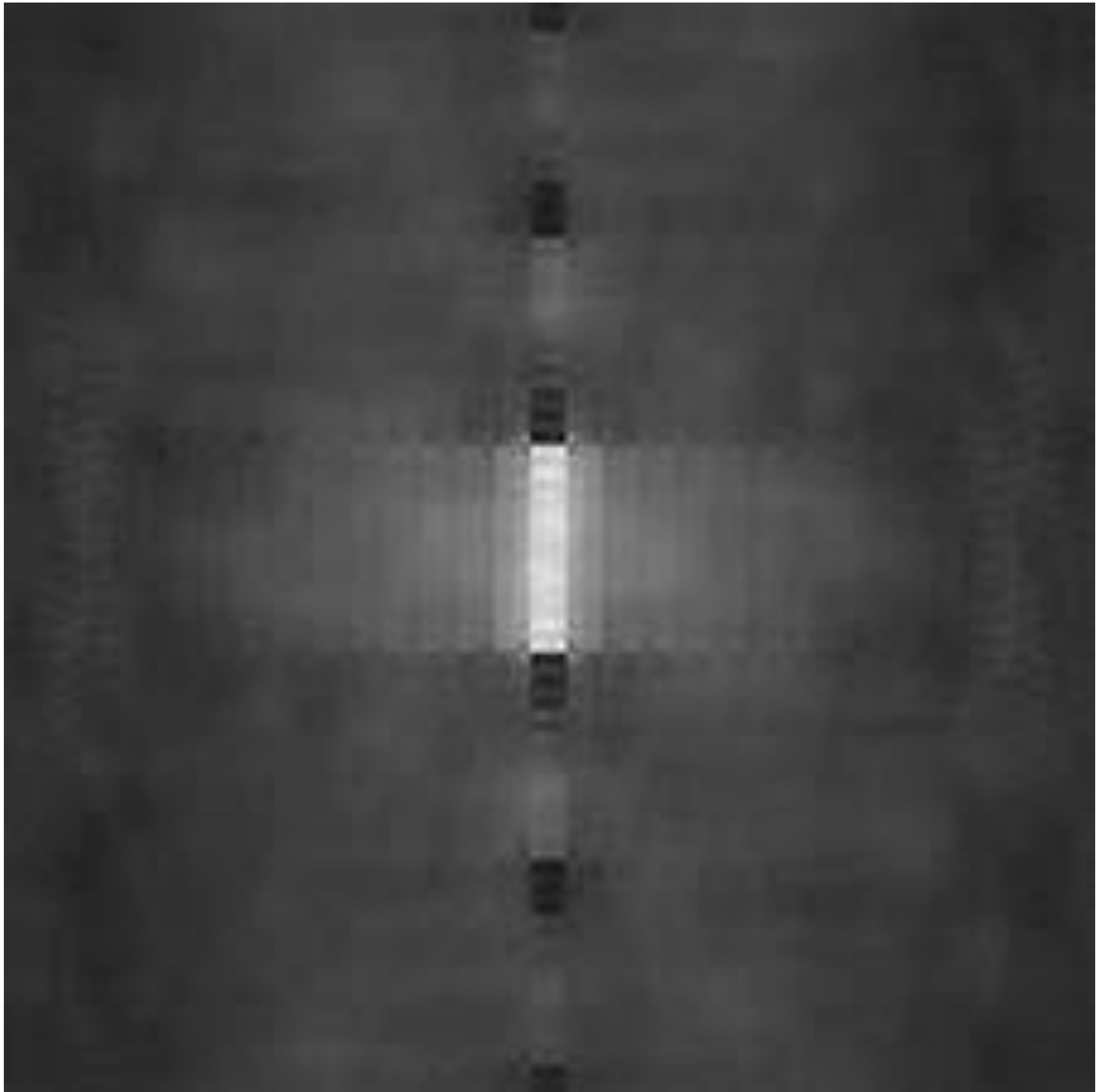
using spectrum of original & phase ∠ of

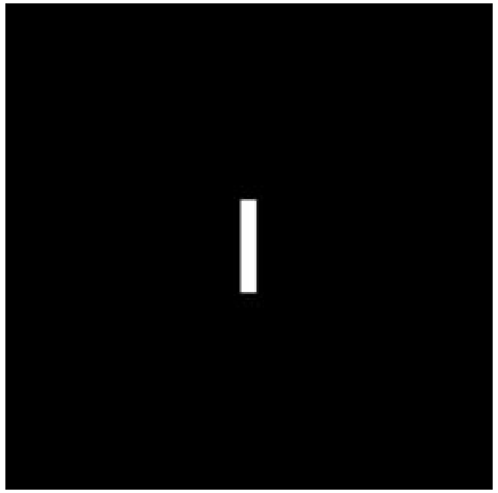




using spectrum of original & phase ∠ of







2-D Circular Convolution

2-D Circular Convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x-m, y-n)$$

2-D Circular Convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x-m, y-n)$$

- For $x = 1, 2, 3, \ldots, M - 1$ *and* $y = 1, 2, 3, \ldots, N - 1$

2-D Circular Convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x-m, y-n)$$

- For $x = 1, 2, 3, \ldots, M - 1$ *and* $y = 1, 2, 3, \ldots, N - 1$

2-D Convolution Theorem

2-D Convolution Theorem

$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

2-D Convolution Theorem

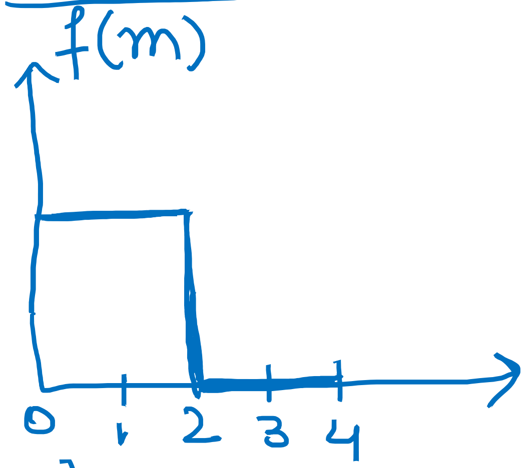
$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

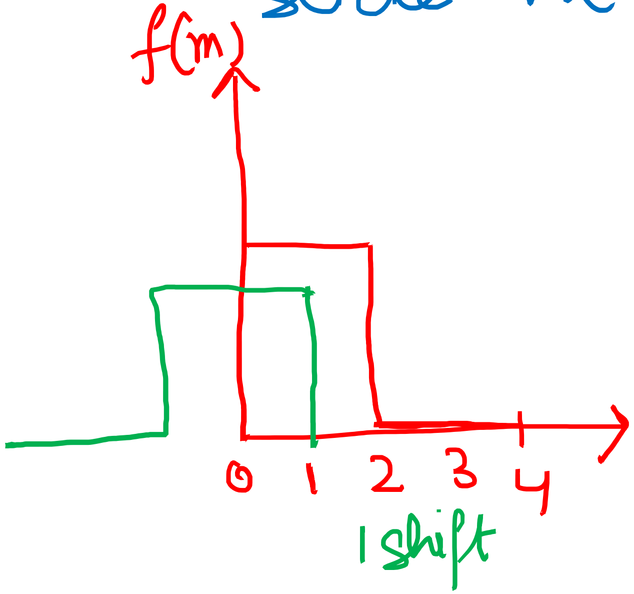
$$f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$$

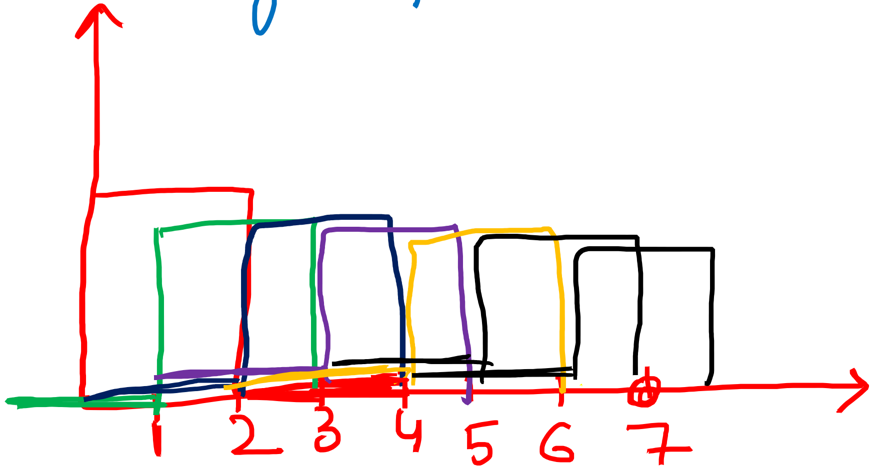
Linear Convolution

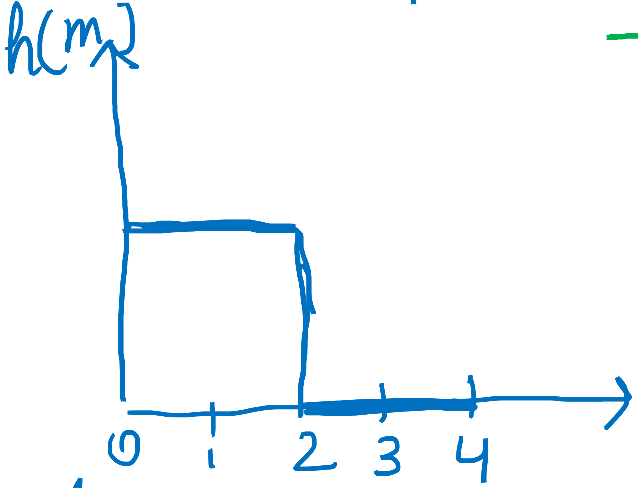
x = shift $\quad x = 1$

slide $h(-m)$ Through $f(m)$



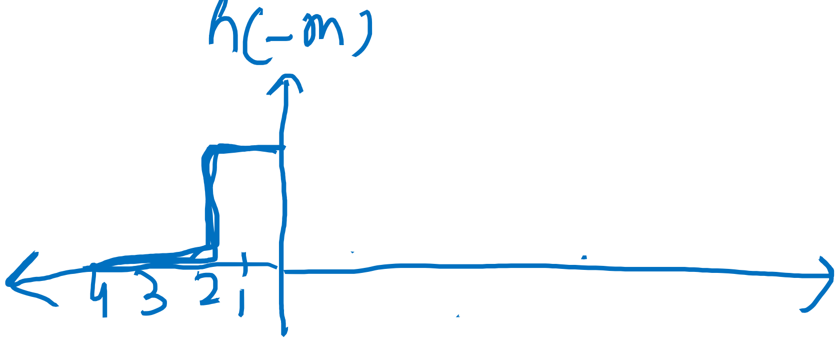




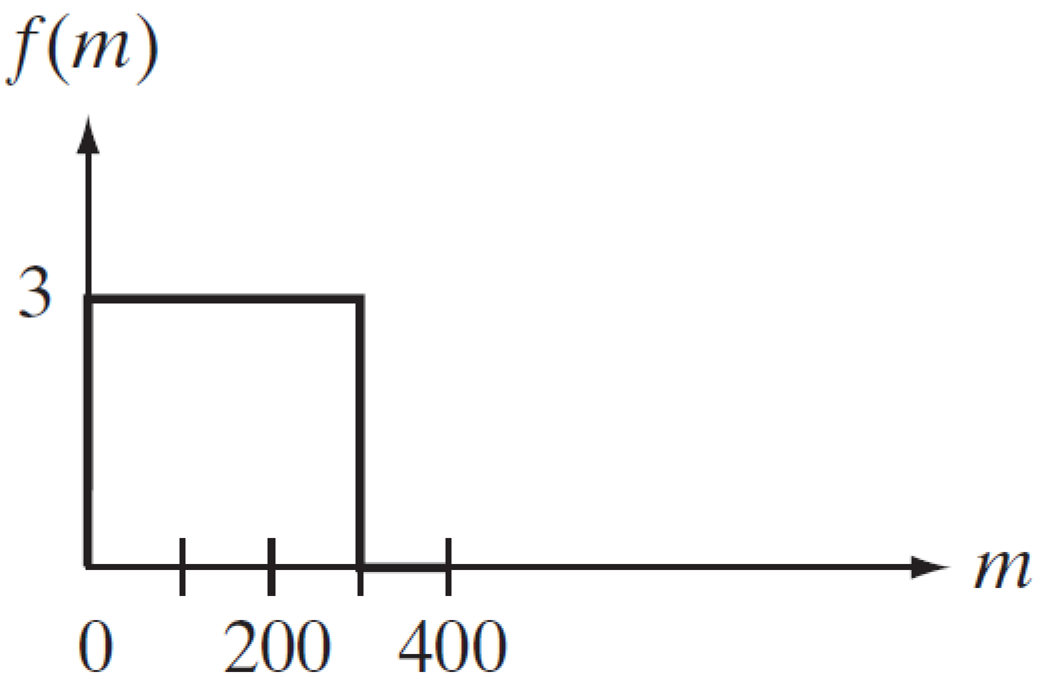


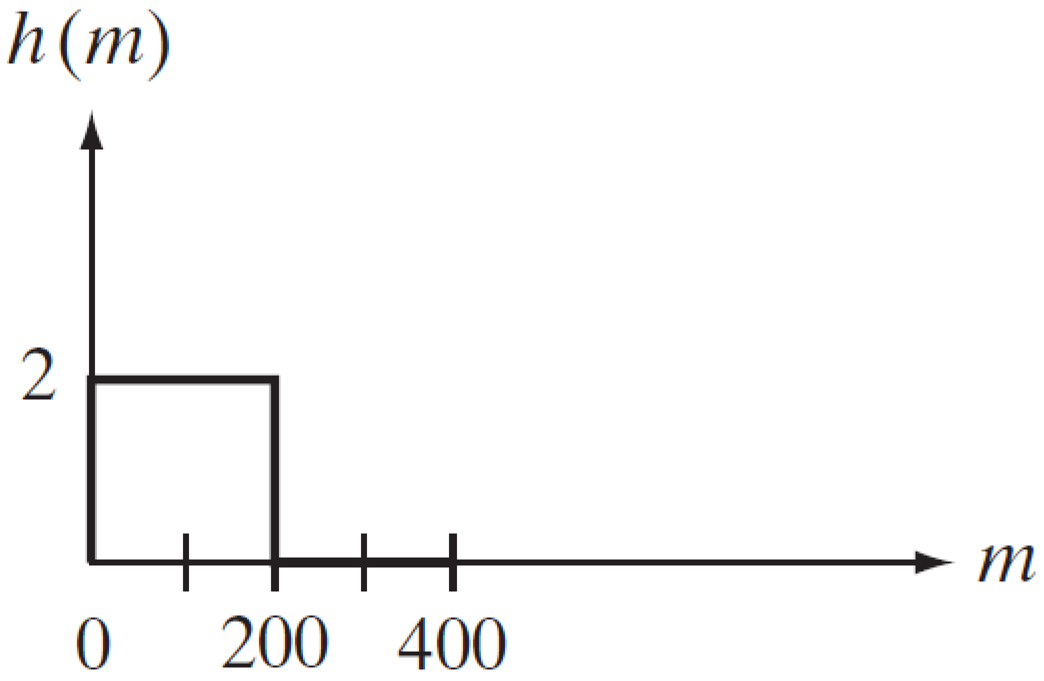
range of x req $0 - 7$

$$\underbrace{m_f + n_h - 1}$$



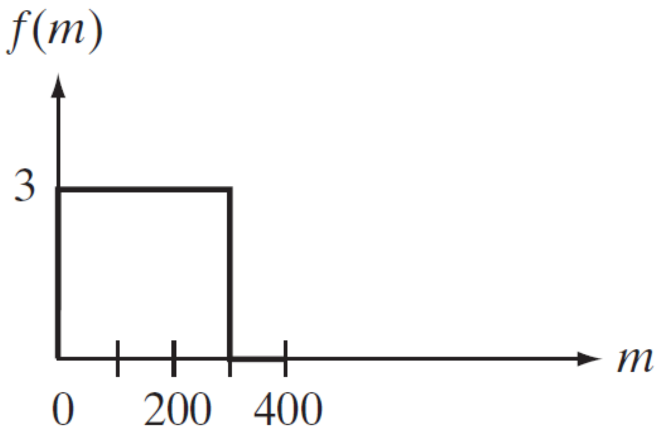
Consider 1-D Case

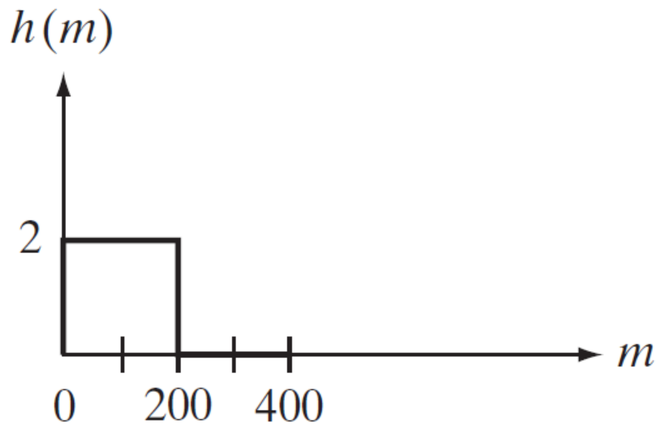


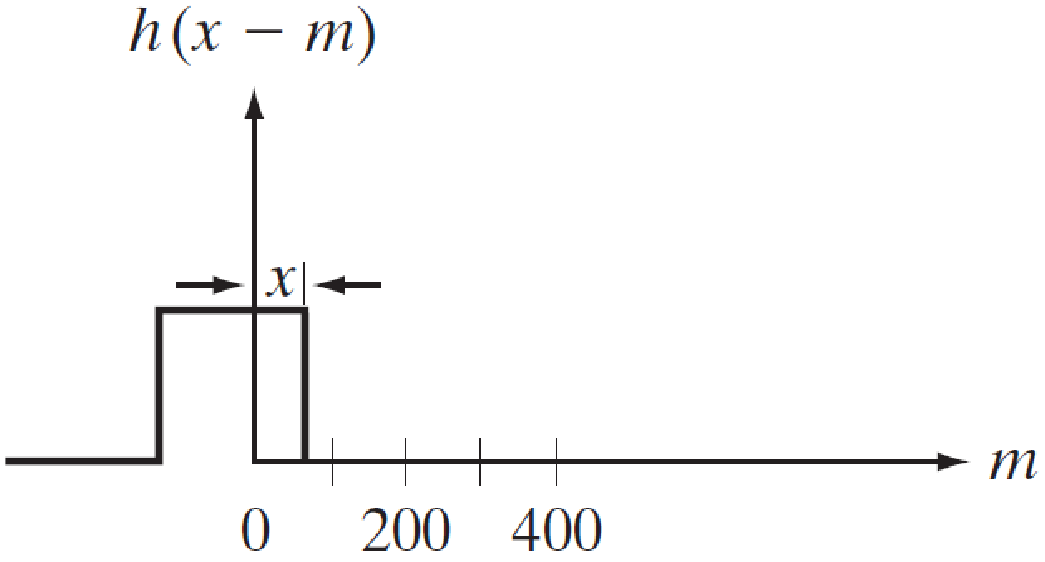


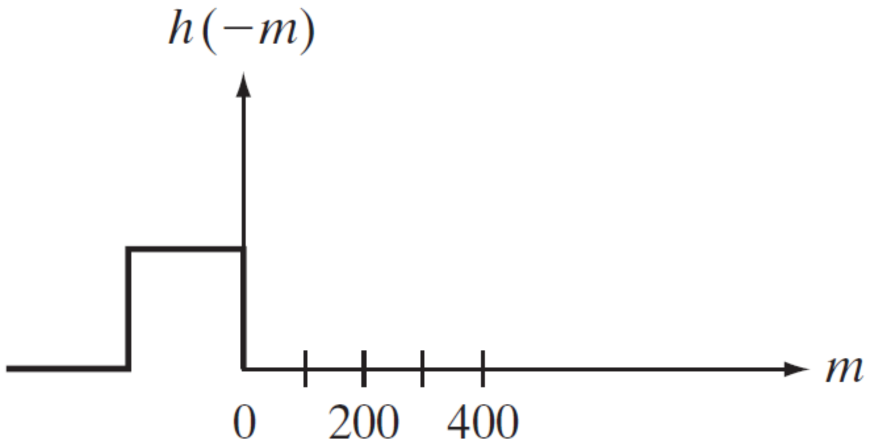
$$f(x) \star h(x) = \sum_{m=0}^{399} f(x)h(x - m)$$

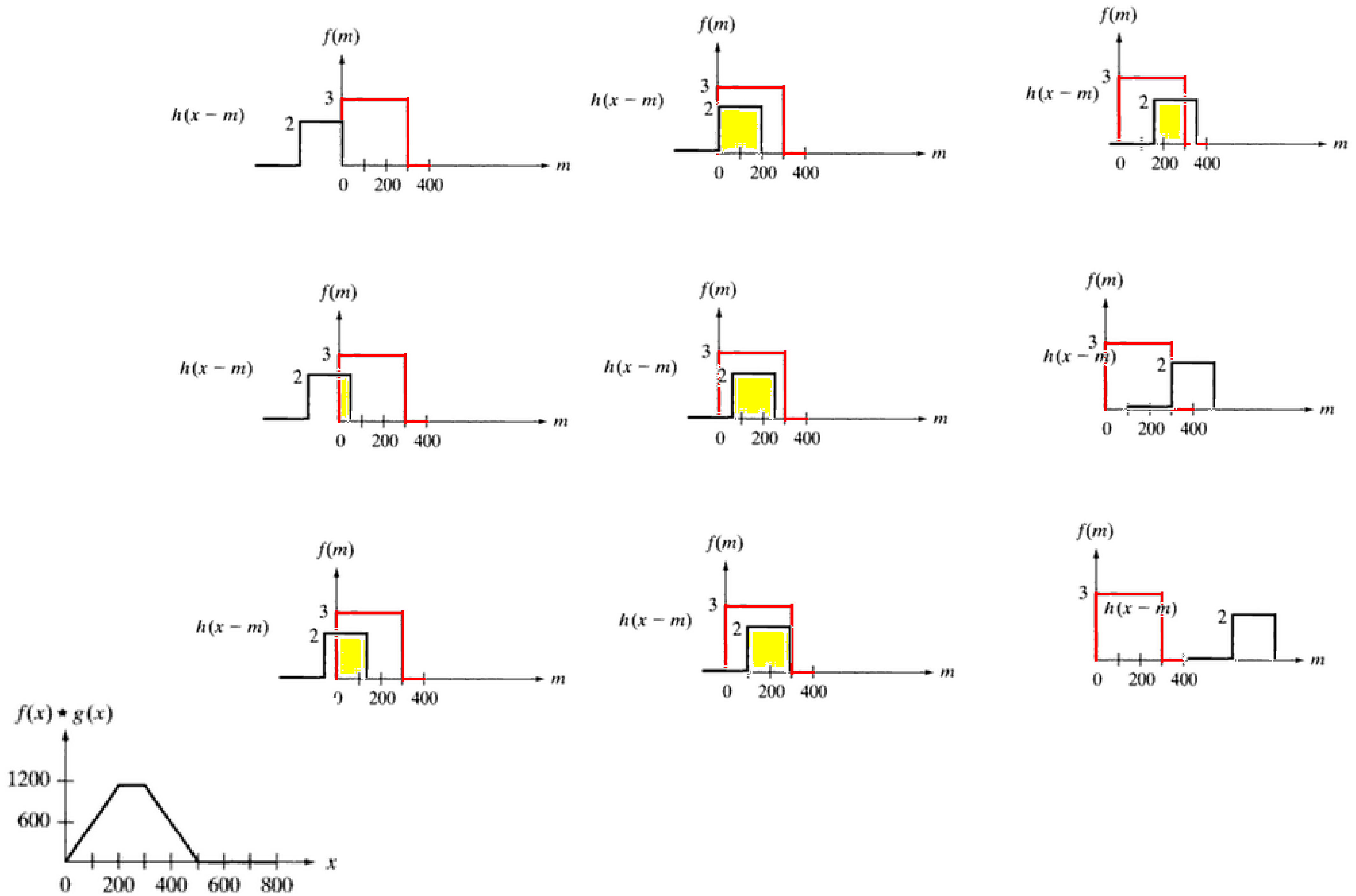
$$f(x) \star h(x) = \sum_{m=0}^{399} f(x)h(x - m)$$











f(m)
h(x − m)
3
2
0
200
400
m
f(x) * g(x)
1200
600
0
200
400
600
800
x

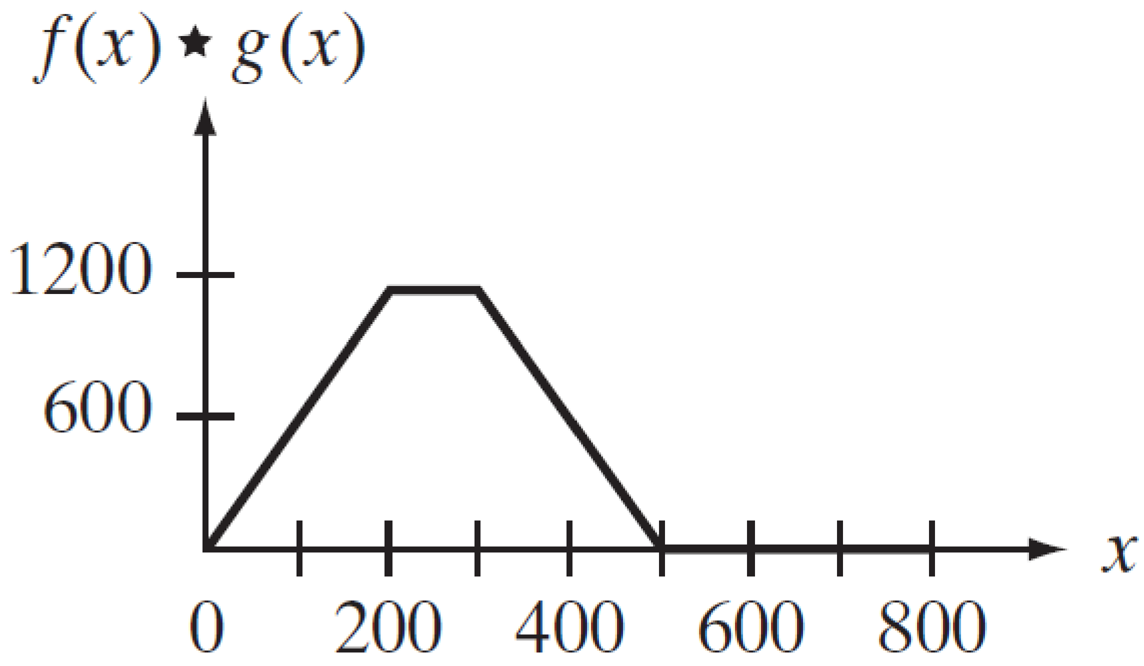
Convolution Result

- the procedure consists of

(1) Mirroring *h* about the origin (i.e., rotating it by 180°)

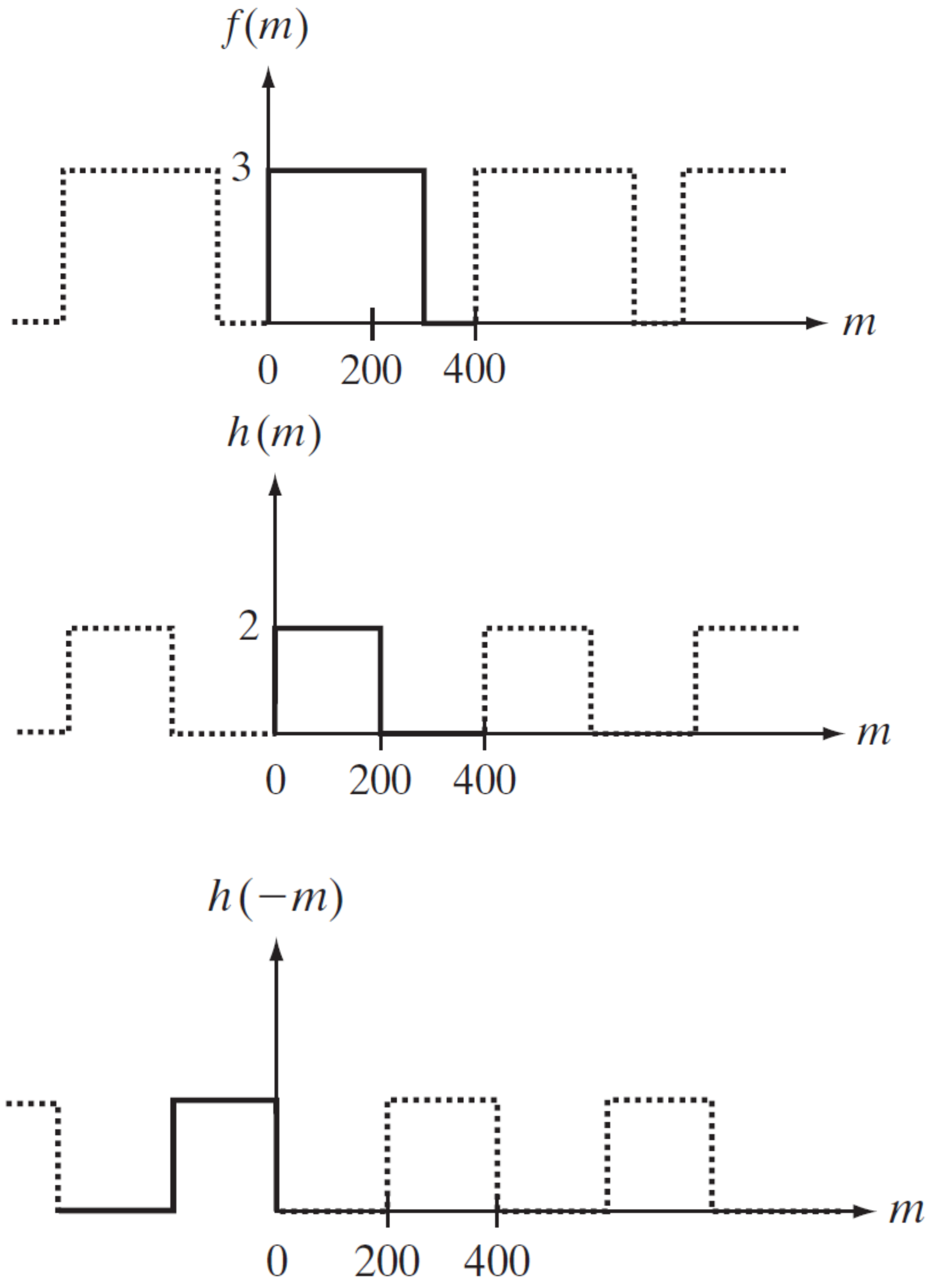
(2) translating the mirrored function by an amount *x*

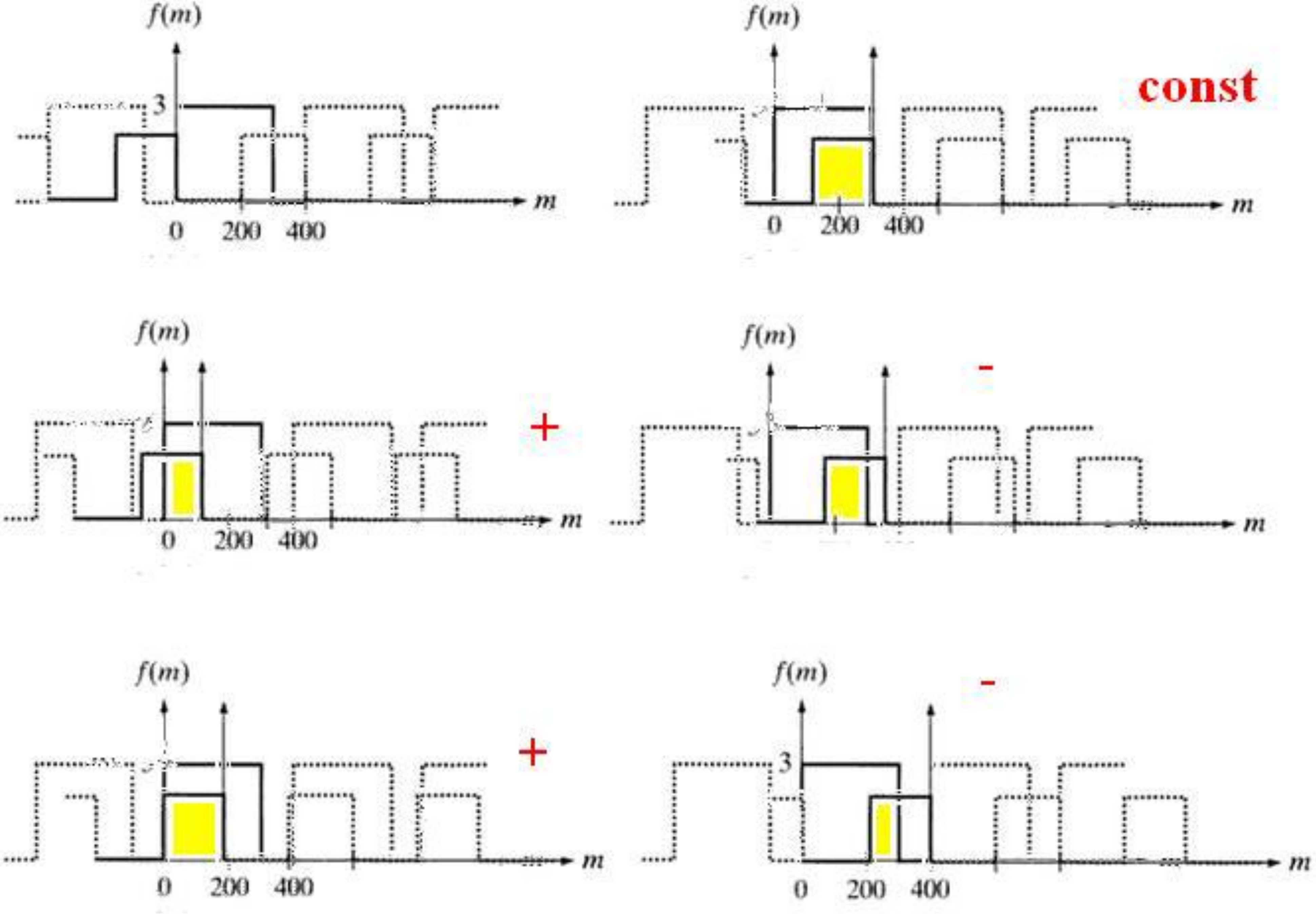
(3) for *each* value *x* of translation, computing the *entire* sum of products in the right side of the preceding equation.



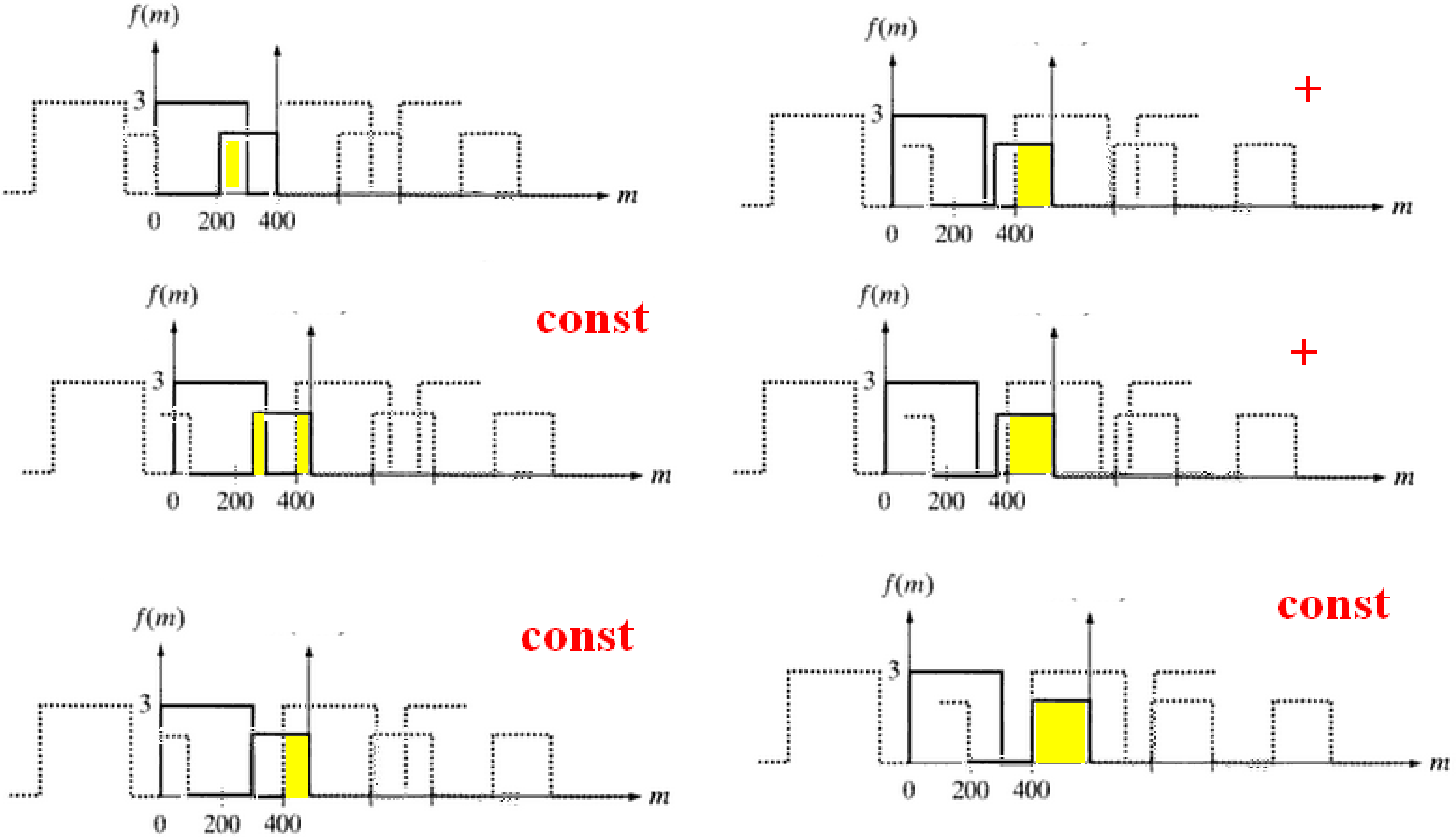
Periodicity of function

- Suppose we calculate the two functions using IDFT then spatial convolution would be periodic in nature



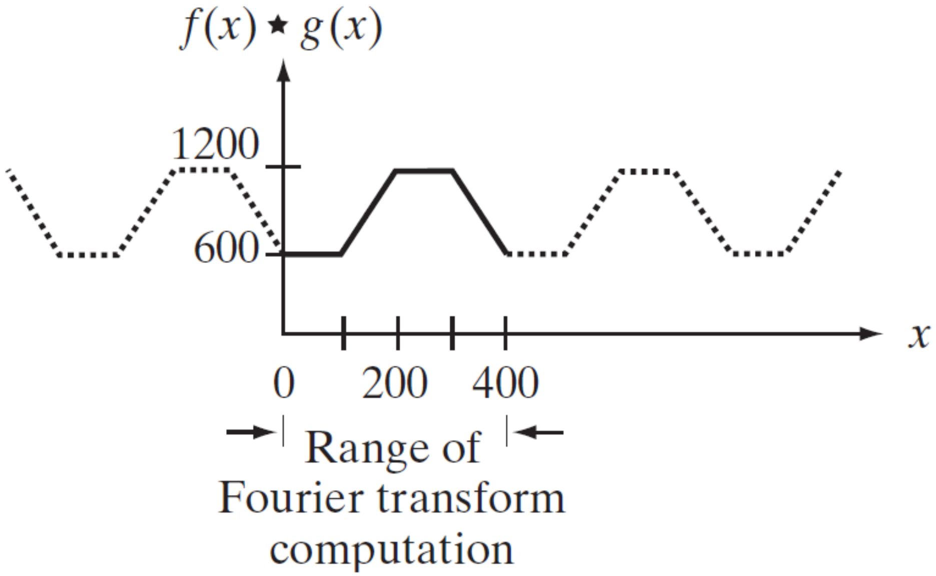


f(m)
3
0
200
400
m
const
+
-
+
-



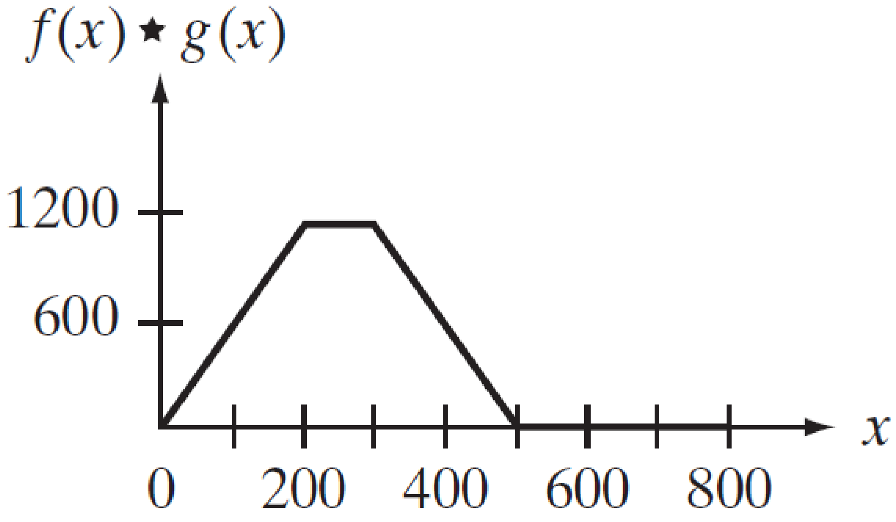
f(m)
3
0
200
400
m
f(m)
3
0
200
400
m
+
f(m)
3
0
200
400
m
const
f(m)
3
0
200
400
m
+
f(m)
3
0
200
400
m
const
f(m)
3
0
200
400
m
const

- Circular Convolution

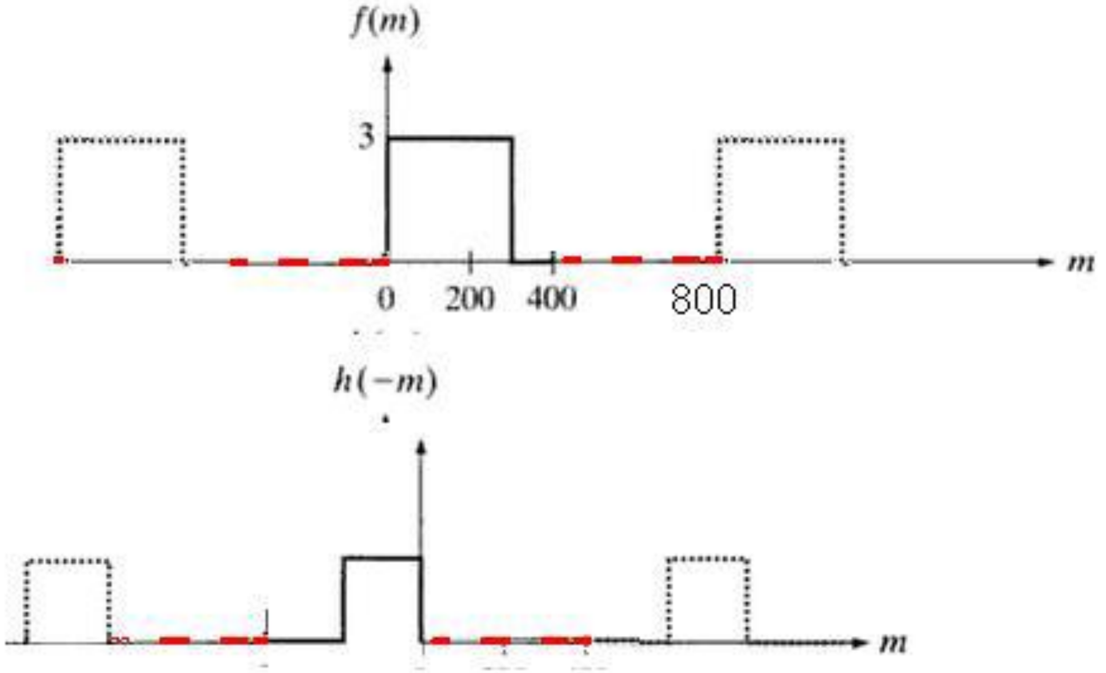


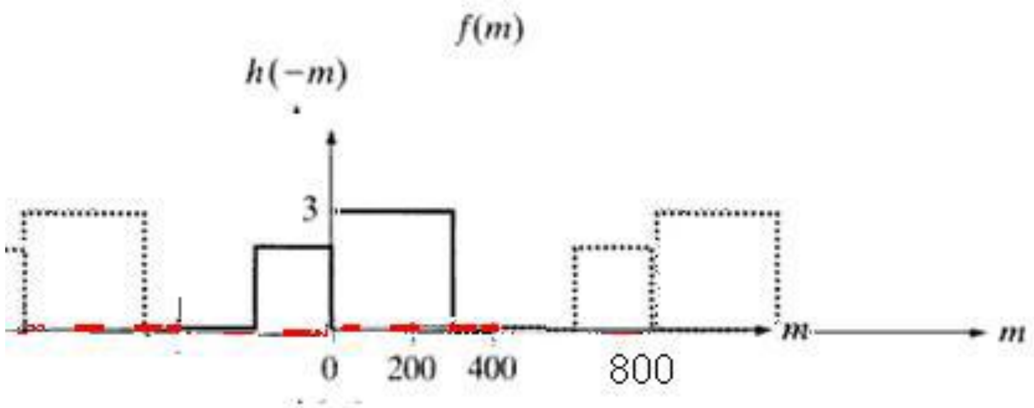
- Wrap around error
- Reason: Closeness in period and hence overlapping
- Solution?

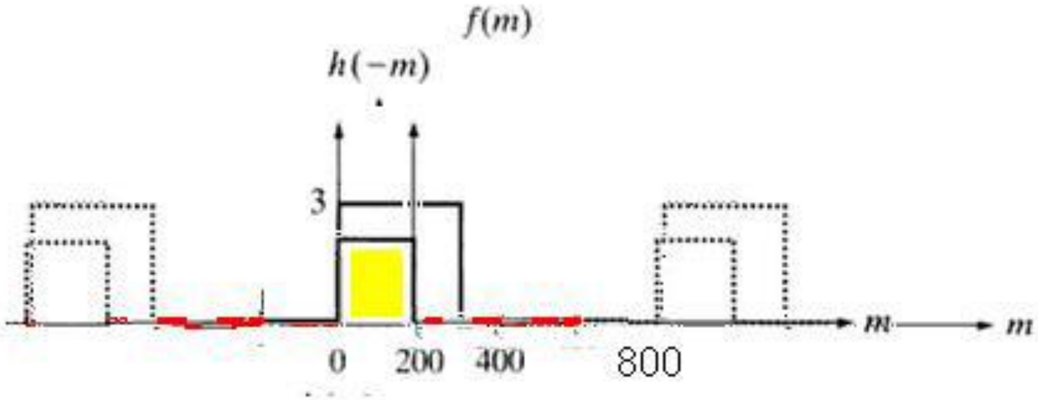
- Linear Convolution

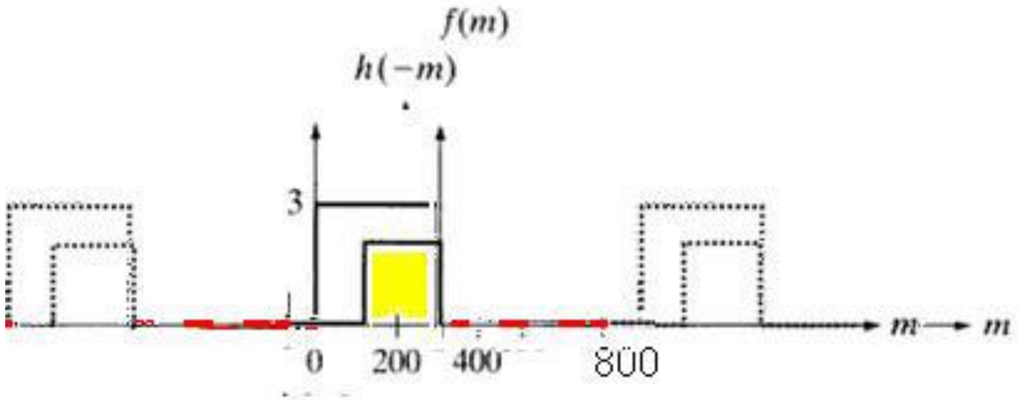


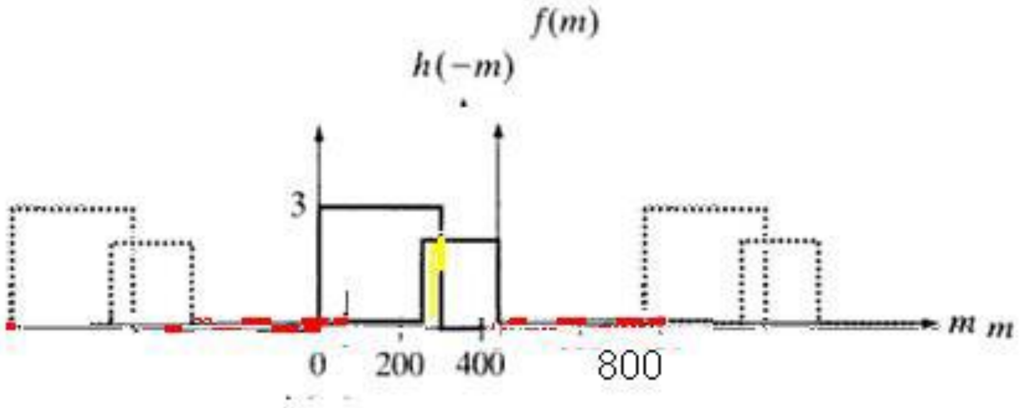
Padding

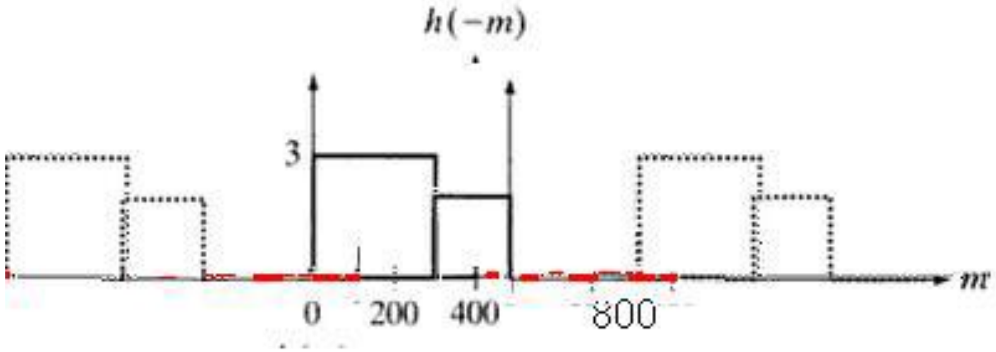


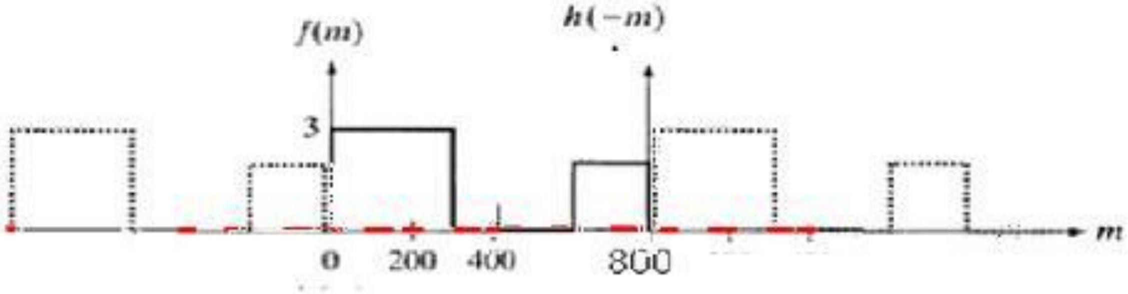

f(m)
h(−m)
3
m
m
0
200
400
800


f(m)
h(−m)
3
m
m
0
200
400
800


f(m)
h(−m)
3
m
m
0
200
400
800


f(m)
h(−m)
3
m
m
0
200
400
800


h(−m)
3
m
0
200
400
800


f(m)
h(−m)
3
m
0
200
400
800

Amount of Padding

- f → size: A
- h → size: B
- New and equal size after padding is P
- In our case value of A, B and P?
- A → 400
- B → 400
- P → 799 (minimum)

$P \geq A + B - 1$

Zero Padding

$$P \geq A + B - 1$$

- 2-D :Let $f(x, y)$ and $h(x, y)$ be two image arrays of sizes A x B and C x D pixels, respectively.

$$f_p(x, y) = \begin{cases} f(x, y) & 0 \leq x \leq A-1 \quad \text{and} \quad 0 \leq y \leq B-1 \\ 0 & A \leq x \leq P \quad \text{or} \quad B \leq y \leq Q \end{cases}$$

$$h_p(x, y) = \begin{cases} h(x, y) & 0 \leq x \leq C-1 \quad \text{and} \quad 0 \leq y \leq D-1 \\ 0 & C \leq x \leq P \quad \text{or} \quad D \leq y \leq Q \end{cases}$$

- with

$$P \geq A + C - 1 \qquad\qquad Q \geq B + D - 1$$

- If both arrays are of the same size, M x N then we require that

$$P \geq 2M - 1$$

$$Q \geq 2N - 1$$

- As rule, DFT algorithms tend to execute faster with arrays of even size, so it is good practice to select P and Q as the smallest even integers that satisfy the preceding equations.
- If the two arrays are of the same size, this means that P and Q are selected as twice the array size.

Summary of 2-D DFT

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M+vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \mathrm{Real}(F); \quad I = \mathrm{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
7) Average value	$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

Summary of 2-D DFT *contd..*

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
✗ 10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
✗ 11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
✗ 12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Summary of 2-D DFT *contd..*

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

X

X