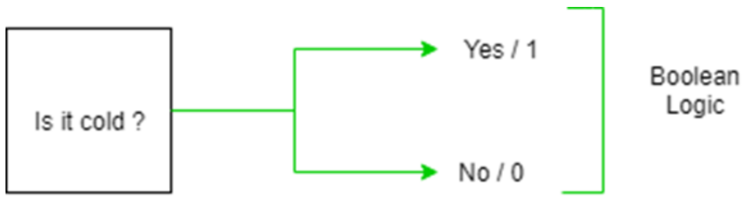
Fuzzy Logic

❖ Introduction

- It is a way to represent uncertainty.
- Uncertainty associated with vagueness, with imprecision and /or with a lack of information regarding a particular element of the problem.
- Fuzziness is found in our thinking, in our decisions, in the way we process information & particularly in our language.
- Statements can be unclear or subject to different interpretations.

  e.g. Phrases like "see you later", "little more", "Don't feel very well" are the fuzzy expressions



- Boolean logic uses sharp distinctions. It forces us to draw lines between members of a class and non-members.
- Fuzzy Set expresses the degree to which an element belongs to a set.

❖ Fuzzy Logic applications

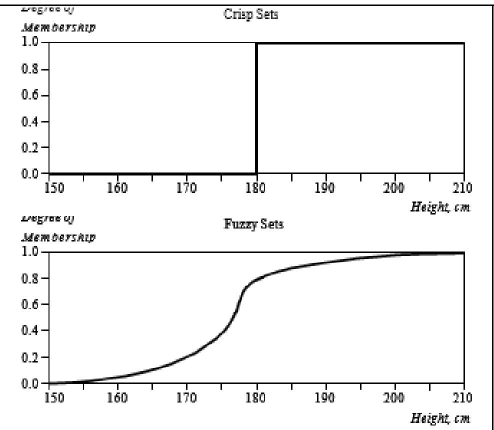
- pattern recognition
- data mining & information retrieval
- automata theory, game theory
- automatic train control
- tunnel digging machinery
- home appliances: washing machines, air conditioners

Crisp vs. Fuzzy Sets

- **The classical example in fuzzy sets is tall men. The elements of the fuzzy set tall men" are all men, but their degrees of membership depend on their height.**

- **For instance, in crisp set, we may say, Tom is tall because his height is 181 cm. If we drew a line at 180 cm, we would find that David, who is 179 cm, is small. Is David really a small man?**

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00



- **The x-axis represents the universe of discourse – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.**

- **The y-axis represents the membership value of the fuzzy set. In our case, the fuzzy set of "tall men" maps height values into corresponding membership values.**

Crisp Set

- **Let *X* be the universe of discourse and its elements be denoted as *x*. In the classical set theory, crisp set *A* of *X* is defined as function $f_A(x)$ called the characteristic function of *A***

$$f_A(x) : X \text{ in } \{0, 1\}, \text{ where } f_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

This set maps universe *X* to a set of two elements. For any element *x* of universe *X*, characteristic function $f_A(x)$ is equal to 1 if *x* is an element of set *A*, and is equal to 0 if *x* is not an element of *A*.

Fuzzy Set

In the fuzzy theory, fuzzy set *A* of universe *X* is defined by function $\mu_A(x)$ called the membership function of set *A*

$\mu_A(x)$: X in [0, 1], where **$\mu_A(x) = 1$ if x is totally in A;**

$\mu_A(x) = 0$ if x is not in A;

$0 < \mu_A(x) < 1$ if x is partly in A.

Fuzzy set A in X is defined by a set of ordered pairs

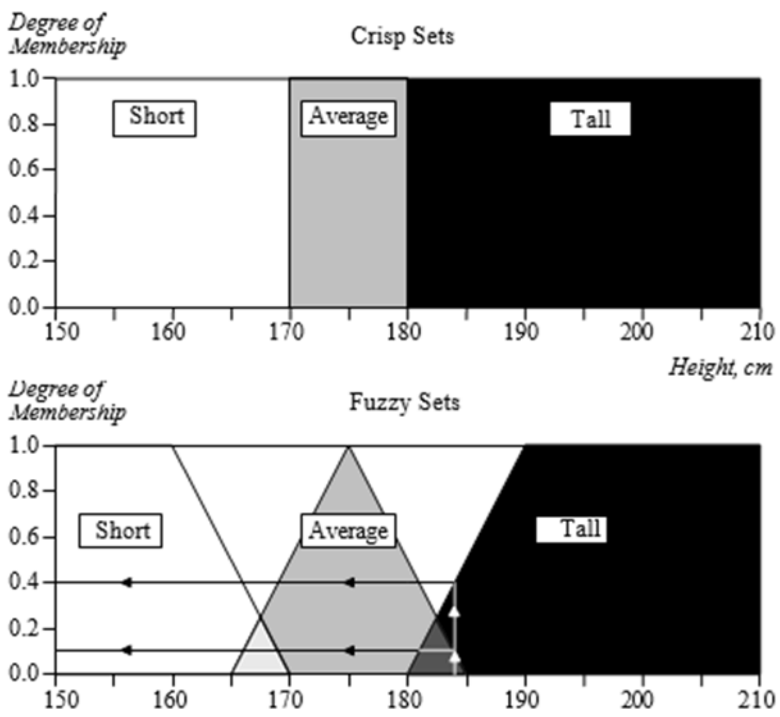
$$A = \{\ (x, \mu_A(x))\ |\ x \in X\ \}$$

- **This definition of set allows a continuum of possible choices. For any element *x* of universe *X*, membership function $\mu_A(x)$ equals the degree to which *x* is an element of set *A*. This degree, a value between 0 and 1, represents the degree of membership, also called membership value, of element *x* in set *A*.**

Fuzzy Set Representation

- **In our "tall men" example, we can define fuzzy sets of *tall*, *short* and *average* men.**
- **The universe of discourse for three defined fuzzy sets consist of all possible values of the men's heights.**

Height	Short	Average	Tall
150	1	0	0
155	1	0	0
160	1	0	0
165	0.45	0	0
170	0	0.45	0
175	0	1	0
180	0	0.45	0
185	0	0	0.45
190	0	0	1
195	0	0	1
200	0	0	1
205	0	0	1
210	0	0	1



- **For example, a man who is 184 cm tall is a member of the *average* men set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall* men set with a degree of 0.4**

❖ <u>Linguistic Variables</u>

- **A linguistic variable is a fuzzy variable. For example, the statement "John is tall" implies that the linguistic variable John takes the linguistic value tall.**
- **In fuzzy expert systems, linguistic variables are used in fuzzy rules.**

For example:

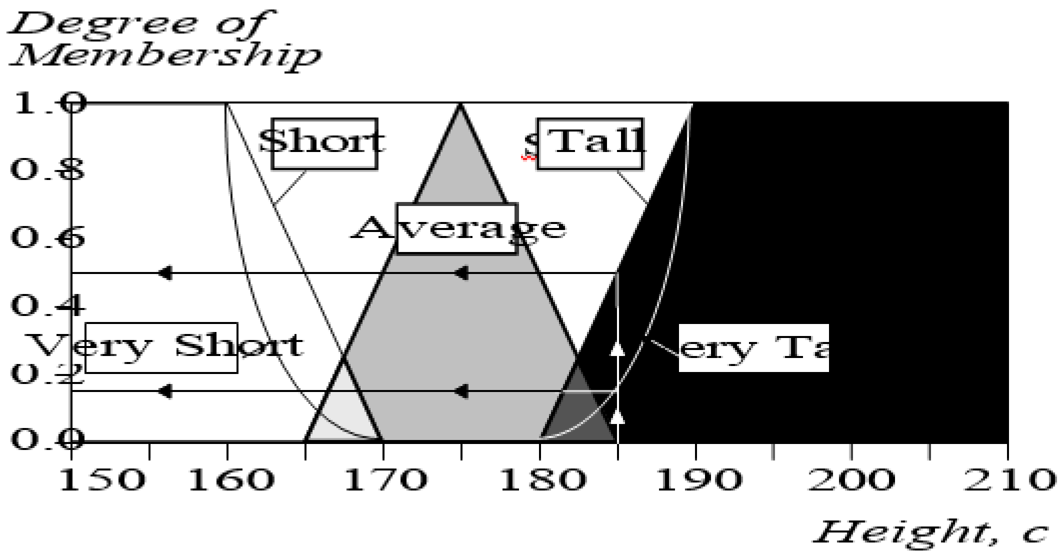
IF wind is strong THEN sailing is good

IF project_duration is long THEN completion_risk is high

IF speed is slow THEN stopping_distance is short

❖ Linguistic Variables and Hedges

- **The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.**
- **A linguistic variable carries with it the concept of fuzzy set qualifiers, called hedges.**
- **Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as very, somewhat, quite, more or less and slightly.**



❖ Fuzzy Sets & Fuzzy Rules

Complement

- **<u>Crisp Sets</u>: Who does not belong to the set?**
- **<u>Fuzzy Sets</u>: How much do elements not belong to the set?**
- **The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.**
- **If A is the fuzzy set, its complement $\neg A$ can be found as follows:**

$$\mu_{\neg A}(x) = 1 - \mu_A(x).$$

Containment

- **Crisp Sets: Which sets belong to which other sets?**
- **Fuzzy Sets: How much sets belong to other sets?**
- **Similar to a Chinese box, a set can contain other sets. The smaller set is called the subset. For example, the set of tall men contains all tall men; very tall men is a subset of tall men. However, the tall men set is just a subset of the set of men.**
- **In crisp sets, all elements of a subset entirely belong to a larger set.**
- **In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in the subset than in the larger set.**

Intersection

- **Crisp Sets: Which element belongs to both sets?**
- **Fuzzy Sets: How much of the element is in both sets?**
- **In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of tall men and the set of fat men is the area where these sets overlap.**
- **In fuzzy sets, an element may partly belong to both sets with different memberships.**
- **A fuzzy intersection is the lower membership in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:**

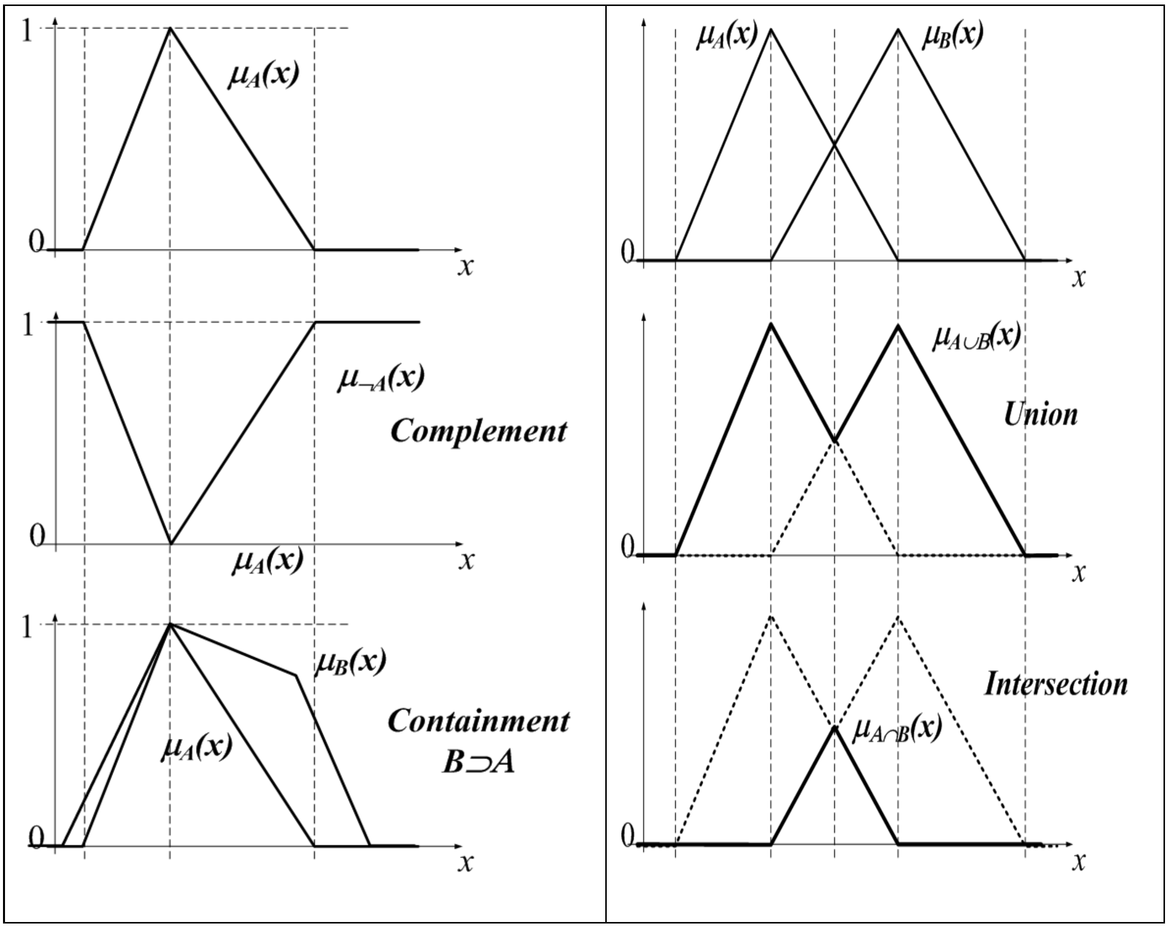
$$\mu_{A\cap B}(x) = \min\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x), \text{ where } x \in X.$$

Union

- **Crisp Sets: Which element belongs to either set?**
- **Fuzzy Sets: How much of the element is in either set?**
- **The union of two crisp sets consists of every element that falls into either set. For example, the union of tall men and fat men contains all men who are tall OR fat.**
- **In fuzzy sets, the union is the reverse of the intersection. That is, the union is the largest membership value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:**

$$\mu_{A\cup B}(x) = \max\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x), \text{ where } x \in X.$$

Operations of Fuzzy Sets



❖ Properties of Fuzzy Sets

Equality

- **Fuzzy set *A* is considered equal to a fuzzy set *B*, IF AND ONLY IF:**

  $\mu_A(x) = \mu_B(x), \forall x \in X$

- **Example: $A = 0.3/1 + 0.5/2 + 1/3$, $B = 0.3/1 + 0.5/2 + 1/3$,**

  therefore $A = B$.

Inclusion

- **Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:**

  $\mu_A(x) \le \mu_B(x), \forall x \in X$

- **Example: Consider $X = \{1, 2, 3\}$ and sets A**

  and B $A = 0.3/1 + 0.5/2 + 1/3$;

  $B = 0.5/1 + 0.55/2 + 1/3$

  then A is a subset of B, or $A \subseteq B$

Cardinality

- **Cardinality of a crisp (non-fuzzy) set Z is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:**

  $card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma\mu_A(x_i)$, **for i=1..n**

- **Example: Consider $X = \{1, 2, 3\}$ and sets A**

  and B $A = 0.3/1 + 0.5/2 + 1/3$;

  $B = 0.5/1 + 0.55/2 + 1/3$

  $card_A = 1.8$

  $card_B = 2.05$

Empty Fuzzy Set

- **A fuzzy set A is empty, IF AND ONLY IF:**

  $\mu_A(x) = 0, \forall x \in X$

- **Example: Consider $X = \{1, 2, 3\}$ and fuzzy set**

  $A = 0/1 + 0/2 + 0/3$,

  then A is *empty.*

Alpha-Cut

- **An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:**

  $A_\alpha = \{\mu_A(x) \geq \alpha, \forall x \in X\}$.

- **Example: Consider $X = \{1, 2, 3\}$ and set $A = 0.3/1 + 0.5/2$**

  $+ 1/3$ then: $A_{0.5} = \{2, 3\}$, $A_{0.1} = \{1, 2, 3\}$, $A_1 = \{3\}$.

Fuzzy Set Normality

- **A fuzzy subset of X is called normal if there exists at least one element $x \in X$ such that $\mu_A(x) = 1$.**
- **A fuzzy subset that is not normal is called subnormal.**
- **All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.**
- **The height of a fuzzy set A is the largest membership grade of an element in**

  A $height(A) = \max_x(\mu_A(x))$.
- **Fuzzy set is called normal if and only if:**

  $height(A) = 1$.

Fuzzy Sets Core and Support

- **Assume A is a fuzzy set over universe of discourse X.**
- **The support of A is the crisp subset of X consisting of all elements with membership grade:**

  $$supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$$
- **The core of A is the crisp subset of X consisting of all elements with membership grade:**

  $$core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$$

❖ Fuzzy Sets Examples

- Consider two fuzzy subsets of the set X, X = {a, b, c, d, e }

  referred to as A and B

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e} and B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

- **Support**:

  $supp(A)$ = {a, b, c, d }

  $supp(B)$ = {a, b, c, d, e }

- **Core**:

  $core(A)$ = {a}

  $core(B)$ = {}

- **Cardinality**:

  $card(A)$ = 1+0.3+0.2+0.8+0 = 2.3

  $card(B)$ = 0.6+0.9+0.1+0.3+0.2 = 2.1

- **Complement**:

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  $\neg A$ = {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

- **Union**:

  $A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

- **Intersection**:

  $A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

- **α-cut**:

  $A_{0.2}$ = {a, b, c, d}

  $A_{0.3}$ = {a, b, d}

  $A_{0.8}$ = {a, d}

  A_{1} = {a}

❖ Fuzzy Rules

- **In 1973, Lotfi Zadeh published his second most influential paper. This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in fuzzy rules.**

- **A fuzzy rule can be defined as a conditional statement in the form:**

IF *x* is *A* THEN *y* is *B*

where *x* and *y* are linguistic variables; and *A* and *B* are linguistic values determined by fuzzy sets on the universe of discourses *X* and *Y*, respectively.

➢ Classical vs. Fuzzy Rules

- **A classical IF-THEN rule uses binary logic, for example,**

Rule 1: IF speed >100 THEN stopping_distance is long
Rule 2: IF speed < 40 THEN stopping_distance is short

- **The variable speed can have any numerical value between 0 and 220 km/h, but the linguistic variable stopping_distance can take either value long or short.**

- **In other words, classical rules are expressed in the black-and-white language of Boolean logic.**

- **We can also represent the stopping distance rules in a fuzzy form:**

Rule 1: IF speed is fast THEN stopping_distance is long
Rule 2: IF speed is slow THEN stopping_distance is short

- **In fuzzy rules, the linguistic variable speed also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as slow, medium and fast.**

- **The universe of discourse of the linguistic variable stopping_distance can be between 0 and 300 m and may include such fuzzy sets as short, medium and long.**

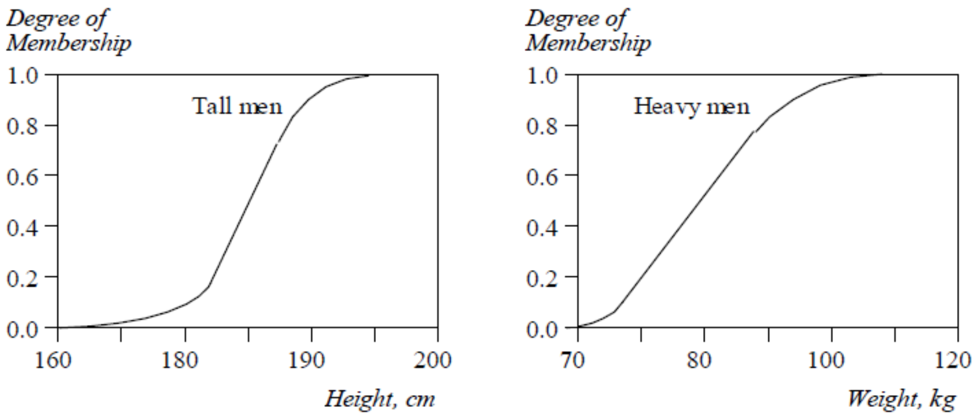
- **Fuzzy rules relate fuzzy sets.**

- **In a fuzzy system, all rules fire to some extent, or in other words they fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.**

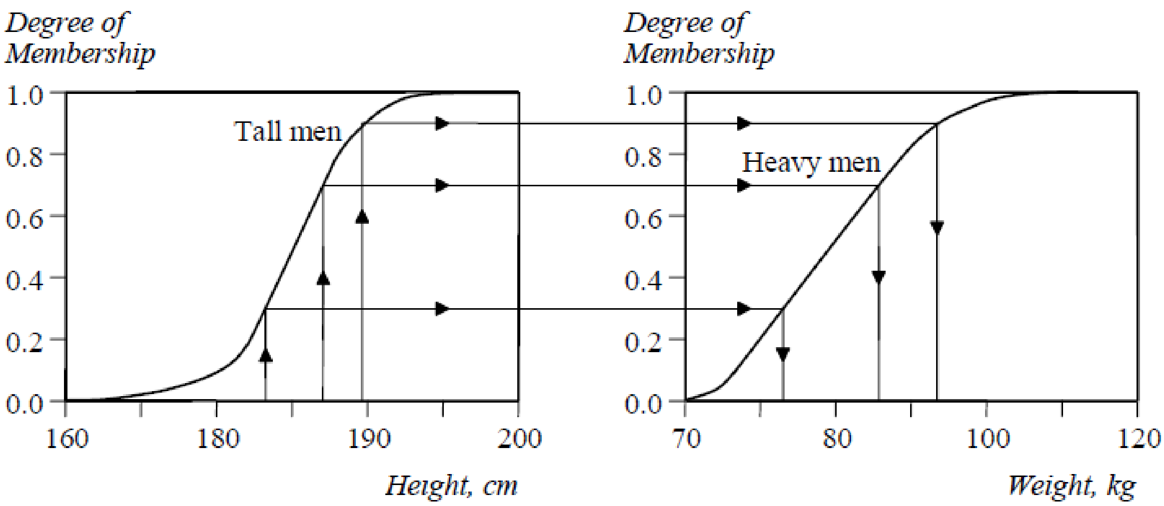
Firing Fuzzy Rules

■ **These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:**

IF height is tall THEN weight is heavy



■ **The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called monotonic selection.**



■ A fuzzy rule can have multiple antecedents, for example:

IF project_duration is long AND project_staffing is large AND project_funding is inadequate
THEN risk is high

IF service is excellent OR food is delicious
THEN tip is generous

■ The consequent of a fuzzy rule can also include multiple parts, for instance:

IF temperature is hot
THEN hot_water is reduced; cold_water is increased

Fuzzy Inference

Mamdani Fuzzy Inference

- The Mamdani-style fuzzy inference process is performed in four steps:

  1. Fuzzification of the input variables
  2. Rule evaluation (inference)
  3. Aggregation of the rule outputs (composition)
  4. Defuzzification.

We examine a simple two-input one-output problem that includes three rules:

Rule: 1	IF x is A3	OR y is B1	THEN z is C1
Rule: 2	IF x is A2	AND y is B2	THEN z is C2
Rule: 3	IF x is A1	THEN z is C3	

Real-life example for these kinds of rules:

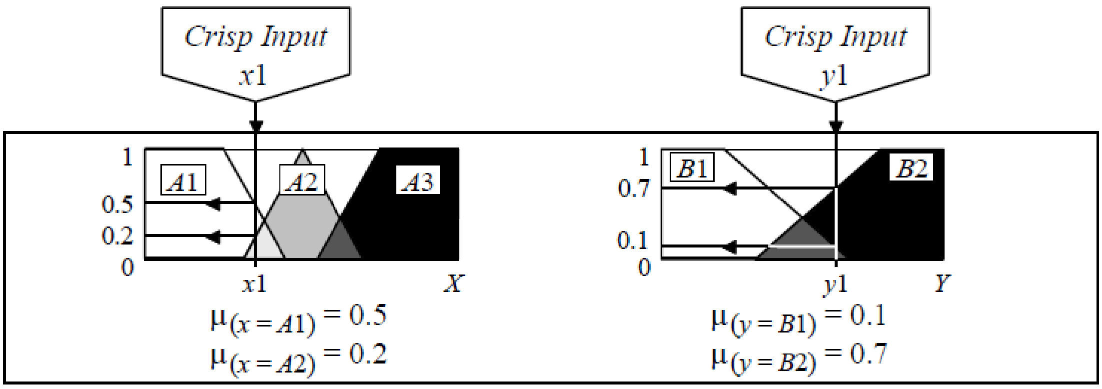
Rule: 1 IF project_funding is adequate OR project_staffing is small THEN risk is low

Rule: 2 IF project_funding is marginal AND project_staffing is large THEN risk is normal

Rule: 3 IF project_funding is inadequate THEN risk is high

Step 1: Fuzzification

The first step is to take the crisp inputs, x1 and y1 (*project funding* and *project staffing*), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.

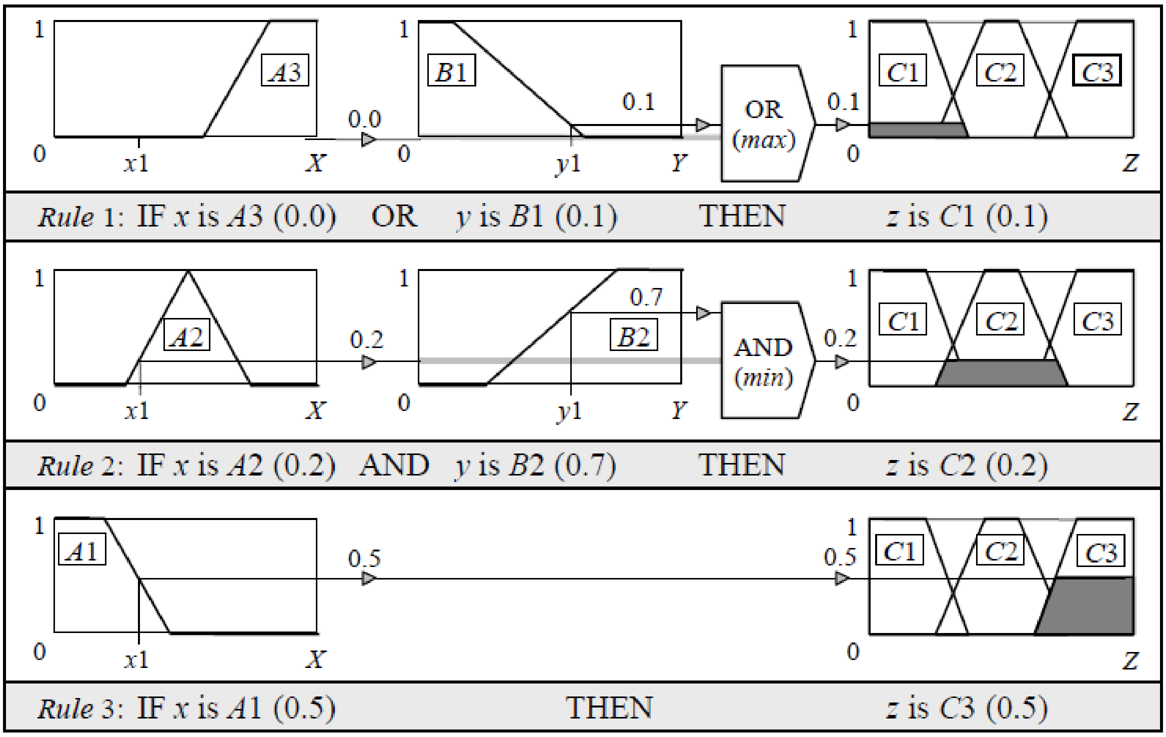
■ If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.

■ RECALL: To evaluate the disjunction of the rule antecedents, we use the **OR** fuzzy operation. Typically, fuzzy expert systems make use of the classical fuzzy operation union:

$$\mu_{A \cup B}(x) = \max\,[\mu_A(x),\ \mu_B(x)]$$

■ Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND** fuzzy operation intersection:

$$\mu_{A \cap B}(x) = \min\,[\mu_A(x),\ \mu_B(x)]$$



■ Now the result of the antecedent evaluation can be applied to the membership function of the consequent.

■ The most common method is to cut the consequent membership function at the level of the antecedent truth. This method is called **clipping** (alpha-cut).

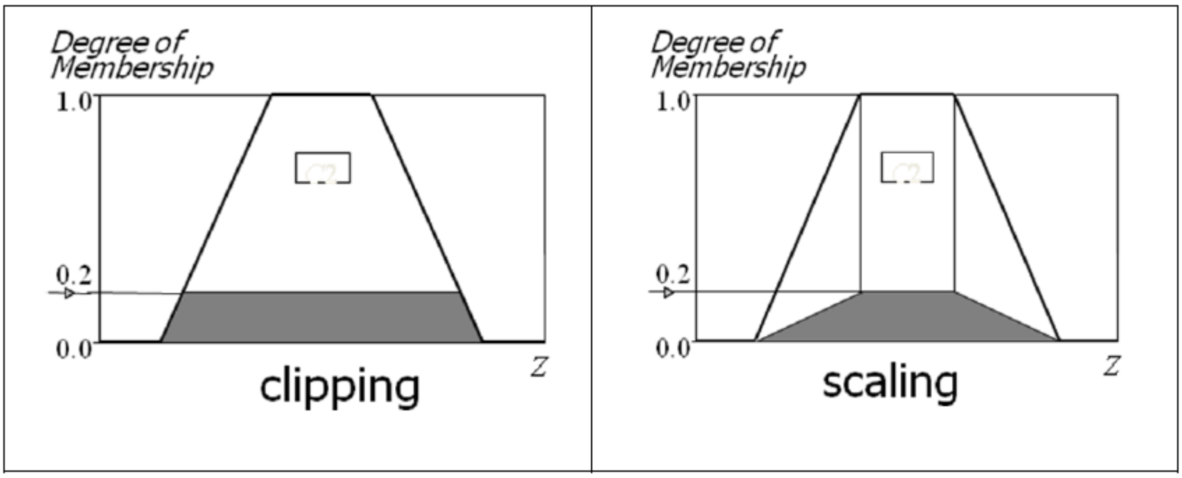
- ■ Since the top of the membership function is sliced, the clipped fuzzy set loses some information.

- ■ However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.

■ While clipping is a frequently used method, **scaling** offers a better approach for preserving the original shape of the fuzzy set.

- ■ The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.

- ■ This method, which generally loses less information, can be very useful in fuzzy expert systems.

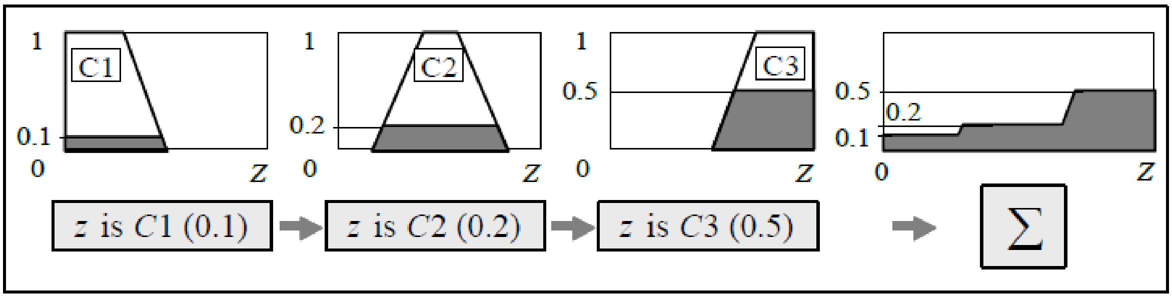


Step 3: Aggregation of the Rule Outputs

■ Aggregation is the process of unification of the outputs of all rules.

■ We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.

■ The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable



Step 4: Defuzzification

■ The last step in the fuzzy inference process is defuzzification.

■ Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.

■ The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.

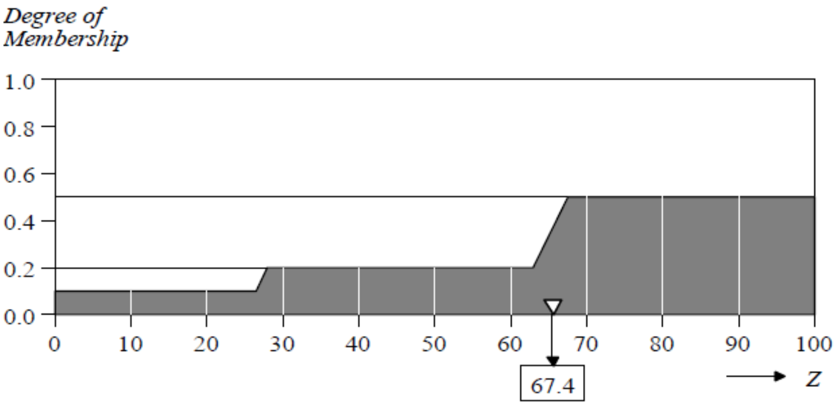
■ There are several defuzzification methods, but probably the most popular one is the **centroid technique**. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity** (**COG**) can be expressed as:

$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

Step 4: Defuzzification

■ Centroid defuzzification method finds a point representing the centre of gravity of the aggregated fuzzy set A, on the interval [a, b].

■ A reasonable estimate can be obtained by calculating it over a sample of points.



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$