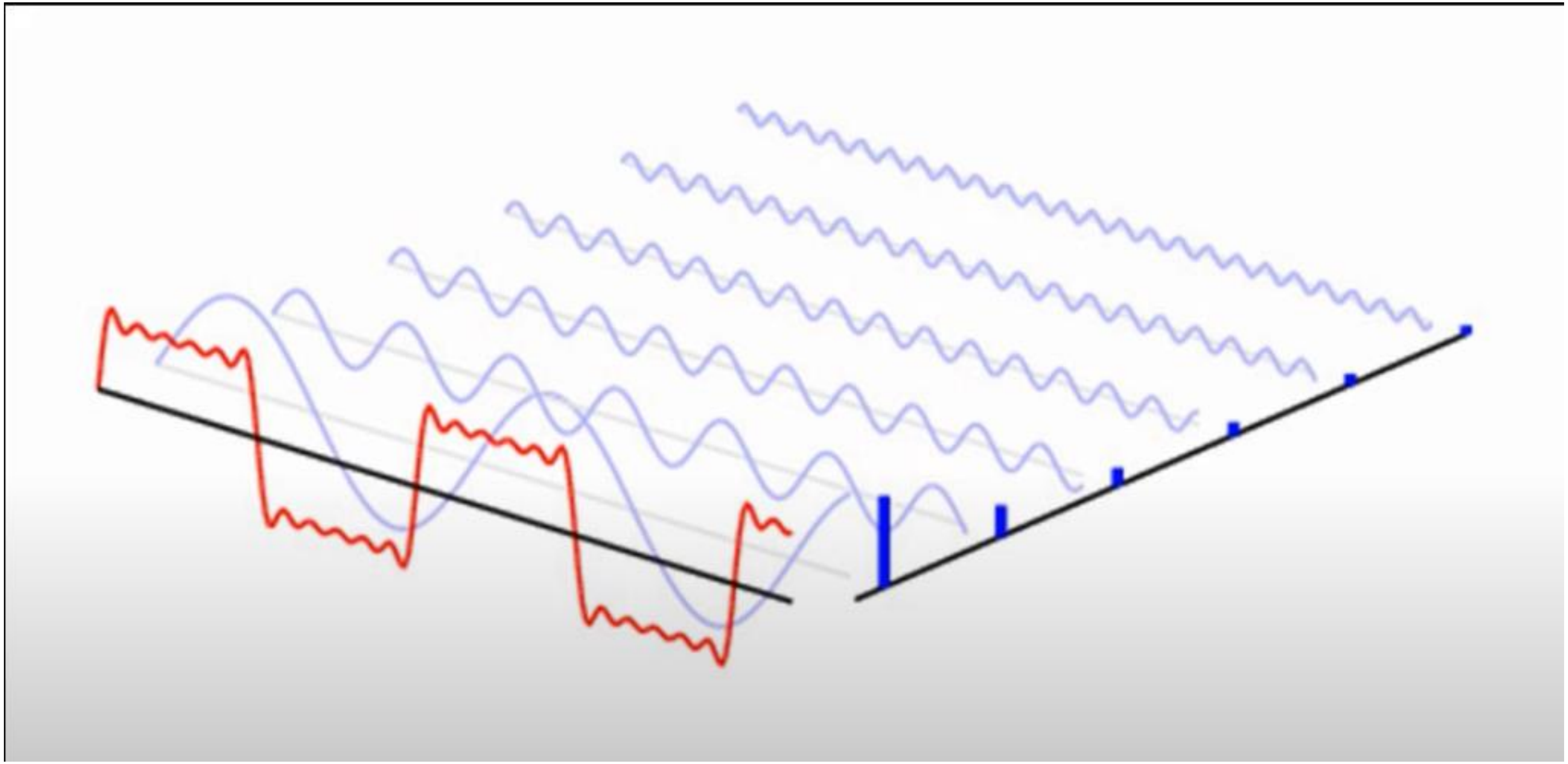
Filtering in Frequency Domain

(Sessional II)

5 lectures

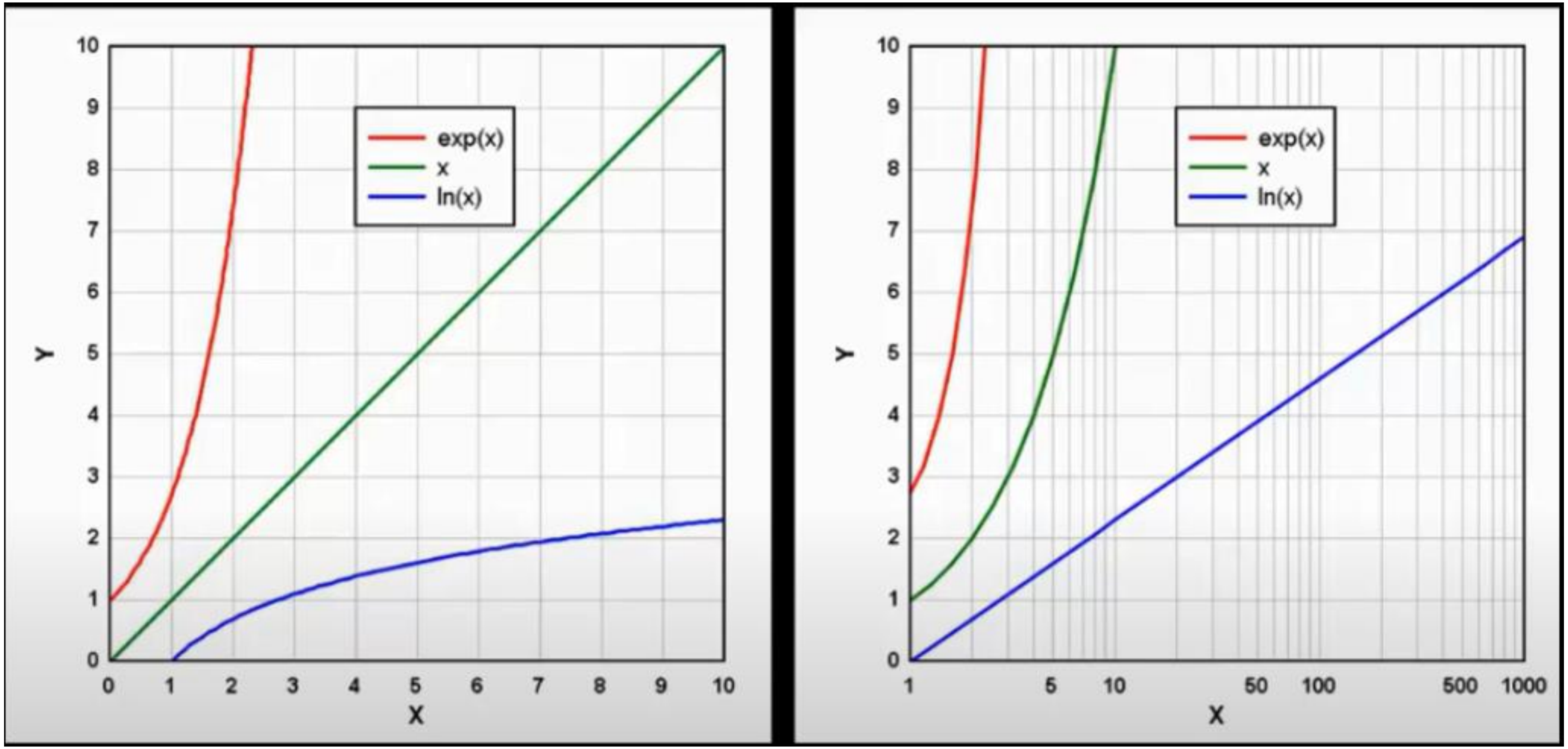
Why we switch to Frequency Domain?

Fourier Transform



Applications

- Reading Text – Captcha
- Transferring books to electronic copies
- Number plate recognition
- Automating the input of handwritten forms


exp(x)
x
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100
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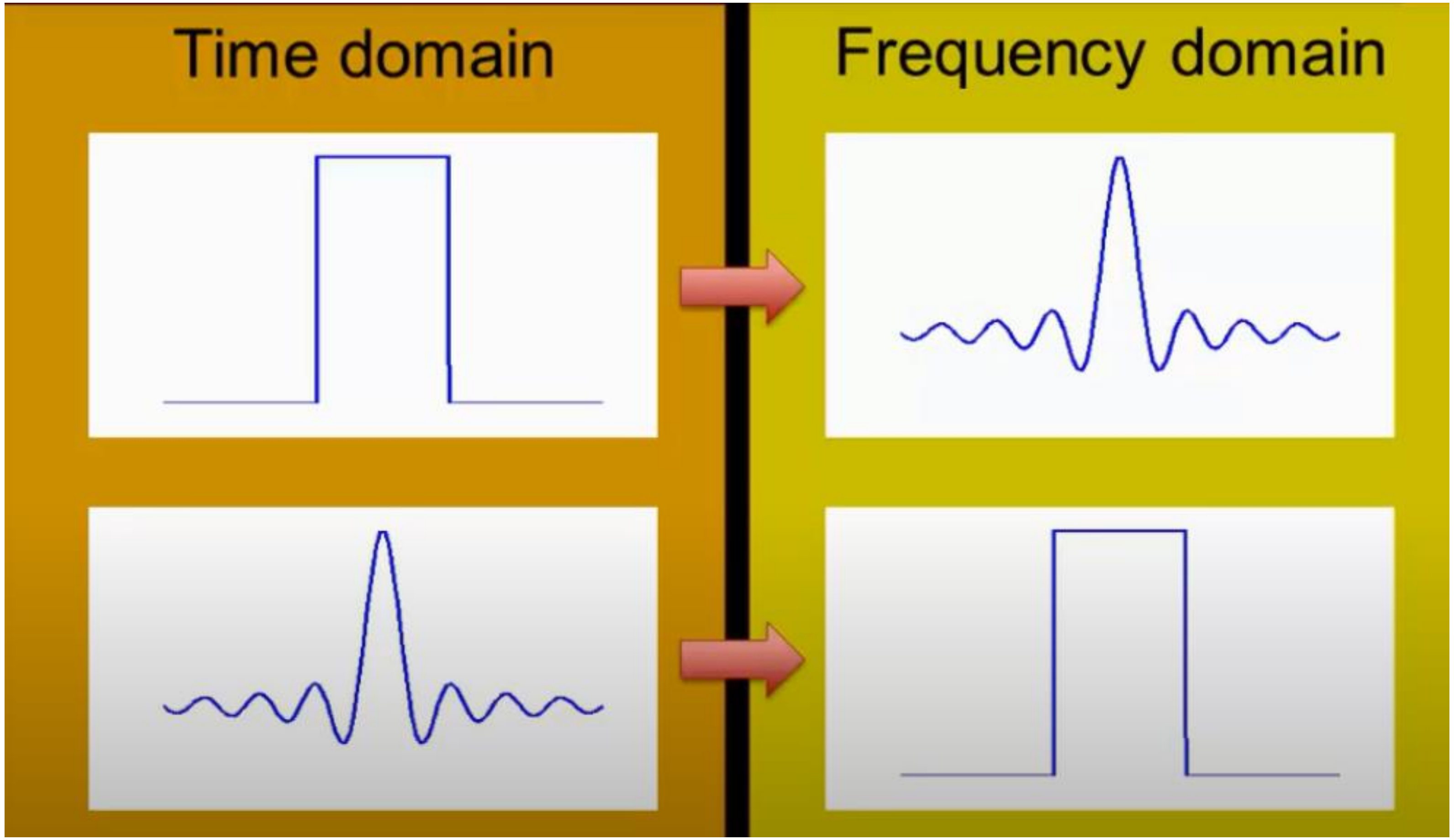
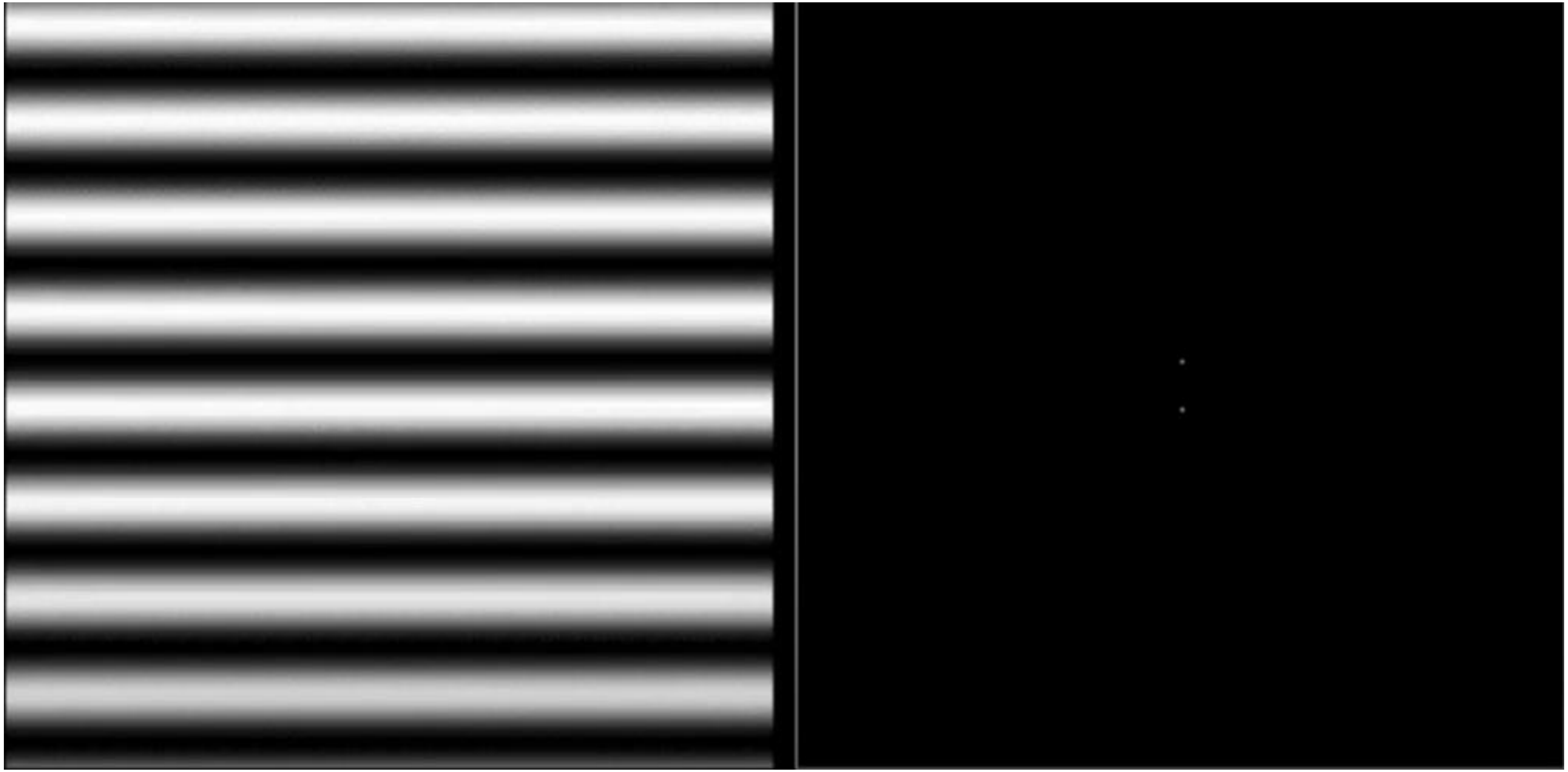
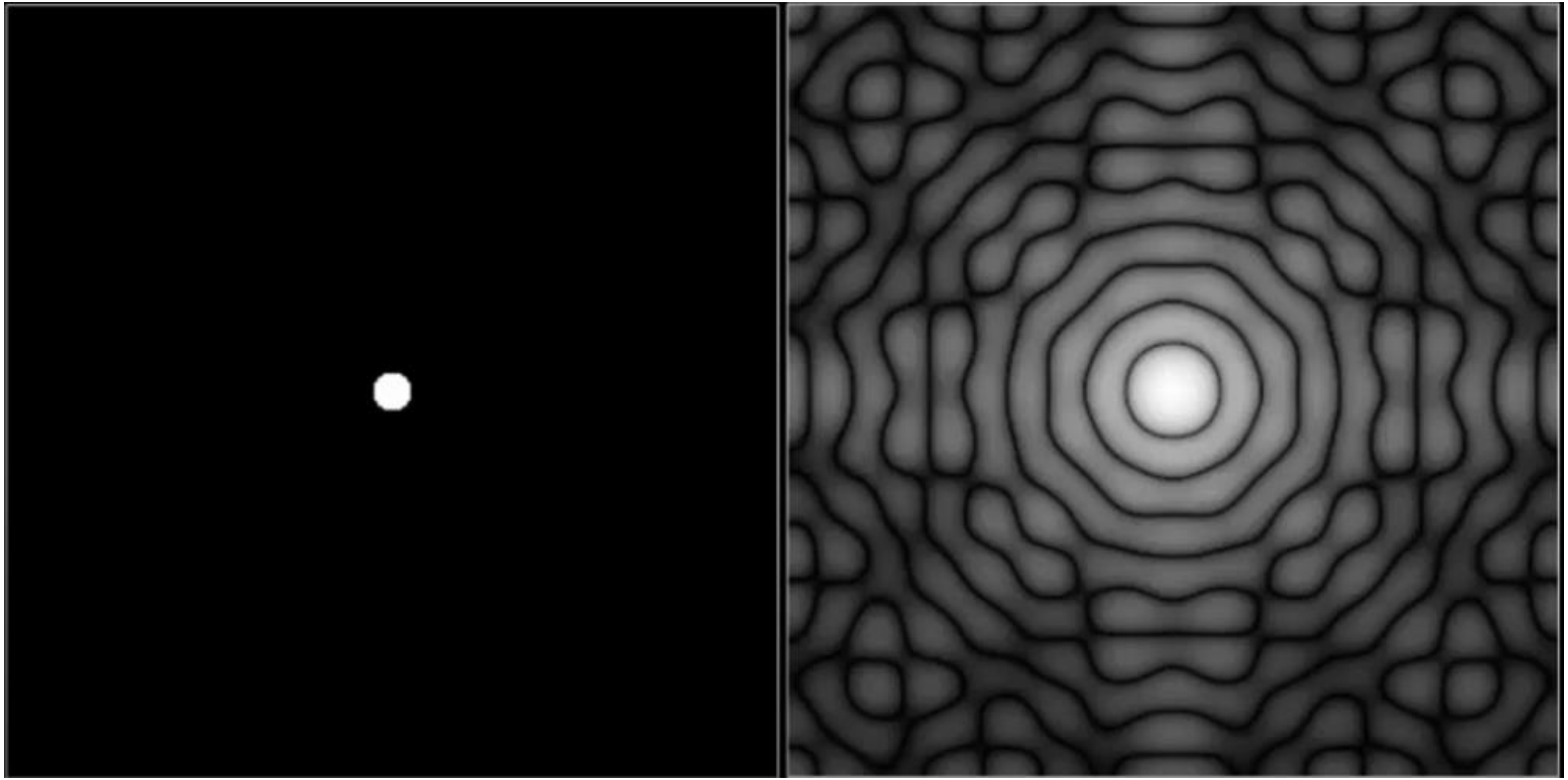
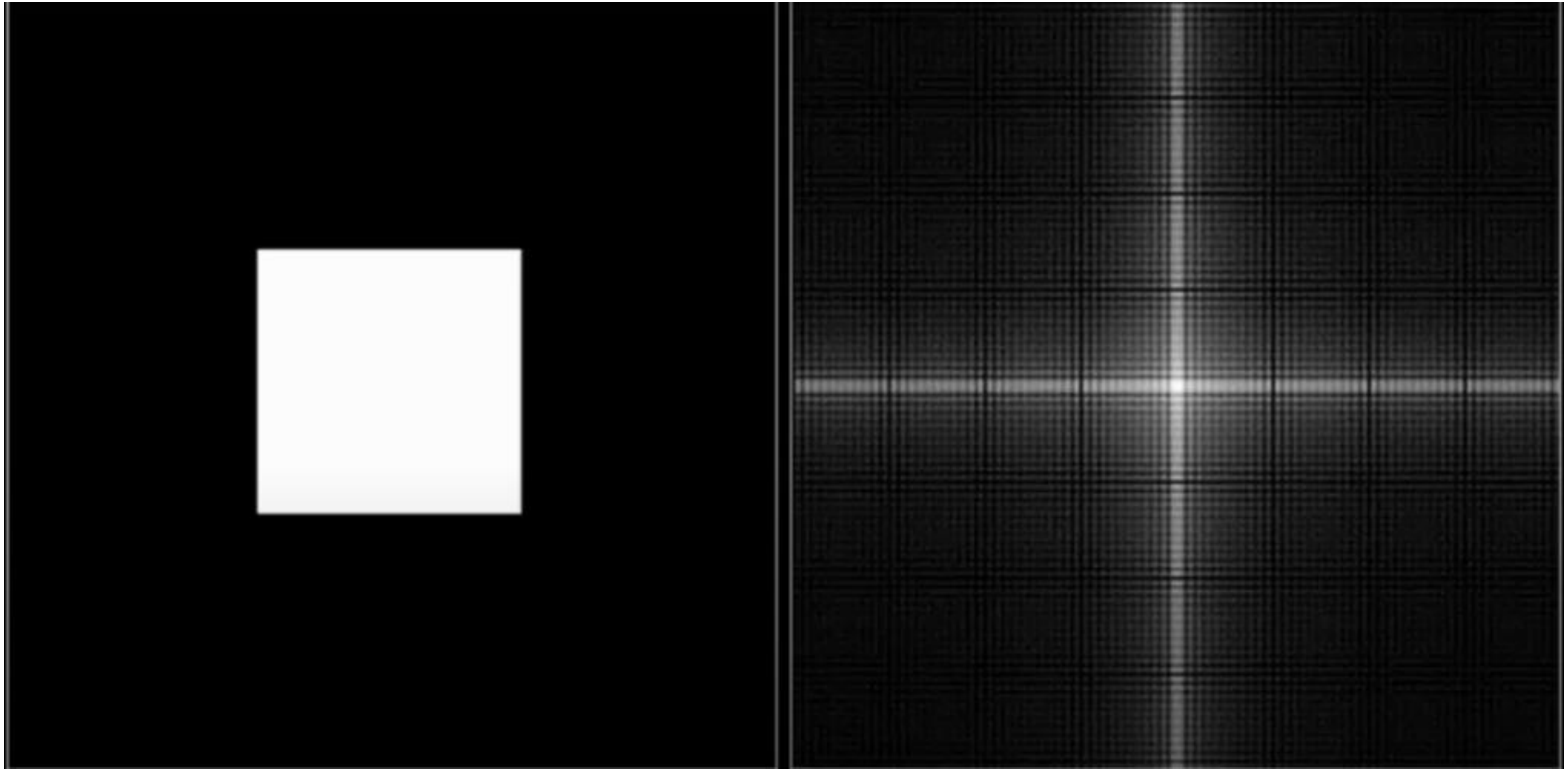
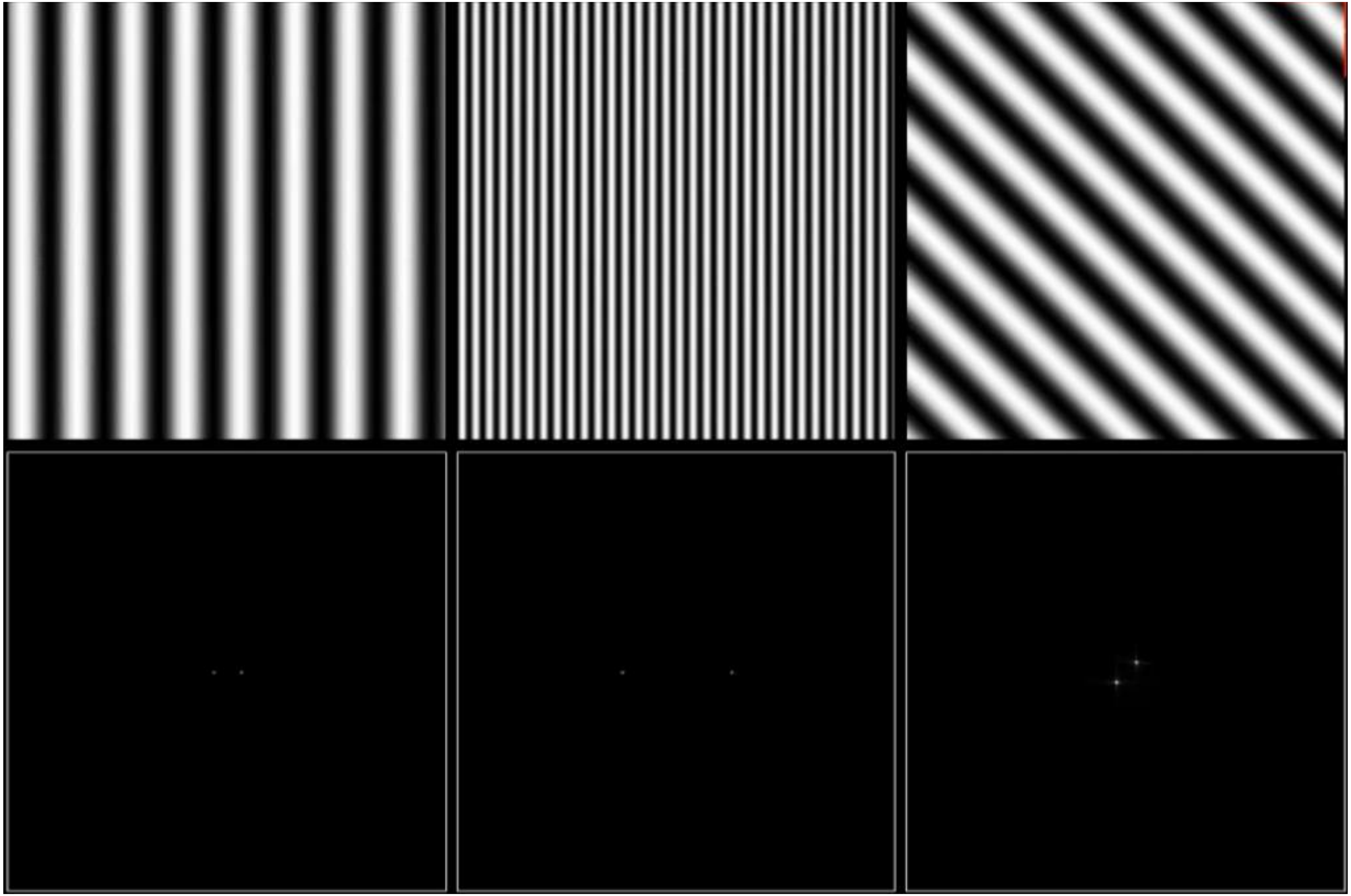

Time domain
Frequency domain









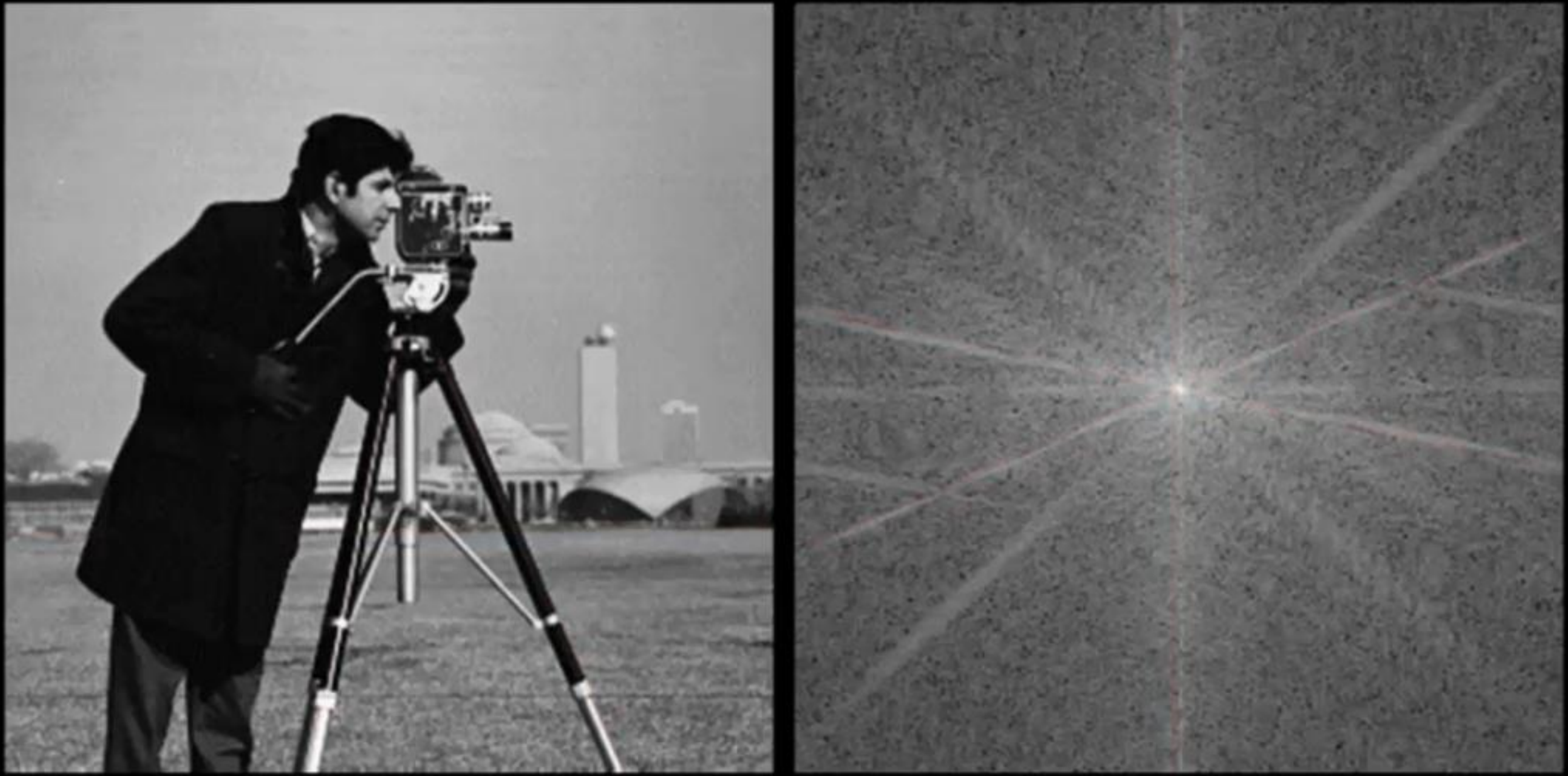
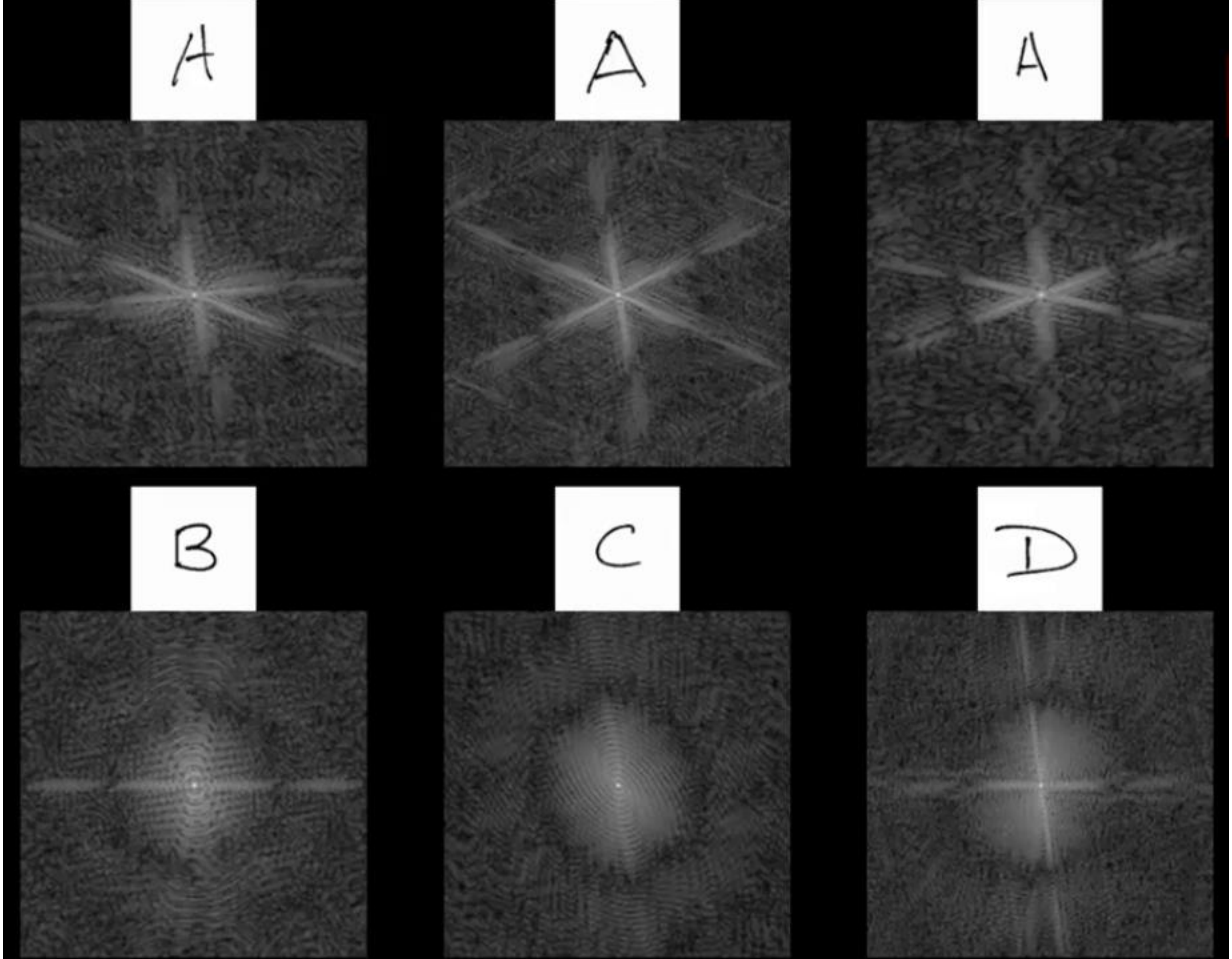
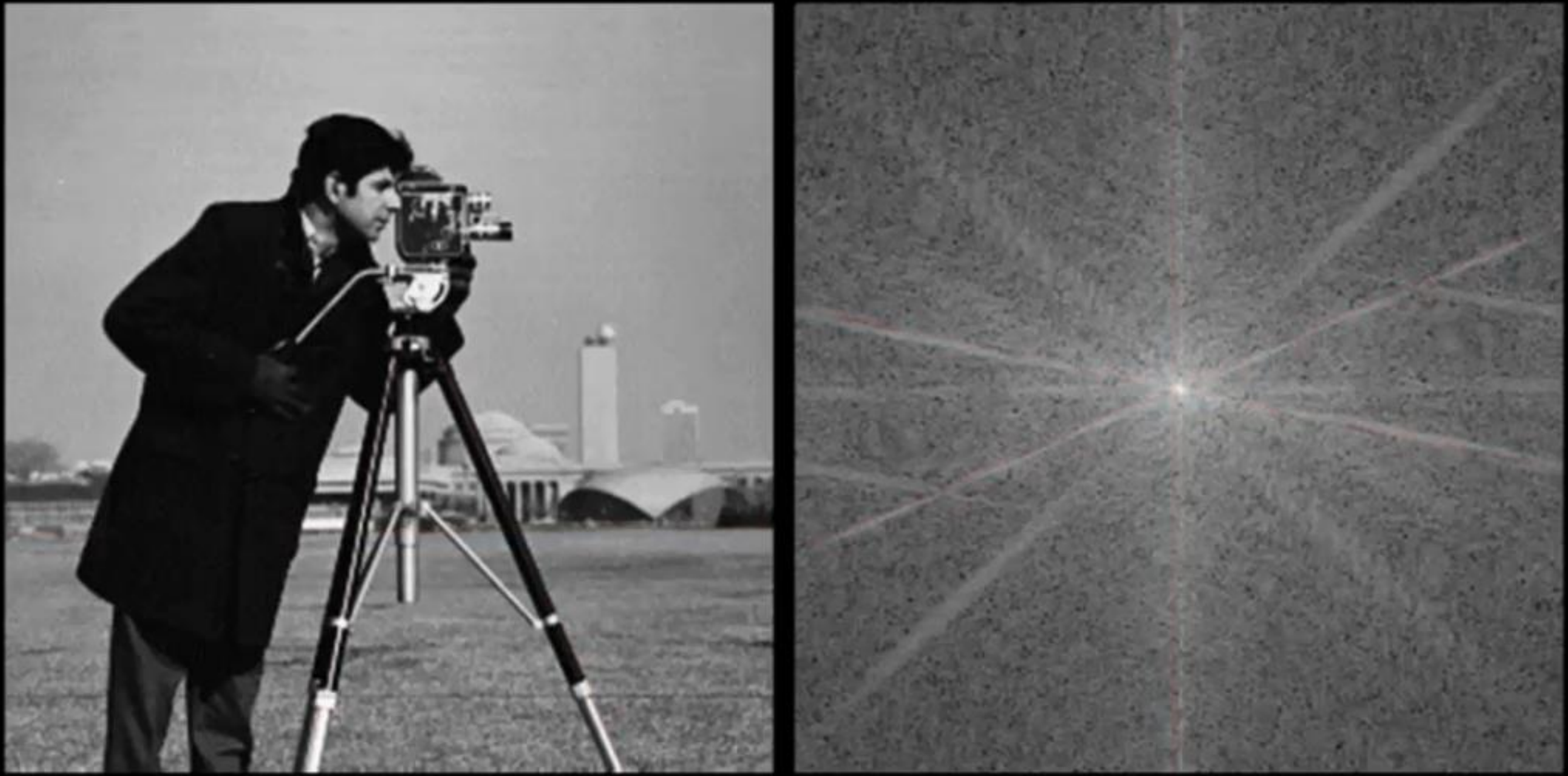


Image: 'cameraman'

https://au.mathworks.com/help/images/examples/deblurring-images-using-the-blind-deconvolution-algorithm.html


A
A
A
B
C
D



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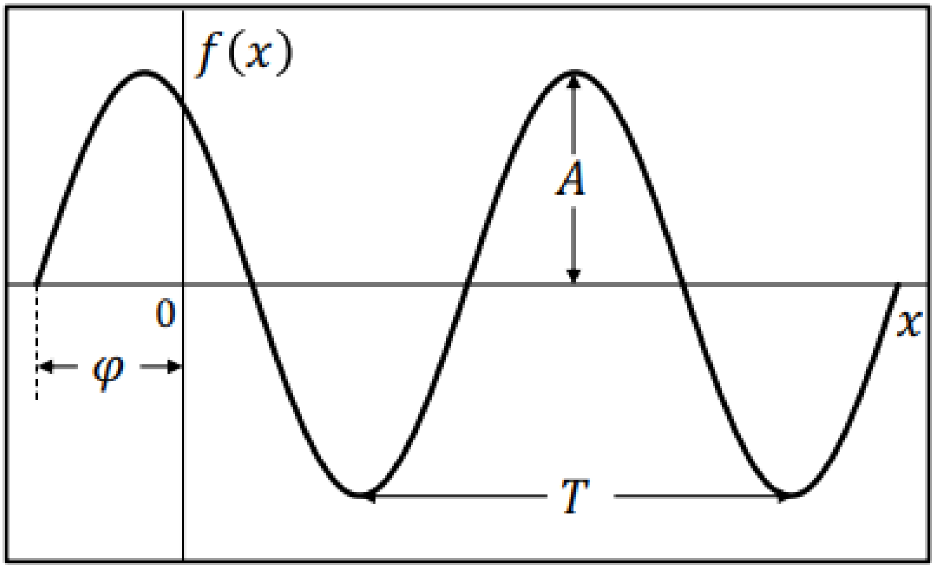
Jean Baptiste Joseph Fourier

(1768-1830)

Any Periodic Function can be rewritten as a Weighted Sum of Infinite Sinusoids of Different Frequencies.

Sinusoid

$$f(x) = A\sin(2\pi ux + \varphi)$$



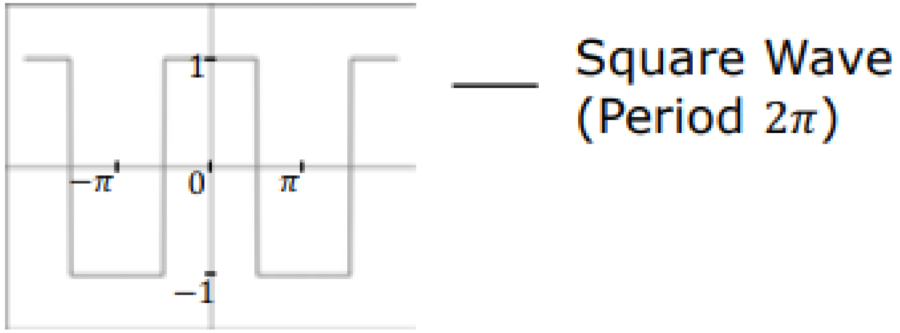
A: Amplitude

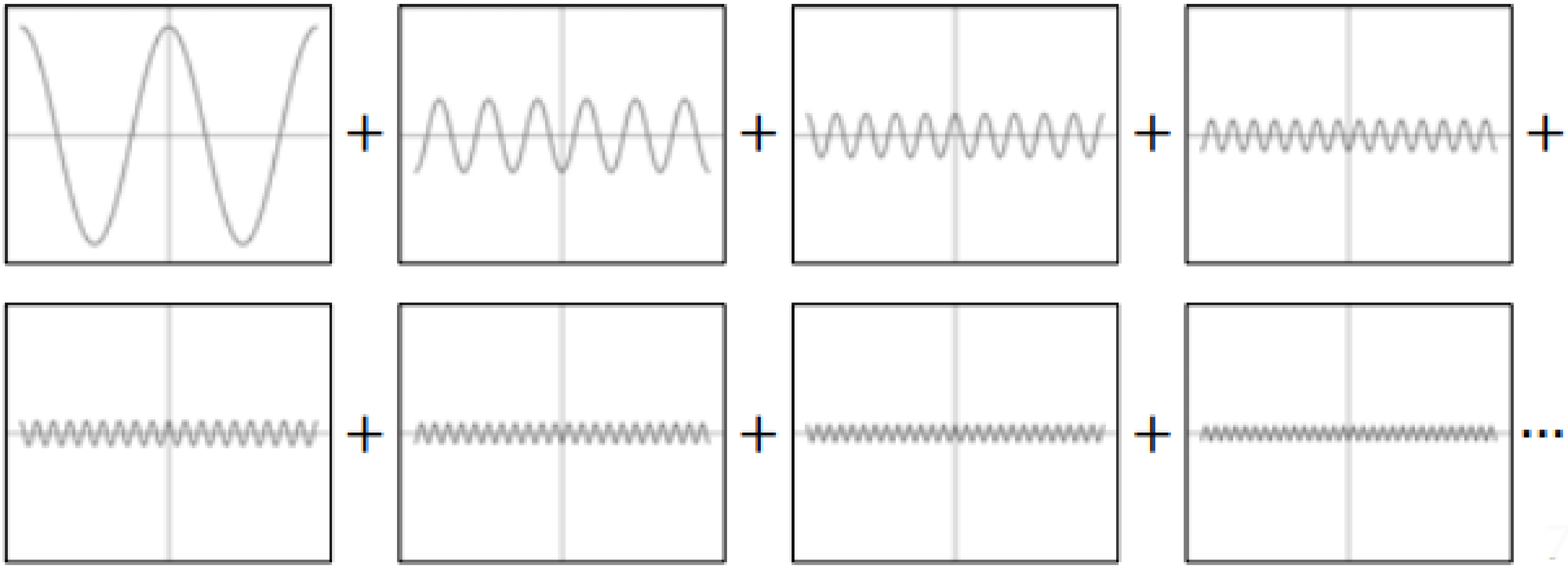
T: Period

φ: Phase

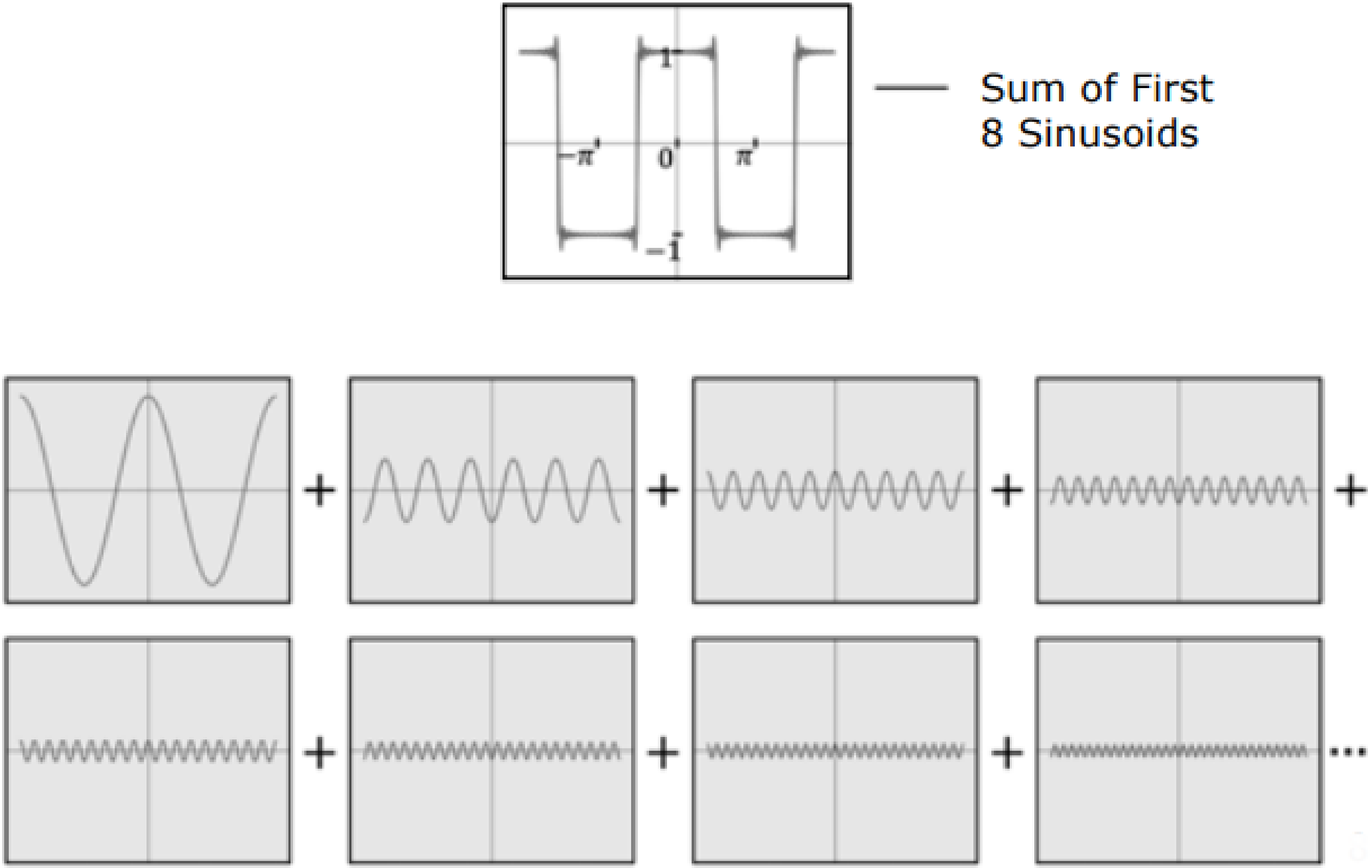
u: Frequency $(1/T)$

Fourier Series

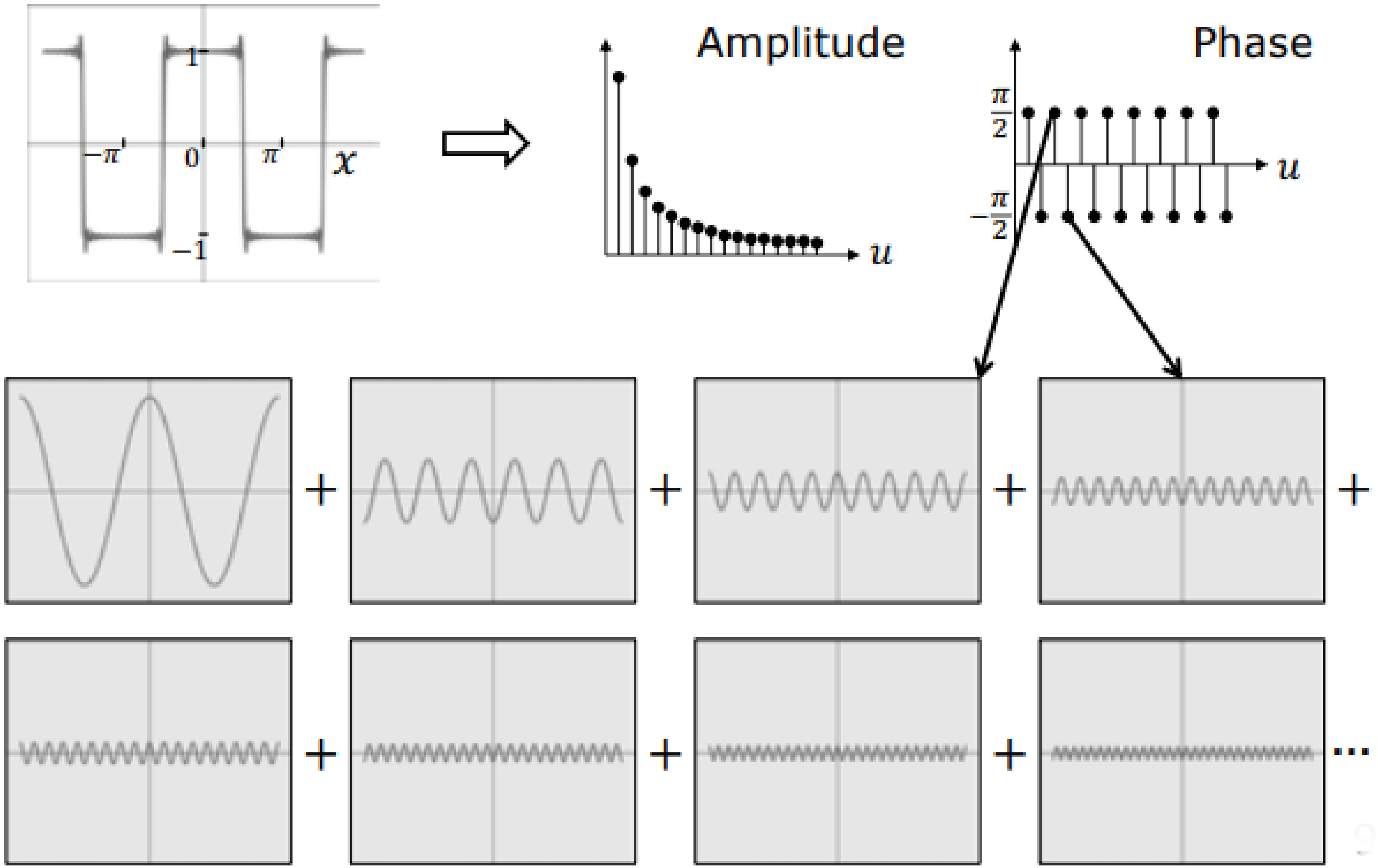




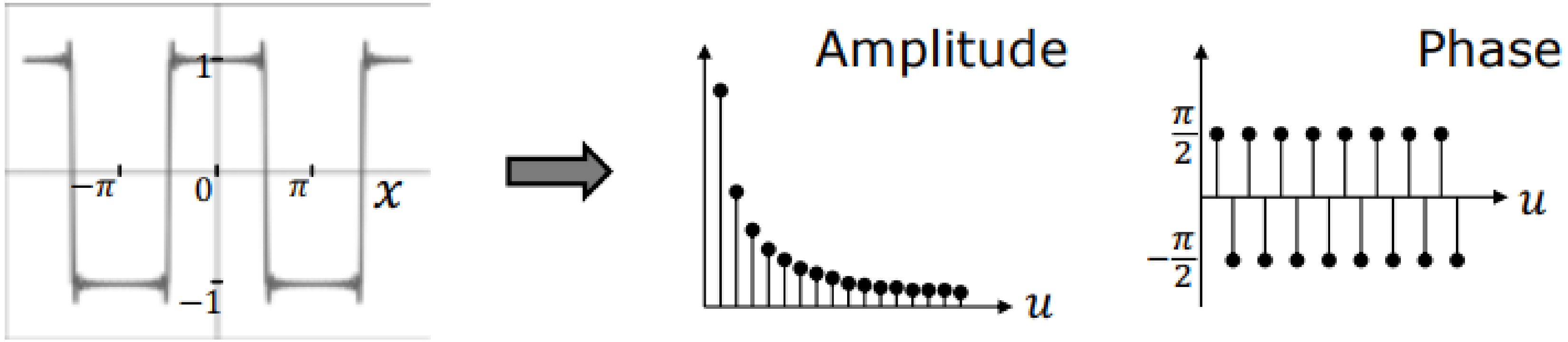
Fourier Series



Frequency Representation of Signal



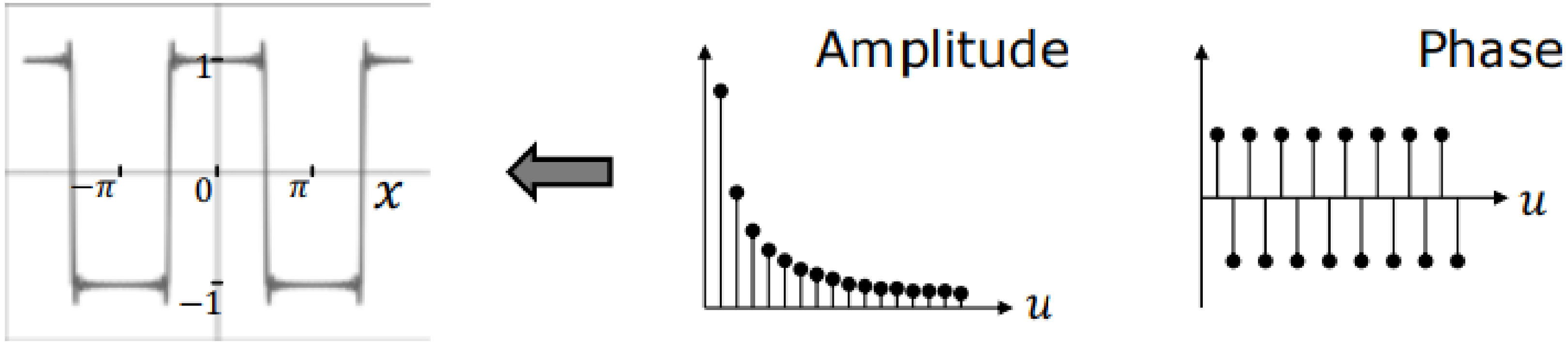
Fourier Transform (FT)



Represents a signal $f(x)$ in terms of Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longrightarrow \boxed{\mathrm{FT}} \longrightarrow F(u)$$

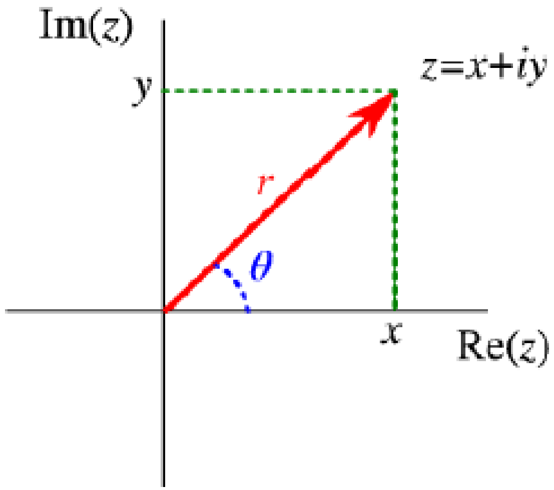
Inverse Fourier Transform (IFT)



Computes the signal $f(x)$ from the Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longleftarrow \boxed{\text{IFT}} \longleftarrow F(u)$$

Maths Primer(Preliminary Concept)



- A complex number *C*, is defined as

$$C = R + jI$$

$$C^* = R - jI$$

- Sometimes, it is useful to represent complex numbers in polar coordinates

$$C = |C|(\cos\theta + j\sin\theta) \qquad |C| = \sqrt{(R^2 + I^2)} \qquad \theta = \arctan(I/R)$$

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad C = |C|e^{j\theta}$$

Fourier Series

- If *f(t)* is a periodic function of a continuous variable *t*, with period *T*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Impulses

- A *unit impulse* of a continuous variable *t* located at t = 0, denoted by δ(*t*), is defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

- And is constrained also to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Sifting Property of Impulses

- Impulse has a sifting property with respect to integration, provided that $f(t)$ is continuous at $t = 0$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

- A more general statement involves an impulse located at an arbitrary point t_0 denoted by $\delta(t - t_0)$,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

Unit Discrete Impulse

- Unit discrete impulse located at x = 0,

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

- And is constrained to satisfy the property

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

- The sifting property of the discrete variable has the form

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

- More generally using a discrete impulse located at $x = x_0$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

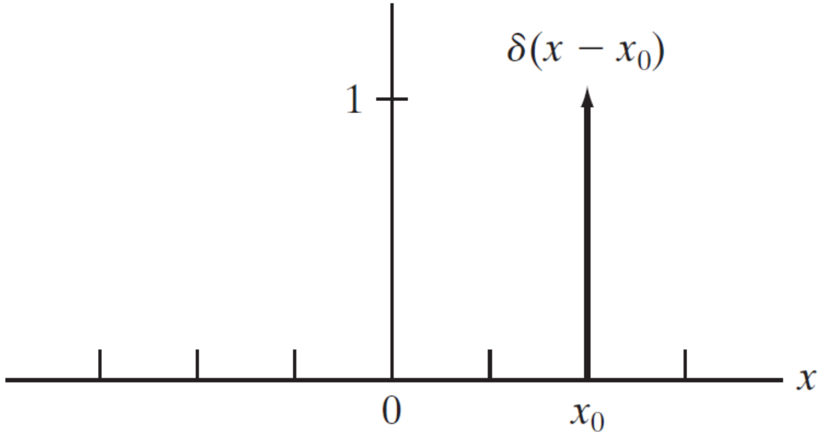


FIGURE 4.2
A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Impulse Train

- Sum of infinitely many periodic impulses ΔT units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

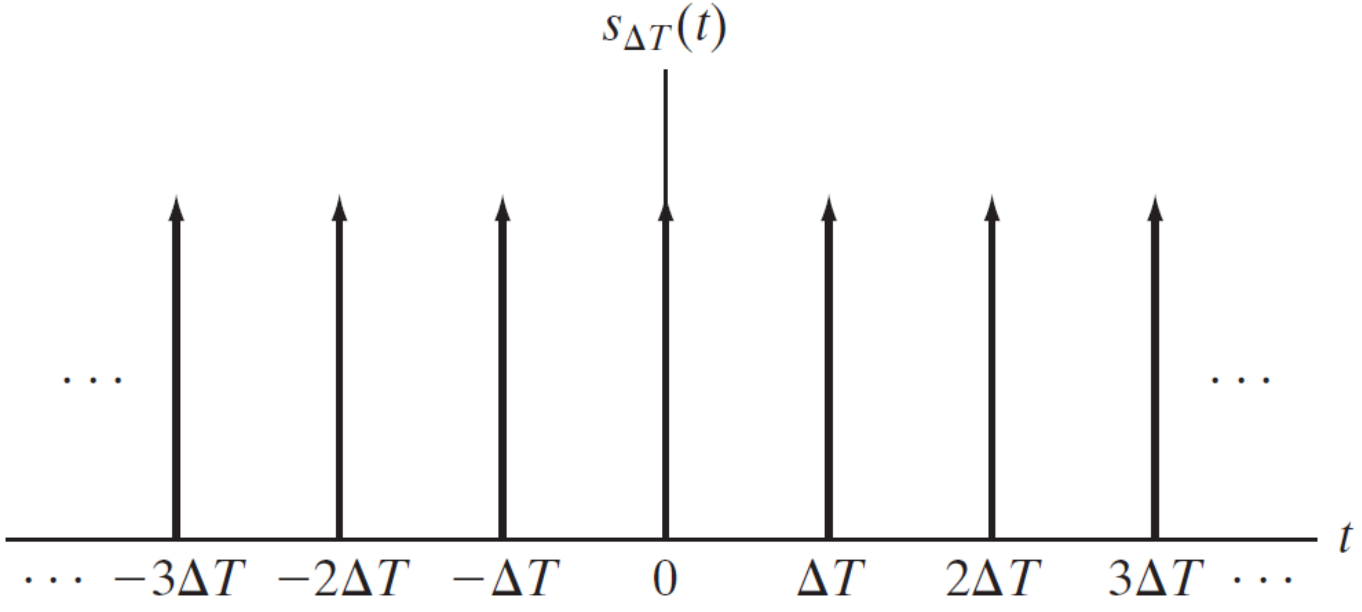


FIGURE 4.3 An impulse train.

Fourier Transform of functions of one continuous variable

- Fourier transform of a continuous function $f(t)$ of a continuous variable t :

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$$

- Inverse Fourier Transform

$$\Im^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}\, d\mu$$

Example

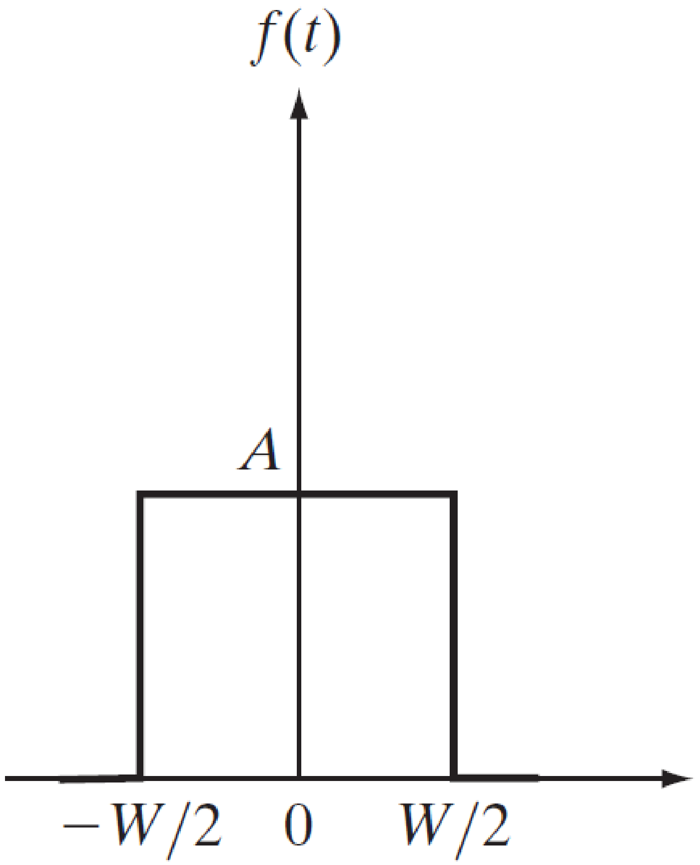
- Example 1: Find Fourier Transform of function shown in figure below

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \qquad \sin\theta = (e^{j\theta} - e^{-j\theta})/2j.$$



- The result in the last step of the preceding expression is known as *sinc* function:

- Where sinc(0) = 1 and sinc(m) = 0 for all other integer values of m.

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$$\operatorname{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

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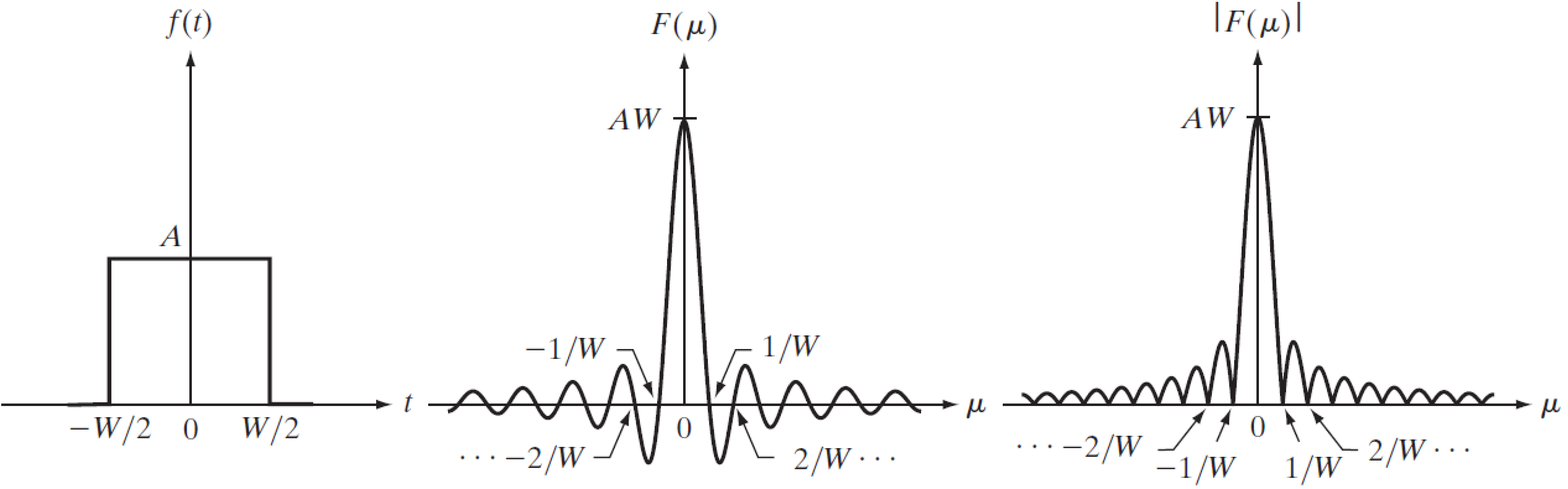

a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

a b c

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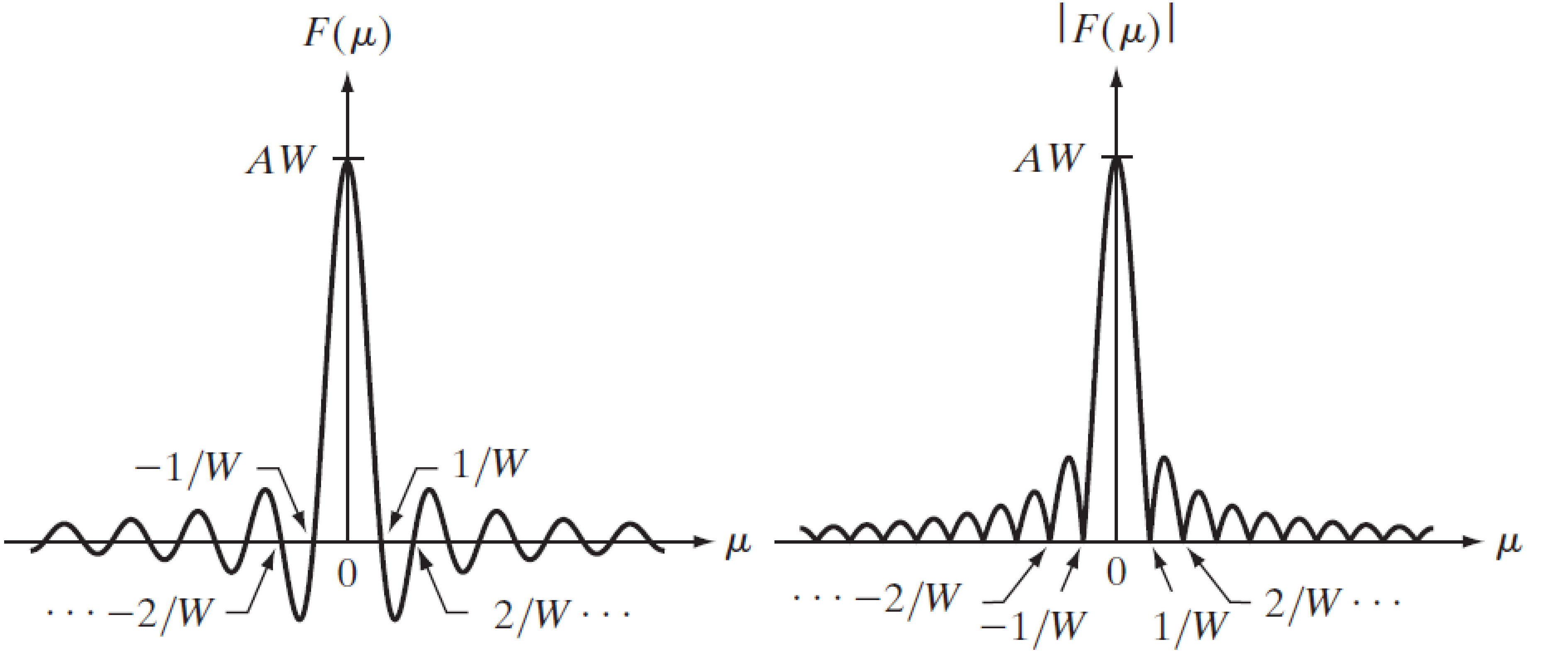
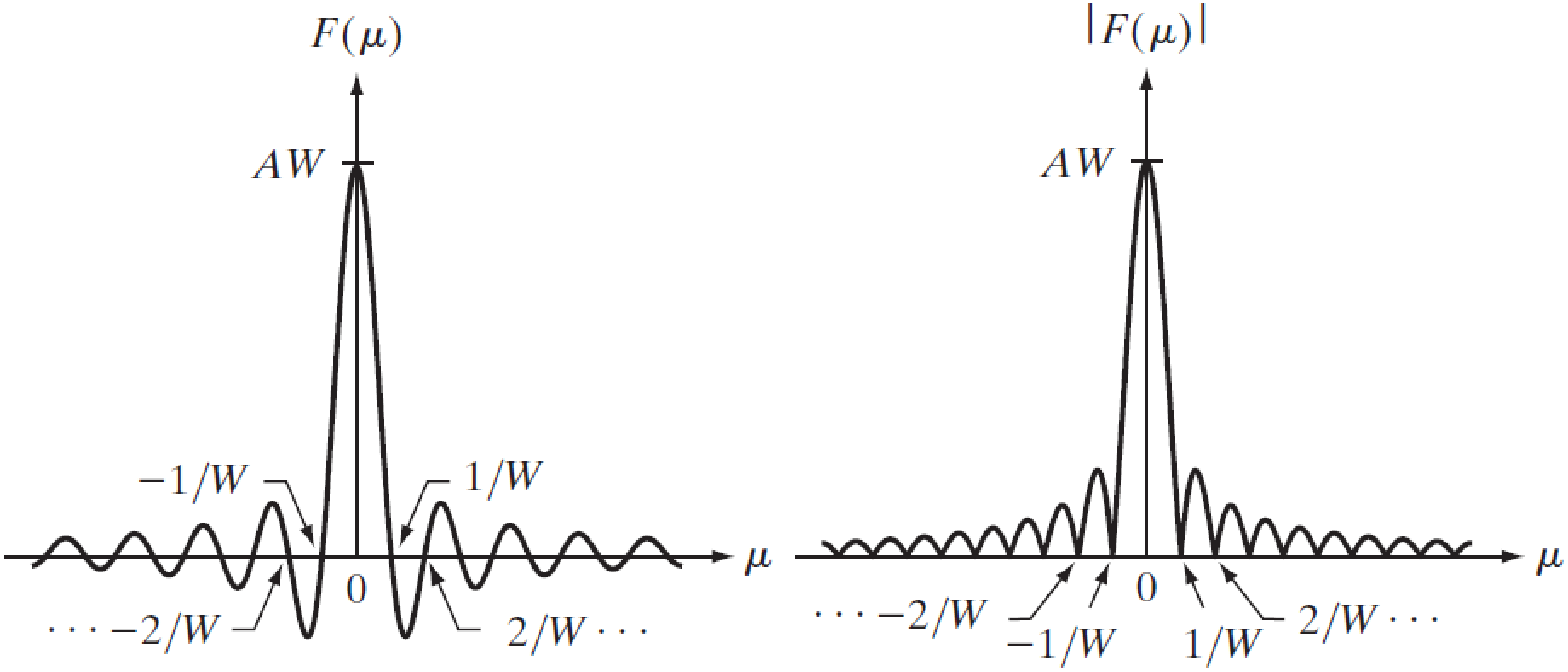


a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

The locations of the zeros of both F(μ) and |F(μ)| are *inversely* proportional to the width, W, of the "box" function



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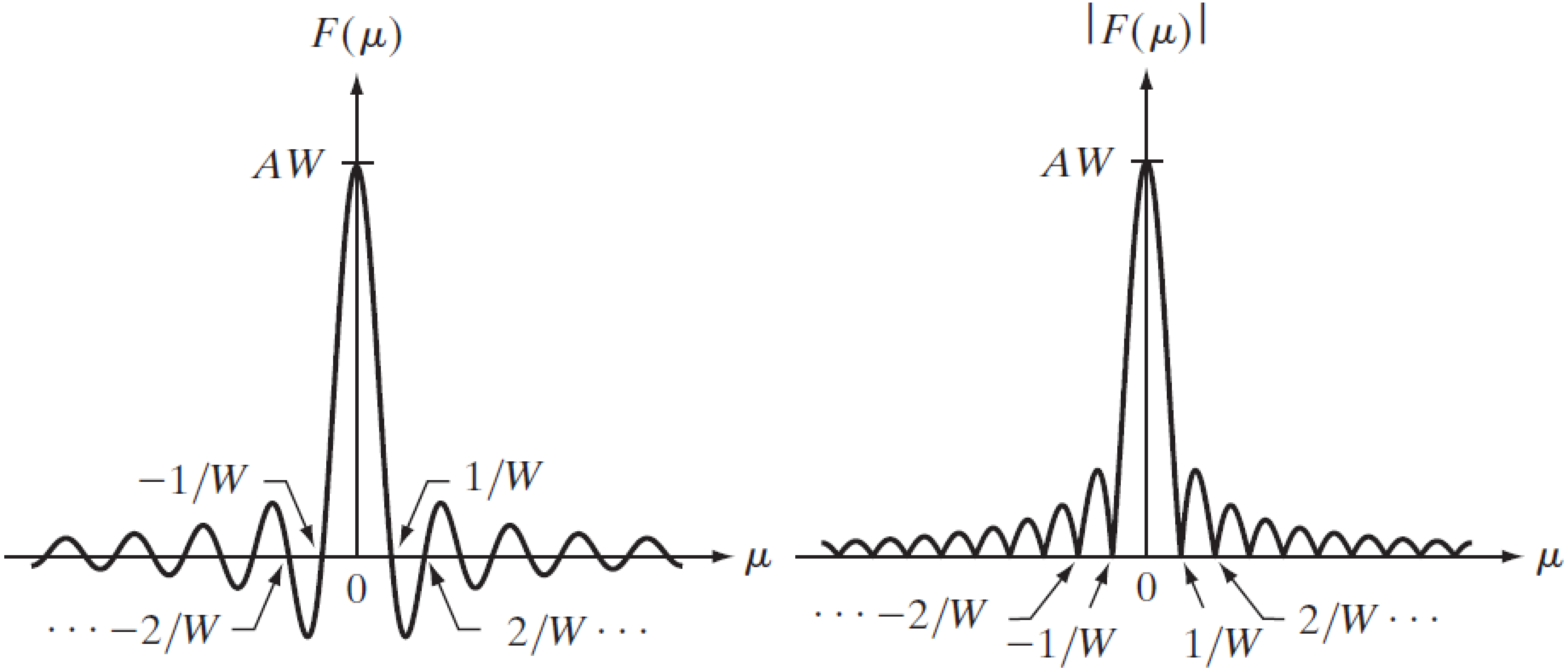
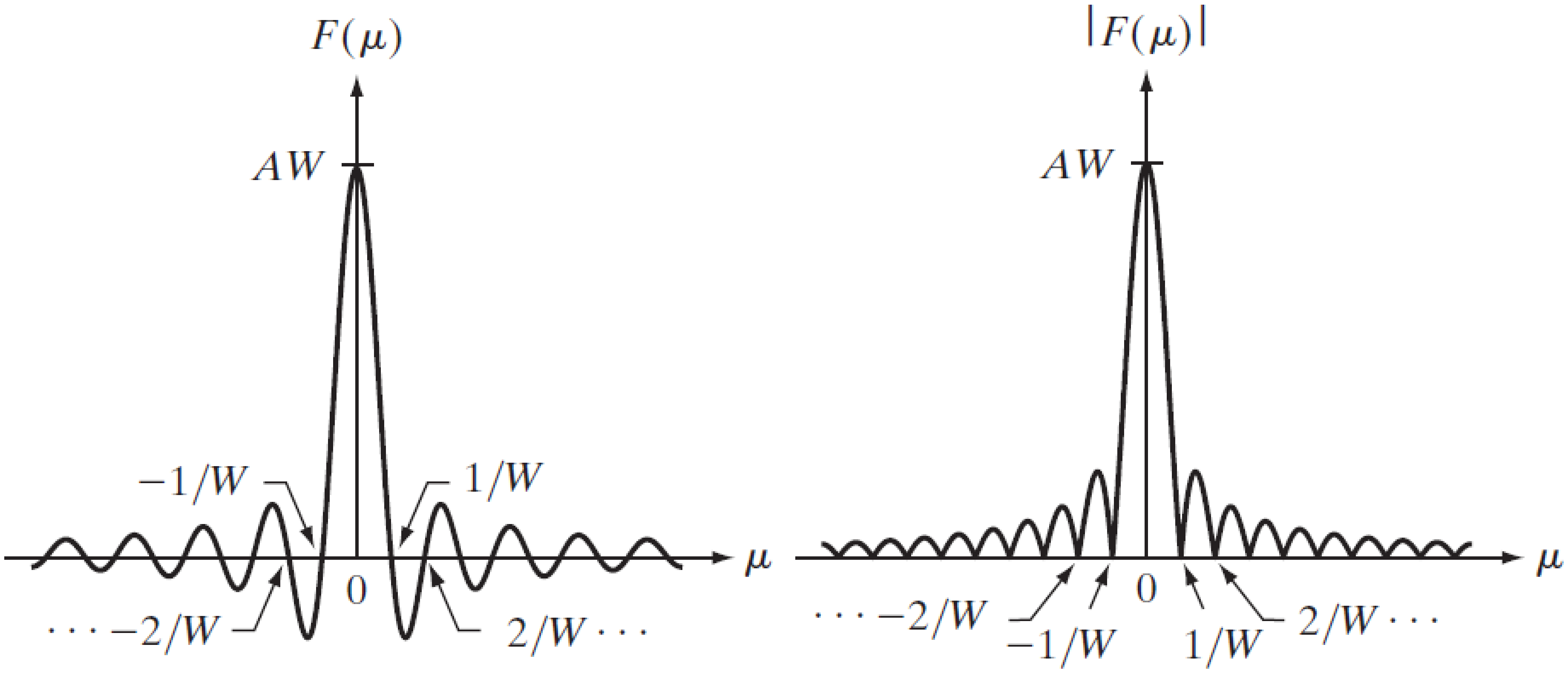


a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

The function extends to infinity for both positive and negative values of μ



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- Example 2: Fourier Transform of a unit impulse located at the origin

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\begin{aligned}
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&= e^{-j2\pi\mu 0} = e^{0} \\
&= 1
\end{aligned}
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Sifting property of impulse function

$$= e^{-j2\pi\mu 0} = e^{0}$$

$$= 1$$

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\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt \\
&= e^{-j2\pi\mu t_0} \\
&= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)
\end{aligned}
$$

$$F(\mu) = \delta(\mu - \mu_0)$$

$$IFT\,[\,F(\mu)] = \int_{-\propto}^{\propto} \delta(\,\mu - \mu_0)\, e^{j2\pi\mu t} d\mu = f(t)$$

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- Example 4: Fourier Transform $S(\mu)$ of an impulse train with period ΔT

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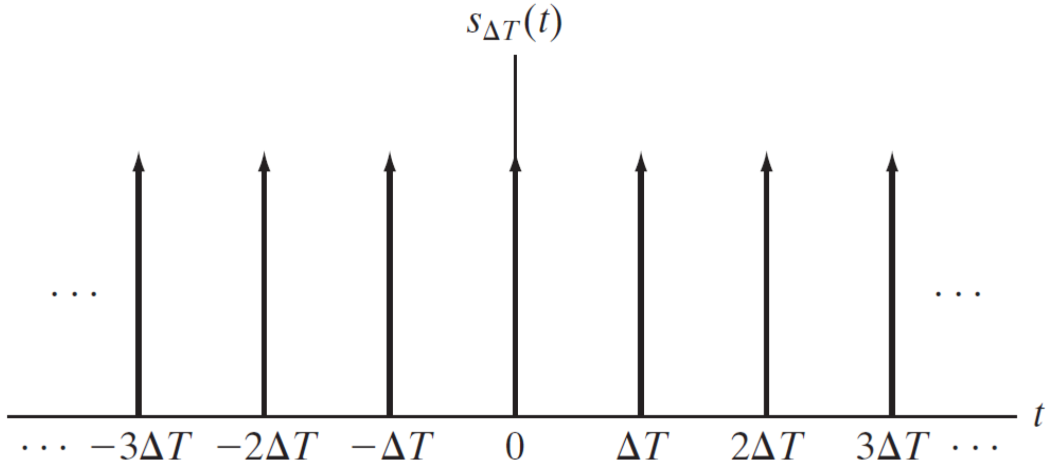

Impulse Train

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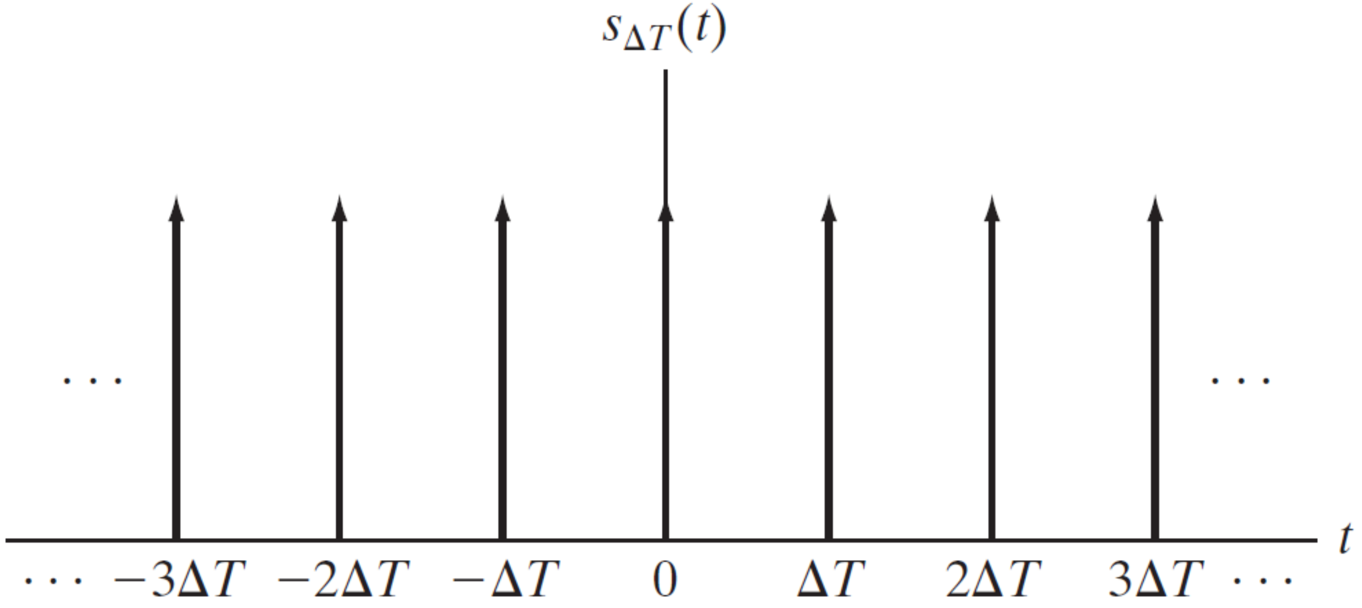

FIGURE 4.3 An impulse train.

Fourier Series

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$$

The Fourier series expansion then becomes

$$
s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{\,j\frac{2\pi n}{\Delta T}t}
$$

So, $S(\mu)$, the Fourier transform of the periodic impulse train $s_{\Delta T}(t)$, is

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&= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}
$$

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This fundamental result tells us that Fourier transform of an impulse train with period ΔT is also an impulse train, whose period is 1/ ΔT

Convolution

Convolution

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

Convolution

- Convolution of the two continuous functions, $f(t)$ and $h(t)$, of one continuous variable t.

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$$\Im\{f(t) \star h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi\mu t}\, dt$$

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Convolution Theorem: Proof

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau},$$

$$\begin{aligned}\Im\{f(t)\star h(t)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt \\ &= \int_{-\infty}^{\infty} f(\tau)\boxed{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,dt\right]}d\tau\end{aligned}$$

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$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau = H(\mu)F(\mu)$$

Convolution Theorem

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Convolution Theorem

- We have,

$$\Im\left\{f(t) \star h(t)\right\} = H(\mu)F(\mu)$$

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$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

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Extension to function of two variables

The 2-D Impulse

Extension to function of two variables

The 2-D Impulse

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Extension to function of two variables

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$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

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Extension to function of two variables
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- The impulse, $\delta(t, z)$, of two continuous variables, t and z is defined as

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Sifting property of 2-D Impulse

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At Origin

Located at (t_0 , z_0)

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Discrete 2-D Impulse

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Discrete 2-D Impulse

- For discrete variables x and y, the 2-D discrete impulse is defined as

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

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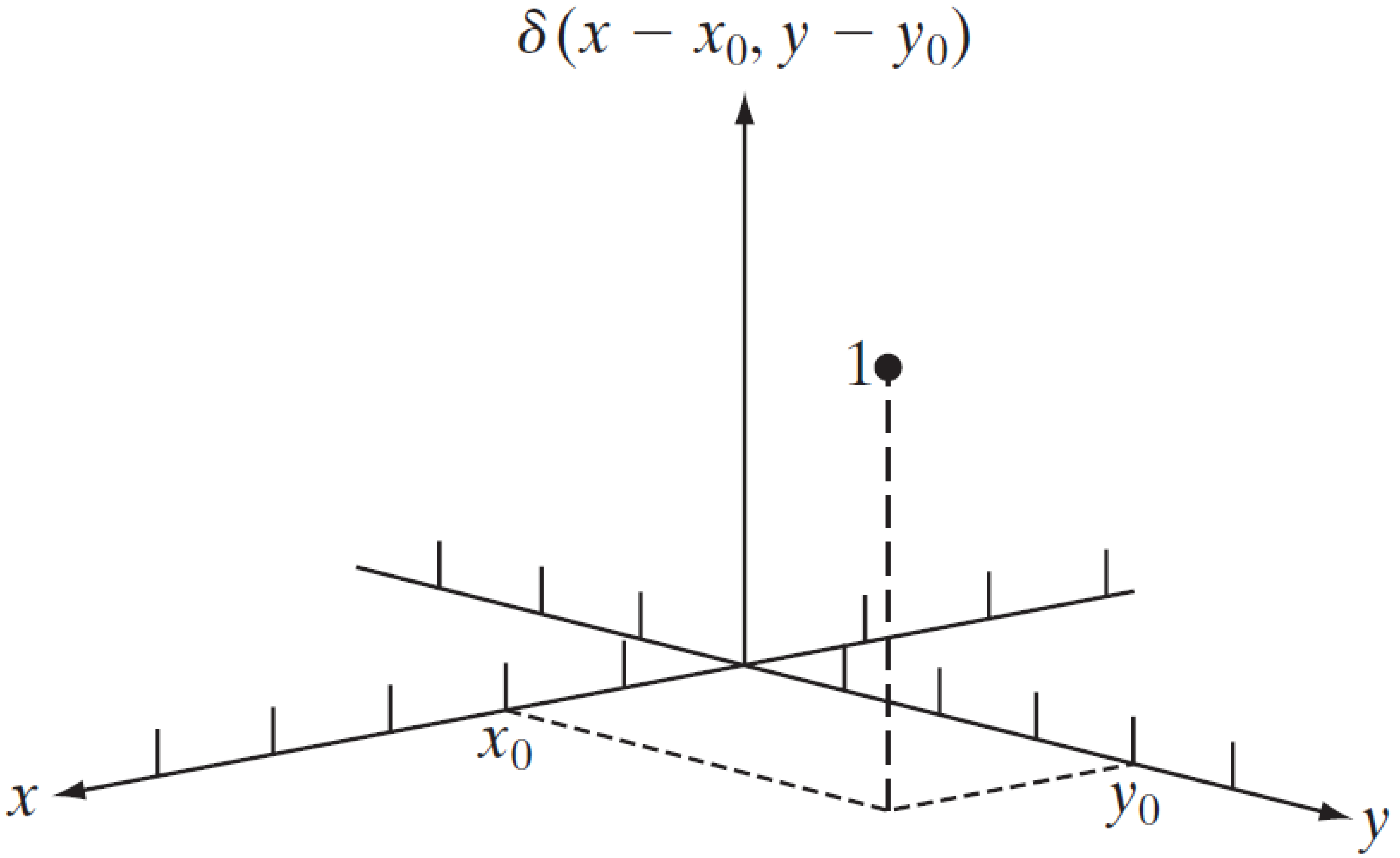

FIGURE 4.12
Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

The 2-D Continuous Fourier transform pair

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- Let $f(t, z)$ be a continuous function of two continuous variables, t and z
- *Fourier Transform of* $\boldsymbol{f(t, z)}$

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$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

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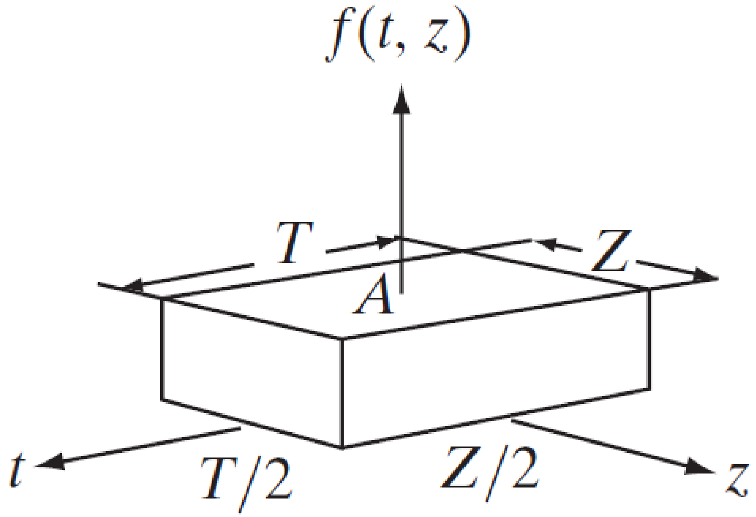

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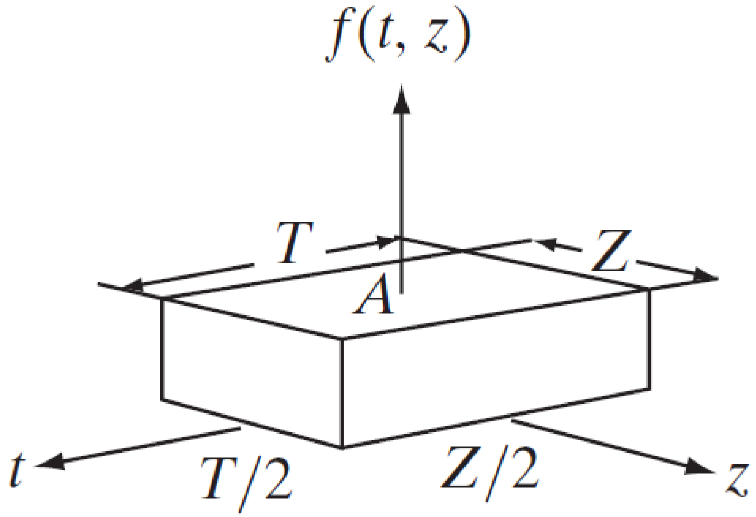

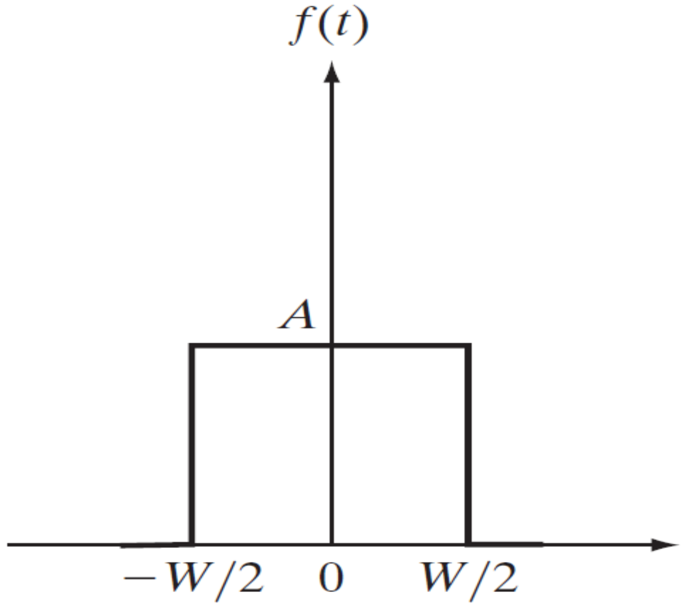

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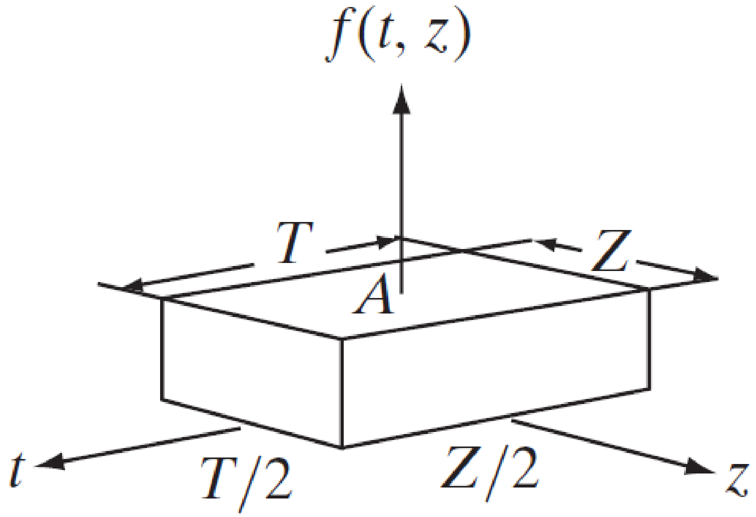

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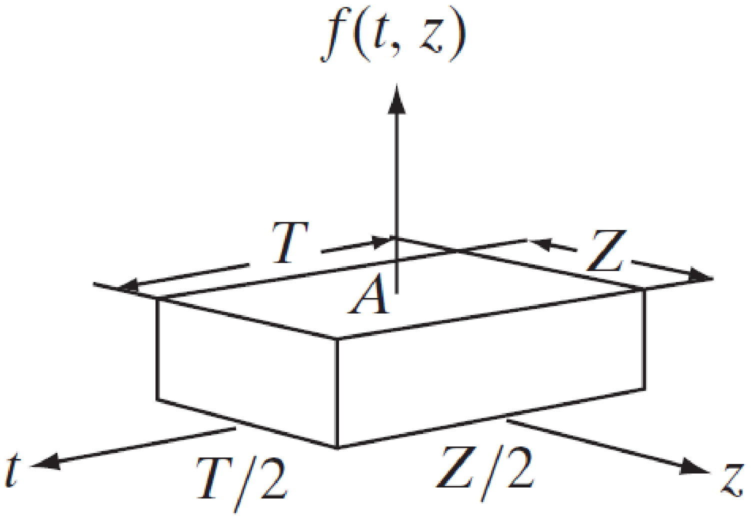

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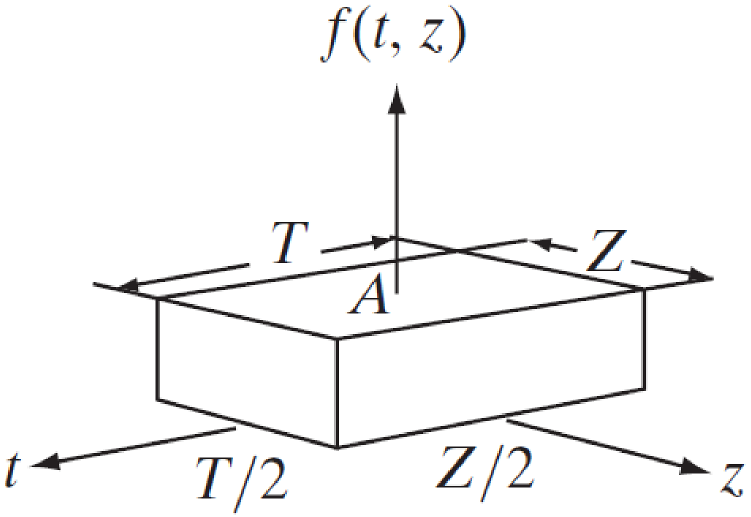

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$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

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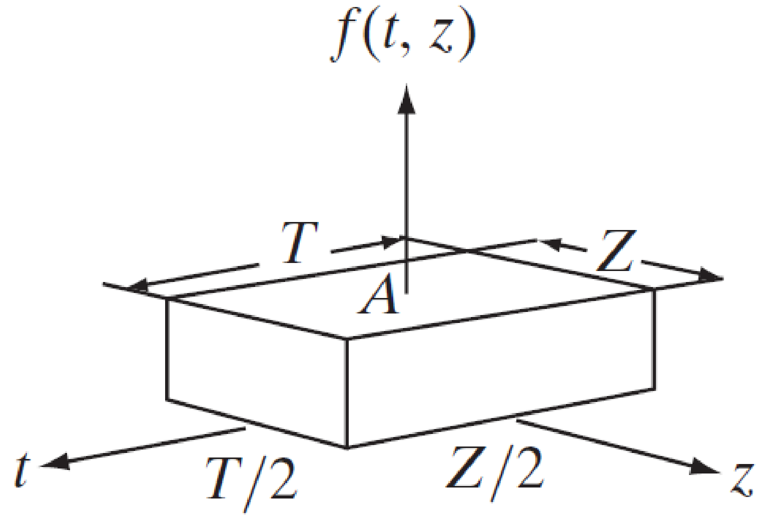

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$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

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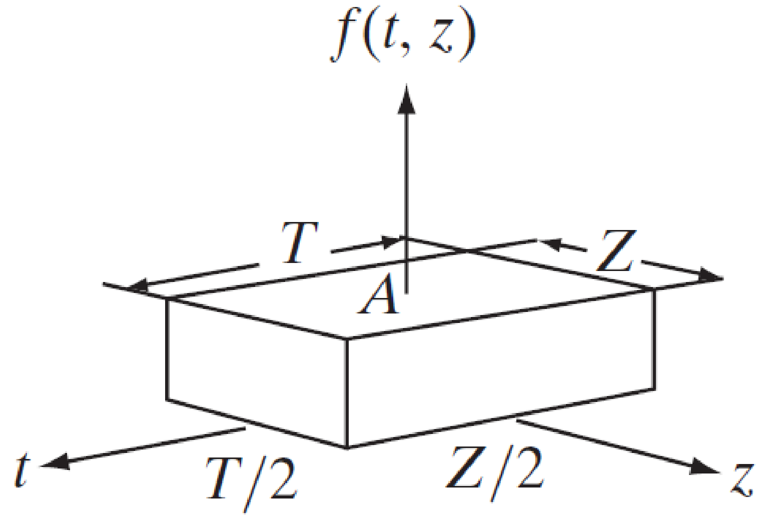

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[Students don’t remember]

Example

- Example 1: Find Fourier Transform of function shown in figure below

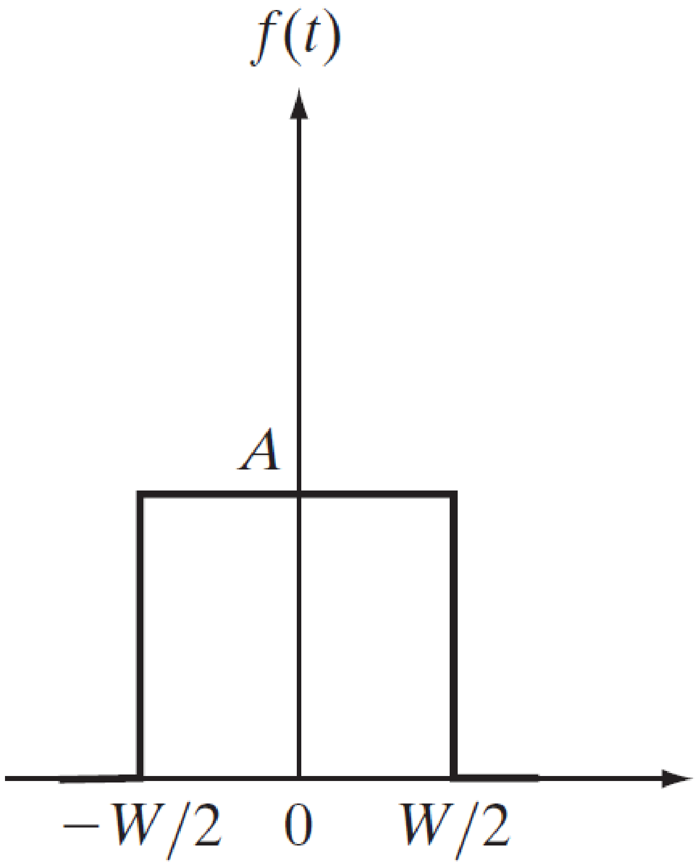
$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

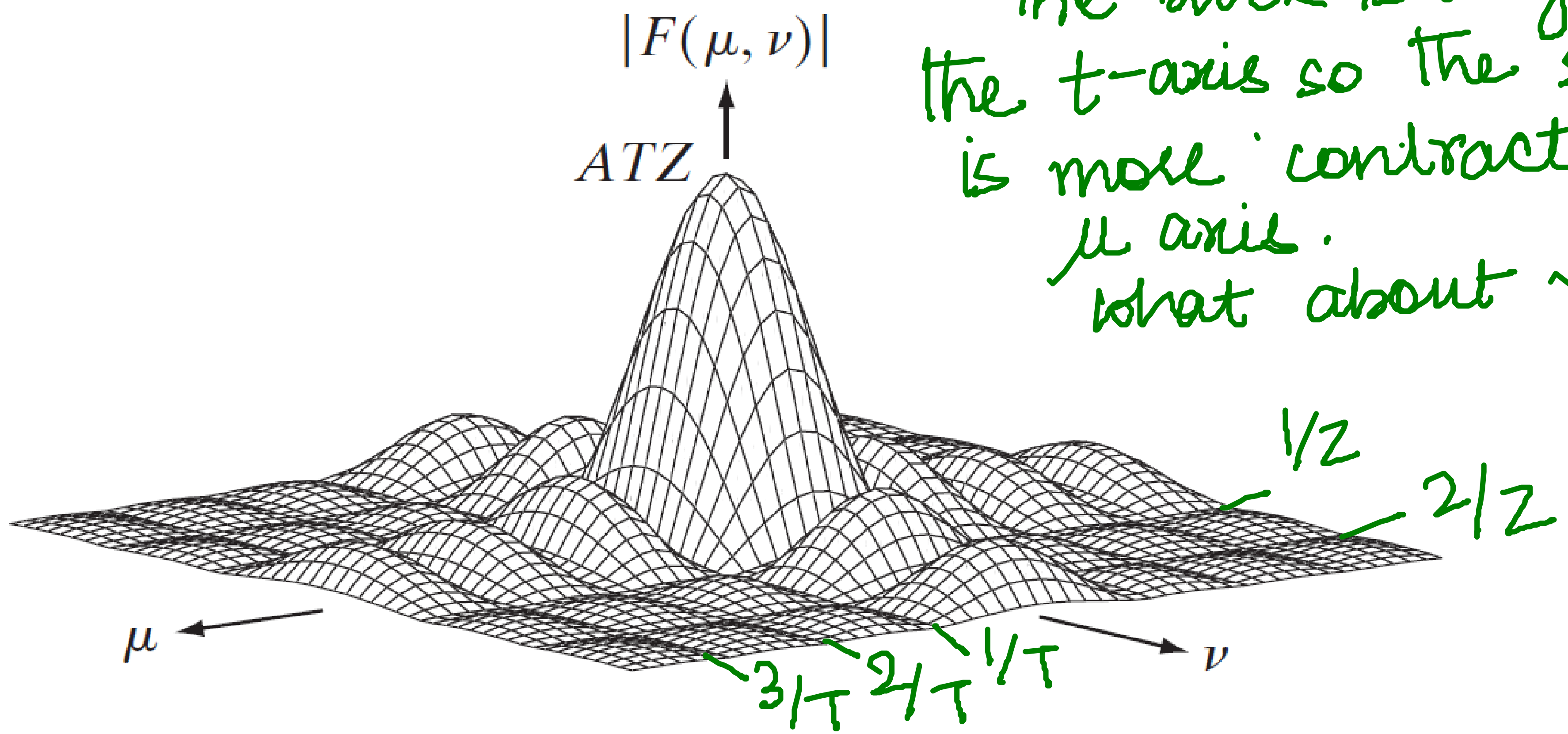
$$\sin\theta = (e^{j\theta} - e^{-j\theta})/2j.$$



- The magnitude (spectrum) is given by the expression

$$|F(\mu,\nu)| = ATZ\left|\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right|\left|\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right|$$

The block is longer along the t-axis so the spectrum is more contracted along μ axis.
What about ν axis?



Sampling and the Fourier Transform of Sampled functions.

Sampling

Sampling and the Fourier Transform of Sampled functions.
Sampling

- Continuous functions have to be converted into a sequence of discrete values before they can be processed in a computer.
- This is accomplished by sampling and quantization.

Sampling and the Fourier Transform of Sampled functions.

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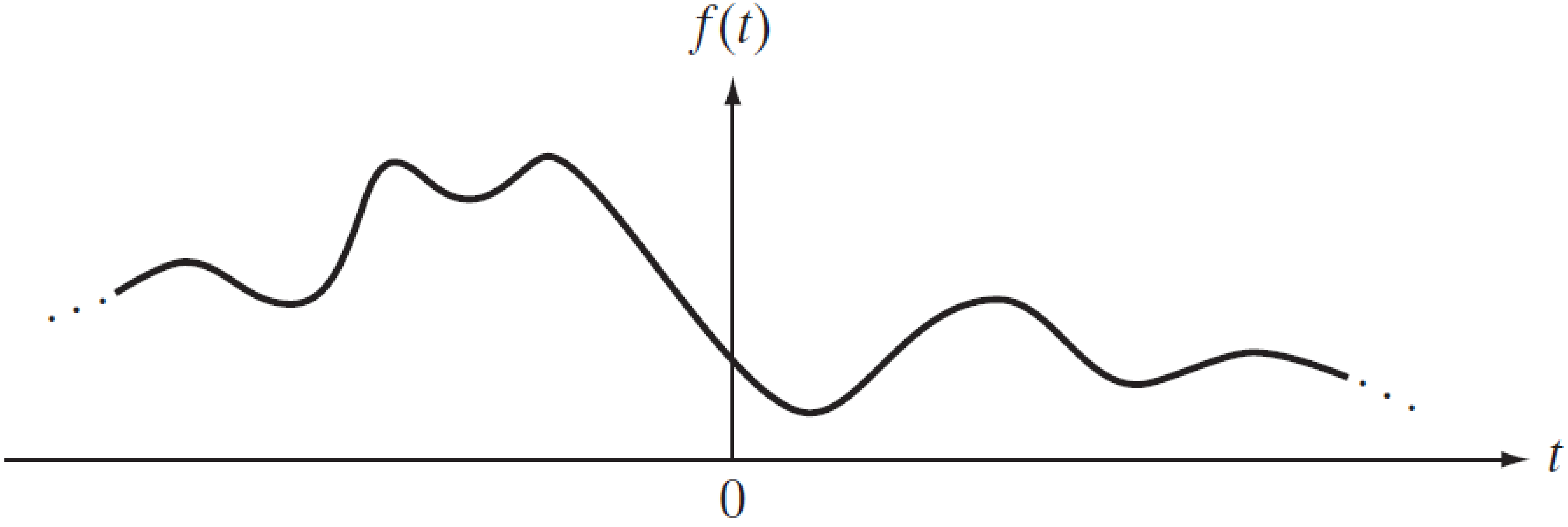
Sampling



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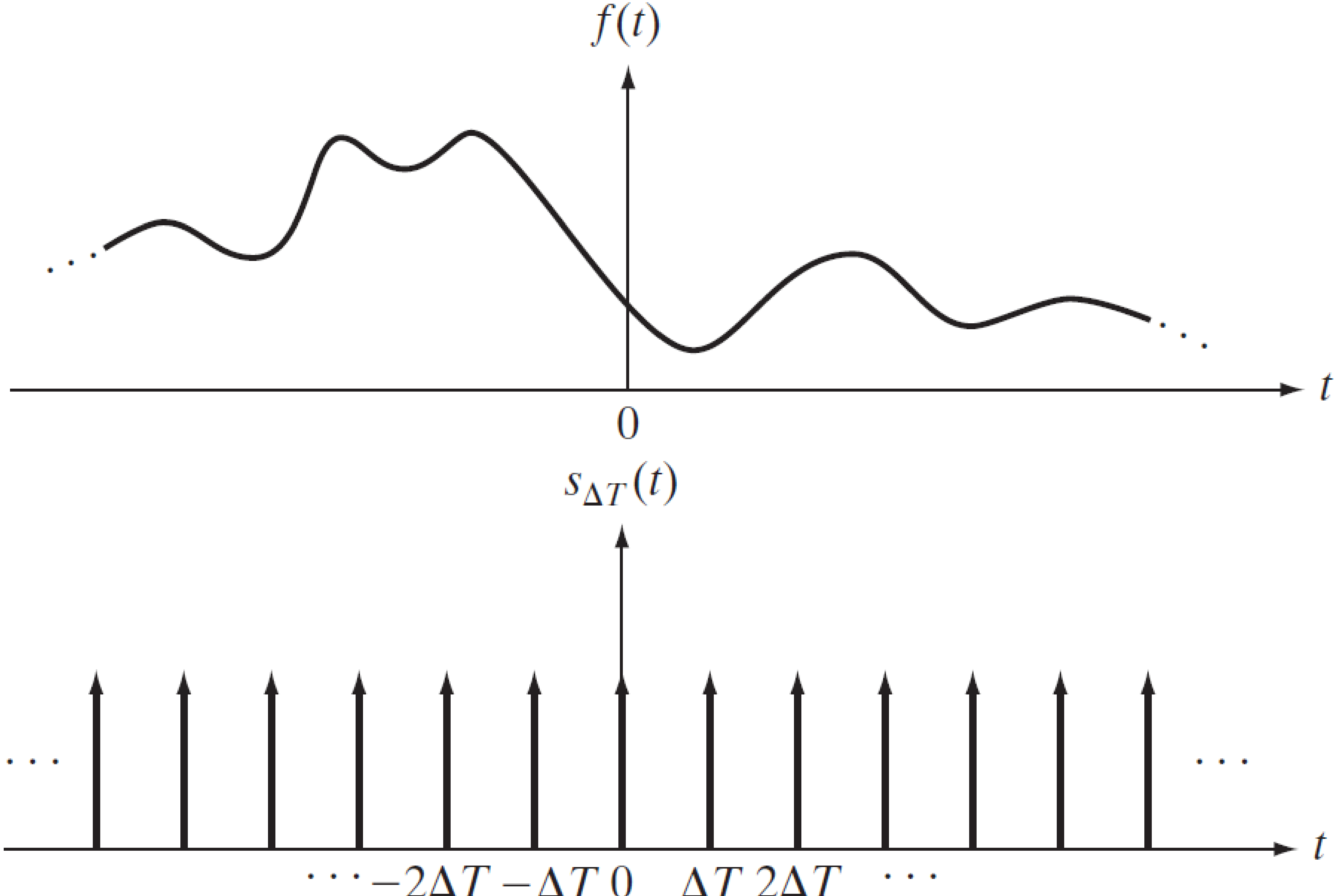
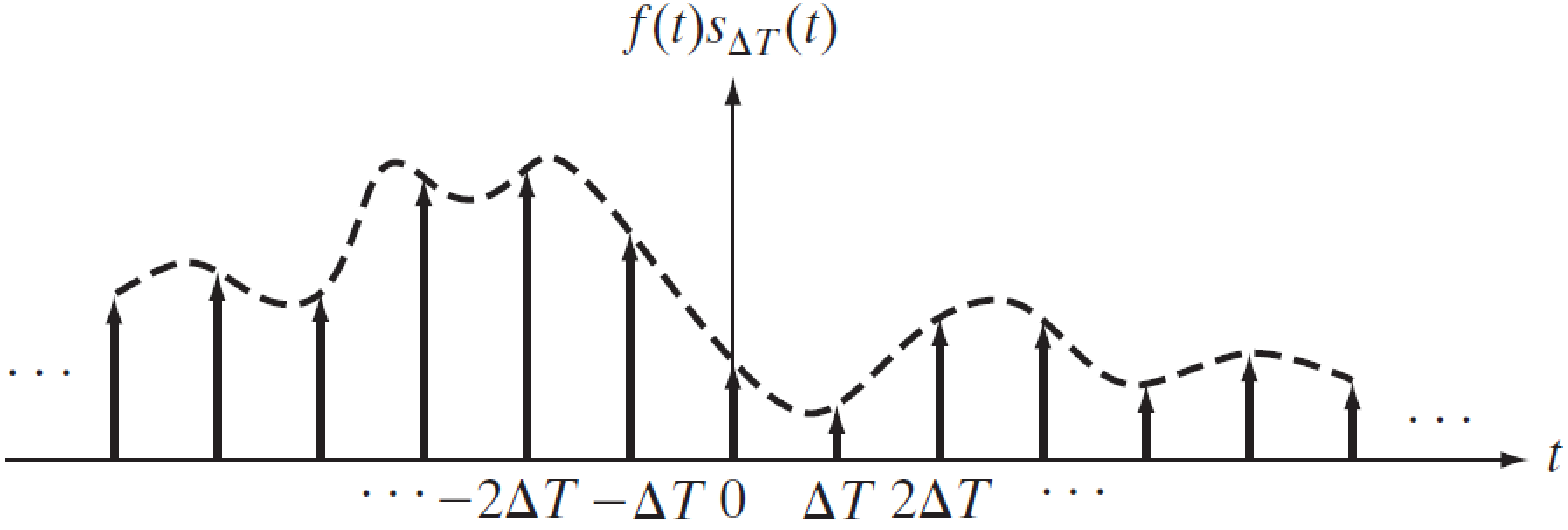
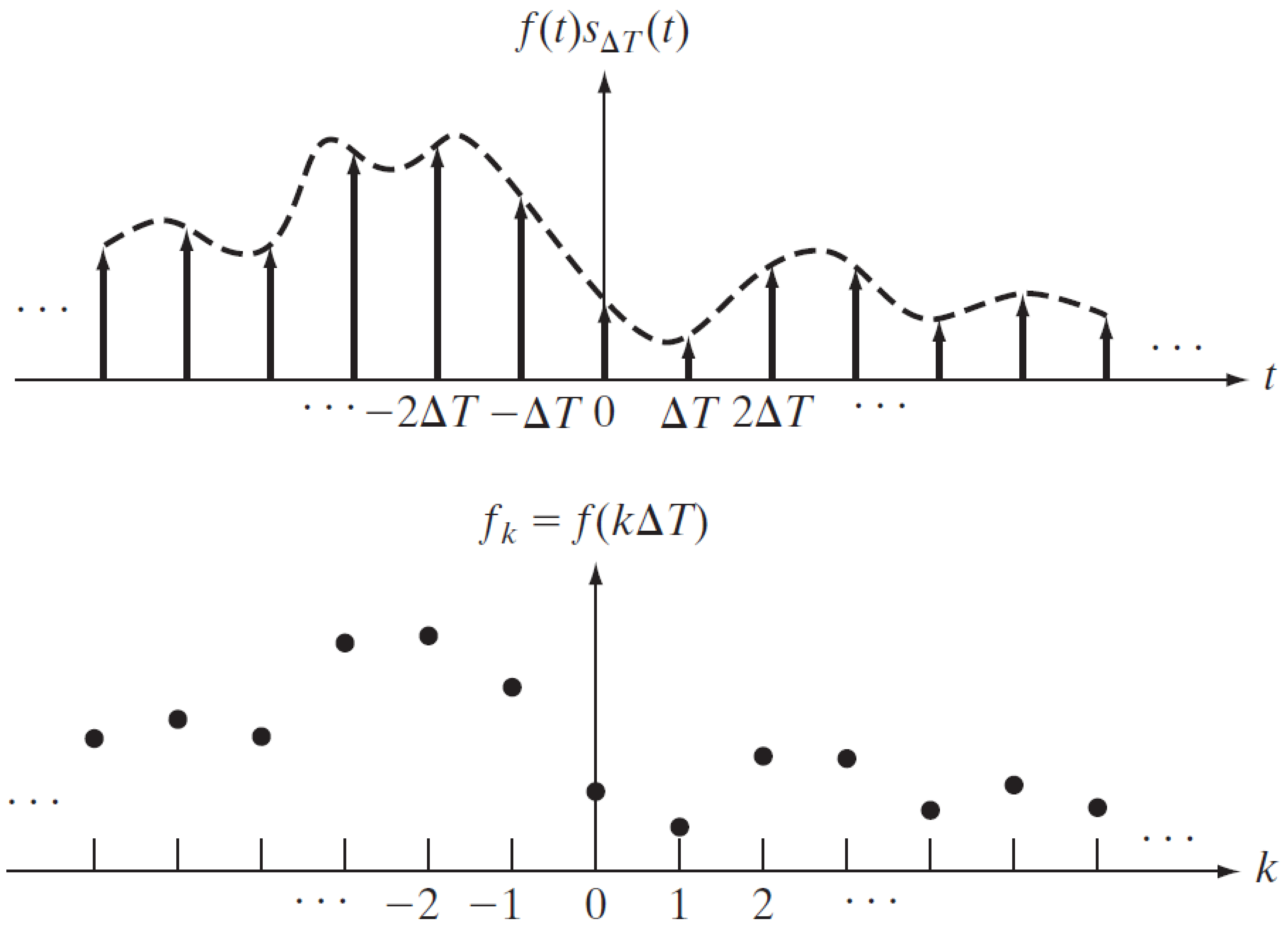




$f(t)s_{\Delta T}(t)$
$\cdots$
$\cdots$
t
$\cdots -2\Delta T \; -\Delta T \; 0 \quad \Delta T \; 2\Delta T \quad \cdots$



$f(t)s_{\Delta T}(t)$
t
$\cdots -2\Delta T\ -\Delta T\ 0\quad \Delta T\ 2\Delta T\quad \cdots$
$f_k = f(k\Delta T)$
k
$\cdots\ -2\ -1\ 0\ 1\ 2\ \cdots$

Sampling of 1-D function

Sampling of 1-D function

- Consider a continuous function, $f\ (t)$, that we wish to sample at uniform intervals ΔT of the independent variable t.
- One way to model sampling is to multiply $f\ (t)$ by a *sampling function* equal to a train of impulses ΔT units apart.

- Where denotes the sampled function.
- The value, f_k of an arbitrary sample in the sequence is given by

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$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

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The Fourier Transform of Sampled Functions

$\tilde{f}(t)$

The Fourier Transform of Sampled Functions

- Let $F(\mu)$ denote the Fourier transform of a continuous function $f(t)$. As discussed in the previous section, the corresponding sampled, $\tilde{f}(t)$ function, is the product of $f(t)$ and an impulse train.

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$$
\begin{aligned}
\tilde{F}(\mu) &= \Im\{\tilde{f}(t)\} \\
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$$

- We know,

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

Reference

$$\widetilde{F}(\mu) = F(\mu) \star S(\mu)$$

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&= \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau
\end{aligned}
$$

$$
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&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}
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$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

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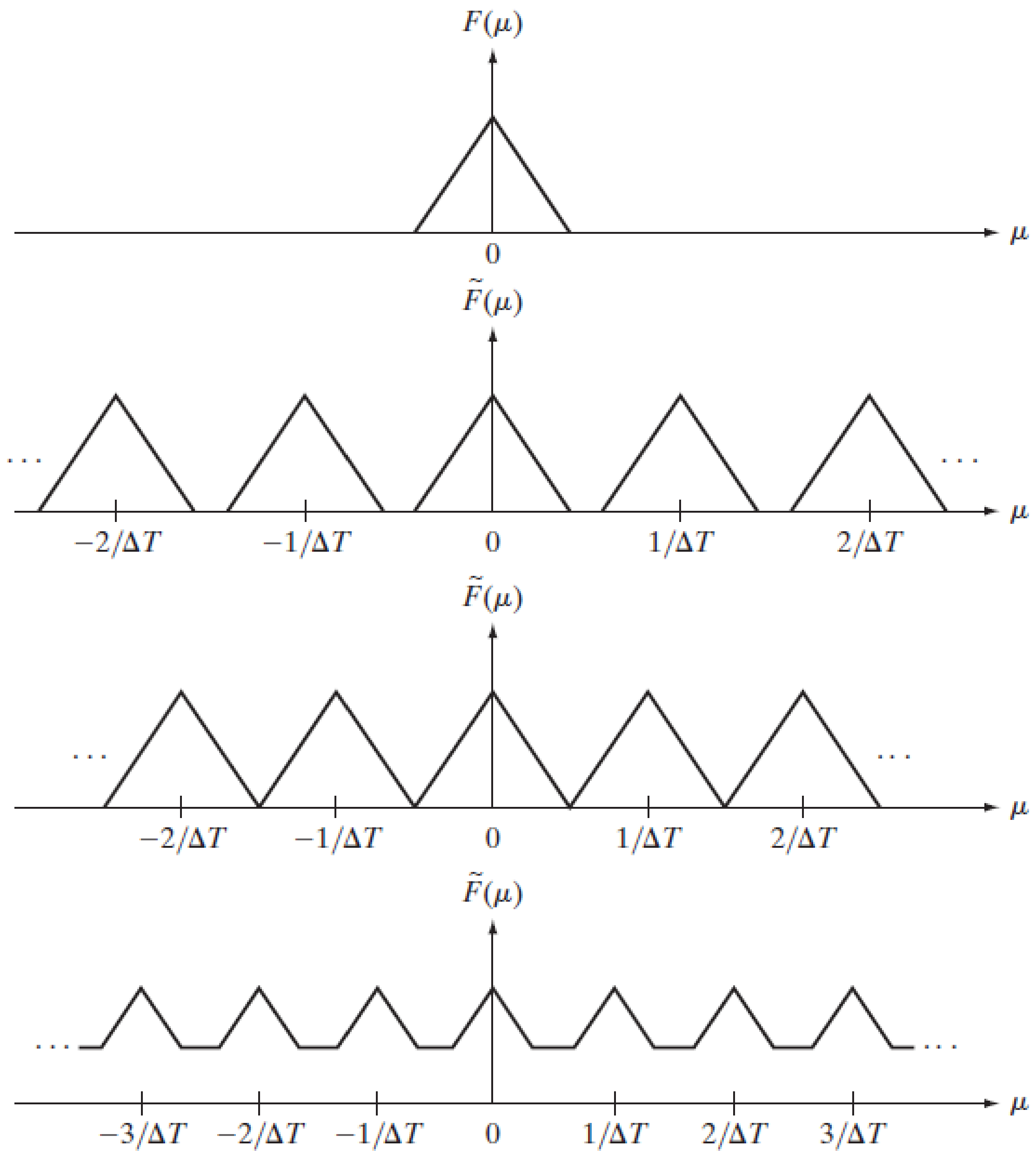
a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.



Sampling Process

Sampling Process

- Establish the conditions under which a continuous function can be *recovered uniquely* from a set of its samples.
- A function $f(t)$ whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{max}, \mu_{max}]$ about the origin is called a *band-limited function*.

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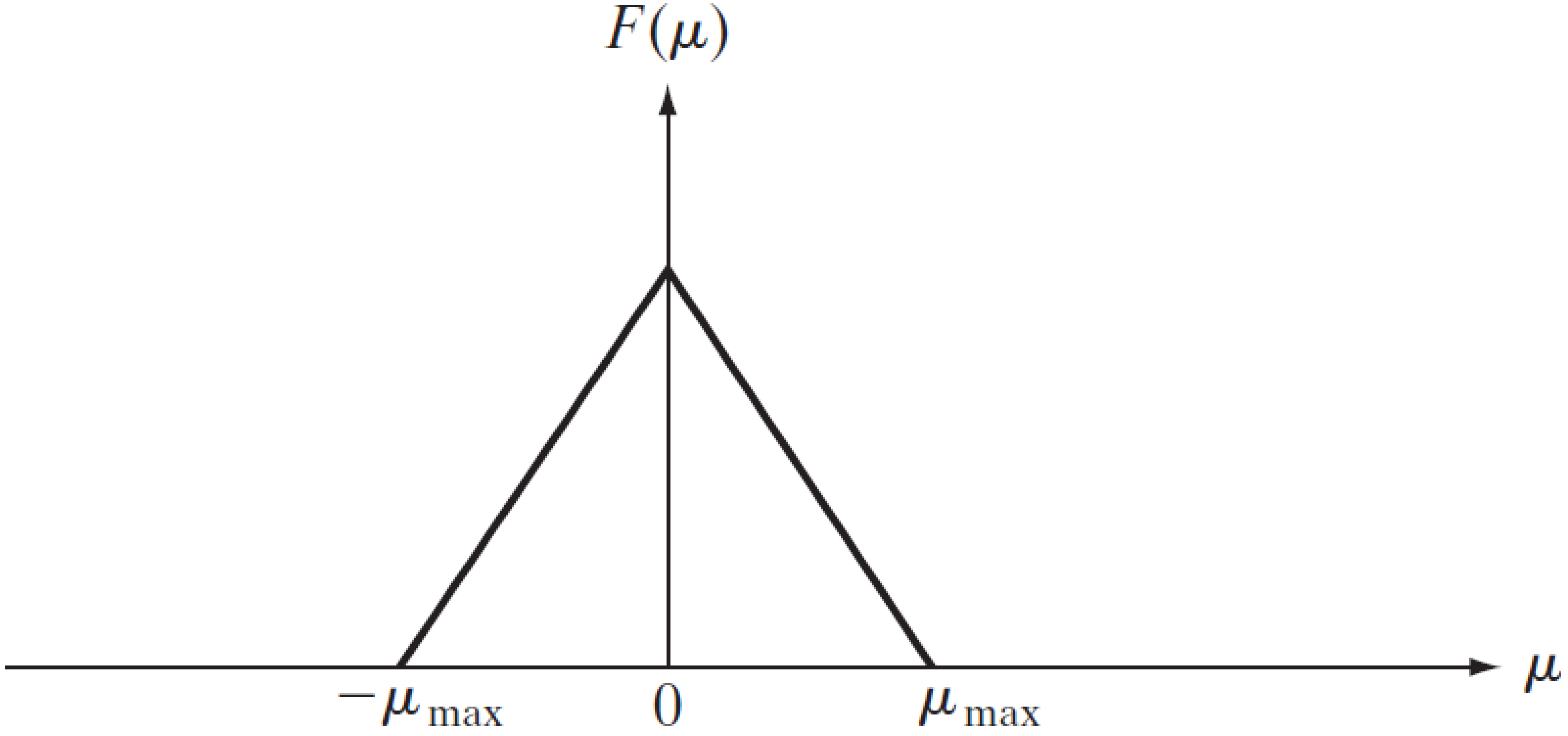
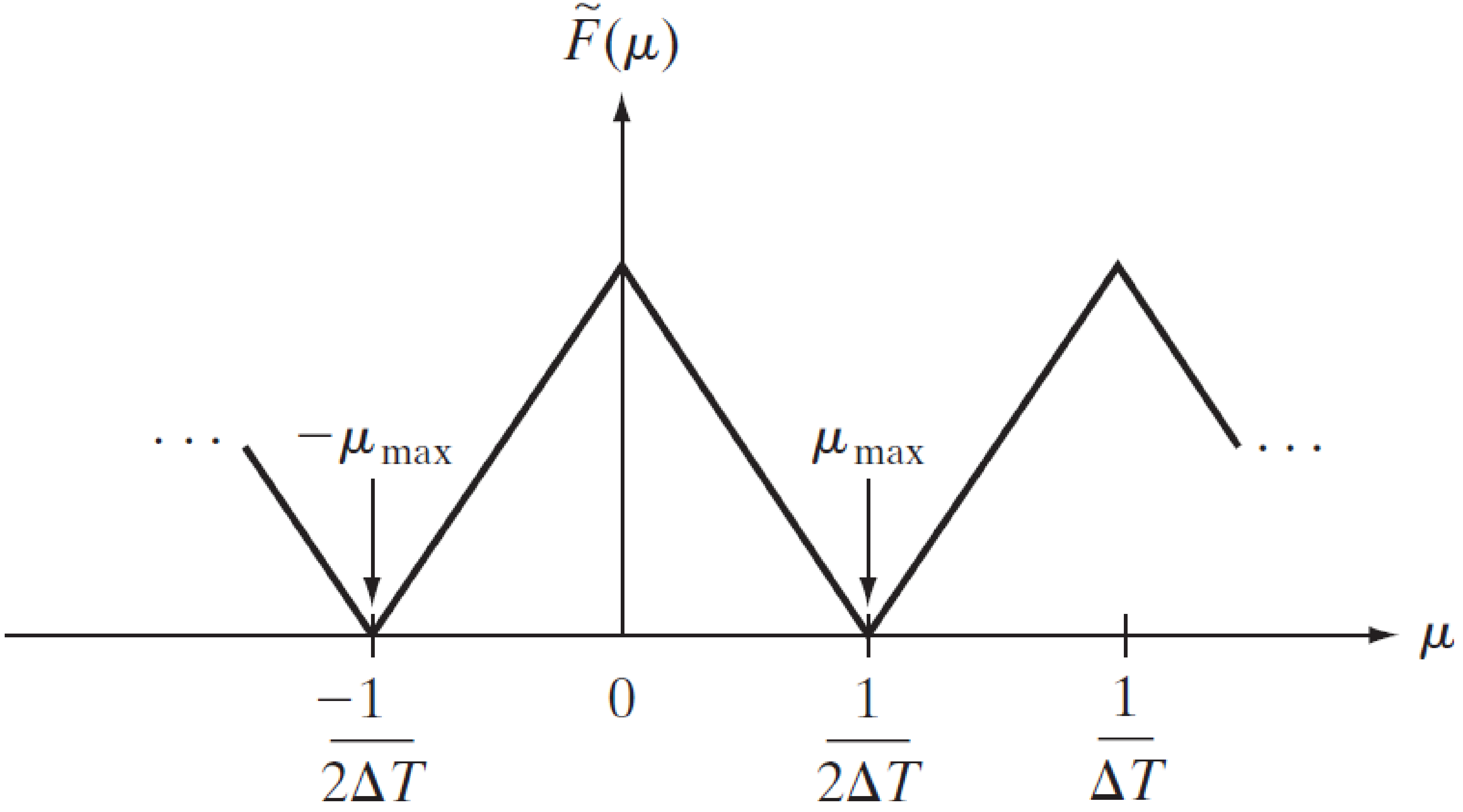
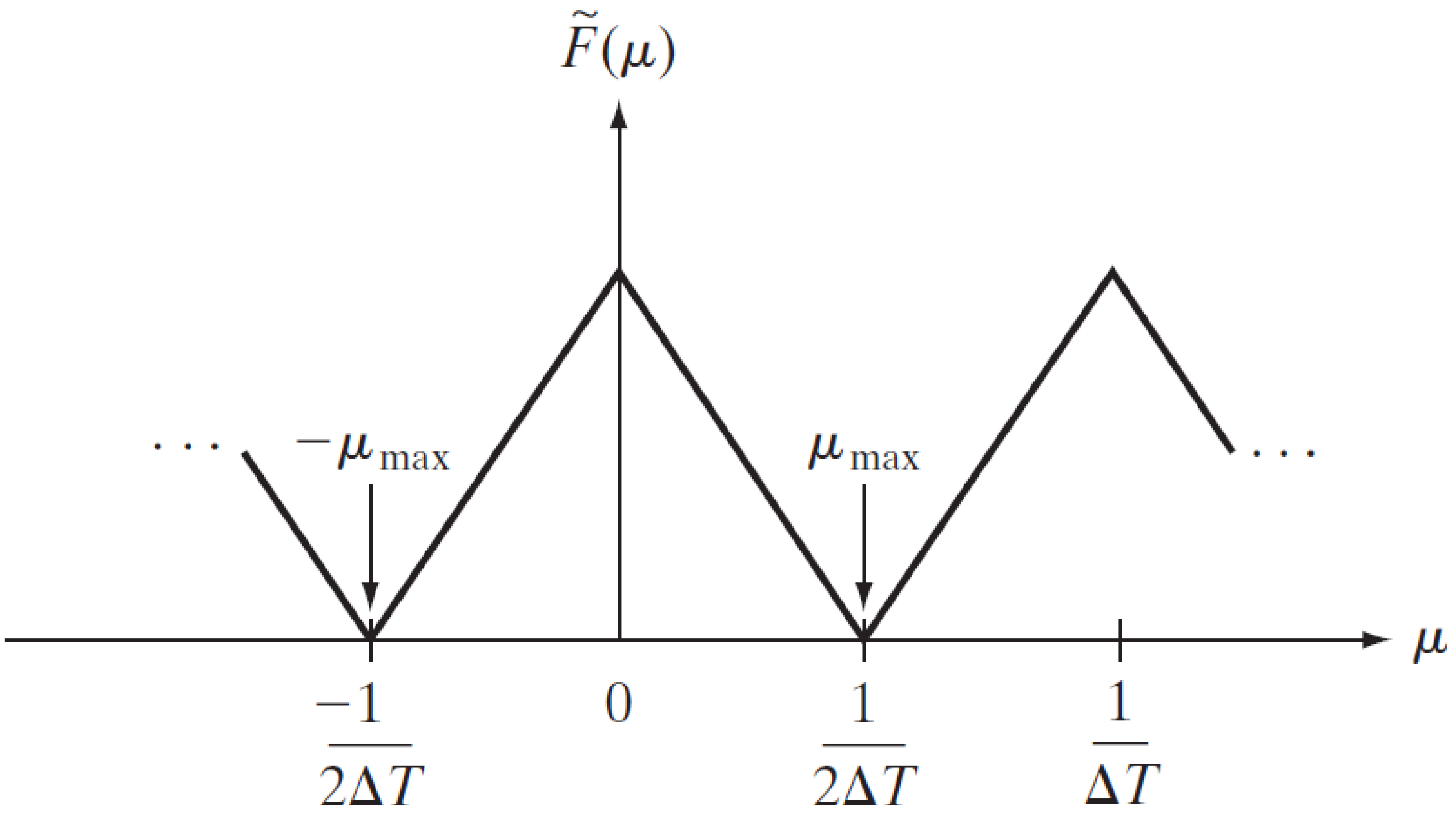
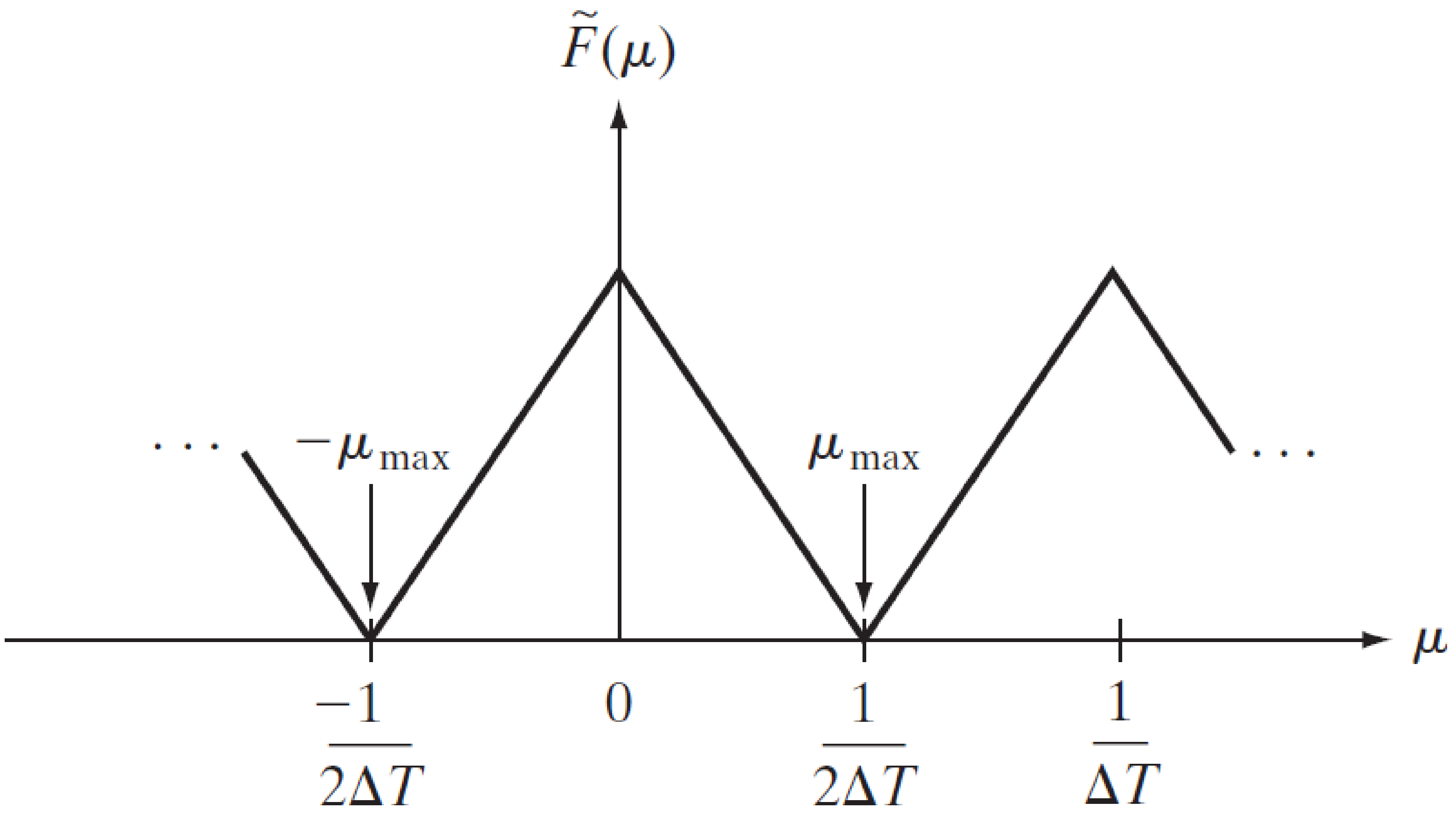




$\tilde{F}(\mu)$
$-\mu_{\max}$
$\mu_{\max}$
. . .
. . .
μ
$\frac{-1}{2\Delta T}$
0
$\frac{1}{2\Delta T}$
$\frac{1}{\Delta T}$



A lower value of $1/\Delta T$ would cause the periods in $\tilde{F}(\mu)$ to merge; a higher value would provide a clean separation between the periods.



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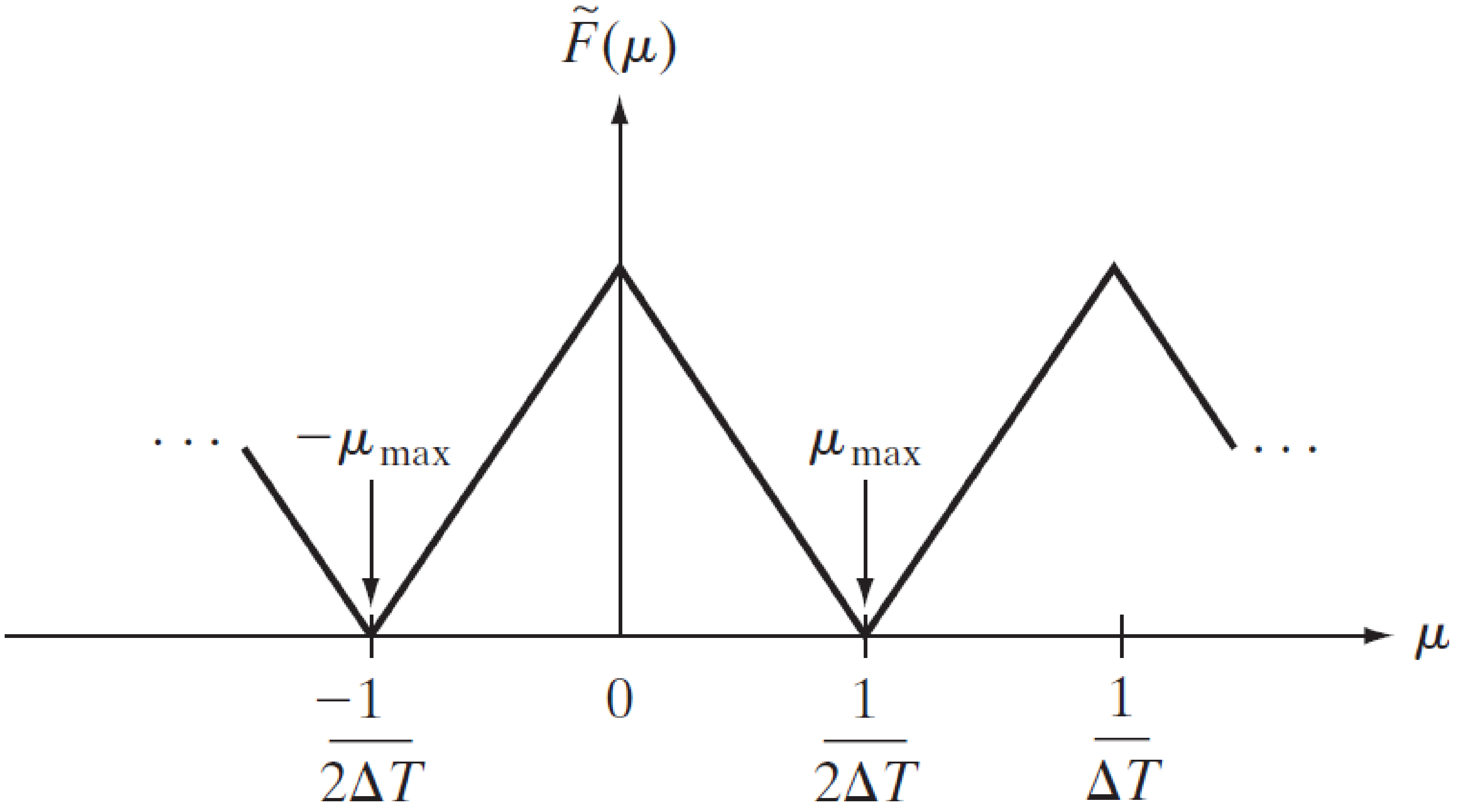
Sufficient Separation is guaranteed if

$$\frac{1}{\Delta T} > 2\mu_{max}$$



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Sufficient Separation is guaranteed if

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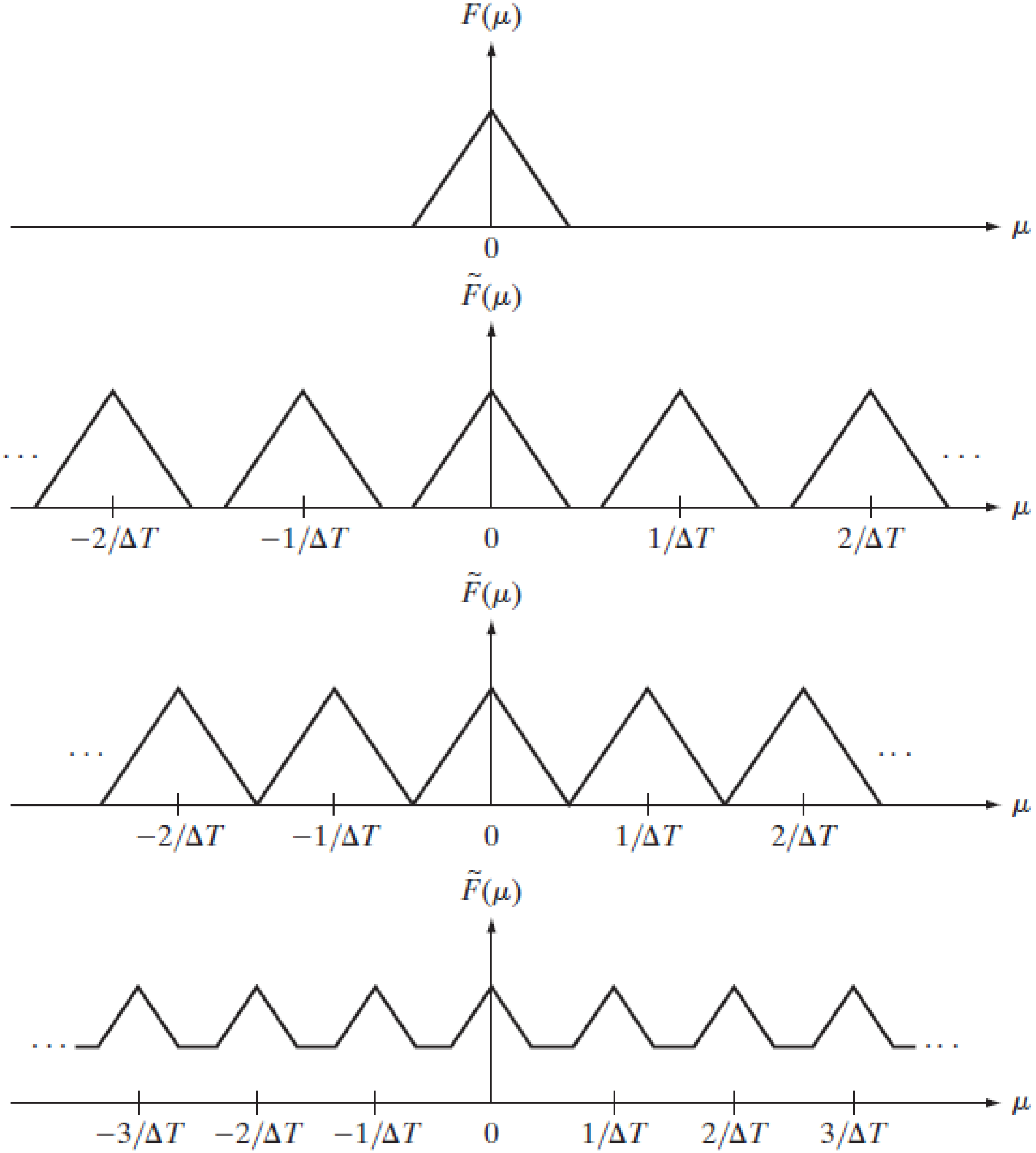
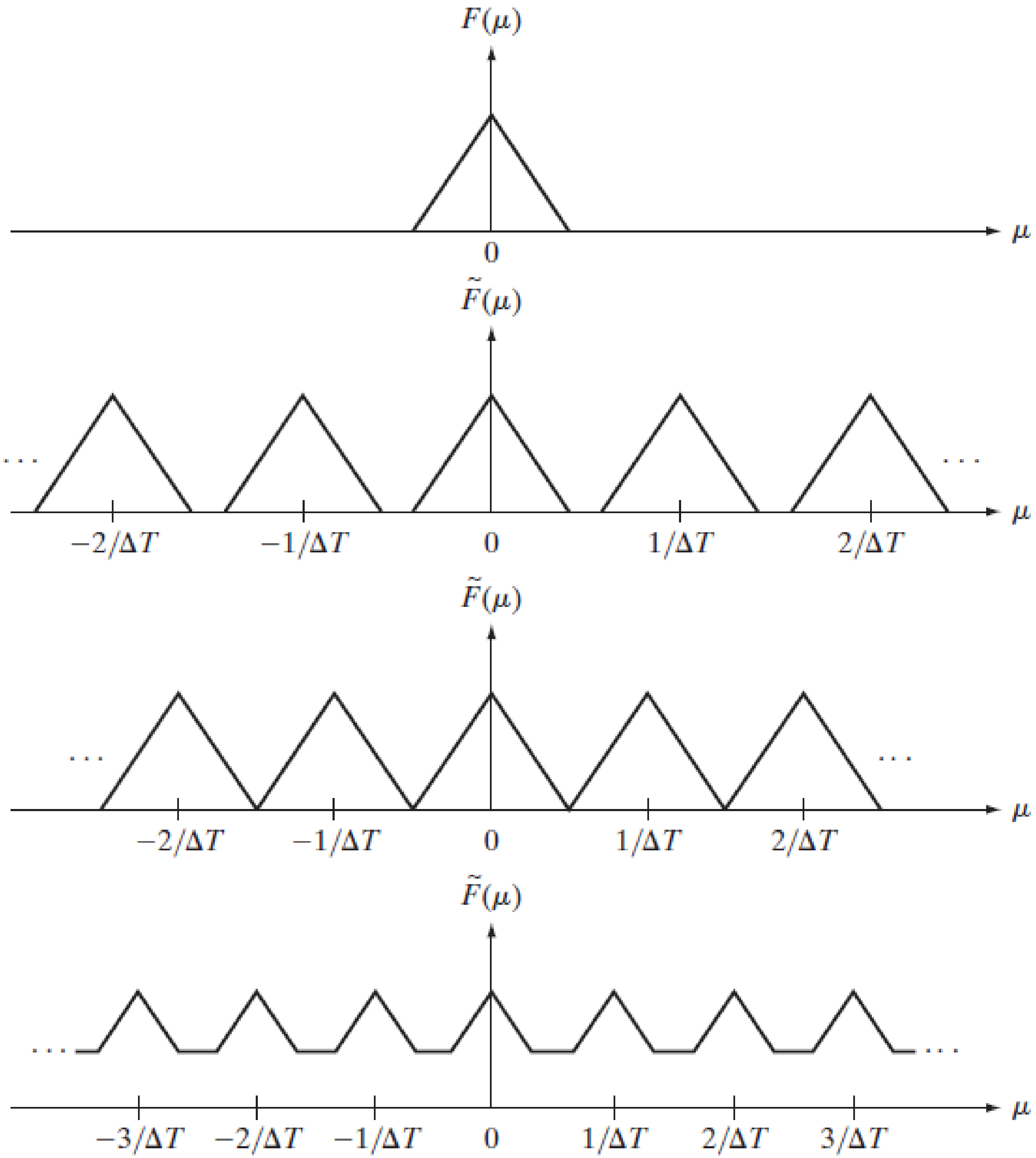
This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function. This result is known as the *sampling theorem.*



$F(\mu)$
μ
0
$\tilde{F}(\mu)$
. . .
. . .
μ
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$\tilde{F}(\mu)$
. . .
. . .
μ
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$\tilde{F}(\mu)$
. . .
. . .
μ
$-3/\Delta T$
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$3/\Delta T$

Oversampled

$$\frac{1}{\Delta T} > 2\mu_{max}$$

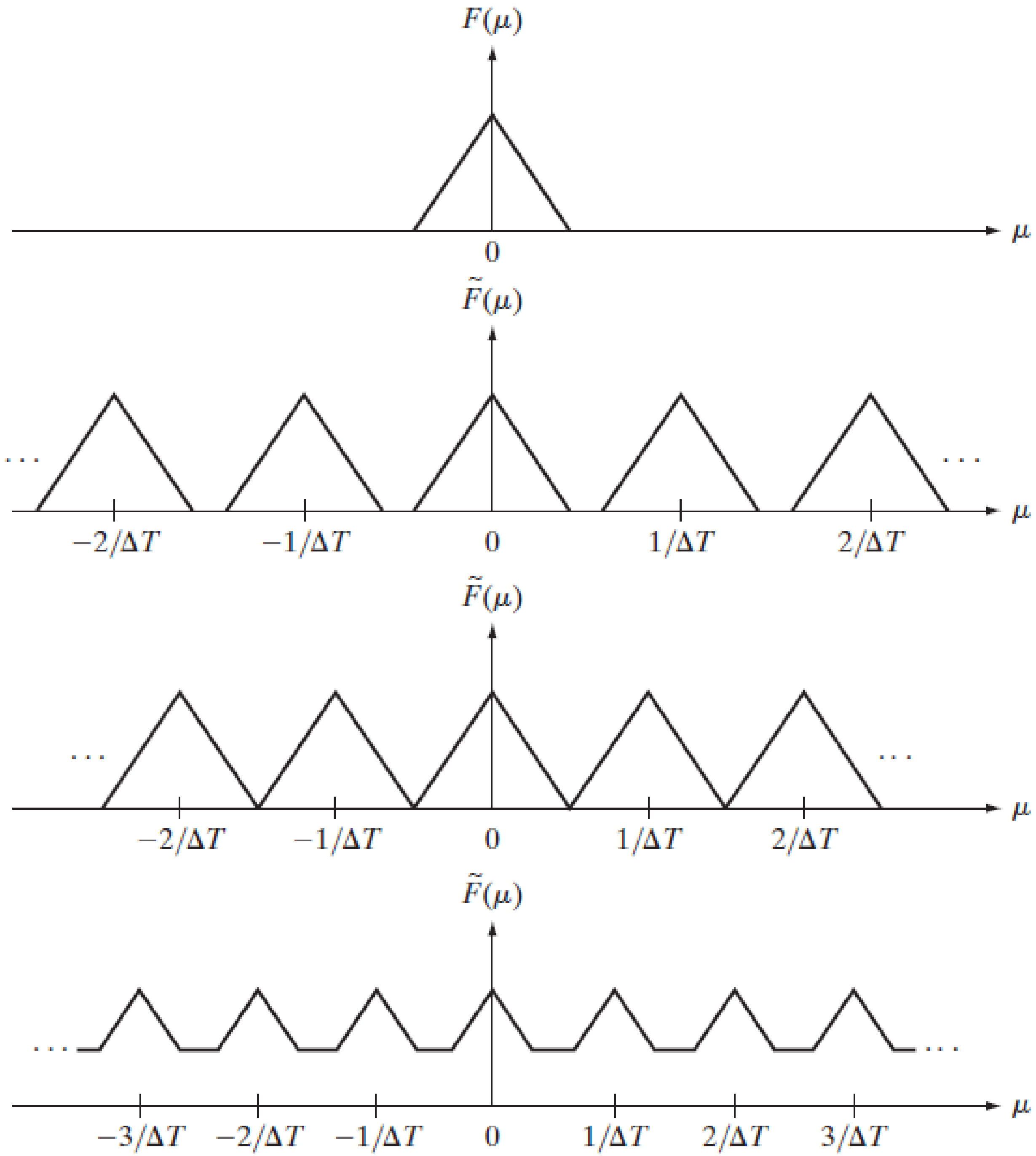


Oversampled

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled

$$\frac{1}{\Delta T} = 2\mu_{max}$$



Oversampled
$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled
$$\frac{1}{\Delta T} = 2\mu_{max}$$

Under sampled
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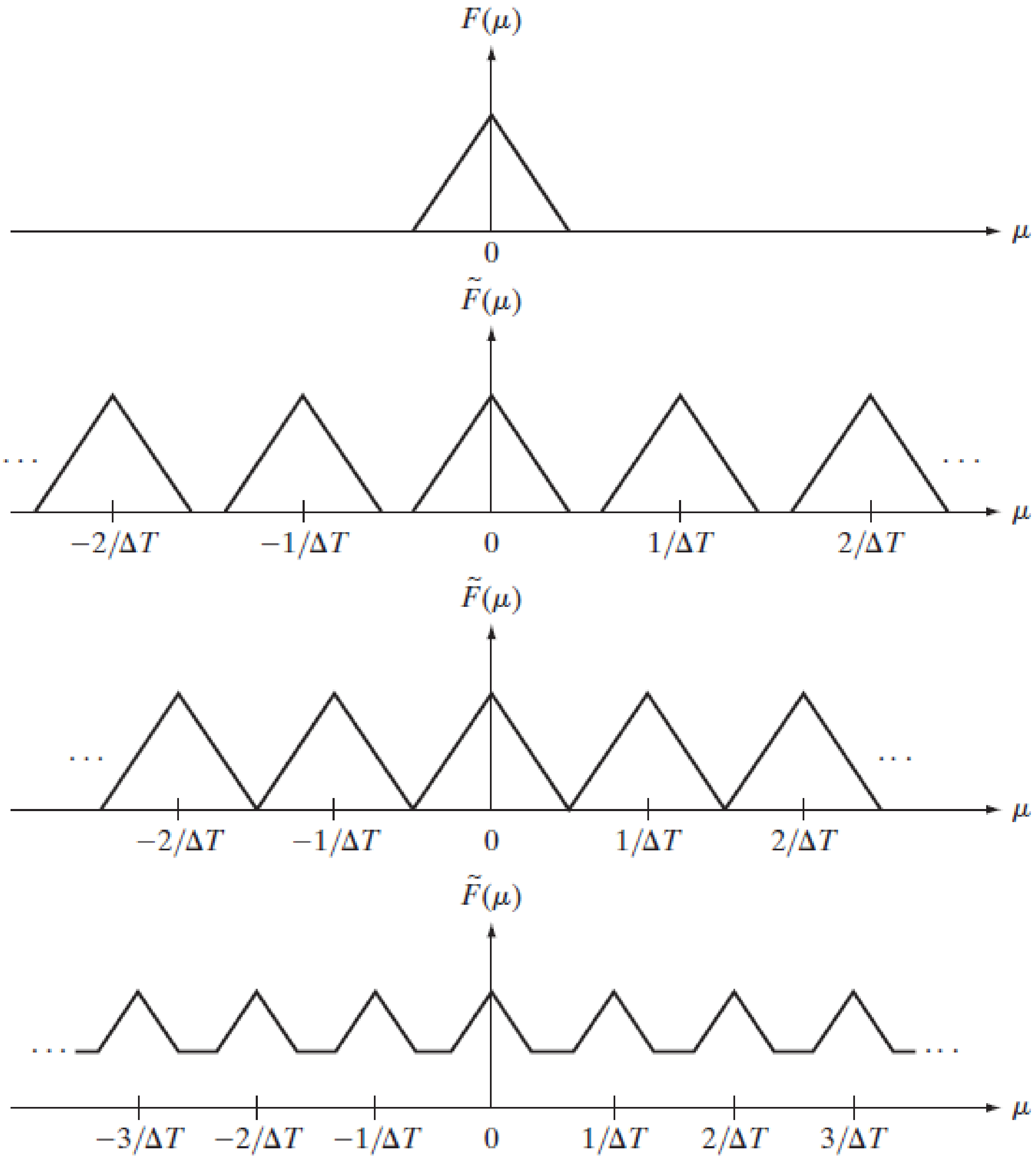


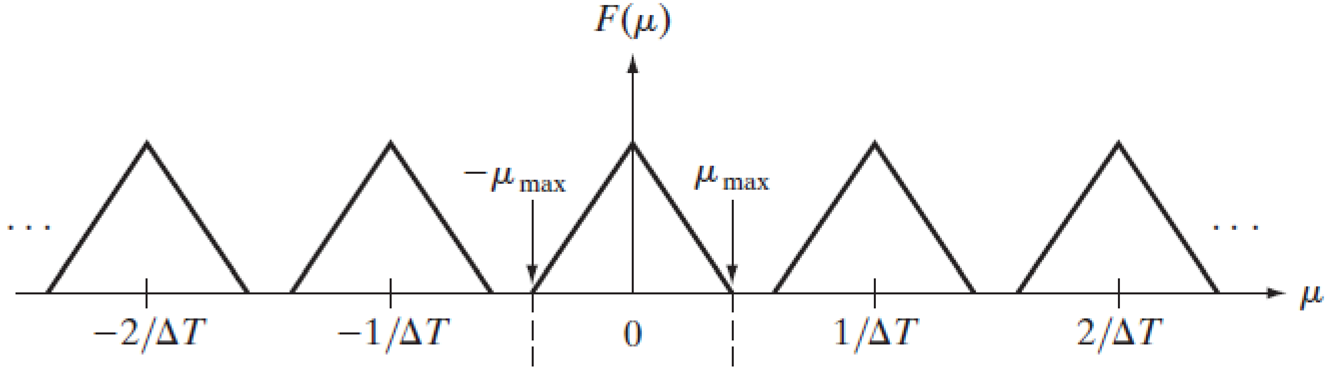
Sampling Theorem

- a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.

Sampling at : $$\frac{1}{\Delta T} = 2\mu_{max}$$ *Nyquist rate*

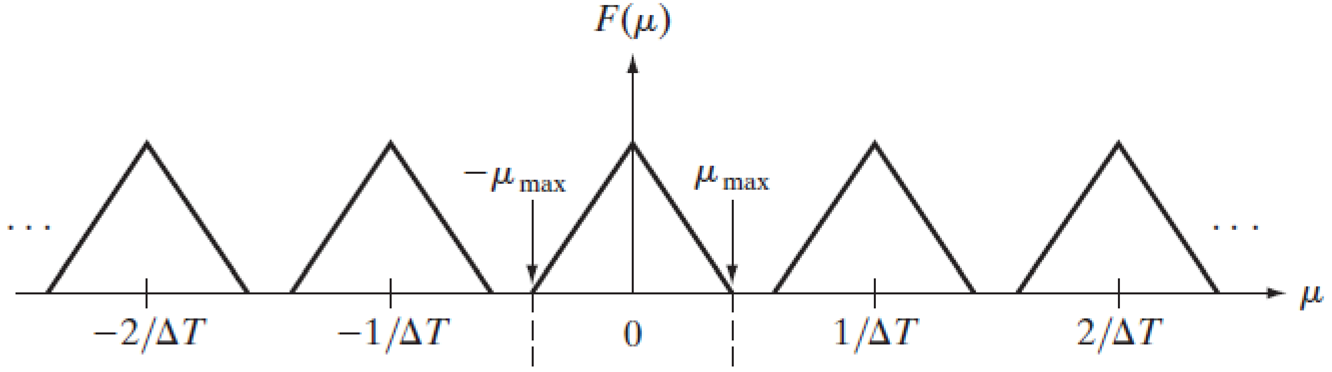
Signal Recovery

Signal Recovery



Signal Recovery

Fig shows the FT of a function sampled at a rate slightly higher than the Nyquist rate.



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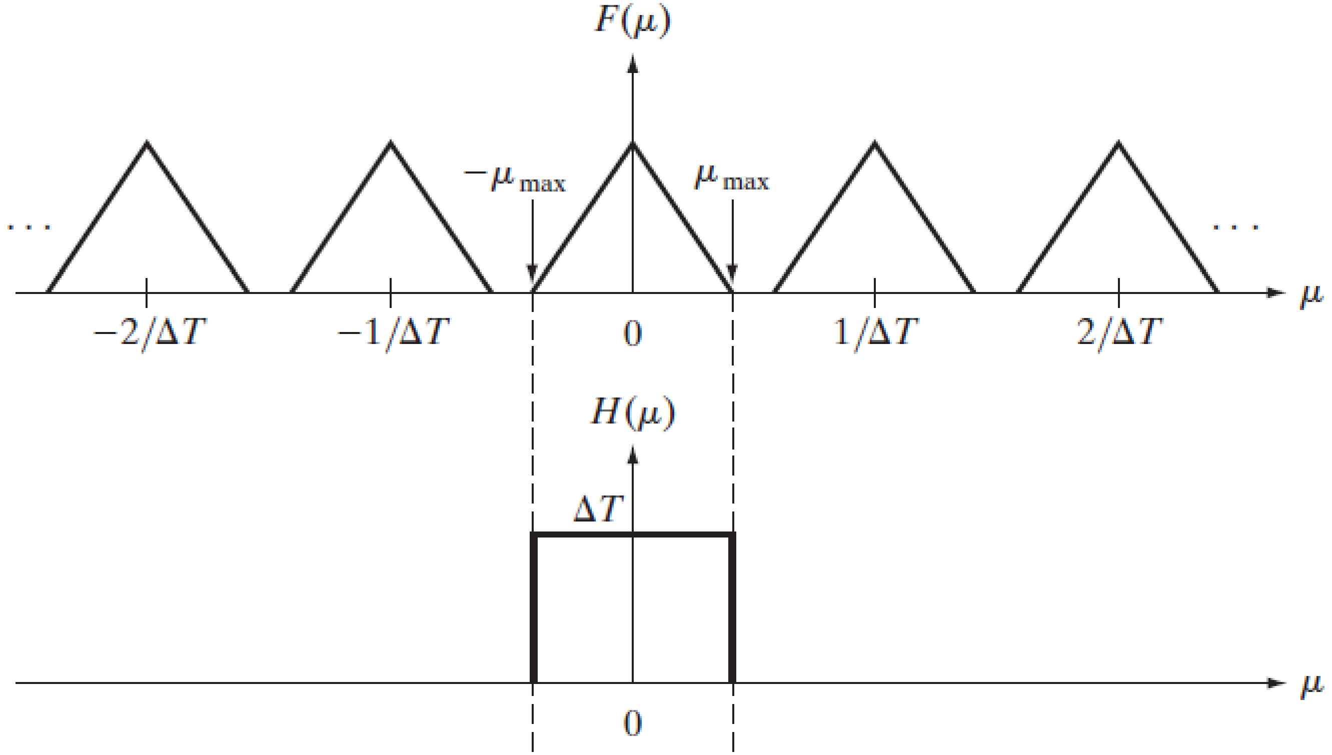
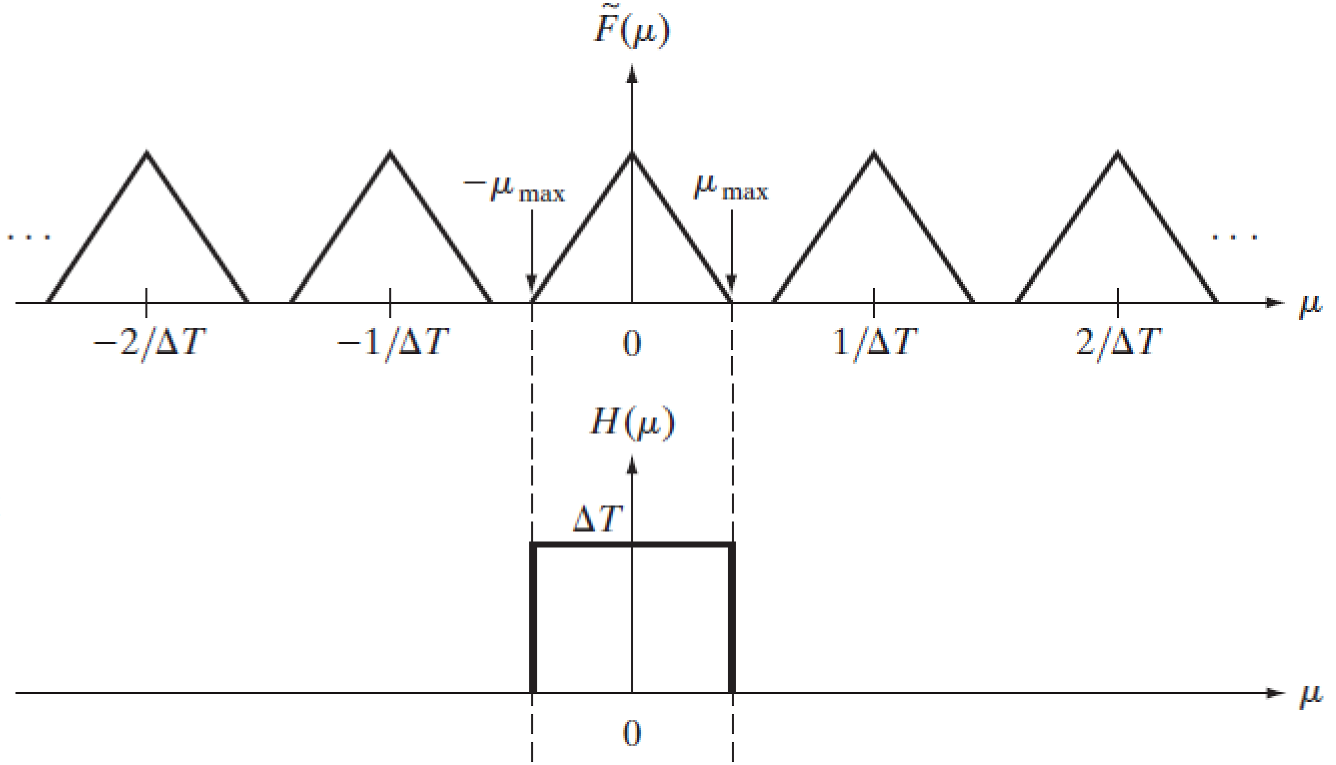


Signal Recovery

Fig shows the FT of a function sampled at a rate slightly higher than the Nyquist rate.

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \leq \mu \leq \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$



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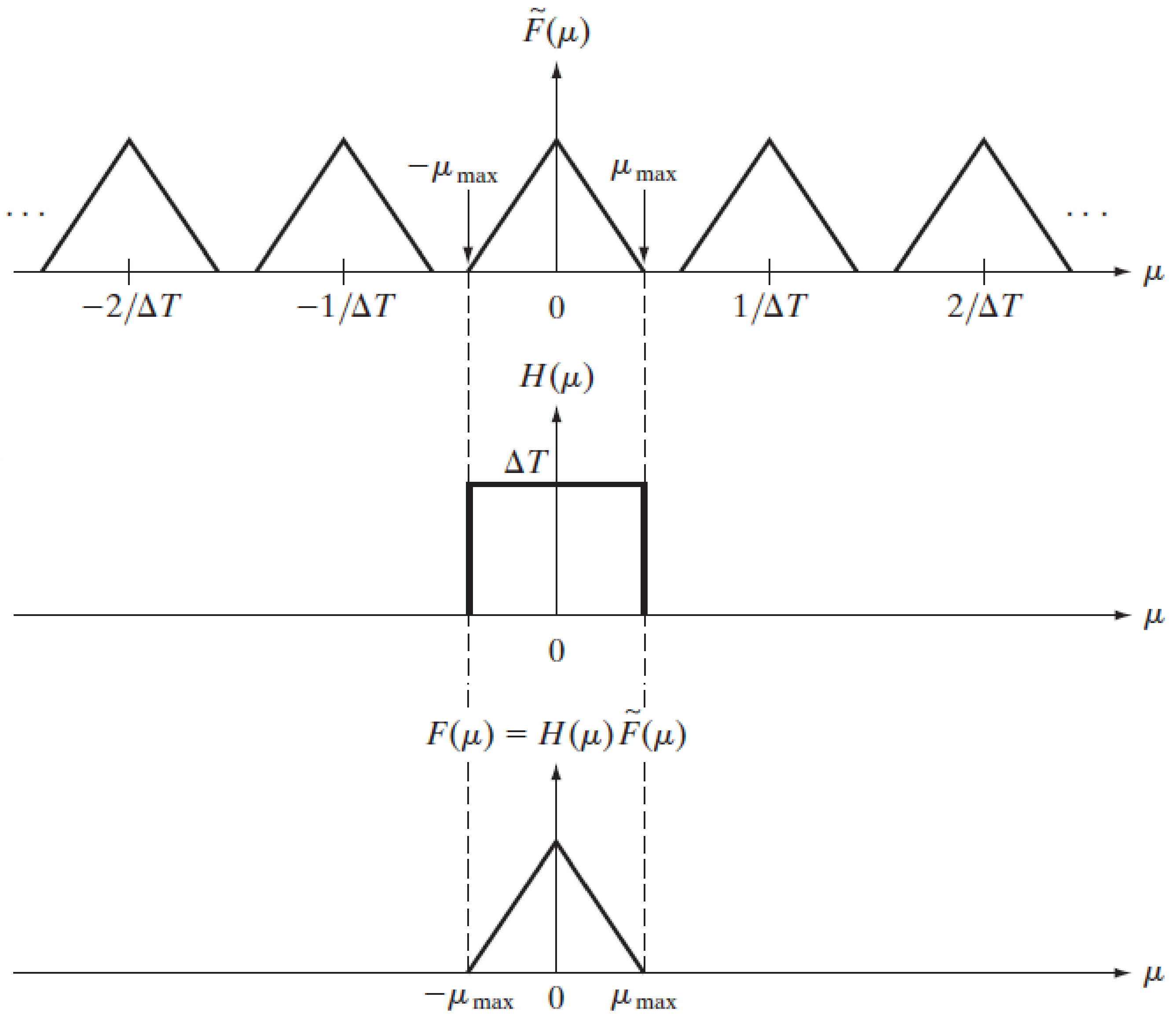

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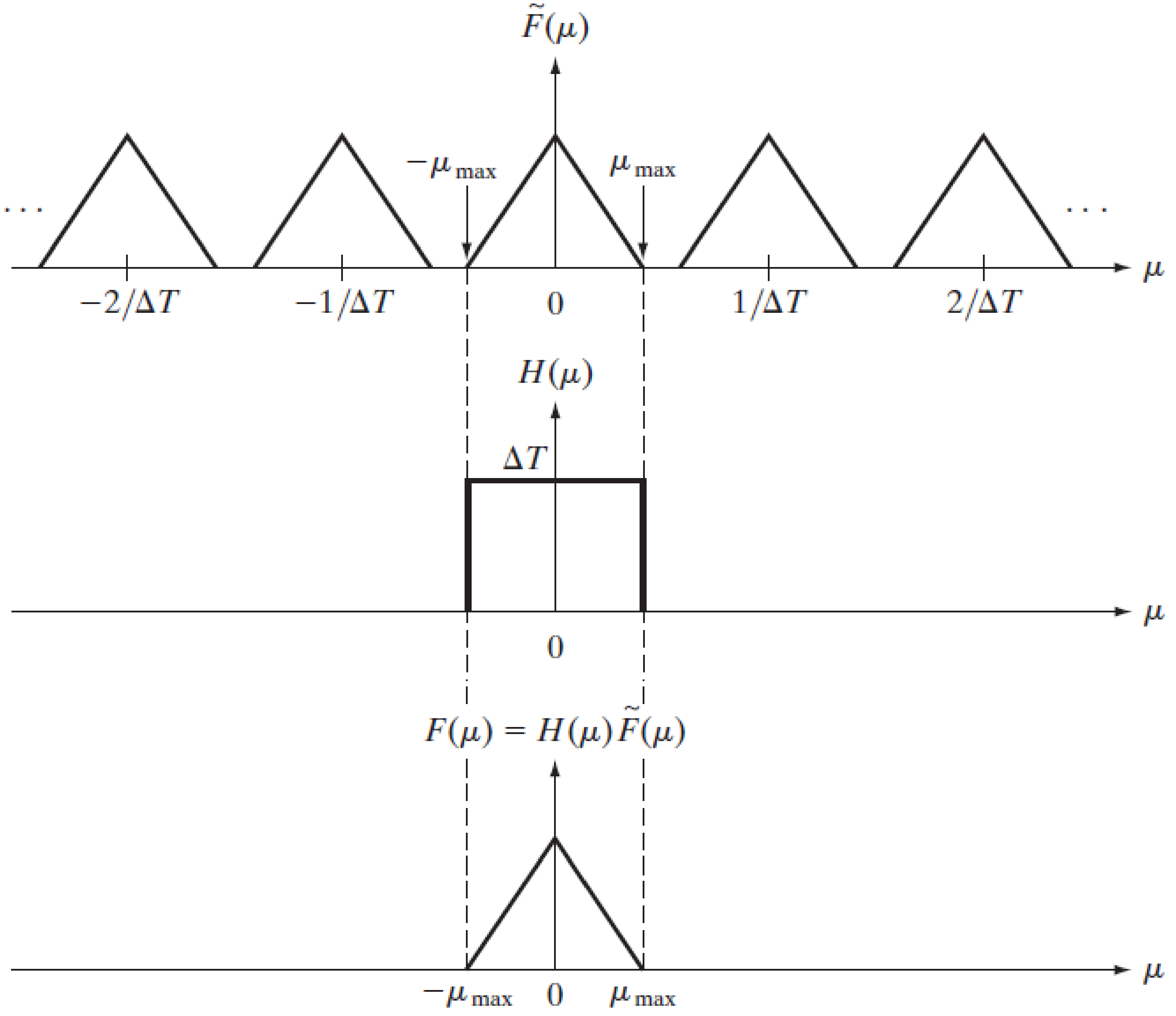

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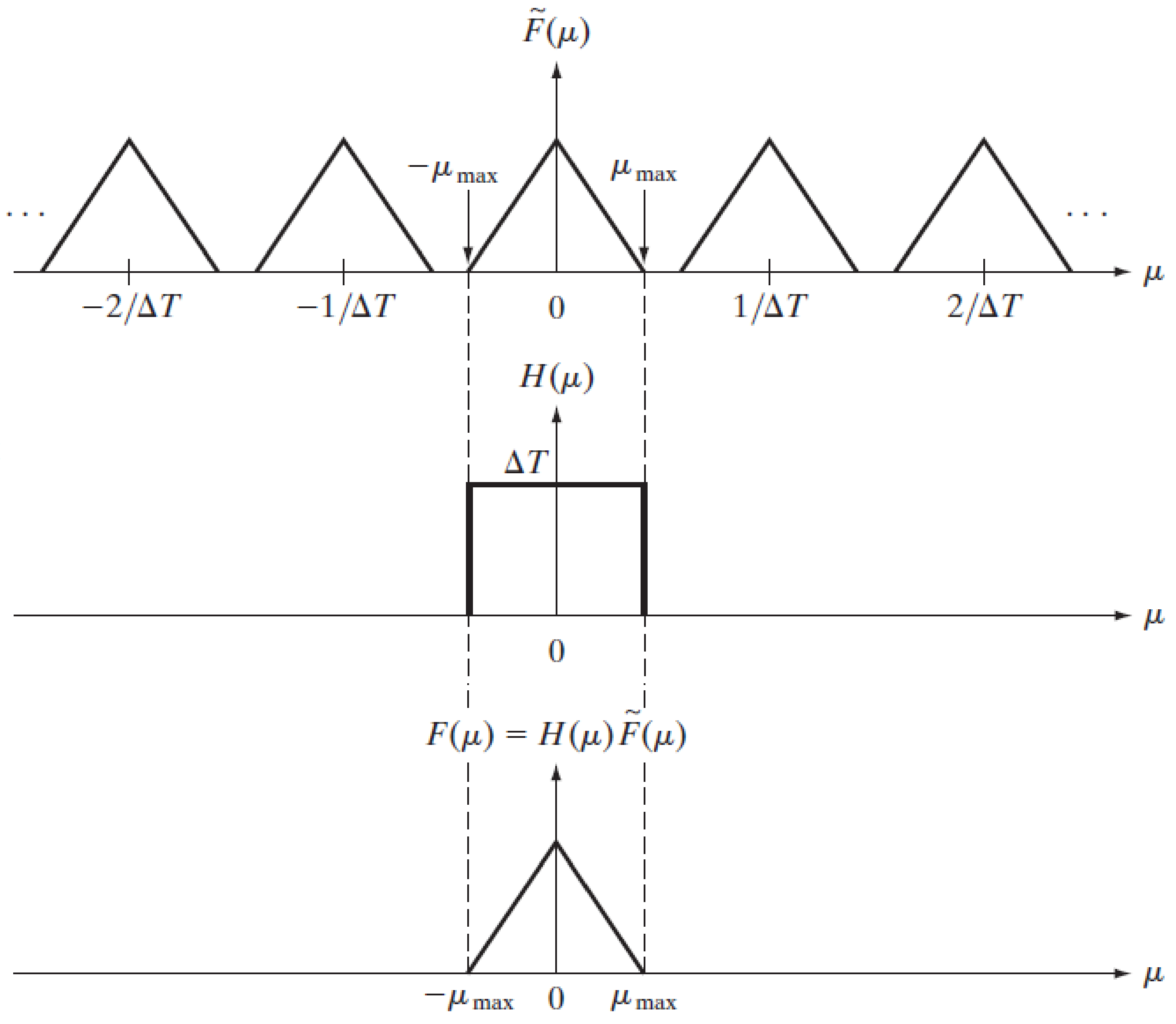

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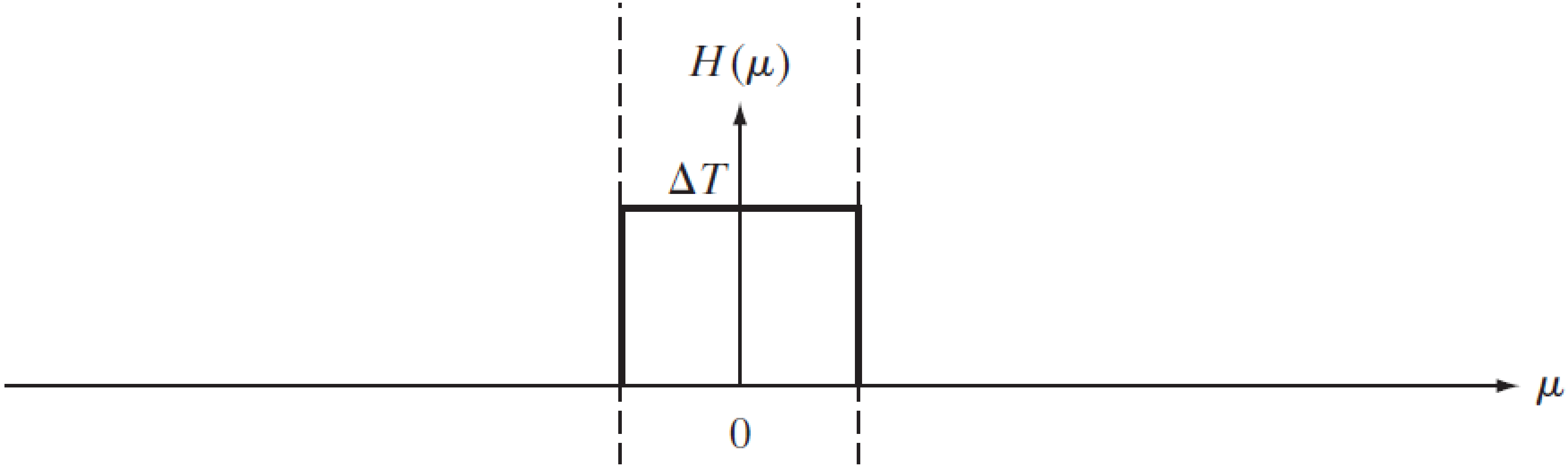
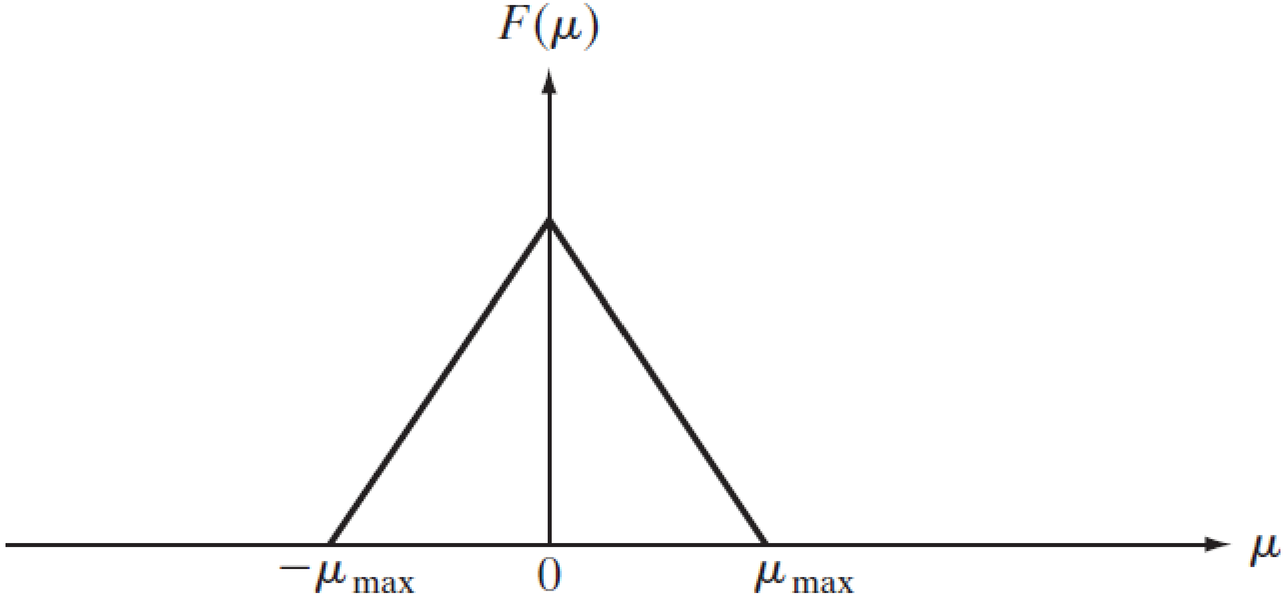
$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$



$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

- H (μ) is called a low pass filter
- It is also called an *ideal* lowpass filter because of its infinitely rapid transitions in amplitude (between 0 and ΔT at location $-\mu_{max}$ and the reverse at μ_{max} .
- This characteristic is not achievable by physical electronic components.
- Also known as **Reconstruction Filters.**



- Assumed : $f(t)$ is band limited to μ_{max}
- Band limited signal can not be time limited. $f(t)$ must extend from –α to α.
- This prevents perfect recovery !!

Aliasing

Aliasing

- What happens if a band-limited function is sampled at a rate that is less than twice its highest frequency?
- The net effect of lowering the sampling rate below the Nyquist rate is that the periods now overlap, and it becomes impossible to isolate a single period of the transform, regardless of the filter used.

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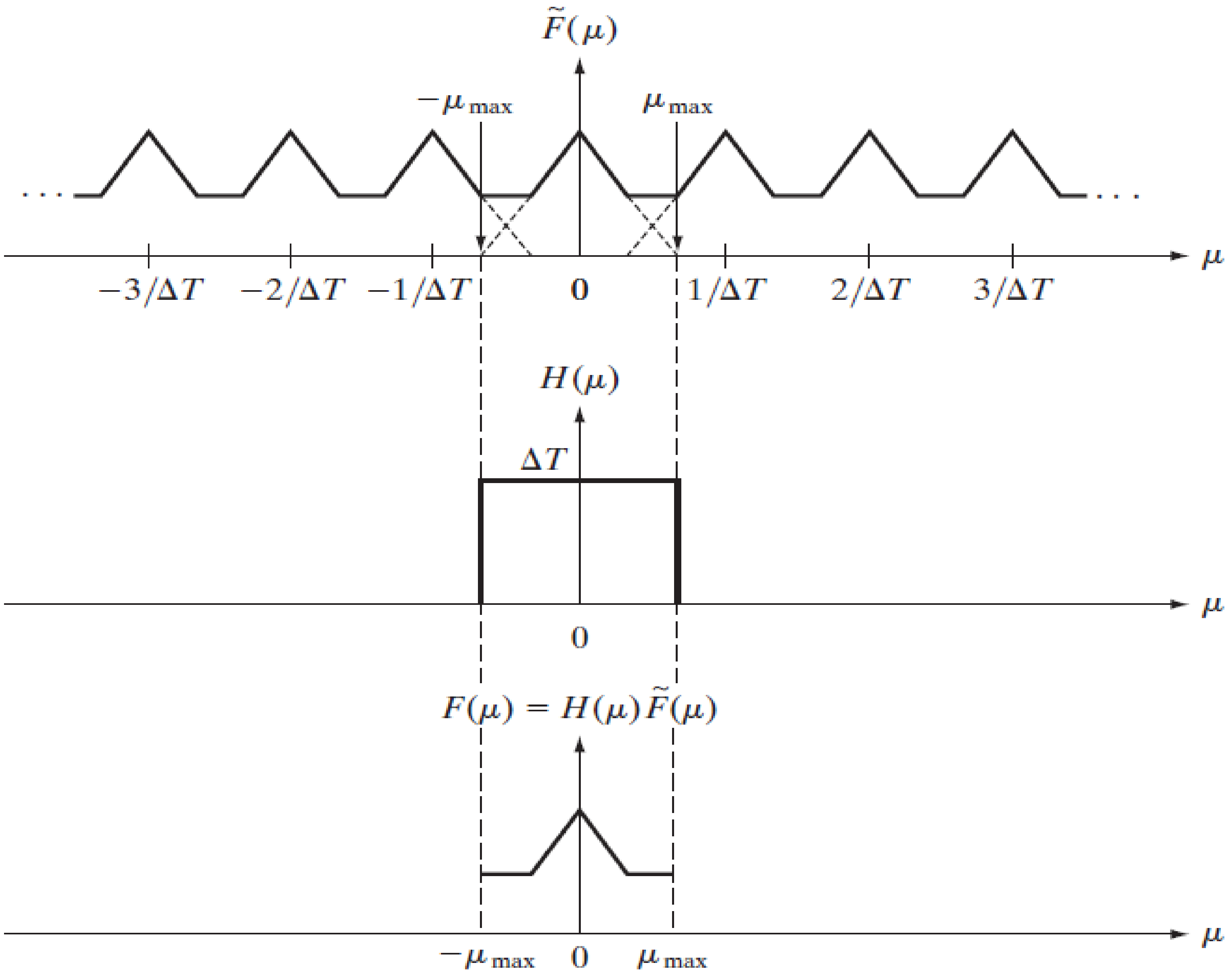

$\tilde{F}(\mu)$
$-\mu_{max}$
μ_{max}
μ
$-3/\Delta T$
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$3/\Delta T$
$H(\mu)$
ΔT
μ
0
$F(\mu) = H(\mu)\tilde{F}(\mu)$
μ
$-\mu_{max}$
0
μ_{max}

- The inverse transform would then yield a corrupted function of t.
- This effect, caused by under-sampling a function, is known as *frequency aliasing* or simply as *aliasing*.
- *aliasing* is a process in which high frequency components of a continuous function "masquerade" as lower frequencies in the sampled function.
- This is consistent with the common use of the term *alias*, which means "a false identity."
- Except for some special case, aliasing is always present in sampled signals.
- Even if the original sampled function is band-limited, infinite frequency components are introduced the moment we limit the duration of the function, which we always have to do in practice.
- In the next slide we see the example for same.

Why no function of finite duration can be band-limited?

$$h(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

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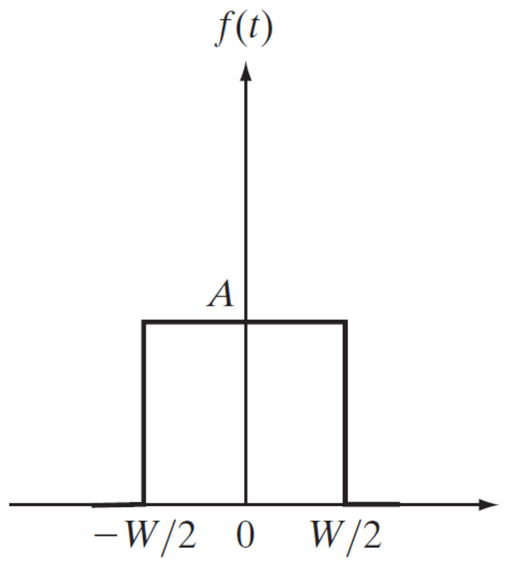
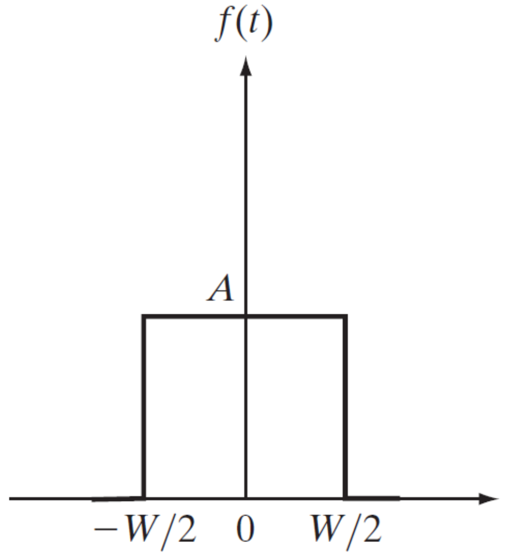


Why no function of finite duration can be band-limited?

- For example, suppose that we want to limit the duration of a bandlimited function $f(t)$ to an interval, say $[0, T]$. We can do this by multiplying $f(t)$ by the function

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$



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- whose transform, has frequency components extending to infinity

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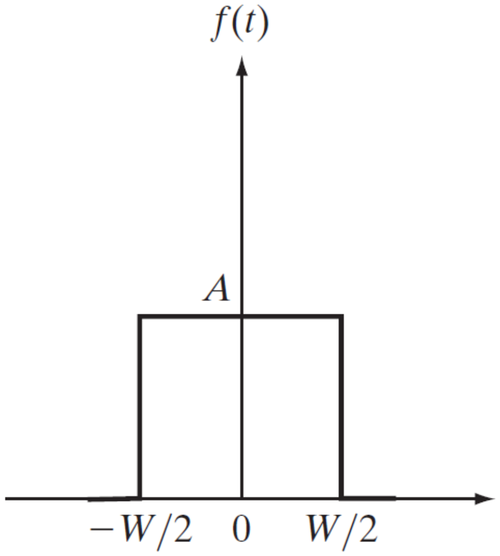

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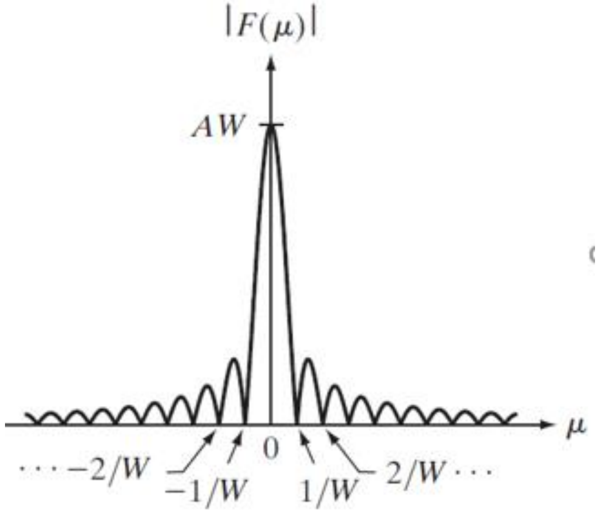
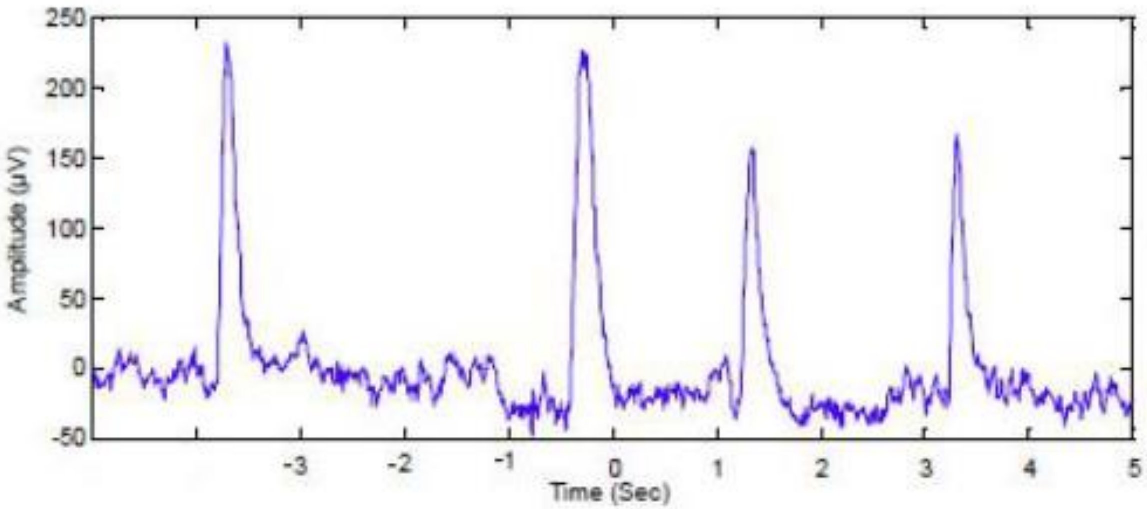
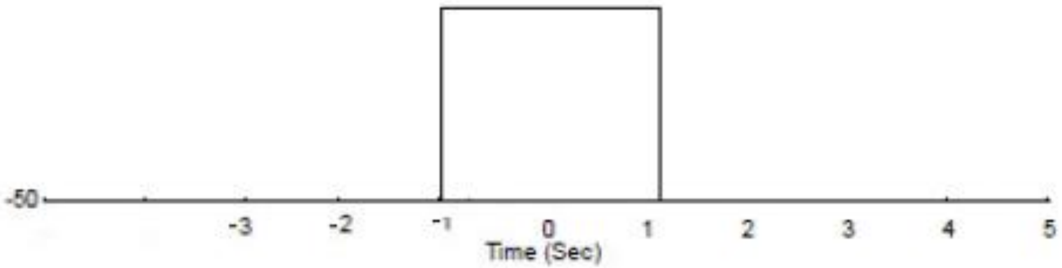


- From the convolution theorem we know that the transform of the product of *h(t)f(t)* is the convolution of the transforms of the functions.
- Even if the transform of $f\,(t)$ is band-limited, convolving it with $H\,(\mu)$ which involves sliding one function across the other, will yield a result with frequency components extending to infinity.
- Therefore, no function of finite duration can be band-limited. Conversely, a function that is band-limited must extend from to –α to α

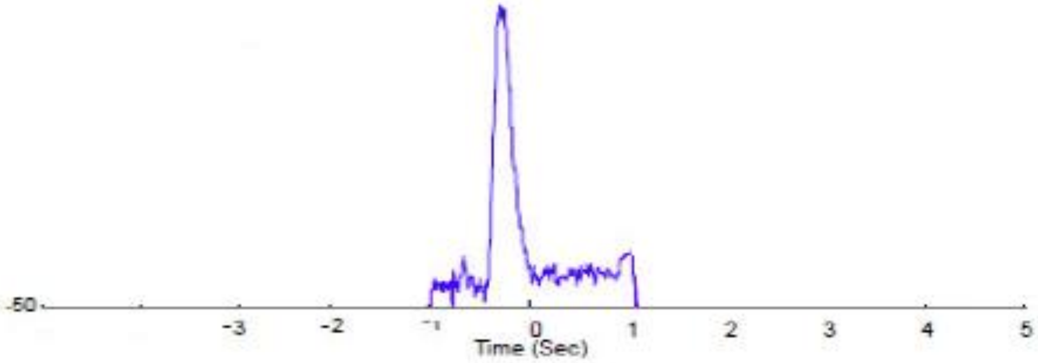
f(t)


250
200
150
100
50
0
-50
Amplitude (μV)
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

h(t)

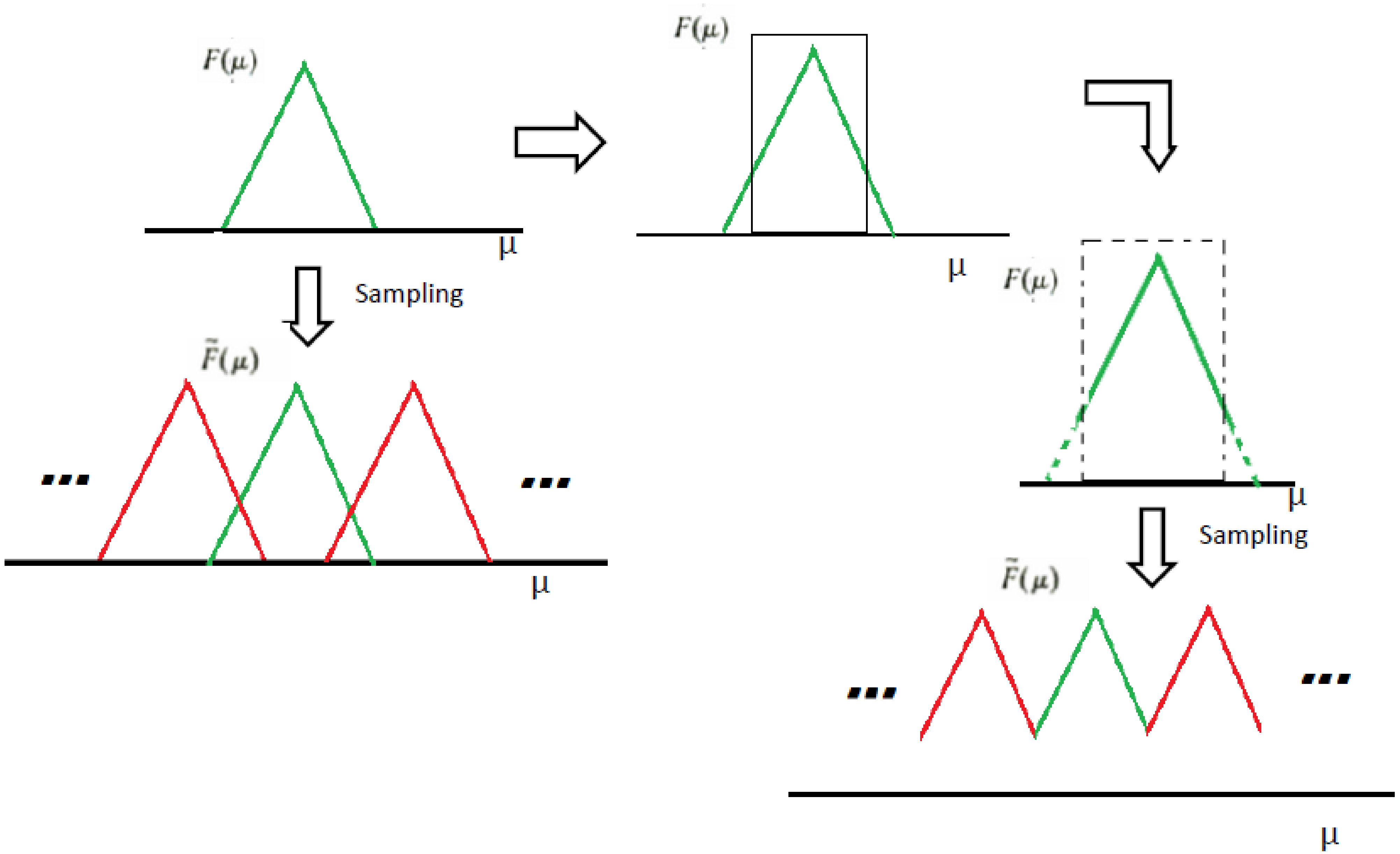

-50
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

f(t)h(t)


-50
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

Anti-Aliasing

- In practice, the effects of aliasing can be *reduced* by smoothing the input function to attenuate its higher frequencies (e.g., by defocusing in the case of an image).
- This process, called *anti-aliasing*, has to be done *before* the function is sampled because aliasing is a sampling issue that cannot be "undone after the fact" using computational techniques.



$F(\mu)$
μ
Sampling
$\tilde{F}(\mu)$
...
...
μ
$F(\mu)$
μ
$F(\mu)$
μ
Sampling
$\tilde{F}(\mu)$
...
...
μ

Illustration of Aliasing

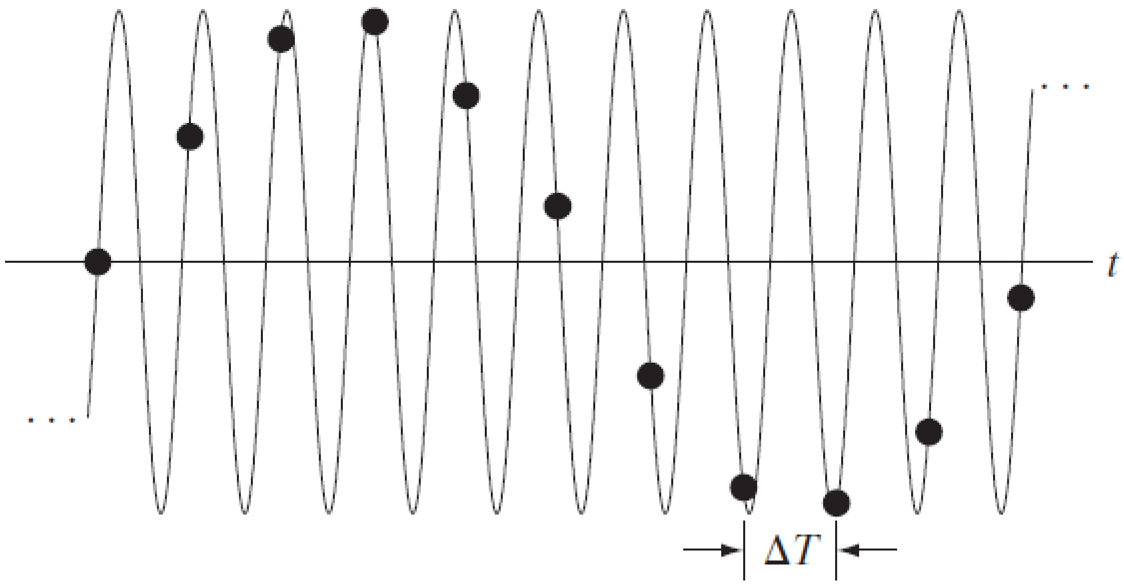
- Sampling Rate $\frac{1}{\Delta T} \geq 2 \ast$ max_frequency

$$\frac{1}{\Delta T} \geq 2 \ast \frac{1}{period}$$

$$\Delta T \leq \frac{period}{2}$$

- Sometimes equal sign is not sufficient
- Sampling rate should exceed twice the highest frequency.

- A pure sine wave extending infinitely in both directions has a single frequency so, it is band-limited.
- Suppose that the sine wave in the figure (ignore the large dots for now) has the equation $\sin(\pi t)$ and that the horizontal axis corresponds to time, t, in seconds.
- The function crosses the axis at t = . . . -1, 0, 1, 2, 3, . . .
- The period P of $\sin(\pi t)$ is 2 s



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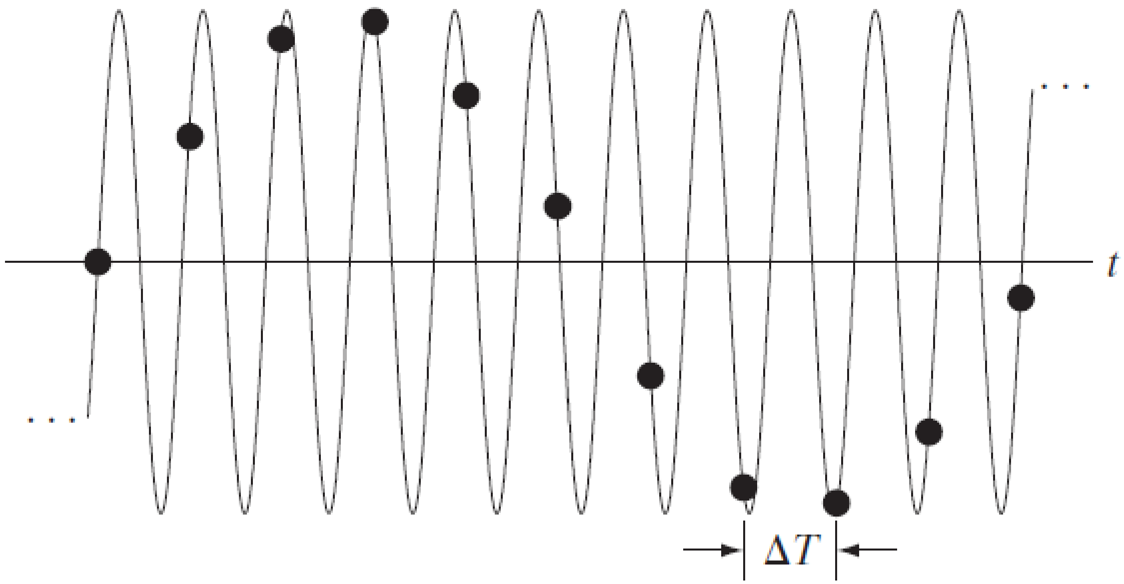

Frequency = 1/P or 1 / 2 cycles / s

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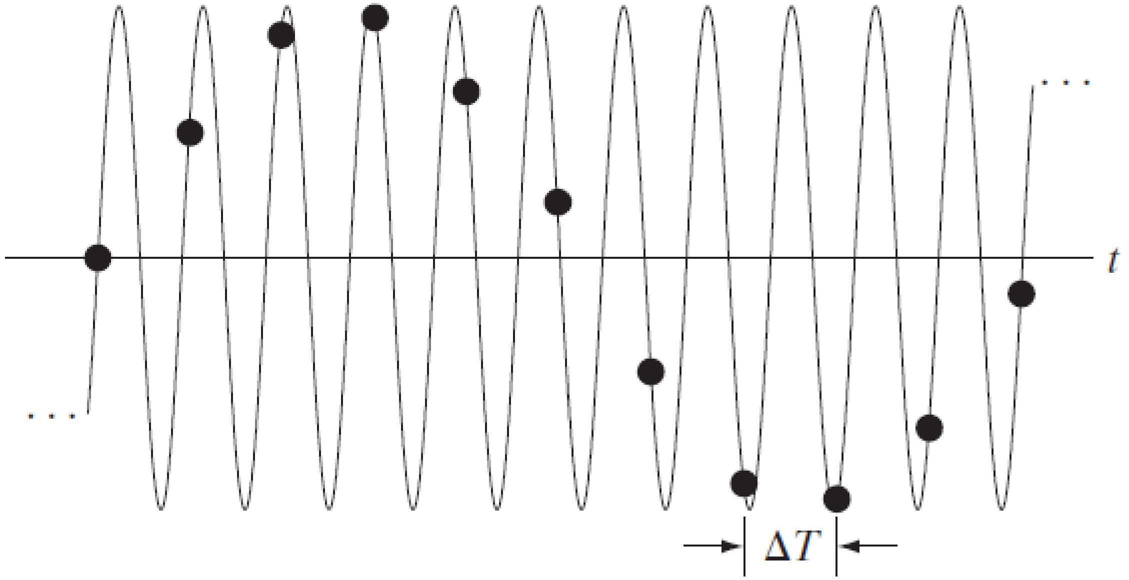

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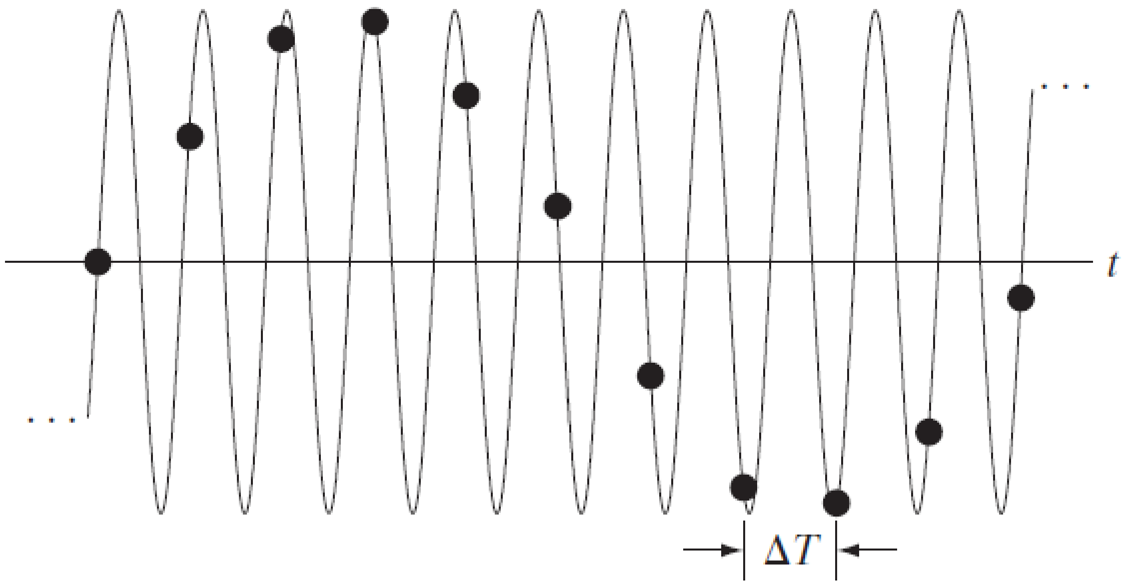

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- The large dots in Figure are samples taken uniformly at a rate of less than 1 sample/s (in fact, the separation between samples exceeds 2 s, which gives a sampling rate lower than 1/2 samples s).
- The sampled signal *looks* like a sine wave, but its frequency is about *one-tenth* the frequency of the original.
- This sampled signal, having a frequency well below anything present in the original continuous function is an example of aliasing.

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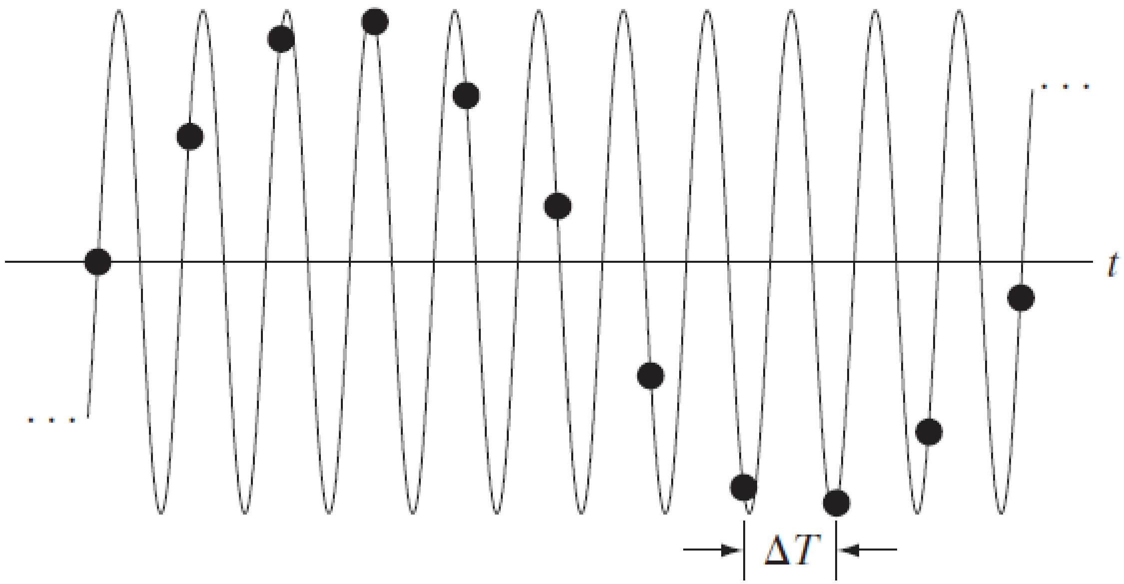

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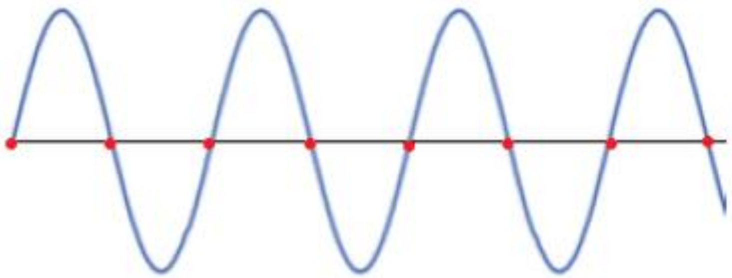

Revision

Revision

- Fourier Transform Pair

- Impulses

- Sifting Property of Impulse

Revision

- Fourier Transform Pair $F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$

$$\Im^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}\, d\mu$$

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$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \qquad \delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

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$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0) \qquad \int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\, dt = f(t_0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\,\delta(x) = f(0) \qquad \sum_{x=-\infty}^{\infty} f(x)\,\delta(x - x_0) = f(x_0)$$

Revision

Revision

- Impulse Train

- Fourier transform of Box Function

- Fourier transform of an impulse train with period ΔT

- Convolution

Revision

- Impulse Train $$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$
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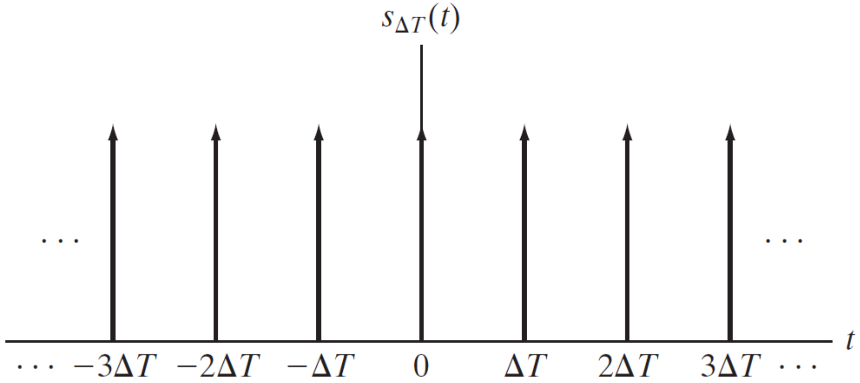

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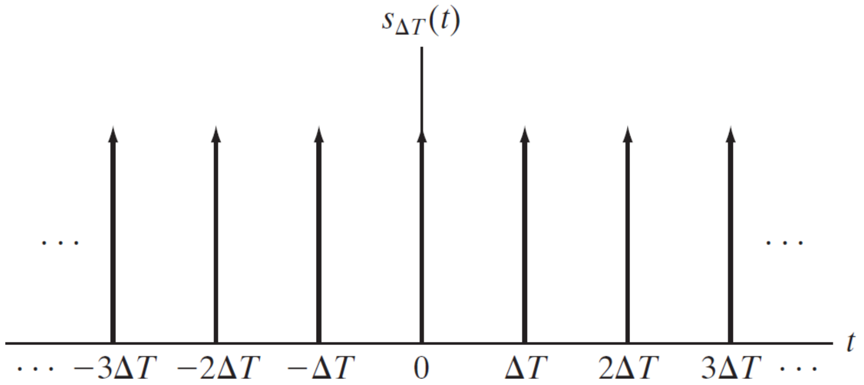

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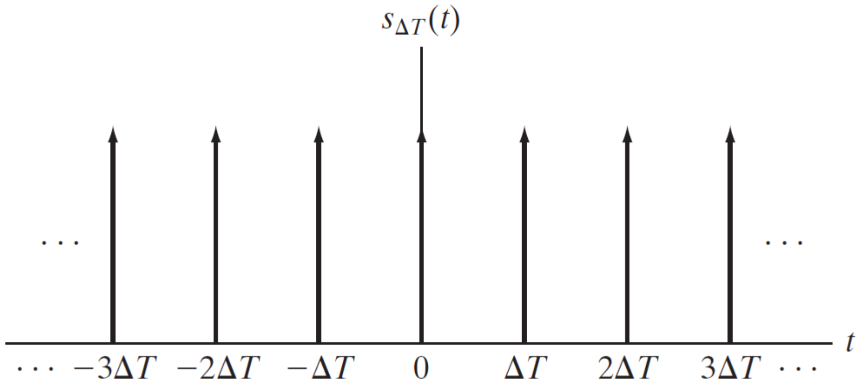

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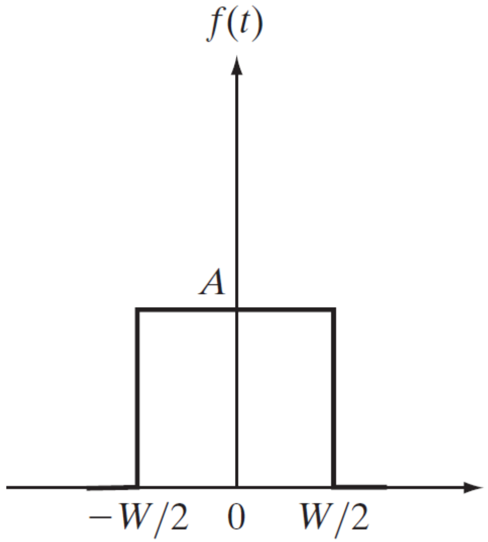

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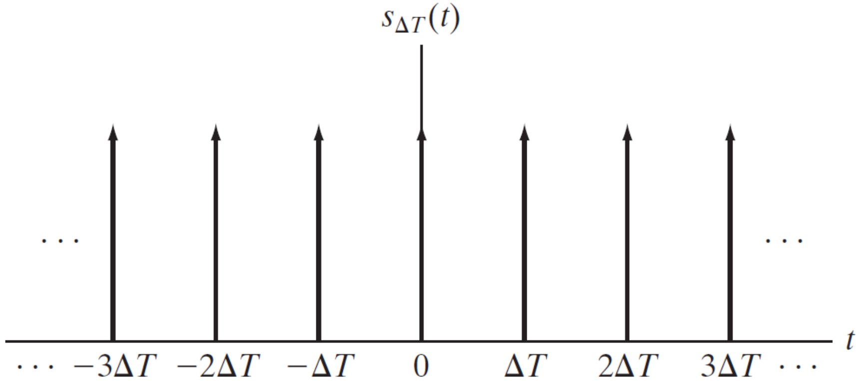
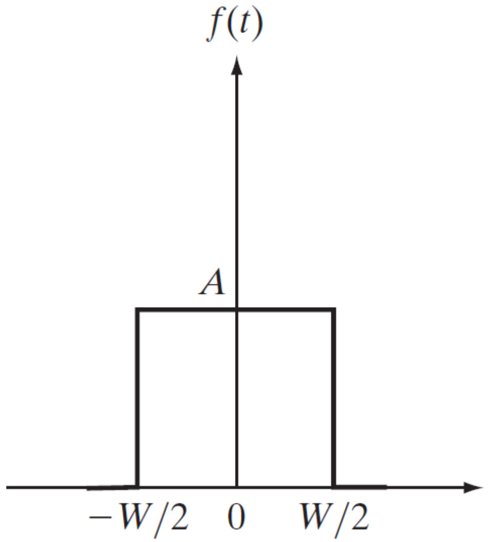


- Fourier transform of Box Function

$$AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$



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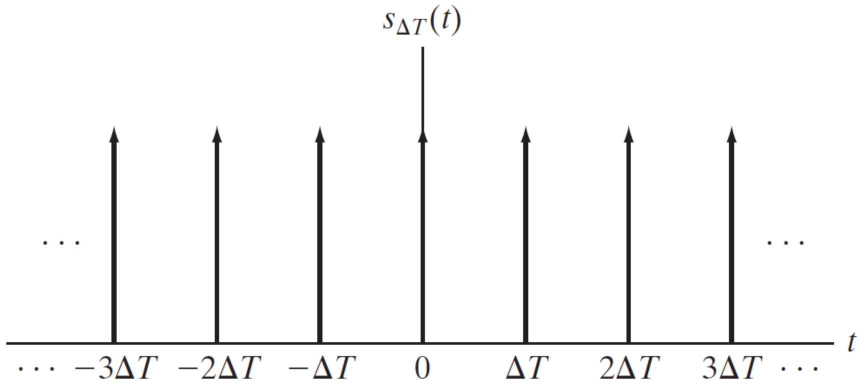

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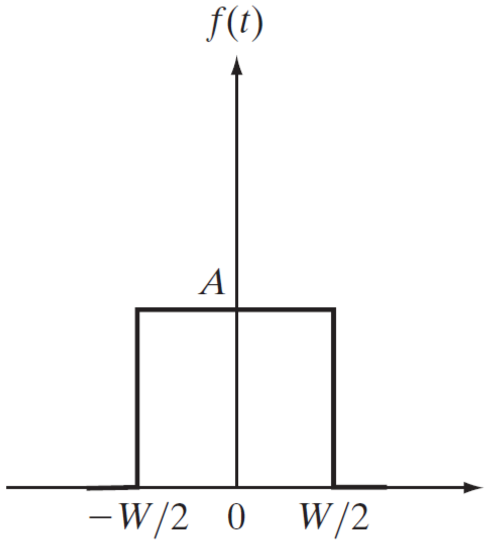

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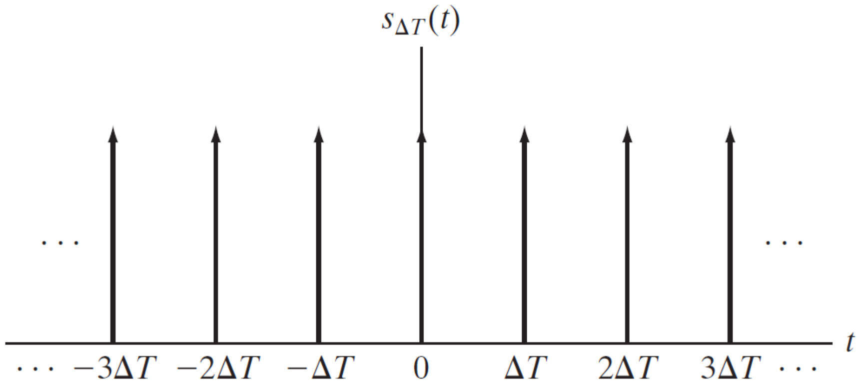

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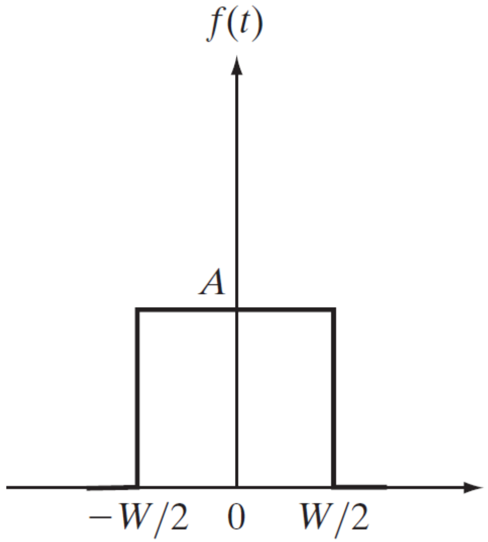

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$$\boxed{S(\mu) = \Im\{s_{\Delta T}(t)\}} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

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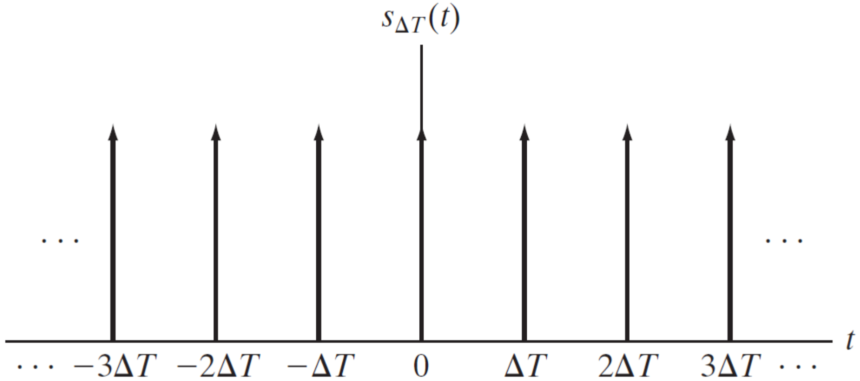

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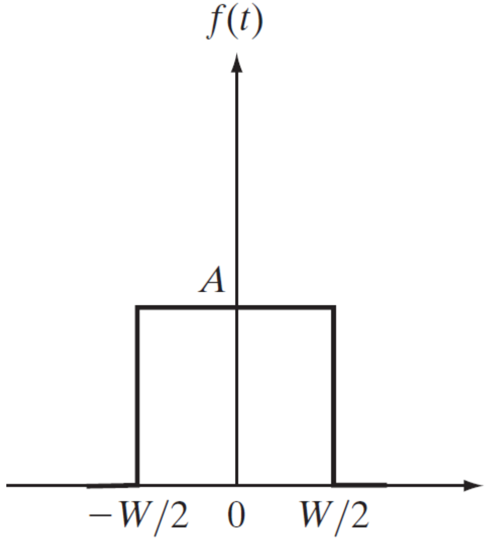

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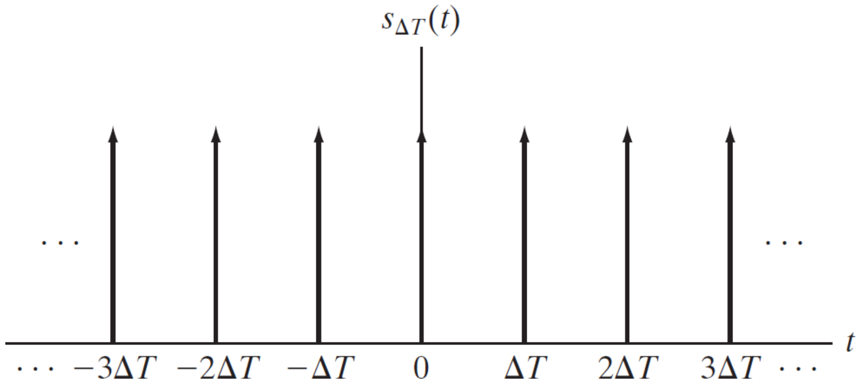

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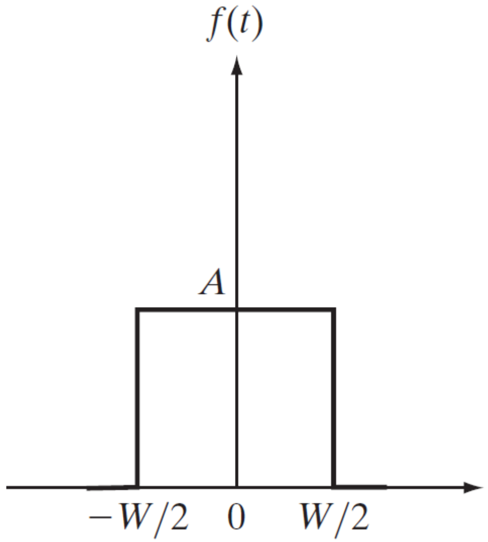

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- Convolution

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

Revision

Revision

- Convolution Theorem

- Extension to 2D
- 2D Impulse

- Sifting Property

Revision

$$f(t) \star h(t) \Leftrightarrow H(\mu) F(\mu)$$

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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t, z)\,dt\,dz = f(0, 0) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t - t_0, z - z_0)\,dt\,dz = f(t_0, z_0)$$

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Revision

Revision

- The 2-D Continuous Fourier transform pair

- Fourier Transform of a 2-D function

Revision

- The 2-D Continuous Fourier transform pair

$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

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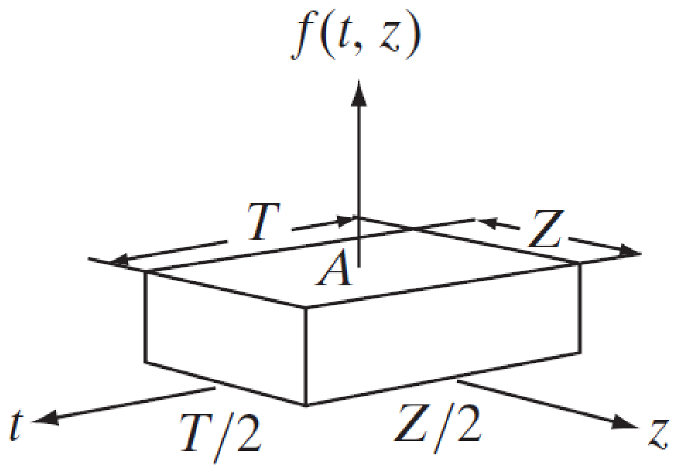

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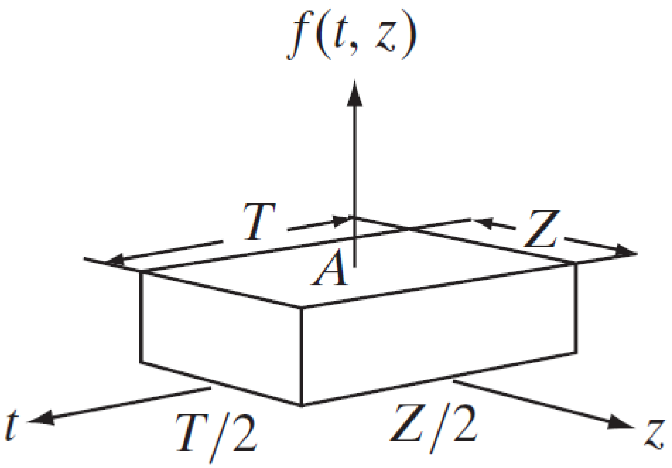

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

Revision

Revision

- Sampling and the Fourier Transform of the Sampled functions

Revision

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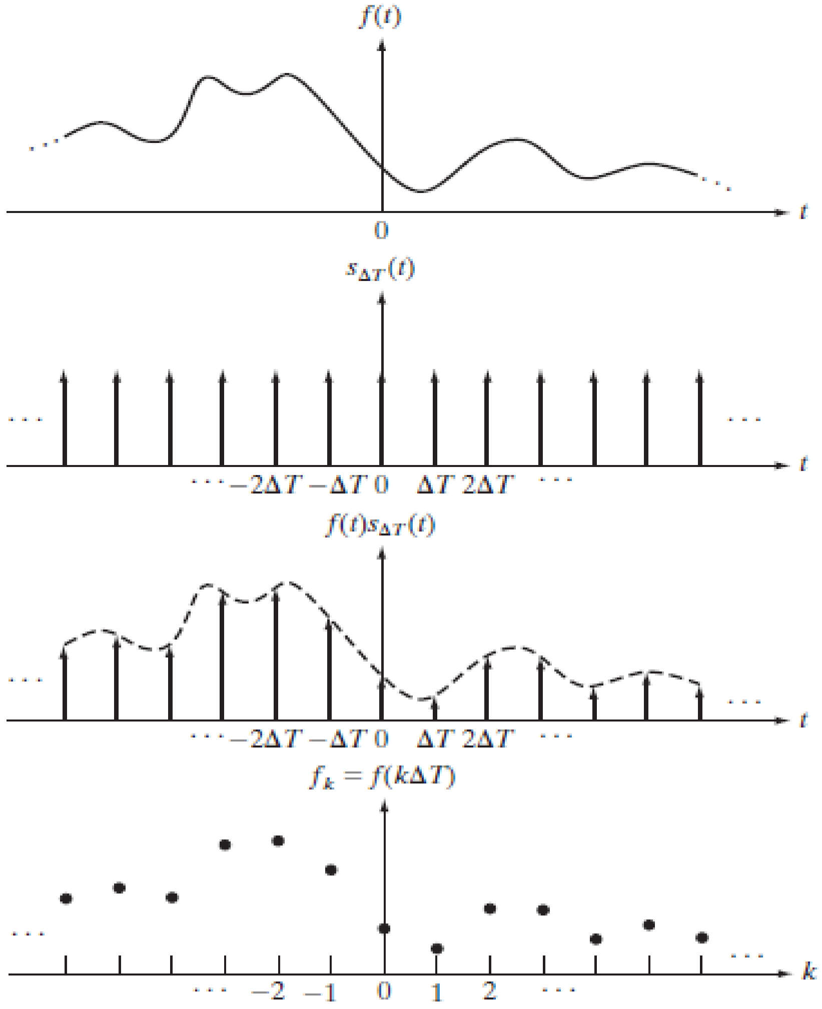

a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

Revision

Revision

- Sampling of 1-D function

- Sampling Theorem:

This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function. This result is known as the sampling theorem.

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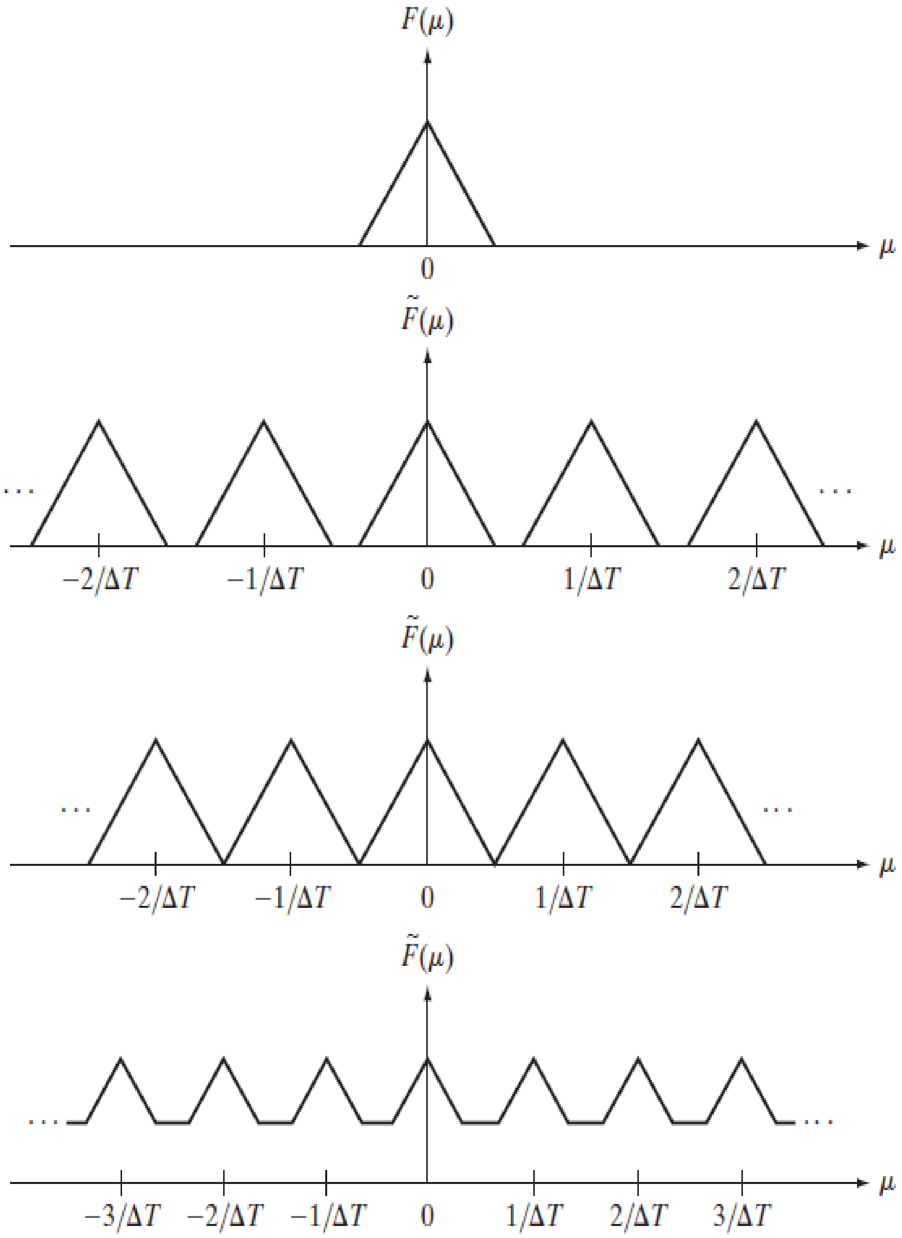
FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.



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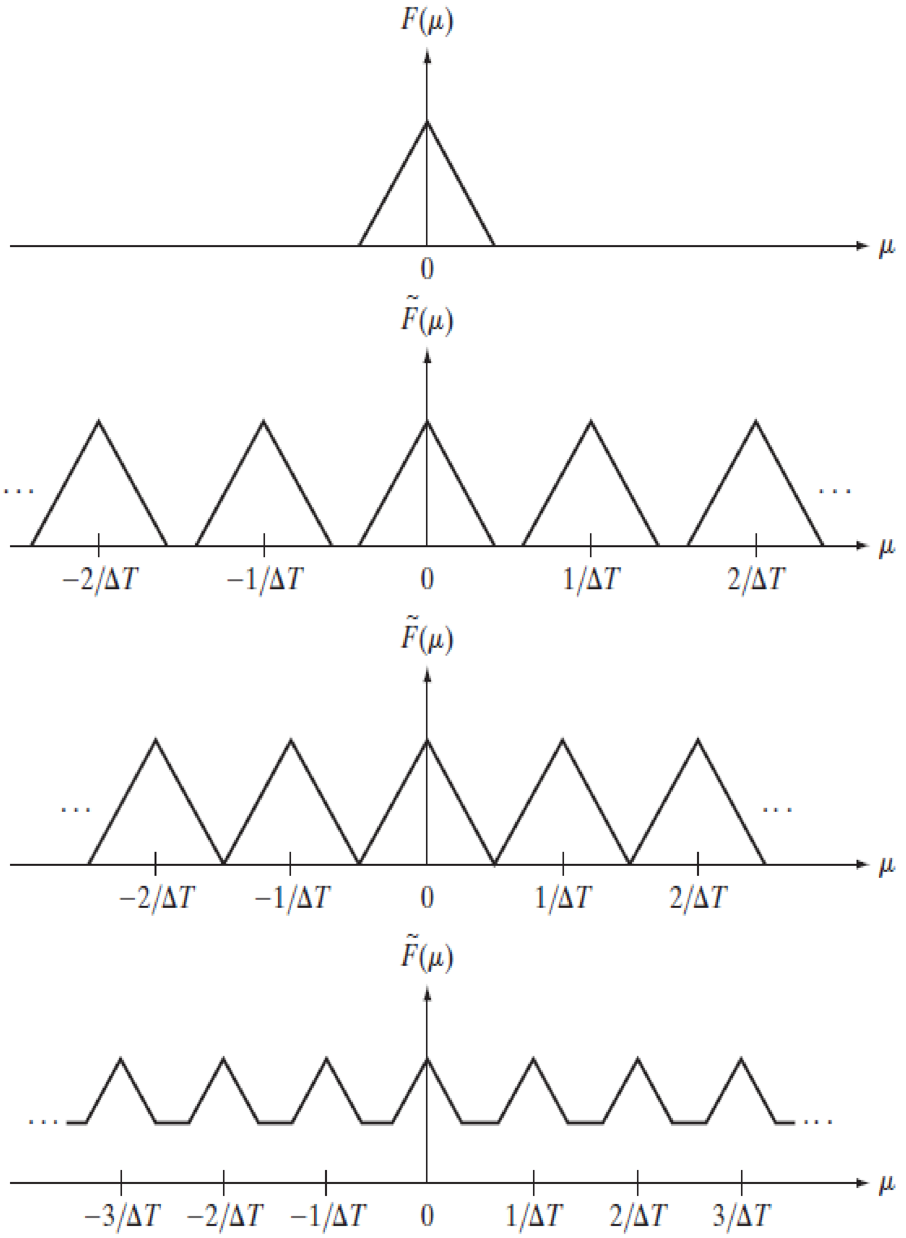

Revision

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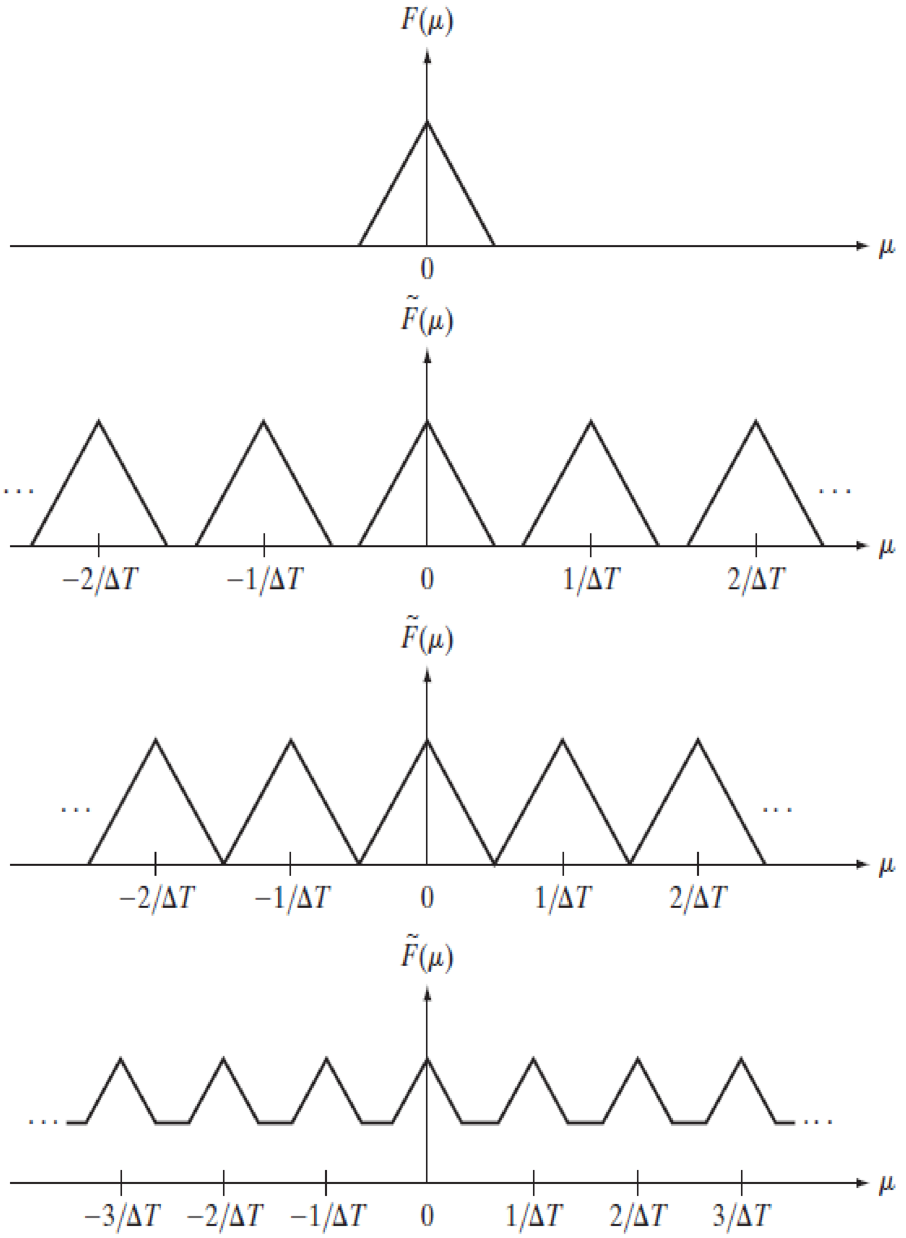

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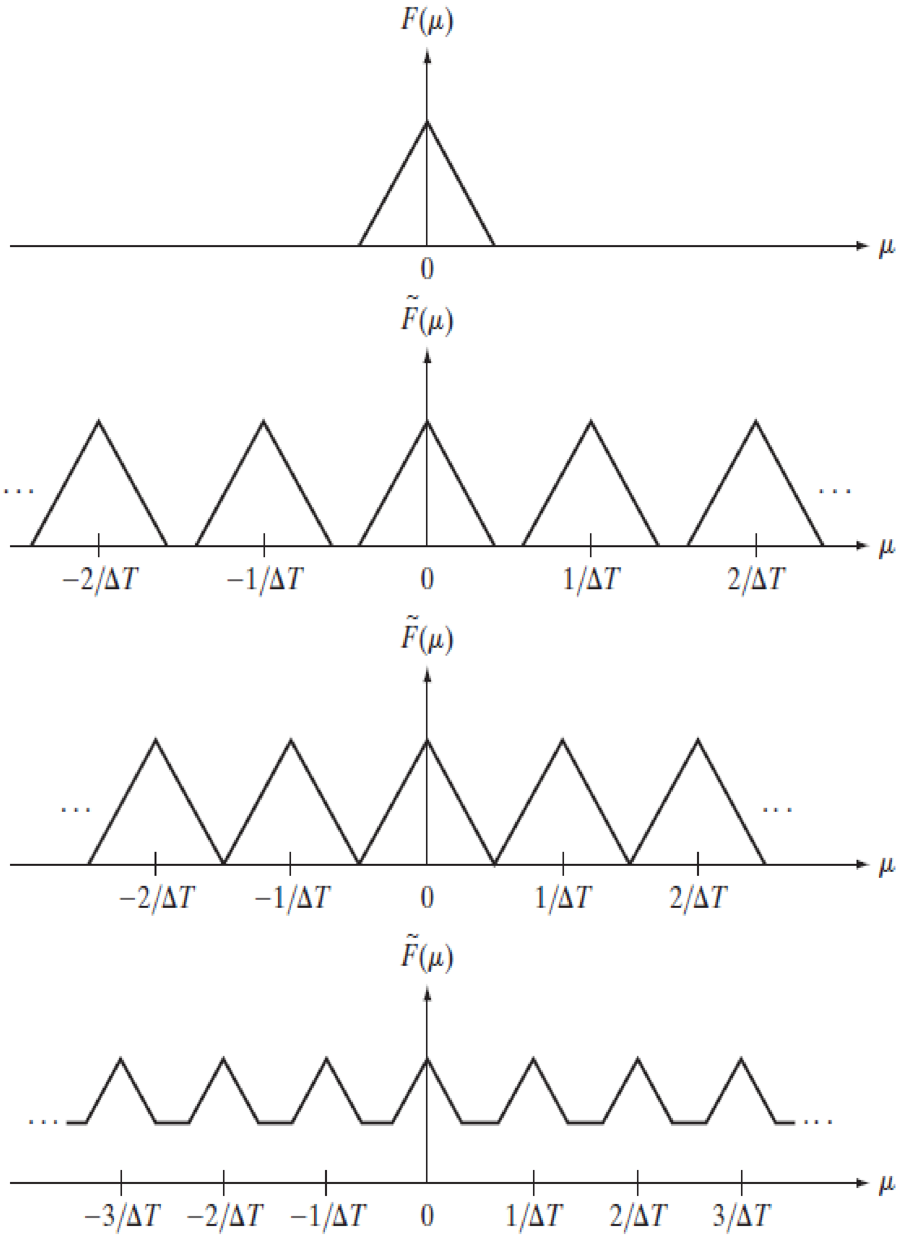

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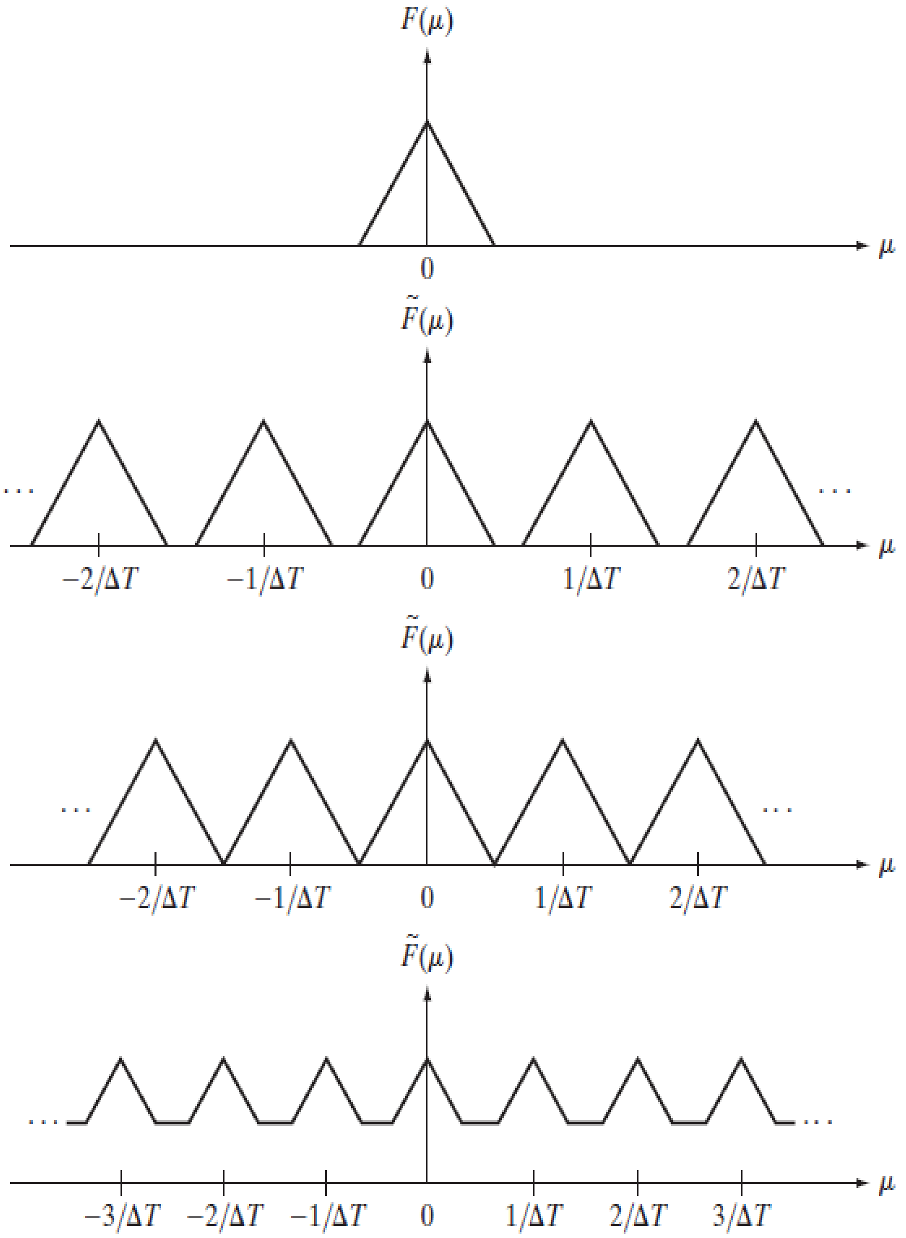

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$$\frac{1}{\Delta T} > 2\mu_{\max}$$

a
b
c
d

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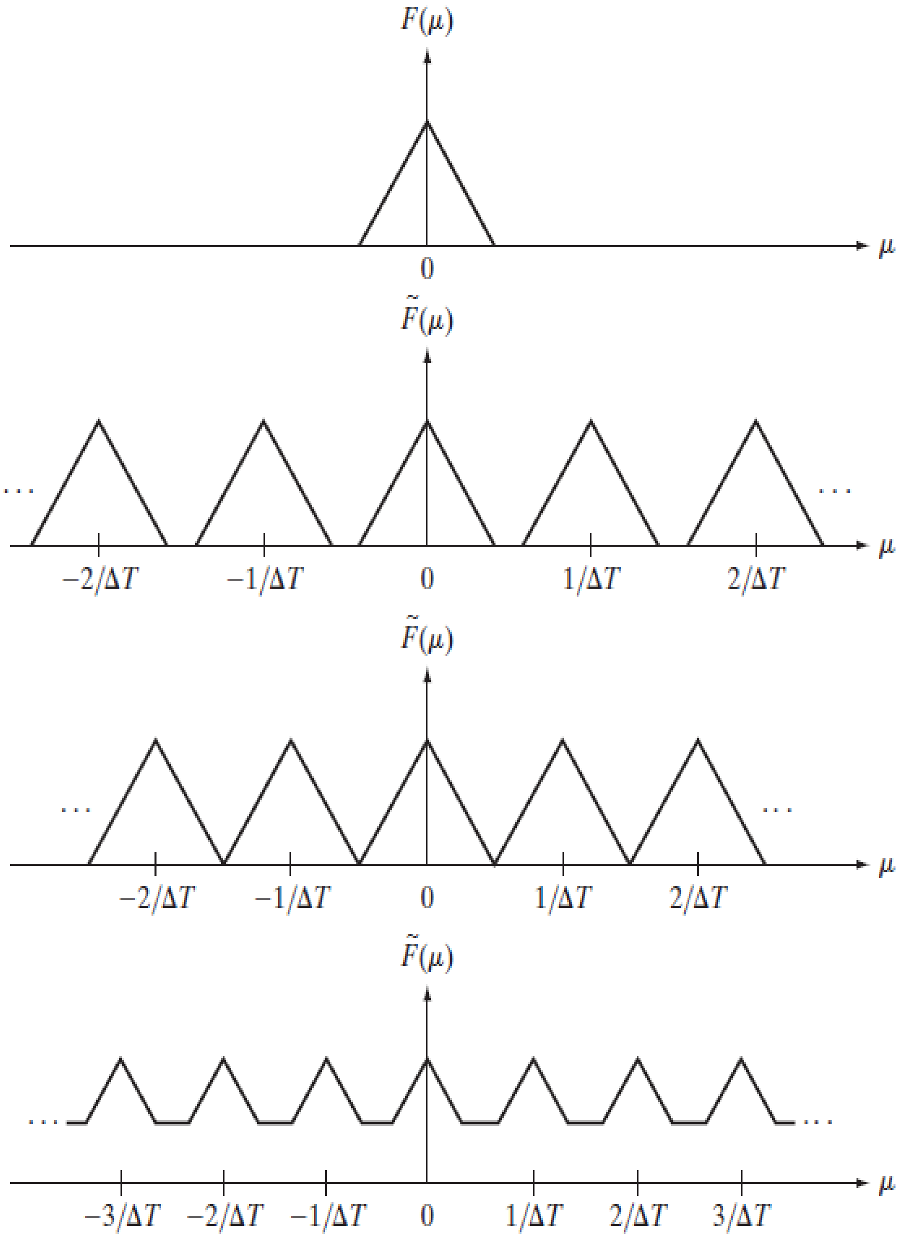
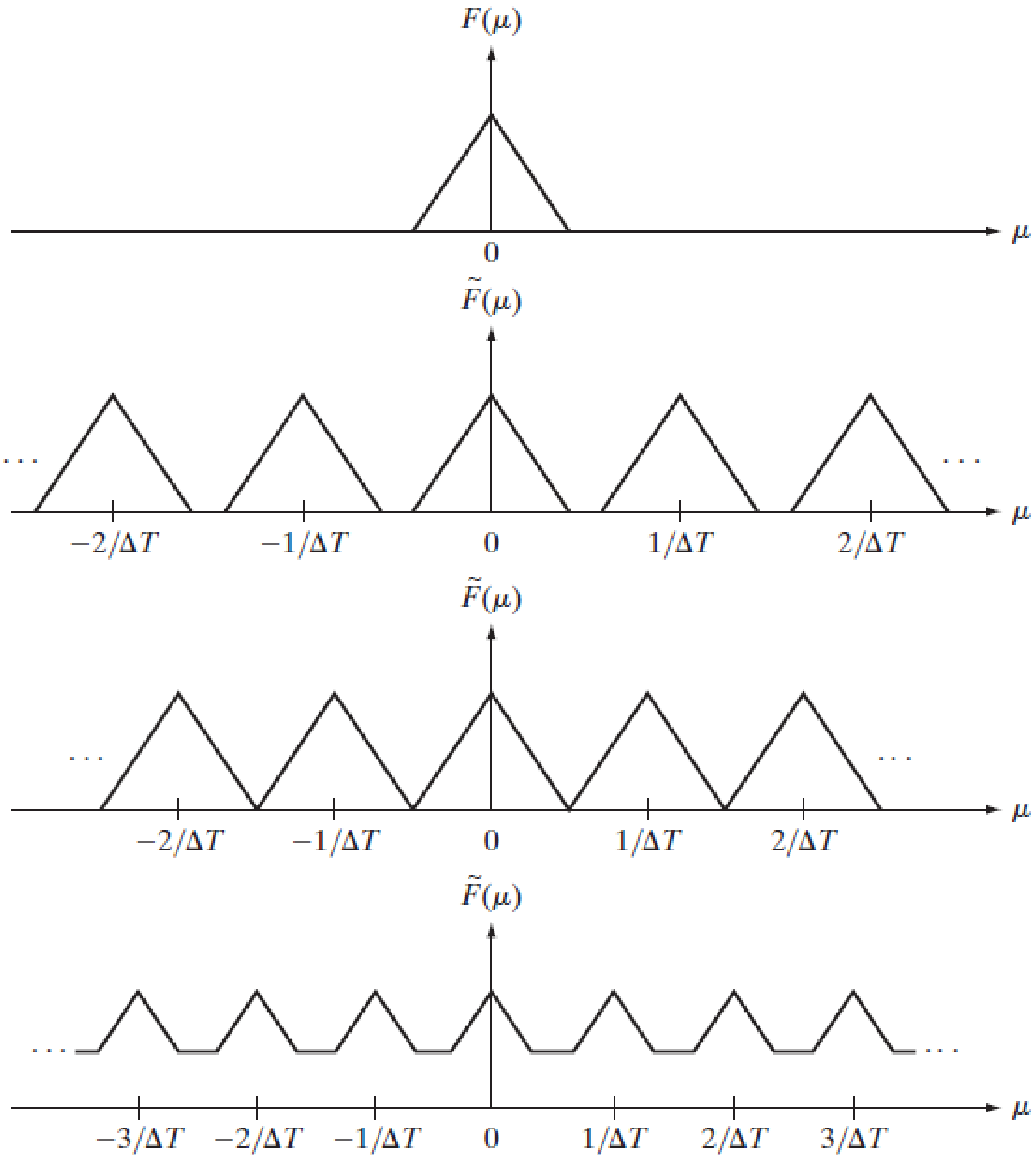
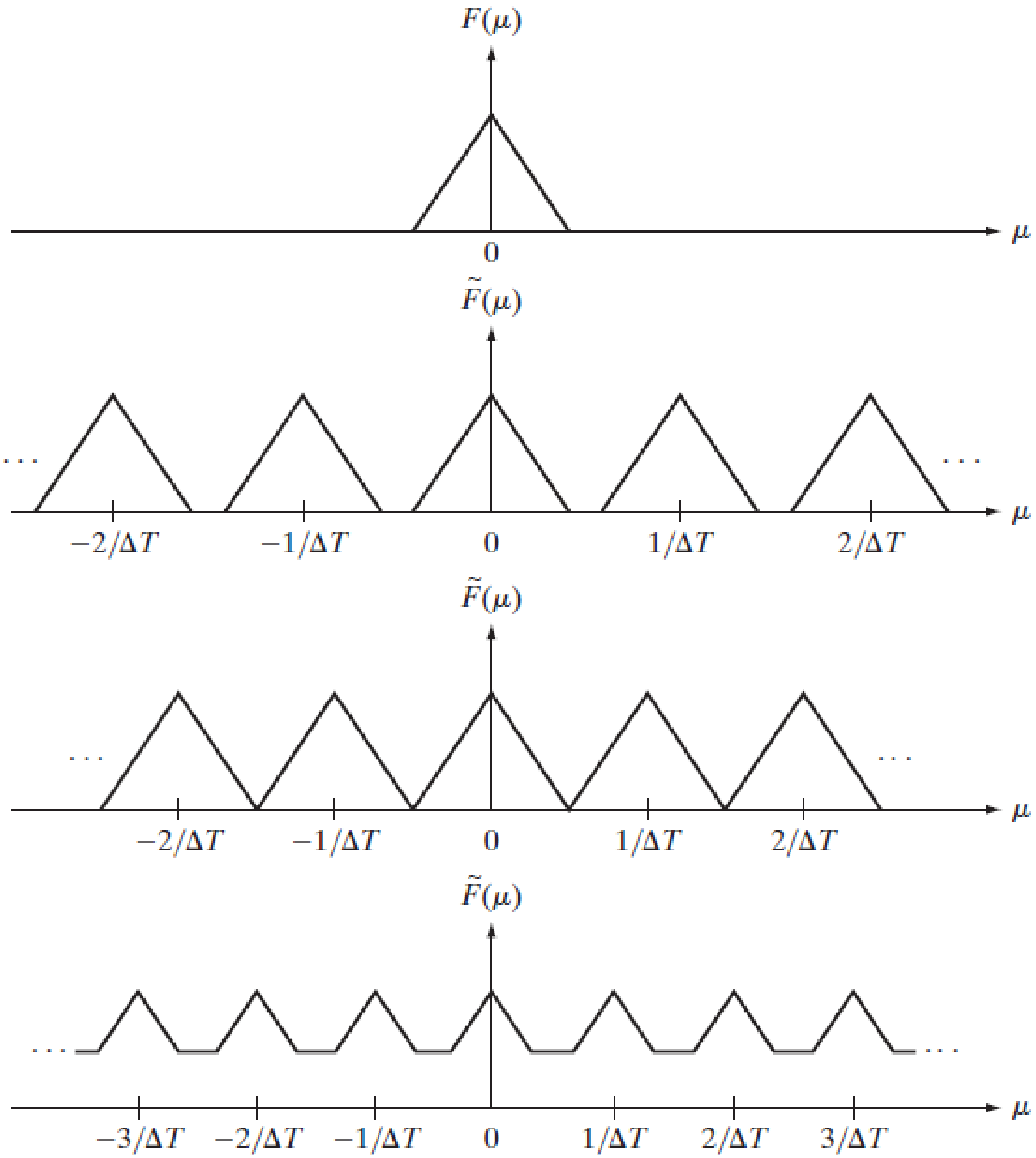



$F(\mu)$
μ
0
$\tilde{F}(\mu)$
$-2/\Delta T$ $-1/\Delta T$ 0 $1/\Delta T$ $2/\Delta T$
μ
$\tilde{F}(\mu)$
$-2/\Delta T$ $-1/\Delta T$ 0 $1/\Delta T$ $2/\Delta T$
μ
$\tilde{F}(\mu)$
$-3/\Delta T$ $-2/\Delta T$ $-1/\Delta T$ 0 $1/\Delta T$ $2/\Delta T$ $3/\Delta T$
μ

Oversampled

$$\frac{1}{\Delta T} > 2\mu_{max}$$

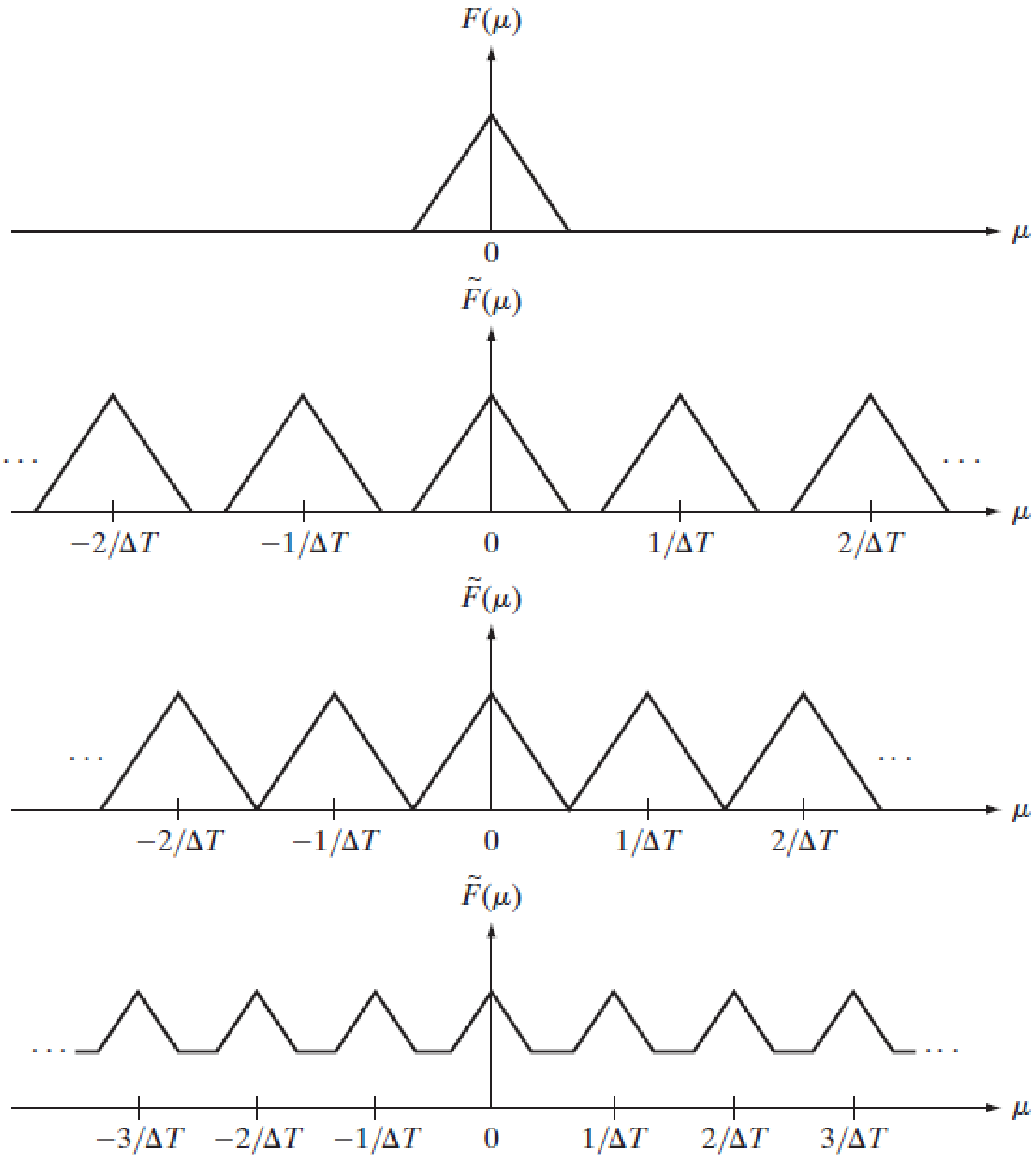


Oversampled

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled

$$\frac{1}{\Delta T} = 2\mu_{max}$$



Oversampled

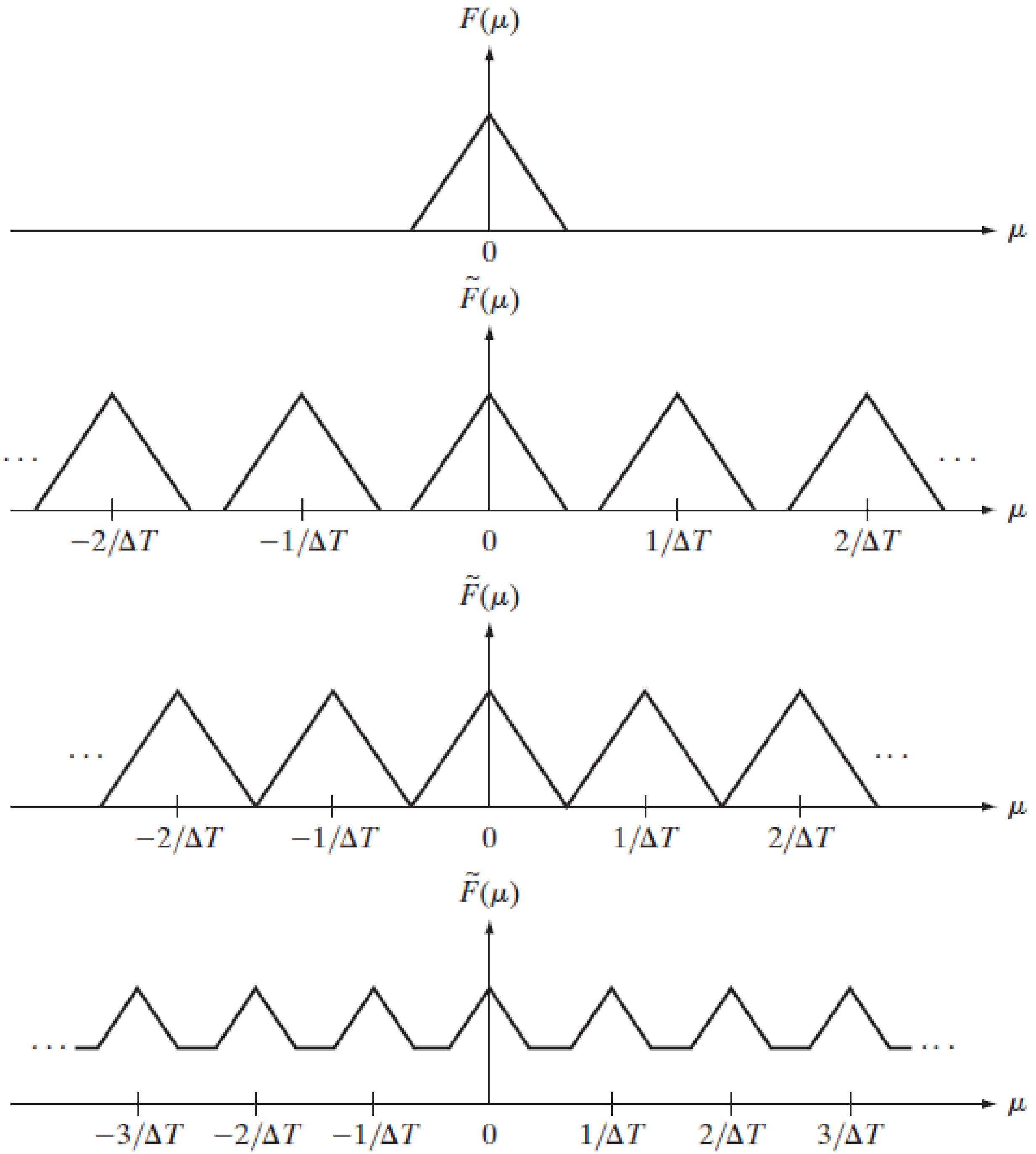
$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled

$$\frac{1}{\Delta T} = 2\mu_{max}$$

Under sampled

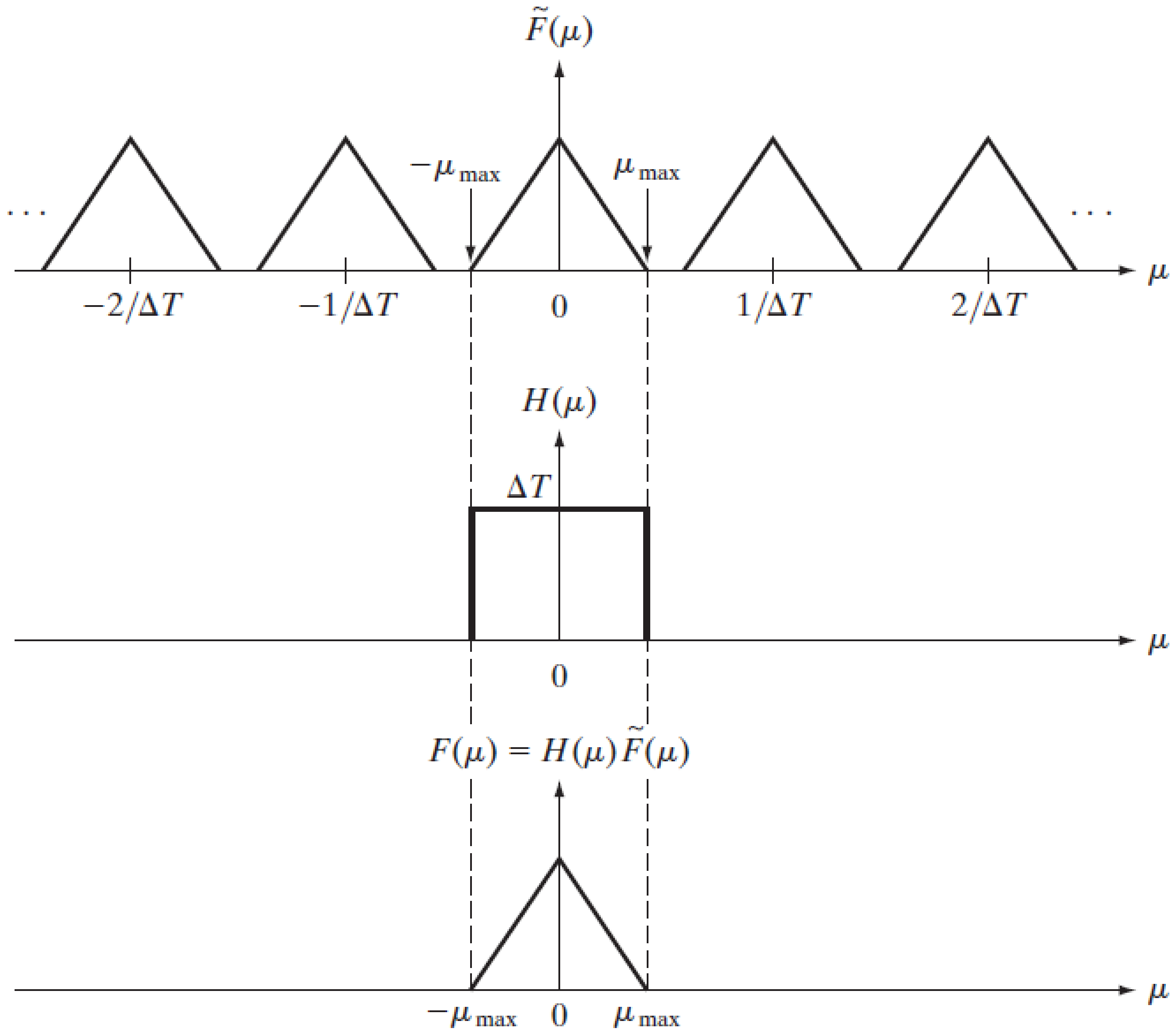
$$\frac{1}{\Delta T} < 2\mu_{max}$$



Revision

- Signal recovery

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$



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- Aliasing: The net effect of lowering the sampling rate below the Nyquist rate is that the periods now overlap, and it becomes impossible to isolate a single period of the transform, regardless of the filter used.
- No function of finite duration can be band-limited. Conversely, a function that is band-limited must extend from to –α to α.
- The effects of aliasing can be *reduced* by smoothing the input function to attenuate its higher frequencies (e.g., by defocusing in the case of an image).
- ***anti-aliasing***, has to be done *before* the function is sampled because aliasing is a sampling issue that cannot be “undone after the fact” using computational techniques.

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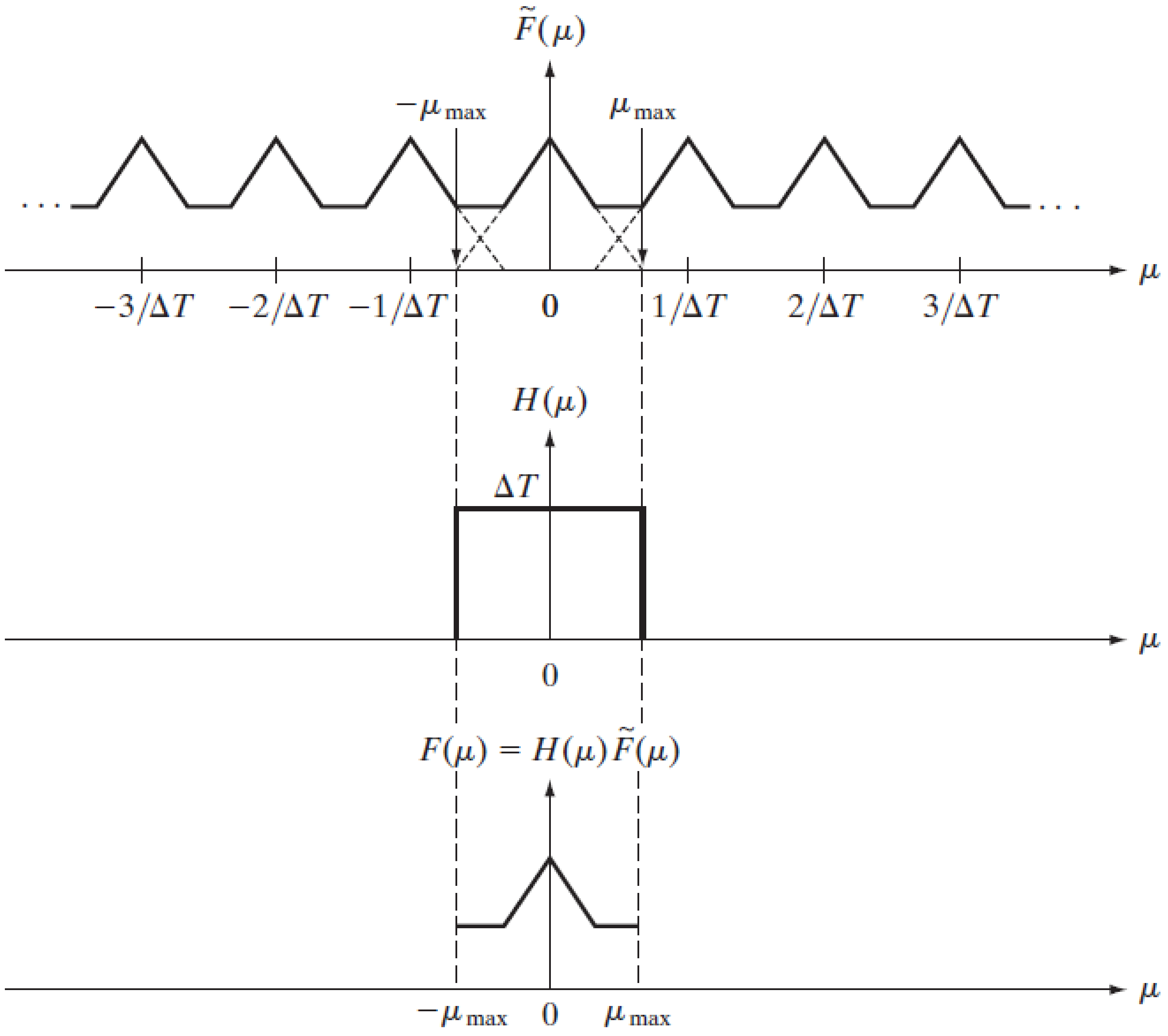

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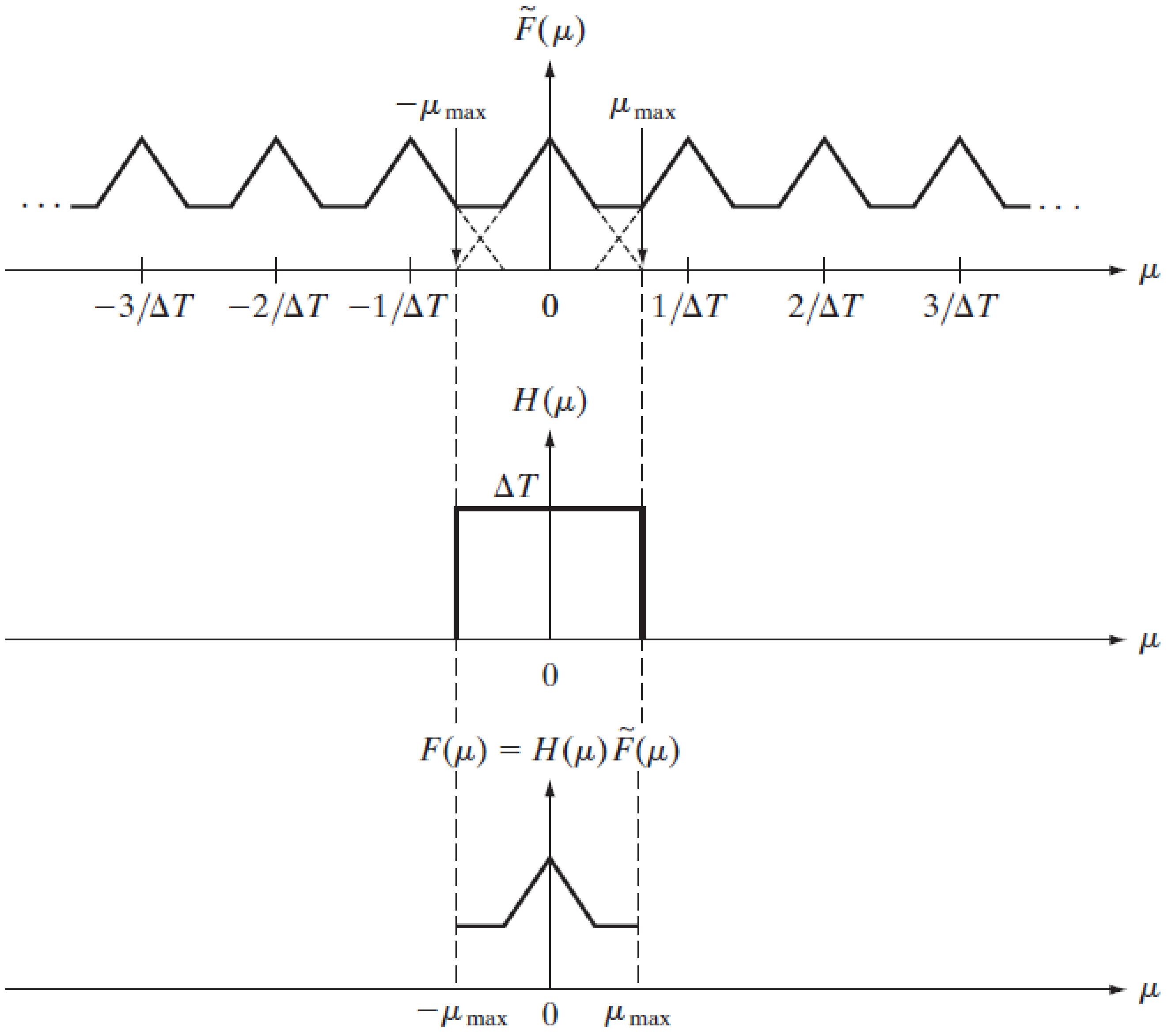

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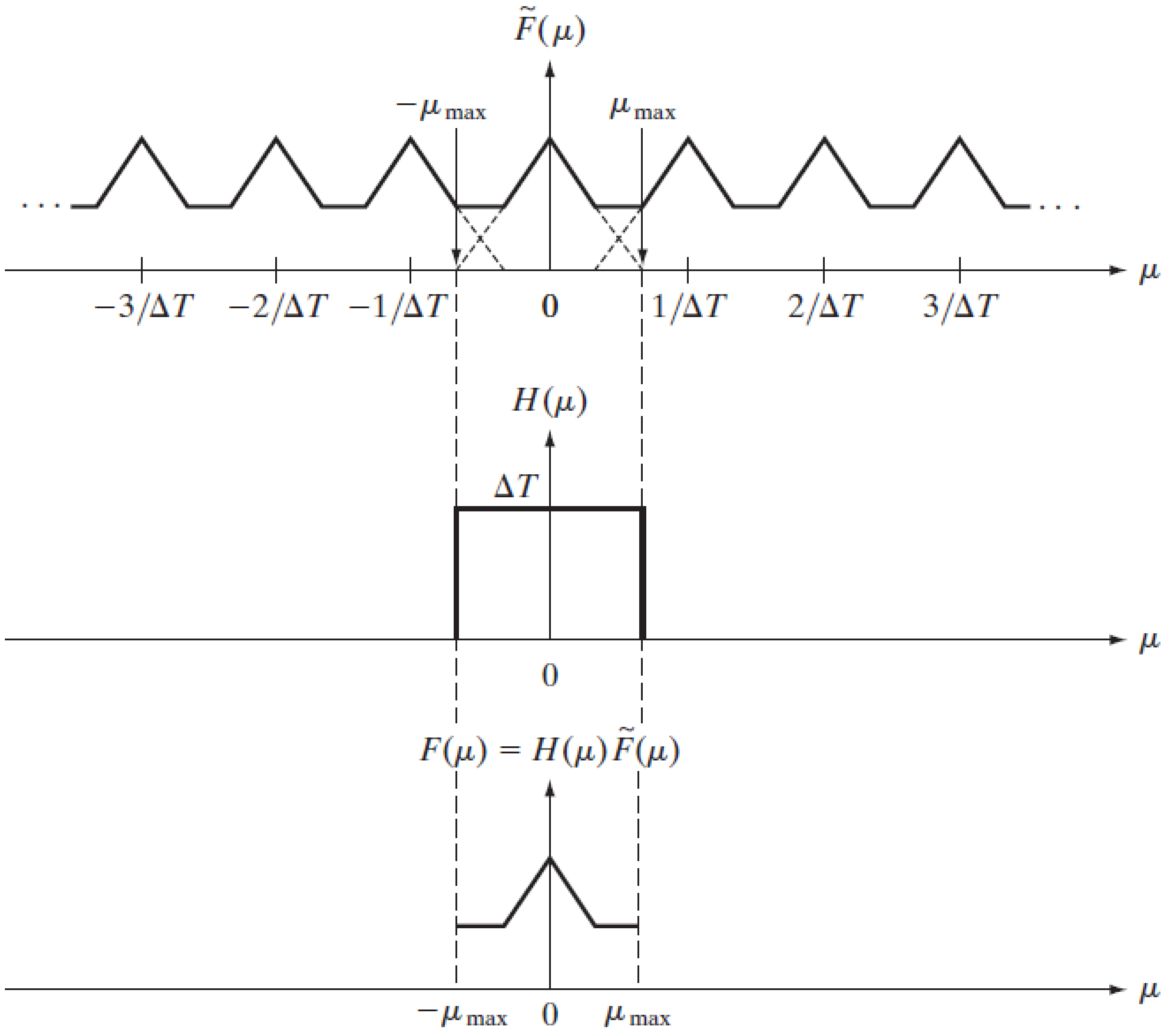

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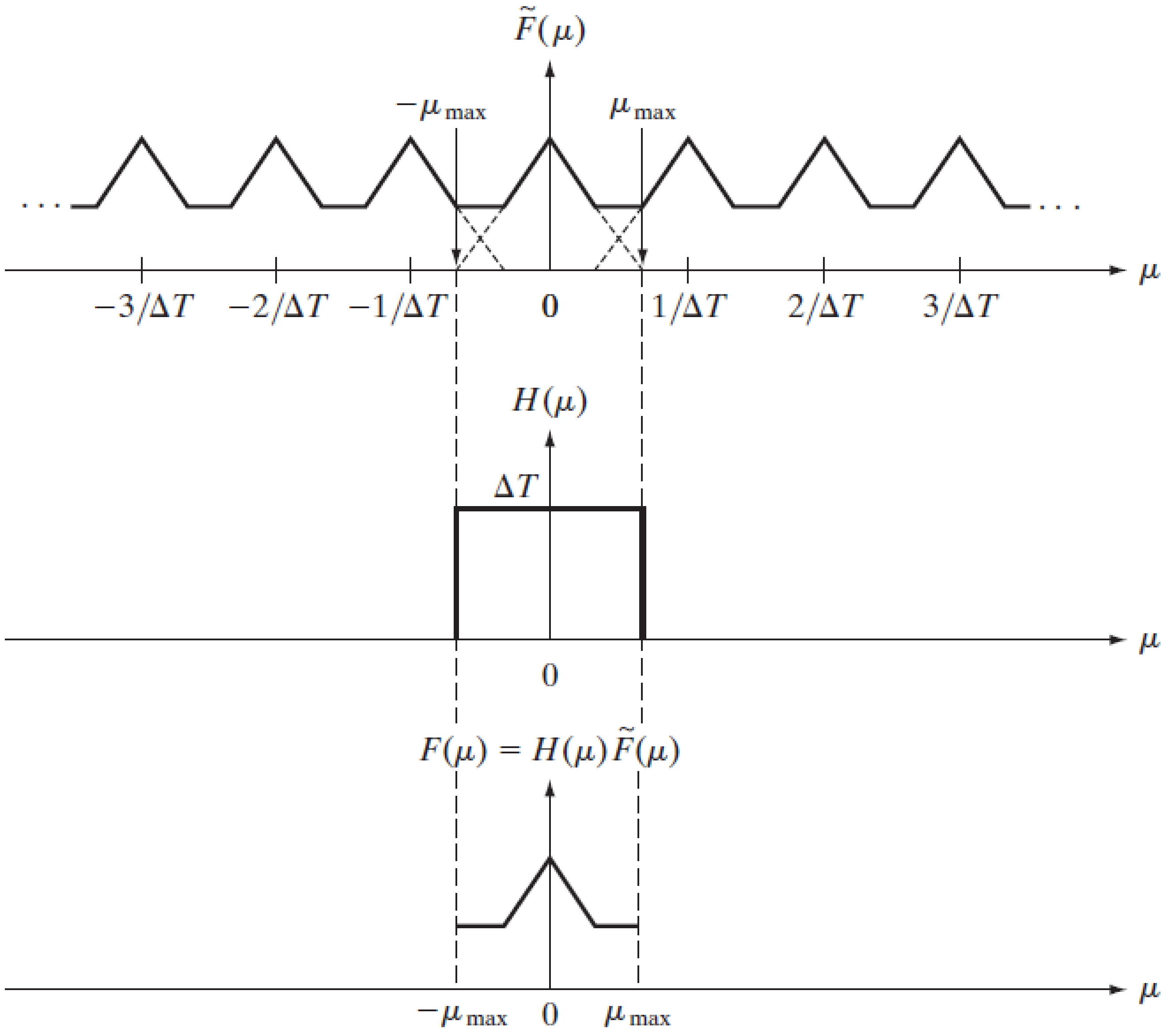

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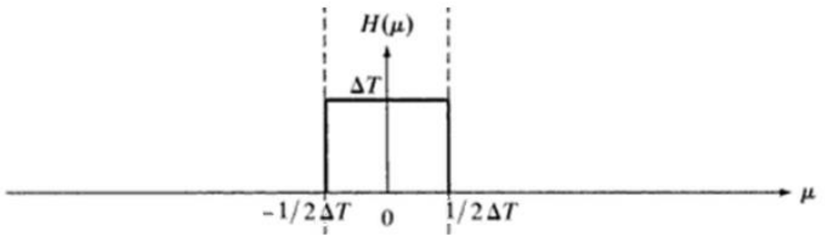
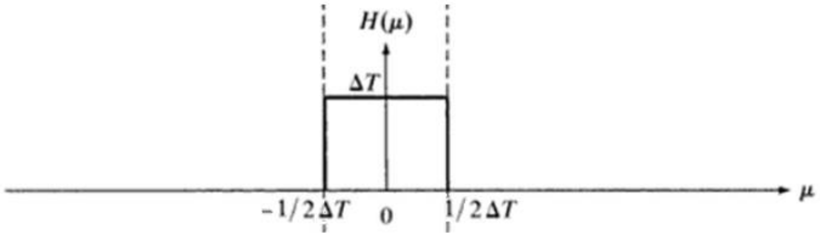
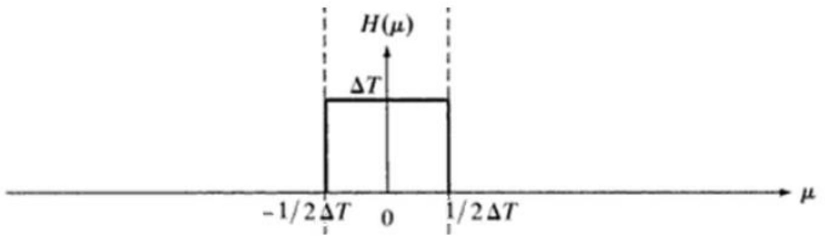

$H(\mu)$
ΔT
μ
$-1/2\Delta T$
0
$1/2\Delta T$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t} d\mu$$

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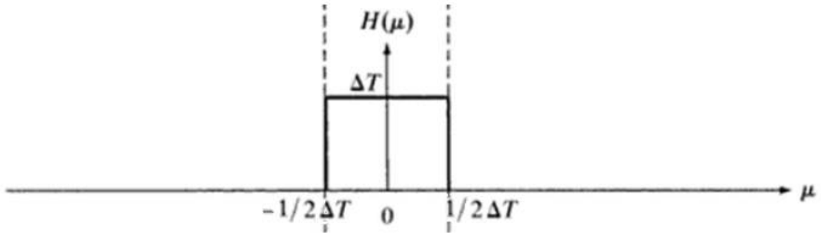

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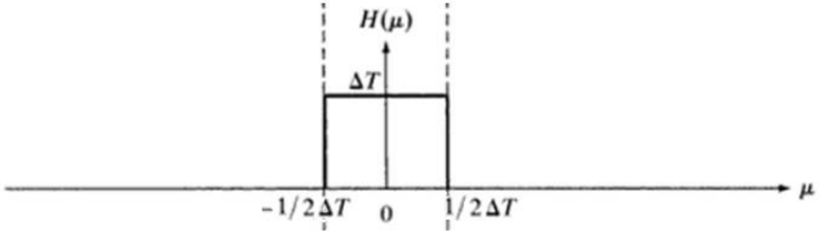

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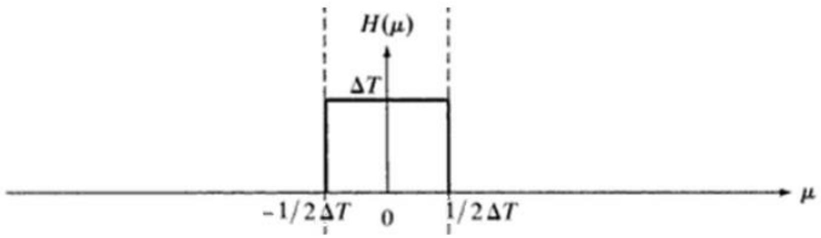

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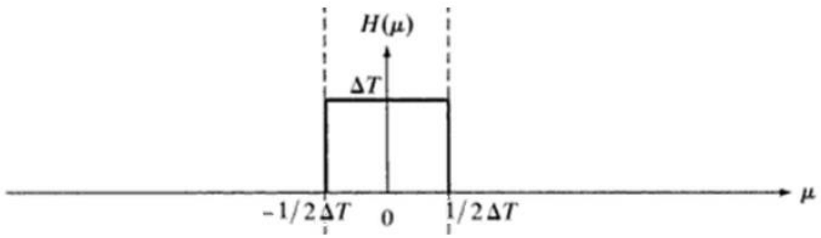

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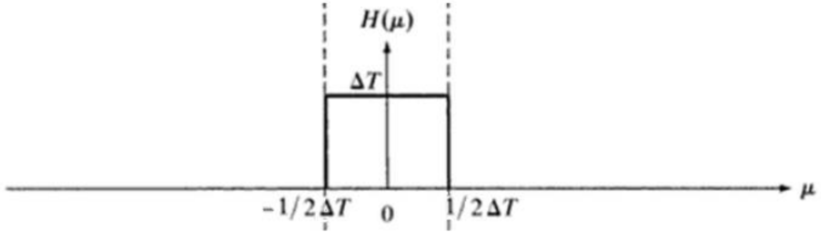

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f(t) &= h(t) \star \tilde{f}(t) \\
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That is, for any $t = k\,\Delta T$ where k is an integer, $f(t)$ is equal to the kth sample $f(k\,\Delta T)$. This follows from above equation because $sinc(0) = 1$ and $sinc(m) = 0$ for any other integer value of m.

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Between sample points, values of (t) are *interpolations* formed by the sum of the *sinc* functions.

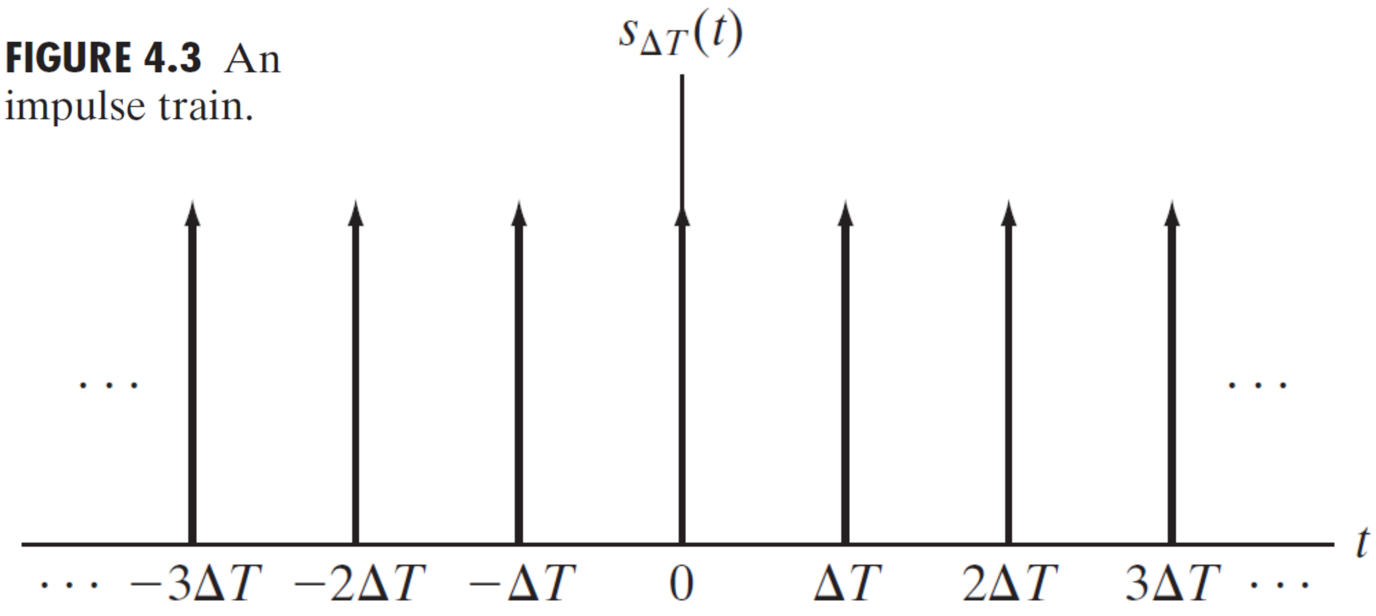
2D Impulse Train

Impulse Train

- Sum of infinitely many periodic impulses ΔT units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

FIGURE 4.3 An impulse train.



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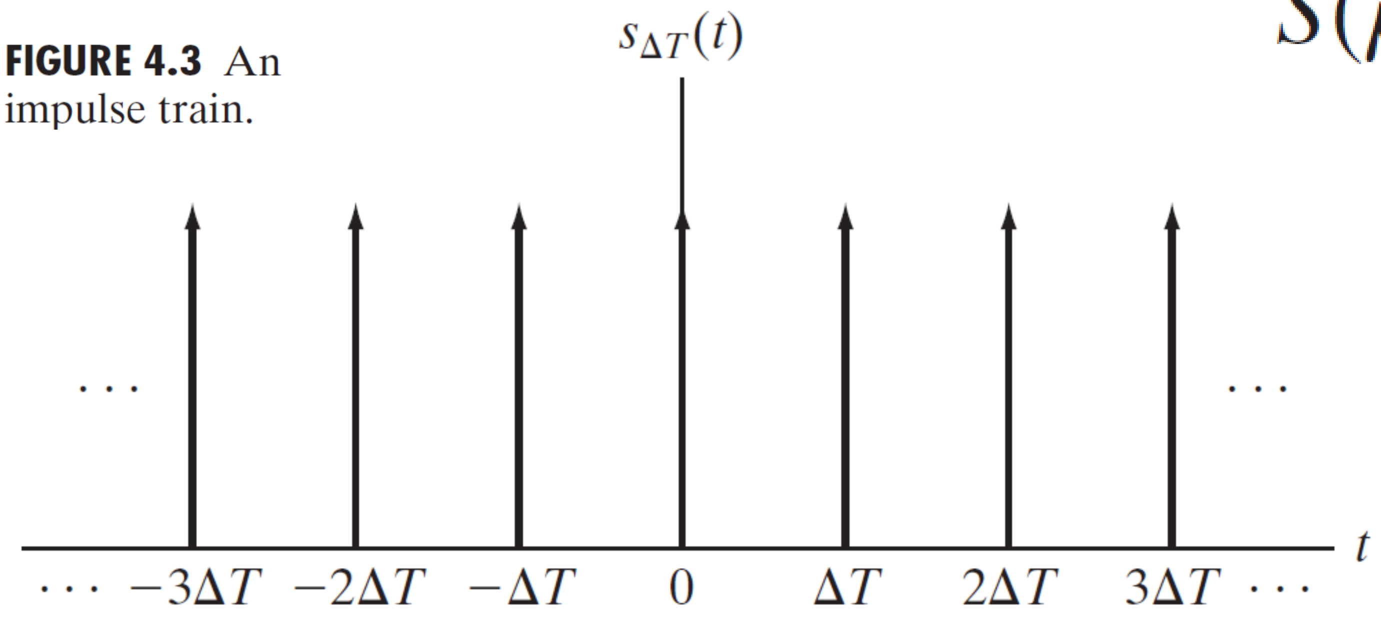

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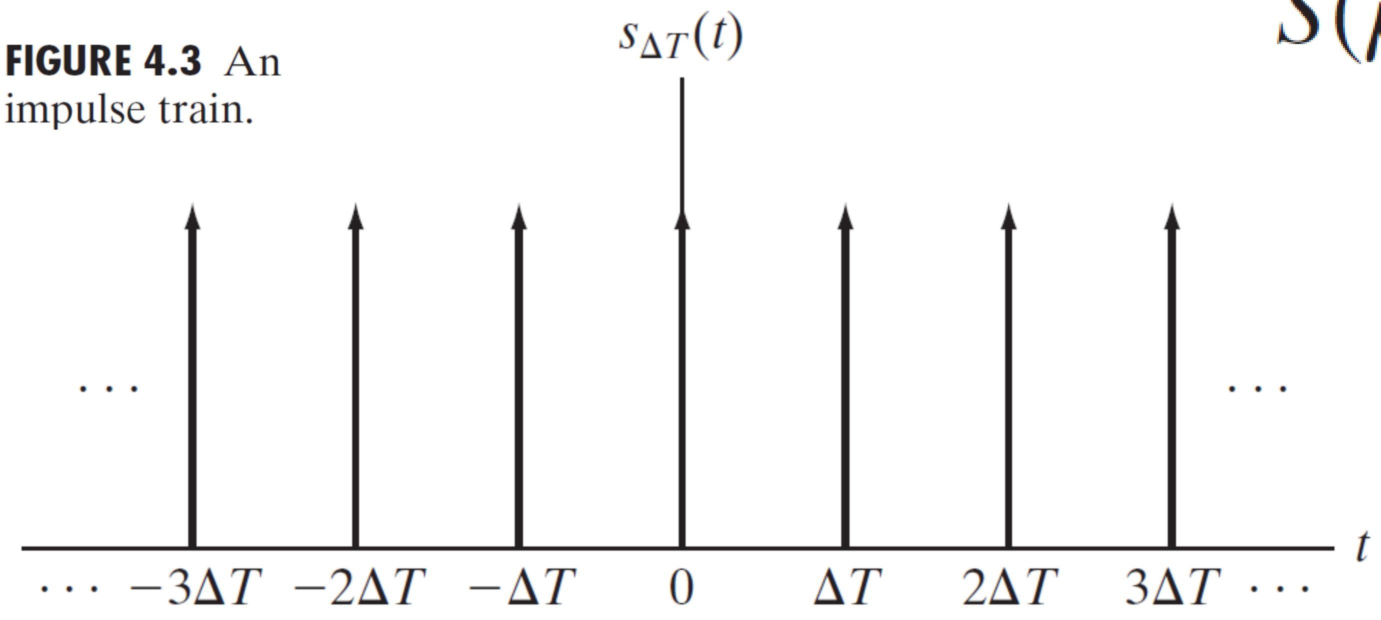

$$S(\mu) = \Im\{s_{\Delta T}(t)\}$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

2D Impulse Train

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

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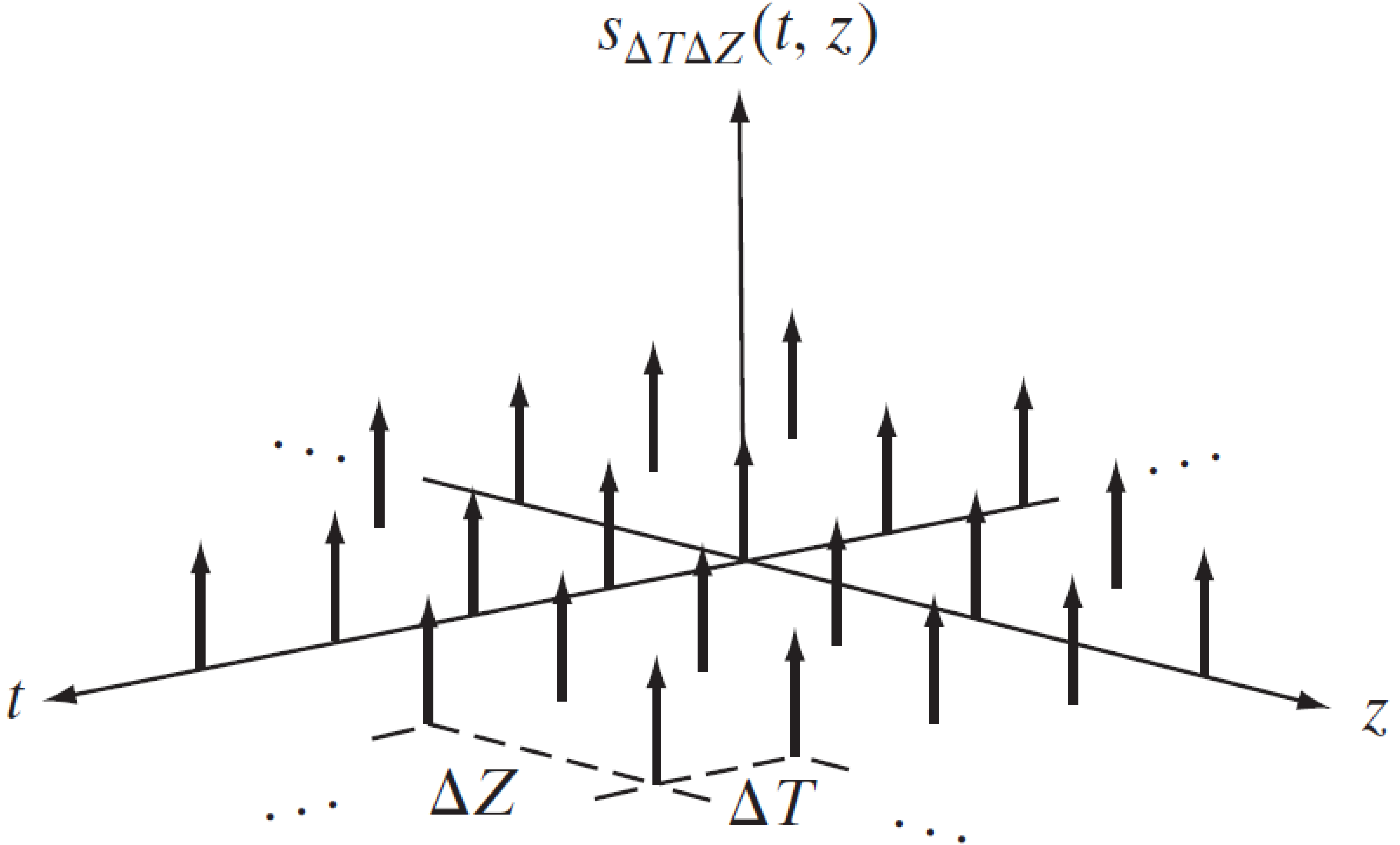

2D Sampling and Sampling Theorem

- As in the 1-D case multiplying $f\ (t,\ z)$ by $s_{\Delta T \Delta Z}\ (t,\ z)$ yields the sampled function.
- Function $f\,(t,\ z)$ is said to be *band-limited* if its Fourier transform is 0 outside a rectangle established by the intervals and $[-\mu_{\max},\ \mu_{\max}]$ and $[-v_{\max},\ v_{\max}]$ that is,

$$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \text{ and } |\nu| \geq \nu_{\max}$$

2D Sampling and Sampling Theorem

- The *two-dimensional sampling theorem* states that a continuous, band-limited function $f(t, z)$ can be recovered with no error from a set of its samples if the sampling rate is

$$\frac{1}{\Delta T} > 2\mu_{\max} \quad \text{and} \quad \frac{1}{\Delta Z} > 2\nu_{\max}$$

- or if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{\max}} \quad \text{and} \quad \Delta Z < \frac{1}{2\nu_{\max}}$$

Aliasing in Images

- Extending the concept of aliasing to images
  - Discussion on several aspects related to image sampling and resampling.
- As in the 1-D case, a continuous function $f(t, z)$ of two continuous variables, t and z, can be band-limited in general only if it extends infinitely in both coordinate directions.
- Because we cannot sample a function infinitely, aliasing is always present in digital images, just as it is present in sampled 1-D functions.
- There are two principal manifestations of aliasing in images:
  - spatial aliasing : due to under - sampling
  - temporal aliasing : related to time intervals between images in a sequence of images.

Aliasing In Images

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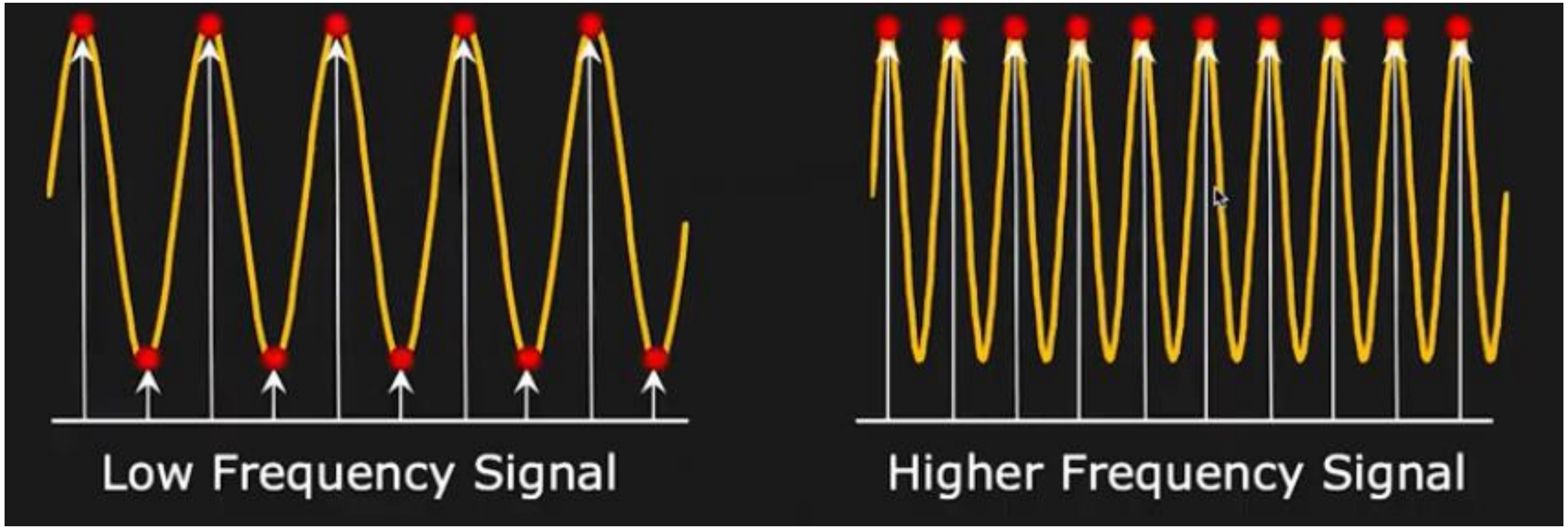

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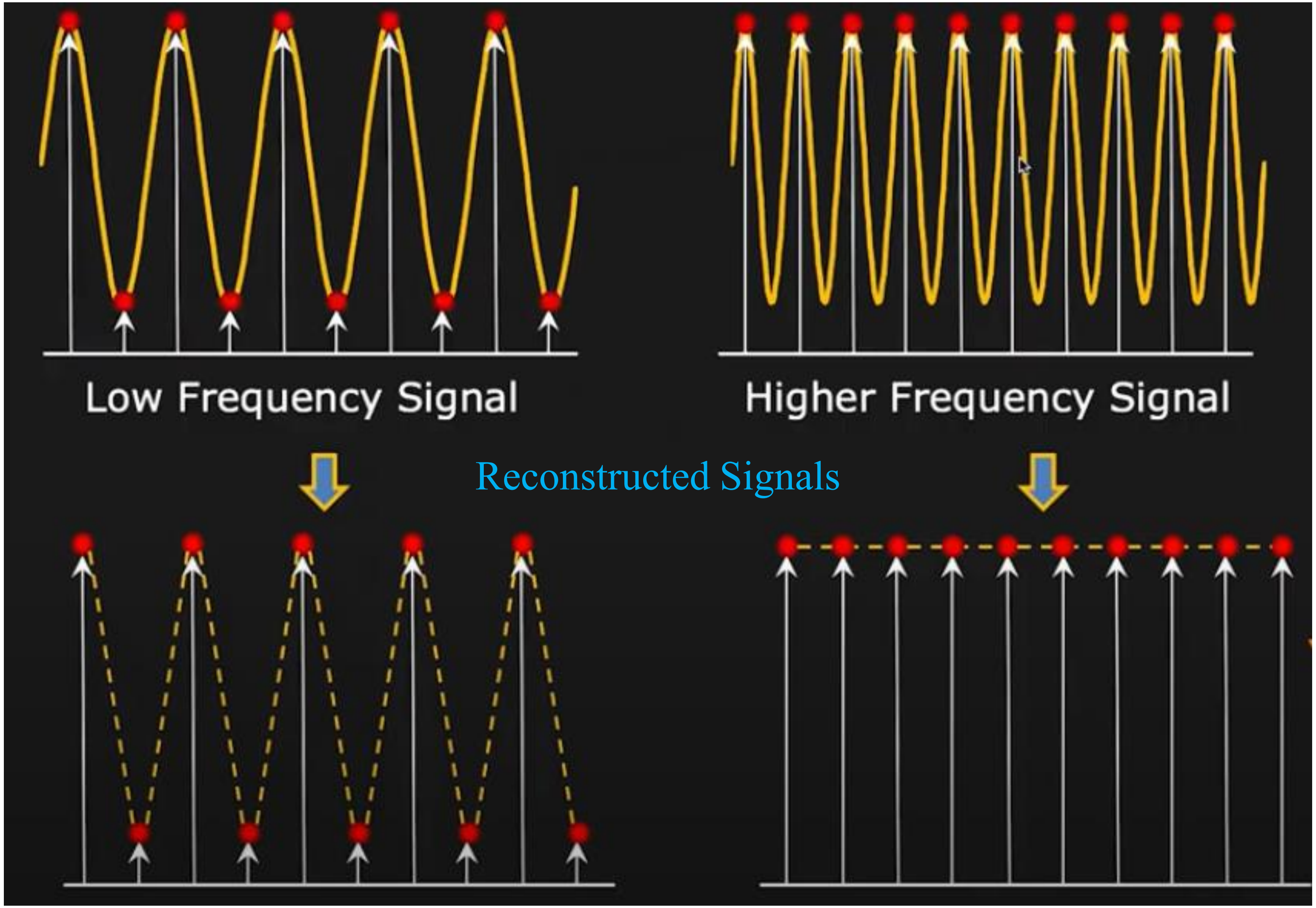
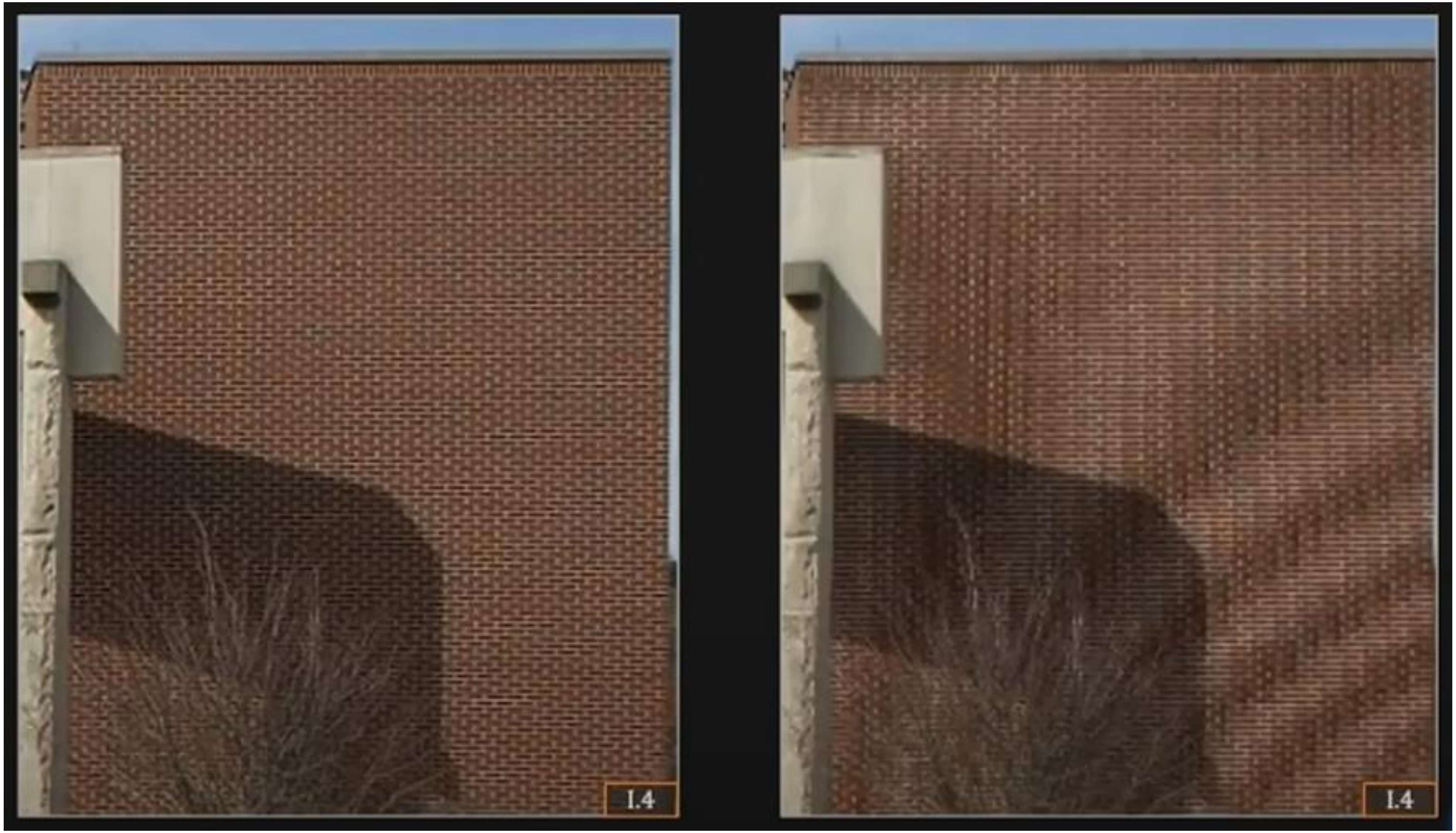




Well Sampled Image

Under – Sampled Image
“Visible artifacts”

When Does Spatial Aliasing Occur?

During image synthesis:

when sampling a continuous (geometric) model to create a raster image, e.g. scan converting a line or polygon.

Sampling: converting a continuous signal to a discrete signal.

During image processing and image synthesis:

when resampling a picture, as in image warping or texture mapping.

Resampling: sampling a discrete signal at a different sampling rate.

Example: "zooming" a picture from n_x by n_y pixels to sn_x by sn_y pixels

$s>1$: called **upsampling** or **interpolation**

can lead to blocky appearance if point sampling is used

$s<1$: called **downsampling** or**decimation**

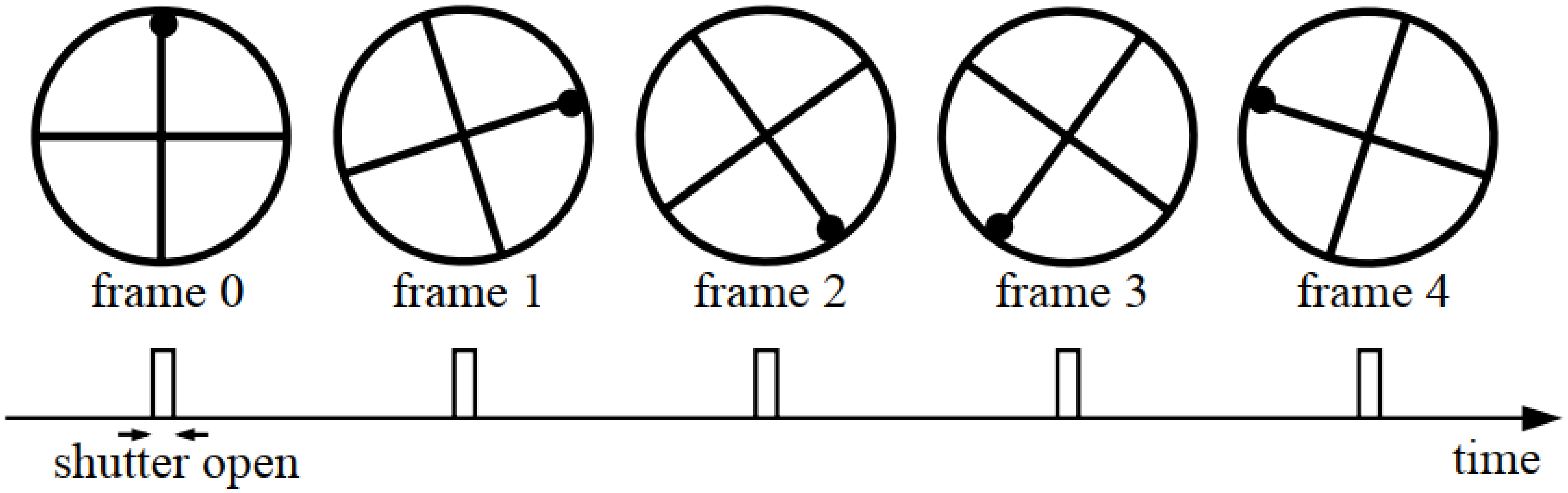
can lead to moire patterns and jaggies

Wagon Wheel Effect

an example of temporal aliasing

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what's happening.

If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):



Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)