AI VIETNAM
All-in-One Course

Multi-layer Perception

Initialization (Advanced)

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Ph.D. in Computer Science

Year 2023

Outline

- Case Studies
- Gradient Vanishing
- Gradient Explosion
- Xavier Glorot Initialization
- Kaiming He Initialization

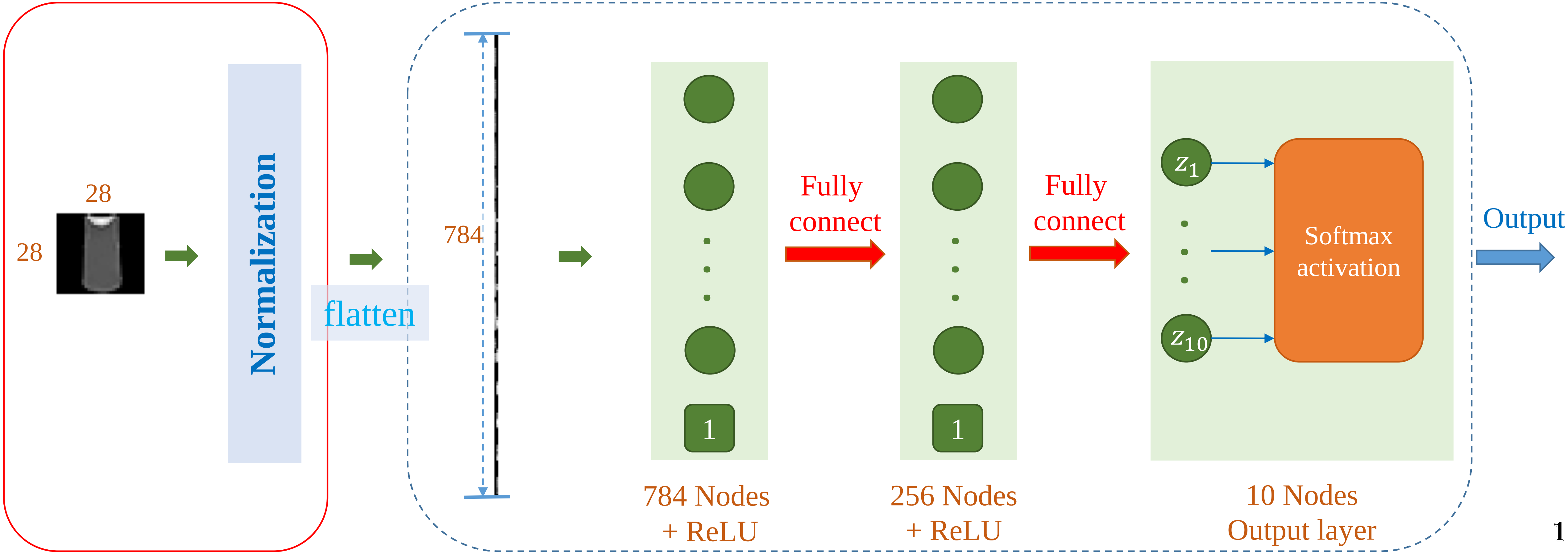
$$X \in [0, 255]$$

$$\text{Normalize}(mean, \text{std})$$

$$\text{Image} = \frac{\text{Image} - mean}{\text{std}}$$

```
transform = transforms.Compose([transforms.ToTensor(),
                                transforms.Normalize((0,),
                                                     (1.0/255,))])

model = nn.Sequential(
    nn.Flatten(), nn.Linear(784, 256),
    nn.ReLU(), nn.Linear(256, 10)
)
```



$$X \in [-1, 1]$$

$$\text{Normalize}(mean, \text{std})$$

$$\text{Image} = \frac{\text{Image} - mean}{\text{std}}$$

```
transform = transforms.Compose([transforms.ToTensor(),
                                transforms.Normalize((0.5,),
                                                     (0.5,))])

model = nn.Sequential(
    nn.Flatten(), nn.Linear(784, 256),
    nn.ReLU(), nn.Linear(256, 10)
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```

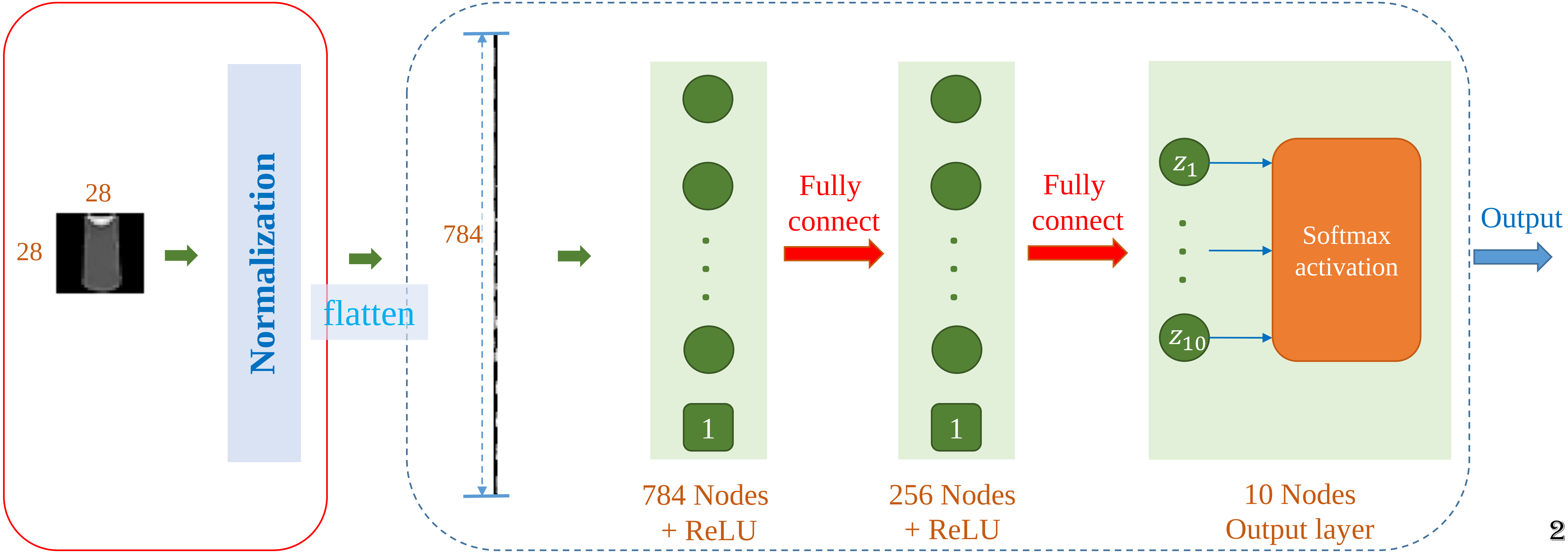

Experimental Results

```python
Compose([transforms.ToTensor(),
         transforms.Normalize((0,),
                              (1.0/255,))])
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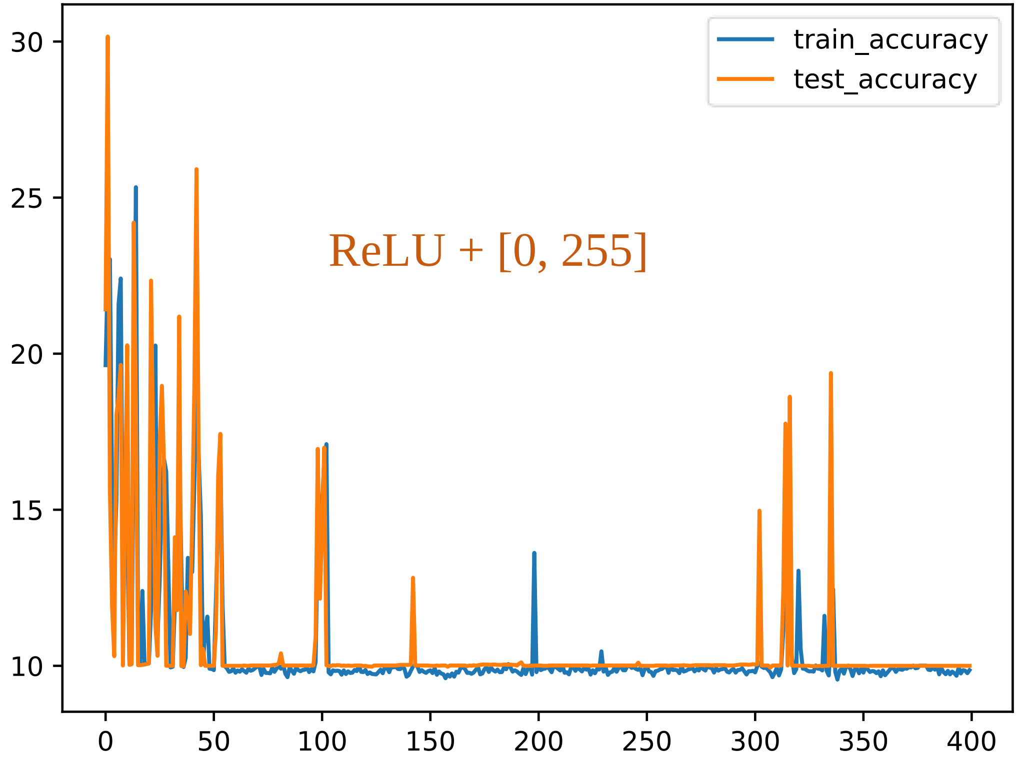

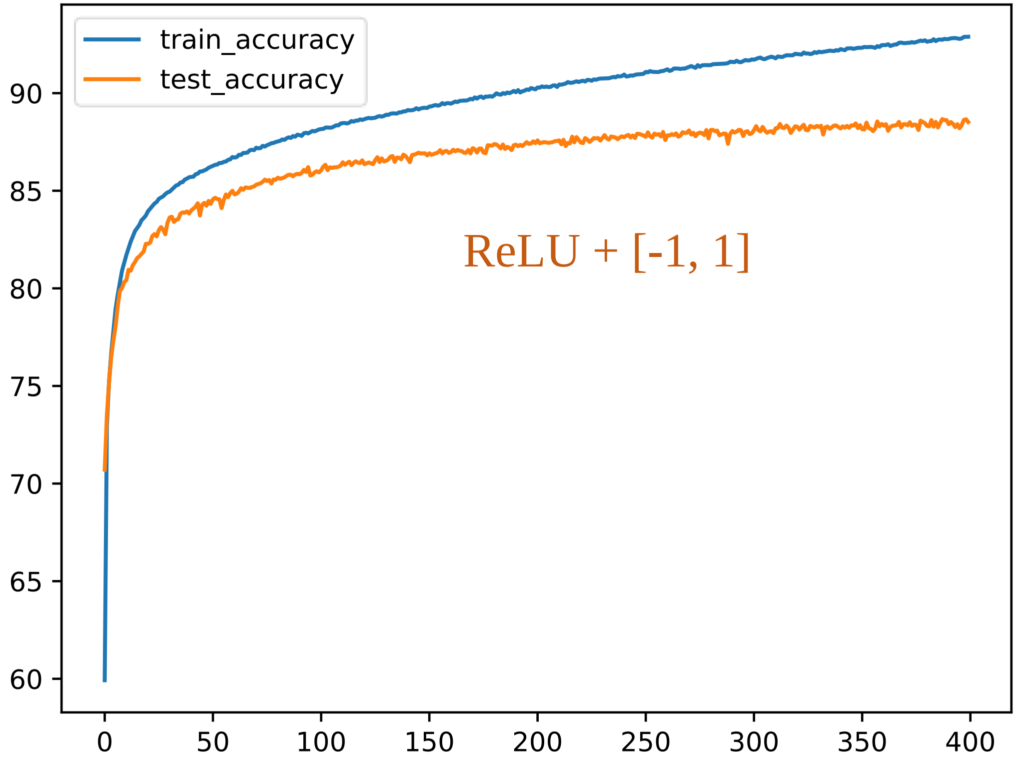

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    nn.Flatten(), nn.Linear(784, 256),
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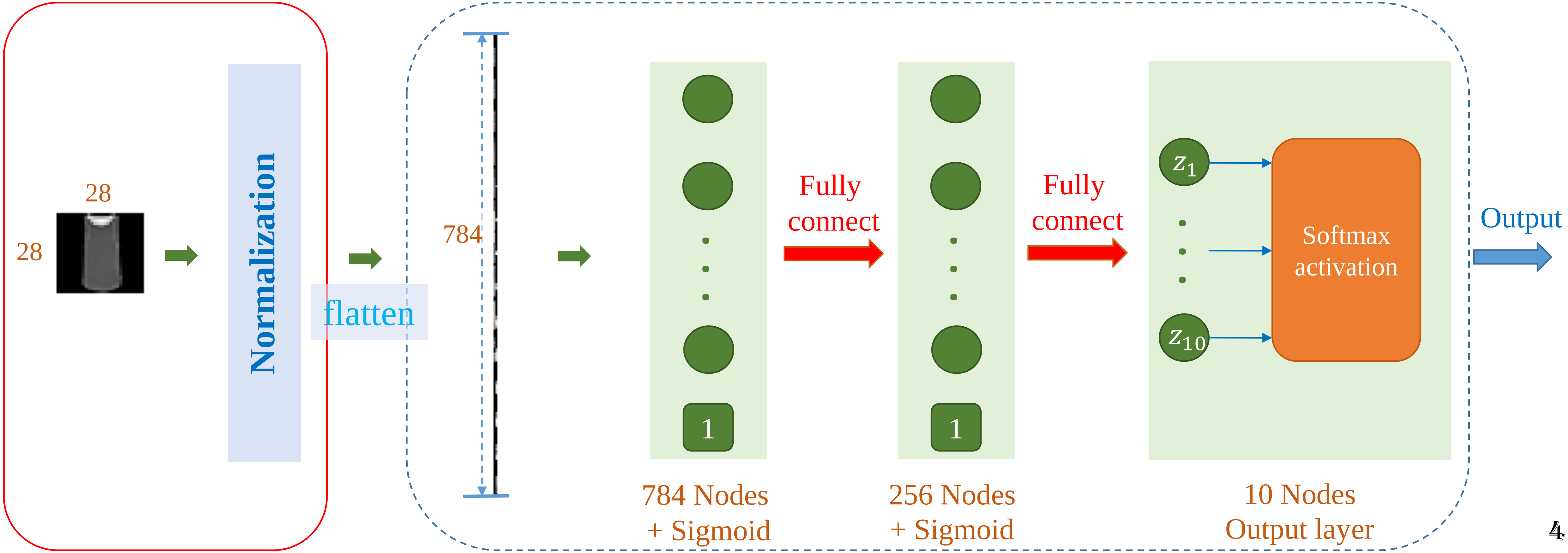

$$X \in [-1, 1]$$

Normalize(*mean*, std)

$$\text{Image} = \frac{\text{Image} - mean}{\text{std}}$$

```python
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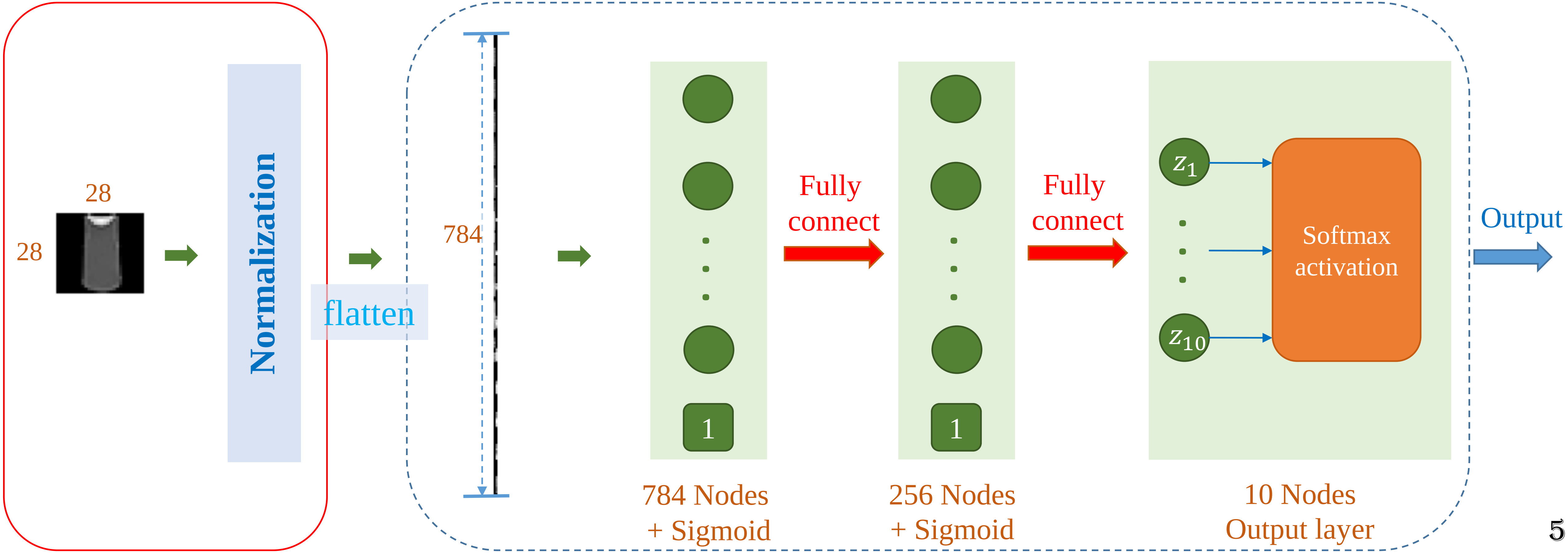

Experimental Results

```
Compose([transforms.ToTensor(),
         transforms.Normalize((0,),
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```

```
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         transforms.Normalize((0.5,),
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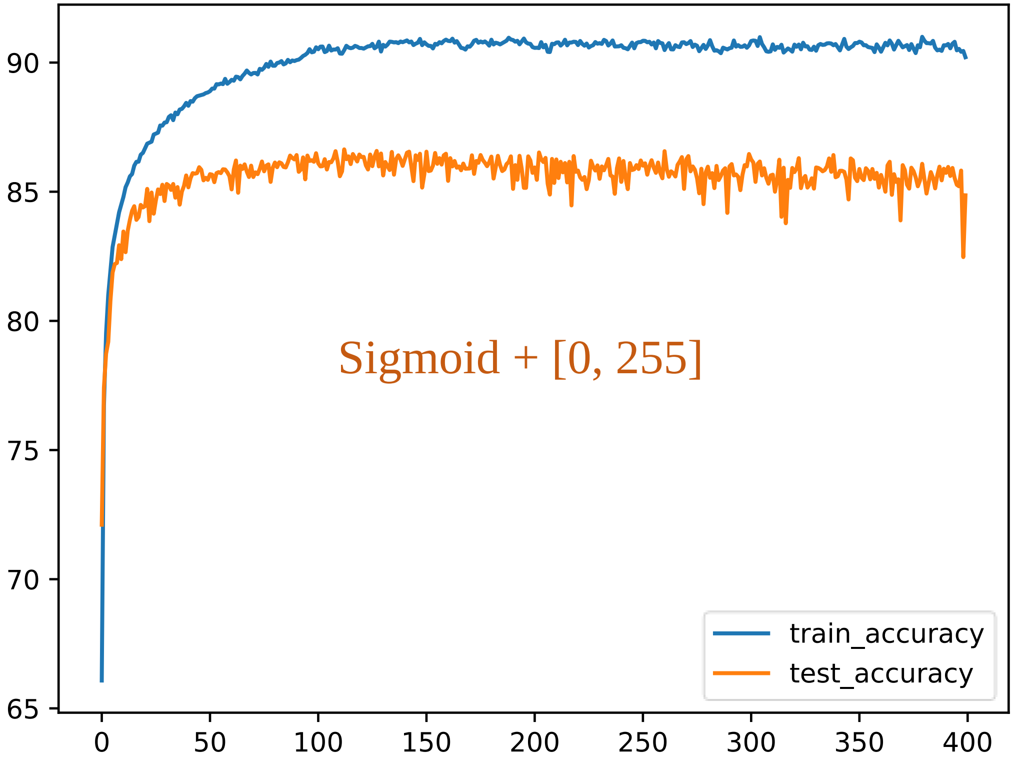

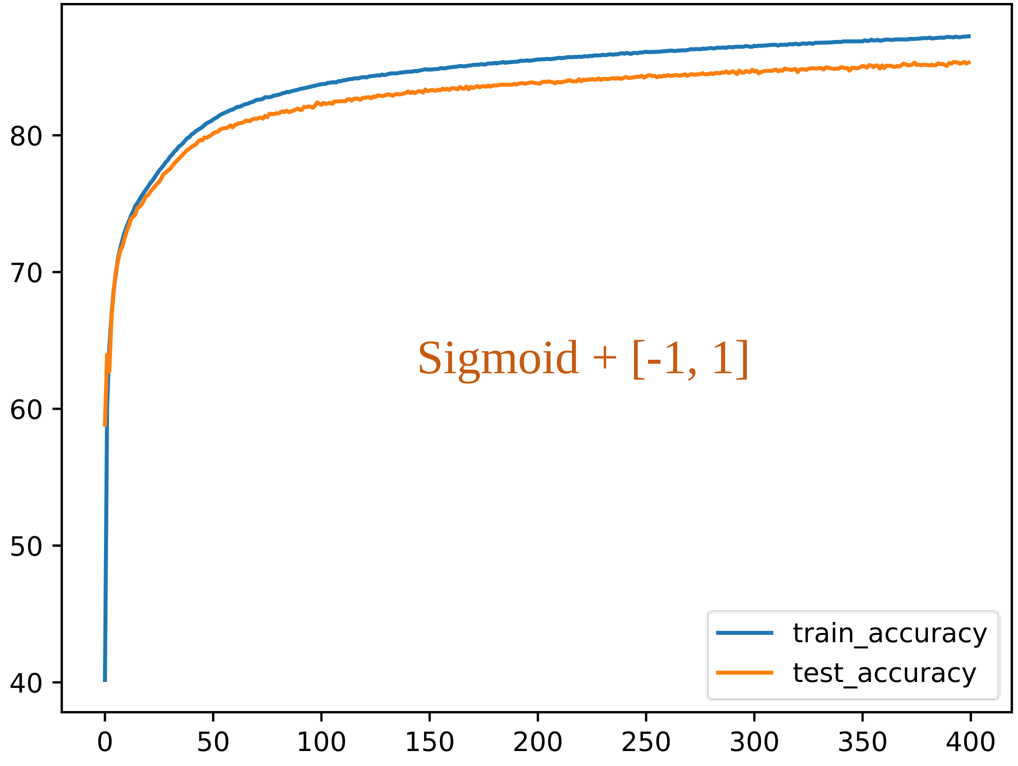

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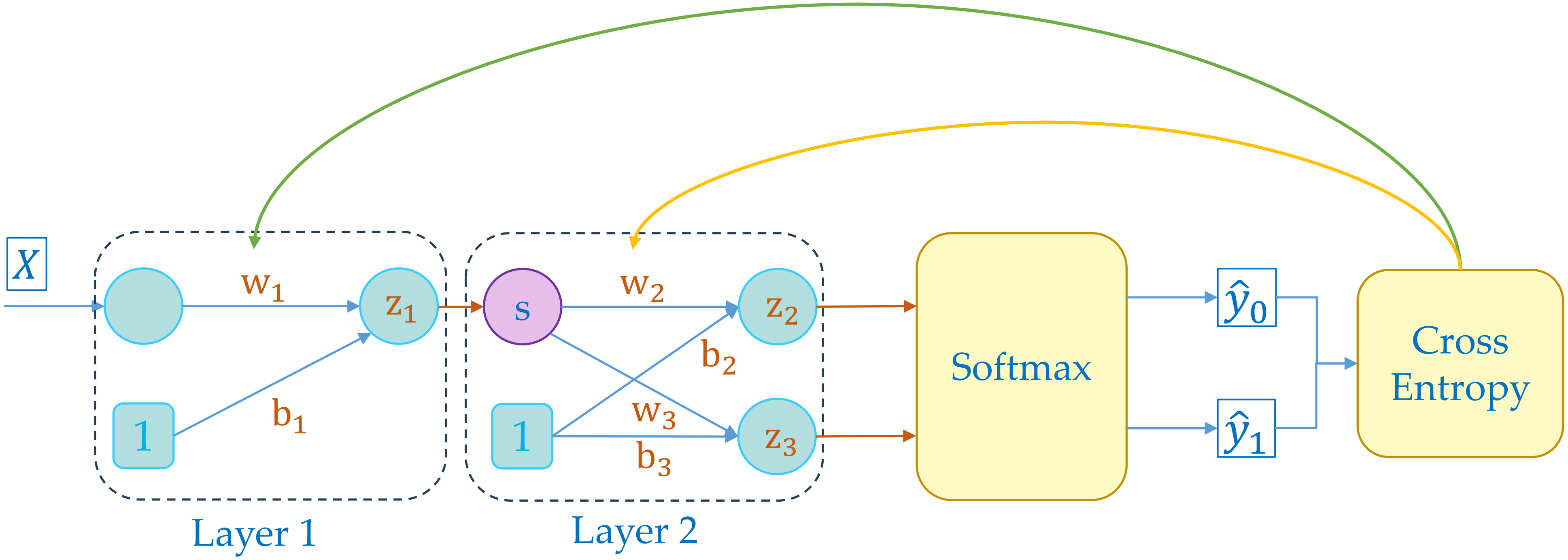
Gradient Vanishing

Large weight initialization



Gradient Vanishing

Large weight initialization

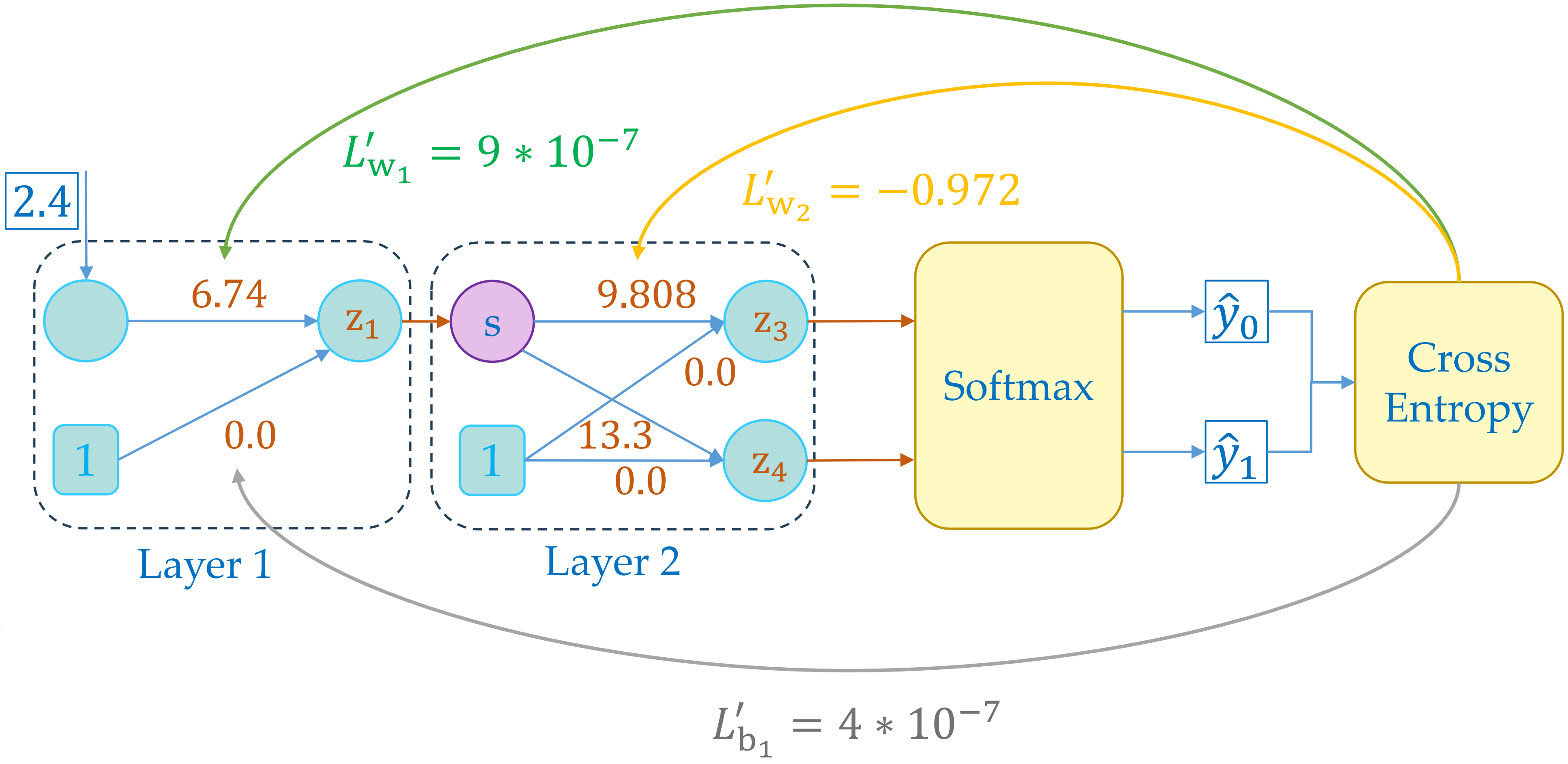
```
linear1 = nn.Linear(1, 1)
linear2 = nn.Linear(1, 2)

init.normal_(linear1.weight,
             mean=0, std=10)
init.normal_(linear2.weight,
             mean=0, std=10)
```

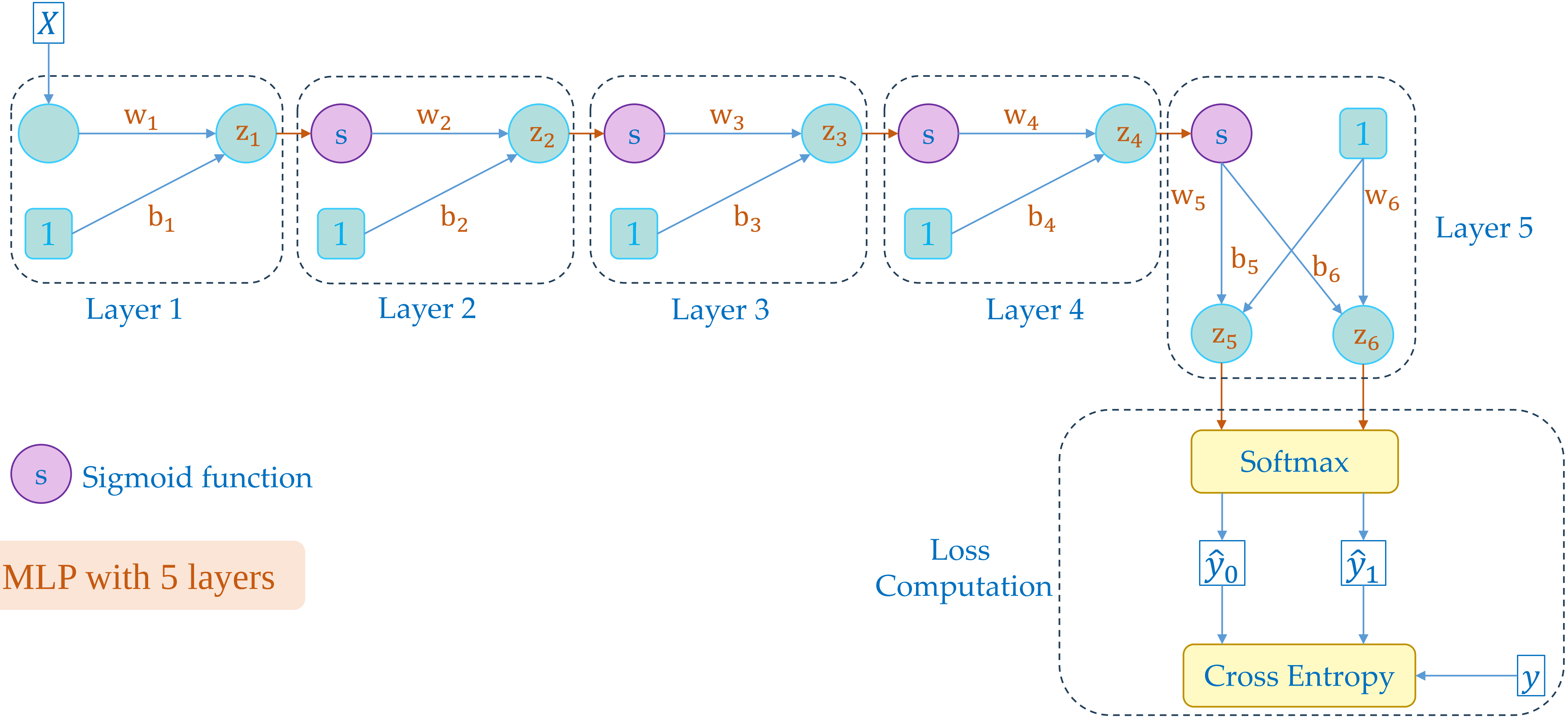
with $\eta = 0.01$

$\eta L'_{w_1} = 9 * 10^{-9}$

$\eta L'_{b_1} = 4 * 10^{-9}$

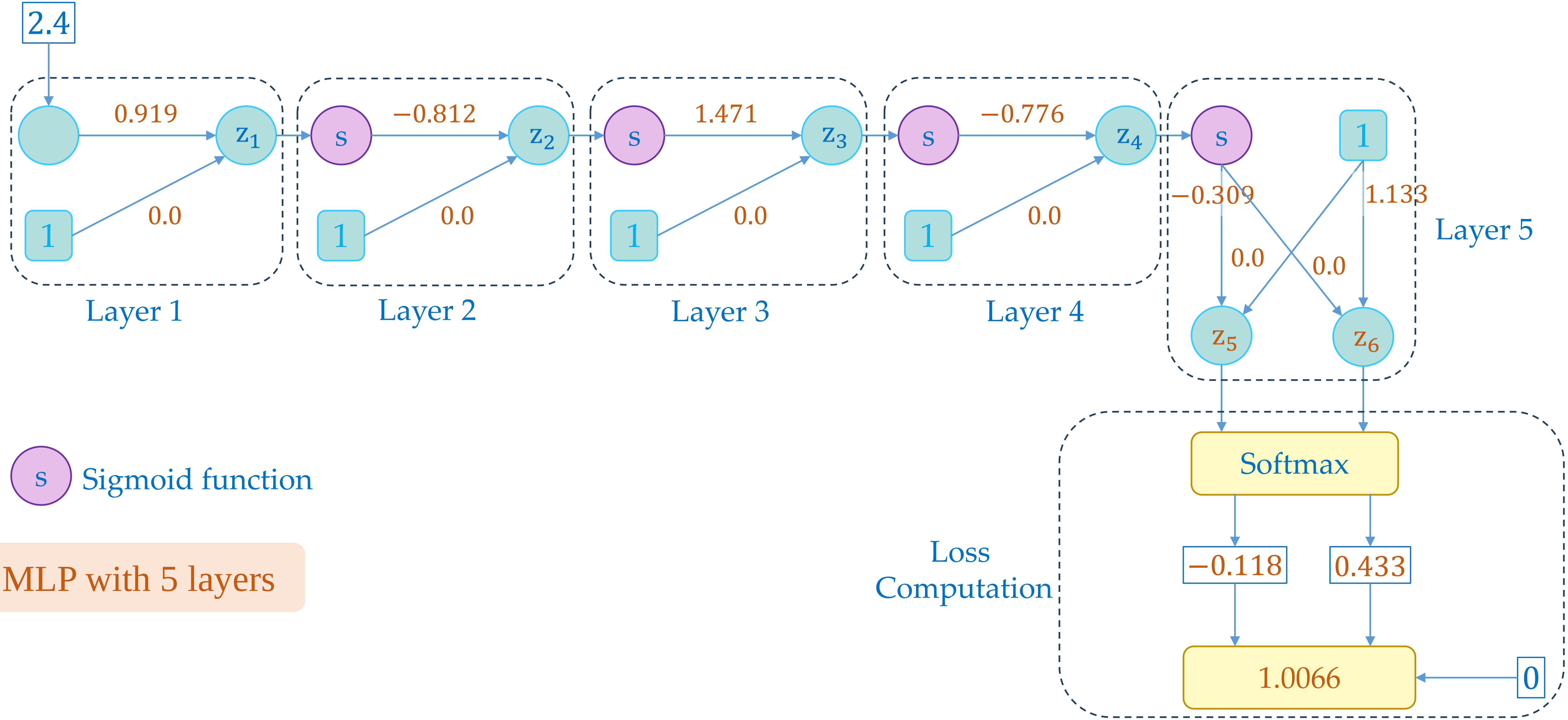


Gradient Vanishing



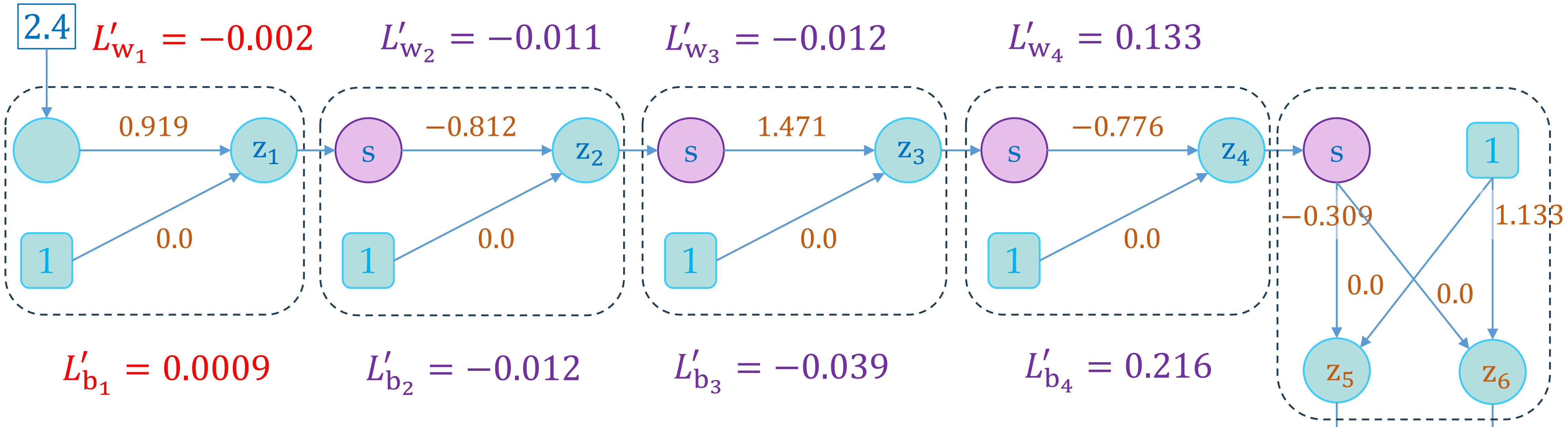
MLP with 5 layers

Gradient Vanishing

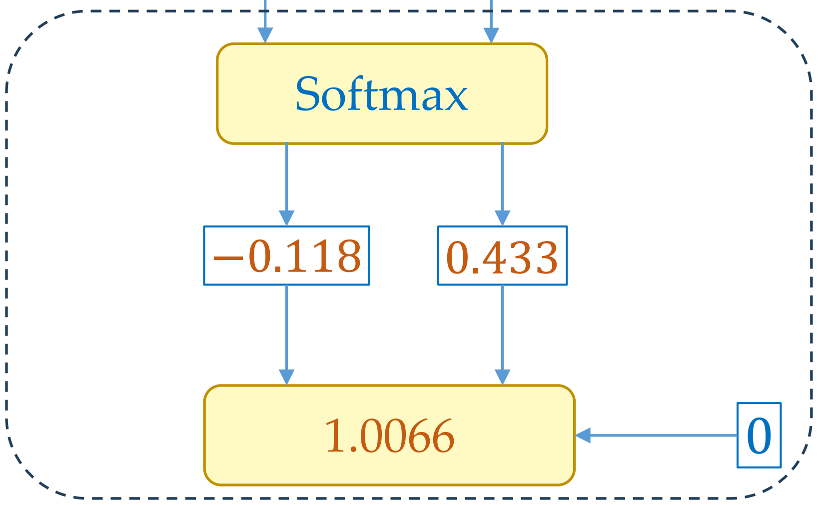


MLP with 5 layers

Gradient Vanishing



Derivative values are too small

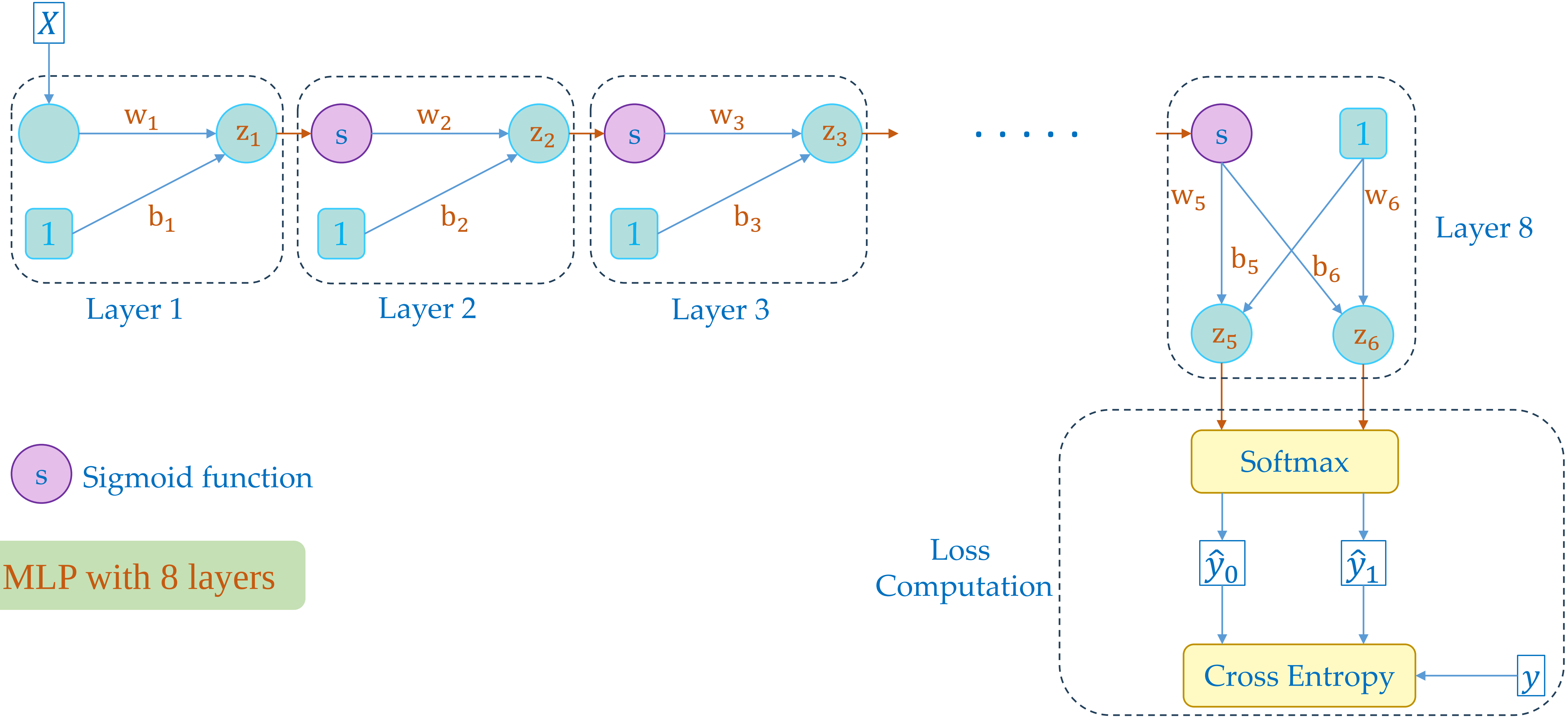


$$
\begin{aligned}
w_1 &= w_1 - \eta L'_{w_1} \\
&= 0.919 - 0.01 * (-0.0002) \\
&= 0.919002
\end{aligned}
$$

MLP with 5 layers

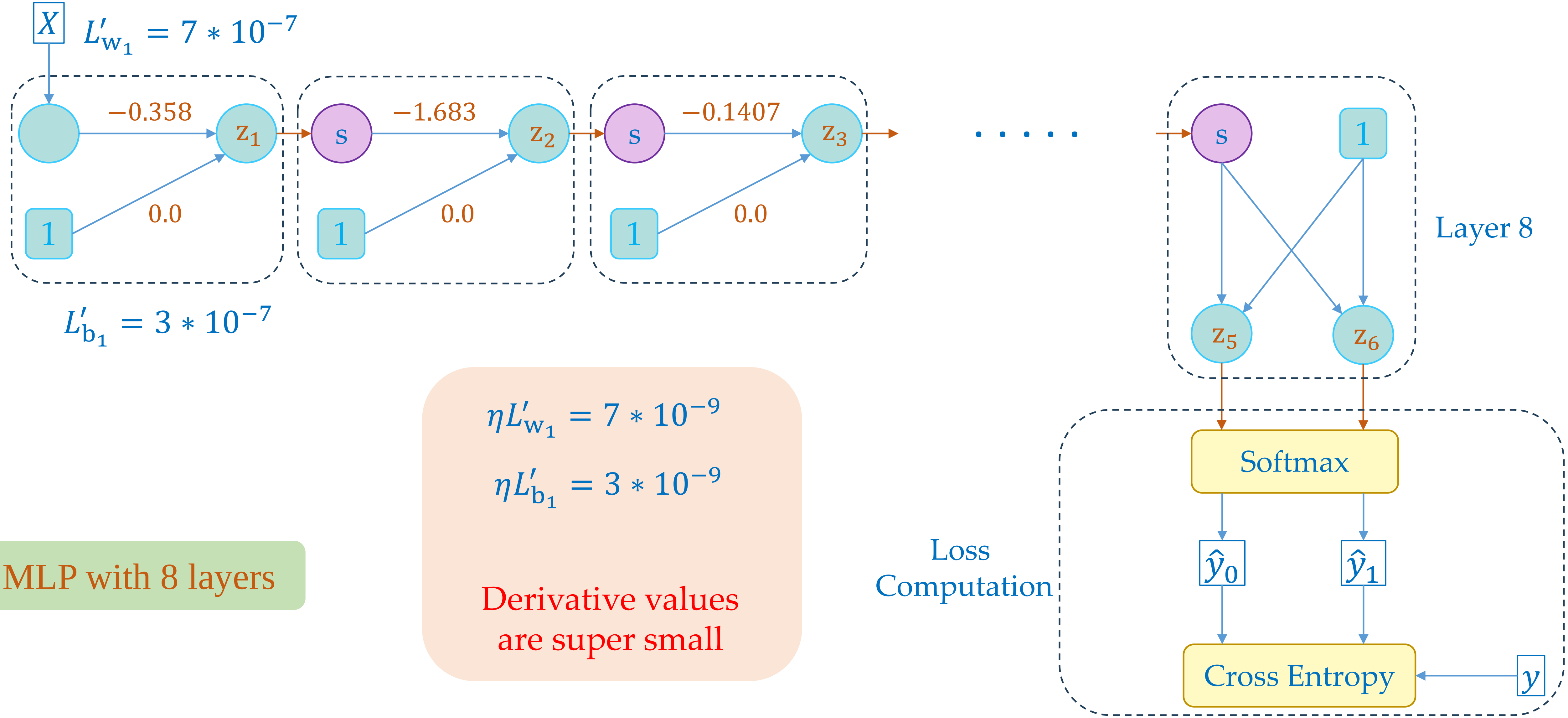
$$b_1 = b_1 - \eta L'_{b_1} = 9 * 10^{-6}$$

Gradient Vanishing



MLP with 8 layers

Gradient Vanishing



$\eta L'_{w_1} = 7 * 10^{-9}$

$\eta L'_{b_1} = 3 * 10^{-9}$

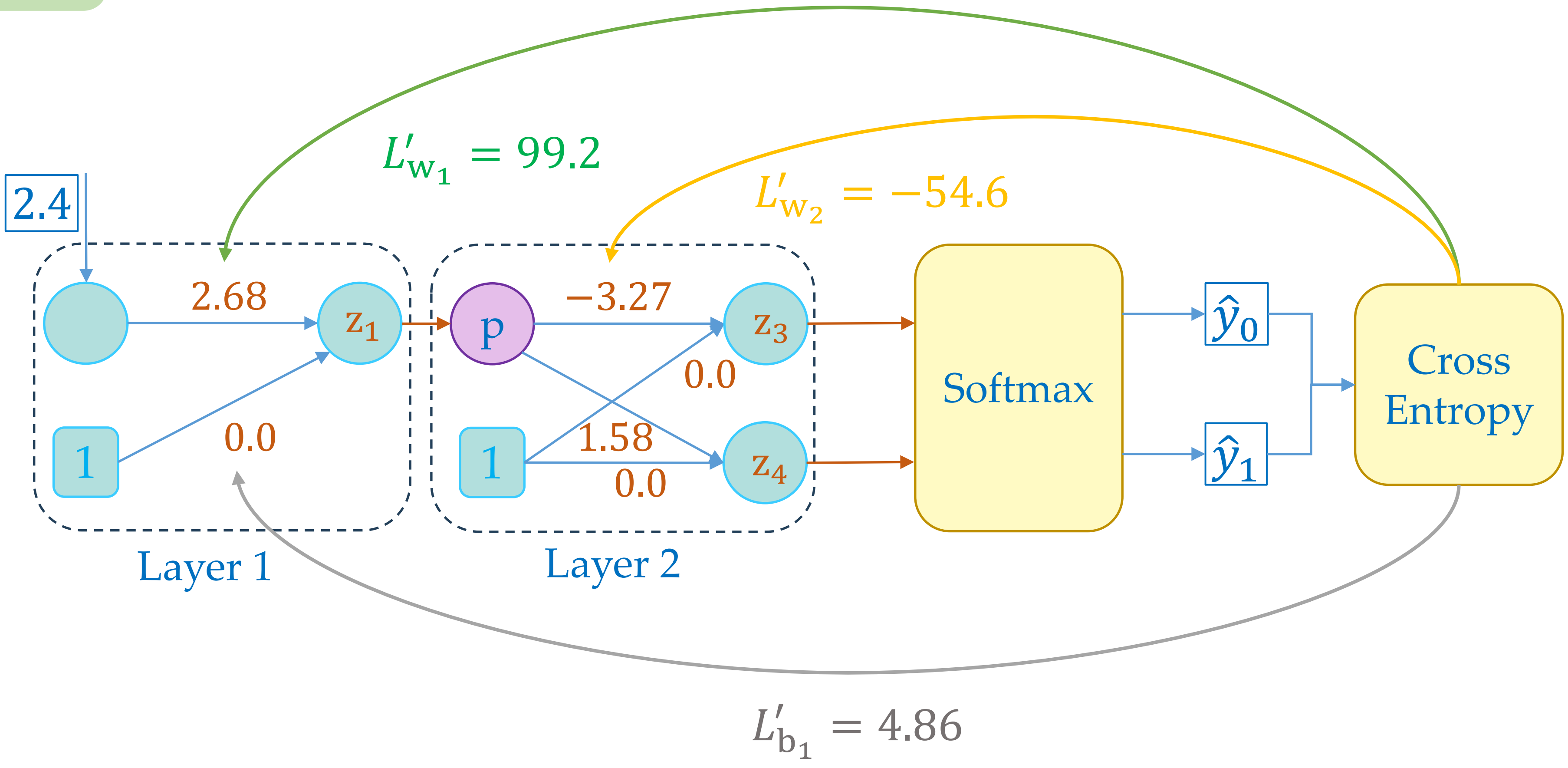
Derivative values are super small

MLP with 8 layers

Gradient Explosion

Large weight initialization and large learning rate

s PReLU function



with $\eta = 10$

$\eta L'_{w_1} = 99$

$\eta L'_{b_1} = 48.6$

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Mean

Data

$$X = \{X_1, \dots, X_N\}$$

Formula

$$E(X) = \sum_{i=1}^{N} X_i P_X(X_i)$$

Given the data

$$X = \{2, 8, 5, 4, 1, 4\}$$

$$N = 6$$

$$P_X(X = 2) = \frac{1}{6} \qquad P_X(X = 4) = \frac{2}{6}$$

$$P_X(X = 8) = \frac{1}{6} \qquad P_X(X = 1) = \frac{1}{6}$$

$$P_X(X = 5) = \frac{1}{6}$$

$$E(X) = 2 \times \frac{1}{6} + 8 \times \frac{1}{6} + 5 \times \frac{1}{6} + 4 \times \frac{2}{6} + 1 \times \frac{1}{6}$$

$$= \frac{2}{6} + \frac{8}{6} + \frac{5}{6} + \frac{8}{6} + \frac{1}{6} = 4$$

Mean

Data

$$X = \{X_1, \dots, X_N\}$$

Formula

$$E(X) = \sum_{i=1}^{N} X_i P_X(X_i)$$

$$E(XY) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i Y_j P(X_i, Y_j)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} X_i Y_j P(X_i) P(Y_j)$$

$$= \sum_{i=1}^{N} X_i P(X_i) \sum_{j=1}^{N} Y_j P(Y_j)$$

$$= E(X)E(Y)$$

Variance

Formula		Example: $X = \{5, 3\ 6, 7, 4\}$
mean	$E(X) = \sum_{i=1}^{N} X_i P_X(X_i)$	$E(X) = 5 \times \frac{1}{5} + 3 \times \frac{1}{5} + 6 \times \frac{1}{5} + 7 \times \frac{1}{5} + 4 \times \frac{1}{5} = 5$
variance	$var(X) = E\left(\left(X - E(X)\right)^2\right) = \sum_{i=1}^{N} \left(X_i - E(X)\right)^2 P_X(X_i)$	$var(X) = \frac{1}{5}[(5-5)^2 + (3-5)^2 + (6-5)^2 + (7-5)^2 + (4-5)^2] = \frac{1}{5}(0+4+1+4+1)=2$
Standard deviation	$\sigma = \sqrt{var(X)}$	$\sigma = \sqrt{var(X)} = 1.41$

Variance

Formula

mean

$$E(X) = \sum_{i=1}^{N} X_i P_X(X_i)$$

variance

$$var(X) = E\left(\left(X - E(X)\right)^2\right)$$
$$= \sum_{i=1}^{N} \left(X_i - E(X)\right)^2 P_X(X_i)$$

Standard deviation

$$\sigma = \sqrt{var(X)}$$

$$var(X) = \sum_{i=1}^{N} \left(X_i - E(X)\right)^2 P_X(X_i)$$
$$= \sum_{i=1}^{N} \left(X_i^2 - 2X_i E(X) + E(X)^2\right) P_X(X_i)$$
$$= \sum_{i=1}^{N} X_i^2 P_X(X_i) - \sum_{i=1}^{N} 2X_i E(X) P_X(X_i)$$
$$+ \sum_{i=1}^{N} E(X)^2 P_X(X_i)$$
$$= E(X^2) - 2E(X)\left[\sum_{i=1}^{N} X_i P_X(X_i)\right] + E(X)^2$$
$$= E(X^2) - \left(E(X)\right)^2$$

Variance

$$var(X) = E(X^2) - \big(E(X)\big)^2$$

$$var(XY) = E(X^2Y^2) - \big(E(XY)\big)^2$$

$$= E(X^2)E(Y^2) - \big(E(X)E(Y)\big)^2$$

$$= \left[var(X) + \big(E(X)\big)^2\right]\left[var(Y) + \big(E(Y)\big)^2\right] - \big(E(X)E(Y)\big)^2$$

$$= var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(Y)\big(E(Y)\big)^2$$

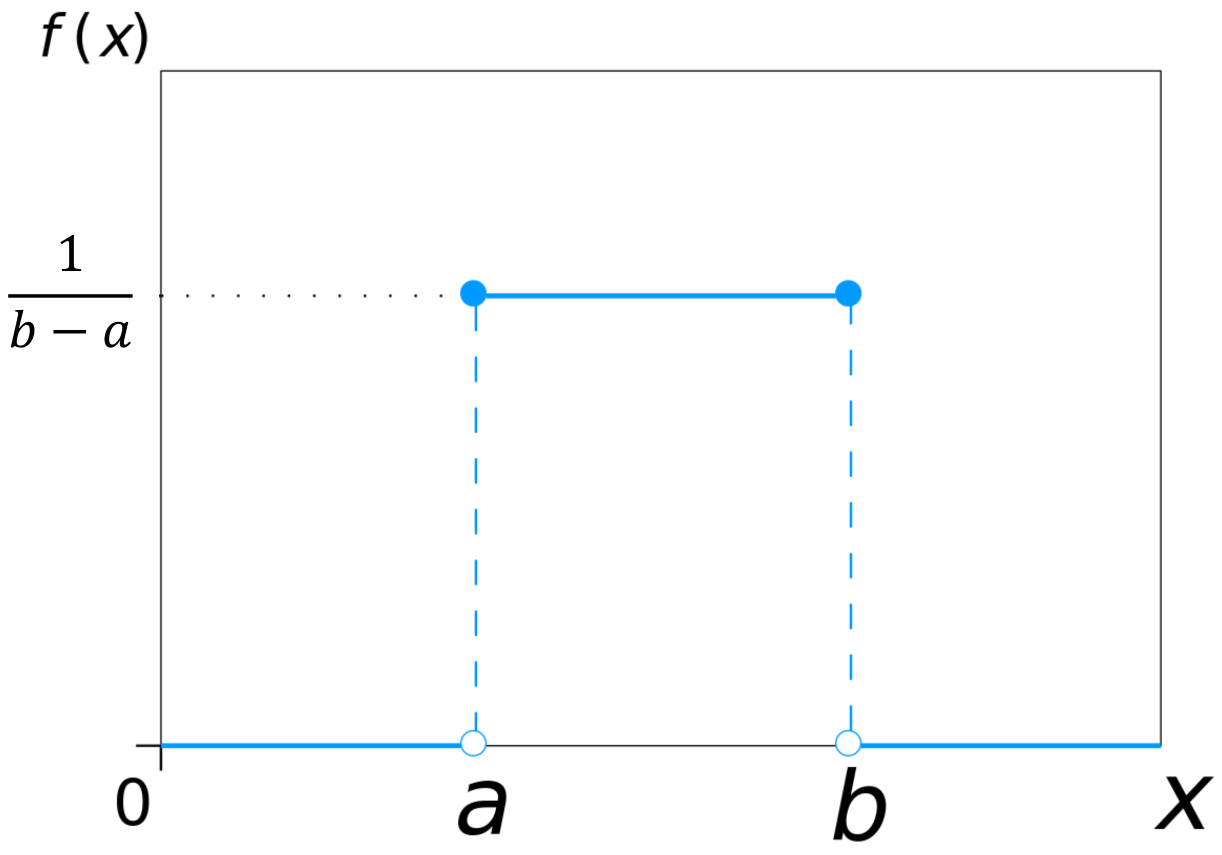
Initialization Methods

Xavier Initialization

Uniform Distribution

$$X \sim U(a, b) \qquad E[X] = \frac{a+b}{2}$$

$$f(x) = \frac{1}{b-a} \qquad var[X] = \frac{(b-a)^2}{12}$$



Initialization Methods

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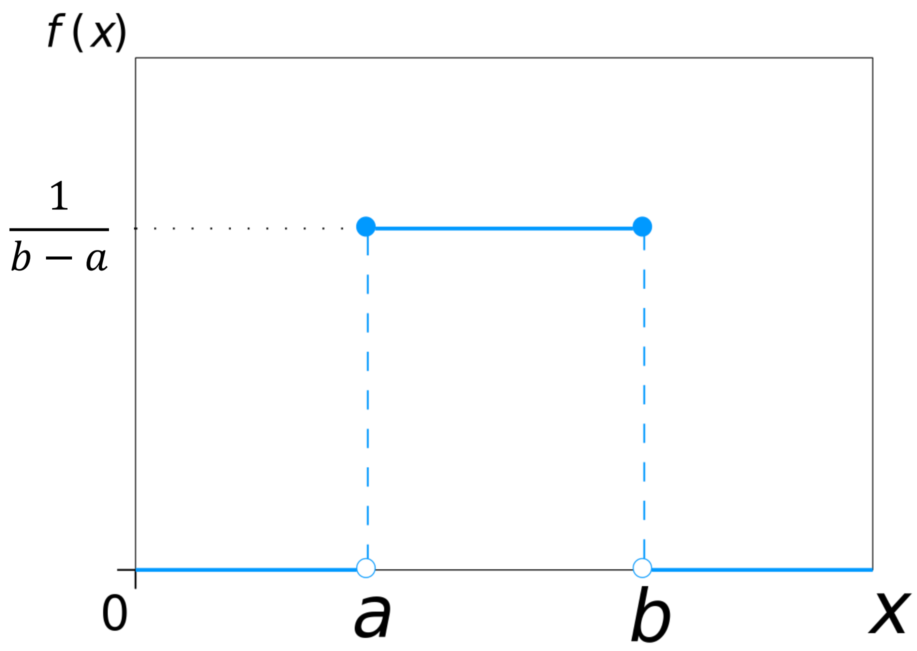

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{x^2}{2(b-a)} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

Initialization Methods

Uniform Distribution

$$X \sim U(a, b) \qquad E[X] = \frac{a+b}{2}$$

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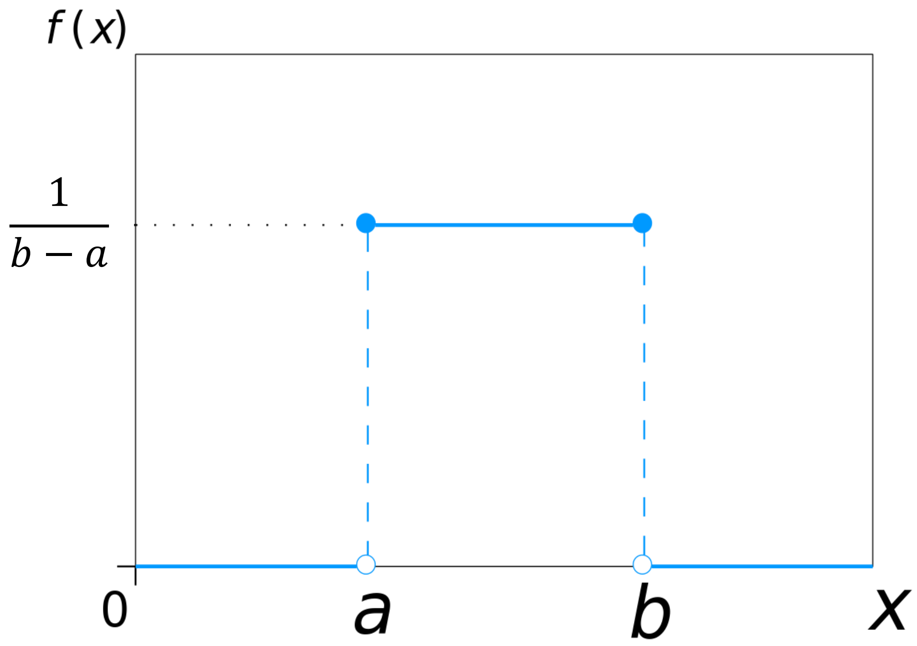


$$var[X] = E\left(\left(X - E(X)\right)^2\right) = \int_{-\infty}^{\infty} \left(x - E(X)\right)^2 f(x)dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\int_a^b x^2 dx - \int_a^b 2x\frac{a+b}{2}dx\right] + \int_a^b \left(\frac{a+b}{2}\right)^2 dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\Big|_a^b - \frac{x^2(a+b)}{2}\Big|_a^b + \left(\frac{a+b}{2}\right)^2 x\Big|_a^b\right]$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3} - \frac{(b^2-a^2)(a+b)}{2} + \left(\frac{a+b}{2}\right)^2 (b-a)\right]$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{2} + \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4(a^2+ab+b^2) - 3(a^2+2ab+b^2)}{12} = \frac{(b-a)^2}{12}$$

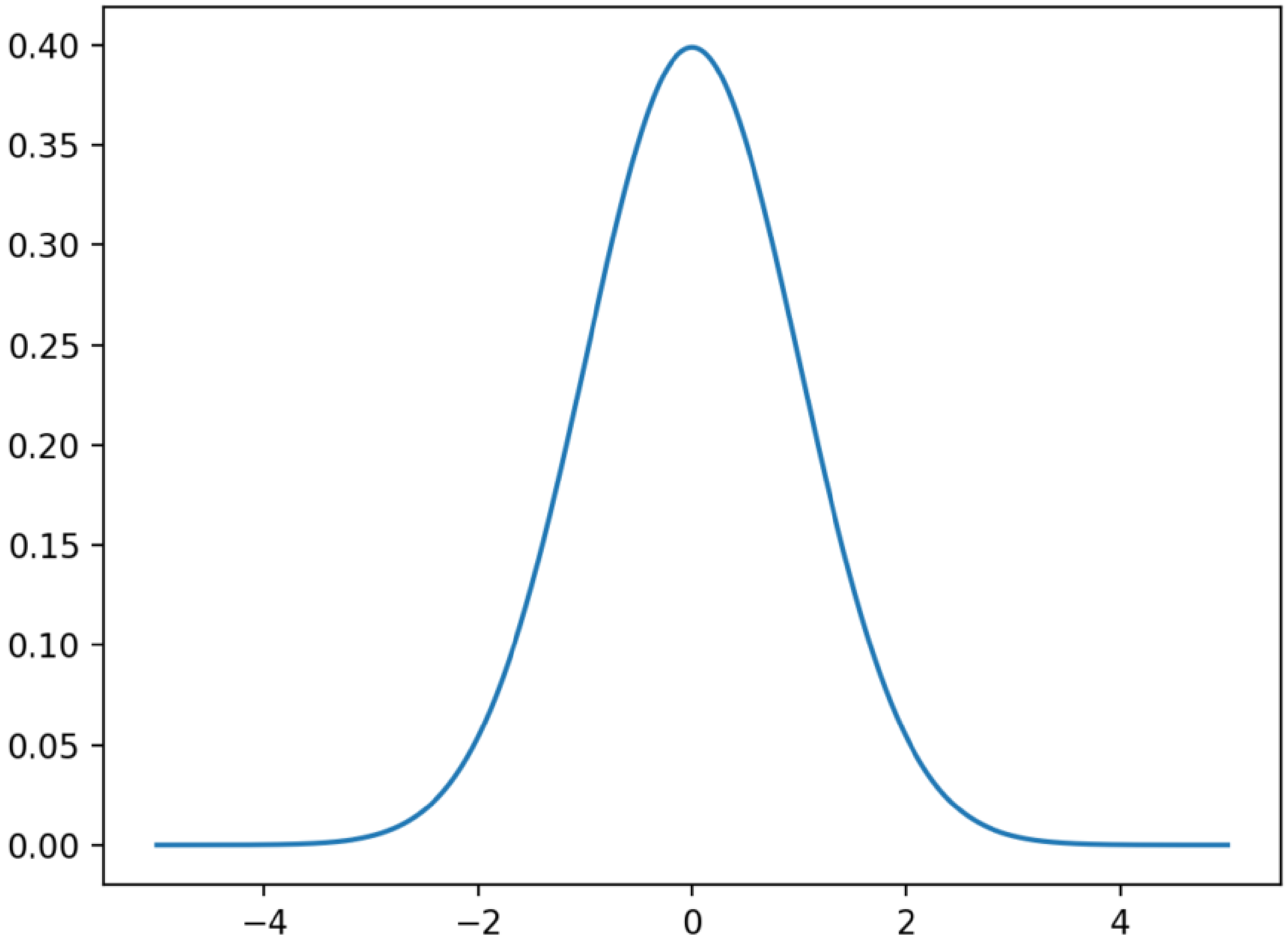
Initialization Methods

Xavier Initialization

Gaussian Distribution

$$X \sim \boldsymbol{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

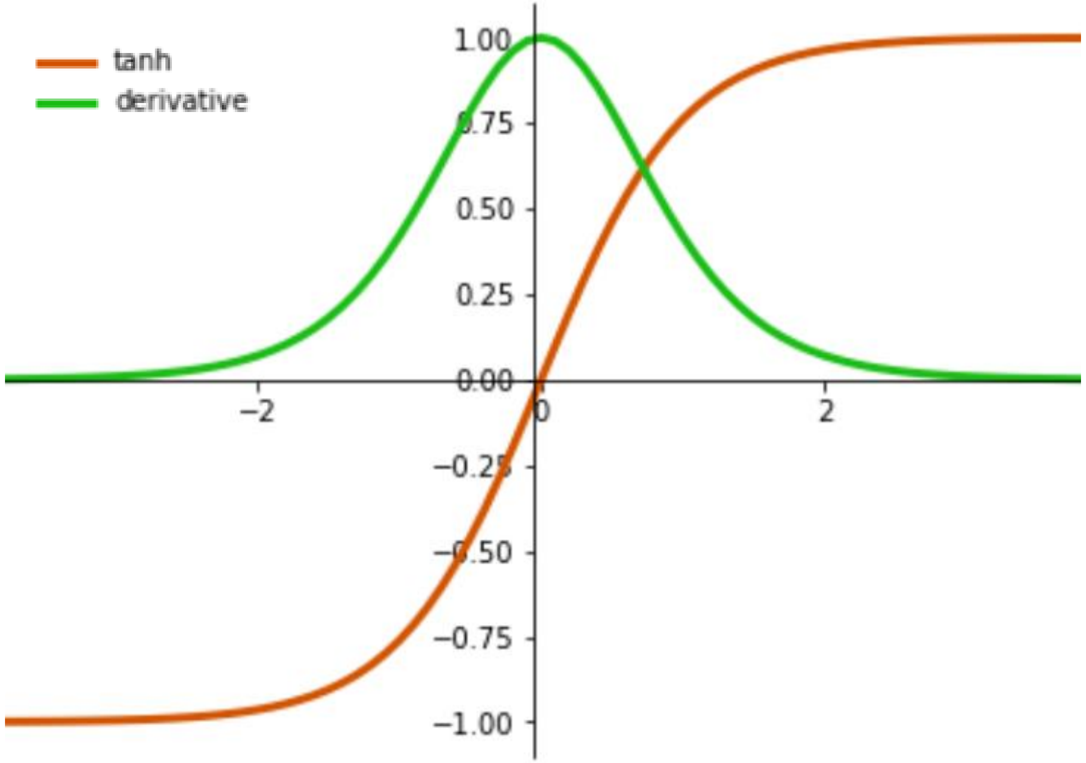


Maclaurin series

Tính giá trị xấp xỉ hàm f(x) cho những giá trị $x \approx 0$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$



$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{e^{2x}+1} = \frac{2}{e^{-2x}+1} - 1$$

$$\tanh(0) = 0$$

$$\tanh'(0) = 1 - tanh^2(0) = 1$$

$$\tanh''(0) = \left(1 - tanh^2(0)\right)'$$

$$= -2tanh(0)\tanh'(0) = 0$$

$$\tanh^{(3)}(0) = \left(-2tanh(0)\tanh'(0)\right)'$$

$$= -2[\tanh'(0)\tanh'(0) + \tanh''(0)tanh(0)] = -2$$

$$\tanh(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \quad \tanh(x) \approx x$$

Maclaurin series

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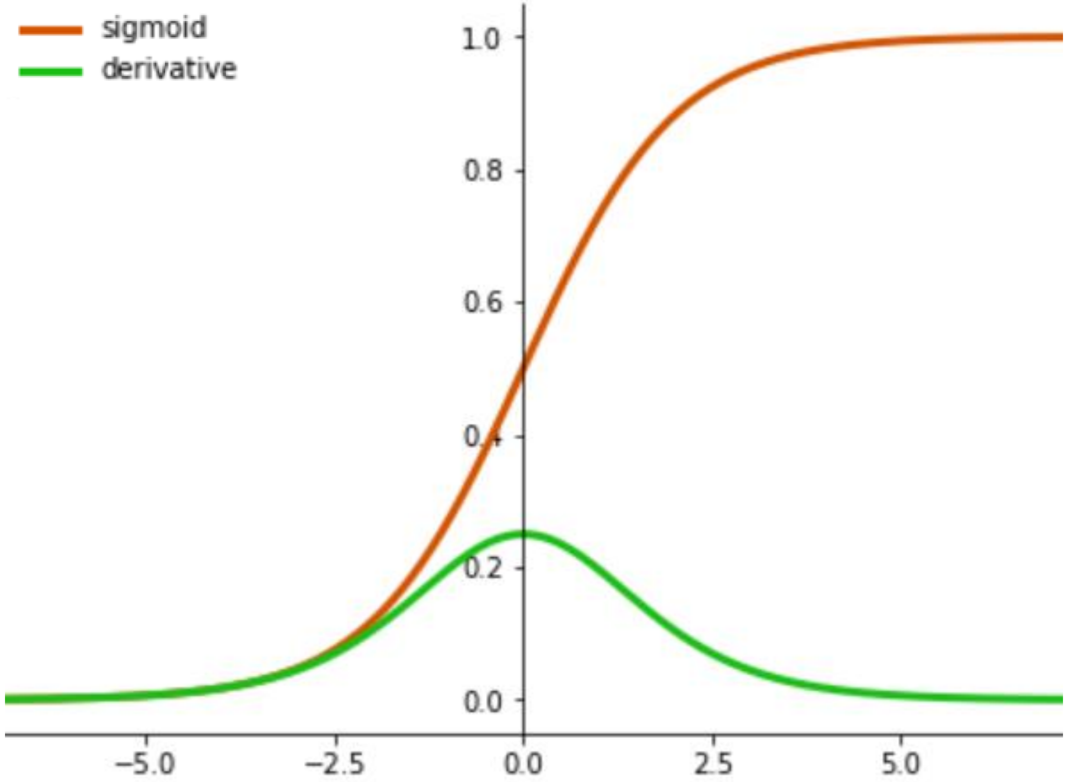


$$\text{sigmoid}(x) = \frac{1}{1+e^{-x}}$$

$$\text{sigmoid}(0) = \frac{1}{2}$$

$$\text{sigmoid}'(0) = \text{sigmoid}(0)\,(1-\text{sigmoid}(0)) = \frac{1}{4}$$

$$\text{sigmoid}''(0) = [\text{sigmoid}(0)\,(1-\text{sigmoid}(0))]'$$

$$= \text{sigmoid}'(0) - 2\,\text{sigmoid}(0)\text{sigmoid}'(0) = 0$$

$$\text{sigmoid}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$= \frac{1}{2} + \frac{x}{4} + \cdots$$

$$\Rightarrow \quad \text{sigmoid}(x) \approx \frac{1}{2} + \frac{x}{4}$$

Initialization Methods

Xavier Initialization

$$E(XY) = E(X)E(Y)$$

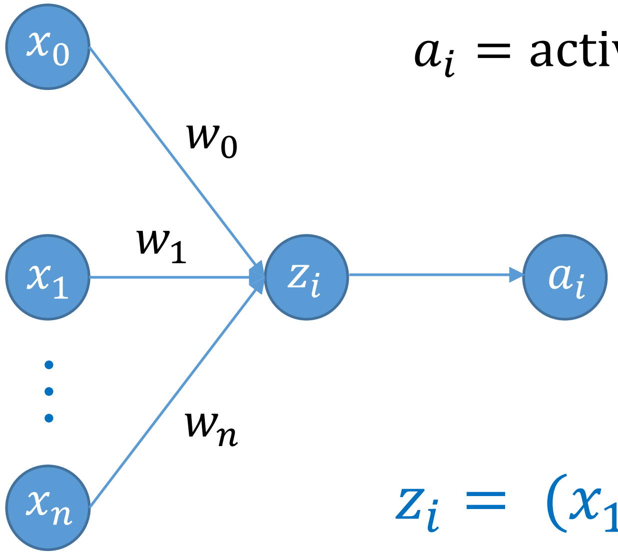
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$a_i = \text{activation}(z_i)$$

$$E(X) = 0$$

$$E(W) = 0$$

$$b = 0$$

$$z_i = (x_1 w_1 + \cdots + x_n w_n + b)$$

$$\text{var}(z_i) = \text{var}(x_1 w_1 + \cdots + x_n w_n + b)$$

$$= n\text{var}(x_i w_i) = n\text{var}(x_i)\text{var}(w_i)$$

$$\text{activation} = \tanh \;\Rightarrow\; a_i = \tanh(z_i) \approx z_i \;\Rightarrow\; \text{var}(a_i) = \text{var}(z_i)$$

$$\text{var(X)} \approx \text{var}(\mathbf{a}) \xrightarrow{\text{iid}} \text{var}(x_i) \approx \text{var}(a_i) \;\Rightarrow\; n\text{var}(w_i) \approx 1$$

$$\Rightarrow\; \text{var}(w_i) \approx \frac{1}{n}$$

Initialization Methods

Xavier Initialization

activation = tanh

$$E(XY) = E(X)E(Y)$$

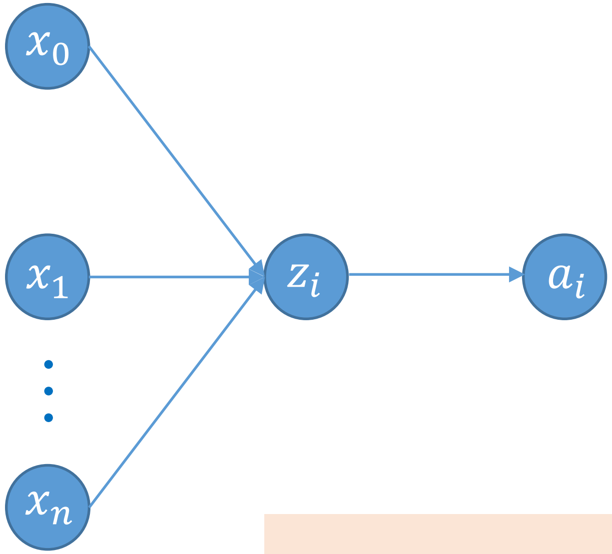
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$\mathrm{var}(w_i) \approx \frac{1}{n}$$

$$w_i \sim U(-r, r)$$

$$var[w_i] = \frac{r^2}{3}$$

$$W_i \sim U\left(-\frac{\sqrt{3}}{\sqrt{n}}, \frac{\sqrt{3}}{\sqrt{n}}\right)$$

Initialization Methods

Xavier Initialization

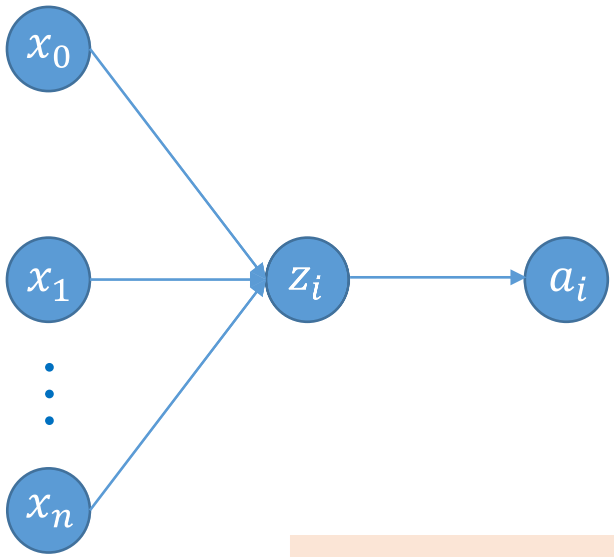
$$\text{activation} = \tanh$$

$$E(XY) = E(X)E(Y)$$

$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Gaussian Distribution

$$X \sim N(0, \sigma^2)$$



$$\text{var}(w_i) \approx \frac{1}{n}$$

$$w_i \sim N(0, \sigma^2)$$

$$\sigma^2 = \frac{1}{n} \Rightarrow \sigma = \frac{1}{\sqrt{n}}$$

$$W_i \sim N\left(0, \frac{1}{n}\right)$$

Initialization Methods

Xavier Initialization

activation = tanh

Uniform Distribution

$$W_{ij} \sim U\left(-\frac{\sqrt{3}}{\sqrt{n}}, \frac{\sqrt{3}}{\sqrt{n}}\right)$$

Gaussian Distribution

$$W_{ij} \sim \boldsymbol{N}\left(0, \frac{1}{n}\right)$$

Initialization Methods

Xavier Initialization

$$E(XY) = E(X)E(Y)$$

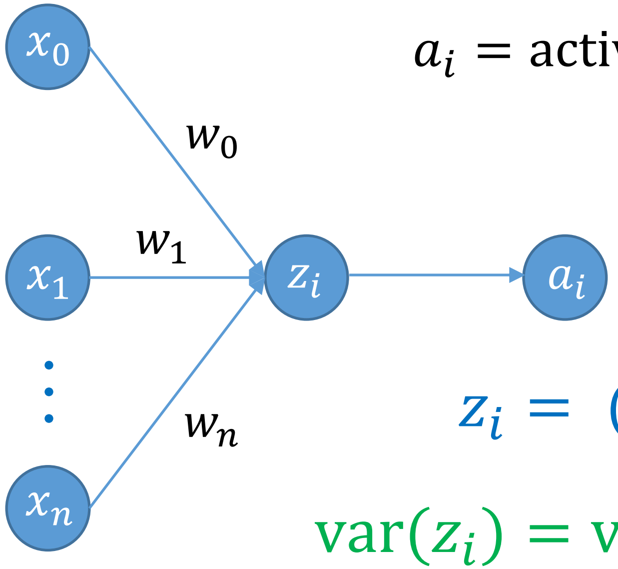
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$a_i = \text{activation}(z_i)$$

$$E(X) = 0$$

$$E(W) = 0$$

$$b = 0$$

$$z_i = (x_1 w_1 + \cdots + x_n w_n + b)$$

$$\text{var}(z_i) = \text{var}(x_1 w_1 + \cdots + x_n w_n + b)$$

$$= n\text{var}(x_i w_i) = n\text{var}(x_i)\text{var}(w_i)$$

$$\text{activation} = \text{sigmoid} \Rightarrow a_i = \text{sigmoid}(z_i) \approx \frac{1}{2} + \frac{z_i}{4}$$

$$\Rightarrow 16\text{var}(a_i) = \text{var}(z_i)$$

$$\text{var(X)} \approx \text{var}(\mathbf{a}) \xrightarrow{\text{iid}} \text{var}(x_i) \approx \text{var}(a_i) \Rightarrow n\text{var}(w_i) \approx 16$$

$$\Rightarrow \text{var}(w_i) \approx \frac{16}{n}$$

Initialization Methods

Xavier Initialization

activation = sigmoid

$$E(XY) = E(X)E(Y)$$

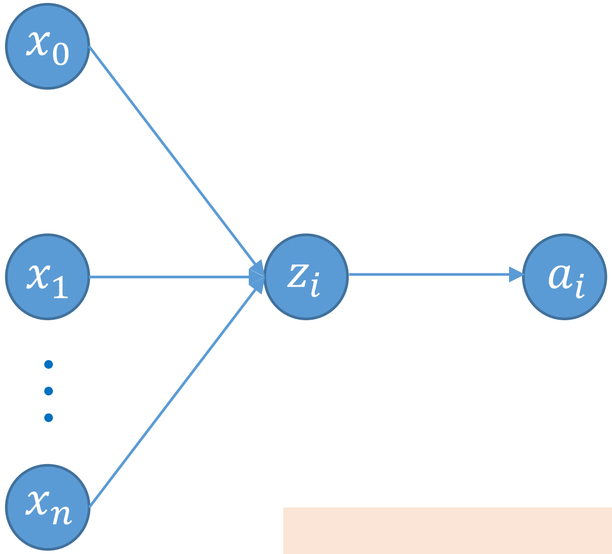
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$\mathrm{var}(w_i) \approx \frac{16}{n}$$

$$w_i \sim U(-r, r)$$

$$var[w_i] = \frac{r^2}{3}$$

$$W_i \sim U\left(-\frac{4\sqrt{3}}{\sqrt{n}}, \frac{4\sqrt{3}}{\sqrt{n}}\right)$$

Initialization Methods

Xavier Initialization

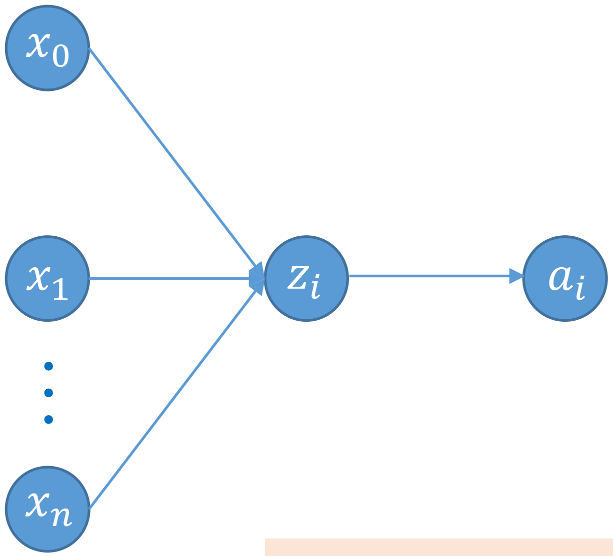
activation = sigmoid

$$E(XY) = E(X)E(Y)$$

$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Gaussian Distribution

$$X \sim N(0, \sigma^2)$$



$$\text{var}(w_i) \approx \frac{16}{n}$$

$$w_i \sim N(0, \sigma^2)$$

$$\sigma^2 = \frac{1}{n}$$

$$W_i \sim N\left(0, \frac{16}{n}\right)$$

Initialization Methods

Kaiming He Initialization

$$E(XY) = E(X)E(Y)$$

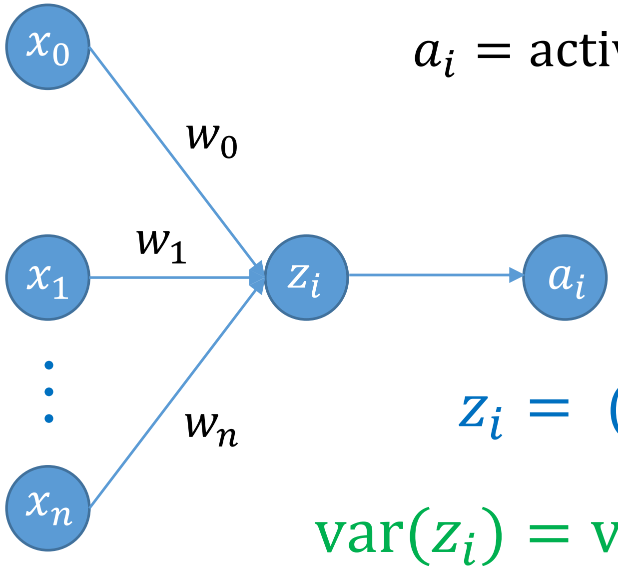
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$a_i = \text{activation}(z_i)$$

$$E(X) = 0$$
$$E(W) = 0$$
$$b = 0$$

$$z_i = (x_1 w_1 + \cdots + x_n w_n + b)$$

$$\text{var}(z_i) = \text{var}(x_1 w_1 + \cdots + x_n w_n + b)$$
$$= n\text{var}(x_i w_i) = n\text{var}(x_i)\text{var}(w_i)$$

activation = relu $\Rightarrow$ $a_i = max(0, z_i)$

$\Rightarrow$ $2\text{var}(a_i) = \text{var}(z_i)$

$\text{var(X)} \approx \text{var}(\mathbf{a})$ $\xrightarrow{\text{iid}}$ $\text{var}(x_i) \approx \text{var}(a_i)$ $\Rightarrow$ $n\text{var}(w_i) \approx 2$

$\Rightarrow$ $\text{var}(w_i) \approx \dfrac{2}{n}$

Initialization Methods

He Initialization

$\text{activation} = \text{he}$

$$E(XY) = E(X)E(Y)$$

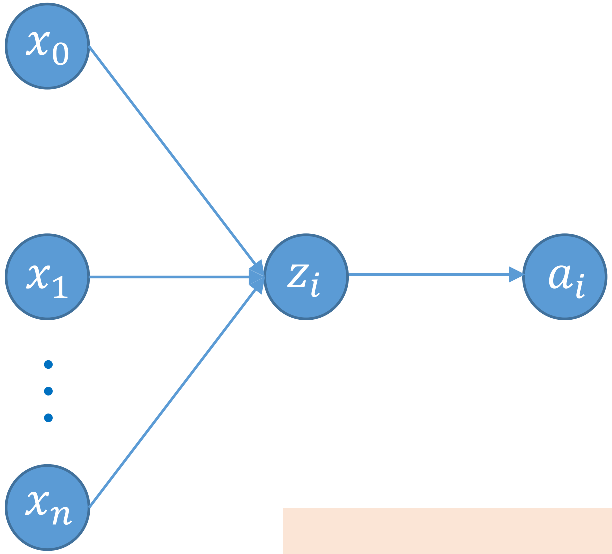
$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Uniform Distribution

$$X \sim U(a, b)$$

$$f(x) = \frac{1}{b-a}$$

$$var[X] = \frac{(b-a)^2}{12}$$



$$\text{var}(w_i) \approx \frac{2}{n}$$

$$w_i \sim U(-r, r)$$

$$var[w_i] = \frac{r^2}{3}$$

$$W_i \sim U\left(-\frac{\sqrt{6}}{\sqrt{n}}, \frac{\sqrt{6}}{\sqrt{n}}\right)$$

Initialization Methods

He Initialization

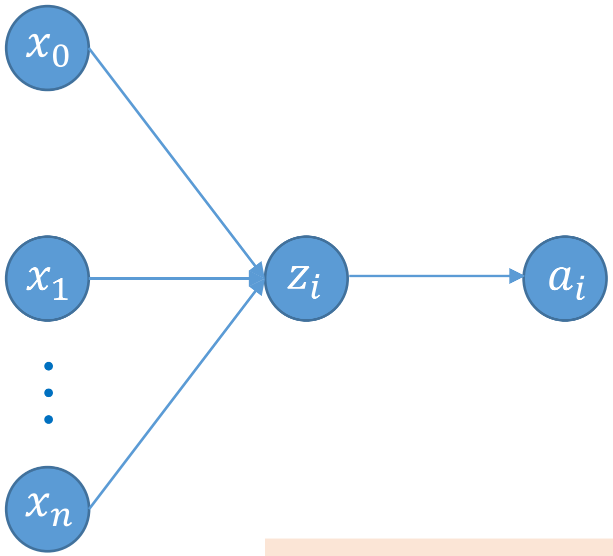
activation = he

$$E(XY) = E(X)E(Y)$$

$$var(XY) = var(X)var(Y) + var(X)\big(E(Y)\big)^2 + var(y)\big(E(X)\big)^2$$

Gaussian Distribution

$$X \sim N(0, \sigma^2)$$



$$\text{var}(w_i) \approx \frac{2}{n}$$

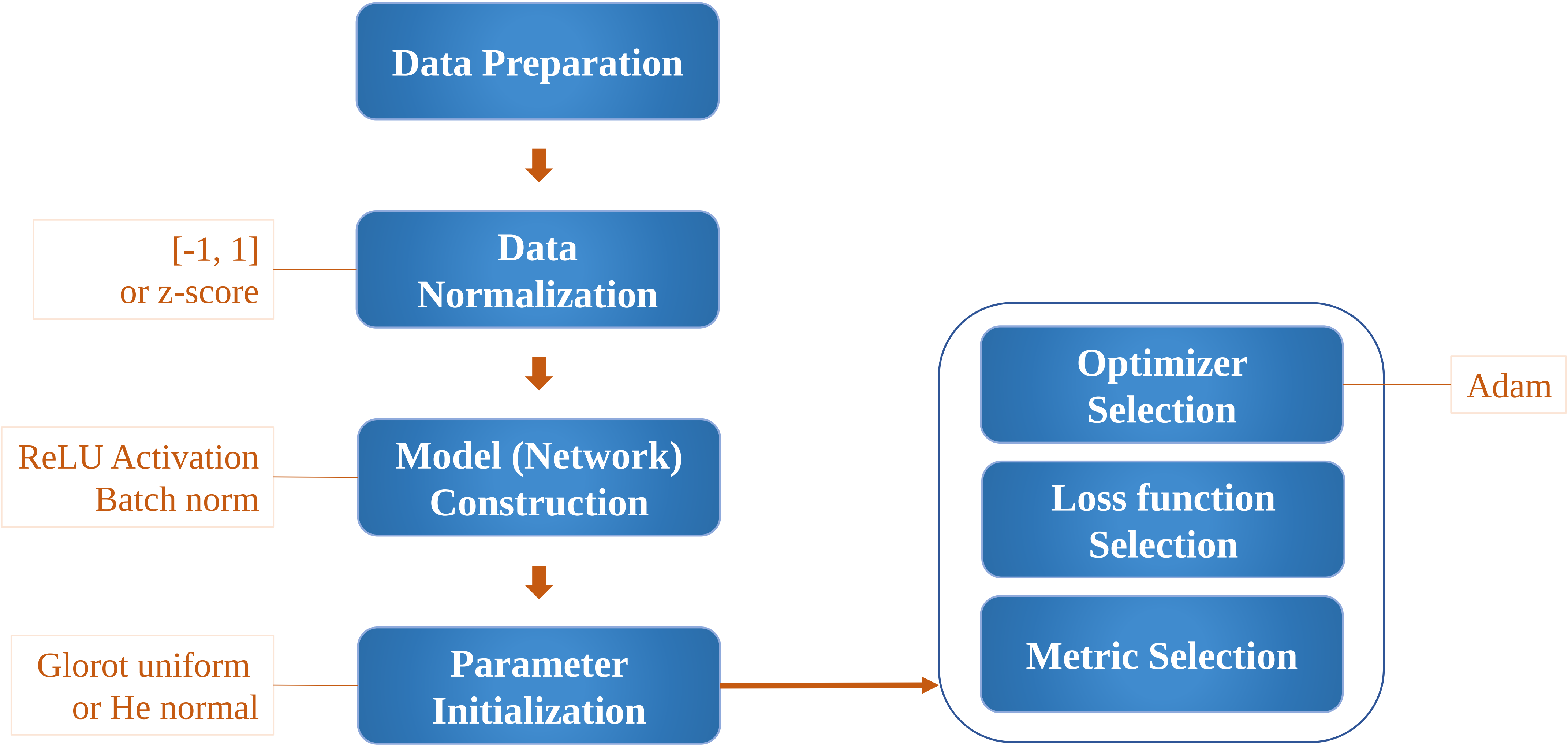
$$w_i \sim N(0, \sigma^2)$$

$$\sigma^2 = \frac{1}{n}$$

$$W_i \sim N\left(0, \frac{2}{n}\right)$$

Summary

Recommendation



Further Reading

Dying ReLU

https://towardsdatascience.com/the-dying-relu-problem-clearly-explained-42d0c54e0d24

Initialization

https://www.deeplearning.ai/ai-notes/initialization/index.html