Water Jet Impact on Different Geometries: Theoretical Analysis

1 Problem Definition and Theoretical Framework

We are tasked with determining the theoretical expressions for the force exerted by a water jet on three surfaces: a flat surface, an inclined surface at 30°, and a hemispherical surface. The force is derived using the principles of conservation of mass (continuity equation) and conservation of momentum (momentum equation). The common parameters for all cases are:

- Flow rate: Q
- Jet exit area: $A = \frac{\pi d^2}{4}$
- Jet velocity: $v = \frac{Q}{A}$
- Water density: ρ

2 Case (a): Flat Surface

2.1 Control Volume and Momentum Analysis

The water jet impacts the flat surface vertically and exits radially. The incoming vertical velocity is $v_{\text{in}} = -v$, and the outgoing velocity is horizontal, with no vertical component.

2.2 Momentum Conservation in the Vertical Direction

The momentum change in the vertical direction is:

$$\Delta(\text{Momentum}_y) = \dot{m}(v_{\text{out}_y} - v_{\text{in}_y}) = \rho Q(0 - (-v)) = \rho Q v$$

The force exerted by the jet on the surface is the change in momentum:

$$F = \rho Q v = \rho Q \left(\frac{Q}{A}\right) = \frac{\rho Q^2}{A}$$

Thus, the force on the flat surface is:

$$F = \frac{\rho Q^2}{A}$$

3 Case (b): Inclined Surface at 30°

3.1 Control Volume and Momentum Analysis

In this case, the water jet impacts the surface at an angle of 30°. The incoming vertical velocity is $v_{\text{in}} = -v$, and the outgoing velocity has both horizontal and vertical components:

$$v_{\text{out}_x} = v \cos\theta \quad \text{and} \quad v_{\text{out}_y} = v \sin\theta$$

where $\theta = 30^\circ$.

3.2 Momentum Conservation in the Vertical Direction

The momentum change in the vertical direction is:

$$F_y = \rho Q(v_{\text{out}_y} - v_{\text{in}_y}) = \rho Q(v \sin\theta - (-v)) = \rho Q v(1 + \sin\theta)$$

Thus, the force on the plate is:

$$F = \rho Q v(1 + \sin\theta) = \rho Q \left(\frac{Q}{A}\right)(1 + \sin\theta) = \frac{\rho Q^2}{A}(1 + \sin\theta)$$

For $\theta = 30°$, $\sin\theta = 0.5$, so:

$$F = \frac{\rho Q^2}{A}(1 + 0.5) = \frac{1.5\rho Q^2}{A}$$

4 Case (c): Hemispherical Surface

4.1 Control Volume and Momentum Analysis

For the hemispherical surface, the water jet is redirected directly upward, resulting in a complete reversal of velocity. The incoming vertical velocity is $v_{\text{in}} = -v$, and the outgoing velocity is $v_{\text{out}} = v$ in the upward direction.

4.2 Momentum Conservation in the Vertical Direction

The momentum change in the vertical direction is:

$$\Delta(\text{Momentum}_y) = \rho Q(v_{\text{out}_y} - v_{\text{in}_y}) = \rho Q(v - (-v)) = 2\rho Q v$$

Thus, the force on the plate is:

$$F = 2\rho Q v = 2\rho Q \left(\frac{Q}{A}\right) = \frac{2\rho Q^2}{A}$$

5 Inclusion of Atmospheric Pressure

The atmospheric pressure acts on both the incoming jet and the outgoing flow. However, since the pressures at the inlet and outlet are equal and atmospheric, their effects cancel out in the momentum equation. For the dry face of the obstacle, atmospheric pressure acts but does not contribute to the net force on the plate because it is balanced by the atmospheric pressure on the control volume boundaries.

6 Summary of Forces

The theoretical forces for each surface geometry are summarized as follows:

- **Flat Surface**: $F = \frac{\rho Q^2}{A}$
- **Inclined Surface (30°)**: $F = \frac{1.5\rho Q^2}{A}$
- **Hemispherical Surface**: $F = \frac{2\rho Q^2}{A}$

7 Introduction

This report presents the calculation of the coefficient C_d for three different obstacles subjected to a water jet: flat, inclined at 30°, and hemispherical surfaces. The coefficient C_d accounts for discrepancies between theoretical predictions and experimental measurements of the force exerted by the jet on the obstacles.

8 Theoretical Background

The coefficient C_d is calculated using:

$$C_d = \frac{F}{\frac{1}{2}\rho v^2 A} \tag{1}$$

where:

- F is the experimental force ($F = m \cdot g$).
- ρ is the density of water ($1000\,\mathrm{kg\,m^{-3}}$).
- v is the velocity of the jet ($v = \frac{Q}{A}$).
- A is the cross-sectional area of the jet ($A = \frac{\pi d^2}{4}$).
- m is the mass required to balance the jet force.
- g is the gravitational acceleration ($9.81\,\mathrm{m\,s^{-2}}$).

9 Experimental Data

The experimental data for each obstacle are provided in Tables 1, 2, and 3.

9.1 Flat Surface

Table 1: Experimental Data for Flat Surface

Trial	Mass m (g)	Q ($\mathrm{L\,h^{-1}}$)
1	50	900
2	100	1100
3	150	1200
4	200	1300
5	250	1450
6	300	1550
7	350	1620
8	400	1700
9	450	1800
10	500	1900

9.2 Inclined Surface

Table 2: Experimental Data for Inclined Surface

Trial	Mass m (g)	Q ($\mathrm{L\,h^{-1}}$)
1	50	800
2	100	925
3	150	1075
4	200	1200
5	250	1300
6	300	1400
7	350	1500
8	400	1535
9	450	1650
10	500	1700

9.3 Hemispherical Surface

Table 3: Experimental Data for Hemispherical Surface

Trial	Mass m (g)	Q ($\mathrm{L\,h^{-1}}$)
1	50	650
2	100	775
3	150	900
4	200	950
5	250	1000
6	300	1100
7	350	1200
8	400	1275
9	450	1350
10	500	1400

10 Calculation of C_d

10.1 Common Parameters

The cross-sectional area of the jet is:

$$A = \frac{\pi d^2}{4} = \frac{\pi(0.008\,\mathrm{m})^2}{4} = 5.0265 \times 10^{-5}\,\mathrm{m}^2 \tag{2}$$

10.2 Methodology

For each trial, we perform the following steps:

1. Convert mass from grams to kilograms: $m_{\mathrm{kg}} = \dfrac{m}{1000}$.
2. Convert flow rate from liters per hour to cubic meters per second: $Q_{\mathrm{m^3/s}} = \dfrac{Q}{3,600,000}$.
3. Calculate the experimental force: $F = m_{\mathrm{kg}} \cdot g$.
4. Calculate the jet velocity: $v = \dfrac{Q_{\mathrm{m^3/s}}}{A}$.

5. Calculate the dynamic pressure force: $\frac{1}{2}\rho v^2 A$.

6. Compute C_d: $C_d = \frac{F}{\frac{1}{2}\rho v^2 A}$.

10.3 Sample Calculations

Below are sample calculations for Trial 1 of each obstacle.

10.3.1 Flat Surface - Trial 1

1. $m_{\text{kg}} = \frac{50}{1000} = 0.05\,\text{kg}$

2. $Q_{\text{m}^3/\text{s}} = \frac{900}{3,600,000} = 2.5 \times 10^{-4}\,\text{m}^3\,\text{s}^{-1}$

3. $F = 0.05\,\text{kg} \times 9.81\,\text{m}\,\text{s}^{-2} = 0.4905\,\text{N}$

4. $v = \frac{2.5 \times 10^{-4}}{5.0265 \times 10^{-5}} = 4.9749\,\text{m}\,\text{s}^{-1}$

5. $\frac{1}{2}\rho v^2 A = \frac{1}{2} \times 1000 \times (4.9749)^2 \times 5.0265 \times 10^{-5} = 0.6231\,\text{N}$

6. $C_d = \frac{0.4905}{0.6231} = 0.787$

10.3.2 Inclined Surface - Trial 1

1. $m_{\text{kg}} = 0.05\,\text{kg}$

2. $Q_{\text{m}^3/\text{s}} = \frac{800}{3,600,000} = 2.2222 \times 10^{-4}\,\text{m}^3\,\text{s}^{-1}$

3. $F = 0.4905\,\text{N}$

4. $v = \frac{2.2222 \times 10^{-4}}{5.0265 \times 10^{-5}} = 4.4244\,\text{m}\,\text{s}^{-1}$

5. $\frac{1}{2}\rho v^2 A = 0.4905\,\text{N}$

6. $C_d = \frac{0.4905}{0.4905} = 1.000$

10.3.3 Hemispherical Surface - Trial 1

1. $m_{\text{kg}} = 0.05\,\text{kg}$

2. $Q_{\text{m}^3/\text{s}} = \frac{650}{3,600,000} = 1.8056 \times 10^{-4}\,\text{m}^3\,\text{s}^{-1}$

3. $F = 0.4905\,\text{N}$

4. $v = \frac{1.8056 \times 10^{-4}}{5.0265 \times 10^{-5}} = 3.5947\,\text{m}\,\text{s}^{-1}$

5. $\frac{1}{2}\rho v^2 A = 0.3245\,\text{N}$

6. $C_d = \frac{0.4905}{0.3245} = 1.512$

10.4 Results Tables

The calculated values of C_d for each obstacle are summarized in Tables 4, 5, and 6.

10.4.1 Flat Surface

Table 4: Calculated C_d for Flat Surface

Trial	v ($\mathrm{m\,s^{-1}}$)	F (N)	$\frac{1}{2}\rho v^2 A$ (N)	C_d
1	4.9749	0.4905	0.6231	0.787
2	6.0805	0.9810	0.9271	1.058
3	6.6332	1.4715	1.1057	1.331
4	7.1859	1.9620	1.2986	1.511
5	8.0140	2.4525	1.6118	1.521
6	8.5667	2.9430	1.8436	1.596
7	8.9565	3.4335	2.0097	1.709
8	9.3963	3.9240	2.2011	1.782
9	9.9497	4.4145	2.4960	1.769
10	10.5024	4.9050	2.8071	1.748

10.4.2 Inclined Surface

Table 5: Calculated C_d for Inclined Surface

Trial	v ($\mathrm{m\,s^{-1}}$)	F (N)	$\frac{1}{2}\rho v^2 A$ (N)	C_d
1	4.4244	0.4905	0.4905	1.000
2	5.1130	0.9810	0.6541	1.499
3	5.9422	1.4715	0.8856	1.662
4	6.6332	1.9620	1.1057	1.774
5	7.1859	2.4525	1.2986	1.889
6	7.7386	2.9430	1.5080	1.951
7	8.2914	3.4335	1.7340	1.980
8	8.4741	3.9240	1.7992	2.181
9	9.1205	4.4145	2.0839	2.119
10	9.3963	4.9050	2.2011	2.229

10.4.3 Hemispherical Surface

Table 6: Calculated C_d for Hemispherical Surface

Trial	v ($\mathrm{m\,s^{-1}}$)	F (N)	$\frac{1}{2}\rho v^2 A$ (N)	C_d
1	3.5947	0.4905	0.3245	1.512
2	4.2844	0.9810	0.4618	2.124
3	4.9749	1.4715	0.6231	2.361
4	5.2512	1.9620	0.6890	2.848
5	5.5274	2.4525	0.7580	3.235
6	6.0805	2.9430	0.9271	3.174
7	6.6332	3.4335	1.1057	3.104
8	7.0155	3.9240	1.2415	3.160
9	7.4684	4.4145	1.3997	3.155
10	7.7386	4.9050	1.5080	3.252

11 Results and Discussion

The C_d values vary with the type of obstacle and the flow rate:

- **Flat Surface**: C_d ranges from approximately 0.79 to 1.75, indicating that the experimental force is slightly less than or comparable to the dynamic pressure force.
- **Inclined Surface**: C_d increases from 1.0 to around 2.23, reflecting the additional momentum change due to the inclination.
- **Hemispherical Surface**: C_d ranges from 1.51 to 3.25, showing a significant increase in force due to the shape causing the jet to spread in all directions.

The results demonstrate that the obstacle geometry significantly affects the coefficient C_d, with more complex shapes leading to higher values.

12 Conclusion

By calculating C_d using the equation $C_d = \frac{F}{\frac{1}{2}\rho v^2 A}$, we have quantified the effect of obstacle shape on the force exerted by a water jet. The hemispherical surface exhibits the highest C_d values due to its geometry causing maximum momentum change. These findings are valuable for engineering applications involving fluid impact on surfaces.