Dynamics Final Lab Project

Jay Parmar, Daniel Duarte, Justin Garcia

August 6, 2025

1 Introduction

In this experiment, wheels for a model car are designed with an understanding of key physical properties that optimize its efficiency in travelling down a gravity ramp. This report concludes that the physical properties that must optimized to reduce the race time include having more mass in the center of the wheel, larger wheel radius, larger vehicle weight, and larger wheel weight. Wheels were designed in CAD by adopting these criteria. They were then printed, tested for mass properties, and timed in a final race. Analysis was performed on the moments of inertia, energy conservation/time estimates, friction, equations of motion, and angular acceleration. A final comparison is made between the estimated and actual time of the race.

2 Preliminary Analysis

2.1 Point mass

To start our first preliminary analysis, a simple model will be used that assumes (1) that the car can be treated as a point mass of total mass M, (2) no kinetic friction of rotation, (3) no initial velocity. Conservation of energy will be used.

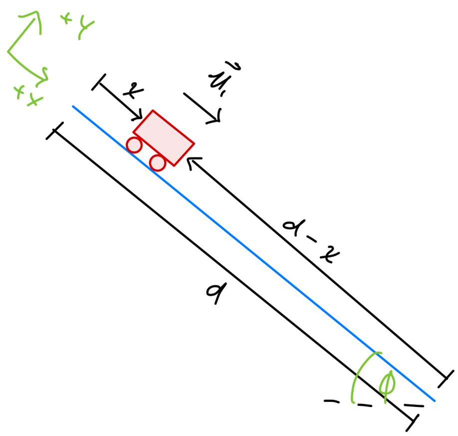

Figure 1: Simple Model Free Body Diagram

$$E(0) = Mgdsin(\phi)$$
$$E(t) = \frac{1}{2}M\dot{x}^2 + Mg(d-x)sin(\phi)$$

Using the statement for energy conservation $E(t) = E(0) \quad \forall \quad t > 0$ yields the following expression for the velocity of the car, $\dot{x}(t)$:

$$\dot{x}(t) = \sqrt{2gsin(\phi)x}$$

From this expression, the **total time the car needs to go down the ramp** would be:

$$\int_0^{t_r} dt = \frac{1}{\sqrt{2gsin(\phi)}} \int_0^d \frac{1}{\sqrt{x}}\, dx$$
$$t_r = \sqrt{\frac{2d}{gsin(\phi)}}$$

Since we assumed no kinetic friction and gravity stops doing work on the car as soon as it reaches the flat section of track, the speed at the end of the track will be equal to:

$$\dot{x}(t \geq t_r) = \sqrt{2dgsin(\phi)}$$

Denoting the length of the flat section as L, the expression for time it will take the car to complete the race will be:

$$t_{race} = \sqrt{\frac{2d}{gsin(\phi)}} + \frac{L}{\sqrt{2dgsin(\phi)}}$$

Given that assumptions made in this analysis simplified the problem significantly, the expression for the time does not include any constraints that can be used for the analysis the car. Therefore, further analysis has to be performed.

2.2 Further Analysis

Now, the car will be treated as composed of four wheels and a body. The following constants will be used:

1. The total mass of the car is $\boldsymbol{M}$
2. The mass of the wheels is $\boldsymbol{m}$
3. The length of the flat section of the track is $\boldsymbol{L}$
4. The radius of the each wheel is $\boldsymbol{R}$

The distance from the top to the track to the center of mass of the car is, $\boldsymbol{x(t)}$. The assumptions used in this model are:

1. Wheels are modeled as solid disks, and they are identical.
2. There is no kinetic friction.
3. The wheels roll without slipping.
4. The center of mass of the car is located in the body of the car, above the wheels.
5. No initial velocity.

The model will be again constructed using conservation of energy.

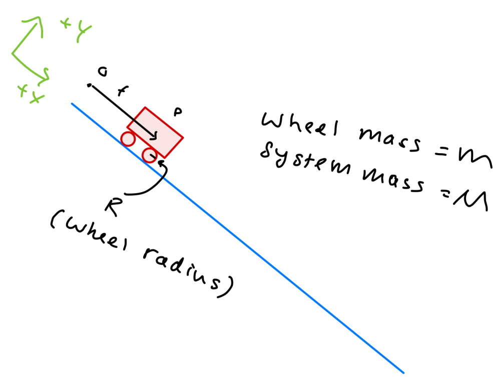

Figure 2: Vehicle Free Body Diagram

$$
\begin{aligned}
E(0) &= Mgdsin(\phi) \\
E(t) &= T_{\text{translational}}(t) + T_{rotational}(t) + V(t) \\
&= \frac{1}{2}(M-4m)\dot{x}^2 + \frac{1}{2}I\omega^2 + Mg(d-x)sin(\phi)
\end{aligned}
$$

The rotational kinetic energy can be simplified using the facts that (1) $\omega = \frac{\dot{x}}{R}$, and (2) $I = 4I_0 = 4mk_0^2$. Thus the expression for $E(t)$ becomes:

$$
E(t) = \frac{1}{2}(M-4m)\dot{x}^2 + \frac{2mk_0^2}{R^2}\dot{x}^2 + Mg(d-x)sin(\phi)
$$

Using the statement for conservation of energy yields the formula for velocity in terms of position:

$$
\dot{x}(t) = \sqrt{\frac{2Mgsin(\phi)x}{M-4m+4m(\frac{k_0}{R})^2}}
$$

The total time it would take to reach the end of the ramp can be determined from this formula.

$$
\begin{aligned}
\int_0^{t_r} dt &= \sqrt{\frac{M-4m+4m(\frac{k_0}{R})^2}{2Mgsin(\phi)}} \int_0^d \frac{1}{\sqrt{x}}\,dx \\
t_r &= \sqrt{\frac{2d(M-4m+4m(\frac{k_0}{R})^2)}{Mgsin(\phi)}}
\end{aligned}
$$

Again, since we assumed no kinetic friction, the velocity for $t \geq t_r$ (when $x = d$) would be:

$$
\dot{x}(t \geq t_r) = \sqrt{\frac{2Mgsin(\phi)d}{M-4m+4m(\frac{k_0}{R})^2}}
$$

Therefore, the time needed to complete the race would be:

$$
\begin{aligned}
t_{race} &= t_r + \frac{L}{\dot{x}(t \geq t_r)} \\
&= \sqrt{\frac{2d[M-4m+4m(\frac{k_0}{R})^2]}{Mgsin(\phi)}} + L\sqrt{\frac{M-4m+4m(\frac{k_0}{R})^2}{2Mgsin(\phi)d}} \\
t_{race} &= \sqrt{\frac{2d[M-4m+4m(\frac{k_0}{R})^2]}{Mgsin(\phi)}}(1+\frac{L}{2d})
\end{aligned}
$$

2.2.1 Conclusions

Even though factors such as kinetic friction between the axles and the wheels, the lack of slipping, et cetera, were not accounted for in this model, it still produces a better understanding of some considerations to take into account in order to reduce the time it would take the car to finish the race. For instance:

1. Maximizing $\boldsymbol{M}$ would yield lower times. This makes intuitive sense, as a higher overall mass will produce a greater weight.
2. Minimizing the term $\boldsymbol{k_0/R}$ will also result in lower race times. This is equivalent to minimizing the moment of inertia of the wheels. One way to do this is to concentrate as much of the wheels' mass near its center of rotation.
3. If the moment of inertia is sufficiently minimized, the positive contribution $4m(\frac{k_0}{R})^2$ to the time will become negligible, and the negative contribution $-4m$ will become a lot more significant. As such, increasing the mass of the wheels, $\boldsymbol{m}$ will result in lower race times.

3 Design and Analysis

3.1 Wheel Design and Justification

From the preliminary analysis, an efficient design for this system appeared to be one which maximized the wheel's overall radius, the mass of the wheel, the mass of the entire car system, and had more of the wheel mass congregated within its inner radius. Based on this design criteria, a prototype was developed in SolidWorks, rendered as follows:

Figure 3: Wheel 1 Design

This wheel incorporates all of the optimized design criteria. It has a relatively large overall diameter of 30mm as desired. This design features a a solid inner diameter of 15mm which accounts for most of the mass of the wheel, achieving the desired majority mass placement in the center of the wheel. The spokes are minimal and there is hollow areas between the inner and outer radii to maintain the desired mass ratio (mass in the center : mass on the diameter). The thin (2mm) profile on the external radius helps increase this ratio. Finally, the face of the wheel contacting the track is relatively narrow at 6mm, helping minimize $\boldsymbol{k_0/R}$. The overall weight of this design is 0.96g.

For testing purposes and use for comparison of the coefficient of kinetic friction and moment of inertia, a second wheel design was made. It shares the same properties as the first, except it has a reduced number of spokes (2 vs. 4) and subsequently a slightly reduced overall weight (0.91 g).

Figure 4: Wheel 2 Design

The wheels were 3D printed using high infill density to maximize weight, tested using a pulley mechanism, and assembled onto the original vehicle body (no additional modifications where made). The final assembly of the car is demonstrated in the following render, where additional weights are added in the center of the car:

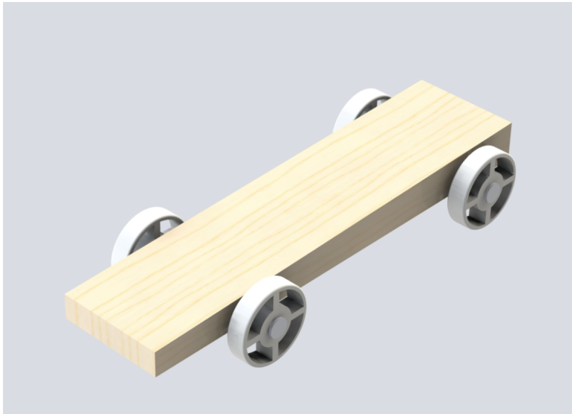

Figure 5: Car Assembly

The total mass of the car is 0.045kg with body dimensions of 7.06"x1.7"x0.5."

3.2 Wheel Inertia and Friction

To find the mass moment of inertia of the wheel design and the coefficient of kinetic friction, a python script was written to analyze some of the aforementioned specifications of the wheel diameter, wheel weight, and test results from the pulley experiments (Code A). The program calculated a moment of inertia of $3.97{*}10^{-}8Nm^2$ and a coefficient of kinetic friction of 0.178.

3.2.1 Angular acceleration of wheel in experiment

The values used to calculate the moment of inertia and the coefficient of kinetic friction were determined through experimental data. As part of the experiment, the following results were proven, as indicated in the Laboratory 2 objectives document.

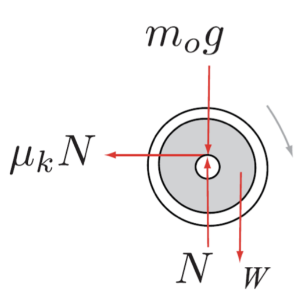

Figure 1: Wheel experiment with applied weight W.

Assuming that the vertical down direction is the $+\hat{\jmath}$ and the horizontal right direction is the $+\hat{\imath}$ direction, the **first result** shows that the angular acceleration of the wheel is constant:

$$\begin{aligned}\sum \vec{\tau} &= \vec{\tau_W} + \vec{\tau_N} + \vec{\tau_g} + \vec{\tau_f} = I_0\vec{\alpha} \\ &= (r_s\hat{\imath}) \times (W\hat{\jmath}) + 0 + 0 + [r_i(-\hat{\jmath}) \times (\mu_k N r_i(-\hat{\imath})] \\ &= (Wr_s - \mu_k N r_i)\hat{k}\end{aligned}$$

Equating the magnitudes both sides yields:

$$\begin{aligned}\|I_0\vec{\alpha}\| &= \|(Wr_s - \mu_k N r_i)\hat{k}\| \\ \alpha &= \frac{Wr_s - \mu_k N r_i}{I_0}\end{aligned}$$

The **second result** allows you to determine the angular acceleration from the number of trials (n_1, n_2) and the time differences $(\Delta t_1, \Delta t_2)$

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{\Delta t_2 - \Delta t_1} \\ &= \frac{2}{\Delta t_2 - \Delta t_1}[2\pi\frac{n_2}{\Delta t_2} - 2\pi\frac{n_1}{\Delta t_1}] \\ &= \frac{4\pi}{\Delta t_2 - \Delta t_1}[\frac{n_2\Delta t_1}{\Delta t_2\Delta t_1} - 2\pi\frac{n_1\Delta t_2}{\Delta t_1\Delta t_2}] \\ \alpha &= 4\pi\frac{n_2\Delta t_1 - n_1\Delta t_2}{\Delta t_1\Delta t_2(\Delta t_2 - \Delta t_1)}\end{aligned}$$

3.3 Derivation of Equations of Motion

To get a more accurate estimation of the time it will take the car to finish the race, a model that includes the effects of friction will be used. Lagrange's equations are best suited for this. The generalized coordinate to be used is $x(t)$. The same variables as before are used, and it makes the following assumptions:

1. Wheels roll without slipping: $\dot{x} = R\dot{\omega}$.

The initial conditions are:

1. (0) = 0
2. x(0) = 0

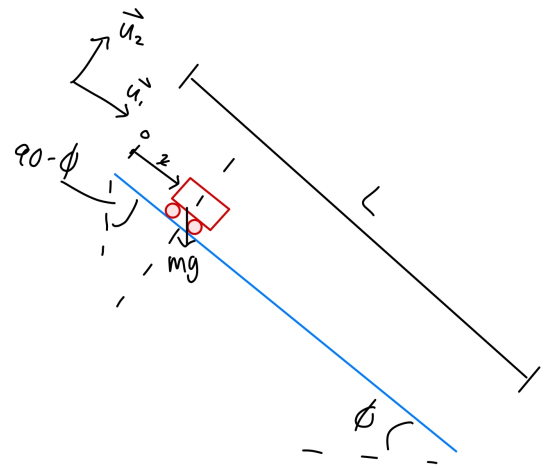

Figure 6: Lagrange Free Body Diagram

First, time-dependent expressions for the kinetic and potential energies, and Lagrangian are found.

$$
\begin{aligned}
T &= \frac{1}{2}(M-4m)\dot{x}^2 + \frac{1}{2}I\dot{\omega}^2 \\
&= \frac{1}{2}(M-4m)\dot{x}^2 + \frac{1}{2}(4I_0)(\frac{\dot{x}}{R})^2 \\
T &= \frac{1}{2}(M-4m)\dot{x}^2 + \frac{2mk_0^2}{R^2}\dot{x}^2 \\
V &= -Mgxsin(\phi) \\
\mathcal{L} &= T - V = \frac{1}{2}(M-4m)\dot{x}^2 + \frac{2mk_0^2}{R^2}\dot{x}^2 + Mgxsin(\phi)
\end{aligned}
$$

Since friction is a non-conservative force, the generalized work has to be found in order to find the generalized forces:

$$
\begin{aligned}
\delta W &= \vec{F_{fr}} \cdot \delta x \\
&= Q_x \delta x \\
Q_x &= -Mgcos(\phi)\mu_k
\end{aligned}
$$

To determine the EOM for $x(t)$ use Lagrange's equation:

$$
\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = Q_x
$$

This yields the following EOM:

$$
x(t) = \frac{Mgt^2(sin(\phi) - \mu_k cos(\phi))}{2[M - 4m + 4m(\frac{k_0}{R})^2]}
$$

3.4 Race Time Estimations

Referring back to the initial symbolic estimation of the race time of the race car, the following equation is recalled:

$$t_{race} = \sqrt{\frac{2d[M - 4m + 4m(\frac{k_0}{R})^2]}{Mgsin(\phi)}}(1 + \frac{L}{2d})$$

Solving for k_0 gives:

$$k_0 = \sqrt{I_0/m} = \frac{\sqrt{3.97 * 10^{-8}}}{0.00096} = 0.0064$$

Substitution for the following values are performed:

1. M = 0.045kg
2. m = 0.00096kg
3. R = 0.0015m
4. d = 2.181m
5. g = 9.81m/s^2
6. L = 2.06m
7. K_0 = 0.0064

A value of 1.97s is obtained. Using the Lagrange equation that account for friction:

$$x(t) = \frac{\mu * t^2 * (sin(\phi) - \mu * cos(\phi))}{2(4mk^2 + \mu - 4m)}$$

Substitution of x(t) = d and solving for x in this equation yields an estimated time of 2.23 seconds.

4 Results and Discussion

Car Weight	Wheel Weight	Wheel Inner Radius	Wheel Outer Radius
0.045kg	0.00096kg	7.5mm	15mm

Friction	Inertia	Expected Time	Average Time
0.178	$3.97*10^{-}8\ Nm^2$	2.23 s	2.32s

As the table shows, the average time it took the car to complete the race was 2.32 seconds, while the time predicted by the most accurate model we were able to produce was 2.23 seconds. This represents a discrepancy of 3.89%, which represents a pretty accurate estimate.

The estimate being lower than the actual time may be attributed by the model's failure to take into account other factors that may slow the car down, such as air resistance, imperfections in the track, slipping while rolling, et cetera. Despite this, the model was able to produce an acceptable estimate.

References

[1] In-class resources, Professor Dowell's lectures/notes, teaching assistants

A Source code

```python
import numpy as np
import pandas as pd
import sympy as sp

data_2 = pd.read_csv('/Users/jaivir/Downloads/data2.csv')
data_4 = pd.read_csv('/Users/jaivir/Downloads/data4.csv')

rs = 15.25 * 0.0001
ri = 0.0075
W1 = 0.002  * 9.81
W2 = 0.005 * 9.81
Wheel = 0.0096  * 9.81

def extract_values(data):
    max_index = len(data) - 1
    half_index = len(data) // 2
    t1 = data['Time'].iloc[min(3, max_index)]
    t2 = data['Time'].iloc[min(17, max_index)]
    y1 = data['Y'].iloc[min(3, max_index)] * -0.0254
    y2 = data['Y'].iloc[min(17, max_index)] * -0.0254
    return t1, t2, y1, y2

t1_1, t2_1, y1_1, y2_1 = extract_values(data_2)
t1_2, t2_2, y1_2, y2_2 = extract_values(data_4)
print(len(data_2))
print(len(data_4))

n1_1 = y1_1 / (2 * np.pi * rs)
n2_1 = y2_1 / (2 * np.pi * rs)
n1_2 = y1_2 / (2 * np.pi * rs)
n2_2 = y2_2 / (2 * np.pi * rs)

alpha1_1 = (4 * np.pi * ((n1_1 * t2_1) - (n2_1 * t1_1))) / ((t1_1*t2_1)*(t2_1-t1_1))
alpha1_2 = (4 * np.pi * ((n1_2 * t2_2) - (n2_2 * t1_2))) / ((t1_2*t2_2)*(t2_2-t1_2))

I0, mu = sp.symbols('I0 mu')

alpha2_1_eq = sp.Eq(((W1 * rs) - (mu * (W1 + Wheel) * ri)) / I0, alpha1_1)
alpha2_2_eq = sp.Eq(((W2 * rs) - (mu * (W2 + Wheel) * ri)) / I0, alpha1_2)

solution = sp.solve((alpha2_1_eq, alpha2_2_eq), (I0, mu))

print("Solution for I0 and mu:", solution)
```

Figure 7: Inertia/Friction Code