TP2 Risk Management

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Libraries

Risk Management: European Options Portfolio

The objective is to implement (part of) the risk management framework for estimating the risk of a book of European call options by taking into account the risk drivers such as underlying and implied volatility.

Data

Load the database Market. Identify the price of the **SP500**, the **VIX index**, the term structure of interest rates (current and past), and the traded options (calls and puts).

```
# load dataset into environment
load(file = here("data_raw", "Market.rda"))

# reassign name and inspect structure of loaded data
mkt <- Market
summary(mkt)
```

```
##       Length Class  Mode
## sp500 3410   xts    numeric
## vix   3410   xts    numeric
## rf      14   -none- numeric
## calls 1266   -none- numeric
## puts  2250   -none- numeric
```

```
str(mkt)
```

```
## List of 5
##  $ sp500:An xts object on 2000-01-03 / 2013-09-10 containing:
##   Data:    double [3410, 1]
##   Index:   Date [3410] (TZ: "UTC")
##  $ vix  :An xts object on 2000-01-03 / 2013-09-10 containing:
##   Data:    double [3410, 1]
##   Index:   Date [3410] (TZ: "UTC")
##  $ rf   : num [1:14, 1] 0.00071 0.00098 0.00128 0.00224 0.00342 ...
##   ..- attr(*, "names")= chr [1:14] "0.00273972602739726" "0.0192307692307692" "0.0833333333333333" "0.25" .
##  $ calls: num [1:422, 1:3] 1280 1370 1380 1400 1415 ...
##   ..- attr(*, "dimnames")=List of 2
##   .. ..$ : NULL
##   .. ..$ : chr [1:3] "K" "tau" "IV"
##  $ puts : num [1:750, 1:3] 1000 1025 1050 1075 1100 ...
##   ..- attr(*, "dimnames")=List of 2
##   .. ..$ : NULL
##   .. ..$ : chr [1:3] "K" "tau" "IV"
```

Let's unpack these into the env. individually:

```
# unpack each of the elements in the mkt list
sp500 <- mkt$sp500
vix <- mkt$vix
Rf <- mkt$rf # risk-free rates
calls <- mkt$calls
puts <- mkt$puts

# assign colname for aesthetic
colnames(sp500) <- "sp500"
colnames(vix) <- "vix"
```

SP500 and VIX

By inspection, we observe that we the SP500 and VIX indices are contained in the `sp500` and `vix` xts objects respectively.

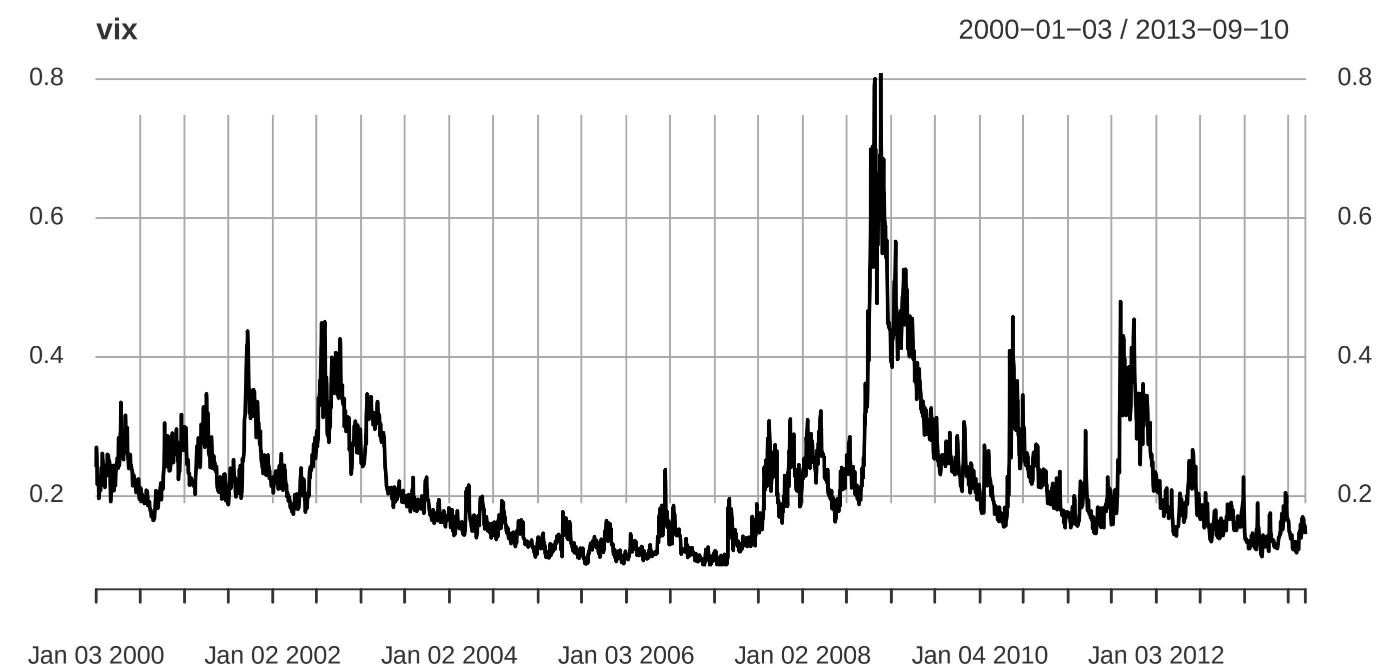
```
# show head of both indexes
head(sp500)
```

```
##              sp500
## 2000-01-03 1455.22
## 2000-01-04 1399.42
## 2000-01-05 1402.11
## 2000-01-06 1403.45
## 2000-01-07 1441.47
## 2000-01-10 1457.60
```

```
head(vix)
```

```
##               vix
## 2000-01-03 0.2421
## 2000-01-04 0.2701
## 2000-01-05 0.2641
## 2000-01-06 0.2573
## 2000-01-07 0.2172
## 2000-01-10 0.2171
```

```
par(mfrow = c(2,1))

# plot both series on top of each other
plot(sp500)
plot(vix)
```



Interest Rates

The **interest rates** are given in the `$rf` attribute. We can see that

```
Rf
```

```
##                [,1]
##  [1,] 0.0007099993
##  [2,] 0.0009799908
##  [3,] 0.0012799317
##  [4,] 0.0022393730
##  [5,] 0.0034170792
##  [6,] 0.0045123559
##  [7,] 0.0043206525
```

```
##  [8,] 0.0064284968
##  [9,] 0.0090558654
## [10,] 0.0117237591
## [11,] 0.0141196498
## [12,] 0.0176131823
## [13,] 0.0207989304
## [14,] 0.0203526819
## attr(,"names")
##  [1] "0.00273972602739726" "0.0192307692307692"  "0.0833333333333333"
##  [4] "0.25"                "0.5"                 "0.75"
##  [7] "1"                   "2"                   "3"
## [10] "4"                   "5"                   "7"
## [13] "10"                  "30"
```

These represent the interest rates at different maturities. The maturities are given as follows:

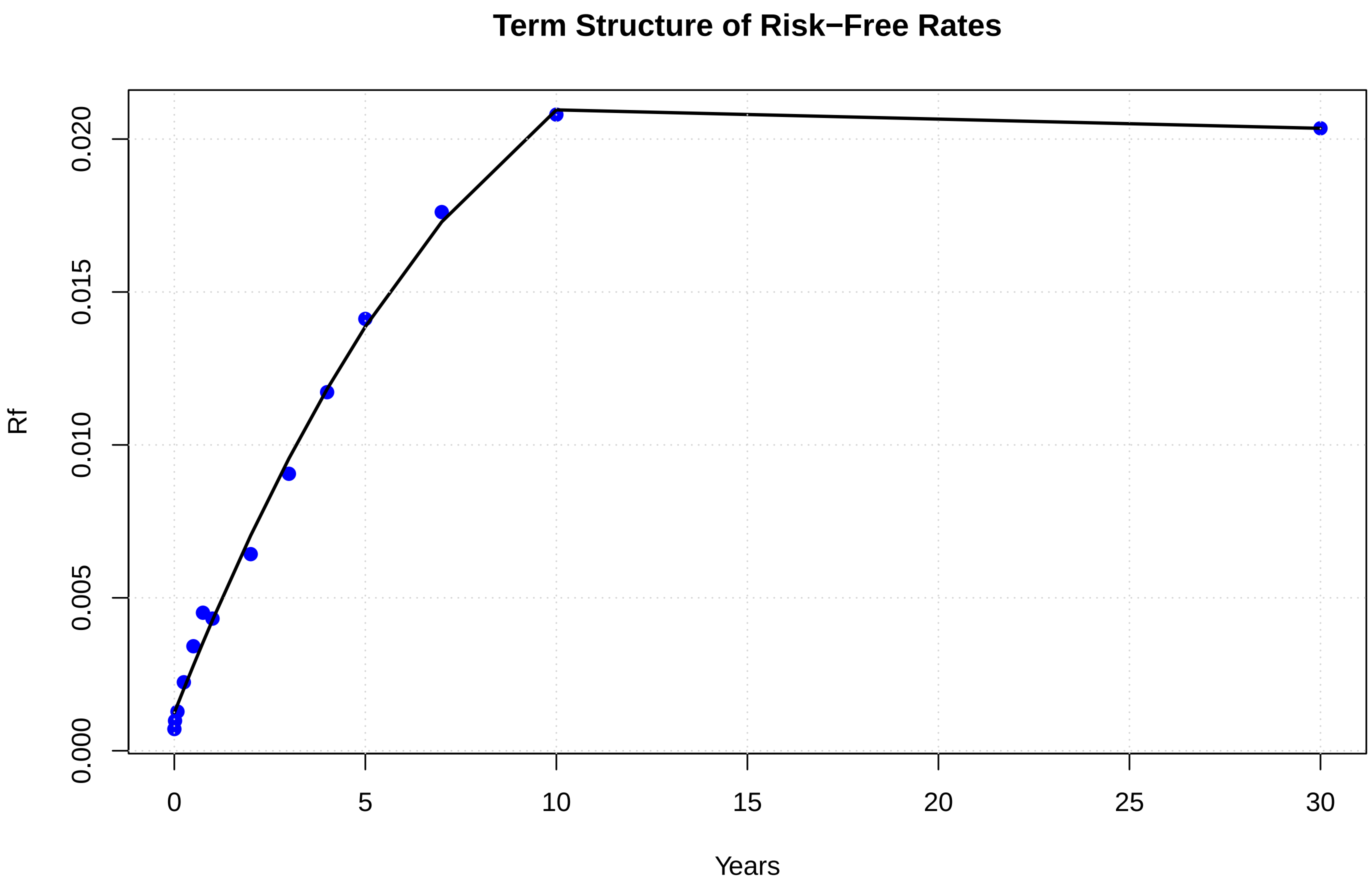
```
r_f <- as.vector(Rf)
names(r_f) <- c("1d","1w", "1m", "3m", "6m", "9m", "1y", "2y", "3y", "4y", "5y", "7y","10y", "30y")
r_f
```

```
##           1d           1w           1m           3m           6m           9m
## 0.0007099993 0.0009799908 0.0012799317 0.0022393730 0.0034170792 0.0045123559
##           1y           2y           3y           4y           5y           7y
## 0.0043206525 0.0064284968 0.0090558654 0.0117237591 0.0141196498 0.0176131823
##          10y          30y
## 0.0207989304 0.0203526819
```

Further, we can pack different sources of information in a matrix:

```
# pack Rf into a matrix with rf, years, and days
rf_mat <- as.matrix(r_f)
rf_mat <- cbind(rf_mat, as.numeric(names(Rf)))
rf_mat <- cbind(rf_mat, rf_mat[, 2]*360)
colnames(rf_mat) <- c("rf", "years", "days")
rf_mat
```

```
##               rf        years          days
## 1d  0.0007099993  0.002739726     0.9863014
## 1w  0.0009799908  0.019230769     6.9230769
## 1m  0.0012799317  0.083333333    30.0000000
## 3m  0.0022393730  0.250000000    90.0000000
## 6m  0.0034170792  0.500000000   180.0000000
## 9m  0.0045123559  0.750000000   270.0000000
## 1y  0.0043206525  1.000000000   360.0000000
## 2y  0.0064284968  2.000000000   720.0000000
## 3y  0.0090558654  3.000000000  1080.0000000
## 4y  0.0117237591  4.000000000  1440.0000000
## 5y  0.0141196498  5.000000000  1800.0000000
## 7y  0.0176131823  7.000000000  2520.0000000
## 10y 0.0207989304 10.000000000  3600.0000000
## 30y 0.0203526819 30.000000000 10800.0000000
```



Calls

The `calls` object displays the different values of K (**Strike Price**), τ (**time to maturity**) and $\sigma = IV$ (**Implied Volatilty**)

```
dim(calls)
```

```
## [1] 422   3
```

```
head(calls)
```

```
##         K        tau        IV
## [1,] 1280 0.02557005 0.7370370
## [2,] 1370 0.02557005 0.9691616
## [3,] 1380 0.02557005 0.9451401
## [4,] 1400 0.02557005 0.5274481
## [5,] 1415 0.02557005 0.5083375
## [6,] 1425 0.02557005 0.4820041
```

Add `days` column for convenience:

```
calls <- cbind(calls, calls[, "tau"]*250)
colnames(calls) <- c("K","tau", "IV", "tau_days")
head(calls)
```

```
##         K        tau        IV tau_days
## [1,] 1280 0.02557005 0.7370370 6.392513
## [2,] 1370 0.02557005 0.9691616 6.392513
## [3,] 1380 0.02557005 0.9451401 6.392513
```

```
## [4,] 1400 0.02557005 0.5274481 6.392513
## [5,] 1415 0.02557005 0.5083375 6.392513
## [6,] 1425 0.02557005 0.4820041 6.392513
```

```
tail(calls)
```

```
##          K      tau        IV tau_days
## [417,] 1925 2.269406 0.1605208 567.3514
## [418,] 1975 2.269406 0.1602093 567.3514
## [419,] 2000 2.269406 0.1559909 567.3514
## [420,] 2100 2.269406 0.1480259 567.3514
## [421,] 2500 2.269406 0.1441222 567.3514
## [422,] 3000 2.269406 0.1519319 567.3514
```

Puts

```
dim(puts)
```

```
## [1] 750   3
```

```
head(puts)
```

```
##         K        tau        IV
## [1,] 1000 0.02557005 1.0144250
## [2,] 1025 0.02557005 1.0083110
## [3,] 1050 0.02557005 0.9622093
## [4,] 1075 0.02557005 0.9170457
## [5,] 1100 0.02557005 0.8728757
## [6,] 1120 0.02557005 0.8381910
```

```
puts <- cbind(puts, puts[, "tau"]*250)
colnames(puts) <- c("K","tau", "IV", "tau_days")
head(puts)
```

```
##         K        tau        IV tau_days
## [1,] 1000 0.02557005 1.0144250 6.392513
## [2,] 1025 0.02557005 1.0083110 6.392513
## [3,] 1050 0.02557005 0.9622093 6.392513
## [4,] 1075 0.02557005 0.9170457 6.392513
## [5,] 1100 0.02557005 0.8728757 6.392513
## [6,] 1120 0.02557005 0.8381910 6.392513
```

```
tail(puts)
```

```
##          K      tau        IV tau_days
## [745,] 1750 2.269406 0.1899088 567.3514
## [746,] 1800 2.269406 0.1698365 567.3514
## [747,] 1825 2.269406 0.1986200 567.3514
## [748,] 1850 2.269406 0.1853406 567.3514
## [749,] 2000 2.269406 0.1520378 567.3514
## [750,] 3000 2.269406 0.2759397 567.3514
```

Pricing a Portfolio of Options

Black-Scholes

Notation:

- S_t = Current value of underlying asset price
- K = Options **strike price**
- T = Option **maturity** (in years)
- t = **time** in years
- $\tau = T - t$ = **Time to maturity**
- r = **Risk-free rate**
- y **Dividend yield**
- $R = r - y$
- σ = **Implied volatility**
- c = **Price Call Option**
- p = **Price Put Option**

Proposition 1 (Black-Scholes Model)**.** Assume the notation before, and let $N(\cdot)$ be the cumulative standard normal distribution function. Under certain assumptions, the Black-Scholes models prices Call and Put options as follows:

$$\begin{cases} C(S_t, t) = Se^{yT}N(d_1) - Ke^{-r\times\tau}N(d_2), \\ \\ P(S_t, t) = Ke^{-r\times\tau}(1 - N(d_2)) - Se^{y\times T}(1 - N(d_1)), \end{cases}$$

where:

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{S_t}{K}\right) + \tau\left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \\ d_2 = d_1 - \sigma\sqrt{\tau} \end{cases}$$

, further the Put Option price corresponds to the **Put-Call parity**, given by:

$$C(S_t, t) + Ke^{-r\times\tau} = P(S_t, t) + S_t$$

Note As here we don't have dividends, then $y = 0$, and so

$$\begin{cases} C(S_t, t) = S_tN(d_1) - Ke^{-r\times\tau}N(d_2), \\ \\ P(S_t, t) = Ke^{-r\times\tau}(1 - N(d_2)) - S_t(1 - N(d_1)), \end{cases}$$

BlackScholes function

```
# source code with options pricign
source(here("code", "OptionPricing.R")) # BlackScholes and Option pricing

# Test: Call Option
S_t = 1540
K = 1600
r = 0.03
tau = 10/360
sigma = 1.05
black_scholes(S_t, K, r, tau, sigma)
```

```
## [1] 80.81672
```

Book of Options

Assume the following book of **European Call Options:**

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1650$ with maturity $T = 20d$
3. **1x** strike $K = 1750$ with maturity $T = 40d$
4. **1x** strike $K = 1800$ with maturity $T = 40d$

Find the price of this book given **the last underlying price** and the **last implied volatility** (take the VIX for all options). Use **Black-Scholes** to price the options. Take the current term structure and **linearly interpolate** to find the corresponding rates. Use 360 days/year for the term structure and **250 days/year** for the maturity of the options.

Nearest values

This function will obtain the two nearest values a, b for a number x in a vector v, such that $a < x < b$.

```
# Test: function used to get two nearest values in a vector (OptionsPricing.R)
days <- rf_mat[, "days"]
get_nearest(40, rf_mat[, "days"]) # nearest day values
```

```
## 1m 3m
## 30 90
```

Linear Interpolation

Given two known values (x_1, y_1) and (x_2, y_2), we can estimate the y-value for some x-value with:

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

```
# Function to interpolate y given two points
interpolate <- function(x,x1=1,y1=1,x2=2,y2=2){
  y1 + (x-x1)*(y2-y1)/(x2-x1)
}
```

Finding the rates through interpolation

The **yield curve** for the given structure of interest rates can be modeled a function $r_f = f(x)$,where x is the number of years. Then, we can interapolate the values from `rf_mat`. This is done in the `price_option()` function under `code/OptionPricing.R`

Example

ex.: **1x** strike $K = 1600$ with maturity $T = 20d$

```
S_t = sp500[length(sp500)] # last price of underlying
IV = vix[length(vix)] # last volatility

## test: specific price (func from OptionPricing.R)
price_option(T=20, K=1600, calls = calls, rf_mat =  rf_mat, stock = NA, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 87.56885
##
## $Put
## [1] NA
```

```
##
## $S
## [1] 1683.99
##
## $K
## [1] 1600
##
## $r_interp
## [1] 0.001264335
##
## $calls
##         K        tau        IV  tau_days
## [1,] 1600 0.02557005 0.1817481  6.392513
## [2,] 1600 0.10228238 0.1701946 25.570595
##
## $rates
##              rf      years      days
## 1w 0.0009799908 0.01923077  6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

Next, using the function above we price the book of options given:

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1650$ with maturity $T = 20d$
3. **1x** strike $K = 1750$ with maturity $T = 40d$
4. **1x** strike $K = 1800$ with maturity $T = 40d$

First, we retrieve the latest value for the underlying (SP500) and the latest implied volatility (VIX):

```
S_t = sp500[length(sp500)] # last price of underlying
IV = vix[length(vix)] # last volatility
```

Then, we price the options accordingly:

```
# First Call Option
price_option(T=20, K=1600, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 87.56885
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1600
##
## $r_interp
## [1] 0.001264335
##
## $calls
##         K        tau        IV  tau_days
## [1,] 1600 0.02557005 0.1817481  6.392513
## [2,] 1600 0.10228238 0.1701946 25.570595
##
## $rates
```

```
##               rf      years      days
## 1w 0.0009799908 0.01923077  6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

```
# Second Call Option
price_option(T=20, K=1650, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 47.70804
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1650
##
## $r_interp
## [1] 0.001264335
##
## $calls
##         K        tau        IV  tau_days
## [1,] 1650 0.02557005 0.1456375  6.392513
## [2,] 1650 0.10228238 0.1448237 25.570595
##
## $rates
##               rf      years      days
## 1w 0.0009799908 0.01923077  6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

```
# Third Call Option
price_option(T=40, K=1750, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 15.25057
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1750
##
## $r_interp
## [1] 0.001721275
##
## $calls
##         K       tau        IV tau_days
## [1,] 1750 0.1022824 0.1047194 25.57059
## [2,] 1750 0.1789947 0.1130030 44.74868
##
## $rates
##              rf      years days
## 1m 0.001279932 0.08333333   30
## 3m 0.002239373 0.25000000   90
```

```
# Fourth Call Option
price_option(T=40, K=1800, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 6.34395
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1800
##
## $r_interp
## [1] 0.001721275
##
## $calls
##         K       tau        IV tau_days
## [1,] 1800 0.1022824 0.1057523 25.57059
## [2,] 1800 0.1789947 0.1044115 44.74868
##
## $rates
##             rf      years days
## 1m 0.001279932 0.08333333   30
## 3m 0.002239373 0.25000000   90
```

One risk driver and Gaussian Model

```
##################################################
###  ---> INSERT YOUR PART FROM HERE (IOAN) <---- ###
############################################ ######
```

Two risk drivers and Gaussian Model

```
####################################################
###  ---> INSERT YOUR PART FROM HERE (ALESSIO) <---- ###
############################################# ######
```

Two risk drivers and copula-marginal model (Student-t and Gaussian Copula)

1. Compute the daily log-returns of the underlying stock
2. Assume the first invariant is generated using a Student-t distribution with $\nu = 10$ df and the second invariant is generated using a Student-t distribution with $\nu = 5$ df.
3. Assume the **normal copula** to merge the marginals.
4. Generate 10000 scenarios for the one-week ahead price for the underlying and the one-week ahead VIX value using the copula.
5. Determine the P&L distribution of the book of options, using the simulated values.
6. Take interpolated rates for the term structure.

Gaussian Copula with two Student-t marginals

A bivariate distribution H can be formed via a copula C from two marginal distributions with CDFs F and G via:

$$H(x, y) = C(F(x), G(y)) = C(F^{-1}(u), G^{-1}(u))$$

with density

$$h(x, y) = c(F(x), G(y))f(x)g(y)$$

The **Gaussian Copula** is given by:

$$C_\rho^{\text{Gauss}}(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)).$$

In this case, a Gaussian copula with two Student-t marginals with CDFs $t(\nu_1)$ with ν_1 degrees of freedom and $t(\nu_2)$ with ν_2 degrees of freedom is given by:

$$C_\rho^{\text{Gauss}}(u, v) = \Phi_\rho(F_{\nu_1}^{-1}(u), F_{\nu_1}^{-1}(v)),$$

where F_{ν_1} and F_{ν_2} are their respective CDFs.

Log-returns

The **discrete returns** are given by:

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t}$$

and the next ahead log-returns are given by:

$$\log(R_{t+1}) = \log(P_{t+1} - P_t) - \log(P_t)$$

```
# load reqruired libraries
library("PerformanceAnalytics")

# calculate returns
sp500_rets <- PerformanceAnalytics::CalculateReturns(sp500, method="log")
vix_rets <- PerformanceAnalytics::CalculateReturns(vix, method="log")

# remove first return
sp500_rets <- sp500_rets[-1]
vix_rets <- vix_rets[-1]

# remove nas
sp500_rets[is.na(sp500_rets)] <- 0
```

```r
vix_rets[is.na(vix_rets)] <- 0

# display
head(sp500_rets)
```

```
##                    sp500
## 2000-01-04 -0.0390992269
## 2000-01-05  0.0019203798
## 2000-01-06  0.0009552461
## 2000-01-07  0.0267299353
## 2000-01-10  0.0111278213
## 2000-01-11 -0.0131486343
```

```r
head(vix_rets)
```

```
##                      vix
## 2000-01-04  0.1094413969
## 2000-01-05 -0.0224644415
## 2000-01-06 -0.0260851000
## 2000-01-07 -0.1694241312
## 2000-01-10 -0.0004605112
## 2000-01-11  0.0357423253
```

Generating the simulation scenarios

Assumptions: - Marginal Student-t distributions - Disregard time dependence in the bootstrapping process

```r
# Load required libraries
library("fGarch")
library("MASS")
library("copula")
library("Matrix")

# random seed for replication
set.seed(123)

# convert to vector since fitting without dependence
sp500_rets_vec <- as.vector(sp500_rets)
vix_rets_vec <- as.vector(vix_rets)

# calculate means and sds for both indices
mu <- c(mean(sp500_rets_vec), mean(vix_rets_vec))
sigma <- c(sd(sp500_rets_vec), sd(vix_rets_vec))

# display
mu
```

```
## [1]  0.00004283042 -0.00014976541
```

```r
sigma
```

```
## [1] 0.01332592 0.06367330
```

```r
## Fit marginals by MLE

# Student-t for sp500
```

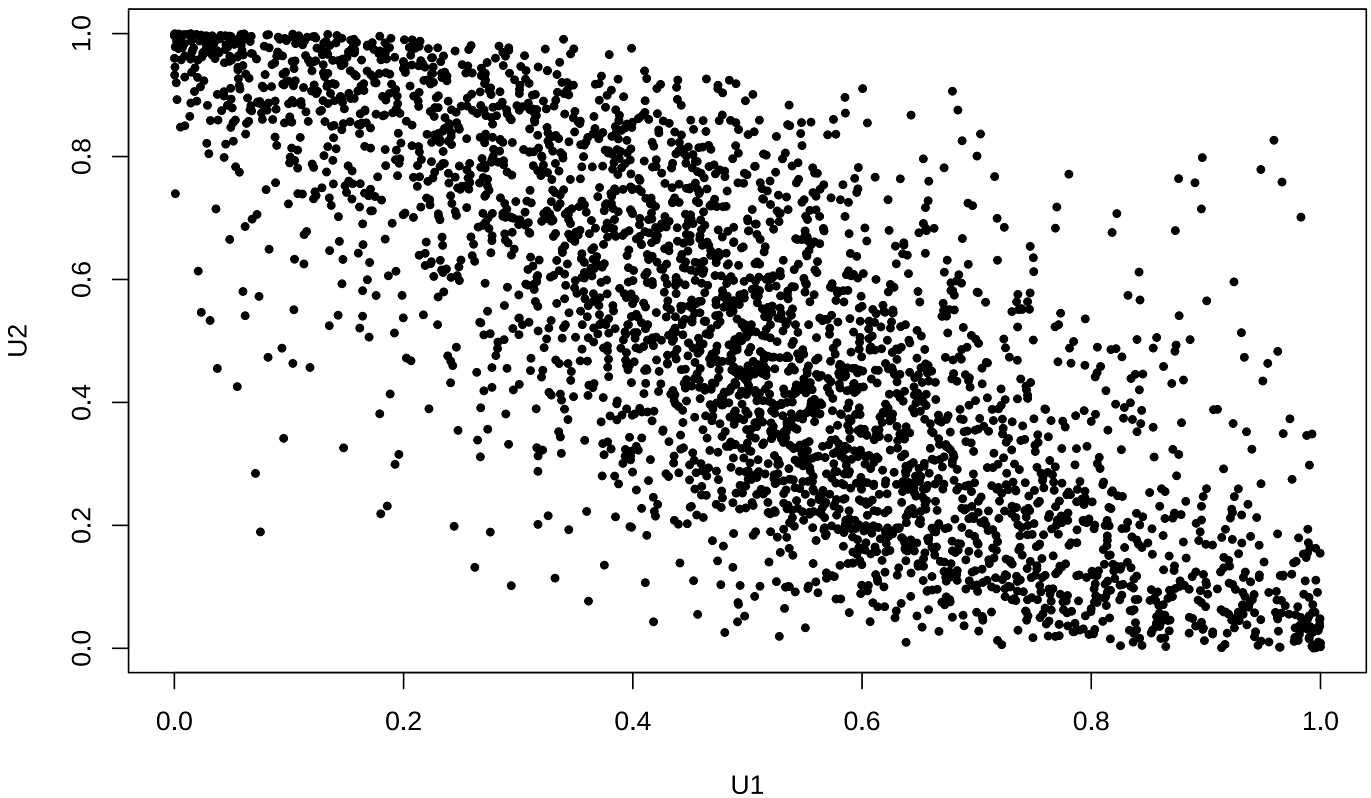
```
fit1 <- suppressWarnings(
  fitdistr(x = sp500_rets_vec,
           densfun = dstd,
           start = list(mean = 0, sd = 1, nu = 10))
  )
theta1 <- fit1$estimate #extract fitted parameters

# Student-t for vix
fit2 <- suppressWarnings(
  fitdistr(x = vix_rets_vec,
           densfun = dstd,
           start = list(mean = 0, sd = 1, nu = 5))
  )
theta2 <- fit2$estimate # extract fitted parameters

# display parameters
theta1
```

```
##         mean           sd           nu
## 0.0004414879 0.0156603739 2.6953920404
```

```
theta2
```

```
##         mean           sd           nu
## -0.003475206  0.064192681  4.230323432
```

```
# Fit Student-t to the marginals
# U1 <- pstd(sp500_rets_vec, mean = theta1[1], sd = theta1[2], nu = theta1[3]) # sp500
# U2 <- pstd(vix_rets_vec,mean = theta2[1], sd = theta2[2], nu = theta2[3]) # vix
U1 <- pstd(sp500_rets_vec, mean = theta1[1], sd = theta1[2], nu = 10) # sp500
U2 <- pstd(vix_rets_vec,mean = theta2[1], sd = theta2[2], nu = 5) # vix
# U1 <- pt(sp500_rets_vec, df = 5) # sp500
# U2 <- pt(vix_rets_vec, df = 10) # vix
U <- cbind(U1, U2) # join into one matrix
plot(U, pch = 20, cex = 0.9)
```

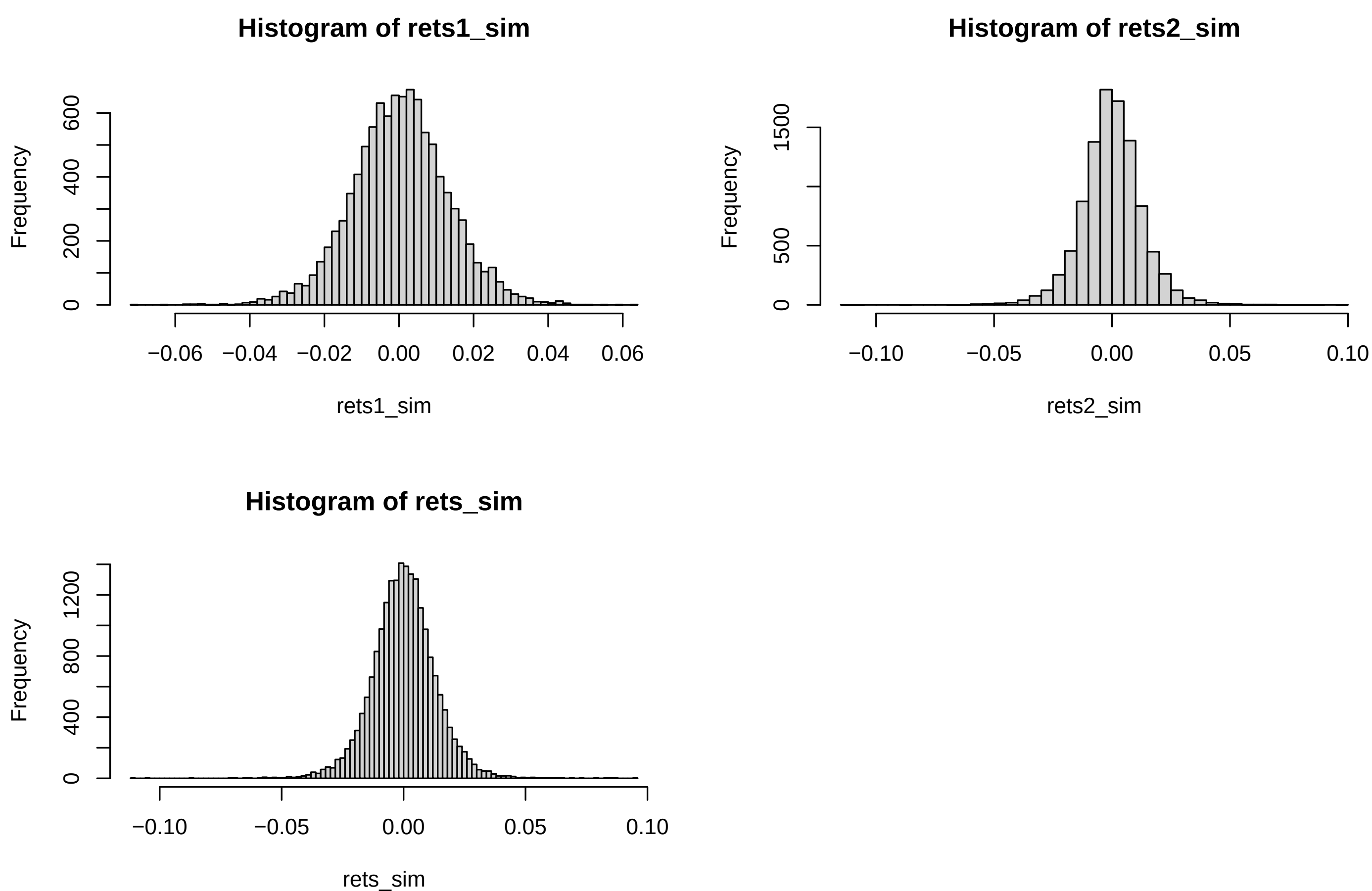



```
# Obtain the best rho for the Gaussian Copula
C <- normalCopula(dim = 2)
fit <- fitCopula(C, data = U, method = "ml")
fit
```

```
## Call: fitCopula(C, data = U, ... = pairlist(method = "ml"))
## Fit based on "maximum likelihood" and 3409 2-dimensional observations.
## Copula: normalCopula
##   rho.1
## -0.7984
## The maximized loglikelihood is 1494
## Optimization converged
```

```
# seed for replication
set.seed(420)

# Simulation parameters
n_sim = 10000 # set number of simulations
# n_ahead = 5 # days ahead to produce samples

# produce simulations from copula
U_sim <- rCopula(n_sim, fit@copula)

# use copula U_sim to reproduce the marginals with student-t distr
# rets1_sim <- qstd(U_sim[,1], mean = mu[1], sd = sigma[1], nu = theta1[3]) # sp500
# rets2_sim <- qstd(U_sim[,2], mean = mu[1], sd = sigma[1], nu = theta2[3]) # vix
rets1_sim <- qstd(U_sim[,1], mean = mu[1], sd = sigma[1], nu = 10) # sp500
rets2_sim <- qstd(U_sim[,2], mean = mu[1], sd = sigma[1], nu = 5) # vix
```

```
rets_sim <- cbind(rets1_sim, rets2_sim)

# visualize
par(mfrow = c(2,2))
hist(rets1_sim, nclass=50)
hist(rets2_sim, nclass=50)
hist(rets_sim, nclass = round(10 * log(n_sim)))
```




```
# random seed for replication
set.seed(69)

##############################
### Setup & Initialization ###
##############################

# Simulation parameters
n_sim = 10000 # set number of simulations
n_ahead = 5 # days ahead to produce samples

# preallocate matrices to store simulations
sim_rets_sp500 <- matrix(NA, nrow = n_sim, ncol=5)
sim_rets_vix <- matrix(NA, nrow = n_sim, ncol=5)

# assign days ahead
colnames(sim_rets_sp500) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
colnames(sim_rets_vix) <- c("T+1", "T+2", "T+3", "T+4", "T+5")

##############################
```

```r
### Running the simulation ###
##############################

# perform n_head days of n_sim scenarios
for(t in 1:n_ahead){

  # Sample 5-days ahead from Gaussian Copula
  U_sim <- rCopula(n_sim, fit@copula)

  # use copula U_sim to reproduce the marginals quantiles F^{-1}(u) with student-t distr
  rets1_sim <- qstd(U_sim[,1], mean = theta1[1], sd = theta1[2], nu = 10) # sp500
  rets2_sim <- qstd(U_sim[,2], mean = theta2[1], sd = theta2[2], nu = 5) # vix
  # rets1_sim <- qt(U_sim[,1], df = 10) # sp500
  # rets2_sim <- qt(U_sim[,2], df = 5) # vix
  rets_sim <- cbind(rets1_sim, rets2_sim)

  # store simulation of log return in matrix
  sim_rets_sp500[ ,t] <- rets1_sim
  sim_rets_vix[ ,t] <- rets2_sim
}

# preview of simulated log returns
head(sim_rets_sp500)
```

```
##                T+1           T+2          T+3           T+4           T+5
## [1,] -0.0009645126  0.0158554079  0.015383149 -0.0174310226  0.0155476192
## [2,]  0.0022227010  0.0178616966  0.003249986  0.0015075435  0.0004961551
## [3,] -0.0202696762  0.0070645564  0.011080134 -0.0163632659  0.0039542793
## [4,]  0.0267344996  0.0190648399 -0.004275895  0.0312856876  0.0004778219
## [5,] -0.0045092785  0.0008870700 -0.005286801  0.0089807462 -0.0122229007
## [6,] -0.0039970206 -0.0002199501 -0.003419139 -0.0004051185  0.0371441443
```

```r
head(sim_rets_vix)
```

```
##              T+1          T+2          T+3         T+4         T+5
## [1,]  0.01231074  0.006644294 -0.005354024  0.01255679 -0.04752175
## [2,] -0.04109607 -0.073223553 -0.020098934 -0.03207569 -0.06300583
## [3,]  0.08429964 -0.030662396 -0.071921523  0.10934242 -0.05145715
## [4,] -0.08896620 -0.032518583  0.020560914 -0.12085679 -0.02129170
## [5,] -0.05179948 -0.017505235  0.022004416 -0.04412445  0.03046923
## [6,]  0.01708910 -0.034281364  0.032441799  0.04104414 -0.11119617
```

Computing Prices from Returns

Next, we crate a function to forecast the 5 day ahead prices from the returns. Since:

$$
\begin{aligned}
R_t &= \frac{P_t - P_{t-1}}{P_{t-1}} \\
\implies R_t &= \frac{P_t}{P_{t-1}} - 1 \\
\implies \log(R_t) &= \log\left(\frac{P_t}{P_{t-1}}\right) \\
\implies \log(R_t) &= \log(P_t) - \log(P_{t-1}) \\
\implies \log(P_t) &= \log(R_t) + \log(P_{t-1}) \\
\implies P_t &= \exp(\log(R_t) + \log(P_{t-1})) \\
\implies P_{t+1} &= \exp(\log(R_{t+1}) + \log(P_t))
\end{aligned}
$$

This logic is implemented in the `f_logret_to_price()` function through the `f_next_Pt()` function, under the `code/Utils.R` script.

```
# Obtain Initial values (last value of simulation)
spT <- sp500[length(sp500)][[1]]
vixT <- vix[length(vix)][[1]]

# calculate the price and values from the simulated log-returns
sim_val_mats <- f_logret_to_price(sp_init = spT,
                                  vix_init = vixT,
                                  sim_rets_sp500 = sim_rets_sp500,
                                  sim_rets_vix = sim_rets_vix
                                  )

# unpack matrices
sim_price_sp500 <- sim_val_mats$sp500
sim_vol_vix <- sim_val_mats$vix
```

```
# compare simulated returns with the price
head(sim_rets_sp500)
```

```
##                T+1           T+2          T+3           T+4           T+5
## [1,] -0.0009645126  0.0158554079  0.015383149 -0.0174310226  0.0155476192
## [2,]  0.0022227010  0.0178616966  0.003249986  0.0015075435  0.0004961551
## [3,] -0.0202696762  0.0070645564  0.011080134 -0.0163632659  0.0039542793
## [4,]  0.0267344996  0.0190648399 -0.004275895  0.0312856876  0.0004778219
## [5,] -0.0045092785  0.0008870700 -0.005286801  0.0089807462 -0.0122229007
## [6,] -0.0039970206 -0.0002199501 -0.003419139 -0.0004051185  0.0371441443
```

```
head(sim_price_sp500)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 1682.367 1709.254 1735.751 1705.757 1732.485
## 2 1687.737 1718.154 1723.747 1726.347 1727.204
## 3 1650.200 1661.899 1680.415 1653.142 1659.692
## 4 1729.618 1762.909 1755.387 1811.174 1812.039
## 5 1676.414 1677.901 1669.054 1684.111 1663.651
## 6 1677.272 1676.904 1671.180 1670.503 1733.719
```

```
# compare simualted log rets with volatility
head(sim_rets_vix)
```

```
##              T+1          T+2          T+3         T+4         T+5
## [1,]  0.01231074  0.006644294 -0.005354024  0.01255679 -0.04752175
## [2,] -0.04109607 -0.073223553 -0.020098934 -0.03207569 -0.06300583
## [3,]  0.08429964 -0.030662396 -0.071921523  0.10934242 -0.05145715
## [4,] -0.08896620 -0.032518583  0.020560914 -0.12085679 -0.02129170
## [5,] -0.05179948 -0.017505235  0.022004416 -0.04412445  0.03046923
## [6,]  0.01708910 -0.034281364  0.032441799  0.04104414 -0.11119617
```

```
head(sim_vol_vix)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1 0.1470998 0.1480804 0.1472897 0.1491509 0.1422287
## 2 0.1394498 0.1296037 0.1270248 0.1230150 0.1155035
## 3 0.1580798 0.1533063 0.1426674 0.1591518 0.1511695
## 4 0.1329316 0.1286783 0.1313515 0.1163985 0.1139464
## 5 0.1379651 0.1355711 0.1385873 0.1326051 0.1367077
## 6 0.1478044 0.1428233 0.1475327 0.1537141 0.1375377
```

```r
# save data from the simulated values
save(sim_price_sp500, file=here("data_out", "sim_vol_sp500_student_copula.rda"))
save(sim_vol_vix, file=here("data_out", "sim_vol_vix_student_copula.rda"))
```

Pricing the simulation scenarios

Recall the initial (call) options:

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1605$ with maturity $T = 40d$
3. **1x** strike $K = 1800$ with maturity $T = 40d$

Option Pricing of Simulated Values

Next, we calculate the price of the book of options for the simulated values using the `f_opt_price_simulation()` function under `code/OptionPricing.R`:

```r
# random seed for replication
set.seed(123)

# Initialize strikes and maturities the options
T_vec <- c(20, 20, 40,40) #  maturities
K_vec <- c(1600, 1650, 1750, 1800) # Strikes

# Obtain the obtion prices from simulation values
opt_price_mats <- f_opt_price_simulation(sim_price_sp500 = sim_price_sp500,
                                         sim_vol_vix = sim_vol_vix,
                                         K_vec = K_vec,
                                         T_vec = T_vec,
                                         put=FALSE)
```

```r
# overview of dataframes
head(opt_price_mats$opt1)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1  85.93966 110.69740 136.26860 107.01790 132.81728
## 2  90.24098 118.71906 124.10892 126.59144 127.36517
## 3  60.25326  68.45593  83.19676  60.93087  64.90429
## 4 130.13155 163.09257 155.57730 211.29567 212.15179
## 5  79.84446  80.69383  72.68668  85.73559  67.03312
## 6  81.47264  80.36234  75.33654  74.88091 133.99041
```

```r
head(opt_price_mats$opt2)
```

```
##        T+1       T+2       T+3       T+4       T+5
## 1 46.25238  66.20369  88.70371  62.42688  84.74275
## 2 48.85695  71.80725  76.29910  78.16879  78.37897
## 3 28.82996  33.59097  42.91411  28.16558  29.63988
## 4 82.65190 113.67027 106.33042 161.30988 162.16013
## 5 40.69084  40.69720  34.71388  43.46786  29.67818
## 6 42.91247  41.18801  37.52507  37.34910  85.61179
```

```r
head(opt_price_mats$opt3)
```

```
##         T+1      T+2       T+3       T+4       T+5
## 1 14.810848 22.79605 32.780281 20.869586 28.958731
```

```
## 2 14.568687 21.40844 22.440547 22.011624 20.050466
## 3  9.896975 10.94269 12.611995  9.539164  9.060680
## 4 27.267439 42.16247 38.244662 71.282004 71.171124
## 5 11.490916 11.00432  9.378581 11.127012  7.444574
## 6 13.679359 12.19921 11.448791 12.070888 28.306100
```

```
head(opt_price_mats$opt4)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1  6.123012 10.265706 15.823742  9.093513 13.050811
## 2  5.750180  8.622494  8.911881  8.380779  6.947708
## 3  4.015923  4.362723  4.812672  3.739971  3.319742
## 4 11.997642 20.224478 17.932947 37.896755 37.291731
## 5  4.272045  3.942236  3.273257  3.813760  2.371687
## 6  5.586715  4.680559  4.407937  4.806311 12.402286
```

Distribution of the Profit and Loss for the Book Of Options

Recall the profit functions for European options:

Parameters

Parameters: - S: Spot price (current) - S_0: Spot price at the beginnin of the option - S_T: Spot price at maturity - T: Maturity of option - K: Strike price - c: Price of Call option - p: Price of Put option

Profit at Maturity

The profit functions of a long call and a long put are given by:

$$\pi^{\text{Long Call}} = \max(S_T - K, 0) - c$$
$$\pi^{\text{Long Put}} = \max(K - S_T, 0) - p$$

Calculating the profits

For each of the simulated prices and resulting premiums, we want to calculate the profit generated at each simulation timestep. The function used is `f_pl_simulation()`, found under `code/OptionPricing.R`.

```
# Initialize strikes and maturities the options
T_vec <- c(20, 20, 40,40) #  maturities
K_vec <- c(1600, 1650, 1750, 1800) # Strikes

# Compute the profits and loses for the simulation  from the simulated option premiums
PL_mats <- f_pl_simulation(sim_price_sp500 = sim_price_sp500,
                           opt_price_mats = opt_price_mats,
                           K_vec = K_vec)
```

```
# display profit matrices
head(PL_mats$PL1)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 156.9949 187.2824 204.0350 281.2273 266.0588
## 2 152.6936 179.2607 216.1946 261.6537 271.5109
## 3 182.6813 229.5238 257.1068 327.3143 333.9718
## 4 112.8030 134.8872 184.7263 176.9495 186.7242
## 5 163.0901 217.2859 267.6169 302.5096 331.8429
## 6 161.4620 217.6174 264.9670 313.3643 264.8856
```

```r
head(PL_mats$PL2)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 146.6822 181.7761 201.5999 275.8183 264.1333
## 2 144.0776 176.1725 214.0045 260.0764 270.4971
## 3 164.1046 214.3888 247.3895 310.0796 319.2362
## 4 110.2827 134.3095 183.9731 176.9353 186.7159
## 5 152.2438 207.2826 255.5897 294.7773 319.1979
## 6 150.0221 206.7918 252.7785 300.8961 263.2643
```

```r
head(PL_mats$PL3)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 78.12374 125.1837 157.5233 217.3756 219.9173
## 2 78.36590 126.5713 167.8630 216.2336 228.8256
## 3 83.03762 137.0371 177.6916 228.7060 239.8154
## 4 65.66715 105.8173 152.0589 166.9632 177.7049
## 5 81.44367 136.9754 180.9250 227.1182 241.4315
## 6 79.25523 135.7806 178.8548 226.1743 220.5699
```

```r
head(PL_mats$PL4)
```

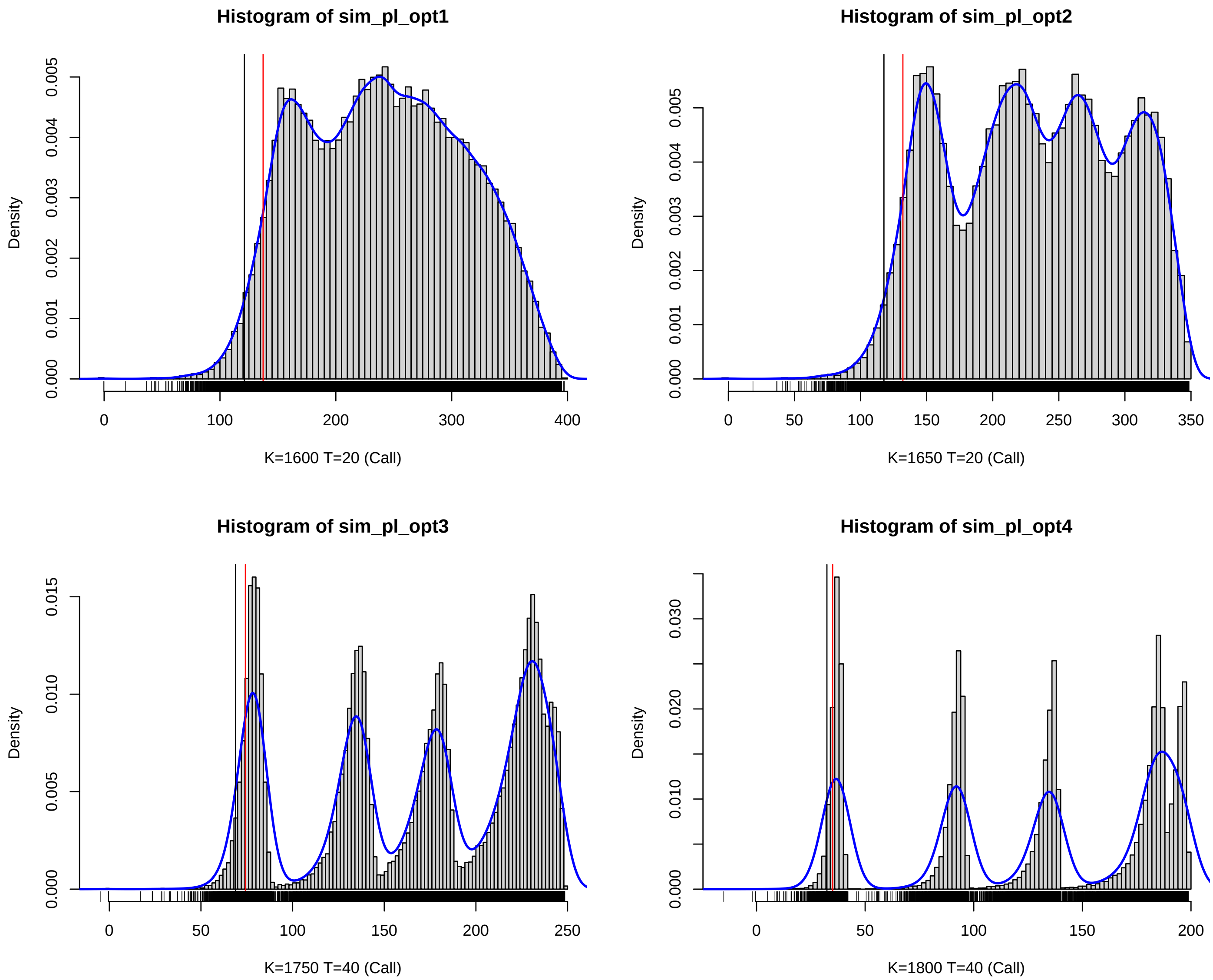
```
##        T+1      T+2      T+3      T+4      T+5
## 1 36.81158 87.71407 124.4798 179.1517 185.8252
## 2 37.18441 89.35728 131.3917 179.8644 191.9283
## 3 38.91867 93.61705 135.4909 184.5052 195.5563
## 4 30.93695 77.75529 122.3706 150.3484 161.5843
## 5 38.66255 94.03754 137.0303 184.4314 196.5044
## 6 37.34788 93.29921 135.8956 183.4389 186.4738
```

Distribution of Options P/L

Next, using all the simulated profits and losses for each of the options, we display a histogram for the distribution for each of the options, for the aggregated 5 days of simulation:

```r
# flatten the matrices 5-days ahead simulated P/L for the three options
sim_pl_opt1 <- as.vector(PL_mats$PL1)
sim_pl_opt2 <- as.vector(PL_mats$PL2)
sim_pl_opt3 <- as.vector(PL_mats$PL3)
sim_pl_opt4 <- as.vector(PL_mats$PL4)

# Compute the 95% VaR and 95% ES
opt1_VaR_ES <- f_VaR_ES(sim_pl_opt1, alpha = 0.05)
opt2_VaR_ES <- f_VaR_ES(sim_pl_opt2, alpha = 0.05)
opt3_VaR_ES <- f_VaR_ES(sim_pl_opt3, alpha = 0.05)
opt4_VaR_ES <- f_VaR_ES(sim_pl_opt4, alpha = 0.05)

# plot the distribution for each of the options
par(mfrow = c(2,2))
hist(sim_pl_opt1, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[1], " T=", T_vec[1], " (Call)"))
lines(density(sim_pl_opt1), lwd=2, col="blue")
abline(v=opt1_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt1_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt1)
```

```
hist(sim_pl_opt2, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[2], " T=", T_vec[2], " (Call)"))
lines(density(sim_pl_opt2), lwd=2, col="blue")
abline(v=opt2_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt2_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt2)


hist(sim_pl_opt3, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[3], " T=", T_vec[3], " (Call)"))
lines(density(sim_pl_opt3), lwd=2, col="blue")
abline(v=opt3_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt3_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt3)

hist(sim_pl_opt4, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[4], " T=", T_vec[4], " (Call)"))
lines(density(sim_pl_opt4), lwd=2, col="blue")
abline(v=opt4_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt4_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt4)
```



These all look like multimodal distributions. The last one, particularly shows a different mode for each of the fives days computed. The 95% VaR (red) and ES (black) are all displayed in the plots.

VaR95

Definition

For a random variable X, the **Value-at-Risk (VaR)** at level α is defined as the α-lower quantile of the distribution of X, thus:

$$VaR_X(\alpha) = F_X^{-1}(1-\alpha)$$

First Option

```r
opt1_VaR_ES$VaR # first option
```

```
## [1] 137.2302
```

```r
opt2_VaR_ES$VaR # second doption
```

```
## [1] 132.0325
```

```
opt3_VaR_ES$VaR # third option
```

```
## [1] 74.2462
```

```
opt4_VaR_ES$VaR # fourth option
```

```
## [1] 35.0578
```

ES95

Expected shortfall is calculated by averaging all of the returns in the distribution that are worse than the VAR of the portfolio at a given level of confidence.

```
# display
opt1_VaR_ES$ES
```

```
## [1] 121.0392
```

```
opt2_VaR_ES$ES
```

```
## [1] 117.6961
```

```
opt3_VaR_ES$ES
```

```
## [1] 68.87072
```

```
opt4_VaR_ES$ES
```

```
## [1] 32.40935
```

Volatility Surface

```
##########################################################
###  ---> INSERT YOUR PART FROM HERE (ALESSIO) <---- ###
############################################### #########
```

Full Approach

1. Filter the volatility clustering of the log-returns of the underlying using a GARCH(1,1) model with Normal innovations. Use the residuals as invariants.
2. Take and AR(1) model for the log-returns of the VIX. Use the residuals as invariants.
3. Use normal marginals for the invariants and a normal copula.
4. Generate draws for the invariants, compute next week (five days) values and reprice the portfolio.
5. Compute the VaR95 and ES95.

Log returns of the underlying

```
# load reqruired libraries
library("PerformanceAnalytics")

# calculate returns
sp500_rets <- PerformanceAnalytics::CalculateReturns(sp500, method="log")
vix_rets <- PerformanceAnalytics::CalculateReturns(vix, method="log")

# remove first return
sp500_rets <- sp500_rets[-1]
vix_rets <- vix_rets[-1]

# remove nas
sp500_rets[is.na(sp500_rets)] <- 0
vix_rets[is.na(vix_rets)] <- 0

# display
head(sp500_rets)
```

```
##                    sp500
## 2000-01-04 -0.0390992269
## 2000-01-05  0.0019203798
## 2000-01-06  0.0009552461
## 2000-01-07  0.0267299353
## 2000-01-10  0.0111278213
## 2000-01-11 -0.0131486343
```

```
head(vix_rets)
```

```
##                      vix
## 2000-01-04  0.1094413969
## 2000-01-05 -0.0224644415
## 2000-01-06 -0.0260851000
## 2000-01-07 -0.1694241312
## 2000-01-10 -0.0004605112
## 2000-01-11  0.0357423253
```

GARCH(1,1) Model

Model specification

$$
\begin{aligned}
y_t &= \epsilon_t \sigma_t, \\
\sigma_t^2 &= \omega \; + \; \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\epsilon_t &\overset{i.i.d.}{\sim} \mathcal{N}(0,1),
\end{aligned}
$$

Mean and variance

$$
\begin{aligned}
\mathbb{E}[Y_t] &\approx 0 \\
\mathbb{V}ar[Y_t] = \mathbb{E}[\epsilon_t^2] = \mathbb{E}[\sigma_t^2] &= \frac{\omega}{(1-\alpha-\beta)}
\end{aligned}
$$

Stationarity Conditions

$$
\begin{aligned}
\omega &\geq 0 \\
\alpha\ ,\ \beta &> 0 \\
\alpha + \beta &< 1 \quad \text{(Covariance-Stationary)}
\end{aligned}
$$

VaR

$$
VaR_Y(\alpha) = \Phi^{-1}(1-\gamma)\sigma_t,
$$

Log-likelihood

$$
\ln L(\theta|\mathbf{y}) = -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T}\ln\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{y_t^2}{\sigma_t^2}.
$$

Volatility clustering of the log-returns of the underlying with GARCH(1,1)

Which indicates a high level of autocorrelation in the returns.

Fitting the GARCH(1,1)

```
# source code for garch
source(here("code", "GARCH.R")) # GARCH model implementation

# Estimate the GARCH(1,1) model
fit_garch <- f_optim_garch(sp500_rets)
```

```
## Aside: If we had used the MSGARCH package

# load MSGARCH
library("MSGARCH")
# GARCH with NOrmal innovations
garch_n <- MSGARCH::CreateSpec(variance.spec = list(model = c("sGARCH")),
                               distribution.spec = list(distribution = c("norm")))
fit_garch_n <- MSGARCH::FitML(spec = garch_n, data = sp500_rets)

#check the fit
summary(fit_garch_n)
```

```
## Specification type: Single-regime
## Specification name: sGARCH_norm
## Number of parameters in variance model: 3
## Number of parameters in distribution: 0
## ------------------------------------------
## Fitted parameters:
##          Estimate Std. Error  t value  Pr(>|t|)
## alpha0_1   0.0000     0.0000   4.7392 1.073e-06
## alpha1_1   0.0859     0.0294   2.9253 1.721e-03
## beta_1     0.9035     0.0038 240.8889    <1e-16
## ------------------------------------------
## LL: 10660.12
## AIC: -21314.24
## BIC: -21295.8374
## ------------------------------------------
```

Inpect the parameters

```
# extract parameters (omega, alpha, beta)
theta_hat_garch <- fit_garch$theta_hat
theta_hat_garch
```

```
## [1] 0.000001461511 0.090037405420 0.903175089840
```

Verify stationarity

```
# make sure stationarity is satisfied
sum(theta_hat_garch[2:3])
```

```
## [1] 0.9932125
```

Mean Squared Error

```
# MSE ?
sqrt(theta_hat_garch[1] / (1 - sum(theta_hat_garch[2:3]))) * sqrt(250)
```

```
## [1] 0.2320149
```

```
# sd of returns annualized?
sd(sp500_rets) * sqrt(250)
```

```
## [1] 0.2107013
```

Residuals

The residuals are given by:

$$\hat{\epsilon}_t = \frac{y_t}{\hat{\sigma}_t}$$

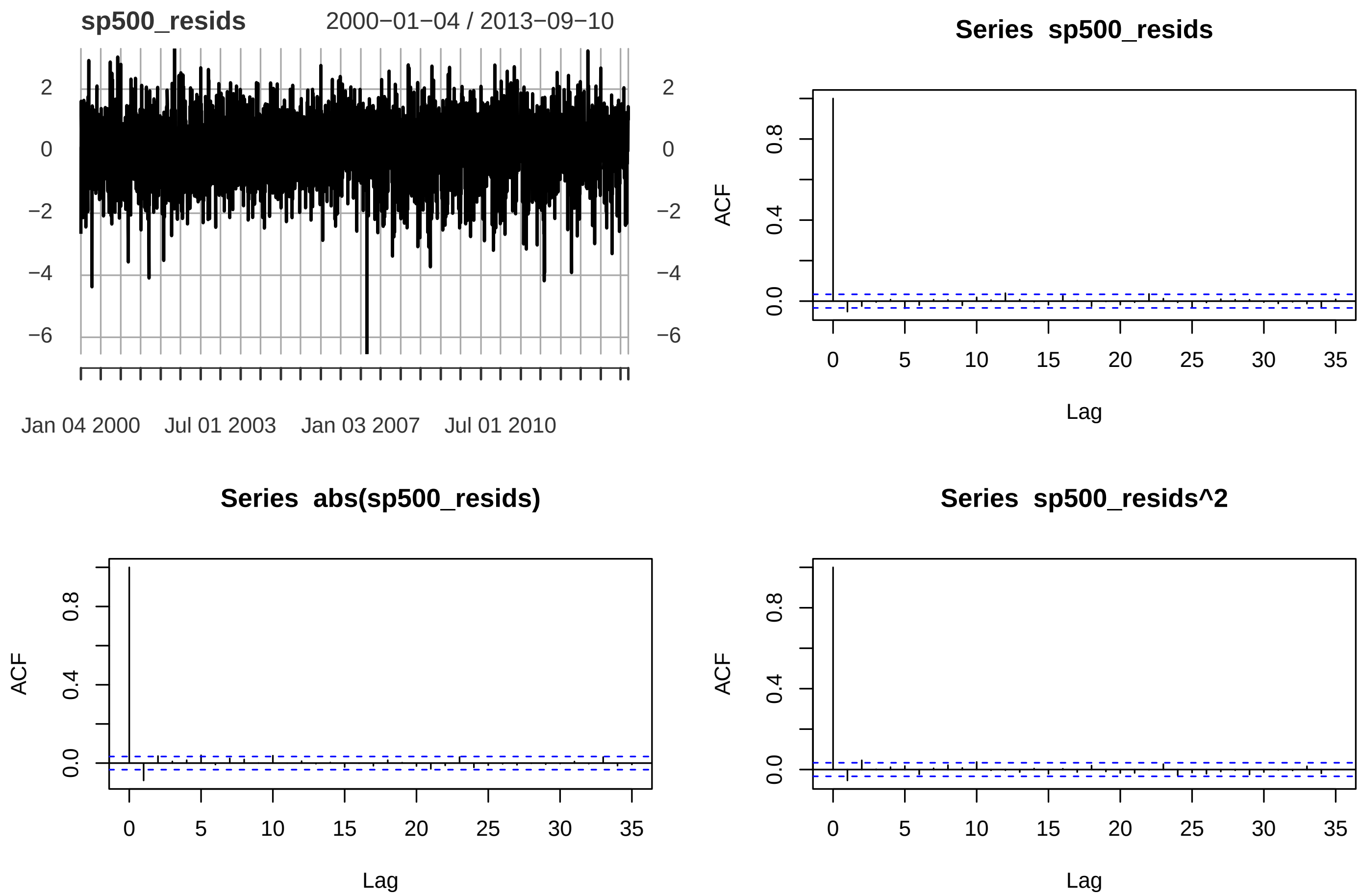
```
# extrct the residuals
sp500_resids <- fit_garch$eps_hat

# inspect their mean and variance
mean(sp500_resids)
```

```
## [1] 0.005801314
```

```
sd(sp500_resids)
```

```
## [1] 0.9908629
```

```
# Look at dependence in the residuals
par(mfrow = c(2,2))

# Eps_hat = Innovations Series
plot(sp500_resids, pch = 20)

# autocorr of innovations
acf(sp500_resids)

# autocorr of the absolute values
acf(abs(sp500_resids))

# autocorr of the variance of the innovations
acf(sp500_resids^2)
```



Fitting GARCH(1,1) with mean

AR(1) for the log-returns of the VIX

First-order Autoregressive Process AR(1)

- Let $\{\varepsilon_t\}$ be a mean-zero white noise process with variance σ^2.
- Consider a process $\{X_t\}$, independent of $\{\varepsilon_t\}$.
- Let ϕ be constant.

The **AR(1) process** satisfies:

$$X_t = \phi X_{t-1} + \varepsilon_t$$

It can be shown that:

$$\mu_X(t) = \mathbb{E}[X_t] = \phi \mu_X(t-1) = 0 \quad ,\forall t$$

when the process is stationary, and the autocovariance function $\gamma_X(h)$ with alg h and autocorrelation $\rho_X(h)$ are given by

$$\gamma_X(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2} \quad \text{and} \quad \rho_X(h) = \phi^{|h|}$$

VIX log-returns

```
library("forecast")
# Construct an AR(1) model to the vix
vix_ar1 <- ar(vix_rets, order.max = 1)
vix_ar1$ar # phi coefficient
```

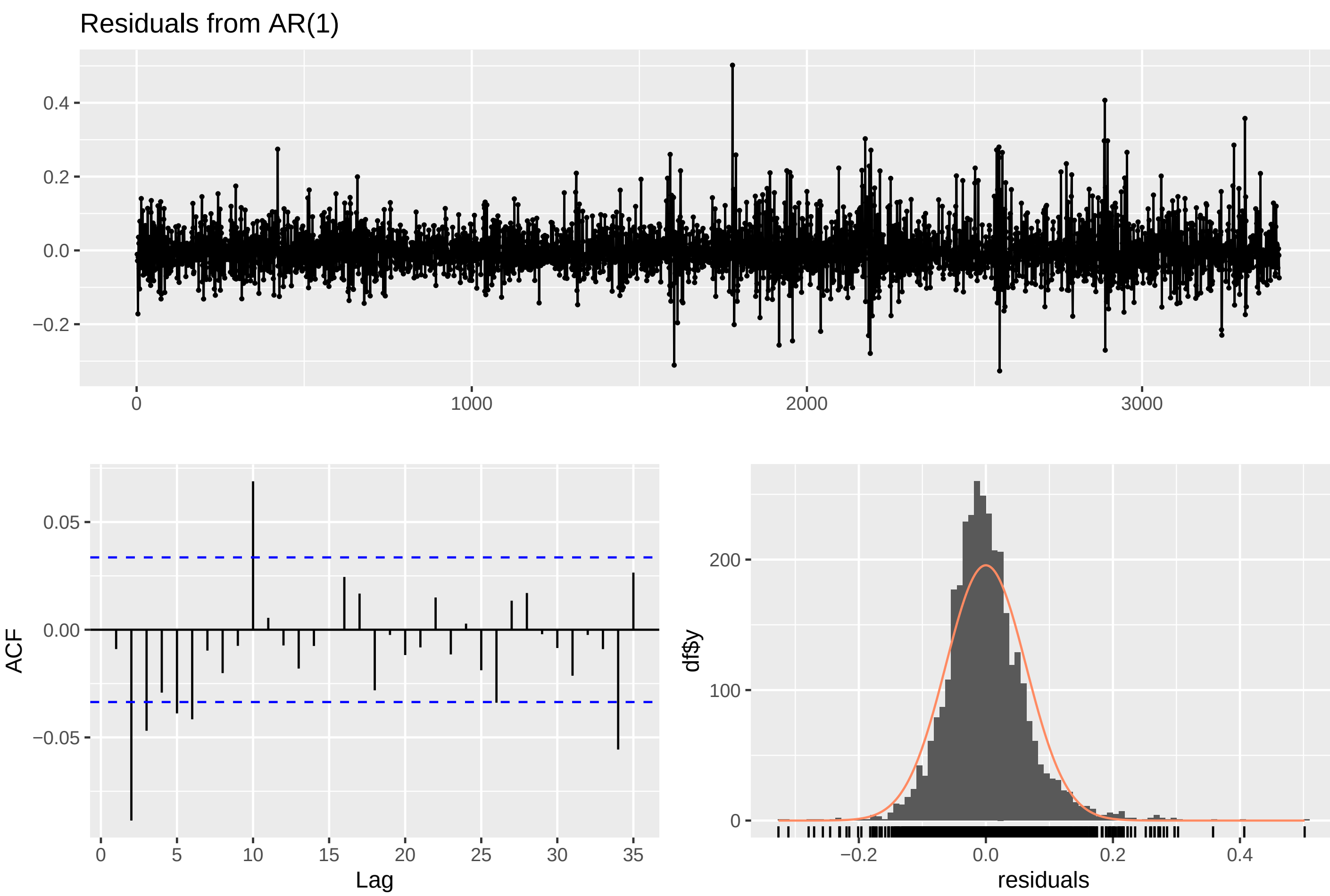
```
## [1] -0.1074941
```

Stationarity of the residuals & underlying normality

```
# extract the residuals
vix_resids <- vix_ar1$resid
vix_resids[1] <- 0 # first residual is NA
head(vix_resids)
```

```
## [1]  0.00000000 -0.01053427 -0.02833403 -0.17206226 -0.01850675  0.03585869
```

```
# comes from the forecast package
checkresiduals(vix_ar1, main="Residuals for AR(1) Model")
```




```
##
##  Ljung-Box test
##
## data:  Residuals from AR(1)
## Q* = 66.683, df = 10, p-value = 0.0000000001929
##
## Model df: 0.   Total lags used: 10
```

Normal Copula with Normal Marginals for the Invariants

Bivariate Gaussian Copula

Recall that the bivariate Gaussian copula is given by:

$$C_\rho^{\text{Gauss}}(u,v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)). \quad \Longleftarrow \quad H(x,y) = C(F(x), G(y))$$

$$C_\rho^{\text{Gauss}}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dxdy$$

Gaussian marginals to the invariants

```
# invariants are the residuals
sp500_resids <- as.vector(sp500_resids)
vix_resids <- as.vector(vix_resids)

# display some values
head(sp500_resids, 10)
```

```
##  [1] -2.66453978  0.10514445  0.05487059  1.61107285  0.62756665 -0.76350023
##  [7] -0.26044462  0.74944189  0.67144427 -0.44500488
```

```
head(vix_resids, 10)
```

```
##  [1]  0.00000000 -0.01053427 -0.02833403 -0.17206226 -0.01850675  0.03585869
##  [7]  0.01900603 -0.04896233 -0.10447530  0.07897068
```

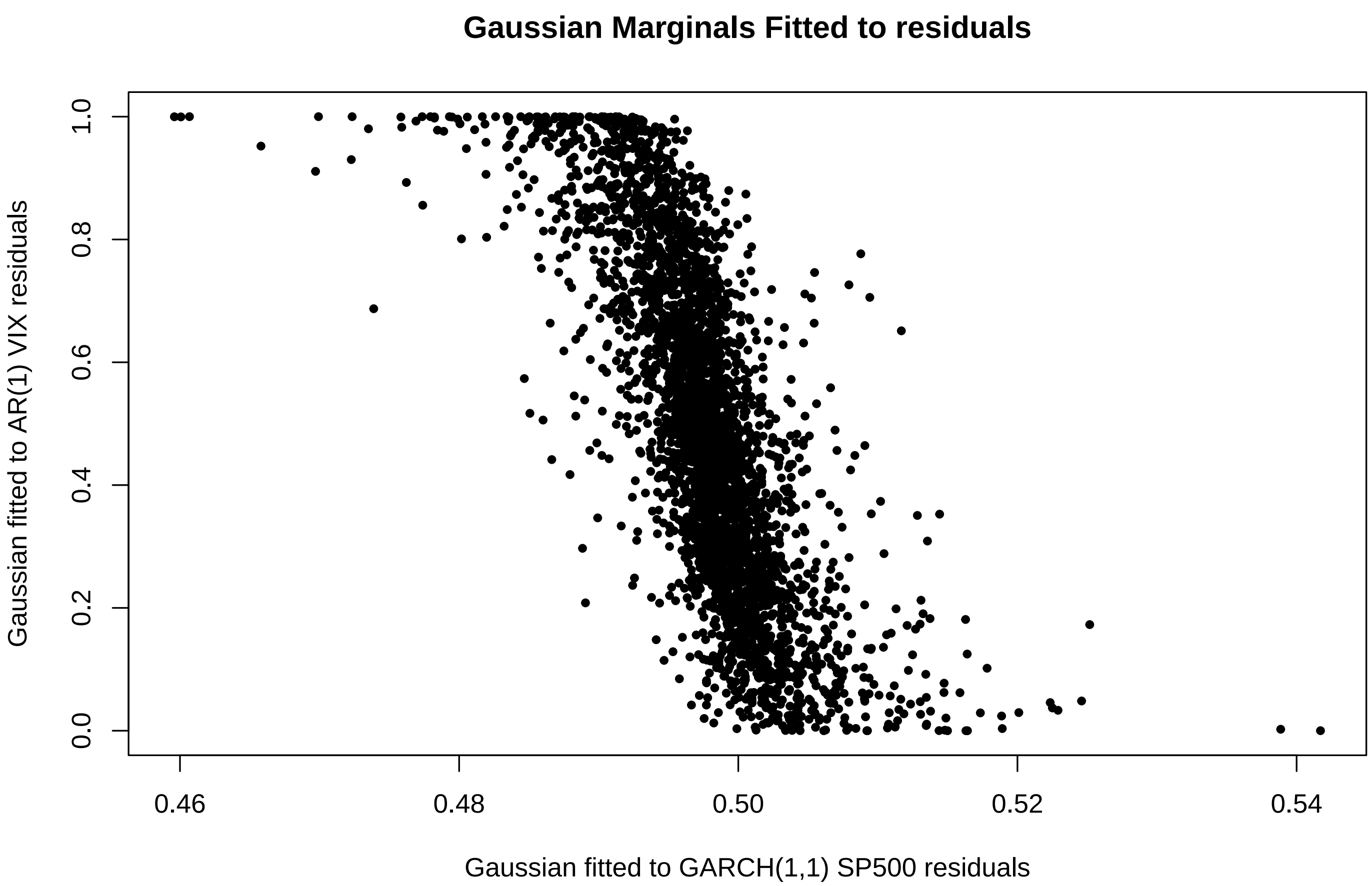
```
library("MASS")
## Fit marginals by MLE

# Gaussian for sp500 invariants (from the GARCH(1,1))
fit1 <- suppressWarnings(
  fitdistr(x = sp500_resids,
           densfun = dnorm,
           start = list(mean = 0, sd = 1))
  )
theta1 <- fit1$estimate #extract fitted parameters

# Gaussian for vix invariants (from the AR(1))
fit2 <- suppressWarnings(
  fitdistr(x = vix_resids,
           densfun = dnorm,
           start = list(mean = 0, sd = 1))
  )
theta2 <- fit2$estimate # extract fitted parameters

# display parameters
theta1
```

```
##        mean          sd
## 0.005801451 0.990717432
```

```
theta2
```

```
##           mean             sd
## -0.00004052605  0.06327442562
```

```
# Fit a Gaussian to the marginals
U1 <- pnorm(sp500_rets_vec, mean = theta1[1], sd = theta1[2]) # sp500
U2 <- pnorm(vix_rets_vec,mean = theta2[1], sd = theta2[2]) # vix
U <- cbind(U1, U2) # join into one matrix
plot(U,
     pch = 20, cex = 0.9,
     main="Gaussian Marginals Fitted to residuals",
     xlab="Gaussian fitted to GARCH(1,1) SP500 residuals",
     ylab="Gaussian fitted to AR(1) VIX residuals"
     )
```



Fitting the Gaussian Copula

```
# Obtain the best rho for the Gaussian Copula
C <- normalCopula(dim = 2)
fit <- fitCopula(C, data = U, method = "ml")
fit
```

```
## Call: fitCopula(C, data = U, ... = pairlist(method = "ml"))
## Fit based on "maximum likelihood" and 3409 2-dimensional observations.
## Copula: normalCopula
##   rho.1
## -0.2006
## The maximized loglikelihood is 4.903
## Optimization converged
```

Simulating the invariants with the Copula

```r
# random seed for replication
set.seed(69)

##############################
### Setup & Initialization ###
##############################

# Simulation parameters
n_sim = 10000 # set number of simulations
n_ahead = 5 # days ahead to produce samples

# preallocate matrices to store simulations
sim_inv_sp500 <- matrix(NA, nrow = n_sim, ncol=5)
sim_inv_vix <- matrix(NA, nrow = n_sim, ncol=5)

# assign days ahead
colnames(sim_inv_sp500) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
colnames(sim_inv_vix) <- c("T+1", "T+2", "T+3", "T+4", "T+5")

##############################
### Running the simulation ###
##############################

# perform n_head days of n_sim scenarios
for(t in 1:n_ahead){

  # Sample n_sim scenarios from Gaussian Copula
  U_sim <- rCopula(n_sim, fit@copula)

  # use copula U_sim to reproduce the marginals quantiles F^{-1}(u) with Gaussian distr
  inv1_sim <- qnorm(U_sim[,1], mean = theta1[1], sd = theta1[2]) # sp500
  inv2_sim <- qnorm(U_sim[,2], mean = theta2[1], sd = theta2[2]) # vix
  invs_sim <- cbind(rets1_sim, rets2_sim)

  # store simulation of log return in matrix
  sim_inv_sp500[ ,t] <- inv1_sim
  sim_inv_vix[ ,t] <- inv2_sim
}

# preview of simulated invariants
head(sim_inv_sp500)
```

```
##              T+1        T+2         T+3        T+4        T+5
## [1,]  0.04447154  1.5691517  1.39371665 -1.4835644  0.9565560
## [2,] -0.22933227  0.9195288  0.09356549 -0.2042606 -0.6108582
## [3,] -1.03976298  0.3455542  0.31955037 -0.5236770 -0.1662329
## [4,]  1.51949269  1.4194440 -0.18535447  1.6314713 -0.1877315
## [5,] -0.98740133 -0.1065783 -0.26676884  0.3869440 -0.8275658
## [6,] -0.19608631 -0.3949083  0.02257609  0.4012918  2.1015336
```

```r
head(sim_inv_vix)
```

```
##              T+1          T+2         T+3         T+4         T+5
## [1,]  0.02303111  0.055872097  0.03438314 -0.01742967 -0.03425186
## [2,] -0.05770198 -0.065635901 -0.02089352 -0.04514607 -0.09491143
## [3,]  0.08084674 -0.028569376 -0.08021689  0.11747472 -0.06914887
```

```
## [4,] -0.06615843 -0.002383062  0.02820027 -0.09207113 -0.03004906
## [5,] -0.09149873 -0.022648918  0.02798077 -0.04514606  0.02429271
## [6,]  0.02318232 -0.053178235  0.04957536  0.07071302 -0.07137518
```

Transforming back the invariants to returns

From GARCH(1,1) residuals to SP500 returns

$$\hat{\epsilon}_t = \frac{y_t}{\hat{\sigma}_t} \implies \hat{y}_t = \hat{\epsilon}_t \hat{\sigma}_t$$

and

$$\begin{cases} \sigma_t^2 = \omega \ + \ \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\ \hat{y_t} = \hat{\epsilon}_t \hat{\sigma}_t \end{cases}$$

$$\begin{cases} \sigma_{T+1}^2 = \omega \ + \ \alpha y_T^2 + \beta \sigma_T^2 \\ \hat{y_{T+1}} = \hat{\epsilon}_{T+1} \hat{\sigma}_{T+1} \\ \vdots \\ \sigma_{T+t}^2 = \omega \ + \ \alpha y_{T+t-1}^2 + \beta \sigma_{T+t-1}^2 \\ \hat{y_{T+t}} = \hat{\epsilon}_{T+t} \cdot \hat{\sigma}_{T+t} \end{cases}$$

First, obtain last conditional variance available (up to time T):

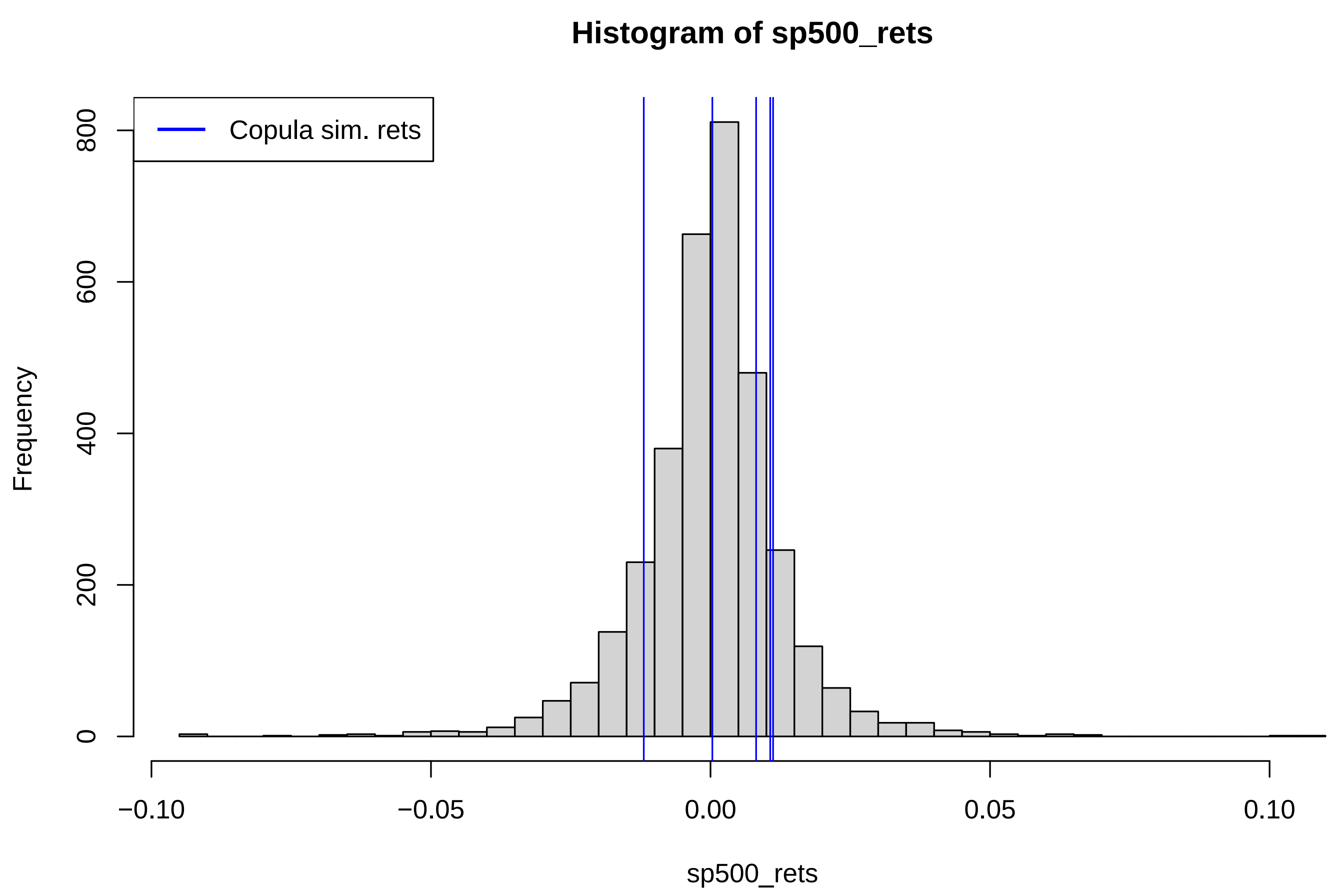
```r
# load source code with GARCh custom functions
source(here("code", "GARCH.R")) # display the pdf through a 3-d chart

# data from up to T
y <- sp500_rets_vec
sig2 <- fit_garch$sig2_hat # vector of sig2 from GARCH
theta <- fit_garch$theta_hat # GARCH parameters

# initial parameters
y_prev <- y[length(y)] # last sp500 observation
sig2_prev <- sig2[length(sig2)] # last sig2_T

# residuals forecasted from copula (invariants)
garch_resids_next <- sim_inv_sp500
resids_next <- garch_resids_next[1, ] # example vector of residuals for prediction

# obtain 5days-ahead prediction for variance
sig2_forecast <- f_forecast_y(theta = theta,
                              sig2_prev = sig2_prev,
                              y_prev = y_prev,
                              resids_next = resids_next)

sig2_forecast
```

```
## $resids_next
##         T+1         T+2         T+3         T+4         T+5
##  0.04447154  1.56915167  1.39371665 -1.48356435  0.95655596
##
## $sig2_next
## [1] 0.00005469191 0.00005086762 0.00005868089 0.00006472350 0.00007274435
##
## $y_next
## [1]  0.0003288847  0.0111914311  0.0106763511 -0.0119354110  0.0081584945
```

```
# apply to all rows and pack into a matrix
sp500_sim_rets_full <- t(apply(sim_inv_sp500, 1, function(x){f_forecast_y(theta=theta,
                                                                          sig2_prev = sig2_prev,
                                                                          y_prev = y_prev,
                                                                          resids_next = x)$y_next}))
colnames(sp500_sim_rets_full) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
head(sp500_sim_rets_full)
```

```
##                T+1           T+2           T+3          T+4          T+5
## [1,]  0.0003288847  0.0111914311  0.0106763511 -0.011935411  0.008158495
## [2,] -0.0016960033  0.0065742685  0.0006715923 -0.001415663 -0.004098909
## [3,] -0.0076894607  0.0025900801  0.0023221298 -0.003689649 -0.001145947
## [4,]  0.0112372528  0.0111971934 -0.0015391388  0.013046763 -0.001620878
## [5,] -0.0073022255 -0.0007951261 -0.0019197751  0.002696617 -0.005611657
## [6,] -0.0014501362 -0.0028215149  0.0001568720  0.002694088  0.013752217
```

```
# example 5-days ahead simulation vs actual values:
hist(sp500_rets, nclass=30)
abline(v=sig2_forecast$y_next, col="blue")
legend(x="topleft",
       legend = c("Copula sim. rets"),
       col = c("blue"),
       lwd=rep(2, time=2))
```



From AR(1) residuals to VIX observations

The AR(1) model specifies

$$X_t = \phi X_{t-1} + \varepsilon_t$$

therefore for the step ahead predictions

$$\begin{cases} x_{T+1} = \phi x_T + \varepsilon_T \\ x_{T+2} = \phi x_{T+1} + \varepsilon_{T+1} \\ \vdots \\ x_{T+t} = \phi x_{T+t-1} + \varepsilon_{T+t-1} \end{cases}$$

Transforming the simulated returns into SP500 prices and VIX values

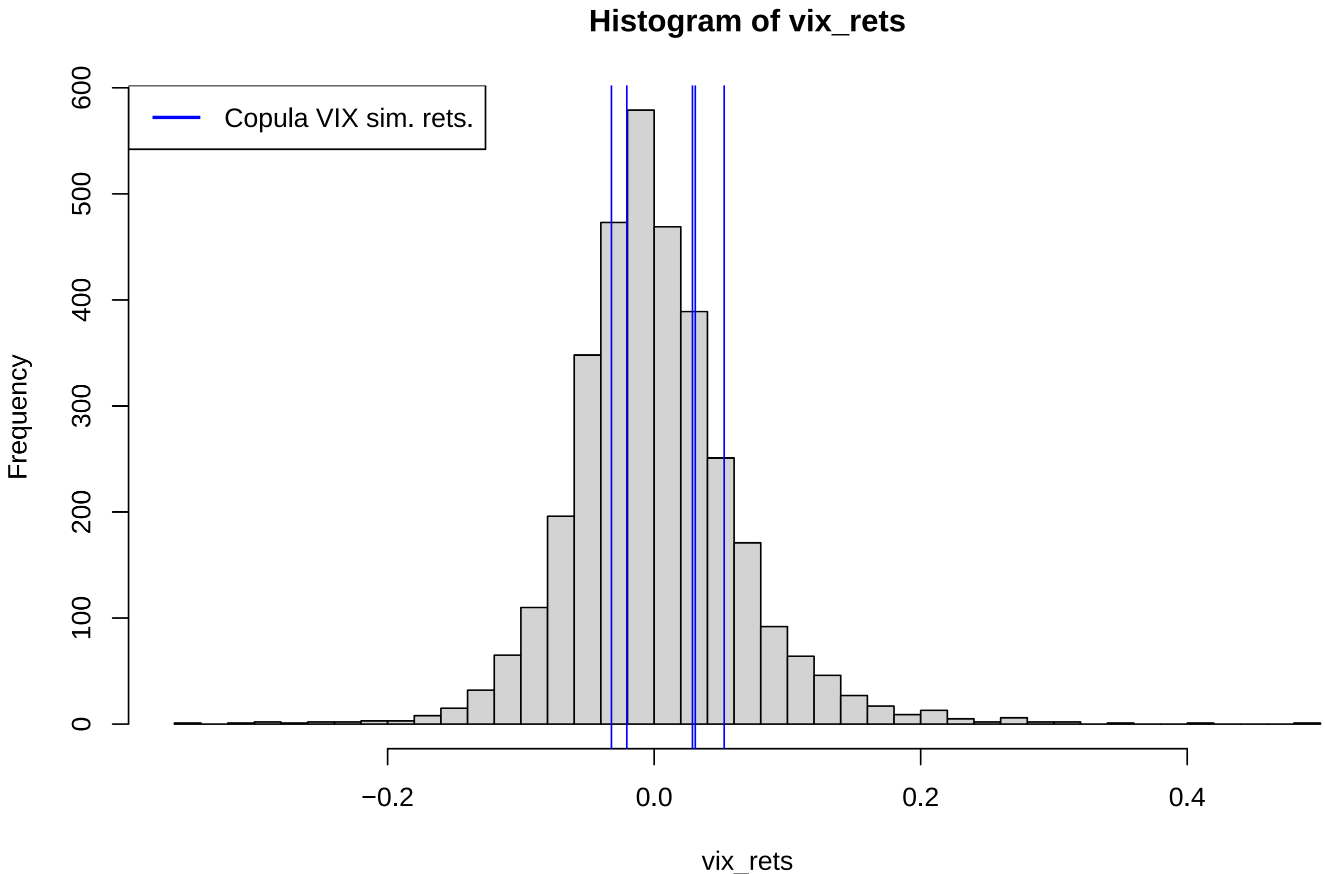
```
# data from up to T (VIX)
x <- vix_rets_vec

# initial parameters
phi <- vix_ar1$ar
x_prev <- x[length(x)] # last vix observation

# residuals forecasted from copula (vix invariants)
ar1_resids_next <- sim_inv_vix
ar1_res_next <- ar1_resids_next[1, ] # example vector

# forecast the vix values using the copula simulated residuals
ex_vix_forecast <- f_forecast_x(phi=phi, x_prev = x_prev, resids_next = ar1_res_next)
ex_vix_forecast
```

```
## [1]  0.03087567  0.05255314  0.02873398 -0.02051841 -0.03204625
```

```
# apply to all rows and pack into a matrix
vix_sim_rets_full <- t(apply(sim_inv_vix, 1, function(x){f_forecast_x(phi=phi,
                                                                     x_prev = x_prev,
                                                                     resids_next = x)}))
colnames(vix_sim_rets_full) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
head(vix_sim_rets_full)
```

```
##              T+1          T+2         T+3         T+4         T+5
## [1,]  0.03087567  0.052553143  0.02873398 -0.02051841 -0.03204625
## [2,] -0.04985742 -0.060276522 -0.01441415 -0.04359663 -0.09022505
## [3,]  0.08869130 -0.038103171 -0.07612103  0.12565729 -0.08265629
## [4,] -0.05831387  0.003885337  0.02778262 -0.09505760 -0.01983093
## [5,] -0.08365417 -0.013656585  0.02944877 -0.04831163  0.02948593
## [6,]  0.03102688 -0.056513443  0.05565022  0.06473095 -0.07833338
```

```
# example 5-days ahead simulation vs actual values:
hist(vix_rets, nclass=40)
abline(v=ex_vix_forecast, col="blue") # one simulation
legend(x="topleft",
       legend = c("Copula VIX sim. rets."),
       col = c("blue"),
       lwd=rep(2, time=2))
```



Transforming returns back to SP500 prices and VIX values

```
# Obtain Initial values (last value of simulation)
spT <- sp500[length(sp500)][[1]]
vixT <- vix[length(vix)][[1]]

# calculate the price and values from the simulated log-returns
sim_val_mats_full <- f_logret_to_price(sp_init = spT,
                                      vix_init = vixT,
                                      sim_rets_sp500 = sp500_sim_rets_full,
                                      sim_rets_vix = vix_sim_rets_full
                                      )

# unpack matrices
sp500_sim_price_full <- sim_val_mats_full$sp500
vix_sim_vol_full <- sim_val_mats_full$vix
```

```
# compare simulated returns with the price
head(sp500_sim_rets_full)
```

```
##                T+1           T+2           T+3          T+4          T+5
## [1,]  0.0003288847  0.0111914311  0.0106763511 -0.011935411  0.008158495
## [2,] -0.0016960033  0.0065742685  0.0006715923 -0.001415663 -0.004098909
## [3,] -0.0076894607  0.0025900801  0.0023221298 -0.003689649 -0.001145947
## [4,]  0.0112372528  0.0111971934 -0.0015391388  0.013046763 -0.001620878
## [5,] -0.0073022255 -0.0007951261 -0.0019197751  0.002696617 -0.005611657
## [6,] -0.0014501362 -0.0028215149  0.0001568720  0.002694088  0.013752217
```

```
head(sp500_sim_price_full)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 1684.544 1703.502 1721.787 1701.359 1715.296
## 2 1681.136 1692.225 1693.362 1690.966 1684.049
## 3 1671.091 1675.425 1679.320 1673.135 1671.219
## 4 1703.020 1722.196 1719.548 1742.129 1739.308
## 5 1671.738 1670.409 1667.205 1671.707 1662.353
## 6 1681.550 1676.812 1677.075 1681.599 1704.885
```

```
# compare simualted log rets with volatility
head(vix_sim_rets_full)
```

```
##              T+1          T+2         T+3         T+4         T+5
## [1,]  0.03087567  0.052553143  0.02873398 -0.02051841 -0.03204625
## [2,] -0.04985742 -0.060276522 -0.01441415 -0.04359663 -0.09022505
## [3,]  0.08869130 -0.038103171 -0.07612103  0.12565729 -0.08265629
## [4,] -0.05831387  0.003885337  0.02778262 -0.09505760 -0.01983093
## [5,] -0.08365417 -0.013656585  0.02944877 -0.04831163  0.02948593
## [6,]  0.03102688 -0.056513443  0.05565022  0.06473095 -0.07833338
```

```
head(vix_sim_vol_full)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1 0.1498562 0.1579422 0.1625464 0.1592452 0.1542229
## 2 0.1382333 0.1301473 0.1282848 0.1228121 0.1122166
## 3 0.1587756 0.1528396 0.1416370 0.1606013 0.1478604
## 4 0.1370693 0.1376029 0.1414795 0.1286502 0.1261241
## 5 0.1336396 0.1318269 0.1357668 0.1293636 0.1332348
## 6 0.1498789 0.1416436 0.1497496 0.1597636 0.1477264
```

Pricing the simulation scenarios

Recall the initial (call) options:

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1605$ with maturity $T = 40d$
3. **1x** strike $K = 1800$ with maturity $T = 40d$

Option Pricing of Simulated Values

Same as before, we calculate the price of the book of options for the simulated values using the `f_opt_price_simulation()` function under `code/OptionPricing.R`:

```
# random seed for replication
set.seed(123)

# Initialize strikes and maturities the options
T_vec <- c(20, 20, 40,40) #  maturities
K_vec <- c(1600, 1650, 1750, 1800) # Strikes

# Obtain the obtion prices from simulation values
opt_price_mats_full <- f_opt_price_simulation(sim_price_sp500 = sp500_sim_price_full,
                                              sim_vol_vix = vix_sim_vol_full,
                                              K_vec = K_vec,
                                              T_vec = T_vec,
                                              put=FALSE)
```

```r
# overview of dataframes
head(opt_price_mats_full$opt1)
```

```
##          T+1       T+2       T+3       T+4       T+5
## 1  88.12166 105.77293 123.09876 103.30486 116.20379
## 2  84.11933  93.67377  94.50170  91.84806  84.70578
## 3  77.23641  79.93141  82.12692  77.78522  74.61420
## 4 104.44075 122.86101 120.26090 142.32576 139.47357
## 5  75.30684  73.65459  70.81417  74.04991  65.60863
## 6  85.44634  80.18381  80.73205  85.18549 105.94356
```

```r
head(opt_price_mats_full$opt2)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 48.21438 62.86143 77.77960 60.12340 70.40484
## 2 43.96031 50.33030 50.47168 47.41020 40.21253
## 3 40.81414 41.84976 41.99636 40.11689 36.13479
## 4 60.25694 76.10114 73.75878 93.33145 90.35781
## 5 36.88126 35.10595 33.09037 34.31082 28.35230
## 6 46.14419 40.93505 41.73635 45.62296 61.01254
```

```r
head(opt_price_mats_full$opt3)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1 15.983934 23.204893 30.708491 21.761400 24.967726
## 2 12.642155 13.364089 12.895659 10.818594  7.086030
## 3 14.435450 13.858234 12.148520 14.003073 10.756745
## 4 18.479334 24.989384 24.464055 30.341565 27.894229
## 5  9.669601  8.768552  8.539126  7.976019  6.702534
## 6 15.193991 11.945160 13.225169 15.976066 19.757453
```

```r
head(opt_price_mats_full$opt4)
```

```
##        T+1       T+2       T+3       T+4       T+5
## 1 6.806268 10.943184 15.468670 10.007180 11.464947
## 2 4.804589  4.817726  4.492091  3.407689  1.767350
## 3 6.292417  5.766854  4.570096  5.941050  3.998499
## 4 7.581813 10.925431 10.749666 13.060487 11.447923
## 5 3.364087  2.908896  2.862845  2.468675  2.029383
## 6 6.405588  4.530340  5.307950  6.924876  8.360180
```

Distribution of the Profit and Loss for the Book Of Options

Calculating the profits

For each of the simulated prices and resulting premiums, we want to calculate the profit generated at each simulation timestep. The function used is `f_pl_simulation()`, found under `code/OptionPricing.R`.

```r
# Initialize strikes and maturities the options
T_vec <- c(20, 20, 40,40) #  maturities
K_vec <- c(1600, 1650, 1750, 1800) # Strikes

# Compute the profits and loses for the simulation  from the simulated option premiums
PL_mats_full <- f_pl_simulation(sim_price_sp500 = sp500_sim_price_full,
                                opt_price_mats = opt_price_mats_full,
                                K_vec = K_vec)
```

```
# display profit matrices
head(PL_mats_full$PL1)
```

```
##        T+1      T+2       T+3      T+4      T+5
## 1 43.50442 54.94830  65.58802 152.9507 139.1810
## 2 47.50675 67.04747  94.18508 164.4075 170.6790
## 3 54.38967 80.78982 106.55986 178.4704 180.7705
## 4 27.18534 37.86022  68.42588 113.9298 115.9112
## 5 56.31925 87.06665 117.87261 182.2057 189.7761
## 6 46.17975 80.53742 107.95473 171.0701 149.4412
```

```
head(PL_mats_full$PL2)
```

```
##        T+1      T+2       T+3      T+4      T+5
## 1 33.41170 47.85980  60.90718 146.1322 134.9799
## 2 37.66578 60.39093  88.21509 158.8454 165.1722
## 3 40.81194 68.87147  96.69042 166.1387 169.2500
## 4 21.36914 34.62009  64.92800 112.9241 115.0269
## 5 44.74483 75.61528 105.59641 171.9448 177.0324
## 6 35.48189 69.78618  96.95043 160.6326 144.3722
```

```
head(PL_mats_full$PL3)
```

```
##          T+1        T+2       T+3      T+4      T+5
## 1 -15.983934 -12.483660  7.978288 84.49419 80.41702
## 2 -12.642155  -2.642856 25.791120 95.43699 98.29872
## 3 -14.435450  -3.137001 26.538259 92.25251 94.62800
## 4 -18.479334 -14.268151 14.222724 75.91402 77.49052
## 5  -9.669601   1.952681 30.147653 98.27957 98.68221
## 6 -15.193991  -1.223927 25.461610 90.27952 85.62729
```

```
head(PL_mats_full$PL4)
```

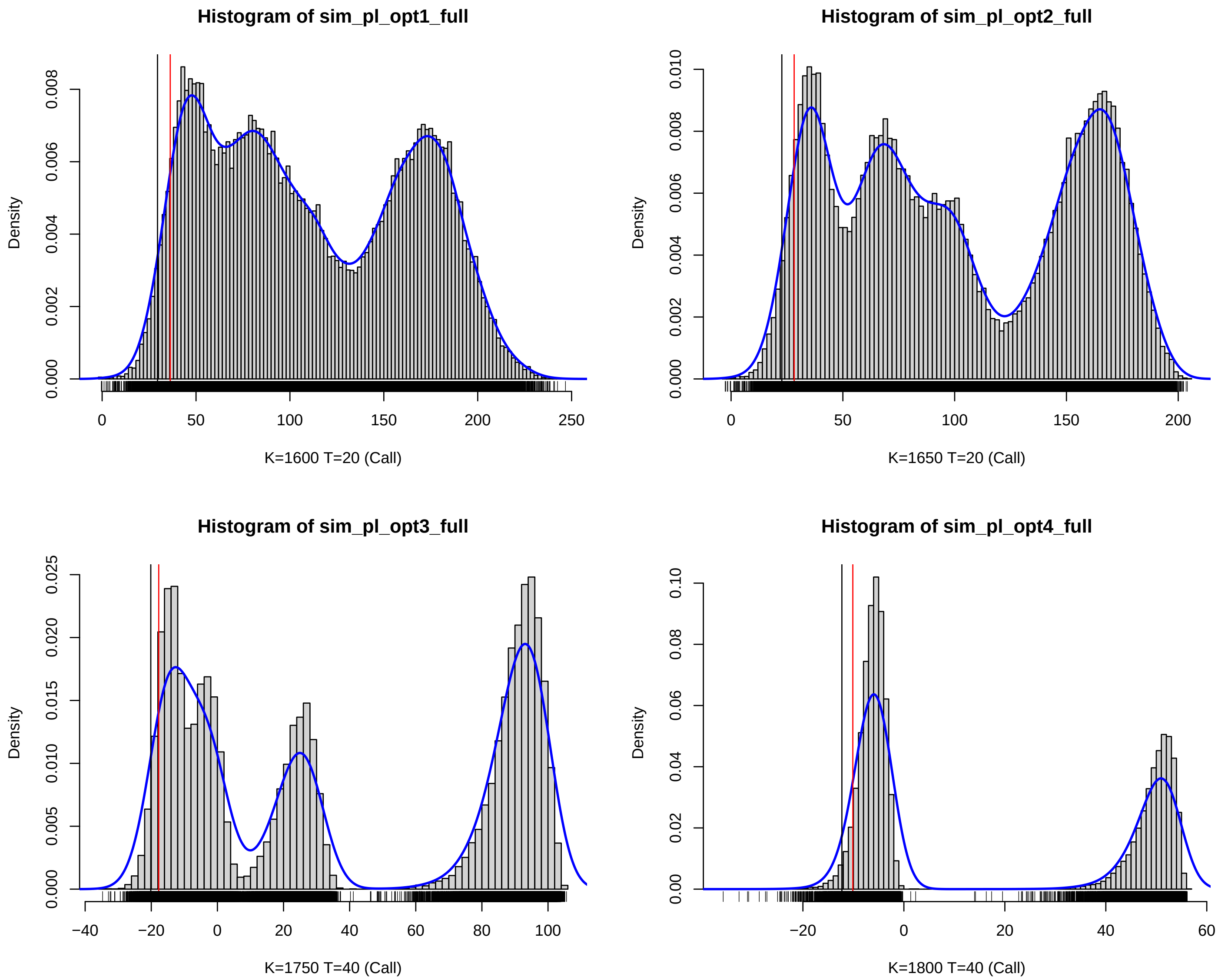
```
##         T+1        T+2        T+3      T+4      T+5
## 1 -6.806268 -10.943184 -15.468670 46.24841 43.91980
## 2 -4.804589  -4.817726  -4.492091 52.84790 53.61740
## 3 -6.292417  -5.766854  -4.570096 50.31454 51.38625
## 4 -7.581813 -10.925431 -10.749666 43.19510 43.93682
## 5 -3.364087  -2.908896  -2.862845 53.78691 53.35536
## 6 -6.405588  -4.530340  -5.307950 49.33071 47.02457
```

Distribution of Options P/L

Next, using all the simulated profits and losses for each of the options, we display a histogram for the distribution for each of the options, for the aggregated 5 days of simulation:

```
# flatten the matrices 5-days ahead simulated P/L for the three options
sim_pl_opt1_full <- as.vector(PL_mats_full$PL1)
sim_pl_opt2_full <- as.vector(PL_mats_full$PL2)
sim_pl_opt3_full <- as.vector(PL_mats_full$PL3)
sim_pl_opt4_full <- as.vector(PL_mats_full$PL4)

# Compute the 95% VaR and 95% ES
opt1_full_VaR_ES <- f_VaR_ES(sim_pl_opt1_full, alpha = 0.05)
opt2_full_VaR_ES <- f_VaR_ES(sim_pl_opt2_full, alpha = 0.05)
opt3_full_VaR_ES <- f_VaR_ES(sim_pl_opt3_full, alpha = 0.05)
opt4_full_VaR_ES <- f_VaR_ES(sim_pl_opt4_full, alpha = 0.05)
```

```
# plot the distribution for each of the options
par(mfrow = c(2,2))

# distribution of first option
hist(sim_pl_opt1_full, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[1], " T=", T_vec[1], " (Call)"))
lines(density(sim_pl_opt1_full), lwd=2, col="blue")
abline(v=opt1_full_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt1_full_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt1_full)

# distribution of second option
hist(sim_pl_opt2_full, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[2], " T=", T_vec[2], " (Call)"))
lines(density(sim_pl_opt2_full), lwd=2, col="blue")
abline(v=opt2_full_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt2_full_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt2_full)

# distribution of third option
hist(sim_pl_opt3_full, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[3], " T=", T_vec[3], " (Call)"))
lines(density(sim_pl_opt3_full), lwd=2, col="blue")
abline(v=opt3_full_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt3_full_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt3_full)

# distribution of fourth option
hist(sim_pl_opt4_full, nclass = round(10 * log(n_sim)),
     probability = TRUE, xlab=paste0("K=", K_vec[4], " T=", T_vec[4], " (Call)"))
lines(density(sim_pl_opt4_full), lwd=2, col="blue")
abline(v=opt4_full_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt4_full_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt4_full)
```


Histogram of sim_pl_opt1_full
Density
K=1600 T=20 (Call)
Histogram of sim_pl_opt2_full
Density
K=1650 T=20 (Call)
Histogram of sim_pl_opt3_full
Density
K=1750 T=40 (Call)
Histogram of sim_pl_opt4_full
Density
K=1800 T=40 (Call)