

# TP2 Risk Management

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## Libraries

## Risk Management: European Options Portfolio

The objective is to implement (part of) the risk management framework for estimating the risk of a book of European call options by taking into account the risk drivers such as underlying and implied volatility.

## Data

Load the database Market. Identify the price of the **SP500**, the **VIX index**, the term structure of interest rates (current and past), and the traded options (calls and puts).

```
# load dataset into environment
load(file = here("data_raw", "Market.rda"))

# reassign name and inspect structure of loaded data
mkt <- Market
summary(mkt)
```

```
##           Length Class  Mode
## sp500 3410    xts    numeric
## vix   3410    xts    numeric
## rf      14 -none- numeric
## calls 1266 -none- numeric
## puts  2250 -none- numeric
```

```
str(mkt)
```

```
## List of 5
## $ sp500:An 'xts' object on 2000-01-03/2013-09-10 containing:
##   Data: num [1:3410, 1] 1455 1399 1402 1403 1441 ...
##   Indexed by objects of class: [Date] TZ: UTC
##   xts Attributes:
##   NULL
## $ vix :An 'xts' object on 2000-01-03/2013-09-10 containing:
##   Data: num [1:3410, 1] 0.242 0.27 0.264 0.257 0.217 ...
##   Indexed by objects of class: [Date] TZ: UTC
##   xts Attributes:
##   NULL
## $ rf : num [1:14, 1] 0.00071 0.00098 0.00128 0.00224 0.00342 ...
##   .. attr(*, "names")= chr [1:14] "0.00273972602739726" "0.0192307692307692" "0.0833333333333333" "0.25" ...
## $ calls: num [1:422, 1:3] 1280 1370 1380 1400 1415 ...
##   .. attr(*, "dimnames")=List of 2
##   .. ..$ : NULL
##   .. ..$ : chr [1:3] "K" "tau" "IV"
```

```
## $ puts : num [1:750, 1:3] 1000 1025 1050 1075 1100 ...
##    .- attr(*, "dimnames")=List of 2
##    .. ..$ : NULL
##    .. ..$ : chr [1:3] "K" "tau" "IV"
```

Let's unpack these into the env. individually:

```
# unpack each of the elements in the mkt list
sp500 <- mkt$sp500
vix <- mkt$vix
Rf <- mkt$rf # risk-free rates
calls <- mkt$calls
puts <- mkt$puts

# assign colname for aesthetic
colnames(sp500) <- "sp500"
colnames(vix) <- "vix"
```

## SP500 and VIX

By inspection, we observe that the SP500 and VIX indices are contained in the `sp500` and `vix` xts objects respectively.

```
# show head of both indexes
head(sp500)
```

```
##           sp500
## 2000-01-03 1455.22
## 2000-01-04 1399.42
## 2000-01-05 1402.11
## 2000-01-06 1403.45
## 2000-01-07 1441.47
## 2000-01-10 1457.60
```

```
head(vix)
```

```
##           vix
## 2000-01-03 0.2421
## 2000-01-04 0.2701
## 2000-01-05 0.2641
## 2000-01-06 0.2573
## 2000-01-07 0.2172
## 2000-01-10 0.2171
```

## Interest Rates

The **interest rates** are given in the `$rf` attribute. We can see that

```
Rf
```

```
##           [,1]
## [1,] 0.0007099993
## [2,] 0.0009799908
## [3,] 0.0012799317
## [4,] 0.0022393730
## [5,] 0.0034170792
## [6,] 0.0045123559
## [7,] 0.0043206525
## [8,] 0.0064284968
```

```
## [9,] 0.0090558654
## [10,] 0.0117237591
## [11,] 0.0141196498
## [12,] 0.0176131823
## [13,] 0.0207989304
## [14,] 0.0203526819
## attr(,"names")
## [1] "0.00273972602739726" "0.0192307692307692" "0.0833333333333333"
## [4] "0.25" "0.5" "0.75"
## [7] "1" "2" "3"
## [10] "4" "5" "7"
## [13] "10" "30"
```

These represent the interest rates at different maturities. The maturities are given as follows:

```
r_f <- as.vector(Rf)
names(r_f) <- c("1d", "1w", "1m", "2m", "3m", "6m", "9m", "1y", "3y", "4y", "5y", "7y", "10y", "30y")
r_f
```

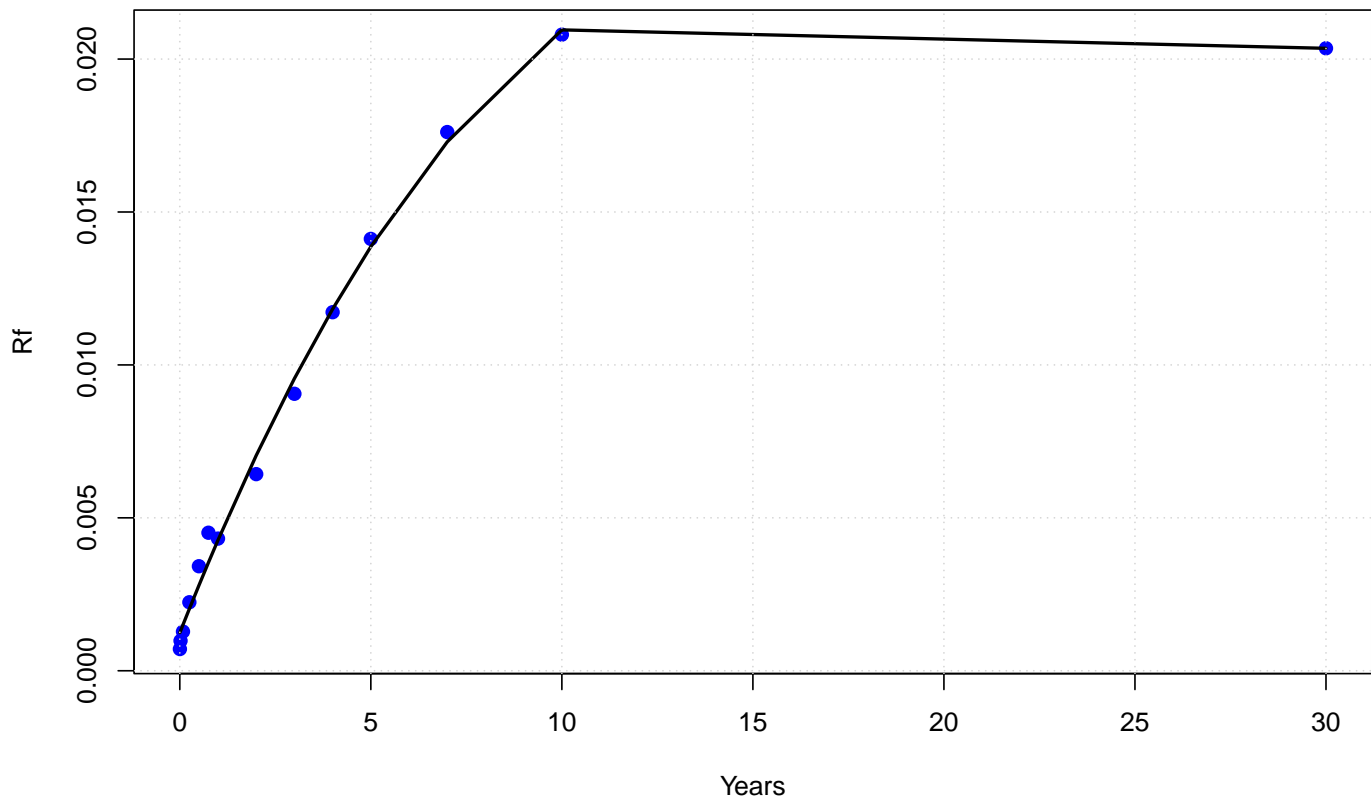
```
##          1d          1w          1m          2m          3m          6m
## 0.0007099993 0.0009799908 0.0012799317 0.0022393730 0.0034170792 0.0045123559
##          9m          1y          3y          4y          5y          7y
## 0.0043206525 0.0064284968 0.0090558654 0.0117237591 0.0141196498 0.0176131823
##          10y          30y
## 0.0207989304 0.0203526819
```

Further, we can pack different sources of information in a matrix:

```
# pack Rf into a matrix with rf, years, and days
rf_mat <- as.matrix(r_f)
rf_mat <- cbind(rf_mat, as.numeric(names(Rf)))
rf_mat <- cbind(rf_mat, rf_mat[, 2]*360)
colnames(rf_mat) <- c("rf", "years", "days")
rf_mat
```

```
##          rf          years          days
## 1d 0.0007099993 0.002739726 0.9863014
## 1w 0.0009799908 0.019230769 6.9230769
## 1m 0.0012799317 0.083333333 30.0000000
## 2m 0.0022393730 0.250000000 90.0000000
## 3m 0.0034170792 0.500000000 180.0000000
## 6m 0.0045123559 0.750000000 270.0000000
## 9m 0.0043206525 1.000000000 360.0000000
## 1y 0.0064284968 2.000000000 720.0000000
## 3y 0.0090558654 3.000000000 1080.0000000
## 4y 0.0117237591 4.000000000 1440.0000000
## 5y 0.0141196498 5.000000000 1800.0000000
## 7y 0.0176131823 7.000000000 2520.0000000
## 10y 0.0207989304 10.000000000 3600.0000000
## 30y 0.0203526819 30.000000000 10800.0000000
```

### Term Structure of Risk-Free Rates



### Calls

The `calls` object displays the different values of  $K$  (**Strike Price**),  $\tau$  (**time to maturity**) and  $\sigma = IV$  (**Implied Volatility**)

```
dim(calls)
```

```
## [1] 422  3
```

```
head(calls)
```

```
##      K      tau      IV
## [1,] 1280 0.02557005 0.7370370
## [2,] 1370 0.02557005 0.9691616
## [3,] 1380 0.02557005 0.9451401
## [4,] 1400 0.02557005 0.5274481
## [5,] 1415 0.02557005 0.5083375
## [6,] 1425 0.02557005 0.4820041
```

Add days column for convenience:

```
calls <- cbind(calls, calls[, "tau"]*250)
colnames(calls) <- c("K", "tau", "IV", "tau_days")
head(calls)
```

```
##      K      tau      IV tau_days
## [1,] 1280 0.02557005 0.7370370 6.392513
## [2,] 1370 0.02557005 0.9691616 6.392513
## [3,] 1380 0.02557005 0.9451401 6.392513
```

```
## [4,] 1400 0.02557005 0.5274481 6.392513
## [5,] 1415 0.02557005 0.5083375 6.392513
## [6,] 1425 0.02557005 0.4820041 6.392513
```

```
tail(calls)
```

```
##           K      tau      IV tau_days
## [417,] 1925 2.269406 0.1605208 567.3514
## [418,] 1975 2.269406 0.1602093 567.3514
## [419,] 2000 2.269406 0.1559909 567.3514
## [420,] 2100 2.269406 0.1480259 567.3514
## [421,] 2500 2.269406 0.1441222 567.3514
## [422,] 3000 2.269406 0.1519319 567.3514
```

## Puts

```
dim(puts)
```

```
## [1] 750  3
```

```
head(puts)
```

```
##           K      tau      IV
## [1,] 1000 0.02557005 1.0144250
## [2,] 1025 0.02557005 1.0083110
## [3,] 1050 0.02557005 0.9622093
## [4,] 1075 0.02557005 0.9170457
## [5,] 1100 0.02557005 0.8728757
## [6,] 1120 0.02557005 0.8381910
```

```
puts <- cbind(puts, puts[, "tau"]*250)
colnames(puts) <- c("K", "tau", "IV", "tau_days")
head(puts)
```

```
##           K      tau      IV tau_days
## [1,] 1000 0.02557005 1.0144250 6.392513
## [2,] 1025 0.02557005 1.0083110 6.392513
## [3,] 1050 0.02557005 0.9622093 6.392513
## [4,] 1075 0.02557005 0.9170457 6.392513
## [5,] 1100 0.02557005 0.8728757 6.392513
## [6,] 1120 0.02557005 0.8381910 6.392513
```

```
tail(puts)
```

```
##           K      tau      IV tau_days
## [745,] 1750 2.269406 0.1899088 567.3514
## [746,] 1800 2.269406 0.1698365 567.3514
## [747,] 1825 2.269406 0.1986200 567.3514
## [748,] 1850 2.269406 0.1853406 567.3514
## [749,] 2000 2.269406 0.1520378 567.3514
## [750,] 3000 2.269406 0.2759397 567.3514
```

## Pricing a Portfolio of Options

### Black-Scholes

Notation:

- $S_t$  = Current value of underlying asset price
- $K$  = Options **strike price**
- $T$  = Option **maturity** (in years)
- $t$  = **time** in years
- $\tau = T - t$  = **Time to maturity**
- $r$  = **Risk-free rate**
- $y$  **Dividend yield**
- $R = r - y$
- $\sigma$  = **Implied volatility**
- $c$  = **Price Call Option**
- $p$  = **Price Put Option**

**Proposition 1** (Black-Scholes Model). Assume the notation before, and let  $N(\cdot)$  be the cumulative standard normal distribution function. Under certain assumptions, the Black-Scholes models prices Call and Put options as follows:

$$\begin{cases} C(S_t, t) = S e^{yT} N(d_1) - K e^{-r \times \tau} N(d_2), \\ P(S_t, t) = K e^{-r \times \tau} (1 - N(d_2)) - S e^{y \times T} (1 - N(d_1)), \end{cases}$$

where:

$$\begin{cases} d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \\ d_2 = d_1 - \sigma\sqrt{\tau} \end{cases}$$

, further the Put Option price corresponds to the **\*\*Put-Call parity\*\***, given by:

$$C(S_t, t) + K e^{-r \times \tau} = P(S_t, t) + S_t$$

**Note** As here we don't have dividends, then  $y = 0$ , and so

$$\begin{cases} C(S_t, t) = S_t N(d_1) - K e^{-r \times \tau} N(d_2), \\ P(S_t, t) = K e^{-r \times \tau} (1 - N(d_2)) - S_t (1 - N(d_1)), \end{cases}$$

### Implementation

```
get_d1 <- function(S_t, K, tau, r, sigma){
  ### Compute d1 for the Black-Scholes model
  # INPUTS
  # S_t: Current value of underlying asset price
  # K: Strike Price
  # tau: T- t, where T=maturity, and t=current time
  # r: risk-free rate
  # sigma Implied volatility (i.e. sigma)

  num <- (log(S_t/K) - tau*(r + 0.5*sigma**2)) # numerator
  denom <- sigma * sqrt(tau) # denominator

  return(num/denom)
```

```

}

get_d2 <- function(d1, sigma, tau){
  ### Compute d2 for the Black-Scholes model
  # INPUTS
  # d1: d1 factor calculated by the get_d1 function
  # tau: T- t, where T=maturity, and t=current time
  # sigma Implied volatility (i.e. sigma)

  return(d1 - sigma * sqrt(tau))
}

# Function to implement the Black-Scholes model
black_scholes <- function(S_t, K, r, tau, sigma, put=FALSE){
  # Calculates a Call (or Option) price using Black-Scholes
  # INPUTS
  # S_t: [numeric] Current value of underlying asset price
  # K: [numeric] Strike Price
  # r: [numeric] risk-free rate
  # tau: [numeric] T- t, where T=maturity, and t=current time
  # sigma: [numeric] Implied volatility (i.e. sigma)
  # put: [logical] if TRUE, calculate a Put, if FALSE, calculate a Call.
  # FALSE by default (Call).
  #
  # OUTPUTS:
  # P or C: [numeric] Option value according to Black-scholes

  # calculate d1 & d2
  d1 <- get_d1(S_t, K, tau, r, sigma)
  d2 <- get_d2(d1, sigma, tau)

  if(put==TRUE){
    # calculate a Put option
    P <- S_t * pnorm(d1) - K*exp(r*tau) * pnorm(d2)
    P <- as.numeric(P)
    return(P)
  }
  # else calculate a Call option (default)
  C <- K*exp(r*tau)*(1 - pnorm(d2)) - S_t * (1 - pnorm(d1))
  return( as.numeric(C) )
}

# Test: Call Option
S_t = 100
K = 1600
r = 0.03
tau = 10/360
sigma = 1.05
black_scholes(S_t, K, r, tau, sigma)

```

```
## [1] 1501.334
```

## Book of Options

Assume the following book of **European Call Options**:

- 1x strike  $K = 1600$  with maturity  $T = 20d$
- 1x strike  $K = 1605$  with maturity  $T = 40d$

3. **1x** strike  $K = 1800$  with maturity  $T = 40d$

Find the price of this book given **the last underlying price** and the **last implied volatility** (take the VIX for all options). Use **Black-Scholes** to price the options. Take the current term structure and **linearly interpolate** to find the corresponding rates. Use 360 days/year for the term structure and **250 days/year** for the maturity of the options.

### Nearest values

This function will obtain the two nearest values  $a, b$  for a number  $x$  in a vector  $v$ , such that  $a < x < b$ .

```
# Obtain the two nearest values of x in vec.
get_nearest<- function(x, vec){
  # find all the numbers that are bigger and smaller than x in vec
  bigger <- vec >= x
  smaller <- vec <= x

  # filter only values with TRUE
  bigger <- bigger[bigger == TRUE]
  smaller <- smaller[smaller == TRUE]

  # obtain the indexes for the left and upper bound
  a_idx <- length(smaller)
  b_idx <- length(smaller)+1

  # retrieve values from original vector
  a <- vec[a_idx]
  b <- vec[b_idx]

  # return the retrieved values
  return( c(a,b) )
}

# Test
days <- rf_mat[, "days"]
get_nearest(40, rf_mat[, "days"]) # nearest day values
```

```
## 1m 2m
## 30 90
```

### Linear Interpolation

Given two known values  $(x_1, y_1)$  and  $(x_2, y_2)$ , we can estimate the  $y$ -value for some  $x$ -value with:

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

```
# Function to interpolate y given two points
interpolate <- function(x,x1=1,y1=1,x2=2,y2=2){
  y1 + (x-x1)*(y2-y1)/(x2-x1)
}
```

### Finding the rates through interpolation

The **yield curve** for the given structure of interest rates can be modeled a function  $r_f = f(x)$ , where  $x$  is the number of years. Then, we can interpolate the values as follows:



```
# Interest rates
```

```
rf_mat
```

```
##           rf           years           days
## 1d  0.0007099993  0.002739726  0.9863014
## 1w  0.0009799908  0.019230769  6.9230769
## 1m  0.0012799317  0.083333333  30.0000000
## 2m  0.0022393730  0.250000000  90.0000000
## 3m  0.0034170792  0.500000000  180.0000000
## 6m  0.0045123559  0.750000000  270.0000000
## 9m  0.0043206525  1.000000000  360.0000000
## 1y  0.0064284968  2.000000000  720.0000000
## 3y  0.0090558654  3.000000000  1080.0000000
## 4y  0.0117237591  4.000000000  1440.0000000
## 5y  0.0141196498  5.000000000  1800.0000000
## 7y  0.0176131823  7.000000000  2520.0000000
## 10y 0.0207989304 10.000000000  3600.0000000
## 30y 0.0203526819 30.000000000 10800.0000000
```

```
head(calls)
```

```
##           K           tau           IV tau_days
## [1,] 1280 0.02557005 0.7370370 6.392513
## [2,] 1370 0.02557005 0.9691616 6.392513
## [3,] 1380 0.02557005 0.9451401 6.392513
## [4,] 1400 0.02557005 0.5274481 6.392513
## [5,] 1415 0.02557005 0.5083375 6.392513
## [6,] 1425 0.02557005 0.4820041 6.392513
```

ex.: 1x strike  $K = 1600$  with maturity  $T = 20d$

```
price_option <- function(T, K, calls, rf_mat, stock=NA, put=FALSE){
  # Calculates
  # INPUTS
  # T:      [numeric] maturity of option (in days)
  # K:      [numeric] Strike Price
  # calls:  [matrix] matrix containing information about tau and IV for different strike prices
  # rf_mat: [matrix] matrix containing risk-free term structure
  # stock:  [xts OR zoo like object] object containing stock prices for a single stock
  # put:    [logical] if TRUE, calculate a Put, if FALSE, calculate a Call.
  #        FALSE by default (Call).
  #
  # OUTPUTS:
  # LIST containing:
  #   - P or C: [numeric] Option value according to Black-scholes and available information
  #   - r_interp: [numeric] Interpolated risk-free rate given risk-free term structure
  #   - calls [matrix] relevant set of calls information
  #   - rates [matrix] relevant set of risk-free rates used for the interpolation

  # Inputs
  tau = T/250 # days --> years
  days_calls <- calls[, "tau_days"] # extract days column
  days_rf <- rf_mat[, "days"] # extract days from rf_mat

  # extract the calls values
  ab <- get_nearest(T, days_calls) # search lower and upper nearest days to T
  valid_days <- calls[, "tau_days"] == ab[1] | calls[, "tau_days"] == ab[2] # where match
  calls_sub <- calls[ valid_days, ] # subset valid rows
```

```

calls_sub <- calls_sub[calls_sub[, "K"] == K, ] # subset matching K

# test whether matrix is empty (i.e. no matching K found)
if(all(is.na(calls_sub))){
  warning("No values matching K in Calls data\n")
}

# extract interpolated risk rates
ab <- get_nearest(T, days_rf) # obtain nearest days to T available in rf_mat
valid_days_rf <- rf_mat[, "days"] == ab[1] | rf_mat[, "days"] == ab[2] # where match
rates <- rf_mat[valid_days_rf, ] # subset for valid days

# interpolate risk free rate for Option given maturity
r <- interpolate(tau,
                x1=rates[1,2],
                y1=rates[1,1],
                x2=rates[2,2],
                y2=rates[2,1])

# retrieve implied volatility for option
if(is.matrix(calls_sub)){
  # average between lower and upper values
  sigma <- (calls_sub[1, "IV"] + calls_sub[2, "IV"])/2
} else{
  # retrieve from numeric vector (single match)
  sigma <- calls_sub["IV"]
}

# retrieve last price for option from VIx
S_t <- as.numeric( stock[length(stock)])

# Calculate Option price
if(put==TRUE){
  C <- NA
  P <- black_scholes(S_t, K, r, tau, sigma, put=TRUE)
} else{
  C <- black_scholes(S_t, K, r, tau, sigma, put=FALSE)
  P <- NA
}

# pack everything into a List and return
return(list(Call = C,
            Put = P,
            r_interp = r,
            calls = calls_sub, # subset of calls used
            rates = rates # subset of rates used
            ))
}

```

Next, using the function above we price the book of options given:

1. 1x strike  $K = 1600$  with maturity  $T = 20d$
2. 1x strike  $K = 1605$  with maturity  $T = 40d$
3. 1x strike  $K = 1800$  with maturity  $T = 40d$

*# First Call Option*

```
price_option(T=20, K=1600, calls, rf_mat, stock = vix)
```

```
## $Call
## [1] 1600.017
##
## $Put
## [1] NA
##
## $r_interp
## [1] 0.001264335
##
## $calls
##      K      tau      IV tau_days
## [1,] 1600 0.02557005 0.1817481 6.392513
## [2,] 1600 0.10228238 0.1701946 25.570595
##
## $rates
##      rf      years      days
## 1w 0.0009799908 0.01923077 6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

*# Second Call Option*

```
price_option(T=40, K=1605, calls, rf_mat, stock = vix)
```

```
## $Call
## [1] 1605.297
##
## $Put
## [1] NA
##
## $r_interp
## [1] 0.001721275
##
## $calls
##      K      tau      IV tau_days
## 1605.0000000 0.1022824 0.1676923 25.5705949
##
## $rates
##      rf      years      days
## 1m 0.001279932 0.08333333 30
## 2m 0.002239373 0.25000000 90
```

*# Third Call Option*

```
price_option(T=40, K=1800, calls, rf_mat, stock = vix)
```

```
## $Call
## [1] 1800.35
##
## $Put
## [1] NA
##
## $r_interp
## [1] 0.001721275
##
## $calls
##      K      tau      IV tau_days
## [1,] 1800 0.1022824 0.1057523 25.57059
```

```
## [2,] 1800 0.1789947 0.1044115 44.74868
##
## $rates
##           rf           years days
## 1m 0.001279932 0.08333333    30
## 2m 0.002239373 0.25000000    90
```