# TP2 Risk Management

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**Libraries**

## Risk Management: European Options Portfolio

The objective is to implement (part of) the risk management framework for estimating the risk of a book of European call options by taking into account the risk drivers such as underlying and implied volatility.

### Data

Load the database Market. Identify the price of the **SP500**, the **VIX index**, the term structure of interest rates (current and past), and the traded options (calls and puts).

```
# load dataset into environment
load(file = here("data_raw", "Market.rda"))

# reassign name and inspect structure of loaded data
mkt <- Market
summary(mkt)
```

```
##       Length Class  Mode
## sp500 3410   xts    numeric
## vix   3410   xts    numeric
## rf      14   -none- numeric
## calls 1266   -none- numeric
## puts  2250   -none- numeric
```

```
str(mkt)
```

```
## List of 5
##  $ sp500:An xts object on 2000-01-03 / 2013-09-10 containing:
##   Data:    double [3410, 1]
##   Index:   Date [3410] (TZ: "UTC")
##  $ vix  :An xts object on 2000-01-03 / 2013-09-10 containing:
##   Data:    double [3410, 1]
##   Index:   Date [3410] (TZ: "UTC")
##  $ rf   : num [1:14, 1] 0.00071 0.00098 0.00128 0.00224 0.00342 ...
##   ..- attr(*, "names")= chr [1:14] "0.00273972602739726" "0.0192307692307692" "0.0833333333333333" "0.25" .
##  $ calls: num [1:422, 1:3] 1280 1370 1380 1400 1415 ...
##   ..- attr(*, "dimnames")=List of 2
##   .. ..$ : NULL
##   .. ..$ : chr [1:3] "K" "tau" "IV"
##  $ puts : num [1:750, 1:3] 1000 1025 1050 1075 1100 ...
##   ..- attr(*, "dimnames")=List of 2
##   .. ..$ : NULL
##   .. ..$ : chr [1:3] "K" "tau" "IV"
```

Let's unpack these into the env. individually:

```r
# unpack each of the elements in the mkt list
sp500 <- mkt$sp500
vix <- mkt$vix
Rf <- mkt$rf # risk-free rates
calls <- mkt$calls
puts <- mkt$puts

# assign colname for aesthetic
colnames(sp500) <- "sp500"
colnames(vix) <- "vix"
```

### SP500 and VIX

By inspection, we observe that we the SP500 and VIX indices are contained in the `sp500` and `vix` xts objects respectively.

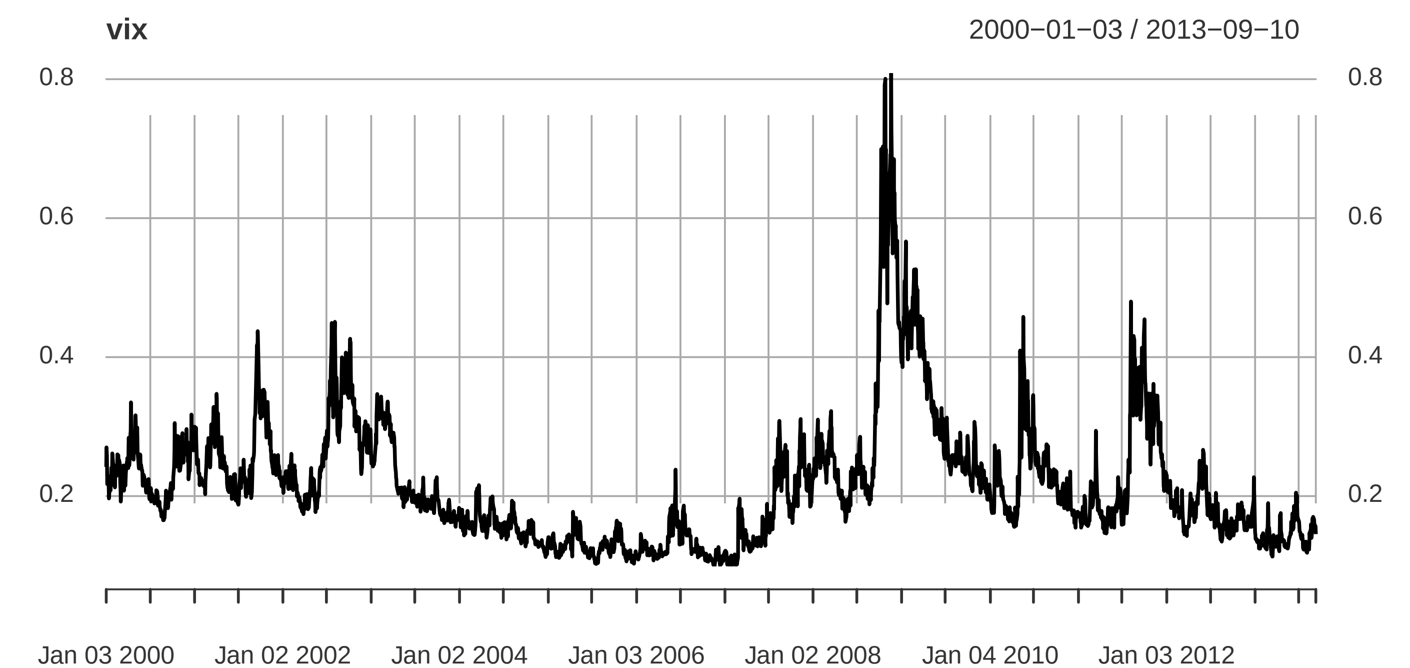
```r
# show head of both indexes
head(sp500)
```

```
##               sp500
## 2000-01-03 1455.22
## 2000-01-04 1399.42
## 2000-01-05 1402.11
## 2000-01-06 1403.45
## 2000-01-07 1441.47
## 2000-01-10 1457.60
```

```r
head(vix)
```

```
##               vix
## 2000-01-03 0.2421
## 2000-01-04 0.2701
## 2000-01-05 0.2641
## 2000-01-06 0.2573
## 2000-01-07 0.2172
## 2000-01-10 0.2171
```

```r
par(mfrow = c(2,1))

# plot both series on top of each other
plot(sp500)
plot(vix)
```


### Interest Rates

The **interest rates** are given in the `$rf` attribute. We can see that

```
Rf
```

```
##                [,1]
##  [1,] 0.0007099993
##  [2,] 0.0009799908
##  [3,] 0.0012799317
##  [4,] 0.0022393730
##  [5,] 0.0034170792
##  [6,] 0.0045123559
##  [7,] 0.0043206525
```

```
##  [8,] 0.0064284968
##  [9,] 0.0090558654
## [10,] 0.0117237591
## [11,] 0.0141196498
## [12,] 0.0176131823
## [13,] 0.0207989304
## [14,] 0.0203526819
## attr(,"names")
##  [1] "0.00273972602739726" "0.0192307692307692"  "0.0833333333333333"
##  [4] "0.25"                "0.5"                 "0.75"
##  [7] "1"                   "2"                   "3"
## [10] "4"                   "5"                   "7"
## [13] "10"                  "30"
```

These represent the interest rates at different maturities. The maturities are given as follows:

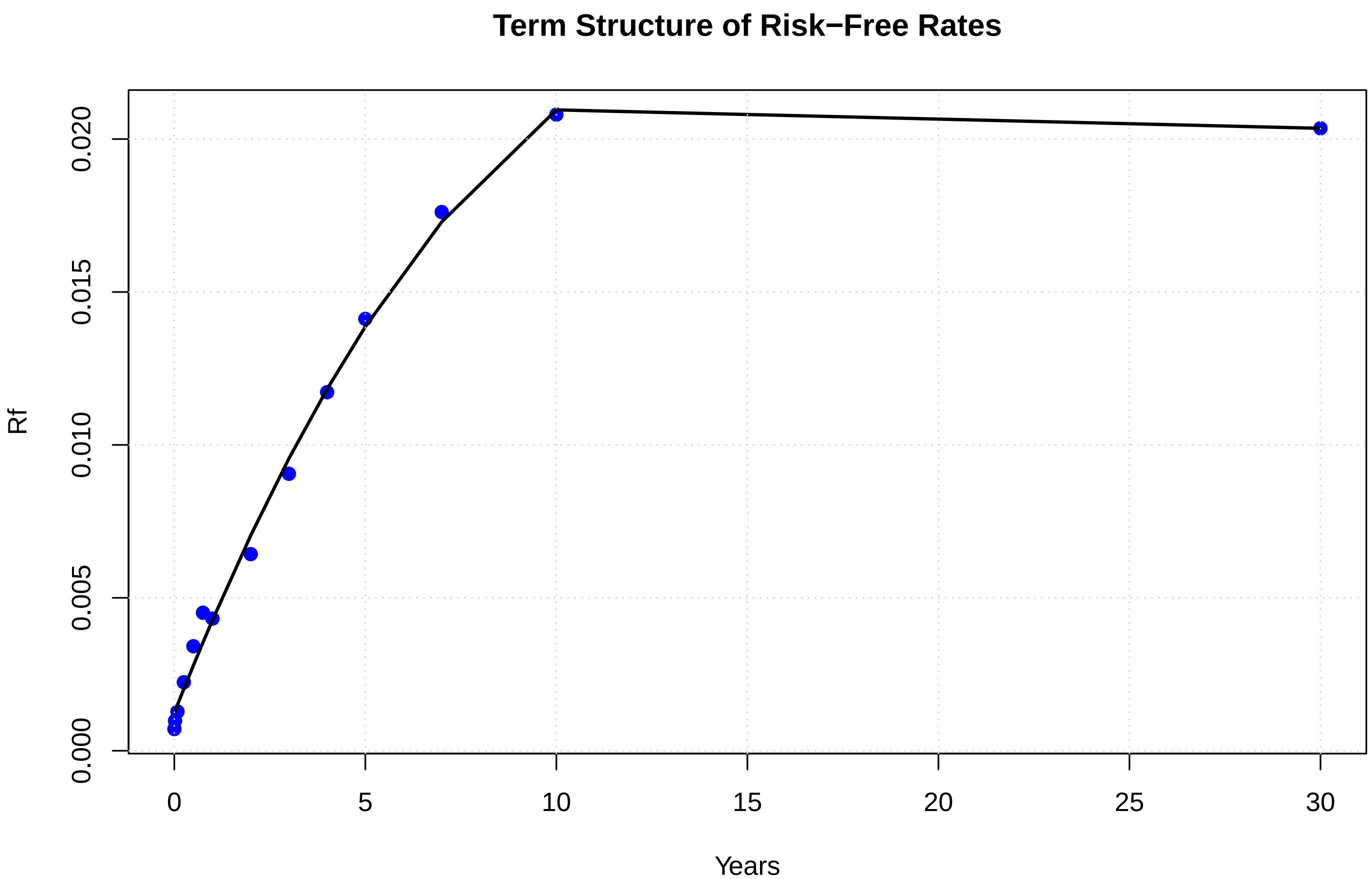
```r
r_f <- as.vector(Rf)
names(r_f) <- c("1d","1w", "1m", "3m", "6m", "9m", "1y", "2y", "3y", "4y", "5y", "7y","10y", "30y")
r_f
```

```
##           1d           1w           1m           3m           6m           9m
## 0.0007099993 0.0009799908 0.0012799317 0.0022393730 0.0034170792 0.0045123559
##           1y           2y           3y           4y           5y           7y
## 0.0043206525 0.0064284968 0.0090558654 0.0117237591 0.0141196498 0.0176131823
##          10y          30y
## 0.0207989304 0.0203526819
```

Further, we can pack different sources of information in a matrix:

```r
# pack Rf into a matrix with rf, years, and days
rf_mat <- as.matrix(r_f)
rf_mat <- cbind(rf_mat, as.numeric(names(Rf)))
rf_mat <- cbind(rf_mat, rf_mat[, 2]*360)
colnames(rf_mat) <- c("rf", "years", "days")
rf_mat
```

```
##               rf        years          days
## 1d  0.0007099993  0.002739726     0.9863014
## 1w  0.0009799908  0.019230769     6.9230769
## 1m  0.0012799317  0.083333333    30.0000000
## 3m  0.0022393730  0.250000000    90.0000000
## 6m  0.0034170792  0.500000000   180.0000000
## 9m  0.0045123559  0.750000000   270.0000000
## 1y  0.0043206525  1.000000000   360.0000000
## 2y  0.0064284968  2.000000000   720.0000000
## 3y  0.0090558654  3.000000000  1080.0000000
## 4y  0.0117237591  4.000000000  1440.0000000
## 5y  0.0141196498  5.000000000  1800.0000000
## 7y  0.0176131823  7.000000000  2520.0000000
## 10y 0.0207989304 10.000000000  3600.0000000
## 30y 0.0203526819 30.000000000 10800.0000000
```




### Calls

The `calls` object displays the different values of $K$ (**Strike Price**), $\tau$ (**time to maturity**) and $\sigma = IV$ (**Implied Volatilty**)

```
dim(calls)
```

```
## [1] 422   3
```

```
head(calls)
```

```
##         K        tau        IV
## [1,] 1280 0.02557005 0.7370370
## [2,] 1370 0.02557005 0.9691616
## [3,] 1380 0.02557005 0.9451401
## [4,] 1400 0.02557005 0.5274481
## [5,] 1415 0.02557005 0.5083375
## [6,] 1425 0.02557005 0.4820041
```

Add `days` column for convenience:

```
calls <- cbind(calls, calls[, "tau"]*250)
colnames(calls) <- c("K","tau", "IV", "tau_days")
head(calls)
```

```
##         K        tau        IV tau_days
## [1,] 1280 0.02557005 0.7370370 6.392513
## [2,] 1370 0.02557005 0.9691616 6.392513
## [3,] 1380 0.02557005 0.9451401 6.392513
```

```
## [4,] 1400 0.02557005 0.5274481 6.392513
## [5,] 1415 0.02557005 0.5083375 6.392513
## [6,] 1425 0.02557005 0.4820041 6.392513
```

```
tail(calls)
```

```
##           K      tau        IV tau_days
## [417,] 1925 2.269406 0.1605208 567.3514
## [418,] 1975 2.269406 0.1602093 567.3514
## [419,] 2000 2.269406 0.1559909 567.3514
## [420,] 2100 2.269406 0.1480259 567.3514
## [421,] 2500 2.269406 0.1441222 567.3514
## [422,] 3000 2.269406 0.1519319 567.3514
```

### Puts

```
dim(puts)
```

```
## [1] 750   3
```

```
head(puts)
```

```
##         K        tau        IV
## [1,] 1000 0.02557005 1.0144250
## [2,] 1025 0.02557005 1.0083110
## [3,] 1050 0.02557005 0.9622093
## [4,] 1075 0.02557005 0.9170457
## [5,] 1100 0.02557005 0.8728757
## [6,] 1120 0.02557005 0.8381910
```

```
puts <- cbind(puts, puts[, "tau"]*250)
colnames(puts) <- c("K","tau", "IV", "tau_days")
head(puts)
```

```
##         K        tau        IV tau_days
## [1,] 1000 0.02557005 1.0144250 6.392513
## [2,] 1025 0.02557005 1.0083110 6.392513
## [3,] 1050 0.02557005 0.9622093 6.392513
## [4,] 1075 0.02557005 0.9170457 6.392513
## [5,] 1100 0.02557005 0.8728757 6.392513
## [6,] 1120 0.02557005 0.8381910 6.392513
```

```
tail(puts)
```

```
##           K      tau        IV tau_days
## [745,] 1750 2.269406 0.1899088 567.3514
## [746,] 1800 2.269406 0.1698365 567.3514
## [747,] 1825 2.269406 0.1986200 567.3514
## [748,] 1850 2.269406 0.1853406 567.3514
## [749,] 2000 2.269406 0.1520378 567.3514
## [750,] 3000 2.269406 0.2759397 567.3514
```

## Pricing a Portfolio of Options

### Black-Scholes

Notation:

- $S_t$ = Current value of underlying asset price
- $K$ = Options **strike price**
- $T$ = Option **maturity** (in years)
- $t$ = **time** in years
- $\tau = T - t$ = **Time to maturity**
- $r$ = **Risk-free rate**
- $y$ **Dividend yield**
- $R = r - y$
- $\sigma$ = **Implied volatility**
- $c$ = **Price Call Option**
- $p$ = **Price Put Option**

**Proposition 1** (Black-Scholes Model)**.** Assume the notation before, and let $N(\cdot)$ be the cumulative standard normal distribution function. Under certain assumptions, the Black-Scholes models prices Call and Put options as follows:

$$\begin{cases} C(S_t, t) = Se^{yT} N(d_1) - Ke^{-r \times \tau} N(d_2), \\ \\ P(S_t, t) = Ke^{-r\times\tau}(1 - N(d_2)) - Se^{y\times T}(1 - N(d_1)), \end{cases}$$

where:

$$\begin{cases} d_1 = \dfrac{\ln\left(\dfrac{S_t}{K}\right) + \tau\left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \\ d_2 = d_1 - \sigma\sqrt{\tau} \end{cases}$$

, further the Put Option price corresponds to the **Put-Call parity**, given by:

$$C(S_t, t) + Ke^{-r\times\tau} = P(S_t, t) + S_t$$

**Note** As here we don't have dividends, then $y = 0$, and so

$$\begin{cases} C(S_t, t) = S_t N(d_1) - Ke^{-r \times \tau} N(d_2), \\ \\ P(S_t, t) = Ke^{-r\times\tau}(1 - N(d_2)) - S_t(1 - N(d_1)), \end{cases}$$

### BlackScholes function

```
# source code with options pricign
source(here("code", "OptionPricing.R")) # BlackScholes and Option pricing

# Test: Call Option
S_t = 1540
K = 1600
r = 0.03
tau = 10/360
sigma = 1.05
black_scholes(S_t, K, r, tau, sigma)
```

```
## [1] 80.81672
```

## Book of Options

Assume the following book of **European Call Options:**

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1605$ with maturity $T = 40d$
3. **1x** strike $K = 1800$ with maturity $T = 40d$

Find the price of this book given **the last underlying price** and the **last implied volatility** (take the VIX for all options). Use **Black-Scholes** to price the options. Take the current term structure and **linearly interpolate** to find the corresponding rates. Use 360 days/year for the term structure and **250 days/year** for the maturity of the options.

### Nearest values

This function will obtain the two nearest values $a, b$ for a number $x$ in a vector $v$, such that $a < x < b$.

```
# Test: function used to get two nearest values in a vector (OptionsPricing.R)
days <- rf_mat[, "days"]
get_nearest(40, rf_mat[, "days"]) # nearest day values
```

```
## 1m 3m
## 30 90
```

### Linear Interpolation

Given two known values $(x_1, y_1)$ and $(x_2, y_2)$, we can estimate the $y$-value for some $x$-value with:

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

```
# Function to interpolate y given two points
interpolate <- function(x,x1=1,y1=1,x2=2,y2=2){
  y1 + (x-x1)*(y2-y1)/(x2-x1)
}
```

### Finding the rates through interpolation

The **yield curve** for the given structure of interest rates can be modeled a function $r_f = f(x)$ ,where $x$ is the number of years. Then, we can interapolate the values as follows:

```
# Interest rates
rf_mat
```

```
##                rf        years          days
## 1d  0.0007099993  0.002739726     0.9863014
## 1w  0.0009799908  0.019230769     6.9230769
## 1m  0.0012799317  0.083333333    30.0000000
## 3m  0.0022393730  0.250000000    90.0000000
## 6m  0.0034170792  0.500000000   180.0000000
## 9m  0.0045123559  0.750000000   270.0000000
## 1y  0.0043206525  1.000000000   360.0000000
## 2y  0.0064284968  2.000000000   720.0000000
## 3y  0.0090558654  3.000000000  1080.0000000
## 4y  0.0117237591  4.000000000  1440.0000000
## 5y  0.0141196498  5.000000000  1800.0000000
## 7y  0.0176131823  7.000000000  2520.0000000
## 10y 0.0207989304 10.000000000  3600.0000000
## 30y 0.0203526819 30.000000000 10800.0000000
```

```
head(calls)
```

```
##         K        tau        IV tau_days
## [1,] 1280 0.02557005 0.7370370 6.392513
## [2,] 1370 0.02557005 0.9691616 6.392513
## [3,] 1380 0.02557005 0.9451401 6.392513
## [4,] 1400 0.02557005 0.5274481 6.392513
## [5,] 1415 0.02557005 0.5083375 6.392513
## [6,] 1425 0.02557005 0.4820041 6.392513
```

ex.: **1x** strike $K = 1600$ with maturity $T = 20d$

```
S_t = sp500[length(sp500)] # last price of underlying
IV = vix[length(vix)] # last volatility

## test: specific price (func from OptionPricing.R)
price_option(T=20, K=1600, calls = calls, rf_mat =  rf_mat, stock = NA, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 87.56885
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1600
##
## $r_interp
## [1] 0.001264335
##
## $calls
##         K        tau        IV  tau_days
## [1,] 1600 0.02557005 0.1817481  6.392513
## [2,] 1600 0.10228238 0.1701946 25.570595
##
## $rates
##               rf      years      days
## 1w 0.0009799908 0.01923077  6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

Next, using the function above we price the book of options given:

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1605$ with maturity $T = 40d$
3. **1x** strike $K = 1800$ with maturity $T = 40d$

First, we retrieve the latest value for the underlying (SP500) and the latest implied volatility (VIX):

```
S_t = sp500[length(sp500)] # last price of underlying
IV = vix[length(vix)] # last volatility
```

Then, we price the options accordingly:

```
# First Call Option
price_option(T=20, K=1600, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 87.56885
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1600
##
## $r_interp
## [1] 0.001264335
##
## $calls
##         K        tau        IV  tau_days
## [1,] 1600 0.02557005 0.1817481  6.392513
## [2,] 1600 0.10228238 0.1701946 25.570595
##
## $rates
##              rf      years      days
## 1w 0.0009799908 0.01923077  6.923077
## 1m 0.0012799317 0.08333333 30.000000
```

```
# Second Call Option
price_option(T=40, K=1605, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 90.22871
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1605
##
## $r_interp
## [1] 0.001721275
##
## $calls
##            K          tau           IV     tau_days
## 1605.0000000    0.1022824    0.1676923   25.5705949
##
## $rates
##             rf      years days
## 1m 0.001279932 0.08333333   30
## 3m 0.002239373 0.25000000   90
```

```
# Third Call Option
price_option(T=40, K=1800, calls=calls, rf_mat=rf_mat, S_t = S_t, IV = IV)
```

```
## $Call
## [1] 6.34395
##
## $Put
## [1] NA
##
## $S
## [1] 1683.99
##
## $K
## [1] 1800
##
## $r_interp
## [1] 0.001721275
##
## $calls
##         K       tau        IV tau_days
## [1,] 1800 0.1022824 0.1057523 25.57059
## [2,] 1800 0.1789947 0.1044115 44.74868
##
## $rates
##             rf      years days
## 1m 0.001279932 0.08333333   30
## 3m 0.002239373 0.25000000   90
```

## Two risk drivers and copula-marginal model (Student-t and Gaussian Copula)

1. Compute the daily log-returns of the underlying stock
2. Assume the first invariant is generated using a Student-t distribution with $\nu = 10$ df and the second invariant is generated using a Student-t distribution with $\nu = 5$ df.
3. Assume the **normal copula** to merge the marginals.
4. Generate 10000 scenarios for the one-week ahead price for the underlying and the one-week ahead VIX value using the copula.
5. Determine the P&L distribution of the book of options, using the simulated values.
6. Take interpolated rates for the term structure.

### Gaussian Copula with two Student-t marginals

A bivariate distribution $H$ can be formed via a copula $C$ from two marginal distributions with CDFs $F$ and $G$ via:

$$H(x, y) = C(F(x), G(y)) = C(F^{-1}(u), G^{-1}(u))$$

with density

$$h(x, y) = c(F(x), G(y))f(x)g(y)$$

The **Gaussian Copula** is given by:

$$C_\rho^{\text{Gauss}}(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)).$$

In this case, a Gaussian copula with two Student-t marginals with CDFs $t(\nu_1)$ with $\nu_1$ degrees of freedom and $t(\nu_2)$ with $\nu_2$ degrees of freedom is given by:

$$C_\rho^{\text{Gauss}}(u, v) = \Phi_\rho(F_{\nu_1}^{-1}(u), F_{\nu_1}^{-1}(v)),$$

where $F_{\nu_1}$ and $F_{\nu_2}$ are their respective CDFs.

#### Log-returns

The **discrete returns** are given by:

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t}$$

and the next ahead log-returns are given by:

$$\log(R_{t+1}) = \log(P_{t+1} - P_t) - \log(P_t)$$

```
# load reqruired libraries
library("PerformanceAnalytics")

# calculate returns
sp500_rets <- PerformanceAnalytics::CalculateReturns(sp500, method="log")
vix_rets <- PerformanceAnalytics::CalculateReturns(vix, method="log")

# remove first return
sp500_rets <- sp500_rets[-1]
vix_rets <- vix_rets[-1]

# remove nas
sp500_rets[is.na(sp500_rets)] <- 0
vix_rets[is.na(vix_rets)] <- 0

# display
head(sp500_rets)
```

```
##                    sp500
## 2000-01-04 -0.0390992269
## 2000-01-05  0.0019203798
## 2000-01-06  0.0009552461
## 2000-01-07  0.0267299353
## 2000-01-10  0.0111278213
## 2000-01-11 -0.0131486343
```

```
head(vix_rets)
```

```
##                      vix
## 2000-01-04  0.1094413969
## 2000-01-05 -0.0224644415
## 2000-01-06 -0.0260851000
## 2000-01-07 -0.1694241312
## 2000-01-10 -0.0004605112
## 2000-01-11  0.0357423253
```

#### Generating the simulation scenarios

Assumptions: - Marginal Student-t distributions - Disregard time dependence in the bootstrapping process

```
# Load required libraries
library("fGarch")
```

```
## NOTE: Packages 'fBasics', 'timeDate', and 'timeSeries' are no longer
## attached to the search() path when 'fGarch' is attached.
##
## If needed attach them yourself in your R script by e.g.,
##         require("timeSeries")
```

```
##
## Attaching package: 'fGarch'

## The following object is masked from 'package:TTR':
##
##     volatility
```

```
library("MASS")
library("copula")
library("Matrix")

# random seed for replication
set.seed(123)

# convert to vector since fitting without dependence
sp500_rets_vec <- as.vector(sp500_rets)
vix_rets_vec <- as.vector(vix_rets)

# calculate means and sds for both indices
mu <- c(mean(sp500_rets_vec), mean(vix_rets_vec))
sigma <- c(sd(sp500_rets_vec), sd(vix_rets_vec))

# display
mu
```

```
## [1]  0.00004283042 -0.00014976541
```

```
sigma
```

```
## [1] 0.01332592 0.06367330
```

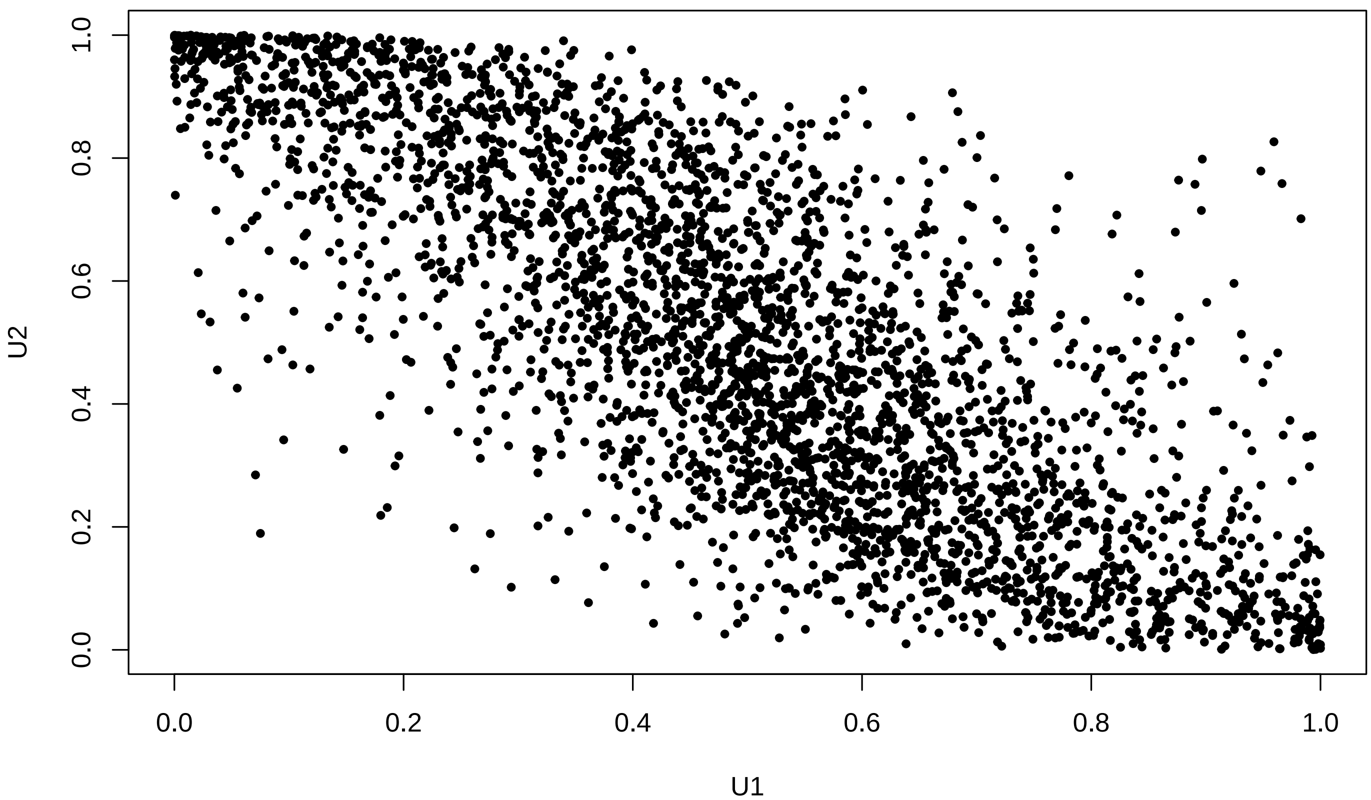
```
## Fit marginals by MLE

# Student-t for sp500
fit1 <- suppressWarnings(
  fitdistr(x = sp500_rets_vec,
           densfun = dstd,
           start = list(mean = 0, sd = 1, nu = 10))
  )
theta1 <- fit1$estimate #extract fitted parameters

# Student-t for vix
fit2 <- suppressWarnings(
  fitdistr(x = vix_rets_vec,
           densfun = dstd,
           start = list(mean = 0, sd = 1, nu = 5))
  )
theta2 <- fit2$estimate # extract fitted parameters

# display parameters
theta1
```

```
##         mean           sd           nu
## 0.0004414879 0.0156603739 2.6953920404
```

```
theta2
```

```
##         mean           sd           nu
## -0.003475206  0.064192681  4.230323432
```

```
# Fit Student-t to the marginals
# U1 <- pstd(sp500_rets_vec, mean = theta1[1], sd = theta1[2], nu = theta1[3]) # sp500
# U2 <- pstd(vix_rets_vec,mean = theta2[1], sd = theta2[2], nu = theta2[3]) # vix
U1 <- pstd(sp500_rets_vec, mean = theta1[1], sd = theta1[2], nu = 10) # sp500
U2 <- pstd(vix_rets_vec,mean = theta2[1], sd = theta2[2], nu = 5) # vix
# U1 <- pt(sp500_rets_vec, df = 5) # sp500
# U2 <- pt(vix_rets_vec, df = 10) # vix
U <- cbind(U1, U2) # join into one matrix
plot(U, pch = 20, cex = 0.9)
```

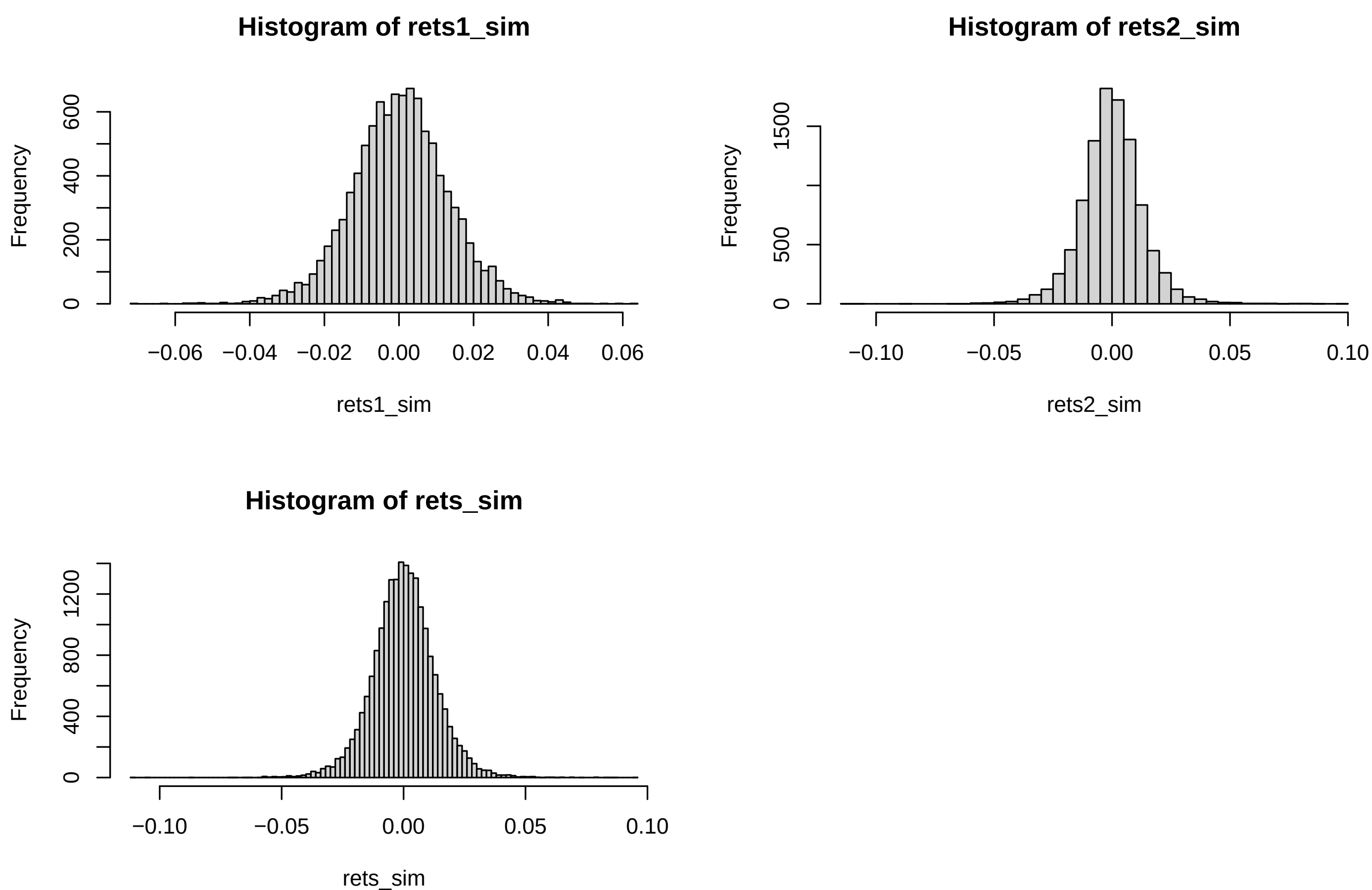



```
# Obtain the best rho for the Gaussian Copula
C <- normalCopula(dim = 2)
fit <- fitCopula(C, data = U, method = "ml")
fit
```

```
## Call: fitCopula(C, data = U, ... = pairlist(method = "ml"))
## Fit based on "maximum likelihood" and 3409 2-dimensional observations.
## Copula: normalCopula
##   rho.1
## -0.7984
## The maximized loglikelihood is 1494
## Optimization converged
```

```
# seed for replication
set.seed(420)

# Simulation parameters
n_sim = 10000 # set number of simulations
```

```r
# n_ahead = 5 # days ahead to produce samples

# produce simulations from copula
U_sim <- rCopula(n_sim, fit@copula)

# use copula U_sim to reproduce the marginals with student-t distr
# rets1_sim <- qstd(U_sim[,1], mean = mu[1], sd = sigma[1], nu = theta1[3]) # sp500
# rets2_sim <- qstd(U_sim[,2], mean = mu[1], sd = sigma[1], nu = theta2[3]) # vix
rets1_sim <- qstd(U_sim[,1], mean = mu[1], sd = sigma[1], nu = 10) # sp500
rets2_sim <- qstd(U_sim[,2], mean = mu[1], sd = sigma[1], nu = 5) # vix
rets_sim <- cbind(rets1_sim, rets2_sim)

# visualize
par(mfrow = c(2,2))
hist(rets1_sim, nclass=50)
hist(rets2_sim, nclass=50)
hist(rets_sim, nclass = round(10 * log(n_sim)))
```




```r
# random seed for replication
set.seed(69)

##############################
### Setup & Initialization ###
##############################

# Simulation parameters
n_sim = 10000 # set number of simulations
n_ahead = 5 # days ahead to produce samples
```

```
# preallocate matrices to store simulations
sim_rets_sp500 <- matrix(NA, nrow = n_sim, ncol=5)
sim_rets_vix <- matrix(NA, nrow = n_sim, ncol=5)

# assign days ahead
colnames(sim_rets_sp500) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
colnames(sim_rets_vix) <- c("T+1", "T+2", "T+3", "T+4", "T+5")

##############################
### Running the simulation ###
##############################

# perform n_head days of n_sim scenarios
for(t in 1:n_ahead){

  # Sample 5-days ahead from Gaussian Copula
  U_sim <- rCopula(n_sim, fit@copula)

  # use copula U_sim to reproduce the marginals quantiles F^{-1}(u) with student-t distr
  rets1_sim <- qstd(U_sim[,1], mean = theta1[1], sd = theta1[2], nu = 10) # sp500
  rets2_sim <- qstd(U_sim[,2], mean = theta2[1], sd = theta2[2], nu = 5) # vix
  # rets1_sim <- qt(U_sim[,1], df = 10) # sp500
  # rets2_sim <- qt(U_sim[,2], df = 5) # vix
  rets_sim <- cbind(rets1_sim, rets2_sim)

  # store simulation of log return in matrix
  sim_rets_sp500[ ,t] <- rets1_sim
  sim_rets_vix[ ,t] <- rets2_sim
}

# preview of simulated log returns
head(sim_rets_sp500)
```

```
##                T+1           T+2          T+3           T+4           T+5
## [1,] -0.0009645126  0.0158554079  0.015383149 -0.0174310226  0.0155476192
## [2,]  0.0022227010  0.0178616966  0.003249986  0.0015075435  0.0004961551
## [3,] -0.0202696762  0.0070645564  0.011080134 -0.0163632659  0.0039542793
## [4,]  0.0267344996  0.0190648399 -0.004275895  0.0312856876  0.0004778219
## [5,] -0.0045092785  0.0008870700 -0.005286801  0.0089807462 -0.0122229007
## [6,] -0.0039970206 -0.0002199501 -0.003419139 -0.0004051185  0.0371441443
```

```
head(sim_rets_vix)
```

```
##              T+1          T+2          T+3         T+4         T+5
## [1,]  0.01231074  0.006644294 -0.005354024  0.01255679 -0.04752175
## [2,] -0.04109607 -0.073223553 -0.020098934 -0.03207569 -0.06300583
## [3,]  0.08429964 -0.030662396 -0.071921523  0.10934242 -0.05145715
## [4,] -0.08896620 -0.032518583  0.020560914 -0.12085679 -0.02129170
## [5,] -0.05179948 -0.017505235  0.022004416 -0.04412445  0.03046923
## [6,]  0.01708910 -0.034281364  0.032441799  0.04104414 -0.11119617
```

**Computing Prices from Returns**

Next, we crate a function to forecast the 5 day ahead prices from the returns. Since:

$$
\begin{aligned}
R_t &= \frac{P_t - P_{t-1}}{P_{t-1}} \\
\implies R_t &= \frac{P_t}{P_{t-1}} - 1 \\
\implies \log(R_t) &= \log\left(\frac{P_t}{P_{t-1}}\right) \\
\implies \log(R_t) &= \log(P_t) - \log(P_{t-1}) \\
\implies \log(P_t) &= \log(R_t) + \log(P_{t-1}) \\
\implies P_t &= \exp(\log(R_t) + \log(P_{t-1})) \\
\implies P_{t+1} &= \exp(\log(R_{t+1}) + \log(P_t))
\end{aligned}
$$

Since:

$$
\exp\log(R_t) + \log(P_{t-1}) = \log(R_t \cdot P_{t-1})
$$

```r
# Obtain Initial values (last value of simulation)
spT <- sp500[length(sp500)][[1]]
vixT <- vix[length(vix)][[1]]

# Initialize empty matrices for the simulated sp500 and vix values
sim_val_mats <- initialize_sim_mats(sim_rets_sp500,
                                    lnames = c("sp500", "vix"), # <- this function comes from Utils.R
                                    num_mats = 2
                                    )

# Initialize the first prices
sim_val_mats$sp500[, 1] <- f_next_Pt(spT, sim_rets_sp500[, 1])
sim_val_mats$vix[, 1] <- f_next_Pt(vixT, sim_rets_vix[, 1])

# for each day ahead
for(t in 2:n_ahead){
  # obtain the values for P_{t-1}
  Pt_prev_sp500 <- sim_val_mats$sp500[, t-1]
  Pt_prev_vix <- sim_val_mats$vix[, t-1]

  # extract current returns R_{t}
  Rt_sp500 <- sim_rets_sp500[, t]
  Rt_vix <- sim_rets_vix[, t]

  # compute and assign next price ahead using current returns
  sim_val_mats$sp500[, t] <- f_next_Pt(Pt_prev_sp500, Rt_sp500)
  sim_val_mats$vix[, t] <- f_next_Pt(Pt_prev_vix, Rt_vix)

}

# unpack matrices
sim_price_sp500 <- sim_val_mats$sp500
sim_vol_vix <- sim_val_mats$vix
```

```r
# compare simulated returns with the price
head(sim_rets_sp500)
```

```
##                T+1          T+2          T+3           T+4          T+5
## [1,] -0.0009645126 0.0158554079  0.015383149 -0.0174310226 0.0155476192
## [2,]  0.0022227010 0.0178616966  0.003249986  0.0015075435 0.0004961551
## [3,] -0.0202696762 0.0070645564  0.011080134 -0.0163632659 0.0039542793
## [4,]  0.0267344996 0.0190648399 -0.004275895  0.0312856876 0.0004778219
```

```
## [5,] -0.0045092785  0.0008870700 -0.005286801  0.0089807462 -0.0122229007
## [6,] -0.0039970206 -0.0002199501 -0.003419139 -0.0004051185  0.0371441443
```

```
head(sim_price_sp500)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 1682.367 1709.254 1735.751 1705.757 1732.485
## 2 1687.737 1718.154 1723.747 1726.347 1727.204
## 3 1650.200 1661.899 1680.415 1653.142 1659.692
## 4 1729.618 1762.909 1755.387 1811.174 1812.039
## 5 1676.414 1677.901 1669.054 1684.111 1663.651
## 6 1677.272 1676.904 1671.180 1670.503 1733.719
```

```
# compare simualted log rets with volatility
head(sim_rets_vix)
```

```
##              T+1          T+2          T+3         T+4         T+5
## [1,]  0.01231074  0.006644294 -0.005354024  0.01255679 -0.04752175
## [2,] -0.04109607 -0.073223553 -0.020098934 -0.03207569 -0.06300583
## [3,]  0.08429964 -0.030662396 -0.071921523  0.10934242 -0.05145715
## [4,] -0.08896620 -0.032518583  0.020560914 -0.12085679 -0.02129170
## [5,] -0.05179948 -0.017505235  0.022004416 -0.04412445  0.03046923
## [6,]  0.01708910 -0.034281364  0.032441799  0.04104414 -0.11119617
```

```
head(sim_vol_vix)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1 0.1470998 0.1480804 0.1472897 0.1491509 0.1422287
## 2 0.1394498 0.1296037 0.1270248 0.1230150 0.1155035
## 3 0.1580798 0.1533063 0.1426674 0.1591518 0.1511695
## 4 0.1329316 0.1286783 0.1313515 0.1163985 0.1139464
## 5 0.1379651 0.1355711 0.1385873 0.1326051 0.1367077
## 6 0.1478044 0.1428233 0.1475327 0.1537141 0.1375377
```

### Pricing the simulation scenarios

Recall the initial (call) options:

1. **1x** strike $K = 1600$ with maturity $T = 20d$
2. **1x** strike $K = 1605$ with maturity $T = 40d$
3. **1x** strike $K = 1800$ with maturity $T = 40d$

### Option Pricing of Simulated Values

Next, we calculate the price of the book of options for the simulated values.

```
# random seed for replication
set.seed(123)

# Initialize empty matrices to store the simulated option prices (aka premiums)
opt_price_mats <- initialize_sim_mats(sim_price_sp500,
                                      lnames = c("opt1", "opt2", "opt3"),
                                      num_mats = 3
                                      )

#  maturities for each of the options
T1 <- 20
```

```
T2 <- 40
T3 <- 40

# Strikes for the options
K1 <- 1600
K2 <- 1605
K3 <- 1800

# looop through simulated prices (n_ahead days)
for(t in 1:n_ahead){

  # extract simulated prices for sp500 at T+t
  prices_t <- sim_price_sp500[, t]

  # extract implied volatility from vix at T+t
  vols_t <- sim_vol_vix[, t]

  # price first Call option
  c1_vec <- prc_opt(T1-t, K1, calls, rf_mat, prices_t, vols_t)
  opt_price_mats$opt1[ ,t] <- c1_vec
  # print(cbind(prices_t, vols_t, c1_vec)) # <-- uncomment for debugging

  # price first Call option
  c2_vec <- prc_opt(T2-t, K2, calls, rf_mat, prices_t, vols_t)
  opt_price_mats$opt2[ ,t] <- c2_vec

  # price first Call option
  c3_vec <- prc_opt(T3-t, K3, calls, rf_mat, prices_t, vols_t)
  opt_price_mats$opt3[ ,t] <- c3_vec
}
```

```
# overview of dataframes
head(opt_price_mats$opt1)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1  85.93966 110.69740 136.26860 107.01790 132.81728
## 2  90.24098 118.71906 124.10892 126.59144 127.36517
## 3  60.25326  68.45593  83.19676  60.93087  64.90429
## 4 130.13155 163.09257 155.57730 211.29567 212.15179
## 5  79.84446  80.69383  72.68668  85.73559  67.03312
## 6  81.47264  80.36234  75.33654  74.88091 133.99041
```

```
head(opt_price_mats$opt2)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1  88.90248 111.24440 134.68187 107.77097 130.79380
## 2  91.92460 116.94248 121.69327 123.66209 123.86035
## 3  67.26520  74.03584  85.82386  68.09352  70.64490
## 4 128.02224 159.29957 152.04729 206.61873 207.44600
## 5  82.45366  82.87864  76.06279  86.86910  70.78117
## 6  84.99456  83.39449  79.42470  79.69059 131.55401
```

```
head(opt_price_mats$opt3)
```

```
##         T+1       T+2       T+3       T+4       T+5
## 1  6.123012 10.265706 15.823742  9.093513 13.050811
## 2  5.750180  8.622494  8.911881  8.380779  6.947708
```

```
## 3  4.015923  4.362723  4.812672  3.739971  3.319742
## 4 11.997642 20.224478 17.932947 37.896755 37.291731
## 5  4.272045  3.942236  3.273257  3.813760  2.371687
## 6  5.586715  4.680559  4.407937  4.806311 12.402286
```

### Distribution of the Profit and Loss for the Book Of Options

Recall the profit functions for European options:

#### Parameters

**Parameters:** - $S$: Spot price (current) - $S_0$: Spot price at the beginnin of the option - $S_T$: Spot price at maturity - $T$: Maturity of option - $K$: Strike price - $c$: Price of Call option - $p$: Price of Put option

#### Profit at Maturity

The profit functions of a long call and a long put are given by:

$$\pi^{\text{Long Call}} = \max(S_T - K, 0) - c$$
$$\pi^{\text{Long Put}} = \max(K - S_T, 0) - p$$

#### Calculating the profits

For each of the simulated prices and resulting premiums, we want to calculate the profit generated at each simulation timestep:

```r
# Matrices of profit and loss for each of the options simulations
PL_mats <- initialize_sim_mats(sim_price_sp500,
                               lnames = c("PL1", "PL2", "PL3"),
                               num_mats = 3
                               )

# Calculate profit for all simulated options at each day ahead
for(t in 1:n_ahead){

  #spot price of underlying at day T+t
  spot <- sim_price_sp500[, t]

  # Option profit for K1 at time T+t with premiums c1
  c1 <- opt_price_mats$opt1[, t] # extract the premiums
  PL_mats$PL1[,t] <- option_profit(S=spot, K=K1, c=c1)$call_profit

  # Option profit for K1 at time T+t with premiums c1
  c2 <- opt_price_mats$opt2[, t] # extract the premiums
  PL_mats$PL2[,t] <- option_profit(S=spot, K=K2, c=c2)$call_profit

  # Option profit for K1 at time T+t with premiums c1
  c3 <- opt_price_mats$opt3[, t] # extract the premiums
  PL_mats$PL3[,t] <- option_profit(S=spot, K=K2, c=c3)$call_profit
}
```

```r
# display profit matrices
head(PL_mats$PL1)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 156.9949 187.2824 204.0350 281.2273 266.0588
## 2 152.6936 179.2607 216.1946 261.6537 271.5109
```

```
## 3 182.6813 229.5238 257.1068 327.3143 333.9718
## 4 112.8030 134.8872 184.7263 176.9495 186.7242
## 5 163.0901 217.2859 267.6169 302.5096 331.8429
## 6 161.4620 217.6174 264.9670 313.3643 264.8856
```

```
head(PL_mats$PL2)
```

```
##        T+1      T+2      T+3      T+4      T+5
## 1 149.0321 181.7354 200.6217 275.4742 263.0822
## 2 146.0100 176.0373 213.6103 259.5831 270.0157
## 3 170.6694 218.9439 249.4797 315.1517 323.2311
## 4 109.9124 133.6802 183.2563 176.6265 186.4300
## 5 155.4809 210.1011 259.2408 296.3761 323.0949
## 6 152.9400 209.5853 255.8789 303.5546 262.3220
```

```
head(PL_mats$PL3)
```

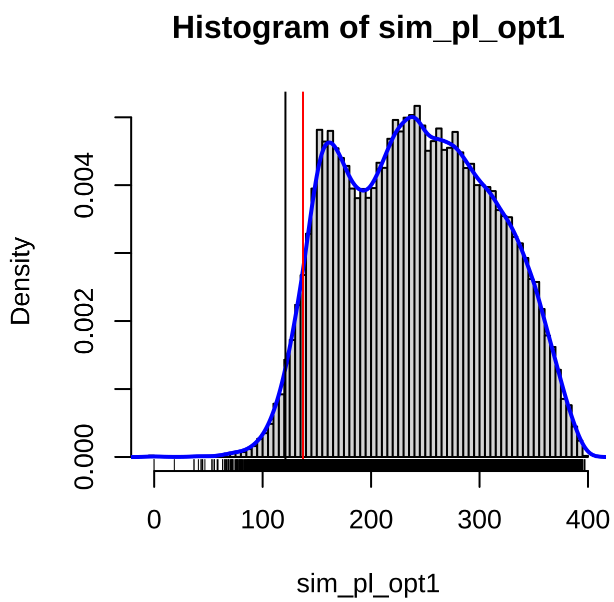
```
##        T+1      T+2      T+3      T+4      T+5
## 1 231.8116 282.7141 319.4798 374.1517 380.8252
## 2 232.1844 284.3573 326.3917 374.8644 386.9283
## 3 233.9187 288.6170 330.4909 379.5052 390.5563
## 4 225.9369 272.7553 317.3706 345.3484 356.5843
## 5 233.6625 289.0375 332.0303 379.4314 391.5044
## 6 232.3479 288.2992 330.8956 378.4389 381.4738
```

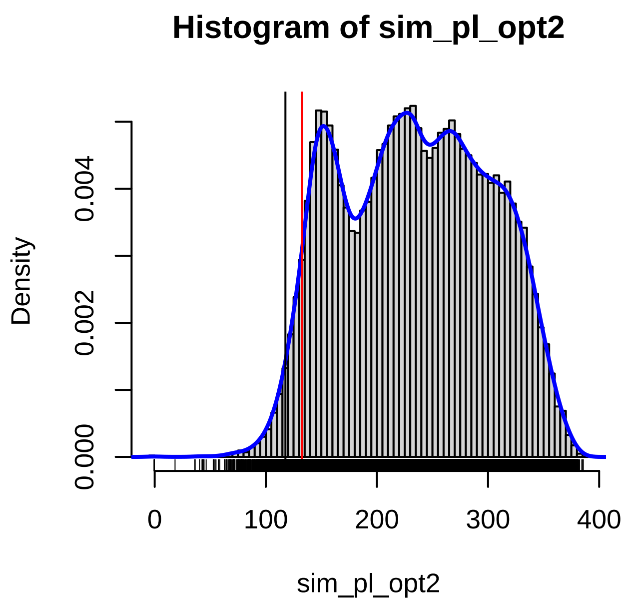
### Distribution of Options P/L

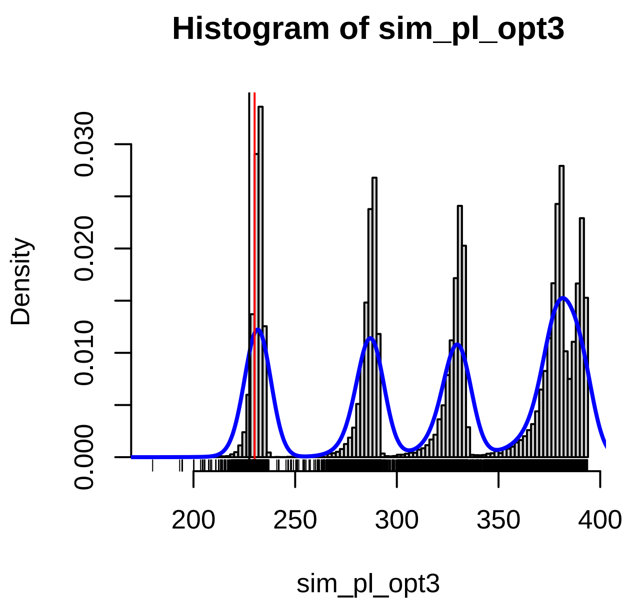
Next, using all the simulated profits and losses for each of the options, we display a histogram for the distribution for each of the options, for the aggregated 5 days of simulation:

```
# flatten the matrices 5-days ahead simulated P/L for the three options
sim_pl_opt1 <- as.vector(PL_mats$PL1)
sim_pl_opt2 <- as.vector(PL_mats$PL2)
sim_pl_opt3 <- as.vector(PL_mats$PL3)

# Compute the 95% VaR and 95% ES
opt1_VaR_ES <- f_VaR_ES(sim_pl_opt1, alpha = 0.05)
opt2_VaR_ES <- f_VaR_ES(sim_pl_opt2, alpha = 0.05)
opt3_VaR_ES <- f_VaR_ES(sim_pl_opt3, alpha = 0.05)

# plot the distribution for each of the options
par(mfrow = c(2,2))
hist(sim_pl_opt1, nclass = round(10 * log(n_sim)), probability = TRUE)
lines(density(sim_pl_opt1), lwd=2, col="blue")
abline(v=opt1_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt1_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt1)


hist(sim_pl_opt2, nclass = round(10 * log(n_sim)), probability = TRUE)
lines(density(sim_pl_opt2), lwd=2, col="blue")
abline(v=opt2_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt2_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt2)


hist(sim_pl_opt3, nclass = round(10 * log(n_sim)), probability = TRUE)
lines(density(sim_pl_opt3), lwd=2, col="blue")
```

```
abline(v=opt3_VaR_ES$VaR, col="red") # 95% VaR
abline(v=opt3_VaR_ES$ES, col="black") # expected shortfall
rug(sim_pl_opt3)
```




**VaR95**

**Definition**

For a random variable $X$, the **Value-at-Risk (VaR)** at level $\alpha$ is defined as the $\alpha$-lower quantile of the distribution of X, thus:

$$VaR_X(\alpha) = F_X^{-1}(1-\alpha)$$

**First Option**

```r
opt1_VaR_ES$VaR # first option
```

```
## [1] 137.2302
```

```r
opt2_VaR_ES$VaR # second doption
```

```
## [1] 132.5263
```

```r
opt3_VaR_ES$VaR # third option
```

```
## [1] 230.0578
```

### ES95

Expected shortfall is calculated by averaging all of the returns in the distribution that are worse than the VAR of the portfolio at a given level of confidence.

```r
# display
opt1_VaR_ES$ES
```

```
## [1] 121.0392
```

```r
opt2_VaR_ES$ES
```

```
## [1] 117.6614
```

```r
opt3_VaR_ES$ES
```

```
## [1] 227.4093
```

## Full Approach

1. Filter the volatility clustering of the log-returns of the underlying using a GARCH(1,1) model with Normal innovations. Use the residuals as invariants.
2. Take and AR(1) model for the log-returns of the VIX. Use the residuals as invariants.
3. Use normal marginals for the invariants and a normal copula.
4. Generate draws for the invariants, compute next week (five days) values and reprice the portfolio.
5. Compute the VaR95 and ES95.

### Log returns of the underlying

```r
# load reqruired libraries
library("PerformanceAnalytics")

# calculate returns
sp500_rets <- PerformanceAnalytics::CalculateReturns(sp500, method="log")
vix_rets <- PerformanceAnalytics::CalculateReturns(vix, method="log")

# remove first return
sp500_rets <- sp500_rets[-1]
vix_rets <- vix_rets[-1]
```

```r
# remove nas
sp500_rets[is.na(sp500_rets)] <- 0
vix_rets[is.na(vix_rets)] <- 0

# display
head(sp500_rets)
```

```
##                    sp500
## 2000-01-04 -0.0390992269
## 2000-01-05  0.0019203798
## 2000-01-06  0.0009552461
## 2000-01-07  0.0267299353
## 2000-01-10  0.0111278213
## 2000-01-11 -0.0131486343
```

```r
head(vix_rets)
```

```
##                      vix
## 2000-01-04  0.1094413969
## 2000-01-05 -0.0224644415
## 2000-01-06 -0.0260851000
## 2000-01-07 -0.1694241312
## 2000-01-10 -0.0004605112
## 2000-01-11  0.0357423253
```

### GARCH(1,1) Model

#### Model specification

$$
\begin{aligned}
y_t &= \epsilon_t \sigma_t, \\
\sigma_t^2 &= \omega \ + \ \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\epsilon_t &\overset{i.i.d.}{\sim} \mathcal{N}(0,1),
\end{aligned}
$$

#### Mean and variance

$$
\begin{aligned}
\mathbb{E}[Y_t] &\approx 0 \\
\mathbb{V}ar[Y_t] &= \mathbb{E}[\epsilon_t^2] = \mathbb{E}[\sigma_t^2] = \frac{\omega}{(1-\alpha-\beta)}
\end{aligned}
$$

#### Stationarity Conditions

$$
\begin{aligned}
\omega &\geq 0 \\
\alpha \ , \ \beta &> 0 \\
\alpha + \beta &< 1 \quad \text{(Covariance-Stationary)}
\end{aligned}
$$

#### VaR

$$
VaR_Y(\alpha) = \Phi^{-1}(1-\gamma)\sigma_t,
$$

#### Log-likelihood

$$
\ln L(\theta|\mathbf{y}) = -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T} \ln \sigma_t^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{y_t^2}{\sigma_t^2}.
$$

## Volatility clustering of the log-returns of the underlying with GARCH(1,1)

Which indicates a high level of autocorrelation in the returns.

### Fitting the GARCH(1,1)

```
# source code for garch
source(here("code", "GARCH.R")) # GARCH model implementation

# Estimate the GARCH(1,1) model
fit_garch <- f_optim(sp500_rets)
```

### Inpect the parameters

```
# extract parameters (omega, alpha, beta)
theta_hat_garch <- fit_garch$theta_hat
theta_hat_garch
```

```
## [1] 0.000001461511 0.090037405420 0.903175089840
```

### Verify stationarity

```
# make sure stationarity is satisfied
sum(theta_hat_garch[2:3])
```

```
## [1] 0.9932125
```

### Mean Squared Error

```
# MSE ?
sqrt(theta_hat_garch[1] / (1 - sum(theta_hat_garch[2:3]))) * sqrt(250)
```

```
## [1] 0.2320149
```

```
# sd of returns annualized?
sd(sp500_rets) * sqrt(250)
```

```
## [1] 0.2107013
```

#### ***Residuals***

The residuals are given by:

$$\hat{\epsilon}_t = \frac{y_t}{\hat{\sigma}_t}$$

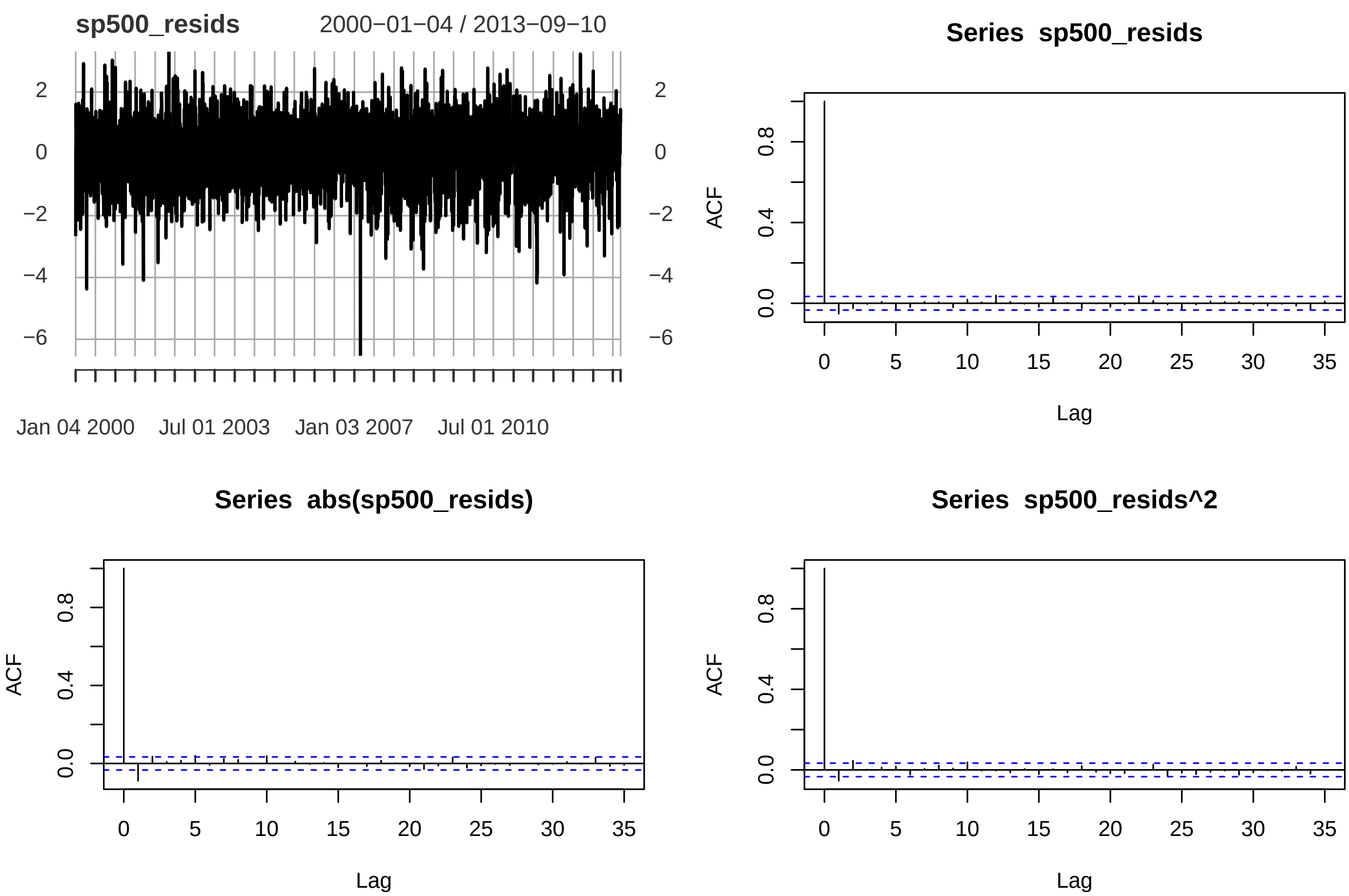
```
# extrct the residuals
sp500_resids <- fit_garch$eps_hat

# inspect their mean and variance
mean(sp500_resids)
```

```
## [1] 0.005801314
```

```
sd(sp500_resids)
```

```
## [1] 0.9908629
```

```
# Look at dependence in the residuals
par(mfrow = c(2,2))

# Eps_hat = Innovations Series
plot(sp500_resids, pch = 20)

# autocorr of innovations
acf(sp500_resids)

# autocorr of the absolute values
acf(abs(sp500_resids))

# autocorr of the variance of the innovations
acf(sp500_resids^2)
```




## AR(1) for the log-returns of the VIX

### First-order Autoregressive Process AR(1)

- Let $\{\varepsilon_t\}$ be a mean-zero white noise process with variance $\sigma^2$.
- Consider a process $\{X_t\}$, independent of $\{\varepsilon_t\}$.
- Let $\phi$ be constant.

The **AR(1) process** satisfies:

$$X_t = \phi X_{t-1} + \varepsilon_t$$

It can be shown that:

$$\mu_X(t) = \mathbb{E}[X_t] = \phi\mu_X(t-1) = 0 \quad , \forall t$$

when the process is stationary, and the autocovariance function $\gamma_X(h)$ with alg $h$ and autocorrelation $\rho_X(h)$ are given by

$$\gamma_X(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2} \quad \text{and} \quad \rho_X(h) = \phi^{|h|}$$

### VIX log-returns

```r
library("forecast")
```

```
## Warning: package 'forecast' was built under R version 4.2.3
```

```r
# Construct an AR(1) model to the vix
vix_ar1 <- ar(vix_rets, order.max = 1)
vix_ar1$ar # phi coefficient
```

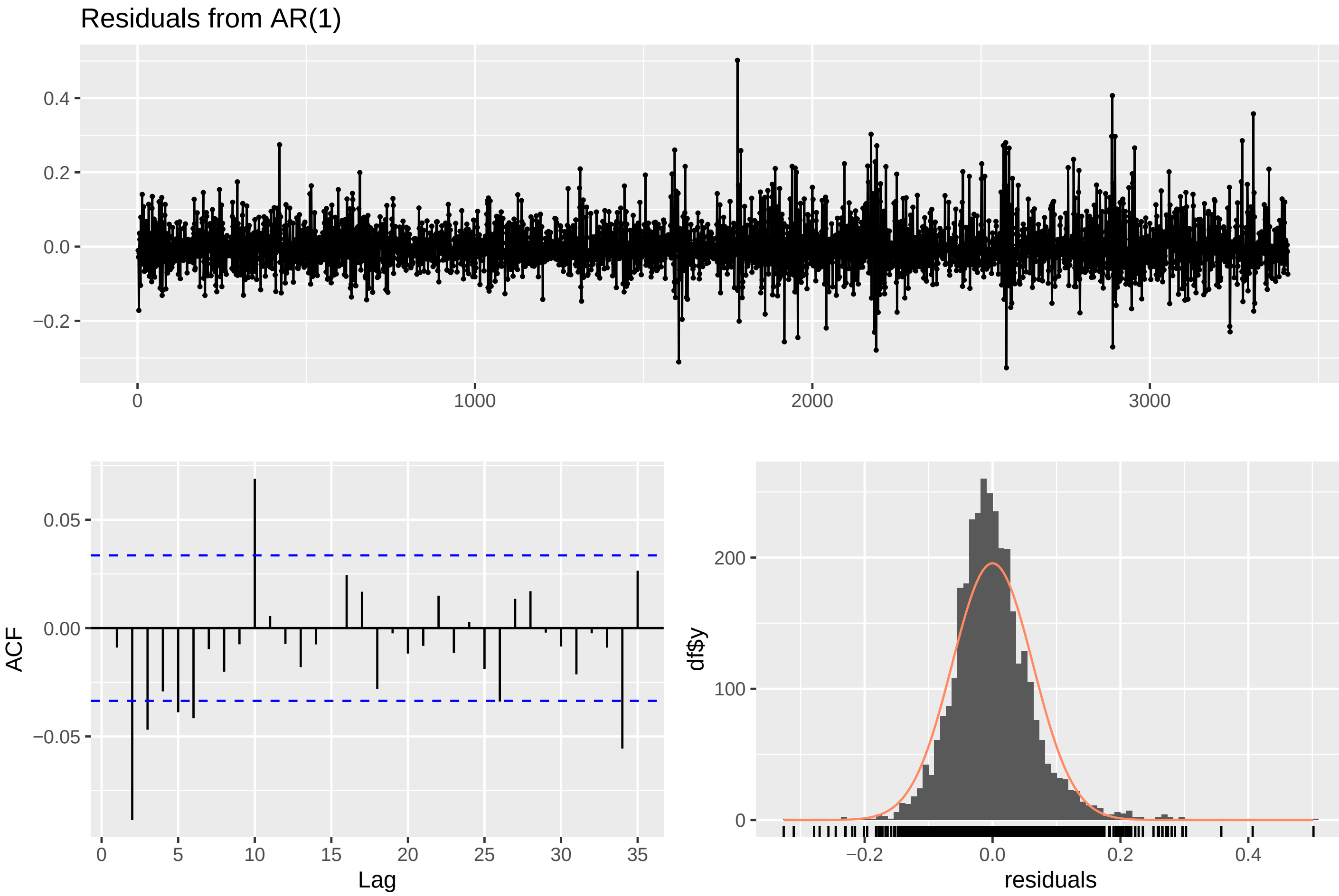
```
## [1] -0.1074941
```

### Stationarity of the residuals & underlying normality

```r
# extract the residuals
vix_resids <- vix_ar1$resid
vix_resids[1] <- 0 # first residual is NA
head(vix_resids)
```

```
## [1]  0.00000000 -0.01053427 -0.02833403 -0.17206226 -0.01850675  0.03585869
```

```
checkresiduals(vix_ar1, main="Residuals for AR(1) Model")
```




```
##
##  Ljung-Box test
##
## data:  Residuals from AR(1)
## Q* = 66.683, df = 10, p-value = 0.0000000001929
##
## Model df: 0.   Total lags used: 10
```

## Normal Copula with Normal Marginals for the Invariants

### Bivariate Gaussian Copula

Recall that the bivariate Gaussian copula is given by:

$$C_\rho^{\text{Gauss}}(u,v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)). \quad \Longleftarrow \quad H(x,y) = C(F(x), G(y))$$

$$C_\rho^{\text{Gauss}}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dxdy$$

### Gaussian marginals to the invariants

```r
# invariants are the residuals
sp500_resids <- as.vector(sp500_resids)
vix_resids <- as.vector(vix_resids)

# display some values
head(sp500_resids, 10)
```

```
##  [1] -2.66453978  0.10514445  0.05487059  1.61107285  0.62756665 -0.76350023
##  [7] -0.26044462  0.74944189  0.67144427 -0.44500488
```

```r
head(vix_resids, 10)
```

```
##  [1]  0.00000000 -0.01053427 -0.02833403 -0.17206226 -0.01850675  0.03585869
##  [7]  0.01900603 -0.04896233 -0.10447530  0.07897068
```

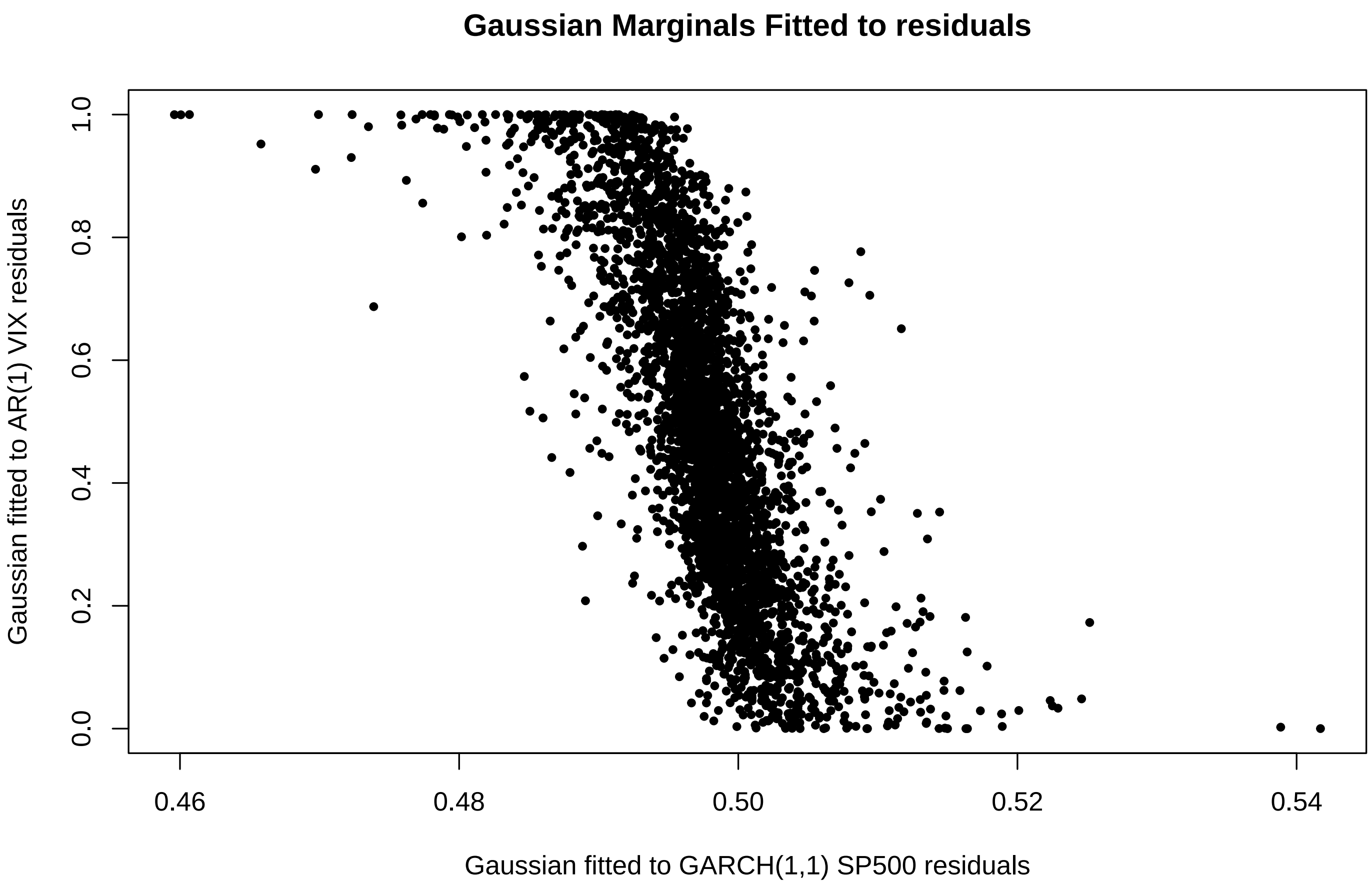
```r
## Fit marginals by MLE

# Gaussian for sp500
fit1 <- suppressWarnings(
  fitdistr(x = sp500_resids,
           densfun = dnorm,
           start = list(mean = 0, sd = 1))
  )
theta1 <- fit1$estimate #extract fitted parameters

# Gaussianfor vix
fit2 <- suppressWarnings(
  fitdistr(x = vix_resids,
           densfun = dnorm,
           start = list(mean = 0, sd = 1))
  )
theta2 <- fit2$estimate # extract fitted parameters

# display parameters
theta1
```

```
##        mean          sd
## 0.005801451 0.990717432
```

```r
theta2
```

```
##           mean             sd
## -0.00004052605  0.06327442562
```

```r
# Fit a Gaussian to the marginals
U1 <- pnorm(sp500_rets_vec, mean = theta1[1], sd = theta1[2]) # sp500
U2 <- pnorm(vix_rets_vec,mean = theta2[1], sd = theta2[2]) # vix
U <- cbind(U1, U2) # join into one matrix
plot(U,
     pch = 20, cex = 0.9,
     main="Gaussian Marginals Fitted to residuals",
     xlab="Gaussian fitted to GARCH(1,1) SP500 residuals",
     ylab="Gaussian fitted to AR(1) VIX residuals"
     )
```




#### Fitting the Gaussian Copula

```
# Obtain the best rho for the Gaussian Copula
C <- normalCopula(dim = 2)
fit <- fitCopula(C, data = U, method = "ml")
fit
```

```
## Call: fitCopula(C, data = U, ... = pairlist(method = "ml"))
## Fit based on "maximum likelihood" and 3409 2-dimensional observations.
## Copula: normalCopula
##   rho.1
## -0.2006
## The maximized loglikelihood is 4.903
## Optimization converged
```

#### Simulating the invariants with the Copula

```
# random seed for replication
set.seed(69)

##############################
### Setup & Initialization ###
##############################

# Simulation parameters
```

```r
n_sim = 10000 # set number of simulations
n_ahead = 5 # days ahead to produce samples

# preallocate matrices to store simulations
sim_inv_sp500 <- matrix(NA, nrow = n_sim, ncol=5)
sim_inv_vix <- matrix(NA, nrow = n_sim, ncol=5)

# assign days ahead
colnames(sim_inv_sp500) <- c("T+1", "T+2", "T+3", "T+4", "T+5")
colnames(sim_inv_vix) <- c("T+1", "T+2", "T+3", "T+4", "T+5")

##############################
### Running the simulation ###
##############################

# perform n_head days of n_sim scenarios
for(t in 1:n_ahead){

  # Sample n_sim scenarios from Gaussian Copula
  U_sim <- rCopula(n_sim, fit@copula)

  # use copula U_sim to reproduce the marginals quantiles F^{-1}(u) with Gaussian distr
  inv1_sim <- qnorm(U_sim[,1], mean = theta1[1], sd = theta1[2]) # sp500
  inv2_sim <- qnorm(U_sim[,2], mean = theta2[1], sd = theta2[2]) # vix
  invs_sim <- cbind(rets1_sim, rets2_sim)

  # store simulation of log return in matrix
  sim_inv_sp500[ ,t] <- inv1_sim
  sim_inv_vix[ ,t] <- inv2_sim
}

# preview of simulated invariants
head(sim_inv_sp500)
```

```
##              T+1        T+2         T+3        T+4        T+5
## [1,]  0.04447154  1.5691517  1.39371665 -1.4835644  0.9565560
## [2,] -0.22933227  0.9195288  0.09356549 -0.2042606 -0.6108582
## [3,] -1.03976298  0.3455542  0.31955037 -0.5236770 -0.1662329
## [4,]  1.51949269  1.4194440 -0.18535447  1.6314713 -0.1877315
## [5,] -0.98740133 -0.1065783 -0.26676884  0.3869440 -0.8275658
## [6,] -0.19608631 -0.3949083  0.02257609  0.4012918  2.1015336
```

```r
head(sim_inv_vix)
```

```
##              T+1          T+2         T+3         T+4         T+5
## [1,]  0.02303111  0.055872097  0.03438314 -0.01742967 -0.03425186
## [2,] -0.05770198 -0.065635901 -0.02089352 -0.04514607 -0.09491143
## [3,]  0.08084674 -0.028569376 -0.08021689  0.11747472 -0.06914887
## [4,] -0.06615843 -0.002383062  0.02820027 -0.09207113 -0.03004906
## [5,] -0.09149873 -0.022648918  0.02798077 -0.04514606  0.02429271
## [6,]  0.02318232 -0.053178235  0.04957536  0.07071302 -0.07137518
```

**Transforming back the invariants to prices**

**From GARCH(1,1) residuals to SP500 observations**

$$\hat{\epsilon}_t = \frac{y_t}{\hat{\sigma}_t} \implies \hat{y}_t = \hat{\epsilon}_t \hat{\sigma}_t$$

and

$$\sigma_t^2 = \omega \ + \ \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2$$

First, obtain last conditional variance available (up to time T):

```
# extrct the residuals
sp500_resids <- fit_garch$eps_hat

# compute the
```

**From AR(1) residuals to VIX observations**

$$X_t = \phi X_{t-1} + \varepsilon_t$$

**Other stuff**

$$\begin{aligned}
\vec{\alpha}^* &= \underset{\vec{\alpha}}{\arg\min} \ \sum_{t=1}^{T} \left| \sigma_t^{observed} - \sigma(m.\tau) \right| \\
&= \underset{\alpha_1,\alpha_2,\alpha_3,\alpha_4}{\arg\min} \ \sum_{t=1}^{T} \left| \sigma_t^{observed} - \left( \alpha_1 + \alpha_2(m-1)^2 + \alpha_3(m-1)^3 + \alpha_4\sqrt{\tau} \right) \right|
\end{aligned}$$