Assignment1

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MATH 545: ASSG1

Assignment 1

Question 1

Let $\{Z_t\}$ be an indpeendent sequence of random variables, $\mathbb{E}[Z_t] = 0$, $\mathbb{V}ar[Z_t] = \sigma^2 < \infty$, and $a, b, c \in \mathbb{R}$. Determine whether the following processes are stationary or no, and if so, find the autocovariance function.

a) $X_t = a + bZ_t + cZ_{t-2}$

$$\mathbb{E}[X_t] = a + b\mathbb{E}[Z_t] + c\mathbb{E}[Z_{t-2}] = a = \mu_x \perp\!\!\!\perp t$$

$$Cov[X_t, X_{t+h}]$$
$$= Cov[a + bZ_t + cZ_{t-2}, a + bZ_{t+h} + cZ_{t+h-2}]$$

$$= bCov[Z_t, a + bZ_{t+h} + cZ_{t+h-2}]$$
$$+cCov[Z_{t-2}, a + bZ_{t+h} + cZ_{t+h-2}]$$

$$= b(bCov[Z_{t+h}, Z_t] + cCov[Z_{t+h-2}, Z_t])$$
$$+c(bCov[Z_{t+h}, Z_{t-2}] + cCov[Z_{t+h-2}, Z_{t-2}]$$

case h=0

$$= b(bCov[Z_t, Z_t] + cCov[Z_{t-2}, Z_t])$$
$$+c(bCov[Z_t, Z_{t-2}] + cCov[Z_{t+h}, Z_{t-2}]$$

$$= b(bCov[Z_t, Z_t] + c\underbrace{Cov[Z_{t-2}, Z_t]}_{=0})$$
$$+c(b\underbrace{Cov[Z_t, Z_{t-2}]}_{=0} + cCov[Z_{t+h}, Z_{t-2}]$$
$$= b^2\sigma^2 + c^2\sigma^2 = \sigma^2(b^2 + c^2)$$

case h= ± 2

$$= b(bCov[Z_t, Z_t] + c\underbrace{Cov[Z_{t-2}, Z_t]}_{=0})$$
$$+c(b\underbrace{Cov[Z_t, Z_{t-2}]}_{=0} + cCov[Z_{t+h}, Z_{t-2}]$$
$$= bc\sigma^2$$

h $\neq 0, \pm 2$

All Zs have different indices, so they are all just 0.

Covariance function

The process **is stationary**, and has autocovariance function

$$\Upsilon(h) = \begin{cases} \sigma^2(b^2 + c^2) \text{ , } h = 0 \\ bc\sigma^2 \text{ , } h = \pm 2 \\ 0 \text{ , } else \end{cases}$$

b)

$$X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$$

$$\mathbb{E}[X_t] = \cos(ct)\mathbb{E}[Z_1] + \sin(ct)\mathbb{E}[Z_2] = 0 = \mu_X \perp\!\!\!\perp t$$

$$Cov[X_t, X_{t+h}] = \mathbb{E}[X_t X_{t+h}]$$
$$\mathbb{E}[(Z_1 \cos(ct) + Z_2 \sin(ct))(Z_1 \cos(c(t+h)) + Z_2 \sin(c(t+h)))]$$

$$= \begin{matrix} \cos(ct)\cos(ct+ch)\mathbb{E}[Z_1^2] \\ +\underbrace{\cos(ct)\sin(ct+ch)\mathbb{E}[Z_1 Z_2]}_{=0} \\ +\underbrace{\sin(ct)\cos(ct+ch)\mathbb{E}[Z_1 Z_2]}_{=0} \\ +\cos(ct)\cos(ct+ch)\mathbb{E}[Z_2^2] \end{matrix}$$

$$= \begin{matrix} \frac{\sigma^2}{2}\Big\{ \cos(ct - ct - ch) + \cos(ct + ct + ch) \\ + \cos(ct - ct - ch) - \cos(ct + ct + ch)\Big\} \end{matrix}$$

$$\sigma^2 cos(ch) = \Upsilon_X(h) \perp\!\!\!\perp t, \forall h, t$$

c)

$$X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$$
$$\mathbb{E}[X_t] = \cos(ct)\mathbb{E}[Z_t] + \sin(ct)\mathbb{E}[Z_{t-1}] = 0 = \mu_X \perp\!\!\!\perp t$$

$$Cov[X_t, X_{t+h}] = \mathbb{E}[X_t X_{t+h}]$$

$$\begin{gathered}\mathbb{E}[(Z_t\cos(ct)+Z_{t-1}\sin(ct))\\(Z_{t+h}\cos(c(t+h))+Z_{t+h-1}\sin(c(t+h))]\end{gathered}$$

$$= \begin{gathered}\cos(ct)\cos(ct+ch)\mathbb{E}[Z_tZ_{t+h}]\\+\cos(ct)\sin(ct+ch)\mathbb{E}[Z_tZ_{t+h-1}]\\+\sin(ct)\cos(ct+ch)\mathbb{E}[Z_{t-1}Z_{t+h}]\\+\cos(ct)\cos(ct+ch)\mathbb{E}[Z_{t-1}Z_{t+h-1}]\end{gathered}$$

$$\begin{gathered}\tfrac{1}{2}\Big\{[\cos(ct)+\cos(2th+ch)]\mathbb{E}[Z_tZ_{t+h}]\\+[\sin(2th+ch)-\sin(ch)]\mathbb{E}[Z_tZ_{t+h-1}]\\+[\sin(2th+ch)+\sin(ch)]\mathbb{E}[Z_{t-1}Z_{t+h}]\\+[\cos(ct)-\cos(2th+ch)]\mathbb{E}[Z_{t-1}Z_{t+h-1}]\Big\}\end{gathered}$$

Now, this process is **not stationary**, as we have

case h=1

$$\begin{gathered}\tfrac{1}{2}\Big\{\underbrace{[\cos(ct)+\cos(2t+c)]\mathbb{E}[Z_tZ_{t+1}]}_{=0}\\+[\sin(2t+c)-\sin(c)]\mathbb{E}[Z_tZ_t]\\+\underbrace{[\sin(2t+c)+\sin(c)]\mathbb{E}[Z_{t-1}Z_{t+t}]}_{=0}\\+\underbrace{[\cos(ct)-\cos(2th+ch)]\mathbb{E}[Z_{t-1}Z_t]}_{=0}\Big\}\\=\tfrac{1}{2}[\sin(2t-c)-\sin(c)]\sigma^2\end{gathered}$$

Which is not independent of t, so the process is **not stationary**.

d)

$$X_t = a + bZ_0$$

$$\begin{gathered}\mathbb{E}[X_t]=a=\mu_X \perp\!\!\!\perp t\\Cov(X_t,X_{t+h})=Cov(a+bZ_0,a+bZ_0)\end{gathered}$$

$$\mathbb{V}ar[a+bZ_0]=b^2\mathbb{V}ar[Z_0]=b^2\sigma^2 \perp\!\!\!\perp t$$

Covariance function

The process **is stationary** and we have covariance function

$$\Upsilon_X(h)=b^2\sigma^2 \perp\!\!\!\perp t$$

e)

$$X_t = Z_0 cos(ct)$$

$$\mathbb{E}[X_t] = cos(ct)\mathbb{E}[Z_0] = 0 = \mu_X \perp\!\!\!\perp t$$

But we have that

$$Cov(X_t, X_{t+h}) = \mathbb{E}[X_t X_{t+h}]$$

$$= \mathbb{E}[(Z_0 cos(ct))\ (Z_0 cos(c(t+h)))]$$

$$= cos(ct)cos(c(t+h))\mathbb{E}[Z_0^2]$$

$$= cos(ct)cos(ct+ch)\sigma^2$$

$$cos(ct - ct - ch) + cos(ct + ct + ch)\sigma^2 = cos(ch) + cos(2ct + ch)\sigma^2$$

Which depends on t, so the process is **not stationary**.

f)

$$X_t = Z_t Z_{t-1}$$

$$\mathbb{E}[X_t] = \mathbb{E}[Z Z_{t-1}] = 0 = \mu_X \perp\!\!\!\perp t$$

$$Cov(X_t, X_{t+h}) = \mathbb{E}[X_t X_{t+h}]$$

$$\mathbb{E}[Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}]$$

case h = 0

$$\mathbb{E}[Z_t^2]\mathbb{E}[Z_{t-1}^2] = \sigma^4$$

case h $\neq 0$

$$Cov(X_t, X_{t+h}) = 0$$

Covariance function

The process **is stationary** with covariance function

$$\Upsilon_X(h) = \begin{cases} \sigma^4, h = 0 \\ 0, h \neq 0 \end{cases}$$

Which is independent of t, so the process **is stationary**.

2) Assume that

- $\{X_t\}$ is a seasonal series of monthly observation
- Has **seasonal component** $s_t = s_{t+4}$
- Weakly stationary series of random derivations $\{Y_t\}$

(a) Let $X_t = a + bt + s_t + Y_y$, with a global trend and additive seasonality. Prove the seasonally differencied seires $\nabla_4 X_t$ is **weakly stationary**.

Proof

Recall that

$$\nabla_d X_t = X_t - X_{t-d} = (1 - B^d) X_t$$

$$\begin{gathered} \nabla_4 X_t = X_t - X_{t-4} \\ = (a + bt + s_t + Y_t) - (a + b(t-4) + s_{t-4} + Y_{t-4}) \\ = 4b + \underbrace{(s_t - s_{t-4})}_{=0} + (Y_t - Y_{t-4}) \\ = 4b + (Y_t - Y_{t-4}) \end{gathered}$$

$$\mathbb{E}[\nabla_4 X_t] = \mathbb{E}[4b + (Y_t - Y_{t-4})] = 4b = \mu_x \perp\!\!\!\perp t$$

$$\begin{gathered} Cov(\nabla_4 X_t, \nabla_4 X_{t+h}) \\ = Cov(4b + (Y_t - Y_{t-4}), 4b + (Y_{t+h} - Y_{t+h-4})) \end{gathered}$$

$$\begin{gathered} = Cov(Y_t, Y_{t+h}) - Cov(Y_t, Y_{t+h-4}) \\ -Cov(Y_{t-4}, Y_{t+h}) + Cov(Y_{t-4}, Y_{t+h-4}) \end{gathered}$$

case h=0

$$\begin{gathered} = Cov(Y_t, Y_t) - \underbrace{Cov(Y_t, Y_{t-4})}_{=0} \\ -\underbrace{Cov(Y_{t-4}, Y_t)}_{=0} + Cov(Y_{t-4}, Y_{t-4}) \end{gathered}$$

$$\Upsilon_Y(t,t) + \Upsilon_Y(t-4, t-4) = 2\Upsilon_Y(0) \perp\!\!\!\perp t$$

Case $\mathbf{h} = \pm 4$

Both cases are follow a similar calculation.

$$= \begin{array}{c} Cov(Y_t, Y_{t+4}) - Cov(Y_t, Y_t) \\ -Cov(Y_{t-4}, Y_{t+4}) + Cov(Y_{t-4}, Y_t) \end{array}$$

$$\begin{gathered} \Upsilon_Y(t, t+4) - \Upsilon_Y(t,t) \\ -\Upsilon_Y(t-4, t+4) + \Upsilon_Y(t-4, t) \\ \Upsilon_Y(4) - \Upsilon_Y(8) - \Upsilon_Y(0) \perp\!\!\!\perp t \end{gathered}$$

More generally

$$\begin{gathered} = Cov(Y_t, Y_{t+h}) - Cov(Y_t, Y_{t+h-4}) \\ -Cov(Y_{t-4}, Y_{t+h}) + Cov(Y_{t-4}, Y_{t+h-4}) \end{gathered}$$

$$\begin{gathered} = \Upsilon_Y(h) - \Upsilon_Y(h-4) - \Upsilon_Y(h+4) + \Upsilon_Y(h) \\ = 2\Upsilon_Y(h) - \Upsilon_Y(h+4) - \Upsilon_Y(h-4) \perp\!\!\!\perp t \end{gathered}$$

So the process **is stationary**.

b) Let

$$X_t = (a + bt)s_t + Y_t$$

have a global trend and multiplicative seasonality. Find a differencing operator that makes X_t weakly stationary.

Solution

Recall that

$$\triangledown_d X_t = X_t - X_{t-d} = (1 - B^d)X_t$$

So that

$$\begin{gathered} \triangledown_4 X_t \\ = [(a + bt)s_t + Y_y] - [(a + b(t-4))s_{t-4} + Y_{t-4}] \\ = 4bs_{t-4} + (Y_t - Y_{t-4}) \end{gathered}$$

$$\begin{gathered} \triangledown_4(\triangledown_4 X_t) = \triangledown_4 X_t - \triangledown_4 X_{t-4} \\ = 4bs_{t-4} + Y_t - Y_{t-4} \\ -4bs_{t-8} - Y_t + Y_{t-8} \\ = Y_t - 2Y_{t-4} + Y_{t-8} \end{gathered}$$

SO we have that $\triangledown_4(\triangledown_4 X_t)$ is stationary with covariance

$$
\begin{gathered}
Cov(\nabla_4(\nabla_4 X_t), \nabla_4(\nabla_4 X_{t+h})) \\
= Cov(Y_t - 2Y_{t-4} + Y_{t-8}, Y_{t+h} - 2Y_{t+h-4} + Y_{t+h-8}) \\
= Cov(Y_t, Y_{t+h}) - 2Cov(Y_t, Y_{t+h-4}) \\
+Cov(Y_t, Y_{t+h-8}) - 2Cov(Y_{t-4}, Y_{t+h}) \\
+4Cov(Y_{t+h}, Y_{t+h-4}) - 2Cov(Y_{t-4}, Y_{t+h-8}) \\
+Cov(Y_{t-8}, Y_{t+h}) - 2Cov(Y_{t-8}, Y_{t+h-4}) \\
+Cov(Y_{t-8}, Y_{t+h-8})
\end{gathered}
$$

Expressing as covariance functions with lag h, and combining terms, we obtain

$$
= 6\Upsilon_y(h) - 4\Upsilon_y(h+4) - 4\Upsilon_y(h-4) + \Upsilon_y(h+8) - \Upsilon_y(h-8) = \Upsilon_X(h) \perp\!\!\!\perp t
$$

So the process **is stationary**.

Question 3

Consider the $MA(m)$ process, with $\theta = 1/(m+1)$, such that

$$
X_t = \frac{1}{m+1} \sum_{k-0}^{m} Z_{t-k}
$$

where $\{Z_t\} \sim WN(0, \sigma^2)$. Show that the ACF of this process is

$$
\rho(h) = \begin{cases} \frac{m+1-h}{m+1}, h = 0, 1, \dots \\ 0, h > m \\ \rho(-h), h < 0 \end{cases}
$$

Proof

$$
\mathbb{E}[X_t] = \frac{1}{m+1} \sum_{k=0}^{m} \mathbb{E}[Z_{t-k}] - 0
$$

$$
Cov(X_t, X_{t+h}) = Cov\left(\frac{1}{m+1} \sum_{k=0}^{m} Z_{t-k}, \frac{1}{m+1} \sum_{k=0}^{m} Z_{t+h-k}\right)
$$

$$
\sum_{i=0}^{m} \sum_{j=0}^{m} \left(\frac{1}{m+1}\right) \left(\frac{1}{m+1}\right) Cov(Z_{t-i}, Z_{t+h-j})
$$

$$
= \left(\frac{1}{m+1}\right)^2 \sum_{i=0}^{m} \sum_{j=0}^{m} Cov(Z_{t-i}, Z_{t+h-j})
$$

Case h = 0

$$\left(\frac{1}{m+1}\right)^2 \sum_{i=0}^{m}\sum_{j=0}^{m} Cov(Z_{t-i}, Z_{t-j})$$

$$\left(\frac{1}{m+1}\right)^2 \left[\sum_{i=0}^{m} \underbrace{\mathbb{V}ar(Z_{t-i}, Z_{t-i})}_{=\sigma^2} + \sum_{\substack{k=0 \\ i\neq j}}^{m}\sum_{k=0}^{m} \underbrace{Cov(Z_{t-i}, Z_{t-j})}_{=0}\right] = \frac{\sigma^2}{m+1}$$

Case h > m

$$\implies Cov(Z_{t-i}, Z_{t+h-j}) = 0$$

And as $0 \leq i, j, \leq m \implies t - i < t + h - j, \forall h > m$, then

$$= \left(\frac{1}{m+1}\right)^2 \sum_{i=0}^{m}\sum_{j=0}^{m} Cov(Z_{t-i}, Z_{t+h-j}) = 0$$

Case $0 < h \leq m$

$$\implies Cov(Z_{t-i}, Z_{t+h-j}) = 0 = \begin{cases} \sigma^2, \text{ if } t-i = t+h-j \\ 0, \text{ else} \end{cases}$$

$$\implies = \left(\frac{1}{m+1}\right)^2 \sum_{\substack{i=0 \\ i+h=j}}^{m}\sum_{j=0}^{m} Cov(Z_{t-i}, Z_{t+h-j}) = \left(\frac{1}{m+1}\right)^2 (m+1-h)\sigma^2$$

So that

$$\Upsilon_X(h) = \begin{cases} \sigma^2 \frac{(m+1-h)}{(m+1)^2}, h = 1, \ldots, m \\ \frac{\sigma^2}{m+1}, h = 0 \\ 0, \ h > m \end{cases}$$

$$\implies \rho_X(h) = \begin{cases} \frac{m+1-h}{m+1}, h = 1, \ldots, m \\ 0, \ h > m \\ \rho(-h), \ h \leq 0 \end{cases}$$

as when $h < 0 \implies \rho_X(h) = \rho_X(-|h|) = \rho_X(-h)$. ■

Question 4

(a) Let $\{Y_t\}$ be a white noise process and let

$$X_t = m_t + Y_t$$

where $m_t = a + bt$. Find the mean and variance of $\hat{m}_t$

$$\hat{m}_t = \frac{1}{2q+1} \sum_{i=t-q}^{t+q} X_t$$

and show that $\hat{m}_t$ is an unbiased estimator of $a + bt$ with lower variance than X_t for estimating the m_t.

Proof

$$\mathbb{E}[\hat{m}_t] = \frac{1}{2q+1} \sum_{i=t-q}^{t+q} \mathbb{E}[X_t] = \frac{1}{2q+1} \sum_{i=t-q}^{t+q} \mathbb{E}[m_t] + \underbrace{\mathbb{E}[Y_t]}_{=0}$$

$$= (a+bt)\frac{1}{2q+1}(2q+1) = a + bt$$

So $\hat{m}_t$ is **unbiased**.

$$\mathbb{V}ar[\hat{m}_t] = \mathbb{V}ar\left[\frac{1}{2q+1} \sum_{i=t-q}^{t+q} X_t\right] = \left(\frac{1}{2q+1}\right)^2 \mathbb{V}ar\left[\sum_{i=t-q}^{t+q} (m_t + Y_t)\right]$$

$$= \left(\frac{1}{2q+1}\right)^2 \mathbb{V}ar\left[\sum_{i=t-q}^{t+q} Y_t\right] \overset{Y_i \perp\!\!\!\perp Y_j}{=} \left(\frac{1}{2q+1}\right)^2 \sum_{i=t-q}^{t+q} \mathbb{V}ar[Y_t]$$

$$\frac{1}{(2q+1)^2}(2q+1)\sigma^2 = \frac{\sigma^2}{(2q+1)^2} \leq \mathbb{V}ar[X_t] = \mathbb{V}ar[m_t + Y_t] = \mathbb{V}ar[Y_t] = \sigma^2$$

■

(b) Let $\{Y_t\} \sim WN(0, \sigma^2)$. Show that a linear filter $\{a_j\}$ applied to $X_t = m_t + Y_t$ so that

$$\hat{m}_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}$$

will yield an unbiased estimate of m_t for all k-th degree polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$ if and only if the two following conditions hold:

i) $\sum_j a_j = 1$ and
ii) $\sum_j j^r a_j = 0$ for $r = 1, \ldots, k$

Proof

We use induction on k for both directions.

(<=)

Base Case

Let $k = 1$, then

$$m_t = \sum_{i=0}^{1} c_i t^i = c_0 + c_1 t$$

Now note that

$$\hat{m}_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j} = \sum_{j=-\infty}^{\infty} a_j m_{t-j} + a_j Y_{t-j}$$

By Fubini-Totelli we can interchange the Expectation and the infinite series given that the product measure converges (proof skipped, assumed for the rest of the proof), and so

$$\implies \mathbb{E}[\hat{m}_t] = \sum_{j=-\infty}^{\infty} \left(a_j(c_0 + c_1(t-j)) + a_j \underbrace{\mathbb{E}[Y_t]}_{=0} \right) = c_0 \sum_{j=-\infty}^{\infty} a_j + c_1 \sum_{j=-\infty}^{\infty} a_j(t-j)$$

By conditions (i) and (ii)

$$= c_0 \underbrace{\sum_{j=-\infty}^{\infty} a_j}_{=1} + c_1 t \underbrace{\sum_{j=-\infty}^{\infty} a_j}_{=1} - c_1 \underbrace{\sum_{j=-\infty}^{\infty} a_j j}_{=0} = c_0 + c_1 t = m_t$$

Inductive step

Assume the conditions i) and ii) and the claim are true for all $1 \leq k$, we show it also holds for $k+1$.

$$\mathbb{E}[\hat{m}_t] = \sum_{j=-\infty}^{\infty} a_j \mathbb{E}[X_{t-j}] = \sum_{j=-\infty}^{\infty} a_j \mathbb{E}[m_{t-j}] + a_j \mathbb{E}[[Y_{t-j}]$$

$$= \sum_{j=-\infty}^{\infty} a_j m_{t-j} = \sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k+1} c_i (t-j)^i \right)$$

$$= c_0 \sum_{j=-\infty}^{\infty} a_j + \sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k} c_i (t-j)^i + c_{k+1}(t-j)^{k+1} \right)$$

$$= c_0 \underbrace{\sum_{j=-\infty}^{\infty} a_j}_{=1} + \underbrace{\sum_{i=1}^{k} c_i \sum_{j=-\infty}^{\infty} a_j (t-j)^i}_{\text{(II)}} + \underbrace{c_{k+1} \sum_{j=-\infty}^{\infty} a_j (t-j)^{k+1}}_{\text{(III)}}$$

Now, by the induction hypothesis, we have that (I) + (II) are just $\hat{m}_t$, which is unbiased for the polynomial $c_0 + c_1 t \cdots + c_k t^k$, that is

$$\sum_{j=-\infty}^{\infty} a_j m_{t-j} = \sum_{i=0}^{k} c_i \sum_{j=-\infty}^{\infty} a_j (t-j)^i = \sum_{i=0}^{k} c_i t^i$$

So we just have to show left the the term (III)=$c_{k+1}t^{k+1}$ and we are done. We have that

$$\text{(III)} = c_{k+1} \sum_{j=-\infty}^{\infty} a_j (t-j)^{k+1} = c_{k+1} \sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k} \binom{k+1}{i} t^i (-1)^{k-i+1} (j)^{k-i+1} \right)$$

where the last term comes from the binomial expansion of $(t-j)^i$. Reorgnaizing terms, we have

$$= c_{k+1} t^{t+1} \left(\sum_{i=0}^{k} \binom{k+1}{i} t^i (-1)^{k-i+1} \left(\sum_{j=-\infty}^{\infty} a_j j^{k-i+1} \right) \right)$$

but as $0 \le i \le k+1 \implies 0 \le k-i+1 \le k+1$,so we have that

$$= c_{k+1} \Bigg\{ \left[\underbrace{\binom{k+1}{0} t^0 (-1)^{k+1} \underbrace{\sum_{j=-\infty}^{\infty} a_j j^{k+1}}_{=0 \text{ by (ii)}}}_{=0} \right] + \left[\underbrace{\sum_{i=0}^{k} \binom{k+1}{i} t^i (-1)^{k-i+1} \underbrace{\sum_{j=-\infty}^{\infty} a_j j^{k-i+1}}_{=0 \text{ by (ii)}}}_{=0} \right]$$

$$+ \left[\underbrace{\binom{k+1}{k+1}}_{=1} t^{k+1} \underbrace{(-1)^{k-(k+1)+1}}_{=1} \left(\underbrace{\sum_{j=-\infty}^{\infty} a_j j^{k-(k+1)+1}}_{=1} \right) \right] \Bigg\} = c_{k+1} t^{k+1}$$

So putting all the terms together gives

$$\mathbb{E}[\hat{m}_t] = \sum_{i=0}^{k+1} c_i t^i = m_t$$

(=>)

Let

$$\hat{m}_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}$$

and assume that $\mathbb{E}[\hat{m}_t] = m_t$, $\forall m_t = \sum_{i=0}^{k} c_i t^i$.

Base Case

For $k = 1 \implies m_t = c_0 + c_1 t$, then as $\mathbb{E}[Y_t] = 0, \forall t$

$$\mathbb{E}[\hat{m}_t] = m_t \implies \sum_{j=-\infty}^{\infty} a_j m_{t-j} = m_t$$

$$\iff \sum_{j=-\infty}^{\infty} a_j (c_0 + c_1 t) = m_t$$

$$\iff c_0 \sum_{j=-\infty}^{\infty} a_j + c_1 t \sum_{j=-\infty}^{\infty} a_j - c_1 \sum_{j=-\infty}^{\infty} a_j j = c_0 + c_1 t$$

Two polynomials are equal iff their coefficients are equal, so this implies that

$$\sum_{j=-\infty}^{\infty} a_j = 1 \ \& \ \sum_{j=-\infty}^{\infty} a_j j = 0 \implies \sum_{j=-\infty}^{\infty} a_j j^r = 0, r = 1$$

Induction Step

Assume that the statement is true for $1 \leq k$, then

$$\mathbb{E}[\hat{m}_t] = m_t \implies \sum_{j=-\infty}^{\infty} a_j m_{t-j} = m_t$$

$$\sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k+1} c_i (t-j)^i \right) = \sum_{i=0}^{k+1} c_i t^i$$

$$\iff \underbrace{\sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k} c_i (t-j)^i \right)}_{(I)} + \underbrace{c_{k+1} \sum_{j=-\infty}^{\infty} a_j (t-j)^{k+1}}_{(III)} = \underbrace{\sum_{i=0}^{k} c_i t^i}_{(II)} + \underbrace{c_{k+1} t^{k+1}}_{(IV)}$$

By the induction hypothesis, we have that (I)=(II), and

$$\text{(I)} = \text{(II)} \implies \sum_{j=-\infty}^{\infty} a_j = 1 \ \& \ \sum_{j=-\infty}^{\infty} a_j j^r = 0, \ r = 1, \ldots, k$$

So if (III) = (IV), we are done. Note that

$$\text{(III)} = c_{k+1} \sum_{j=-\infty}^{\infty} a_j (t-j)^{k+1} = c_{k+1} \sum_{j=-\infty}^{\infty} a_j \left(\sum_{i=0}^{k+1} \binom{k+1}{i} t^i (-1)^{k-i+1} (j)^{k-i+1} \right)$$

$$= c_{k+1} \left(\sum_{i=0}^{k+1} \binom{k+1}{i} t^i (-1)^{k-i+1} \left(\sum_{j=-\infty}^{\infty} a_j j^{k-i+1} \right) \right)$$

$$
= c_{k+1} \left\{ \sum_{j=-\infty}^{\infty} a_j \binom{k+1}{k+1} t^{k+1} + \underbrace{\sum_{i=0}^{k} \binom{k+1}{i} t^i (-1)^{k-i+1} \underbrace{\left(\sum_{j=-\infty}^{\infty} a_j j^{k-i+1} \right)}_{=0}}_{=0} + (-1)^{k+1} \sum_{j=-\infty}^{\infty} a_j j^{k+1} \right\}
$$

Where the second term is 0 follows by (ii) and the I.H., as the for the exponent of j^{k-i+1}, we have that $1 \leq i \leq k \implies 1 \leq k-i+1 \leq k$. Then , we have

$$
c_{k+1} t^{k+1} \sum_{j=-\infty}^{\infty} a_j + c_{k+1} (-1)^{k+1} \sum_{j=-\infty}^{\infty} a_j j^{k+1} = c_k t^{t+1} \implies \sum_{j=-\infty}^{\infty} a_j j^{k+1} = 0
$$

This completes the proof. ■

Question 5

We will analyse hypothetical sales data for company X measure in successive 4-week periods from 1995-1998.

Recall the general time-series decomposition which is of the form

$$
X_t = m_t + s_t + Y_t
$$

where Y_t is a random noise with $\mathbb{E}[Y_t] = 0$ and $\mathbb{V}ar[y] = \sigma^2 < -\infty$

Libraries

```
## Warning: package 'tidyverse' was built under R version 3.6.2

## -- Attaching packages --------------------------- tidyverse 1.3.0 --

## v ggplot2 3.2.1     v purrr   0.3.3
## v tibble  2.1.3     v dplyr   0.8.3
## v tidyr   1.0.2     v stringr 1.4.0
## v readr   1.3.1     v forcats 0.4.0

## Warning: package 'ggplot2' was built under R version 3.6.1

## Warning: package 'tibble' was built under R version 3.6.1

## Warning: package 'tidyr' was built under R version 3.6.2

## Warning: package 'purrr' was built under R version 3.6.1

## Warning: package 'dplyr' was built under R version 3.6.2
```

```
## -- Conflicts ------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()

## Warning: package 'fpp2' was built under R version 3.6.2

## Loading required package: forecast

## Warning: package 'forecast' was built under R version 3.6.2

## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo

## Registered S3 methods overwritten by 'forecast':
##   method             from
##   fitted.fracdiff    fracdiff
##   residuals.fracdiff fracdiff

## Loading required package: fma

## Warning: package 'fma' was built under R version 3.6.2

## Loading required package: expsmooth

## Warning: package 'expsmooth' was built under R version 3.6.2

## Warning: package 'tibbletime' was built under R version 3.6.2

##
## Attaching package: 'tibbletime'

## The following object is masked from 'package:stats':
##
##     filter

## Warning: package 'tsbox' was built under R version 3.6.2

## Warning: package 'gridExtra' was built under R version 3.6.2

##
## Attaching package: 'gridExtra'

## The following object is masked from 'package:dplyr':
##
##     combine

## Warning: package 'knitr' was built under R version 3.6.2
```

```r
# Load the data
sales_data <- read_excel("Assign1Q5_sales.xlsx")

# Convert the data into a time series object
sales_ts <- ts(sales_data, frequency = 13, start = 1) # 14 4-week preiods per year

str(sales_ts)
```

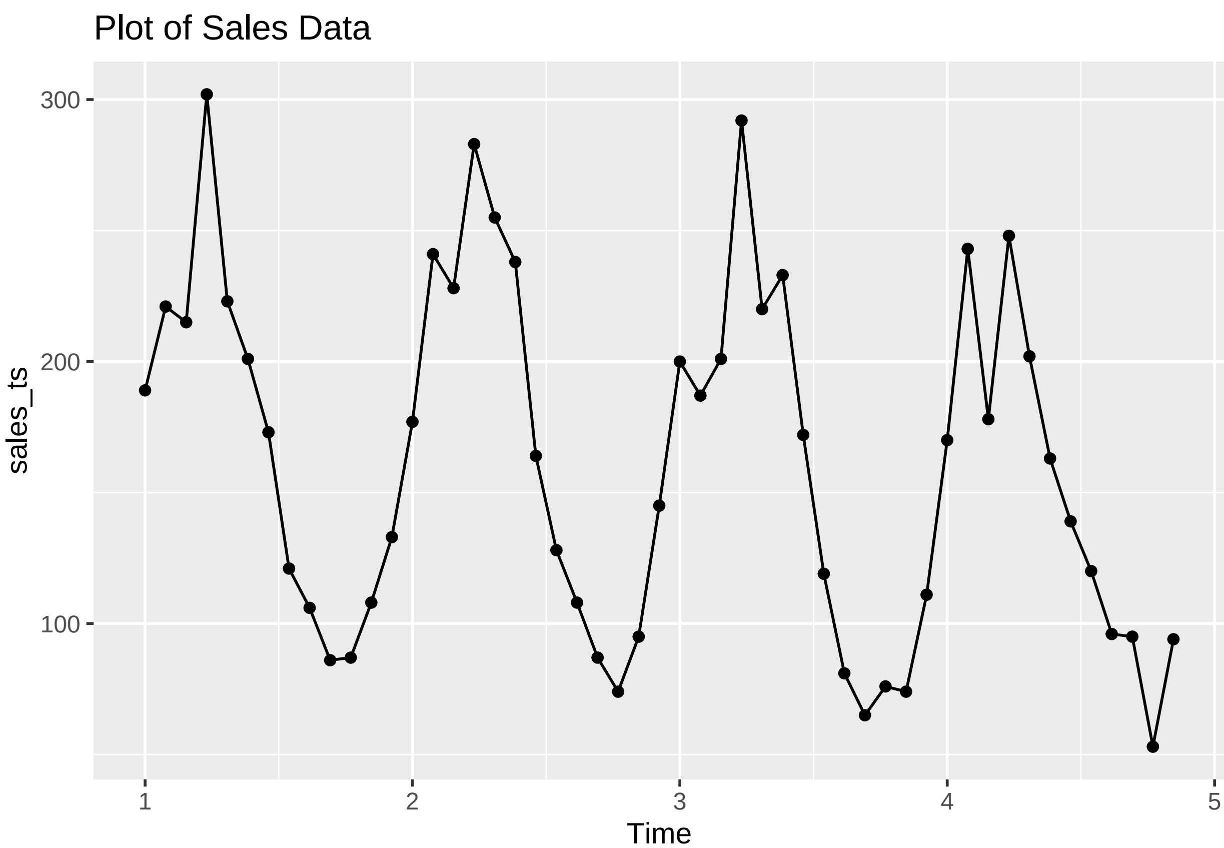
```
## Time-Series [1:51, 1] from 1 to 4.85: 189 221 215 302 223 201 173 121 106 86 ...
## - attr(*, "dimnames")=List of 2
##  ..$ : NULL
##  ..$ : chr "153"
```

a). Plotting and describing the data.

Series plot

We first plot the data

```r
# Generate a plot wih autoplot
autoplot(sales_ts) + geom_point() +
  ggtitle("Plot of Sales Data")
```

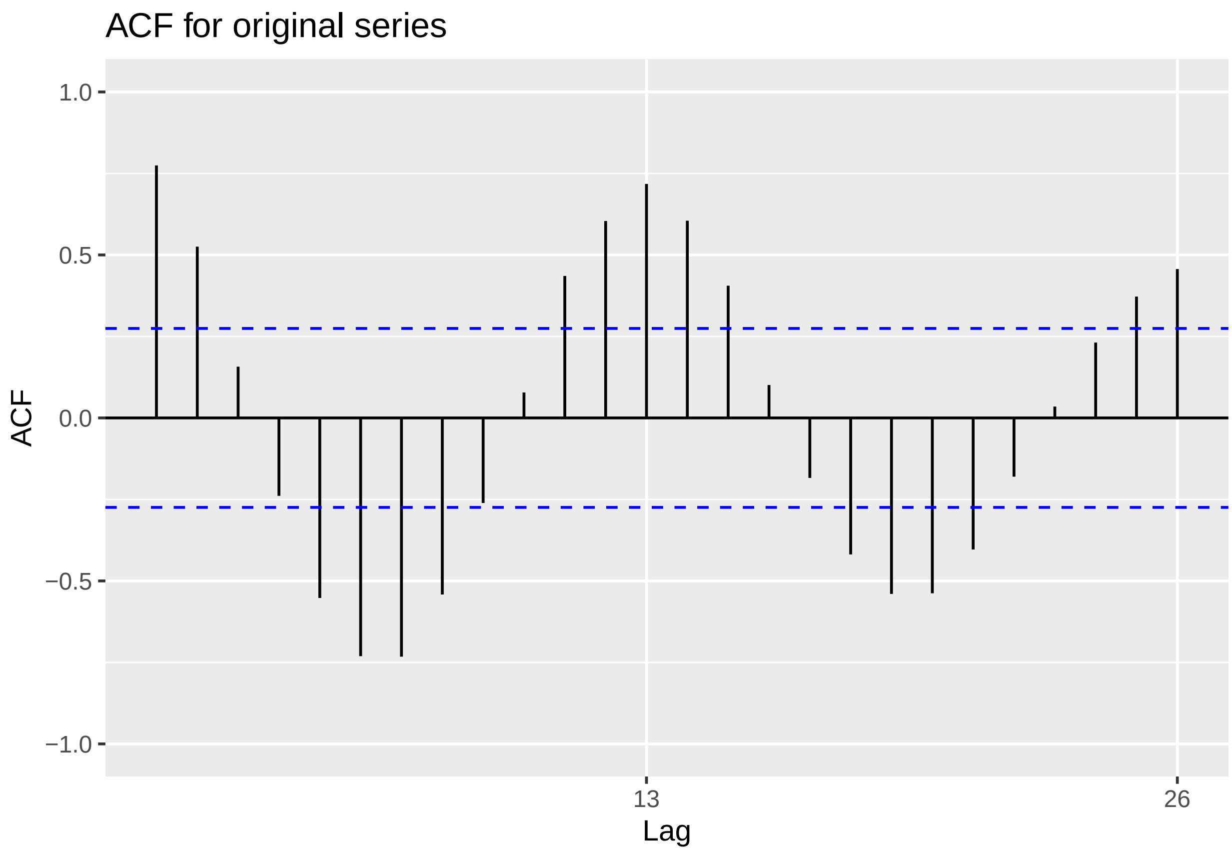


At first sight, we can observe that there is indeed some decreasing trend, along with seasonality at approximatedly each week (because of the peaks).

Autocorrelation plot

In order to determine whether the data is stationary, we may examine the **autocorrelation plot**:

```
# autocorr plot for original  series
ggAcf(sales_ts) + ylim(c(-1,1)) +
  ggtitle("ACF for original series")
```



We have that a sereis is (weakly) **stationary** if

1) $\mathbb{E}[X_t] = \mu_X(t) \perp\!\!\!\perp t$
2) $Cov(X_t, X_s) \perp\!\!\!\perp t \implies Cov(X_t, X_{t+h}) = \Upsilon_X(t, t+h) = \Upsilon_X(0, h) = \Upsilon_X(h), \forall t$

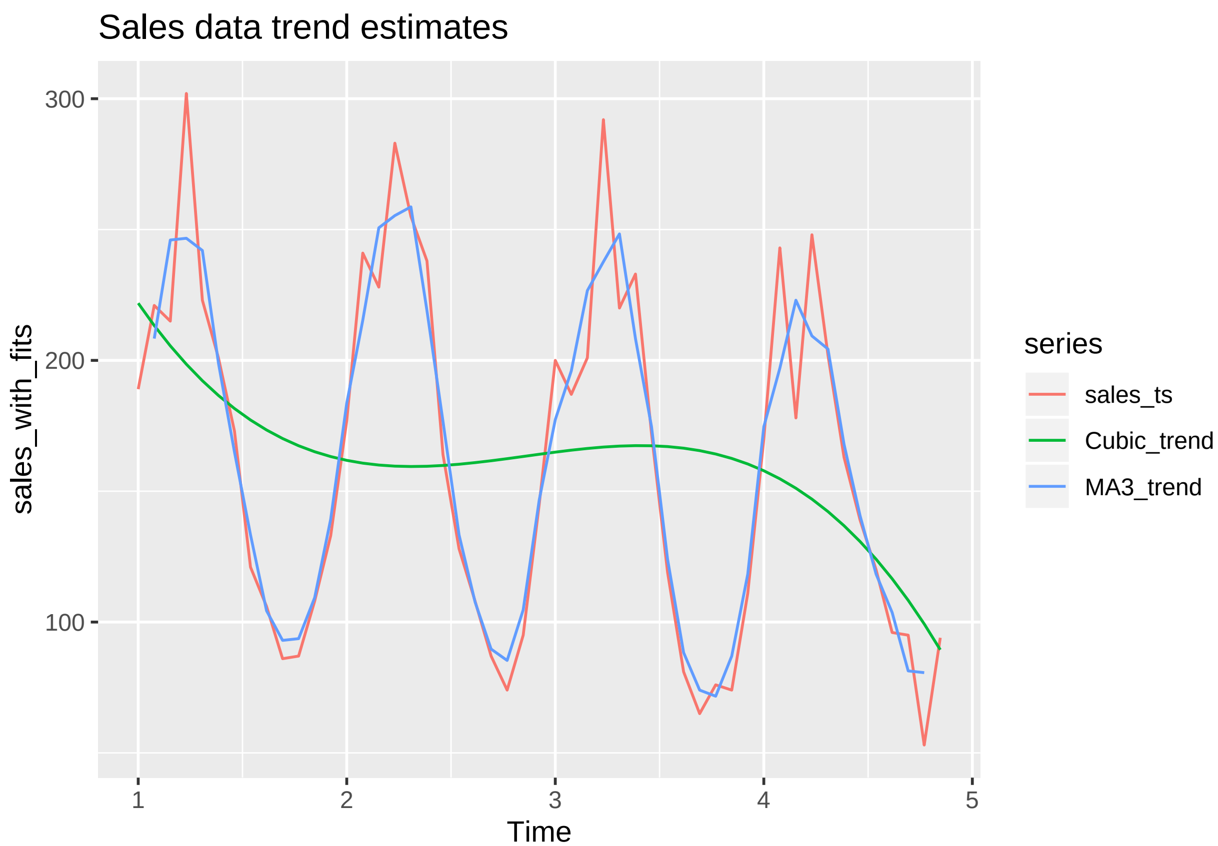
Given this conditions, we see that the auto-correlation plot is dependent on the time, so the series is **not stationary**.

b) Estimating the trend and seasonality

Estimating trend

We first produce multiple estimates of the trend, and compare them.

```r
sales_linear <- tslm(sales_ts ~ trend) ## Fit linear trend
sales_cub <- tslm(sales_ts ~ trend + I(trend^2) + I(trend^3) ) ## Fit quadratic trend
sales_ma7 <- ma(sales_ts, order=7) ## Order 7 moving average
sales_ma3 <- ma(sales_ts, order=3) ## Order 5 moving average

# Bind together the data and fitted trends
sales_with_fits <- cbind(sales_ts,
                         Cubic_trend = fitted(sales_cub),
                         MA3_trend = sales_ma3)

# plot the different fits
autoplot(sales_with_fits) +
  ggtitle("Sales data trend estimates")
```

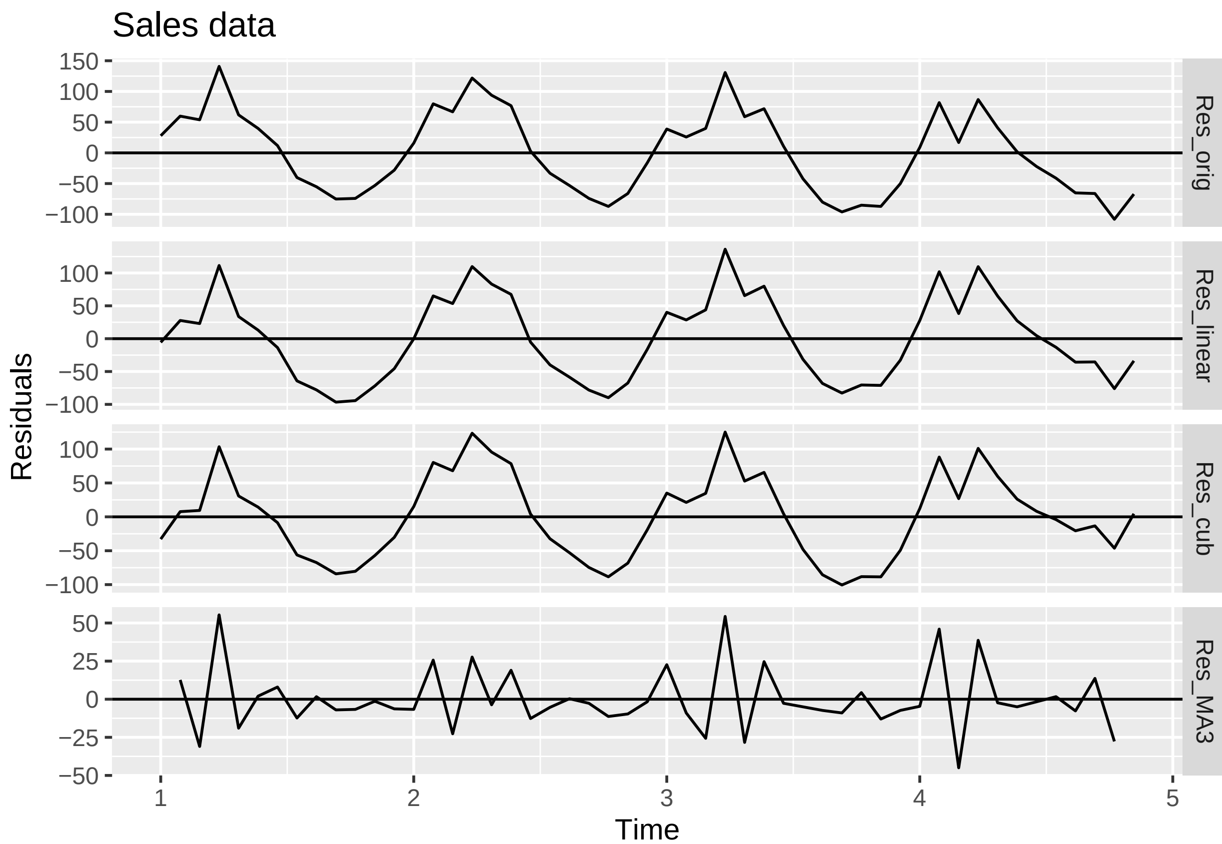


We will choose to apply the order-3 moving average trend estimation, since it it better approximates the curve than the other trends. Nonetheless, "overfitting" the trend estimation will lead to a loss of information and predictive power.

Let's now assess the residuals (eliminating the estimated trend) for the different trend estimations:

```r
# Obtain an object for the residuals
sales_resids <- cbind(Res_orig = sales_ts - mean(sales_ts),
                      Res_linear = sales_ts - fitted(sales_linear),
                      Res_cub = sales_ts - fitted(sales_cub),
                      Res_MA3 = sales_ts - sales_ma3)
```

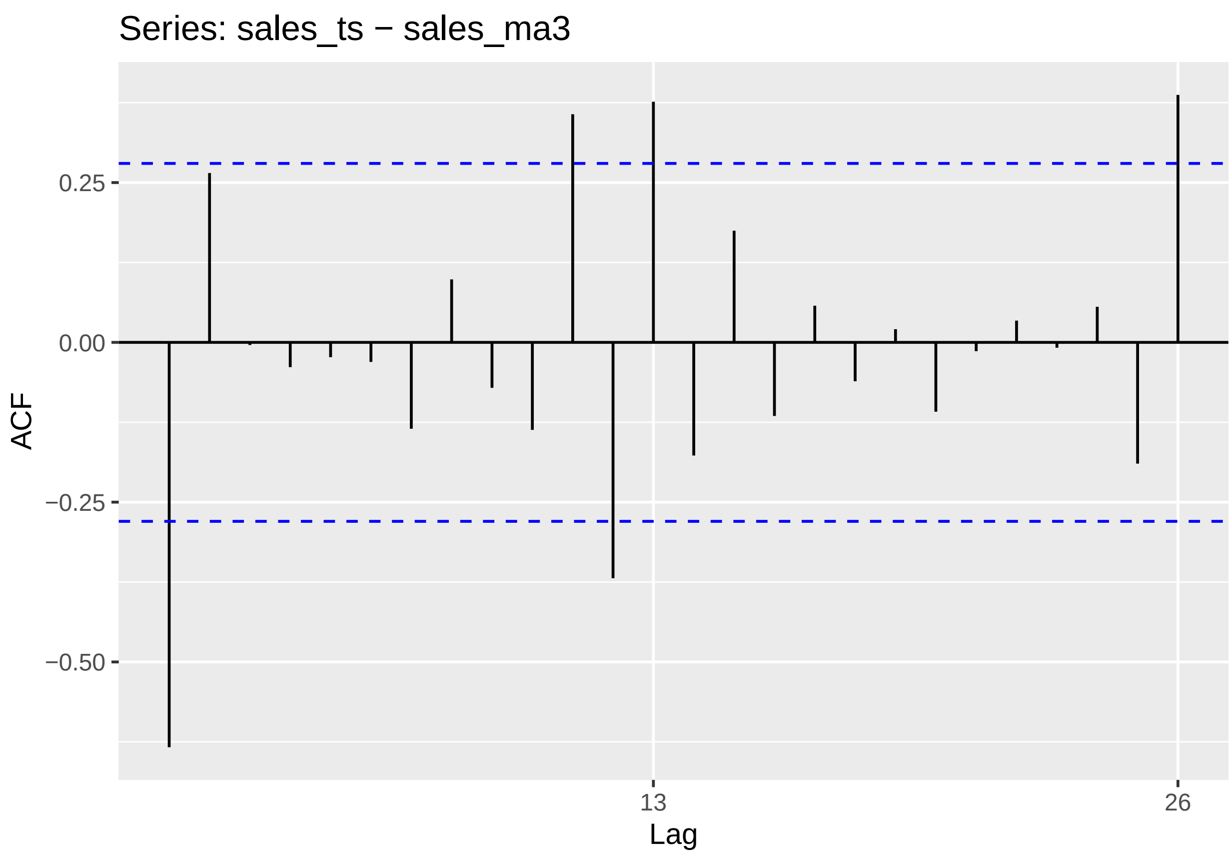
```r
# Plot the residuals
autoplot(sales_resids, facet=TRUE) +
  xlab("Time") + ylab("Residuals") +
  ggtitle("Sales data") +
  geom_hline(yintercept = 0) +
  guides(colour=guide_legend(title="Data series")) +
  scale_colour_manual(values=c("black","red","blue","green"))
```



Indeed, from the residuals plot we observe that substracting the estimated MA5 average yields a process that is not only approximatedly mean-0 , but also starts to look more like a white noise process. However, the seasonality can still be observed. Let's also observe the autocorrelation plot for the data with the order-5 moving average removed:

```r
# Plot the autocorrelation plot
ggAcf(sales_ts - sales_ma3, facet=TRUE)
```

```
## Warning: Ignoring unknown parameters: facet
```



Although we can still clearly see the dependence on time, we observe that the autocorrelation is overall smaller and the process is approximatedly mean 0.

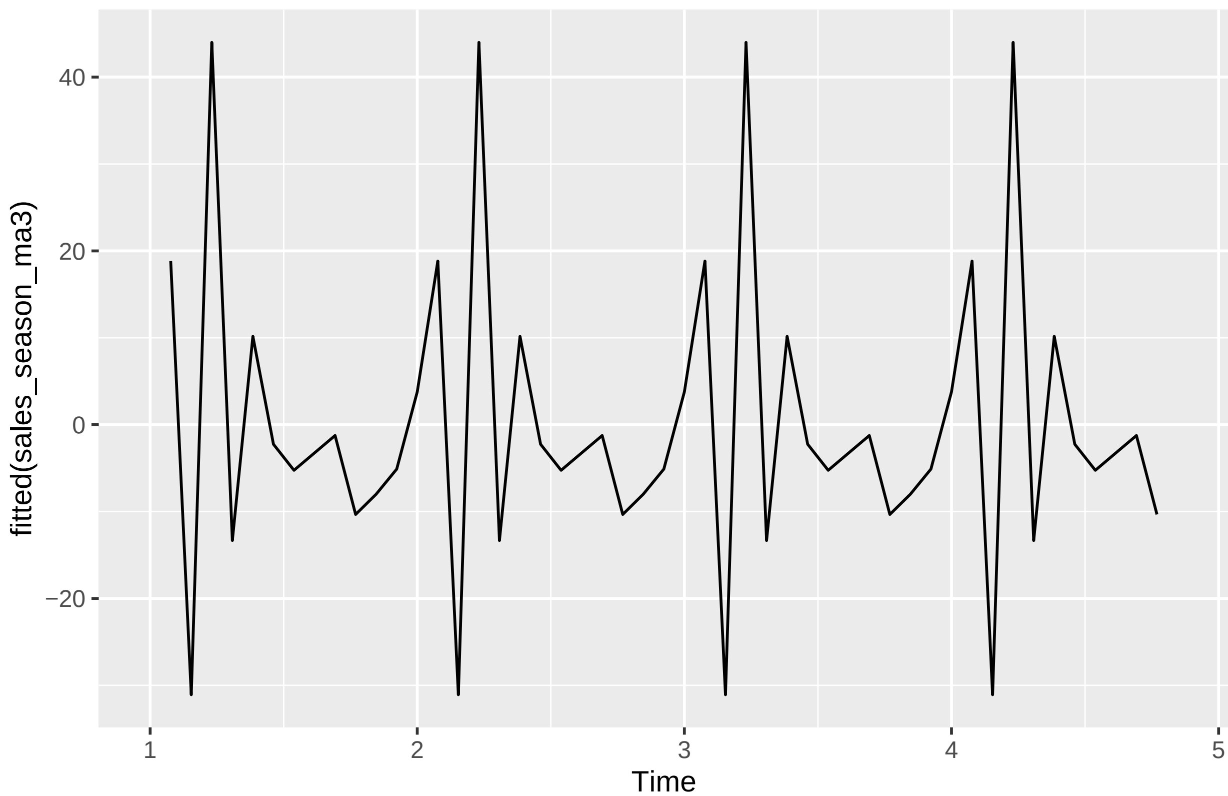
Estimating seasonal component

```r
# frequency obtains the lags
frequency(sales_resids[,"Res_MA3"])
```

```
## [1] 13
```

Recall that we stilll have a frequency of 13 (4-week periods per year); i.e. this is our seasonality lag. We can observe the MA5 residuals model as a function of average.

```r
# Fit a linear model for the diabetes season
sales_season_ma3 <- tslm(Res_MA3 ~ season, data=sales_resids)
autoplot(fitted(sales_season_ma3))
```



There is clearly a seasonal compoinent present in the data.

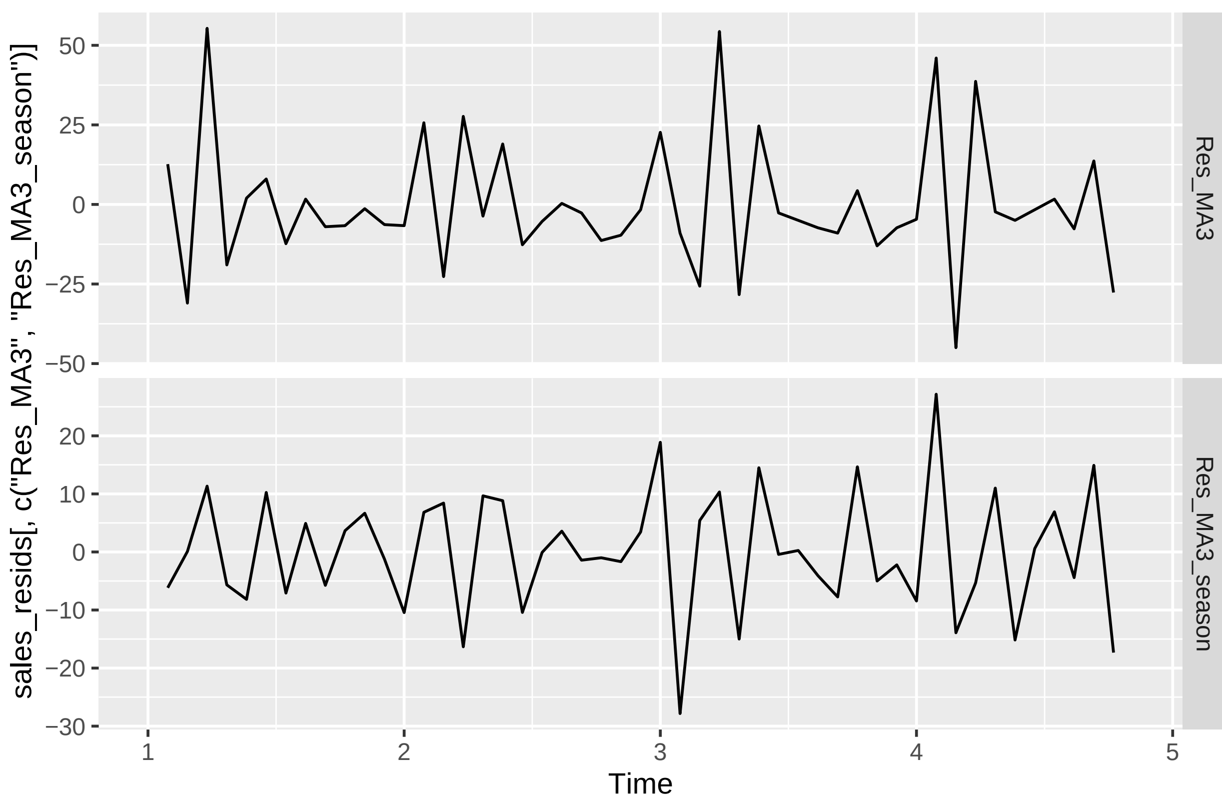
```
# extract the coefficients for the season
kable(coef(sales_season_ma3))
```

	x
(Intercept)	3.777778
season2	15.055556
season3	-34.861111
season4	40.222222
season5	-17.111111
season6	6.388889
season7	-6.027778
season8	-9.027778
season9	-7.027778
season10	-5.027778
season11	-14.111111
season12	-11.777778
season13	-8.888889

Now that we have confirmed a seasonal component, the next step is then to remove it.

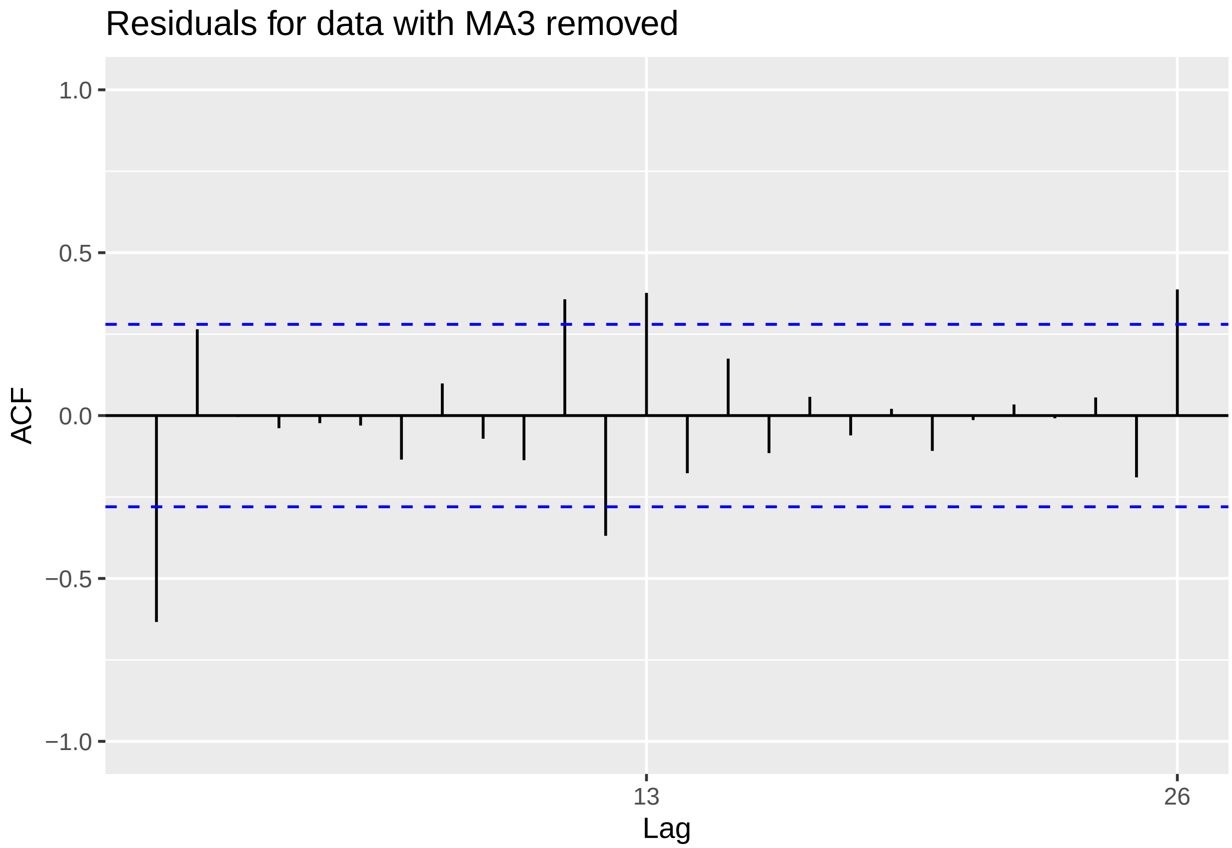
```
#diab_seasonlm2<-tslm(Resid_MA5~season*trend, data=diab_resids)
```

```r
# Obtain an object for the residuals
sales_resids <- cbind(Res_orig = sales_ts - mean(sales_ts),
                      Res_linear = sales_ts - fitted(sales_linear),
                      Res_cub = sales_ts - fitted(sales_cub),
                      Res_MA3 = sales_ts - sales_ma3,
                      Res_MA3_season = (sales_ts - sales_ma3) - fitted(sales_season_ma3))

# plot of the MA5 residuals with and without the seasonal component.
autoplot(sales_resids[,c("Res_MA3","Res_MA3_season")],facet=TRUE)
```

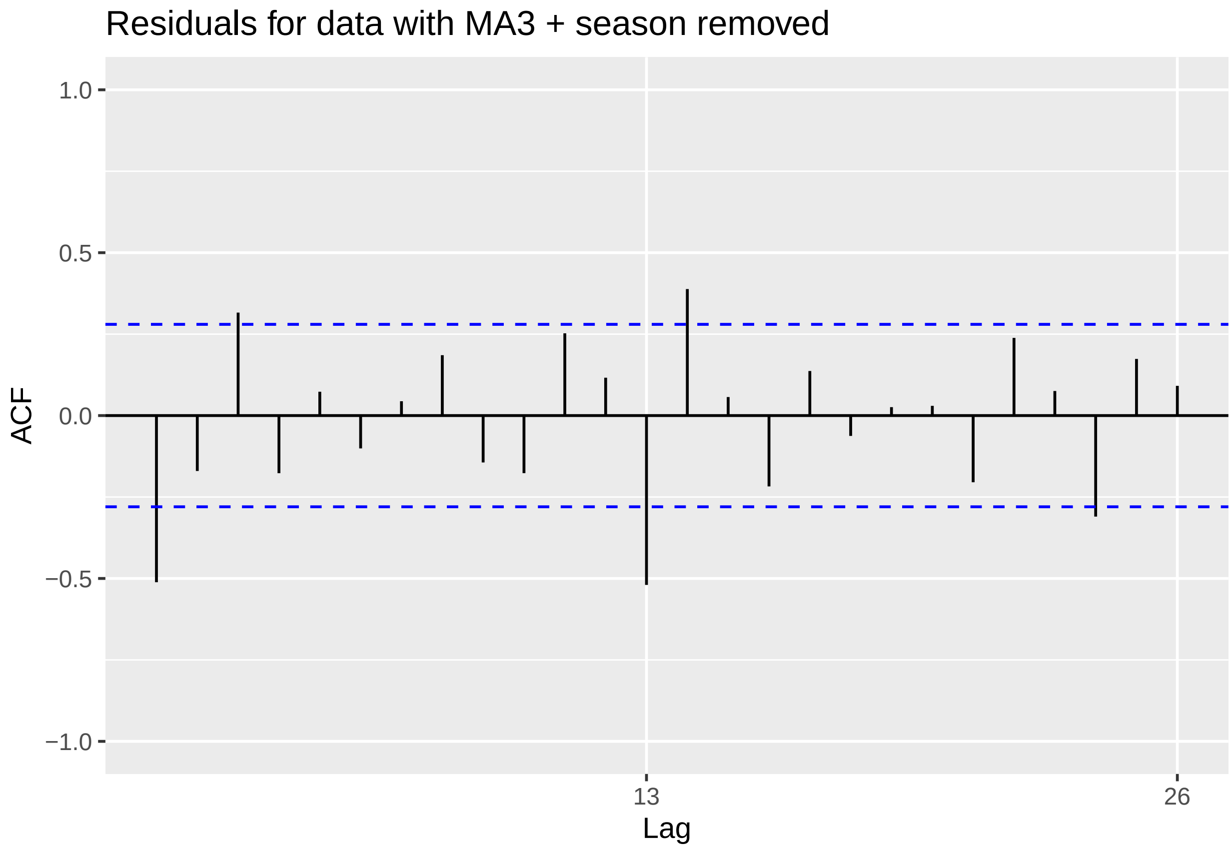


After substracting seasonality, the resulting process looks more like white noise or iid noise, with mean centered towards 0 and most of the seaosnal component being absent. We can confirm this with the autocorrelation plot:

```r
# plot ACF plot without moving average , but trend present.
ggAcf(sales_resids[,"Res_MA3"]) + ylim(c(-1,1)) +
  ggtitle("Residuals for data with MA3 removed")
```

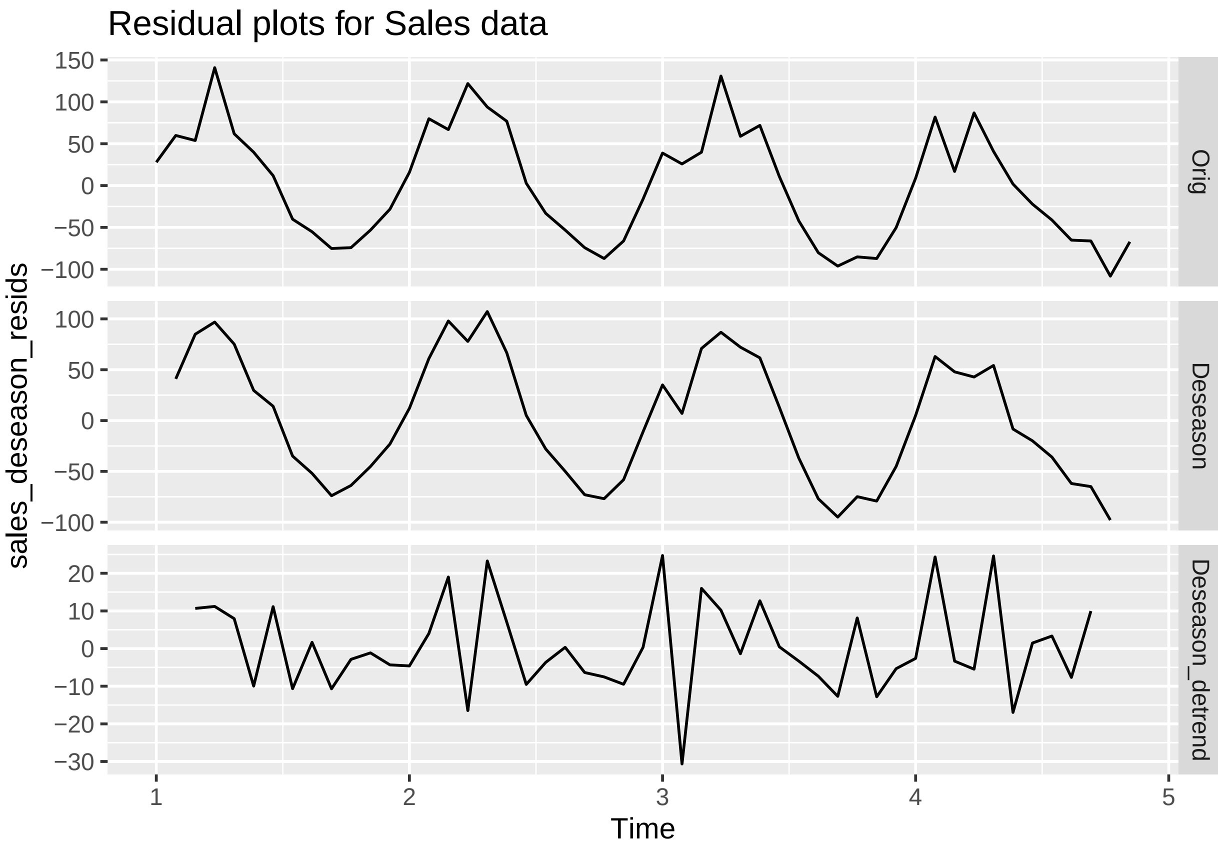



```r
# plot ACF plot without moving average , but trend present.
ggAcf(sales_resids[,"Res_MA3_season"]) + ylim(c(-1,1)) +
  ggtitle("Residuals for data with MA3 + season removed ")
```



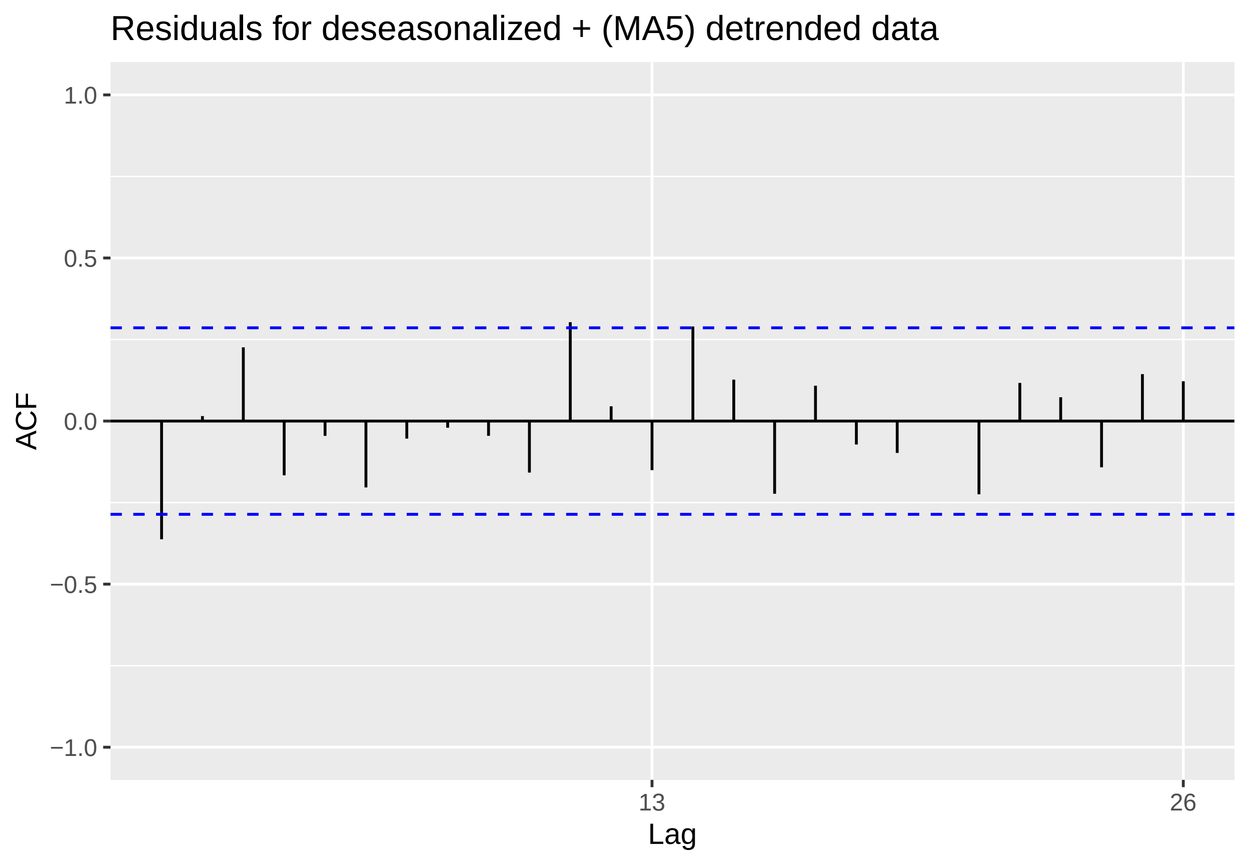
Although the seasonal component is still present, it seems that a good part of it has been removed, and the resulting process tends to have less time dependent autocorrelation.

```
# form deseasonalized data
sales_deseason <- sales_ts - mean(sales_ts) - fitted(sales_season_ma3)

# Re-estimate the trend from the deseasonalized data (using MA5), and substract it from the deseasonali
sales_deseason_detrend_MA3 <- sales_deseason - ma(sales_deseason, 3)

# Create cbind of residuals  for different components
sales_deseason_resids <- cbind(Orig= sales_ts - mean(sales_ts),  # original=detrended
                   Deseason= sales_deseason,
                   Deseason_detrend = sales_deseason_detrend_MA3)

# Plot the different data
autoplot(sales_deseason_resids, facet=TRUE) +
  ggtitle("Residual plots for Sales data")
```



We observe that the deseasonalized + detrended data now looks more like some iid or white noisewith mean zero.

```
ggAcf(sales_deseason_resids[,"Deseason_detrend"]) + ylim(c(-1,1)) +
  ggtitle("Residuals for deseasonalized + (MA5) detrended data ")
```

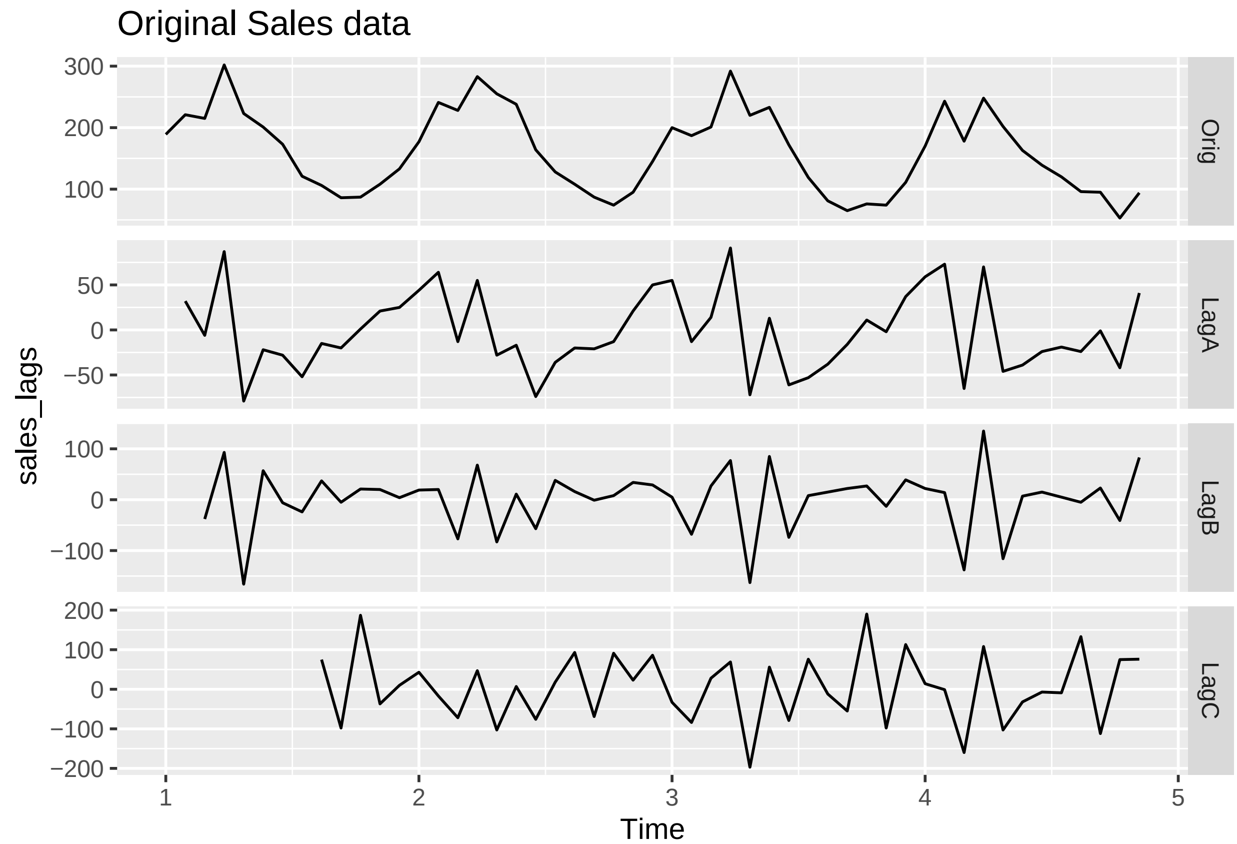


The resulting autocorrelation confirms our statement above.

c) Differencing

Now we will try to eliminate the trend and seasonality using differencing instead.

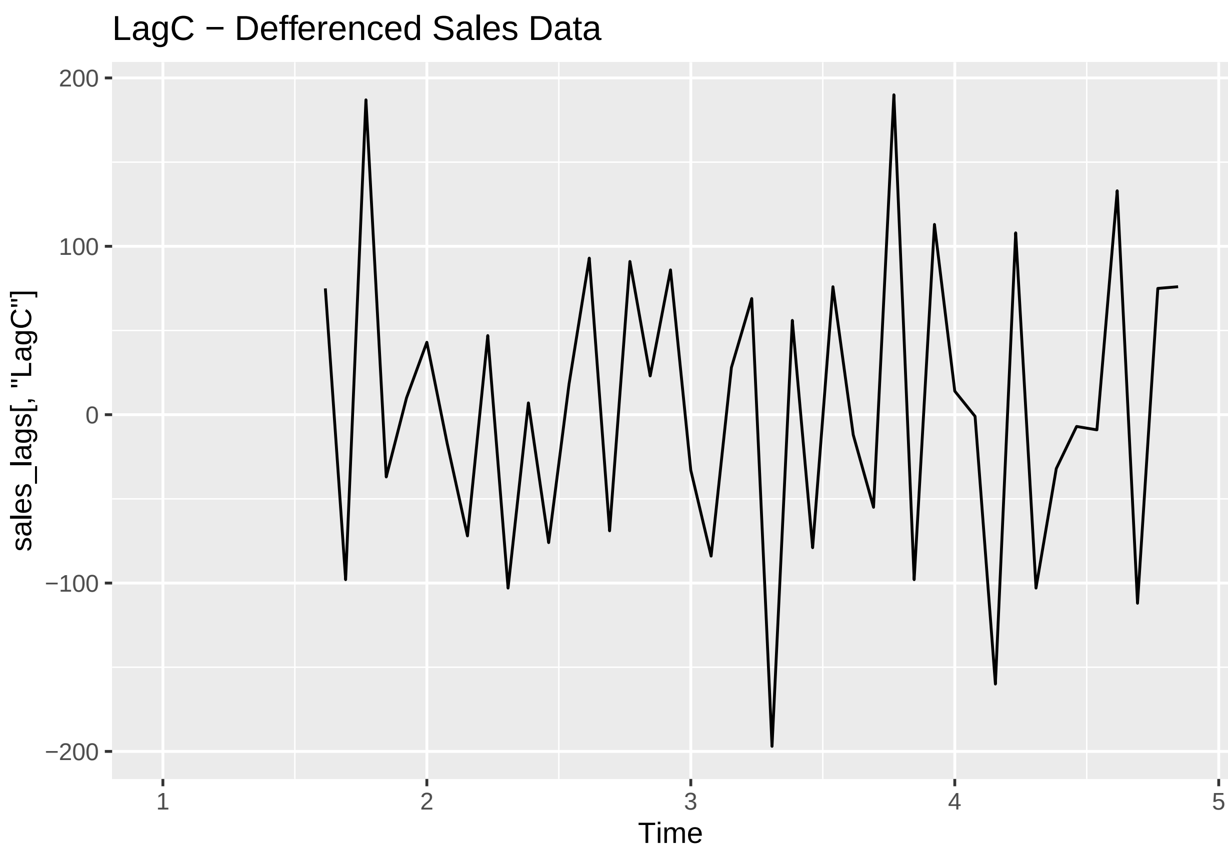
```
# We use a lag 12 difference
sales_lagA <- diff(sales_ts,1)
sales_lagB <- diff(sales_lagA,1)
sales_lagC <- diff(sales_lagB,6)


sales_lags <- cbind(Orig = sales_ts,
                    LagA = sales_lagA,
                    LagB = sales_lagB,
                    LagC = sales_lagC)

# Plot the result
autoplot(sales_lags, facet=TRUE) +
  ggtitle("Original Sales data")
```



Note that although we almost immediately eliminate teh seasonality component and most of the trend, we have also lost some of the data in the process. Therefore, we apply differenceing once again. Differencing with lag 1 , then lag 1, then lag 6 yields a process that ressembles closely to white noise. However, some of the data has been lost.

```r
# Plot the result
autoplot(sales_lags[,"LagC"], facet=TRUE) +
  ggtitle("LagC - Defferenced Sales Data")
```

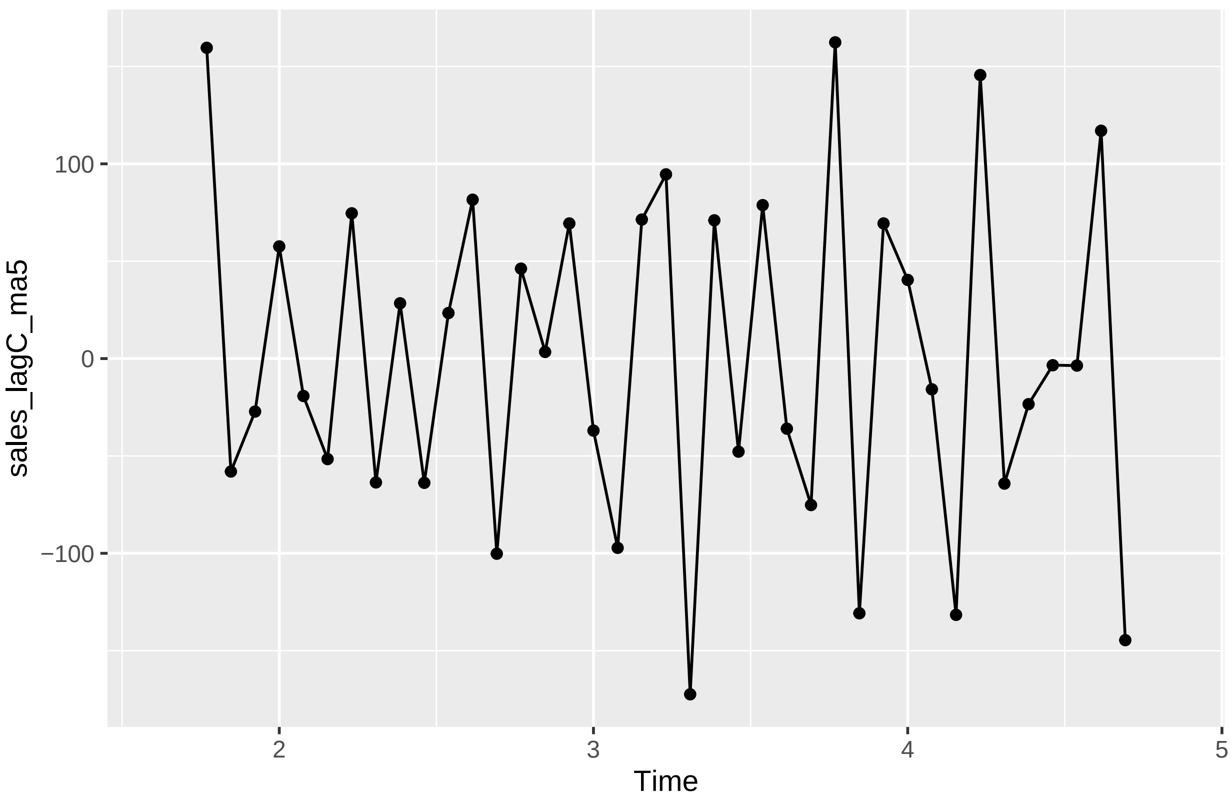
```
## Warning: Ignoring unknown parameters: facet
```



We can now also attempt to remove the remaining trend. Our resulting data with the trend removed looks like this:

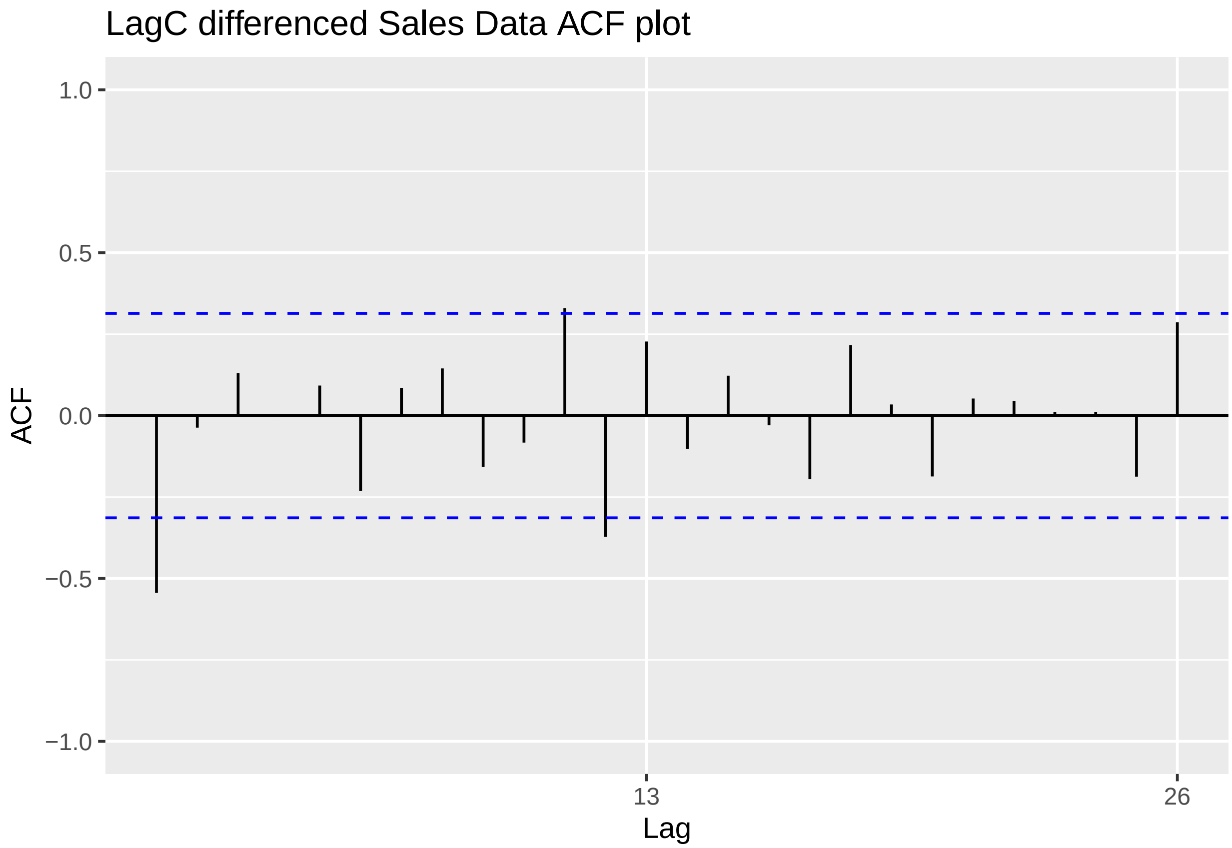
```
# diabetes data, lag12 difference, moving average degree 5
sales_lagC_ma5 <- sales_lagC - ma(sales_lagC, 5)

#plot
autoplot(sales_lagC_ma5) + geom_point()
```

```
## Warning: Removed 4 rows containing missing values (geom_point).
```



Finally, we can observe the ACF plot for this data:

```
ggAcf(sales_lagC_ma5) + ylim(c(-1,1)) +
  ggtitle("LagC differenced Sales Data ACF plot")
```



Indeed, we have suceeded to eliminate most of the seasonality and trend, as the ACF ressembles more to a white noise or iid process.