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260738614 MATH 545

Q3.3 Consider the infinite-order MA process $\{X_t\}$

$$\boxed{X_t = Z_t + c\sum_{j=1}^{\infty} Z_{t-j}}$$

where $c \neq 0$ is constant. Show that this process is not stationary. Also show that the series of first differences $\{Y_t\}$ defined by

$$\boxed{Y_t = X_t - X_{t-1}}$$

is a first-order MA process and it's stationary. Find the ACF of $\{Y_t\}$.

Solution

Since $Z_t \sim WN(0,\sigma^2)$, we have

$$\boxed{E[X_t]} = E\left[Z_t + c\sum_{j=1}^{\infty} Z_{t-j}\right]$$

$$= E[Z_t] + c\sum_{j=1}^{\infty} E[Z_{t-j}] = \boxed{0}$$

Then, let $c_j = \begin{cases} 1, & j=0 \\ c, & j>0 \end{cases}$

$$\gamma_X(h) = E[X_t X_{t+h}]$$

$$= E\left[\left(\sum_{i=0}^{\infty} c_i Z_{t-i}\right)\left(\sum_{j=0}^{\infty} c_j Z_{t-j+h}\right)\right]$$

$$= E\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_i c_j Z_{t-i} Z_{t-j+h}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_i c_j E[Z_{t-i} Z_{t-j+h}]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_i c_j \underbrace{E[Z_{t-i} Z_{t-j+h}]}_{\gamma_Y(h-j+i)}$$

$\gamma_Y(h-j+i) = \sigma^2$ iff $h-j+i=0$

$\Leftrightarrow \boxed{j = h+i}$

$\underline{h=0} \Rightarrow j=i$

$$\Rightarrow \sum_{i=0}^{\infty} c_i^2 E[Z_{t-i}^2] = \sum_{i=0}^{\infty} c_i^2\sigma^2$$

$$= 1 + c^2\sum_{i=1}^{\infty}\sigma^2 = \infty \Rightarrow \{X_t\} \text{ is not stationary}$$

as $E[X_t^2] = +\infty$.

Now let $\{Y_t\}$ be defined by $\boxed{Y_t = X_t - X_{t-1}}$

then, $\boxed{E[Y_t]} = E[X_t] - E[X_{t-1}] = \boxed{0}$ $\forall t$

and,

$$\boxed{Cov(Y_t, Y_{t+h})} = E[(X_t - X_{t-1})(X_{t+h} - X_{t+h-1})]$$

$$= E\left[\left(\sum_{i=0}^{\infty} c_i Z_{t-i} - \sum_{i=0}^{\infty} c_i Z_{t-i-1}\right)\cdot\left(\sum_{j=0}^{\infty} c_j Z_{t-j+h} - \sum_{j=0}^{\infty} c_j Z_{t-j-1+h}\right)\right]$$

$k = j+1$

$$= E\left[\left(Z_t + c\sum_{i=1}^{\infty} Z_{t-i} - Z_{t-1} - c\sum_{i=1}^{\infty} Z_{t-i-1}\right)\left(Z_{t+h} + c\sum_{j=1}^{\infty} Z_{t-j+h} - Z_{t+h-1} - c\sum_{j=1}^{\infty} Z_{t-j-1+h}\right)\right]$$

$$= E\left[\left(Z_t + c\sum_{i=1}^{\infty} Z_{t-i} - Z_{t-1} - c\sum_{i=2}^{\infty} Z_{t-i}\right)\left(Z_{t+h} + c\sum_{j=1}^{\infty} Z_{t-j+h} - Z_{t+h-1} - c\sum_{j=2}^{\infty} Z_{t-j+h}\right)\right]$$

$$= E\left[(Z_t + cZ_{t-1} - Z_{t-1})(Z_{t+h} + cZ_{t+h-1} - Z_{t+h-1})\right]$$

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$$\boxed{= \mathbb{E}\left[\left(Z_t + (c-1)Z_{t-1}\right)\left(Z_{t+h} + (c-1)Z_{t+h-1}\right)\right]}$$

$\underline{h=0}$

$$\gamma_Y(0) = \mathbb{E}\left[\left(Z_t + (c-1)Z_{t-1}\right)^2\right]$$

$$= \mathbb{E}[Z_t^2] + 2(c-1)\mathbb{E}[Z_t Z_{t-1}] + (c-1)^2\mathbb{E}[Z_{t-1}^2]$$

$$= \sigma^2 + (c-1)^2\sigma^2 = \boxed{[1+(c-1)^2]\sigma^2}$$

$\underline{h=\pm 1}$

$$= \mathbb{E}\left[\left(Z_t + (c-1)Z_{t-1}\right)\left(Z_{t+1} + (c-1)Z_t\right)\right]$$

$$= \mathbb{E}[\cancel{Z_t Z_{t+1}}] + (c-1)\mathbb{E}[Z_t^2]$$

$$+ (c-1)\mathbb{E}[\cancel{Z_{t-1} Z_{t+1}}] + (c-1)^2\mathbb{E}[\cancel{Z_{t-1} Z_t}]$$

$$= \boxed{(c-1)\sigma^2}$$

$\underline{h \neq 0, 1}$ $\gamma(h) = 0$

$\Rightarrow \{Y_t\}$ is stationary with ACF

$$\boxed{\gamma_Y(h) = \begin{cases} [1+(c-1)^2]\sigma^2, & h=0 \\ (c-1)\sigma^2, & h=\pm 1 \\ 0, & \text{else} \end{cases}}$$

$$\boxed{\rho(h) = \begin{cases} 1, & h=0 \\ \dfrac{c-1}{1+(c-1)^2}, & h=\pm 1 \\ 0, & h \neq 0,1 \\ \rho(-h), & h \le 0 \end{cases}}$$

Q2.11 Suppose sample of size 100 from AR(1), mean μ, $\phi = .6$, $\sigma^2 = 2$, $\bar{x}_{100} = 0.271$. Construct a 95% C.I. for μ.

Solution

Assuming that $\{X_t\}$ has Gaussian responses, we have that

$$\boxed{\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\sum_{|h|<\infty}\left(1 - \frac{|h|}{n}\right)\gamma(h)\right)}$$

We construct a 95% CI for μ as

$$\boxed{C.I._{95\%}(\mu) = \bar{X} \pm z_{0.975} \cdot \sqrt{\mathrm{Var}(\bar{X})}}$$

Here,

$$\sum_{|h|<\infty} \gamma(h) = \sum_{|h|<\infty} \frac{\sigma^2 \phi^h}{1-\phi^2}$$

$$= \left(1 + 2\sum_{h=1}^{\infty}\phi^h\right)\left(\frac{\sigma^2}{1-\phi^2}\right)$$

$$= \left(1 + 2\phi\sum_{h=1}^{\infty}\phi^{h-1}\right)\left(\frac{\sigma^2}{1-\phi^2}\right) = \left(1 + \frac{2\phi}{1-\phi}\right)\left(\frac{\sigma^2}{1-\phi^2}\right)$$

$$= \left(\frac{1-\phi+2\phi}{1-\phi}\right)\left(\frac{\sigma^2}{1-\phi^2}\right) = \frac{\cancel{(1+\phi)}\,\sigma^2}{(1-\phi)\cancel{(1+\phi)}(1-\phi)^2} \quad \frac{\sigma^2}{(1-\phi)^2}$$

$$\Rightarrow CI_{(1-\alpha)\%}(\mu) = \mu \pm z_{\alpha/2}\left(\frac{1}{\sqrt{n}}\right)\left(\frac{\sigma}{|1-\phi|}\right)$$

$$\Rightarrow C.I._{95\%}(\mu) = 0.271 \pm (1.96)\left(\frac{1}{\sqrt{100}}\right)\frac{2}{|1-0.36|}$$

$$= [-0.422, 0.964]$$

which is compatible with $\mu = 0$ as $0 \in CI(\mu)$ ∎

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Q2.21 Let X_1, X_2, X_4, X_5 be obs. from MA(1)

$\boxed{X_t = Z_t + \theta Z_{t-1}}$, $\{Z_t\} \sim WN(0, \sigma^2)$

a) Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2

Solution

We have to forecast X_{n+h} using $X_1, \dots, X_n$

the best linear predictor with minimum MSE is

$$\boxed{P_n X_{n+h} = \mu + \sum_{i=1}^{n} a_i (X_{n+1-i} - \mu)}$$

where $\underset{\sim}{a}_n$ satisfies $\boxed{\Gamma_n \underset{\sim}{a}_n = \underset{\sim}{\gamma}_n(h)}$

and also given $Y, W_1, \dots, W_n$,

$$\begin{cases} \underset{\sim}{W} = (W_n, \dots, W_1), \ \underset{\sim}{\mu} = (\mu_n, \dots, \mu_1) \\ \underset{\sim}{\gamma} = Cov(Y, \underset{\sim}{W}) = [Cov(Y, W_n), Cov(Y, W_{n-1}), \dots, Cov(Y, W_1)]^T \\ \Gamma = Cov(\underset{\sim}{W}, \underset{\sim}{W}), \ \Gamma_{ij} = Cov(W_{n+1-i}, W_{n+1-j}) \end{cases}$$

Best linear predictor is

$\boxed{P(Y|W) = \mu_Y + a^T(W - \mu_W)}$, where $\boxed{\Gamma \underset{\sim}{a} = \underset{\sim}{\gamma}}$

For MA(1),

$\boxed{X_t = Z_t + \theta Z_{t-1}}$, $\{Z_t\} \sim WN(0, \sigma^2)$

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta, & h = \pm 1 \\ 0, & |h| > 1 \end{cases}$$

a) Best linear estimate of X_3 in terms of X_1 and X_2

Let $Y = X_3$, $W = (X_2, X_1)$

$$\Gamma_{(X_1, X_2)} = \begin{pmatrix} Cov(X_1, X_1) & Cov(X_2, X_1) \\ Cov(X_2, X_1) & Cov(X_2, X_2) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2(1+\theta^2) \end{pmatrix}$$

$$\underset{\sim}{\gamma} = [\gamma(1) \ \gamma(2)]^T$$

$$\Gamma \underset{\sim}{a} = \underset{\sim}{\gamma} \Rightarrow \underset{\sim}{a} = \Gamma^{-1} \underset{\sim}{\gamma}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \left(\sigma^2 \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \sigma^2\theta \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}^{-1} \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\frac{1}{(1+\theta^2)^2 - \theta^2}}_{:= C} \begin{pmatrix} 1+\theta^2 & -\theta \\ -\theta & 1+\theta^2 \end{pmatrix} \begin{bmatrix} \theta \\ 0 \end{bmatrix}$$

$$\boxed{a_1 = \frac{(1+\theta^2)\theta}{(1+\theta^2)^2 - \theta^2}, \ b = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2}}$$

$$\Rightarrow \boxed{P_{X_1, X_2}(X_3) = a_1 X_2 + a_2 X_1}$$

d)

$$\boxed{MSE} = E[(Y - P_W(Y))^2]$$

$$= \boxed{\gamma(0) - a_n^T \underset{\sim}{\gamma}_n(h)}$$

$$= \sigma^2(1+\theta^2) - [a_1 \ a_2] \begin{bmatrix} \sigma^2\theta \\ 0 \end{bmatrix}$$

$$= \boxed{\sigma^2(1+\theta^2) - \frac{(1+\theta^2)\theta^2\sigma^2}{(1+\theta^2)^2 - \theta^2}}$$

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b) Best linear estimate of X_3 in terms of X_4 and X_5

let $Y = X_3$, $W = (X_4, X_5)$

$$\Gamma_{(X_4,X_5)} = \begin{pmatrix} Cov(X_4,X_4) & Cov(X_4,X_5) \\ Cov(X_5,X_4) & Cov(X_4,X_4) \end{pmatrix} = \begin{pmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2(1+\theta^2) \end{pmatrix}$$

$$\underset{\sim}{\gamma} = [\gamma(1), \gamma(2)]^T = [\sigma^2\theta \quad 0]^T$$

So it follows similarly to (a) that

$$\boxed{a_1 = \frac{(1+\theta^2)\theta}{(1+\theta^2)^2 - \theta^2}, \quad b = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2}}$$

$$\Rightarrow \boxed{P_{X_4,X_5}(X_3) = a_1 X_5 + a_2 X_4}$$

and

$$MSE = \gamma(0) - \underset{\sim}{a_n}^T \underset{\sim}{\gamma_n}(h)$$

c) Best linear estimate for X_3 in terms of $W = (X_5, X_4, X_2, X_1)$

$$\Gamma_{(W)} = \begin{bmatrix} Cov(X_5,X_5) & Cov(X_4,X_5) & Cov(X_2,X_5) & Cov(X_1,X_5) \\ Cov(X_5,X_4) & Cov(X_4,X_4) & Cov(X_2,X_4) & Cov(X_1,X_4) \\ Cov(X_5,X_2) & Cov(X_4,X_2) & Cov(X_2,X_2) & Cov(X_1,X_2) \\ Cov(X_5,X_1) & Cov(X_4,X_1) & Cov(X_2,X_1) & Cov(X_1,X_1) \end{bmatrix}$$

$$= \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(3) & \gamma(4) \\ \gamma(1) & \gamma(0) & \gamma(2) & \gamma(3) \\ \gamma(3) & \gamma(2) & \gamma(0) & \gamma(1) \\ \gamma(4) & \gamma(3) & \gamma(1) & \gamma(0) \end{bmatrix}$$

$$\Gamma_{(W)} = \begin{bmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta & 0 & 0 \\ \sigma^2\theta & \sigma^2(1+\theta^2) & 0 & 0 \\ 0 & 0 & \sigma^2(1+\theta^2) & \sigma^2\theta \\ 0 & 0 & \sigma^2\theta & \sigma^2(1+\theta^2) \end{bmatrix}$$

$$\underset{\sim}{\gamma} = \begin{bmatrix} Cov(X_3,X_5) \\ Cov(X_3,X_4) \\ Cov(X_3,X_2) \\ Cov(X_3,X_1) \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^2\theta \\ \sigma^2\theta \\ 0 \end{bmatrix}$$

$$\Gamma \underset{\sim}{a} = \underset{\sim}{\gamma} \Rightarrow \underset{\sim}{a} = \Gamma^{-1}\underset{\sim}{\gamma}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 & 0 & 0 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 & 0 & 0 \\ 0 & 0 & (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ 0 & 0 & \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sigma^2\theta \\ \sigma^2\theta \\ 0 \end{bmatrix}$$

$$\underset{\sim}{a} = \begin{bmatrix} \Gamma^{-1}_{(X_1,X_2)} & 0 \\ 0 & \Gamma^{-1}_{(X_3,X_4)} \end{bmatrix} \begin{bmatrix} 0 \\ \sigma^2\theta \\ \sigma^2\theta \\ 0 \end{bmatrix}$$

$$= \frac{1}{(1+\theta^2)^2 - \theta^2} \begin{bmatrix} 1+\theta^2 & -\theta & 0 & 0 \\ -\theta & 1+\theta^2 & 0 & 0 \\ 0 & 0 & 1+\theta^2 & -\theta \\ 0 & 0 & -\theta & 1+\theta^2 \end{bmatrix} \begin{bmatrix} 0 \\ \theta \\ \theta \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{a_1 = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2} \quad a_2 = \frac{(1+\theta^2)\theta}{(1+\theta^2)^2 - \theta^2} \quad a_3 = \frac{(1+\theta^2)\theta}{(1+\theta^2)^2 - \theta^2} \quad a_4 = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2}}$$

$$\Rightarrow \boxed{P_{X_1,X_2,X_4,X_5}(X_4) = a_1 X_5 + a_2 X_4 + a_3 X_2 + a_4 X_1}$$

↳ Best linear predictor

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d) For a) and b)

$$\boxed{MSE} = E\left[(Y - P_W(Y))^2\right]$$

$$= \boxed{\gamma(0) - a_n^T \underset{\sim}{\gamma}_n(h)}$$

$$= \sigma^2(1+\theta^2) - [a_1 \;\; a_2] \begin{bmatrix} \sigma^2\theta \\ 0 \end{bmatrix}$$

$$= \boxed{\sigma^2(1+\theta^2) - \frac{(1+\theta^2)\theta^2\sigma^2}{(1+\theta^2)^2 - \theta^2}}$$

For c)

$$\boxed{MSE} = E\left[(Y - P_W(Y))^2\right]$$

$$= \boxed{\gamma(0) - a_n^T \underset{\sim}{\gamma}_n(h)}$$

$$= \sigma^2(1+\theta^2) - [a_1 \; a_2 \; a_3 \; a_4] \begin{bmatrix} 0 \\ \sigma^2\theta \\ \sigma^2\theta \\ 0 \end{bmatrix}$$

$$= \boxed{\sigma^2(1+\theta^2) - 2\left(\frac{(1+\theta^2)\theta^2\sigma^2}{(1+\theta^2)^2 - \theta^2}\right)}$$