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Assignment 3

MATH 545

Q 3.5 Let $\{Y_t\}$ be the ARMA plus noise time series defined by

$$\boxed{Y_t = X_t + W_t}$$

where $\{W_t\} \sim WN(0, \sigma_w^2)$, $\{X_t\}$ is the ARMA(p,q) process satisfying.

$$\boxed{\Phi(B)X_t = \Theta(B)Z_t} \quad \boxed{\{Z_t\} \sim WN(0, \sigma_Z^2)}$$

and $\boxed{E[W_s Z_t]=0}$, $\forall s,t$

(a) Show that $\{Y_t\}$ is stationary and find its autocovariance in terms of σ_w^2 and the ACVF of $\{X_t\}$

Solution

Since $\{X_t\}$ is an ARMA(p,q) process, it is stationary, assuming

$$\Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \neq 0, \quad \forall |z|=1$$

also, it can be written as

$\Phi(B)X_t = \Theta(B)Z_t$

$$\Leftrightarrow X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k} + \mu_x$$

$$\Leftrightarrow X_t = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k} + \sum_{j=1}^{p} \phi_j X_{t-j} + \mu_x$$

so that $E[X_t] = \mu_x$, and so

$$\boxed{\mu_y(t) = E[Y_t] = E[X_t] + E[W_t] = \mu_x \perp t}$$

- The ACVF at lag

$\underline{h=0}$

$$\gamma_y(h) = E[(Y_t - \mu_y)(Y_{t+h} - \mu_y)]$$

$$\Rightarrow \boxed{\gamma_y(0) = E[(Y_t - \mu_y)^2]}$$

$$= E[Y_t^2] - 2\mu_y E[Y_t] + \mu_y^2$$

$$= E[(X_t + W_t)^2] - \mu_x^2$$

$$= E[X_t^2] + 2\underbrace{E[X_t W_t]}_{=0} + E[W_t^2] - \mu_x^2$$

$$= E[X_t^2] - \mu_x^2 + \sigma_w^2 = E[(X_t - \mu_x)^2] + \sigma_w^2$$

$$= \boxed{\gamma_x(0) + \sigma_w^2}$$

- $\underline{h>0}$

$$\boxed{\gamma_y(h) = E[(Y_t - \mu_y)(Y_{t+h} - \mu_y)]}$$

$$= E[((X_t - \mu_x) + W_t)((X_{t+h} - \mu_x) + W_{t+h})]$$

$$= \underbrace{E[(X_t - \mu_x)(X_{t+h} - \mu_x)]}_{\to \gamma_x(h)} + \underbrace{E[(X_t - \mu_x)W_{t+h}]}_{=0 \text{ since } W_s \perp Z_t}$$

$$+ \underbrace{E[W_t(X_{t+h} - \mu_x)]}_{=0 \text{ since } W_s \perp Z_t} + \underbrace{E[W_t W_{t+h}]}_{=0 \text{ since } W_i \perp W_j,\ i \neq j}$$

$$= \boxed{\gamma_x(h)}$$

(b) Show that the process $U_t := \Phi(B) Y_t$ is r-correlated, where $r = \max(p,q)$ and hence, by Proposition 2.11 is an MA(r) process.
Conclude that $\{Y_t\}$ is an ARMA(p.r) process

Proof

We have that

$$U_t = \boxed{\Phi(B) Y_t}$$
$$= \Phi(B)(X_t + W_t)$$
$$= \Phi(B) X_t + \Phi(B) W_t$$
$$= \Theta(B) Z_t + \Phi(B) W_t$$
$$= \boxed{Z_t + \sum_{k=1}^{q} \theta_j Z_{t-k} + W_t - \sum_{j=1}^{p} \phi_j W_{t-j}} \quad (1)$$

$\Rightarrow \mathrm{Cov}(U_t, U_s) = 0 \quad \forall\, |t-s| > r := \max(p,q)$

so the process is r-correlated

$\Rightarrow$ it is an MA(r) process, since it only depends on past noise.

This implies that $\exists$ $\boxed{\tilde{\Theta}(z) = 1 - \sum_{j=1}^{r} \tilde{\phi}_j z^j}$

and $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$ such that

$$\boxed{\Phi(B) Y_t = \tilde{\Theta}(B) \varepsilon_t}$$

$\therefore$ $\boxed{\{Y_t\} \sim ARMA(p,r)}$

Q 3.7 Suppose that $\{X_t\}$ is the noninvertible MA(1) process

$$\boxed{X_t = Z_t + \theta Z_{t-1}}, \quad \boxed{\{Z_t\} \sim WN(0, \sigma^2)}$$

where $|\theta| > 1$. Define a new process $\{W_t\}$ as

$$\boxed{W_t = \sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j}}$$

and show that $\{W_t\} \sim WN(0, \sigma_W^2)$.
Express σ_W^2 in terms of θ and σ^2 and show that $\{X_t\}$ has the invertible representation (in terms of $\{W_t\}$)

$$\boxed{X_t = W_t + \frac{1}{\theta} W_{t-1}}$$

Proof

First we show that $W_t \sim WN(0, \sigma_W^2)$

$$\boxed{\mathbb{E}[W_t]} = \mathbb{E}\left[\sum_{j=0}^{\infty} (-\theta)^j X_{t-j}\right] = \sum_{j=0}^{\infty} (-\theta)^j \mathbb{E}[X_{t-j}] = \boxed{0}$$

For $h \geq 0$

$$\gamma_W(h) = \mathbb{E}[W_t W_{t+h}]$$
$$= \mathbb{E}\left[\left(\sum_{j=0}^{\infty} (-\theta)^j X_{t-j}\right)\left(\sum_{k=0}^{\infty} (-\theta)^k X_{t+h-k}\right)\right]$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-\theta)^j (-\theta)^k \mathbb{E}[X_{t-j} X_{t+h-k}]$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-\theta)^j (-\theta)^k \gamma_X(h-k+j)$$

Now, recall that for the MA(1) process, we have that

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta, & h = \pm 1 \\ 0, & |h| > 1 \end{cases}$$

$$= \sum_{j=0}^{\infty}\sum_{k\geq 0}^{\infty} (-\theta)^{-(j+k)} \gamma_X(h-k+j) \quad ①$$

therefore in the expression above,

$$\left.\begin{aligned} &\gamma_X(h-k+j) = \sigma^2(1+\theta^2) \\ &\Leftrightarrow h-k+j=0 \Leftrightarrow j=k-h \end{aligned}\right\} ②$$

$$\left.\begin{aligned} &\gamma_X(h-k+j) = \sigma^2\theta \\ &\Leftrightarrow h-k+j=\pm 1 \Leftrightarrow j=k-h\pm 1 \end{aligned}\right\} ③$$

So that ① can be written as

$$\overset{②}{=} \underbrace{\sum_{k=h}^{\infty} (-\theta)^{-(k-h+k)} \sigma^2(1+\theta^2)}$$

$$\boxed{= \sum_{k=h}^{\infty} (-\theta)^{-(2k-h)} \sigma^2(1+\theta^2)}$$

$$+$$

$$\boxed{\begin{aligned} &\sum_{k=h+1}^{\infty} (-\theta)^{-(k-h-1+k)} \sigma^2\theta + \\ &\sum_{\substack{k=h-1 \\ k\geq 0}}^{\infty} (-\theta)^{-(k-h+1+k)} \sigma^2\theta \end{aligned}} \text{ by } ③$$

So that ① becomes

$$= \sigma^2(1+\theta^2)\sum_{k=h}^{\infty} (-\theta)^{-(2k-h)}$$
$$+ \sigma^2\theta \sum_{k=h+1}^{\infty} (-\theta)^{-(2k-h-1)}$$
$$+ \sigma^2\theta \sum_{\substack{k=h-1 \\ k\geq 0}}^{\infty} (-\theta)^{-(2k-h+1)}$$

$$= \sigma^2(1+\theta^2) \sum_{k=h}^{\infty} (-\theta)^{-(2k-h)}$$
$$+ \sigma^2\theta \sum_{k=h+1}^{\infty} (-\theta)^{-(2k-(h+1))}$$
$$+ \sigma^2\theta \sum_{\substack{k=h-1 \\ k\geq 0}}^{\infty} (-\theta)^{-(2k-(h-1))}$$

$$= \sigma^2(1+\theta^2)(-\theta)^{-h} \sum_{k=h}^{\infty} (-\theta)^{-2(k-h)}$$
$$+ \sigma^2\theta(-\theta)^{-(h+1)} \sum_{k=h+1}^{\infty} (-\theta)^{-2(k-(h+1))}$$
$$+ \sigma^2\theta(-\theta)^{-(h-1)} \sum_{\substack{k=h-1 \\ k\geq 0}}^{\infty} (-\theta)^{-2(k-(h-1))}$$

$$= \boxed{\begin{aligned} &\sigma^2(1+\theta^2)(-\theta)^{-h} \sum_{j=0}^{\infty} (-\theta)^{-2j} \\ &+ \sigma^2\theta(-\theta)^{-(h+1)} \sum_{j=0}^{\infty} (-\theta)^{-2j} \\ &+ \sigma^2\theta(-\theta)^{-(h-1)} \sum_{j=0}^{\infty} (-\theta)^{-2j} \end{aligned}} \quad ④$$

Since $|\theta|>1$, $\left|\frac{1}{(-\theta)^2}\right| < 1$, and so

$$\boxed{\sum_{j=0}^{\infty} (-\theta)^{-2j}} = \sum_{j=0}^{\infty}\left(\frac{1}{\theta^2}\right)^j = \left(\frac{1}{\theta^2}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\theta^2}\right)^{j-1}$$

$$= \frac{\left(\frac{1}{\theta^2}\right)}{1-\frac{1}{\theta^2}} = \boxed{\frac{\theta^2}{\theta^2-1}} \quad ⑤$$

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Then (4) becomes

$$\sigma^2(1+\theta^2)(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}+\sigma^2\theta^2\mathbb{1}\{h=0\}$$

$$+\sigma^2\theta(-\theta)^{-(h+1)}\frac{\theta^2}{\theta^2-1}$$

$$+\sigma^2\theta(-\theta)^{-(h-1)}\frac{\theta^2}{\theta^2-1}$$

$$=\sigma^2(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}+\sigma^2\theta^2(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}$$

$$+\sigma^2\theta(-\theta)^{-h}(-\theta)^{-1}\frac{\theta^2}{\theta^2-1}$$

$$+\sigma^2\theta(-\theta)^{-h}\cdot -\theta\,\frac{\theta^2}{\theta^2-1}$$

$$=\sigma^2(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}\left(\underbrace{1+\theta^2-1-\theta^2}_{=0}\right)$$

$$+\sigma^2\theta^2\cdot\mathbb{1}\{h=0\}$$

$$=\boxed{\sigma^2\theta^2\cdot\mathbb{1}\{h=0\}}$$

$\therefore$ $\{W_t\}\sim WN(0,\sigma_W^2)$, with $\sigma_W^2=\sigma^2\theta^2$.

Now, we can let

$$\boxed{W_t=\sum_{j=0}^{\infty}(-\theta)^{-j}X_{t-j}=\sum_{j=0}^{\infty}\pi_j X_{t-j}}$$

where $\boxed{\pi_j:=(-\theta)^{-j}}$, and so since

$\sum_{j=0}^{\infty}|\pi_j|=\sum_{j=0}^{\infty}\theta^{-j}<+\infty$, then we have that $\{X_t\}\sim MA(1)$ is invertible

and solves $\Phi(B)X_t=\Theta(B)W_t$, where (6)

$$\boxed{\pi(z)=\sum_{j=0}^{\infty}\pi_j z^j=\frac{\Phi(z)}{\Theta(z)}=\xi(B)\Phi(B)}$$

where $\xi(z)=\frac{1}{\Theta(z)}=\sum_{j\geq 0}\theta_j z^j$, so we must have

$$\pi(z)=\sum_{j=0}^{\infty}\pi_j z^j=\sum_{j=0}^{\infty}(-\theta)^{-j}z^j$$

$$=\sum_{j=0}^{\infty}\left(-\frac{z}{\theta}\right)^j=1\cdot\sum_{j=0}^{\infty}\left(\frac{-z}{\theta}\right)^{j-1}=\boxed{\frac{1}{1+\frac{z}{\theta}}=\frac{\Phi(z)}{\Theta(z)}}$$

$$\Rightarrow \Phi(z)=1,\quad \boxed{\Theta(z)=1+\frac{z}{\theta}}\quad (7)$$

So we have in this case

$$\boxed{\begin{gathered}\Phi(B)X_t=\Theta(B)W_t\\ \iff\\ X_t=W_t+\frac{1}{\theta}W_{t-1}\end{gathered}}$$ by (6), (7)

∎

3.9

(a) Calculate the autocovariance function $\gamma(\cdot)$ of the stationary time series

$$Y_t=\mu+Z_t+\theta_1 Z_{t-1}+\theta_{12}Z_{t-12}$$

$$\{Z_t\}\sim WN(0,\sigma^2)$$

Solution

Since Y_t is a kind of MA process, it is stationary, and it can be written as

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$$Y_t = \mu + Z_t + \theta_1 Z_{t-1} + \Theta_{12} Z_{t-12}$$

$\Leftrightarrow$

1. $$\boxed{Y_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}}, \quad \underbrace{\sum_{j=-\infty}^{\infty} |\psi_j| < \infty}_{\text{if } (1+\theta_1+\Theta_{12})<\infty}$$

2. $$\boxed{E(Y_t) = \mu}$$

where

3. $$\boxed{\begin{cases} \psi_0 = 1 \\ \psi_1 = \theta_1 & , \quad \psi_j = 0 \text{ , otherwise} \\ \psi_{12} = \Theta_{12} \end{cases}}$$

Further, since $\{Z_t\} \sim WN(0, \sigma_Z^2)$, from 1

4. $$\boxed{\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2}$$

<u>$h=0$</u>

$$\boxed{\gamma_Y(0)} = \sum_{j=-\infty}^{\infty} \psi_j^2 \sigma^2 = \boxed{\sigma^2(1+\theta_1^2+\Theta_{12}^2)}$$

<u>$|h|=1$</u>

$$\boxed{\gamma_Y(1)} = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+1}$$

$$= \sigma^2(1 \cdot \theta_1 + 0) = \boxed{\sigma^2\theta_1}$$

<u>$|h|=12$</u>

$$\gamma_Y(12) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+12} = \boxed{\sigma^2 \Theta_{12}}$$

<u>$|h|=11$</u>

$$\gamma_Y(11) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+11} = \boxed{\sigma^2 \theta_1 \Theta_{12}}$$

so that

$$\boxed{\gamma_Y(h) = \begin{cases} \sigma^2(1+\theta_1^2+\Theta_{12}^2) & , h=0 \\ \sigma^2\theta_1 & , h=1 \\ \sigma^2\theta_1\Theta_{12} & , h=11 \\ \sigma^2\Theta_{12} & , h=12 \end{cases}}$$

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Q5.3 Consider the AR(2) process $\{X_t\}$ satisfying.

$$\boxed{X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t}, \quad \{Z_t\} \sim WN(0, \sigma^2)$$

a) For what values of ϕ is this process a causal process?

Solution

An ARMA(p,q) process is causal iff

$$\boxed{\Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \neq 0, \ \forall |z| \leq 1},$$

so here $\Phi(z) = 1 - \phi z - \phi^2 z^2 := 0$

$$\Leftrightarrow \quad z^2 + \left(\frac{1}{\phi}\right) z - \left(\frac{1}{\phi^2}\right) = 0$$

$$z_1, z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\left(\frac{1}{\phi}\right) \pm \sqrt{\left(\frac{1}{\phi}\right)^2 - 4(1)\left(-\frac{1}{\phi^2}\right)}}{2\cdot(1)}$$

$$= \frac{-\left(\frac{1}{\phi}\right) \pm \sqrt{1+4}\left(\frac{1}{\phi^2}\right)}{2} = \boxed{\frac{-1 \pm \sqrt{5}}{2\phi}}$$

$$z_1 = \frac{-1+\sqrt{5}}{2\phi} \qquad z_2 = \frac{-1-\sqrt{5}}{2\phi}$$

will lie outside $|z|=1$ if

$$\Rightarrow |z_1| > 1 \Leftrightarrow \left|\frac{-1+\sqrt{5}}{2\phi}\right| > 1$$

$$\Leftrightarrow \left|\frac{-1+\sqrt{5}}{2}\right| > |\phi| \Leftrightarrow \boxed{|\phi| > 0.61}$$

and $|z_2| > 1 \Leftrightarrow \left|\frac{-1-\sqrt{5}}{2\phi}\right| > 1$

$$\Leftrightarrow \left|\frac{-1-\sqrt{5}}{2}\right| > |\phi| \Leftrightarrow \boxed{|\phi| > 1.61}$$

Therefore the process will be causal if $\boxed{|\phi| > 0.619}$ (0.618033...)

b) The following sample moments were computed after observing $X_1, \ldots, X_{200}$.

$$\boxed{\hat{\gamma}(0) = 6.06} \qquad \boxed{\hat{\rho}(1) = 0.687}$$

Find estimates of ϕ and σ^2 by solving the Yule-Walker equations (If you find more than one solution, choose the one that is causal)

Solution

For $m \in \mathbb{Z}$, the Yule-Walker method tells to

- Solve equations $\boxed{\hat{\underset{\sim}{\gamma}}_m = \hat{\Gamma}_m \hat{\underset{\sim}{\phi}}}$, where

$$\boxed{\hat{\phi} = [\phi_1 \cdots \phi_m]} \qquad \boxed{\hat{\Gamma}_m = \left[\hat{\gamma}(i-j)\right]_{i,j=1}^{m}}$$

$$\boxed{\hat{\underset{\sim}{\gamma}} = [\hat{\gamma}(1), \ldots, \hat{\gamma}(m)]}, \text{ so here,}$$

$$\boxed{X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t} \text{ so that}$$

$$\phi = [\phi \ \ \phi^2] \quad \underset{\sim}{\gamma} = [\gamma(1) \ \ \gamma(2)]$$

$$\Gamma = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}$$

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However, this time is easier to simply write down the equations as

$$\begin{cases} \gamma(0) = \sum_{r=1}^{p} \phi_r \gamma(r) + \sigma^2 \\ \gamma(1) = \sum_{r=1}^{p} \phi_r \gamma(|r-1|) \\ \vdots \\ \gamma(p-1) = \sum_{r=1}^{p} \phi_r \gamma(|r-p+1|) \end{cases}$$

$\Rightarrow$

$$\begin{cases} \gamma(0) - \phi\gamma(1) - \phi^2\gamma(2) = \sigma^2 & (1) \\ \gamma(1) - \phi\gamma(0) - \phi^2\gamma(1) = 0 & (2) \\ \gamma(2) - \phi\gamma(1) - \phi^2\gamma(0) = 0 & (3) \end{cases}$$

$(1) := \phi^2 \cdot (3) + (1) \Rightarrow$

$$\begin{cases} -\phi^3\gamma(1) - \phi\gamma(1) - \phi^4\gamma(0) + \gamma(0) = \sigma^2 & (4) \\ \gamma(1) - \phi\gamma(0) - \phi^2\gamma(1) = 0 & (5) \end{cases}$$

From the second equation,

$$\gamma(1)\phi^2 + \gamma(0)\phi - \gamma(1) = 0$$

$$\boxed{\phi} = \frac{-\gamma(0) \pm \sqrt{\gamma(0)^2 + 4\gamma(1)^2}}{2\gamma(1)}$$

$$= -\frac{1}{2\rho(1)} \pm \sqrt{\frac{\gamma(0)^2 + 4\gamma(1)^2}{4\gamma(1)^2}}$$

$$= \boxed{-\frac{1}{2\rho(1)} \pm \sqrt{\frac{1}{4\rho(1)^2} + 1}}$$

$\Rightarrow \phi = \{0.509, -1.965\}$, so take

$\boxed{\hat{\phi} = 0.509}$, since X_t is a causal process.

We're given

$\boxed{\hat{\gamma}(0) = 6.06}$ $\boxed{\hat{\rho}(1) = 0.687}$

so $\hat{\rho}(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \Rightarrow \boxed{\hat{\gamma}(1)} = \hat{\gamma}(0)\hat{\rho}(1) = \boxed{4.16322}$

So plugging into equation (4), we estimate

$$\boxed{\hat{\sigma}^2} = -\hat{\phi}^3\hat{\gamma}(1) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^4\hat{\gamma}(0) + \hat{\gamma}(0) = \boxed{2.985}$$

■

Q 5.4 Two hundred observations of a time series $X_1, \dots, X_{200}$ gave the following sample statistics

sample mean	$\bar{x}_{200} = 3.82$
sample variance	$\hat{\gamma}(0) = 1.15$
sample ACF:	$\hat{\rho}(1) = 0.427$
	$\hat{\rho}(2) = 0.475$
	$\hat{\rho}(3) = 0.169$

(1)

a) Based on these sample statistics, is it reasonable to suppose $\{X_t - \mu\}$ is white noise?

Solution

Suppose $\{X_t - \mu\} \sim WN(0, \sigma^2)$. Then, $\{X_t - \mu\} \perp\!\!\!\perp \{X_s - \mu\}\ \forall s,t$ so that

$$\gamma(h) = \begin{cases} \sigma^2, & h=0 \\ 0, & h>0 \end{cases} \Leftrightarrow \rho(h) = \begin{cases} 1, & h=0 \\ 0, & h>0 \end{cases}$$

and so for $h>0$, $\boxed{\hat{\rho}(h) \dot{\sim} \mathcal{N}(0, \tfrac{1}{n})}$ (2)

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$$L(\phi,\sigma^2)=\frac{1}{(2\pi\sigma^2)\sqrt{r_0 r_1}}\exp\left\{-\frac{1}{2\sigma^2}\left(\frac{(X_1-\hat{X}_1)^2}{r_0}+\frac{(X_2-\hat{X}_2)^2}{r_1}\right)\right\}$$

$$=\frac{1}{(2\pi\sigma^2)\sqrt{r_0 r_1}}\exp\left\{-\frac{1}{2\sigma^2}\left(\frac{\hat{X}_1^2}{r_0}+\frac{(X_2-\hat{X}_2)^2}{r_1}\right)\right\}$$

$$\Rightarrow \boxed{\ell(\phi,\sigma^2)=-\ln(2\pi)-\ln(\sigma^2)-\frac{1}{2}\ln(r_0 r_1) -\frac{1}{2\sigma^2}\left(\frac{X_1^2}{r_0}+\frac{(X_2-\phi X_1)^2}{r_1}\right)}$$

$$\boxed{r_0\cdot r_1}=\left(\frac{1}{1-\phi^2}\right)\left(\frac{1+\phi^2-2\phi^2}{1-\phi^2}\right)$$

$$=\frac{\cancel{(1-\phi^2)}}{(1-\phi^2)^2}=\boxed{\frac{1}{1-\phi^2}}$$

$$\boxed{\ell(\phi,\sigma^2)=-\ln(2\pi)-\ln(\sigma^2)+\frac{1}{2}\ln(1-\phi^2) -\frac{1}{2\sigma^2}\left((1-\phi^2)X_1^2+(X_2-\phi X_1)^2\right)}$$

$\Rightarrow$

- $$\frac{\partial \ell}{\partial \sigma^2}=-\frac{1}{\sigma^2}+\frac{1}{2\sigma^4}\left(\frac{X_1^2}{r_0}+\frac{(X_2-\phi X_1)^2}{r_1}\right):=0$$

$$\Rightarrow \boxed{\hat{\sigma}^2}=\frac{1}{2}\left(\frac{X_1^2}{r_0}+\frac{(X_2-\phi X_1)^2}{r_1}\right)$$

$$=\boxed{\frac{1}{2}\left((1-\phi^2)X_1^2+(X_2-\phi X_1)^2\right)}$$

$$\frac{\partial \ell}{\partial \phi}=\frac{-\phi}{1-\phi^2}-\frac{1}{2\sigma^2}\left(-2\phi X_1^2-2(X_2-\phi X_1)\cdot X_1\right)$$

$$=\frac{-\phi}{1-\phi^2}+\frac{1}{\sigma^2}\left(\cancel{\phi X_1^2}+X_2X_1-\cancel{\phi X_1^2}\right)$$

$$=\frac{-\phi}{1-\phi^2}+\frac{X_1X_2}{\sigma^2}:=0$$

$$\Leftrightarrow \frac{X_1X_2}{\sigma^2}=\frac{\phi}{1-\phi^2}$$

$$\Leftrightarrow (1-\phi^2)X_1X_2=\phi\hat{\sigma}^2$$

$$\Leftrightarrow (1-\phi^2)X_1X_2 = \frac{\phi}{2}\left((1-\phi^2)X_1^2+(X_2-\phi X_1)^2\right)$$

$$\Leftrightarrow 2(1-\phi^2)X_1X_2 = \phi\left(X_1^2-\cancel{\phi^2X_1^2}+X_2^2-2\phi X_1X_2+\cancel{\phi^2X_1^2}\right)$$

$$\Leftrightarrow 2X_1X_2-\cancel{2\phi^2X_1X_2} = \phi(X_1^2-X_2^2)-\cancel{2\phi^2X_1X_2}$$

$$\Rightarrow \boxed{\hat{\phi}=\frac{2X_1X_2}{(X_1^2-X_2^2)}}$$

So for $\hat{\sigma}^2$, just plug this back

A 95% C.I. for $\rho(h)$, $h > 0$ is then

$$\boxed{C.I.(\rho(h)) = \hat{\rho}(h) \pm 1.96 \cdot \left(\frac{1}{\sqrt{n}}\right)}$$

so that

$$C.I.(\rho(1)) = \hat{\rho}(1) \pm \frac{1.96}{\sqrt{200}} = [0.427 \pm 0.139]$$
$$= [0.288, 0.566]$$

$$C.I.(\rho(2)) = \hat{\rho}(2) + \frac{1.96}{\sqrt{200}} = [0.475 \pm 0.139]$$
$$= [0.336, 0.614]$$

$$C.I.(\rho(3)) = \hat{\rho}(3) + \frac{1.96}{\sqrt{200}} = [0.169 \pm 0.139]$$
$$= [0.03, 0.308)$$

So we see that 0 is <u>not</u> in any of these intervals. Therefore its not reasonable to assume that $\{X_t - \mu\} \sim WN(0, \sigma^2)$.

b) Assuming that $\{X_t - \mu\}$ can be modeled as the AR(2) process,

$$\boxed{(X_t - \mu) - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) = Z_t}$$

where $\{Z_t\} \sim iid(0, \sigma^2)$, find estimates of μ, ϕ_1, ϕ_2 and σ^2.

Solution

We can estimate $\boxed{\hat{\mu} = \bar{X}_{200} = 3.82}$

Next, we have the Yule-Walker equations

$$\boxed{\hat{\underset{\sim}{\gamma}}_m = \hat{\Gamma}_m \hat{\underset{\sim}{\phi}}} \quad \text{and} \quad \boxed{\gamma(0) = \underset{\sim}{\phi}^T \underset{\sim}{\gamma}_m + \sigma^2}$$

$$\hat{\phi} = [\phi_1 \cdots \phi_m]^T \qquad \hat{\Gamma}_m = [\hat{\gamma}(i-j)]_{i,j=1}^m$$

$$\hat{\underset{\sim}{\gamma}} = [\hat{\gamma}(1), \ldots, \hat{\gamma}(m)]^T, \text{ but note}$$

$$\boxed{\hat{\underset{\sim}{\gamma}}_m = \hat{\Gamma}_m \hat{\phi} \iff \hat{\underset{\sim}{\rho}}_m = \hat{R}_m \hat{\phi}}$$

$$\hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} \quad \hat{\underset{\sim}{R}}_2 = \begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix} \quad \hat{\underset{\sim}{\rho}}_2 = \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}$$

also,
$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\underset{\sim}{\phi}}^T \hat{\underset{\sim}{\gamma}}_m$$
$$\iff$$
$$\boxed{\hat{\sigma}^2 = \hat{\gamma}(0)\left(1 - \hat{\underset{\sim}{\phi}}^T \hat{\underset{\sim}{R}}_m\right)}$$

so that $\hat{\phi} = \hat{R}_m^{-1} \hat{\rho}_n \iff$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix}^{-1} \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}$$

$$= \frac{1}{1 - \hat{\rho}(1)^2} \begin{bmatrix} 1 & -\hat{\rho}(1) \\ -\hat{\rho}(1) & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{\phi}_1} = \frac{\hat{\rho}(1) - \hat{\rho}(1)\hat{\rho}(2)}{1 - \hat{\rho}(1)^2} = \boxed{0.2742}$$

$$\boxed{\hat{\phi}_2} = \frac{\hat{\rho}(2) - \hat{\rho}(1)^2}{1 - \hat{\rho}(1)^2} = \boxed{0.3574}$$

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$$\boxed{\hat{\sigma}^2} = \hat{\gamma}(0)\left(1 - \hat{\underset{\sim}{\phi}}^T \hat{\underset{\sim}{R}}_n\right)$$

$$= \hat{\gamma}(0)\left(1 - [\hat{\phi}_1 \ \hat{\phi}_2]\begin{bmatrix} 1 & \hat{\rho}(1) \\ \hat{\rho}(1) & 1 \end{bmatrix}\right)$$

$$= \boxed{0.8199}$$

c) Would you conclude that $\mu = 0$?

Solution

We have that if $\{X_t\}$ are Gaussian, then

$$\boxed{\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\sum_{|h|<\infty}\left(1 - \frac{|h|}{n}\right)\gamma(h)\right)}$$

$$\sim \mathcal{N}\left(\mu, \frac{\nu}{n}\right), \quad \boxed{\nu = \sum_{|h|<\infty} \gamma(h)}$$

$$\Rightarrow \boxed{\hat{\nu}} = \sum_{h=-\infty}^{\infty} \hat{\gamma}(h) \approx \hat{\gamma}(-3) + \hat{\gamma}(-2) + \hat{\gamma}(-1) + \hat{\gamma}(0) + \hat{\gamma}(1) + \hat{\gamma}(2) + \hat{\gamma}(3)$$

$$= \boxed{3.61}$$

$$\Rightarrow \boxed{C.I_{95\%}(\mu)} = \hat{\mu} \pm z_{0.025} \cdot \frac{\sqrt{\hat{\nu}}}{\sqrt{n}}$$

$$= 3.82 \pm 1.96\sqrt{\frac{3.61}{200}}$$

$$= \boxed{[3.557, 4.083]}$$

so since $0 \notin C.I_{95\%}(\mu)$, we reject $H_0: \mu = 0$

d) Construct 95% confidence intervals for ϕ_1 and ϕ_2

Solution

We know that for large n and true $AR(p)$ model,

$$\boxed{\hat{\underset{\sim}{\phi}}_p \mathrel{\dot\sim} \mathcal{N}\left(\phi_p, \frac{1}{n}\sigma^2 \Gamma_p^{-1}\right)}$$

so here,

$$\frac{1}{n}\hat{\sigma}^2\begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix}^{-1} \approx \begin{pmatrix} 0.005 & -0.0021 \\ -0.0021 & 0.005 \end{pmatrix}$$

$$\Rightarrow \hat{\phi}_1 \mathrel{\dot\sim} \mathcal{N}(\phi_1, 0.005)$$

$$\hat{\phi}_2 \mathrel{\dot\sim} \mathcal{N}(\phi_2, 0.005)$$

$$\Rightarrow \boxed{C.I_{95\%}(\phi_1)} = \hat{\phi}_1 \pm 1.96\sqrt{0.005}$$

$$= \boxed{[0.135, 0.413]}$$

$$\Rightarrow \boxed{C.I_{95\%}(\phi_2)} = \hat{\phi}_2 \pm 1.96\sqrt{0.005}$$

$$= \boxed{[0.219, 0.497]}$$

e) Assuming that the data were generated from an AR(2), derive estimates of the PACF for all lags $h \geq 1$

Solution

The PACF for an ARMA(p,q) process is

$$\boxed{\begin{cases} \alpha(0) = 1, & h = 0 \\ \alpha(h) = \phi_{hh}, & |h| \geq 1 \end{cases}} \quad \boxed{\phi_{hh} = [\phi_h]_h}$$

is the last component of $\boxed{\underset{\sim}{\phi}_h = \Gamma_h^{-1}\gamma_h}$, so that

$$\Rightarrow \boxed{\begin{cases} \alpha(0) = 1 \\ \hat{\alpha}(1) = \hat{\rho}(1) = 0.427 \\ \hat{\alpha}(2) = \hat{\phi}_2 = 0.358 \end{cases}, \ \hat{\alpha}(h) = 0, \ \forall h > 2}$$

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Q 5.11 Given two observations $\{x_1, x_2\}$ from the causal AR(1) process satisfying

$$\boxed{X_t = \phi X_{t-1} + Z_t}, \quad \{Z_t\} \sim WN(0, \sigma^2)$$

and assuming that $|x_1| \neq |x_2|$, find the maximum likelihood estimates of ϕ and σ^2

Solution

Assume the process were Gaussian, then the general likelihood can be written as

$$L(\Gamma_n) = \frac{1}{(2\pi)^{n/2}|\Gamma_n|^{1/2}} \exp\left\{-\frac{1}{2}\underset{\sim}{X}_n^T \Gamma_n^{-1} \underset{\sim}{X}_n\right\}$$

$\Downarrow$ (innovations)

$$L(\Gamma_n) = \frac{1}{(2\pi)^{n/2}\left(\prod_{j=0}^{n-1} \nu_j\right)^{1/2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{\nu_{j-1}}\right\}$$

$\Downarrow$

$$\boxed{L(\phi, \theta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}\left(\prod_{j=1}^{n} r_{j-1}\right)^{1/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{r_{j-1}}\right]}$$

as

$$\boxed{\begin{aligned}\nu_n &= \mathbb{E}\left[(X_{n+1} - \hat{X}_{n+1})^2\right] \\ &= \sigma^2 \mathbb{E}\left[(W_{n+1} - \hat{W}_{n+1})^2\right] = \sigma^2 r_n\end{aligned}}$$

where $\boxed{\hat{X}_j = \mathbb{E}[X_j | X_{j-1}, \dots, X_1] = P_{j-1} X_j}$

so we have that the innovations

$$\boxed{\begin{cases} X_1 - \hat{X}_1 = X_1 \sim \mathcal{N}(0, \nu_0) \\ X_2 - \hat{X}_2 = X_2 - \phi X_1 \sim \mathcal{N}(0, \nu_1) \end{cases}}$$

where

$$\boxed{\begin{cases} \nu_0 = \sigma^2 r_0 = \mathbb{E}\left[(X_1 - \hat{X}_1)^2\right] \\ \nu_1 = \sigma^2 r_1 = \mathbb{E}\left[(X_2 - \hat{X}_2)^2\right] \end{cases}}$$

But since $X_t \sim AR(1)$, we have

$$\boxed{\gamma_X(h) = \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2}}, \quad |h| \geq 0, \text{ so that}$$

- $\nu_0 = \mathbb{E}[(X_1 - \hat{X}_1)^2] = \mathbb{E}[X_1^2] = \gamma_X(0)$

$$= \frac{\sigma^2}{1 - \phi^2} = \sigma^2 r_1 \Rightarrow \boxed{r_1 = \frac{1}{1 - \phi^2}}$$

- $\nu_1 = \mathbb{E}[(X_2 - \hat{X}_2)^2] = \mathbb{E}[(X_2 - \phi X_1)^2]$

$$= \mathbb{E}[X_2^2] - 2\phi \mathbb{E}[X_2 X_1] + \phi^2 \mathbb{E}[X_1^2]$$

$$= \gamma(0)^2 - 2\phi\gamma(1) + \phi^2 \gamma(0)^2$$

$$= \left(\frac{1}{1 - \phi^2} - 2\frac{\phi^2}{1 - \phi^2} + \frac{\phi^2}{1 - \phi^2}\right)\sigma^2 = r_2 \sigma^2$$

$$\Rightarrow \boxed{r_2 = \frac{1 + \phi^2 - 2\phi^2}{1 - \phi^2}}.$$

So our likelihood becomes

$$L(\phi, \sigma^2) = \frac{1}{(2\pi\sigma^2)\left(\prod_{j=1}^{2} r_{j-1}\right)^{1/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{2} \frac{(X_j - \hat{X}_j)^2}{r_{j-1}}\right]$$

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Q5.12 Derive a cubic equation for the maximum likelihood estimate of the coefficient ϕ of a causal AR(1) process based on the observations $X_1, .., X_n$

Solution

The AR(1) process has the form

$$\boxed{X_t = \phi X_{t-1} + Z_t}$$

assuming $Z_t \sim N(0, \sigma^2)$. We see that the distribution of X_t depends on X_{t-1}, but also,

$$\boxed{E[X_t | X_{t-1}] = \phi X_{t-1}}$$

$$\boxed{V[X_t | X_{t-1}]} = E[(X_t - \phi X_{t-1})^2 | X_{t-1}]$$

$$= \underbrace{E[X_t^2 | X_{t-1}]}_{①} - 2\phi \underbrace{E[X_t X_{t-1} | X_{t-1}]}_{②} + \phi^2 X_{t-1}$$

$$② = 2\phi X_{t-1} E[X_t | X_{t-1}] = \boxed{2\phi^2 X_{t-1}^2}$$

$$① = E[(\phi X_{t-1} - Z_t)^2 | X_{t-1}]$$

$$= E[\phi^2 X_{t-1}^2 | X_{t-1}] + E[Z_t^2]$$

$$= \boxed{\phi^2 X_{t-1}^2 + \sigma^2}$$

$$\Rightarrow \boxed{V[X_t | X_{t-1}]} = \phi^2 \cancel{X_{t-1}^2} + \sigma^2$$

$$- \cancel{2}\phi^2 \cancel{X_{t-1}^2} + \phi^2 \cancel{X_{t-1}} = \boxed{\sigma^2}$$

$$\therefore \boxed{X_t | X_{t-1} \sim N(\phi X_{t-1}, \sigma^2)}$$

Now the likelihood for $X_1, .., X_n$ is

$$\boxed{L = f_{X_1,..,X_n}(x_1, .., x_n)}$$

$$= f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) \cdot f_{X_3|X_1 X_2}(x_3 | x_2, x_1)$$

$$\cdot \cdots \cdot f_{X_n|X_1,..,X_{n-1}}(x_n | x_1, .., x_{n-1})$$

and by conditional independence)

$$= \boxed{f_{X_1}(x_1) \cdot \prod_{i=2}^{n} f_{X_i|X_{i-1}}(x_i | x_i)}$$

Further, we have that

$$E[X_1] = 0, \quad Var[X_1] = \gamma_X(0) = \frac{\sigma^2}{1-\phi^2}$$

so assume that $\boxed{X_1 \sim N\left(0, \frac{\sigma^2}{1-\phi^2}\right)}$

Then the likelihood becomes

$$L(\phi, \sigma^2) = \frac{1}{\left(\frac{2\pi\sigma^2}{1-\phi^2}\right)^{1/2}} \exp\left\{-\frac{1}{2} \frac{X_1^2}{\left(\frac{\sigma^2}{1-\phi^2}\right)}\right\}$$

$$\cdot \prod_{i=2}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} \cdot \exp\left\{-\frac{1}{2} \frac{(X_i - \phi X_{i-1})^2}{\sigma^2}\right\}$$

$$= \frac{(1-\phi^2)^{1/2}}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2} \frac{X_1^2(1-\phi^2)}{\sigma^2}\right\} \cdot$$

$$\cdot \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=2}^{n} (X_i - \phi X_{i-1})^2\right\} =$$

$$\boxed{(1-\phi^2)^{1/2} \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{\frac{X_1^2(1-\phi^2) + \sum_{i=2}^{n} (X_i - \phi X_{i-1})^2}{-2\sigma^2}\right\}}$$

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$$= (1-\phi^2)^{1/2} \left(\frac{1}{2\pi\sigma^2}\right)^{n/2}$$

$$\cdot \exp\left\{-\frac{1}{2\sigma^2}\left(x_1^2(1-\phi^2) + \sum_{i=2}^{n} (X_i - \phi X_{i-1})^2\right)\right\}$$

So the log-likelihood is

$$\boxed{\ell(\phi,\sigma^2) = \frac{1}{2}\log(1-\phi^2) - \frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\left(x_1^2(1-\phi^2) + \sum_{i=2}^{n} (X_i-\phi X_{i-1})^2\right)}$$

So taking derivatives wrt. ϕ.

$$\frac{\partial \ell}{\partial\phi} = -\frac{\phi}{1-\phi^2}$$

$$+ \frac{1}{\sigma^2}\left(\phi x_1^2 + \sum_{i=2}^{n} (X_i-\phi X_{i-1})X_{i-1}\right) := 0$$

$$\Longleftrightarrow$$

$$\boxed{\phi x_1^2 + \sum_{i=2}^{n} X_iX_{i-1} - \phi\sum_{i=2}^{n} X_{i-1}^2 = \frac{\sigma^2\phi}{1-\phi^2}}$$

BUT $\phi\sum_{i=2}^{n} X_{i-1}^2 = \phi x_1^2 + \sum_{i=3}^{n} X_{i-1}^2$

So we have,

$$\sum_{i=2}^{n} X_iX_{i-1} = \frac{\sigma^2\phi}{1-\phi^2} + \phi\sum_{i=3}^{n} X_{i-1}^2$$

$$\Longleftrightarrow$$

$$\underbrace{\sum_{i=2}^{n} X_iX_{i-1}}_{:=a} = \frac{1}{1-\phi^2}\left(\sigma^2\phi + \phi(1-\phi^2)\underbrace{\sum_{i=3}^{n} X_{i-1}^2}_{:=b}\right)$$

$$\Longleftrightarrow$$

$$a - \phi^2 a = \sigma^2\phi + \phi(1-\phi^2)b$$

$$\Longleftrightarrow$$

$$\phi^3 b - \phi b + \sigma^2\phi = a - \phi^2 a$$

$$\Longleftrightarrow$$

$$b\phi^3 + a\phi^2 + (\sigma^2 - b)\phi - a = 0$$

$$\Longleftrightarrow$$

$$\boxed{\left(\sum_{i=3}^{n} X_{i-1}^2\right)\phi^3 + \left(\sum_{i=2}^{n} X_iX_{i-1}\right)\phi^2 + \left(\hat{\sigma}^2 - \sum_{i=3}^{n} X_{i-1}^2\right)\phi - a = 0}$$