A Study on TOURNAMENTS

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Piano della presentazione

Hamilton Paths / Cycles and Tournaments:

- We've just discussed the problem of when a graph contains an Eulerian tour (i.e. a closed walk traversing every edge exactly once) and the problem of when a graph contains an Eulerian trail (i.e. a walk passing through every edge exactly once). The two characterization theorems solved those problems quite satisfactorily.
- Let's now ask the analogous question for vertices: *When does a graph G contain a closed walk that contains every vertex of G exactly once?*
- Tournaments will help us to answer.
- First of all let's briefly recall what a directed graph is.

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Directed Graphs

- We have used graphs to model symmetric relations. However relations need not to be symmetric (for example in the street sweeper problem, relations are not symmetric, the order is important).
- In general, a relation on S can be any set of ordered pairs in $S \times S$. For such relations we need a more general model.
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Directed Graphs

Definition

A directed graph or digraph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

- The first vertex of the ordered pair is the tail of the edge, and the second is the head; together, they are the endpoints. We say that an edge is an edge from its tail to its head. (the terms "head" and "tail" come from the arrows used to draw digraphs; in fact when drawing a digraph we give the curve a direction from the tail to the head).

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Orientation

Definition

- An orientation of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$.
- An oriented graph is an orientation of a simple graph.

We are now ready to introduce tournaments, in fact a tournament is an orientation of a complete graph.

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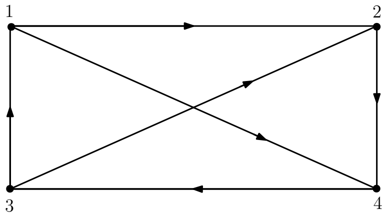
Tournaments

Definition

A tournament is a directed graph obtained by assigning a direction to every edge of the complete graph or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.

For example: a tournament on 4 vertices is:



Tournaments

Observation

- In this context, it only makes sense to consider directed paths and cycles, because every edge is present.
  Hence, a Hamilton cycle is a spanning directed cycle.
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- A Hamilton path/cycle in a graph G is a path/cycle visiting every vertex of G exactly once.
  (Remember that:
  - W is a path if W is a trail whose vertices are distinct.
  - A closed trail whose origin and internal vertices are distinct is a cycle.)
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To determine whether or not a given graph has a Hamilton cycle is much harder than deciding whether it is Eulerian, and no good characterization is known of the graph that do.

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Hamilton path and Tournaments

Theorem 1

Every tournament has a Hamilton path.

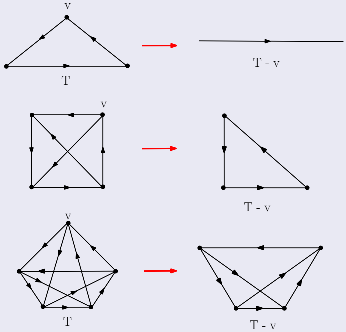
Proof

Let K_n be a complete graph.
We want to show that K_n has a Hamilton path.
We proceed by induction on n.
Base step: The case $n = 2$ is clear: there is only one tournament on two vertices, namely a single directed edge, which is already a Hamilton path.

Hamilton path and Tournaments

Proof

Inductive step: Suppose that the claim holds for all tournaments on n vertices, and let T be a tournament with $n+1$ vertices. Let v be any vertex of T. Then $T - v$ is also a tournament. For example:

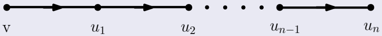


Hamilton path and Tournaments

Proof

Since $T - v$ is also a tournament, $T - v$ has a Hamilton path (for induction hypothesis): $u_1 \to \cdots \to u_n$.
There are two possible cases:

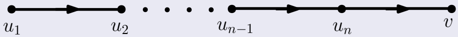
1 If $v \to u_1$, we are done: $v \to u_1 \to \cdots \to u_n$ is a Hamilton path in T.



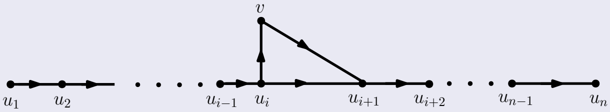
Hamilton path and Tournaments

Proof

2 If $u_1 \to v$, let i be the maximum index such that $u_i \to v$.
If $i = n$ we are done: $u_1 \to \cdots \to u_n \to v$ is a Hamilton path in T.



If not then $v \to u_{i+1}$ and we can take the Hamilton path
$u_1 \to \cdots \to u_i \to v \to u_{i+1} \to \cdots \to u_n$



Strongly connected tournaments

In the theory of directed graph, a graph is said to be strongly connected if every vertex is reachable from every other vertex.

Definition

A directed graph is called strongly connected if there is a path in each direction between each pair of vertices of the graph.

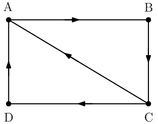
Equivalently:

Let G be a directed graph. G is strongly connected if and only if there are paths from u to v and from v to u for every pair of distinct vertices u and v in G.

Strongly connected tournaments

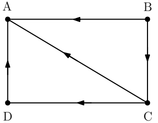
For example:

G =



G is strongly connected

H =



H is not strongly connected (even if H has the same underlying graph of G), in fact there is no path from A to C.

Definition

A tournament is strongly connected if for all u, v there is a directed path from u to v.

Strongly connected tournaments and Hamilton cycles

Theorem

A tournament T is strongly connected if and only if it has a Hamilton cycle.

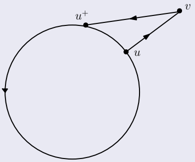
Proof: $\Leftarrow$

If T has a Hamilton cycle, then it is immediately clear that it is strongly connected: for any u and v simply consider the portion of a Hamilton path between u and v.

Strongly connected tournaments and Hamilton cycles

$\Rightarrow$

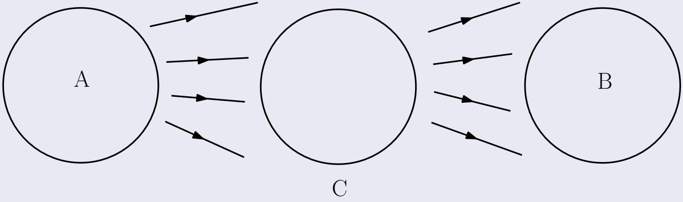
Suppose that T is strongly connected. We want to show that it has a Hamilton cycle.
We prove it by contrapositive.
Suppose T is not Hamiltonian.
Let C be a longest cycle in T and let $v \notin C$.
If C has two consecutive vertices u, u^+ such that $u \to v$ and $v \to u^+$, then there is a longer cycle on the vertex $C \cup \{v\}$ in fact see the following picture:



Strongly connected tournaments and Hamilton cycles

Proof

Otherwise all edges between v and C go in only one direction and we can assume this holds for all $v \notin C$.
Let A be the set of $v \notin C$ such that edges go from v to C, and let B be the set of $v \notin C$ such that edges go from C to v.
See the following picture:



Strongly connected tournaments and Hamilton cycles

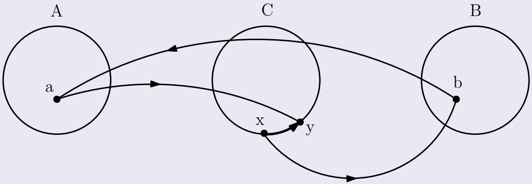
Proof

If one of A or B is empty, it immediately follows that T is not strongly connected: if for example B is empty then there is no path from any vertex of C to any vertex of A.
(because every edge goes from A to C and there is no edge from C to A as required by the definition of "*strongly connected*").

Strongly connected tournaments and Hamilton cycles

Proof

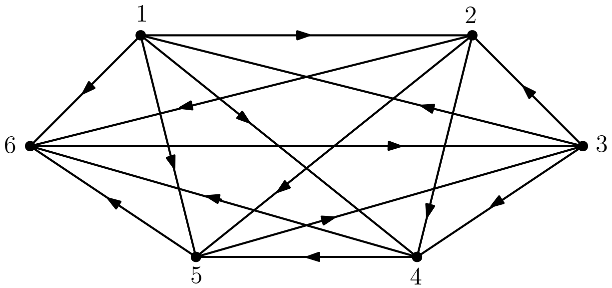
So, suppose both of A and B are nonempty.
There is no edge oriented from a vertex $b \in B$ to a vertex $a \in A$, because if there was we could extend C, contradicting maximality: in fact we could replace any edge $x \to y$ in C with the path
$x \to b \to a \to y$



But then there is no path from any vertex in B to any vertex in A, so again T is not strongly connected.

An application: Ranking the participants in a tournament

Finally let's see an application:
Suppose there's a number of players each play one another in a tennis tournament.
Given the outcomes of the games, how should the participants be ranked?
Consider, for example, the tournament of the following picture:



An application: Ranking the participants in a tournament

- This represents the result of a tournament between six players. For example we see that player 1 beat player 2, 4, 5 and 6 and lost to player 3, and so on.
- One possible approach to ranking the participants would be to find a directed Hamilton path in the tournament (such a path exists by virtue of theorem 1), and then rank according to the position on the path.

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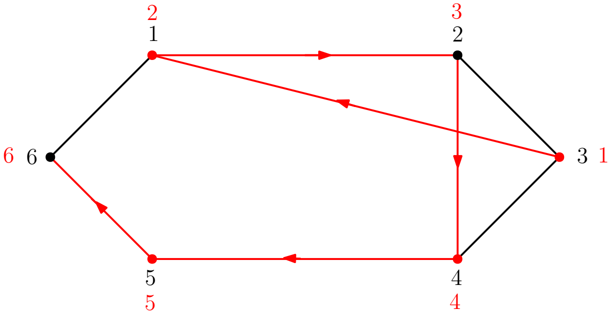
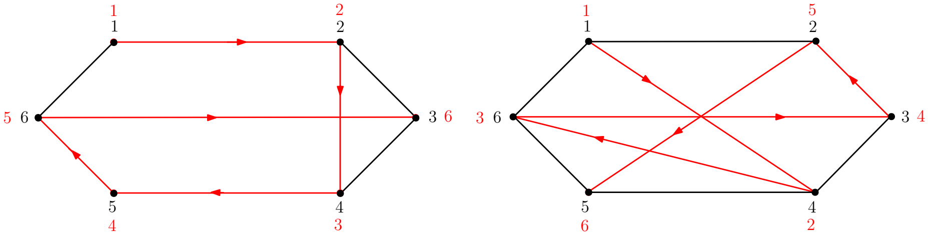
- For instance, the directed Hamilton path $(3, 1, 2, 4, 5, 6)$ would declare player 3 the winner, player 1 runner-up, and so on.



An application: Ranking the participants in a tournament

- This method of ranking, however, does not bear further examination, since a tournament generally has many directed Hamilton paths;
- Our example has $(1, 2, 4, 5, 6, 3)$, $(1, 4, 6, 3, 2, 5)$ and several other.



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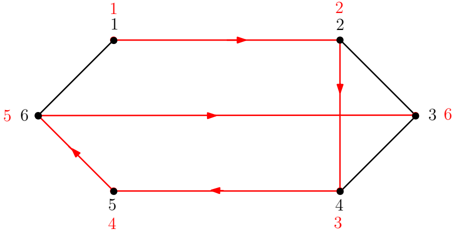

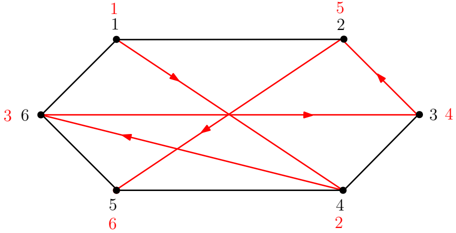

An application: Ranking the participants in a tournament

So a better approach is required: for example we should compute the scores (numbers of games won by each player) and compare them.