Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 7

We list some Fibonacci numbers together with their prime factorization.

n	f_n	factorization
0	0	0
1	1	1
2	1	1
3	2	2
4	3	3
5	5	5
6	8	2^3
7	13	13
8	21	3×7
9	34	2×17
10	55	5×11
11	89	89
12	144	$2^4 \times 3^2$
13	233	233
14	377	13×29
15	610	$2 \times 5 \times 61$
16	987	$3 \times 7 \times 47$
17	1597	1597
18	2584	$2^3 \times 17 \times 19$
19	4181	37×113
20	6765	$3 \times 5 \times 11 \times 41$
21	10946	$2 \times 13 \times 421$
22	17711	89×199
23	28657	28657
24	46368	$2^5 \times 3^2 \times 7 \times 23$

1. We have

n	$\sum_{k=1}^n f_{2k-1}$	$\sum_{k=0}^n f_{2k}$	$\sum_{k=0}^n (-1)^k f_k$	$\sum_{k=0}^n f_k^2$
0	0	0	0	0
1	1	1	-1	1
2	3	4	0	2
3	8	12	-2	$6 = 2 \times 3$
4	21	33	1	$15 = 3 \times 5$
5	55	88	-4	$40 = 5 \times 8$
6	144	232	4	$104 = 8 \times 13$
7	377	609	-9	$273 = 13 \times 21$
n	f_{2n}	$f_{2n+1} - 1$	$-1 + (-1)^n f_{n-1}$	$f_n f_{n+1}$

2. We have

$$f_n = \frac{B^n - L^n}{\sqrt{5}},$$

where

$$B = \frac{1+\sqrt{5}}{2}, \qquad L = \frac{1-\sqrt{5}}{2}.$$

To show that f_n is the integer closest to $B^n/\sqrt{5}$, it suffices to show that

$$-\frac{1}{2} < \frac{L^n}{\sqrt{5}} < \frac{1}{2}.$$

Using $2 < \sqrt{5} < 3$ we have $-1 < L < -1/2$ so $-1 \leq L \leq 1$. Therefore $-1 \leq L^n \leq 1$. Also $1/\sqrt{5} < 1/2$. The result follows.

3. For $m = 2, 3, 4$ consider the Fibonacci sequence $f_0, f_1, \ldots$ modulo m.

n	f_n	f_n mod 2	f_n mod 3	f_n mod 4
0	0	0	0	0
1	1	1	1	1
2	1	1	1	1
3	2	0	2	2
4	3	1	0	3
5	5	1	2	1
6	8	0	2	0
7	13	1	1	1
8	21	1	0	1
9	34	0	1	2
.	.	.	.	.
.	.	.	.	.

The sequence repeats with period 3 (resp. 8) (resp. 6) if $m = 2$ (resp. $m = 3$) (resp. $m = 4$). The result follows.

4. For an integer $n \geq 5$ we have

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ f_{n-1} &= f_{n-2} + f_{n-3} \\ f_{n-2} &= f_{n-3} + f_{n-4} \\ f_{n-3} &= f_{n-4} + f_{n-5}. \end{aligned}$$

In the first equation eliminate f_{n-1} using the second equation and simplify; in the resulting equation eliminate f_{n-2} using the third equation and simplify; in the resulting equation eliminate f_{n-3} using the fourth equation and simplify. The result is

$$f_n = 5f_{n-4} + 3f_{n-5}.$$

Therefore $f_n = 3f_{n-5}$ (mod 5). Consequently $f_n = 0$ (mod 5) if and only if $f_{n-5} = 0$ (mod 5). This together with the initial conditions $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$ shows that $f_n = 0$ (mod 5) if and only if n is divisible by 5.

5. Consider the Fibonacci sequence $f_0, f_1, \ldots$ modulo 7.

n	f_n	f_n mod 7
0	0	0
1	1	1
2	1	1
3	2	2
4	3	3
5	5	-2
6	8	1
7	13	-1
8	21	0
9	34	-1
.	.	.
.	.	.

The table shows that modulo 7 the Fibonacci sequence will repeat with period 16. The pattern of zero/nonzero entries shows that f_n is divisible by 7 if and only if n is divisible by 8.

6, 7. Claim I: f_n, f_{n+1} are relatively prime for $n \geq 1$.
Proof of Claim I: By induction on n. The claim holds for $n = 1$ since $f_1 = 1$ and $f_2 = 1$. Next assume $n \geq 2$. Let x denote a positive integer such that $x|f_n$ and $x|f_{n+1}$. We show $x = 1$. Note that $x|f_{n-1}$ since $f_{n+1} = f_n + f_{n-1}$. By induction f_n and f_{n-1} are relatively prime, so $x = 1$.
Claim II: For $r, s \geq 0$ the expression $f_r f_s + f_{r+1} f_{s+1}$ depends only on $r + s$.
Proof of Claim II: Define $F(r, s) = f_r f_s + f_{r+1} f_{s+1}$. It suffices to show that $F(r, s) = F(r - 1, s + 1)$ provided $r \geq 1$. Note that

$$\begin{aligned} F(r-1, s+1) &= f_{r-1} f_{s+1} + f_r f_{s+2} \\ &= f_{r-1} f_{s+1} + f_r (f_s + f_{s+1}) \\ &= f_r f_s + (f_{r-1} + f_r) f_{s+1} \\ &= f_r f_s + f_{r+1} f_{s+1} \\ &= F(r, s). \end{aligned}$$

Claim III: For $r, s \geq 0$ we have

$$f_r f_s + f_{r+1} f_{s+1} = f_{r+s+1}.$$

Proof of Claim III: By Claim II

$$\begin{aligned} f_r f_s + f_{r+1} f_{s+1} &= f_{r+s} f_0 + f_{r+s+1} f_1 \\ &= f_{r+s+1} \end{aligned}$$

since $f_0 = 0$ and $f_1 = 1$.
Claim IV: For $r, s \geq 1$,

$$\begin{aligned} f_{r+s} &= f_r f_{s-1} + f_s f_{r+1} \\ &= f_s f_{r-1} + f_r f_{s+1}. \end{aligned}$$

Proof of Claim IV: This is a reformulation of Claim IV.
Claim V: For $r, s \geq 1$ consider

$$f_r, f_s, f_{r+s}.$$

If a positive integer x divides at least two of these, then x divides all three.
Proof of Claim V: Assume that $x|f_r$ and $x|f_s$. Then $x|f_{r+s}$ by Claim IV. Next assume that $x|f_r$ and $x|f_{r+s}$. By Claim IV, $x|f_s f_{r+1}$. But x, f_{r+1} are relatively prime, since $x|f_r$ and f_r, f_{r+1} are relatively prime. Therefore $x|f_s$.
Claim VI: Given integers $m, n \geq 1$ such that $n|m$. Then $f_n|f_m$.
Proof of Claim VI: We use induction on $k = m/n$. For $k = 1$ the claim holds. Next assume $k \geq 2$, and consider f_n, f_{m-n}, f_m. By induction $f_n|f_{m-n}$. Applying Claim V with $r = n$, $s = m - n$, $x = f_n$ we find $f_n|f_m$.
Claim VII: Given integers $m, n \geq 1$ with greatest common divisor d. Then f_d is the greatest common divisor of f_m, f_n.
Proof of Claim VII: We use induction on $\min(m, n)$. The claim holds for $\min(m, n) = 1$. Next assume $\min(m, n) \geq 2$. Without loss we may assume $m > n$. We may also assume that n does not divide m; otherwise $d = n$ and we are done since $f_n|f_m$. Divide m by n and consider the remainder:

$$m = qn + r \qquad\qquad 1 \leq r \leq n - 1.$$

Observe that

$$GCD(n, r) = GCD(m, n) = d.$$

By induction and since $r \leq n - 1$,

$$f_d = GCD(n, r).$$

Since $d|m$ and $d|n$ we have $f_d|f_m$ and $f_d|f_n$. Conversely, let x denote a positive integer such that $x|f_m$ and $x|f_n$. We show $x|f_d$. Consider f_m, f_{qn}, f_r. By assumption $x|f_m$. Also $x|f_n$ so $x|f_{qn}$ by Claim VI. Now $x|f_r$ by Claim V. We have $x|f_n$ and $x|f_r$. Therefore $x|f_d$ since $f_d = GCD(f_n, f_r)$. We have shown $f_d = GCD(f_m, f_n)$.

8. By construction $h_0 = 1$ and $h_1 = 2$. We now find h_n for $n \geq 2$. Consider a coloring of the $1 \times n$ chessboard. The first square is colored red or blue. If it is blue, then there are h_{n-1} ways to color the remaining $n - 1$ squares. If it is red, then the second square is blue, and there are h_{n-2} ways to color the remaining $n - 2$ squares. Therefore $h_n = h_{n-1} + h_{n-2}$. Comparing the above data with the Fibonacci sequence we find $h_n = f_{n+2}$.

9. By construction $h_0 = 1$ and $h_1 = 3$. We now find h_n for $n \geq 2$. Consider a coloring of the $1 \times n$ chessboard. The first square is colored red or white or blue. If it is white or blue, then there are h_{n-1} ways to color the remaining $n-1$ squares. If it is red, then the second square is white or blue, and there are h_{n-2} ways to color the remaining $n-2$ squares. Therefore $h_n = 2h_{n-1} + 2h_{n-2}$. To find h_n in closed form, consider the quadratic equation $x^2 = 2x + 2$. By the quadratic formula $x = 1 \pm \sqrt{3}$. We hunt for real numbers a, b such that

$$h_n = a(1+\sqrt{3})^n + b(1-\sqrt{3})^n \qquad\qquad n = 0, 1, 2, \ldots$$

Setting $n = 0, 1$ we find

$$\begin{aligned} 1 &= a + b, \\ 3 &= a(1+\sqrt{3}) + b(1-\sqrt{3}). \end{aligned}$$

Solving these equations for a, b we find

$$a = \frac{\sqrt{3}+2}{2\sqrt{3}}, \qquad\qquad b = \frac{\sqrt{3}-2}{2\sqrt{3}}.$$

Therefore

$$h_n = \frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n \qquad\qquad n = 0, 1, 2, \ldots$$

10. After n months there will be $2f_{n+1}$ pairs of rabbits.

11. (a) Define $Z_n = f_{n-1} + f_{n+1} - l_n$ for $n \geq 1$. One checks $Z_1 = 0$ and $Z_2 = 0$. Also $Z_n = Z_{n-1} + Z_{n-2}$ for $n \geq 3$. Therefore $Z_n = 0$ for $n \geq 1$. The result follows.
(b) Use induction on n. First assume $n = 0$. Then each side equals 4. Next assume $n \geq 1$. By induction

$$\begin{aligned} l_0^2 + l_1^2 + \cdots + l_n^2 &= l_{n-1}l_n + 2 + l_n^2 \\ &= l_n(l_{n-1} + l_n) + 2 \\ &= l_n l_{n+1} + 2. \end{aligned}$$

12. We have

$$(n-1)^3 = n^3 - 3n^2 + 3n - 1$$

so

$$n^3 = (n-1)^3 + 3n^2 - 3n + 1.$$

13. (a) $(1-cx)^{-1}$; (b) $(1+x)^{-1}$; (c) $(1-x)^{\alpha}$; (d) e^x; (e) e^{-x}.

14. (a)

$$\begin{aligned}(x+x^3+x^5+\cdots)^4 &= x^4(1+x^2+x^4+\cdots)^4\\ &= x^4(1-x^2)^{-4}.\end{aligned}$$

(b)

$$(1+x^3+x^6+\cdots)^4 = (1-x^3)^{-4}.$$

(c)

$$(1+x)(1+x+x^2+\cdots)^2 = (1+x)(1-x)^{-2}.$$

(d)

$$(x+x^3+x^{11})(x^2+x^4+x^5)(1+x+x^2+\cdots)^2 = x^3(1+x^2+x^{10})(1+x^2+x^3)(1-x)^{-2}.$$

(e)

$$\begin{aligned}(x^{10}+x^{11}+x^{12}+\cdots)^4 &= x^{40}(1+x+x^2+\cdots)^4\\ &= x^{40}(1-x)^{-4}.\end{aligned}$$

15. We evaluate $\sum_{n=0}^{\infty} n^3x^n$. For $n \geq 0$,

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1}.$$

Recall

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n}{3}x^n &= \sum_{n=3}^{\infty}\binom{n}{3}x^n\\ &= x^3\sum_{n=3}^{\infty}\binom{n}{3}x^{n-3}\\ &= x^3\sum_{n=0}^{\infty}\binom{n+3}{3}x^n\\ &= x^3(1-x)^{-4}.\end{aligned}$$

Similarly,

$$\sum_{n=0}^{\infty}\binom{n}{2}x^n = x^2(1-x)^{-3}, \qquad \sum_{n=0}^{\infty}\binom{n}{1}x^n = x(1-x)^{-2}.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} n^3x^n &= 6x^3(1-x)^{-4} + 6x^2(1-x)^{-3} + x(1-x)^{-2}\\ &= x(x^2+4x+1)(1-x)^{-4}.\end{aligned}$$

16. This is the generating function for the sequence $\{h_n\}_{n=0}^{\infty}$ where h_n is the number of n-combinations of the multiset

$$\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$$

such that (i) e_1 appears at most twice; (ii) e_2 is even and at most 6; (iii) e_3 is even; (iv) e_4 is nonzero.

17. The generating function is

$$\sum_{n=0}^{\infty} h_n x^n = (1 + x^2 + x^4 + \cdots)(1 + x + x^2)(1 + x^3 + x^6 + \cdots)(1 + x).$$

Evaluating this using

$$1 + x^2 + x^4 + \cdots = (1 - x^2)^{-1}, \qquad 1 + x^3 + x^6 + \cdots = (1 - x^3)^{-1}$$

and simplifying, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} h_n x^n &= (1 - x)^{-2} \\ &= \sum_{n=0}^{\infty} (n + 1) x^n. \end{aligned}$$

Therefore $h_n = n + 1$ for $n \geq 0$.

18. Define

$$E_1 = 2e_1, \qquad E_2 = 5e_2, \qquad E_3 = e_3, \qquad E_4 = 7e_4.$$

The scalar h_n is the number of nonnegative integral solutions to

$$E_1 + E_2 + E_3 + E_4 = n,$$

such that (i) E_1 is even; (ii) E_2 is divisible by 5; (iii) E_4 is divisible by 7. The generating function for $\{h_n\}_{n=0}^{\infty}$ is

$$(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x + x^2 + \cdots)(1 + x^7 + x^{14} + \cdots).$$

This simplifies to

$$\frac{1}{1 - x^2} \frac{1}{1 - x^5} \frac{1}{1 - x} \frac{1}{1 - x^7}.$$

19. We have

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{n}{2} x^n &= \sum_{n=2}^{\infty} \binom{n}{2} x^n \\ &= x^2 \sum_{n=2}^{\infty} \binom{n}{2} x^{n-2} \\ &= x^2 \sum_{n=0}^{\infty} \binom{n+2}{2} x^n \\ &= x^2 (1 - x)^{-3}. \end{aligned}$$

20. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{n}{3}x^n &= \sum_{n=3}^{\infty}\binom{n}{3}x^n \\
&= x^3\sum_{n=3}^{\infty}\binom{n}{3}x^{n-3} \\
&= x^3\sum_{n=0}^{\infty}\binom{n+3}{3}x^n \\
&= x^3(1-x)^{-4}.
\end{aligned}$$

21. Define $g(x) = \sum_{n=0}^{\infty} h_n x^n$. We have

$$\sum_{n=1}^{\infty} h_{n-1}x^n = xg(x),$$

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2},$$

$$\sum_{n=0}^{\infty}\binom{n+1}{3}x^n = \frac{x^2}{(1-x)^4}.$$

From the given recurrence we find

$$g(x) - xg(x) = \frac{x^2}{(1-x)^4} + \frac{x}{(1-x)^2}.$$

Therefore

$$\begin{aligned}
g(x) &= \frac{x^2}{(1-x)^5} + \frac{x}{(1-x)^3} \\
&= \sum_{n=0}^{\infty}\binom{n+2}{4}x^n + \sum_{n=0}^{\infty}\binom{n+1}{2}x^n.
\end{aligned}$$

Consequently

$$h_n = \binom{n+2}{4} + \binom{n+1}{2} \qquad\qquad n = 1, 2, 3, \ldots$$

22. The exponential generating function is

$$g^e(x) = \sum_{n=0}^{\infty} n!\frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

23. The exponential generating function is

$$g^e(x) = \sum_{n=0}^{\infty} \alpha(\alpha-1)\cdots(\alpha-n+1)\frac{x^n}{n!} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha.$$

24. (a) We have $g^e(x) = G(x)^k$ where

$$G(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}.$$

(b) We have $g^e(x) = G(x)^k$ where

$$G(x) = \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}.$$

(c) We have $g^e(x) = G_1(x)G_2(x)\cdots G_k(x)$ where for $1 \le r \le k$,

$$G_r(x) = \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \cdots = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{r-1}}{(r-1)!}.$$

(d) We have $g^e(x) = G_1(x)G_2(x)\cdots G_k(x)$ where

$$G_r(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} \qquad (1 \le r \le k).$$

25. For an integer $n \ge 0$, h_n is equal to the number of n-permutations of the multiset

$$\{\infty \cdot R, \infty \cdot W, \infty \cdot B, \infty \cdot G\}$$

such that both (i) R appears an even number of times; (ii) W appears an odd number of times. The exponential generating function is $g^e(x) = G_1(x)G_2(x)G_3(x)G_4(x)$, where

$$G_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2},$$
$$G_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2},$$
$$G_3(x) = G_4(x) = 1 + x + \frac{x^2}{2!} + \cdots = e^x.$$

Using this we obtain

$$g^e(x) = \frac{e^{4x} - 1}{4} = x + \frac{4x^2}{2!} + \frac{4^2x^3}{3!} + \cdots$$

Therefore $h_n = 4^{n-1}$ if $n \ge 1$ and $h_0 = 0$.

26. For an integer $n \ge 0$, h_n is equal to the number of n-permutations of the multiset

$$\{\infty \cdot R, \infty \cdot B, \infty \cdot G, \infty \cdot O\}$$

such that both R and G appears an even number of times. The exponential generating function is $g^e(x) = G_1(x)G_2(x)G_3(x)G_4(x)$, where

$$G_1(x) = G_3(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2},$$
$$G_2(x) = G_4(x) = 1 + x + \frac{x^2}{2!} + \cdots = e^x.$$

Using this we obtain

$$g^e(x) = \frac{(e^{2x}+1)^2}{4} = \frac{e^{4x} + 2e^{2x} + 1}{4}.$$

Therefore $h_n = 4^{n-1} + 2^{n-1}$ if $n \geq 1$ and $h_0 = 1$.

27. Call the number h_n. Then h_n is equal to the number of n-permutations of the multiset

$$\{\infty \cdot 1, \infty \cdot 3, \infty \cdot 5, \infty \cdot 7, \infty \cdot 9\}$$

such that 1 and 3 occur a nonzero even number of times. The exponential generating function is $g^e(x) = G_1(x)G_3(x)G_5(x)G_7(x)G_9(x)$ where

$$G_1(x) = G_3(x) = \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2} - 1,$$
$$G_5(x) = G_7(x) = G_9(x) = e^x.$$

Using this we obtain

$$g^e(x) = \frac{e^x(e^x-1)^4}{4} = \frac{e^{5x} - 4e^{4x} + 6e^{3x} - 4e^{2x} + e^x}{4}.$$

Therefore

$$h_n = \frac{5^n - 4 \times 4^n + 6 \times 3^n - 4 \times 2^n + 1}{4}.$$

28. The exponential generating function is $g^e(x) = \prod_{r=4}^{9} G_r(x)$ where

$$G_4(x) = G_6(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2},$$
$$G_5(x) = G_7(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x - 1,$$
$$G_8(x) = G_9(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

Using this we obtain

$$g^e(x) = \frac{(e^x-1)^2(e^{2x}+1)^2}{4} = \frac{e^{6x} - 2e^{5x} + 3e^{4x} - 4e^{3x} + 3e^{2x} - 2e^x + 1}{4}.$$

Therefore

$$h_n = \frac{6^n - 2 \times 5^n + 3 \times 4^n - 4 \times 3^n + 3 \times 2^n - 2 \times 1}{4}$$

if $n \geq 1$ and $h_0 = 0$.

29. Let h_n, r_n, s_n, t_n denote the number of n-digit numbers that have each digit odd, and the multiplicity of 1 and 3 as shown below:

variable	mult. of 1	mult. of 3
h_n	even	even
r_n	even	odd
s_n	odd	even
t_n	odd	odd

For $n \geq 1$, consider what happens if we remove the first digit of an n-digit number. We find

$$\begin{aligned} h_n &= r_{n-1} + s_{n-1} + 3h_{n-1}, \\ r_n &= t_{n-1} + h_{n-1} + 3r_{n-1}, \\ s_n &= t_{n-1} + h_{n-1} + 3s_{n-1}, \\ t_n &= r_{n-1} + s_{n-1} + 3t_{n-1}. \end{aligned}$$

The initial conditions are

$$h_0 = 1, \qquad r_0 = 0, \qquad s_0 = 0, \qquad t_0 = 0.$$

Using the above data one checks using induction on n that

$$\begin{aligned} h_n &= \frac{5^n + 2 \times 3^n + 1}{4}, \\ r_n = s_n &= \frac{5^n - 1}{4}, \\ t_n &= \frac{5^n - 2 \times 3^n + 1}{4} \end{aligned}$$

for $n \geq 0$. The result follows.

30. Let R_n (resp. r_n) denote the number of ways to color the $1 \times n$ chessboard with colors red, white, and blue, such that red appears with even multiplicity and there is no restriction on blue (resp. blue does not appear). So $h_n = R_n - r_n$. We have $R_0 = 1$, $r_0 = 1$, $h_0 = 0$. Now suppose $n \geq 1$. We show $r_n = 2^{n-1}$. To see this, consider the number of ways to color the $1 \times n$ chessboard with red and white, such that red appears with even multiplicity. We could color squares $2, 3, \ldots, n$ arbitrarily red or white, and then color square 1 red or white in order to make the multiplicity of red even. This shows $r_n = 2^{n-1}$. Now consider what happens if we remove square 1 of the chessboard. We find

$$R_n = 3^{n-1} + R_{n-1}.$$

Therefore

$$R_n = R_0 + 1 + 3 + 3^2 + \cdots + 3^{n-1} = 1 + \frac{3^n - 1}{3 - 1} = \frac{3^n + 1}{2}.$$

Therefore

$$h_n = R_n - r_n = \frac{3^n - 2^n + 1}{2}$$

for $n \geq 1$.

31. One checks by induction on n that $h_n = 0$ if n is even and $h_n = 2^{n-1}$ if n is odd.

32. One checks by induction on n that $h_n = (n+2)!$ for $n \geq 0$.

33. Observe

$$x^3 - x^2 - 9x + 9 = (x-3)(x+3)(x-1).$$

Therefore the general solution is

$$h_n = a3^n + b(-3)^n + c \qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 0$, $h_1 = 1$, $h_2 = 2$ we obtain

$$\begin{aligned} 0 &= a + b + c, \\ 1 &= 3a - 3b + c, \\ 2 &= 9a + 9b + c. \end{aligned}$$

Solving this system we find

$$a = 1/3, \qquad b = -1/12, \qquad c = -1/4.$$

Therefore

$$h_n = \frac{4 \times 3^n - (-3)^n - 3}{12} \qquad n = 0, 1, 2, \ldots$$

34. Observe

$$x^2 - 8x + 16 = (x-4)^2.$$

Therefore the general solution is

$$h_n = (a + bn)4^n \qquad n = 0, 1, 2, \ldots$$

Using $h_0 = -1$, $h_1 = 0$ we obtain

$$a = -1, \qquad b = 1.$$

Therefore

$$h_n = (n-1)4^n \qquad\qquad n = 0, 1, 2, \ldots$$

35. Observe

$$x^3 - 3x + 2 = (x+2)(x-1)^2.$$

Therefore the general solution is

$$h_n = a(-2)^n + bn + c \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 1$, $h_1 = 0$, $h_2 = 0$ we obtain

$$\begin{aligned} 1 &= a + c, \\ 0 &= -2a + b + c, \\ 0 &= 4a + 2b + c. \end{aligned}$$

This yields

$$a = 1/9, \qquad b = -2/3, \qquad c = 8/9.$$

Therefore

$$h_n = \frac{(-2)^n - 6n + 8}{9} \qquad\qquad n = 0, 1, 2, \ldots$$

36. Observe

$$x^4 - 5x^3 + 6x^2 + 4x - 8 = (x-2)^3(x+1).$$

Therefore the general solution is

$$h_n = (an^2 + bn + c)2^n + d(-1)^n \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, $h_3 = 2$ we obtain

$$\begin{aligned} 0 &= c + d, \\ 1 &= 2a + 2b + 2c - d, \\ 1 &= 16a + 8b + 4c + d, \\ 2 &= 72a + 24b + 8c - d. \end{aligned}$$

This yields

$$a = -1/24, \qquad b = 7/72, \qquad c = 8/27, \qquad d = -8/27.$$

Therefore

$$h_n = \frac{(-9n^2 + 21n + 64)2^n - 64(-1)^n}{216} \qquad n = 0, 1, 2, \ldots$$

37. First note that $a_0 = 1$ and $a_1 = 3$. Now assume that $n \geq 2$. We show that $a_n = 2a_{n-1} + a_{n-2}$. Let T_n denote the set of ternary strings of length n that are counted by a_n. For each ternary string s in T_{n-2} the string $22s$ is contained in T_n. Given a ternary string r in T_{n-1} we obtain two ternary strings in T_n as follows: (i) if r begins with 0, then each of $1r, 2r$ is contained in T_n; (ii) if r begins with 1, then each of $0r, 2r$ is contained in T_n; (iii) if r begins with 2, then each of $0r, 1r$ is contained in T_n. Each ternary string in T_n is obtained in exactly one way by the above procedure. Therefore $a_n = 2a_{n-1} + a_{n-2}$. The roots of $x^2 - 2x - 1$ are $1 \pm \sqrt{2}$. Therefore the general solution for a_n is

$$a_n = a(1 + \sqrt{2})^n + b(1 - \sqrt{2})^n \qquad n = 0, 1, 2, \ldots$$

Using $a_0 = 1$ and $a_1 = 3$ we routinely find

$$a = \frac{1 + \sqrt{2}}{2}, \qquad b = \frac{1 - \sqrt{2}}{2}.$$

Therefore

$$a_n = \frac{(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}}{2} \qquad n = 0, 1, 2, \ldots$$

38. (a) $h_n = 3^n$; (b) $h_n = (4 + 5n - n^2)/2$; (c) $h_n = 0$ if n is even and $h_n = 1$ if n is odd; (d) $h_n = 1$; (e) $h_n = 2^{n+1} - 1$.

39. Note that $h_0 = 1$, $h_1 = 1$, $h_2 = 2$. Now assume that $n \geq 3$. We show $h_n = h_{n-1} + h_{n-3}$. Let T_n denote the set of perfect covers of the $1 \times n$ chessboard counted by h_n. For each element of T_{n-1} we can attach a monomino at the left to get an element of T_n that begins with a monomino. For each element of T_{n-3} we can attach a domino followed by a monomino, to get an element of T_n that begins with a domino. Each element of T_n is obtained exactly once by the above procedure. Therefore $h_n = h_{n-1} + h_{n-3}$.

40. One checks $a_0 = 1$ and $a_1 = 3$. Now assume that $n \geq 2$. We show that $a_n = a_{n-1} + 2a_{n-2}$. Let A_n denote the set of ternary strings of length n counted by a_n. Given a ternary string s in A_{n-1} the string $2s$ is contained in A_n. Given a ternary string t in A_{n-2}, each of the strings $02t$, $12t$ are contained in A_n. Each element of A_n is obtained exactly once by the above procedure. Therefore $a_n = a_{n-1} + 2a_{n-2}$. We have $x^2 - x - 2 = (x - 2)(x + 1)$. Therefore the general solution for a_n is

$$a_n = a2^n + b(-1)^n \qquad n = 0, 1, 2, \ldots$$

Using $a_0 = 1$ and $a_1 = 3$ we find

$$a = 4/3, \qquad b = -1/3.$$

Therefore

$$a_n = \frac{2^{n+2} + (-1)^{n+1}}{3} \qquad\qquad n = 0, 1, 2, \ldots$$

41. We have $h_0 = 1$. Now assume that $n \geq 1$. Label the points $1, 2, \ldots, 2n$ clockwise around the circle. Let $M = M_n$ denote the set of matchings of the $2n$ points counted by h_n. For a matching in M let t denote the point matched to point 1. Note that t is even. For $1 \leq s \leq n$ let $M(s)$ denote the set of matchings in M such that point 1 is matched with point $2s$. The sets $\{M(s)\}_{s=1}^n$ partition M, so $|M| = \sum_{s=1}^n |M(s)|$. For $1 \leq s \leq n$ we compute $|M(s)|$. To construct a matching in $M(s)$, there are h_{s-1} ways to match points $2, 3, \ldots, 2s-1$ and there are h_{n-s} ways to match points $2s+1, 2s+2, \ldots, 2n$. Therefore $|M(s)| = h_{s-1}h_{n-s}$. By these comments $h_n = \sum_{s=1}^n h_{s-1}h_{n-s}$. We now show

$$h_n = \frac{1}{n+1}\binom{2n}{n} \qquad\qquad n = 0, 1, 2, \ldots$$

Consider the generating function

$$g(x) = \sum_{n=0}^{\infty} h_n x^n.$$

Using the recursion we obtain

$$xg(x)^2 = g(x) - 1.$$

Using the quadratic formula

$$g(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}.$$

In other words

$$xg(x) = \frac{1 \pm (1-4x)^{1/2}}{2}.$$

Using Newton's binomial theorem this becomes

$$xg(x) = \frac{1 \pm \sum_{n=0}^{\infty} \binom{1/2}{n}(-4)^n x^n}{2}.$$

For this equation at $x = 0$ the left-hand side is zero so the right-hand side is zero. Therefore

$$\begin{aligned} xg(x) &= \frac{1 - \sum_{n=0}^{\infty} \binom{1/2}{n}(-4)^n x^n}{2}, \\ &= -\frac{\sum_{n=1}^{\infty} \binom{1/2}{n}(-4)^n x^n}{2}. \end{aligned}$$

So

$$\begin{aligned} g(x) &= -\frac{\sum_{n=1}^{\infty}\binom{1/2}{n}(-4)^n x^{n-1}}{2} \\ &= -\frac{\sum_{n=0}^{\infty}\binom{1/2}{n+1}(-4)^{n+1} x^{n}}{2}. \end{aligned}$$

Consequently for $n \geq 0$,

$$\begin{aligned} h_n &= -\binom{1/2}{n+1}\frac{(-4)^{n+1}}{2} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

We note

$$h_n = \binom{2n}{n} - \binom{2n}{n-1} \qquad\qquad n = 1, 2, 3, \ldots$$

42. We have $h_0 = 3$ and $h_1 = 16$. For $n \geq 2$,

$$\begin{aligned} h_n - 8h_{n-1} + 16h_{n-2} &= 4h_{n-1} + 4^n - 8h_{n-1} + 16h_{n-2} \\ &= 4^n - 4h_{n-1} + 16h_{n-2} \\ &= 4^n - 4(4h_{n-2} + 4^{n-1}) + 16h_{n-2} \\ &= 0. \end{aligned}$$

The characteristic polynomial is $x^2 - 8x + 16 = (x-4)^2$. Therefore the general solution is

$$h_n = (an + b)4^n \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 3$ and $h_1 = 16$ we find $a = 1$ and $b = 3$. Therefore

$$h_n = (n + 3)4^n \qquad\qquad n = 0, 1, 2, \ldots$$

43. We have $h_0 = 1$ and $h_1 = 10$. Now suppose $n \geq 2$. Using the recursion twice we obtain

$$h_n - 6h_{n-1} + 8h_{n-2} = 0.$$

The characteristic polynomial is $x^2 - 6x + 8 = (x-4)(x-2)$. Therefore the general solution is

$$h_n = a4^n + b2^n \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 1$ and $h_1 = 10$ we find $a = 4$ and $b = -3$. Therefore

$$h_n = 4^{n+1} - 3 \times 2^n \qquad\qquad n = 0, 1, 2, \ldots$$

44. Examining the first few values, it appears that $h_n = 1$ for $n = 0, 1, 2 \ldots$ This is verified by induction.

45. We hunt for solutions of the form

$$h_n = a2^n + bn + c \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 1$, $h_1 = 3$, $h_2 = 8$ we find that

$$a = 3, \qquad b = -1, \qquad c = -2.$$

We now verify that

$$h_n = 3 \times 2^n - n - 2 \qquad\qquad n = 0, 1, 2, \ldots$$

For $n \geq 0$ define $H_n = 3 \times 2^n - n - 2$. One checks that $H_0 = 1$ and

$$H_n = 2H_{n-1} + n \qquad\qquad n = 1, 2, 3, \ldots$$

Therefore $h_n = H_n$ for $n \geq 0$.

46. Noting that $x^2 - 6x + 9 = (x-3)^2$ we hunt for solutions of the form

$$h_n = (an + b)3^n + cn + d \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 1$, $h_1 = 0$, $h_2 = -5$, $h_3 = -24$ we find

$$a = -1/6, \qquad b = -1/2, \qquad c = 1/2, \qquad d = 3/2.$$

Using these values we conjecture

$$h_n = \frac{(3 - 3^n)(3 + n)}{6} \qquad\qquad n = 0, 1, 2, \ldots$$

For $n \geq 0$ let H_n denote the expression on the right-hand side in the above line. One checks $H_0 = 1$, $H_1 = 0$ and

$$H_n = 6H_{n-1} - 9H_{n-2} + 2n \qquad\qquad n = 2, 3, \ldots$$

Therefore $h_n = H_n$ for $n = 0, 1, 2, \ldots$.

47. Noting that $x^2 - 4x + 4 = (x-2)^2$ we hunt for solutions of the form

$$h_n = (an + b)2^n + cn + d \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 1$, $h_1 = 2$, $h_2 = 11$, $h_3 = 46$ we find

$$a = 5, \qquad b = -12, \qquad c = 3, \qquad d = 13.$$

Using these values we conjecture

$$h_n = (5n - 12)2^n + 3n + 13 \qquad\qquad n = 0, 1, 2, \ldots$$

For $n \geq 0$ let H_n denote the expression on the right-hand side in the above line. One checks $H_0 = 1$, $H_1 = 2$ and

$$H_n = 4H_{n-1} - 4H_{n-2} + 3n + 1 \qquad\qquad n = 2, 3, \ldots$$

Therefore $h_n = H_n$ for $n = 0, 1, 2, \ldots$.

48. Define $g(x) = \sum_{n=0}^{\infty} h_n x^n$. Note that $x^r g(x) = \sum_{n=r}^{\infty} h_{n-r} x^n$ for $r \geq 0$.
(a) Using the given information on h_n we find

$$g(x) = x + 4x^2 g(x).$$

Therefore

$$\begin{aligned} g(x) &= \frac{x}{1-4x^2} \\ &= \frac{1}{4(1-2x)} - \frac{1}{4(1+2x)} \\ &= \sum_{n=0}^{\infty} \frac{2^n - (-2)^n}{4} x^n. \end{aligned}$$

Therefore

$$h_n = \frac{2^n - (-2)^n}{4} \qquad\qquad n = 0, 1, 2, \ldots$$

(b) Abbreviate

$$r = \frac{1+\sqrt{5}}{2}, \qquad\qquad s = \frac{1-\sqrt{5}}{2}.$$

Using the given information on h_n we find

$$g(x)(1 - x - x^2) = 1 + 2x.$$

Therefore

$$\begin{aligned} g(x) &= \frac{1+2x}{1-x-x^2} \\ &= \frac{r}{1-rx} + \frac{s}{1-sx} \\ &= \sum_{n=0}^{\infty} (r^{n+1} + s^{n+1}) x^n. \end{aligned}$$

Therefore

$$h_n = r^{n+1} + s^{n+1} \qquad\qquad n = 0, 1, 2, \ldots$$

(c) Using the given information on h_n we find

$$g(x)(1 - x - 9x^2 + 9x^3) = x + x^2.$$

Consequently

$$\begin{aligned} g(x) &= \frac{x+x^2}{1-x-9x^2+9x^3} \\ &= \frac{1}{3(1-3x)} - \frac{1}{12(1+3x)} - \frac{1}{4(1-x)} \\ &= \sum_{n=0}^{\infty} \frac{4\times 3^n - (-3)^n - 3}{12} x^n. \end{aligned}$$

Therefore

$$h_n = \frac{4\times 3^n - (-3)^n - 3}{12} \qquad n = 0, 1, 2, \ldots$$

(d) Using the given information on h_n we find

$$g(x)(1-8x+16x^2) = 8x-1.$$

Consequently

$$\begin{aligned} g(x) &= \frac{8x-1}{1-8x+16x^2} \\ &= \frac{8x-1}{(1-4x)^2} \\ &= (8x-1)\sum_{n=0}^{\infty}(n+1)4^n x^n \\ &= \sum_{n=0}^{\infty}(n-1)4^n x^n. \end{aligned}$$

Therefore

$$h_n = (n-1)4^n \qquad n = 0, 1, 2, \ldots$$

49. For $n \geq 0$ define

$$h_n = \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k} y^k.$$

We show

$$h_n = (x+y)(x+qy)(x+q^2y)\cdots(x+q^{n-1}y).$$

Since $h_0 = 1$ it suffices to show

$$h_n = (x+q^{n-1}y)h_{n-1} \qquad n = 1, 2, 3, \ldots$$

This is obtained using the identity

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^{n-1}\binom{n-1}{k}_q \qquad 1 \le k \le n-1.$$

The above identity is routinely verified.

50. Let E denote the set of extraordinary subsets of $\{1, 2, \ldots, n\}$. For $1 \le k \le n$ let E_k denote the set of elements in E that have cardinality k. The sets $\{E_k\}_{k=1}^n$ partition E so $|E| = \sum_{k=1}^n |E_k|$. For $1 \le k \le n$ we compute $|E_k|$. Consider an element $S \in E_k$. The minimal element of S is k. Therefore S consists of k and a $(k-1)$-subset of $\{k+1, k+2, \ldots, n\}$. There are $\binom{n-k}{k-1}$ ways to choose this $(k-1)$-subset, so $|E_k| = \binom{n-k}{k-1}$. Therefore

$$|E| = \sum_{k=1}^{n} \binom{n-k}{k-1}.$$

Comparing this formula with Theorem 7.1.2 we find $|E| = f_n$.

51. Define $g(x) = \sum_{n=0}^\infty h_n x^n$. Observe

$$\sum_{n=1}^{\infty} h_{n-1}x^n = \sum_{n=0} h_n x^{n+1} = xg(x).$$

Recall

$$\sum_{n=0}^{\infty} nx^n = \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\sum_{n=0}^{\infty}(n+1)x^n = \frac{x}{(1-x)^2}.$$

Observe

$$g(x) - 3xg(x) + \frac{4x}{(1-x)^2} = h_0 = 2.$$

Thus

$$g(x)(1-3x) = 2 - \frac{4x}{(1-x)^2},$$

So

$$\begin{aligned} g(x) &= \frac{2}{1-3x} - \frac{4x}{(1-3x)(1-x)^2} \\ &= \frac{-1}{1-3x} + \frac{3-x}{(1-x)^2} \\ &= -\sum_{n=0}^{\infty} 3^n x^n + 3\sum_{n=0}^{\infty}(n+1)x^n - \sum_{n=0}^{\infty} nx^n \\ &= \sum_{n=0}^{\infty}(2n+3-3^n)x^n. \end{aligned}$$

Therefore

$$h_n = 2n + 3 - 3^n \qquad\qquad n = 0, 1, 2, \ldots$$

52. (a) Note that $h_1 = 11$. We have

$$5^n \;=\; h_n - 2h_{n-1} \qquad\qquad n \geq 1.$$

Replacing n by $n - 1$,

$$5^{n-1} \;=\; h_{n-1} - 2h_{n-2} \qquad\qquad n \geq 2.$$

Combining the above equations we obtain

$$\begin{aligned} 0 &= h_n - 2h_{n-1} - 5(h_{n-1} - 2h_{n-2}) \\ &= h_n - 7h_{n-1} + 10h_{n-2} \qquad\qquad n \geq 2. \end{aligned}$$

For the above homogeneous recurrence the characteristic polynomial is $x^2 - 7x + 10 = (x - 5)(x - 2)$, so it has general solution

$$h_n = a5^n + b2^n \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 3$ and $h_1 = 11$ we find

$$a = 5/3, \qquad\qquad b = 4/3.$$

Therefore

$$h_n = \frac{5^{n+1} + 2^{n+2}}{3} \qquad\qquad n = 0, 1, 2, \ldots$$

(b) Note that $h_1 = 20$. We have

$$5^n \;=\; h_n - 5h_{n-1} \qquad\qquad n \geq 1.$$

Replacing n by $n - 1$,

$$5^{n-1} \;=\; h_{n-1} - 5h_{n-2} \qquad\qquad n \geq 2.$$

Combining the above equations we obtain

$$\begin{aligned} 0 &= h_n - 5h_{n-1} - 5(h_{n-1} - 5h_{n-2}) \\ &= h_n - 10h_{n-1} + 25h_{n-2} \qquad\qquad n \geq 2. \end{aligned}$$

For the above homogeneous recurrence the characteristic polynomial is $x^2 - 10x + 25 = (x - 5)^2$, so it has general solution

$$h_n = (an + b)5^n \qquad\qquad n = 0, 1, 2, \ldots$$

Using $h_0 = 3$ and $h_1 = 20$ we find

$$a = 1, \qquad b = 3.$$

Therefore

$$h_n = (n+3)5^n \qquad n = 0, 1, 2, \ldots$$

53. For $n \geq 0$ we have

$$\begin{aligned} h_n &= 500(1.06)^n + \sum_{k=0}^{n-1} 100(1.06)^k \\ &= 500(1.06)^n + 100\frac{(1.06)^n - 1}{.06}. \end{aligned}$$

The generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$ satisfies

$$g(x) = \left(500 + \frac{100}{.06}\right)\frac{1}{1 - 1.06x} - \frac{100}{.06(1-x)}.$$