Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
Prof: Paul Terwilliger
Selected solutions for Chapter 8

1. See the solution to Problem 41 in Chapter 7.

2. Let P_n denote the set of permutations of the multiset $\{n \cdot 1, n \cdot -1\}$. Let M_n denote the set of $2 \times n$ arrays

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{pmatrix}$$

containing $1, 2, \ldots, 2n$ such that

$$x_{11} < x_{12} < \cdots < x_{1n}, \qquad x_{21} < x_{22} < \cdots < x_{2n}.$$

Given a permutation $a_1 a_2 \cdots a_{2n}$ in P_n, we represent this permutation by an array in M_n as follows. Define the sets

$$R_1 = \{j | 1 \leq j \leq 2n,\ a_j = 1\}, \qquad R_2 = \{j | 1 \leq j \leq 2n,\ a_j = -1\}.$$

Row 1 (resp. row 2) of the array consists of the elements of R_1 (resp. R_2), listed in increasing order. The above representations give a bijection $P_n \to M_n$. Now for a permutation $a_1 a_2 \cdots a_{2n}$ in P_n consider the corresponding array in M_n. The following are equivalent:

(i) the partial sum $a_1 + a_2 + \cdots + a_i \geq 0$ for $1 \leq i \leq 2n$;

(ii) for $1 \leq i \leq n$ the ith 1 comes before the ith -1 in the sequence $a_1 a_2 \cdots a_{2n}$;

(iii) for $1 \leq i \leq n$ the $(1, i)$-entry of the array is less than the $(2, i)$-entry of the array.

The Catalan number C_n counts the number of permutations $a_1 a_2 \cdots a_{2n}$ in P_n that satisfy (i). Therefore C_n counts the number of arrays in M_n that satisfy (iii). The result follows.

3. The multiplication schemes are

$$(((ab)c)d), \qquad ((ab)(cd)), \qquad ((a(bc))d), \qquad (a((bc)d)), \qquad (a(b(cd))).$$

These are in bijection with the triangular decompositions of a convex 5-gon. The bijection is described as follows. Pick an edge of the 5-gon, and label the remaining edges clockwise a, b, c, d. Given one of the above multiplication schemes, attach diagonals to the 5-gon as guided by the parenthesis. The result is a triangular decomposition of the 5-gon.

4. These are readily drawn.

5. We modify the reflection principle discussed in Section 8.1. We first reformulate the problem in terms of ± 1 sequences. Let S denote the set of permutations of the multiset

$\{n\cdot 1, m\cdot -1\}$. Let A denote the set of permutations $a_1a_2\cdots a_{m+n}$ in S such that the partial sum $a_1+\cdots+a_i\geq 0$ for $1\leq i\leq m+n$. We show

$$|A| = \frac{n-m+1}{n+1}\binom{m+n}{m}.$$

Assume that $m\geq 1$; otherwise the result is trivial. Note that

$$\frac{n-m+1}{n+1}\binom{m+n}{m} = \binom{m+n}{m} - \binom{m+n}{m-1}.$$

Let U denote the complement of A in S, so that $|S| = |A|+|U|$. We show

$$|U| = \binom{m+n}{m-1}.$$

Let T denote the set of permutations of the multiset $\{(n+1)\cdot 1, (m-1)\cdot -1\}$. Observe

$$|T| = \binom{m+n}{m-1}.$$

We now display a bijection $f: U\to T$. Given a permutation $a_1a_2\cdots a_{m+n}$ in U, this permuation has at least one negative partial sum. Pick the minimal k such that the kth partial sum is negative. Note that k is odd and $a_k=-1$. Moreover the sequence $a_1a_2\cdots a_{k-1}$ has $(k-1)/2$ 1's and $(k-1)/2$ -1's. Define a sequence $b_1b_2\cdots b_{m+n}$ such that $b_i=-a_i$ for $1\leq i\leq k$ and $b_i=a_i$ for $k+1\leq i\leq m+n$. For the sequence $b_1b_2\cdots b_{m+n}$ the number of 1's and -1's is $n+1$ and $m-1$, respectively. Therefore this sequence is in T. This gives a function $f: U\to T$. By construction f is injective. We now check that f is surjective. Consider a permutation $b_1b_2\cdots b_{m+n}$ in T. For this permutation the number of 1's and -1's is $n+1$ and $m-1$ respectively. Therefore the last partial sum $n+1-(m-1)=2$. Consequently $b_1b_2\cdots b_{m+n}$ has at least one positive partial sum. Pick the minimal k such that the kth partial sum is positive. Note that k is odd and $b_k=1$. Moreover the sequence $b_1b_2\cdots b_{k-1}$ has $(k-1)/2$ 1's and $(k-1)/2$ -1's. Define a sequence $a_1a_2\cdots a_{m+n}$ such that $a_i=-b_i$ for $1\leq i\leq k$ and $a_i=b_i$ for $k+1\leq i\leq m+n$. For the sequence $a_1a_2\cdots a_{m+n}$ the number of 1's and -1's is n and m respectively. Moreover the kth partial sum is -1. Therefore $a_1a_2\cdots a_{m+n}$ is contained in U. By construction f sends $a_1a_2\cdots a_{m+n}$ to $b_1b_2\cdots b_{m+n}$. Therefore f is surjective. We have shown that $f: U\to T$ is a bijection, so $|U|=|T|$. Now

$$|A| = |S|-|U| = |S|-|T| = \binom{m+n}{m} - \binom{m+n}{m-1} = \frac{n-m+1}{n+1}\binom{m+n}{m}.$$

6. The difference table is

$$\begin{array}{ccccccccccccccc}
3 & & 4 & & 9 & & 18 & & 31 & & 48 & & 69 & & \cdots \\
& 1 & & 5 & & 9 & & 13 & & 17 & & 21 & & \cdots & \\
& & 4 & & 4 & & 4 & & 4 & & 4 & & \cdots & & \\
& & & 0 & & 0 & & 0 & & 0 & & \cdots & & & \\
& & & & & & \cdots & & & & & & & &
\end{array}$$

From the diagonal sequence $3, 1, 4, 0, \ldots$ we see that

$$h_n = 3\binom{n}{0} + 1\binom{n}{1} + 4\binom{n}{2} \qquad n = 0, 1, 2, \ldots$$

Therefore

$$\sum_{k=0}^{n} h_k = 3\binom{n+1}{1} + 1\binom{n+1}{2} + 4\binom{n+1}{3} \qquad n = 0, 1, 2, \ldots$$

7. The difference table is

$$\begin{array}{ccccccccccc}
1 & & -1 & & 3 & & 10 & & \cdots \\
& -2 & & 4 & & 7 & & \cdots \\
& & 6 & & 3 & & \cdots \\
& & & -3 & & \cdots \\
& & & & 0 & & \cdots \\
& & & & & \cdots
\end{array}$$

From the diagonal sequence $1, -2, 6, -3, 0, 0, \ldots$ we find

$$h_n = \binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} \qquad n = 0, 1, 2, \ldots$$

Therefore

$$\sum_{k=0}^{n} h_k = \binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4} \qquad n = 0, 1, 2, \ldots$$

8. For the sequence $\{n^5\}_{n=0}^{\infty}$ the difference table is

$$\begin{array}{ccccccccccccc}
0 & & 1 & & 32 & & 243 & & 1024 & & 3125 & & \cdots \\
& 1 & & 31 & & 211 & & 781 & & 2101 & & \cdots \\
& & 30 & & 180 & & 570 & & 1320 & & \cdots \\
& & & 150 & & 390 & & 750 & & \cdots \\
& & & & 240 & & 360 & & \cdots \\
& & & & & 120 & & \cdots \\
& & & & & & 0 & & \cdots
\end{array}$$

From the diagonal sequence $0, 1, 30, 150, 240, 120, 0, 0, \ldots$ we see that

$$n^5 = \binom{n}{1} + 30\binom{n}{2} + 150\binom{n}{3} + 240\binom{n}{4} + 120\binom{n}{5} \qquad n = 0, 1, 2, \ldots$$

So for $n \geq 0$,

$$\sum_{k=0}^{n} k^5 = \binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}$$
$$= \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

9. This is readily checked using Pascal's identity and induction on k.

10. By construction

$$h_n = \sum_{i=0}^{m} c_i \binom{n}{i} \qquad n = 0, 1, 2, \ldots$$

Suppose we are given constants $\{c_i'\}_{i=0}^{m}$ such that

$$h_n = \sum_{i=0}^{m} c_i' \binom{n}{i} \qquad n = 0, 1, 2, \ldots$$

We show $c_i' = c_i$ for $0 \leq i \leq m$. We assume this is not the case, and get a contradiction. Define

$$r = \max\{i | 0 \leq i \leq m,\ c_i' \neq c_i\}.$$

Taking the difference between the above equations,

$$0 = \sum_{i=0}^{r} (c_i' - c_i) \binom{n}{i} \qquad n = 0, 1, 2, \ldots$$

Consider the polynomial

$$f(x) = \sum_{i=0}^{r} (c_i' - c_i) \binom{x}{i}.$$

By construction $0 = f(n)$ for $n = 0, 1, 2, \ldots$. The polynomial $\binom{x}{i}$ has degree exactly i for $i \geq 0$. Therefore the polynomial $f(x)$ has degree exactly r. In particular $f(x)$ is nonzero. A nonzero polynomial has finitely many roots, for a contradiction. Therefore $c_i' = c_i$ for $0 \leq i \leq m$.

11. We have

$$x^8 = [x]_1 + 127[x]_2 + 966[x]_3 + 1701[x]_4 + 1050[x]_5 + 266[x]_6 + 28[x]_7 + [x]_8,$$

where we recall $[x]_k = x(x-1)(x-2)\cdots(x-k+1)$. Therefore

k	0	1	2	3	4	5	6	7	8
$S(8,k)$	0	1	127	966	1701	1050	266	28	1

12. In each case use induction on n along with the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \qquad (0 \le k \le n).$$

13. Write $X = \{1,2,\ldots,p\}$ and $Y = \{1,2,\ldots,k\}$. Let F denote the set of surjective functions $f : X \to Y$. We show $|F| = k!S(n,k)$. To do this we invoke (the proof of) Theorem 8.2.5. Let the set P consist of the partitions of X into k distinguishable boxes $\{B_i\}_{i=1}^k$ such that no box is left empty. By the proof of Theorem 8.2.5, $|P| = k!S(n,k)$. To finish the proof, we display a bijection $F \to P$. Let $f \in F$. So $f : X \to Y$ is a surjective function. Let $\overline{f}$ denote the partition of X into the boxes $\{B_i\}_{i=1}^k$ such that for $1 \le i \le k$, box B_i gets every element of X that f sends to i. No box is left empty since f is surjective; therefore $\overline{f} \in P$. Consider the function $F \to P$, $f \mapsto \overline{f}$. We show that this map is bijective. By construction the map is injective. We check that this map is surjective. Given a partition p of X into boxes $\{B_i\}_{i=1}^k$ such that no box is left empty. Define a function $f : X \to Y$ such that for $j \in X$, $f(j)$ is the label of the box in which p puts j. The function f is surjective since p leaves no box empty. Therefore $f \in F$, and by construction $\overline{f} = p$. We have shown that the function $F \to P$, $f \mapsto \overline{f}$ is surjective and hence a bijection. The result follows.

14. We have

$$\begin{aligned} x^p &= \sum_{t=0}^{p} S(p,t)[x]_t \\ &= \sum_{t=0}^{p} S(p,t)t!\binom{x}{t}. \end{aligned}$$

So for $n \ge 0$,

$$n^p = \sum_{t=0}^{p} S(p,t)t!\binom{n}{t}.$$

Now by Theorem 8.2.3 in the text,

$$\sum_{k=0}^{n} k^p = \sum_{t=0}^{p} S(p,t)t!\binom{n+1}{t+1}.$$

15. Write $X = \{1,2,\ldots,n\}$. Recall that for $0 \le k \le n$, $k!S(n,k)$ counts the number of partitions of X into k distinguishable boxes $\{B_i\}_{i=1}^k$, such that no box is left empty. Let P denote the set of partitions of X into k distinguishable boxes $\{B_i\}_{i=1}^k$ (some of which may be left empty). Note that $|P| = k^n$. For $0 \le t \le k$ let P_t denote the set of partitions in P that leave exactly t boxes nonempty. Then the sets $\{P_t\}_{t=0}^k$ partition P so $|P| = \sum_{t=0}^k |P_t|$. For $0 \le t \le k$ we find $|P_t|$. To construct an element of P_t we proceed in stages:

stage	to do	# choices
1	select t boxes from $\{B_i\}_{i=1}^k$	$\binom{k}{t}$
2	partitition X into the above t boxes, leaving none empty	$t!S(n,t)$

Therefore $|P_t| = \binom{k}{t} t!S(n,t)$. By these comments

$$k^n = |P| = \sum_{t=0}^{k} |P_t| = \sum_{t=0}^{k} \binom{k}{t} t!S(n,t) = \sum_{t=0}^{n} \binom{k}{t} t!S(n,t).$$

16. Using Problem 11,

$$\begin{aligned} B_8 &= \sum_{k=0}^{8} S(8,k) \\ &= 0+1+127+966+1701+1050+266+28+1 \\ &= 4140. \end{aligned}$$

17. Using the recursion

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k) \qquad (0 \le k \le n)$$

we find that for $0 \le k \le n$ the number $s(n,k)$ is the kth entry in row n of the following table:

$$\begin{array}{c} 1 \\ 0 \quad 1 \\ 0 \quad 1 \quad 1 \\ 0 \quad 2 \quad 3 \quad 1 \\ 0 \quad 6 \quad 11 \quad 6 \quad 1 \\ 0 \quad 24 \quad 50 \quad 35 \quad 10 \quad 1 \\ 0 \quad 120 \quad 274 \quad 225 \quad 85 \quad 15 \quad 1 \\ 0 \quad 720 \quad 1764 \quad 1624 \quad 735 \quad 175 \quad 21 \quad 1 \end{array}$$

18. We have

$$\begin{aligned} [n]_5 &= n(n-1)(n-2)(n-3)(n-4) \\ &= 24x - 50x^2 + 35x^3 - 10x^4 + x^5 \end{aligned}$$

and

$$\begin{aligned} [n]_6 &= n(n-1)(n-2)(n-3)(n-4)(n-5) \\ &= -120x + 274x^2 - 225x^3 + 85x^4 - 15x^5 + 1x^6 \end{aligned}$$

and

$$\begin{aligned}[n]_7 &= n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)\\ &= 720x - 1764x^2 + 1624x^3 - 735x^4 + 175x^5 - 21x^6 + x^7.\end{aligned}$$

19. In each case use induction on n along with the recurrence

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k) \qquad (0 \le k \le n).$$

20. By definition

$$[x]_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k.$$

Setting $x = n$ we get

$$n! = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) n^k.$$

For $n = 6$ this becomes

$$6! = 0 \times 1 - 120 \times 6 + 274 \times 6^2 - 225 \times 6^3 + 85 \times 6^4 - 15 \times 6^5 + 1 \times 6^6.$$

21. Routine.

22. (a) The partitions of 6 are

6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111.

Therefore $p_6 = 11$.
(b) The partitions of 7 are

7, 61, 52, 511, 43, 421, 4111, 331, 322, 3211, 31111, 2221, 22111, 211111, 1111111.

Therefore $p_7 = 15$.

23. The maximal partition is n and the minimal partition is $n = 1 + 1 + \cdots + 1$.

24. One checks that the partition n majorizes each partition of n, and the partition $n = 1 + 1 + \cdots + 1$ is majorized by each partition of n. The result follows.

25. We show

$$\prod_{k=1}^{m} (1 - x^{t_k})^{-1} = \sum_{n=0}^{\infty} q_n x^n.$$

Note that for $n \geq 0$, q_n is equal to the number of nonnegative integral solutions $n_1, n_2, \ldots, n_m$ to

$$n_1 t_1 + n_2 t_2 + \cdots + n_m t_m = n.$$

Recall that for $1 \leq k \leq m$,

$$(1 - x^{t_k})^{-1} = 1 + x^{t_k} + x^{2t_k} + \cdots$$

Therefore

$$\begin{aligned}
\prod_{k=1}^{m} (1 - x^{t_k})^{-1} &= \prod_{k=1}^{m} (1 + x^{t_k} + x^{2t_k} + \cdots) \\
&= \left(\sum_{n_1=0}^{\infty} x^{n_1 t_1} \right) \left(\sum_{n_2=0}^{\infty} x^{n_2 t_2} \right) \cdots \left(\sum_{n_m=0}^{\infty} x^{n_m t_m} \right) \\
&= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} x^{n_1 t_1 + n_2 t_2 + \cdots + n_m t_m} \\
&= \sum_{n=0}^{\infty} q_n x^n.
\end{aligned}$$

26. The conjugates are
(a) $12 = 4 + 3 + 2 + 2 + 1$;
(b) $15 = 5 + 3 + 3 + 2 + 1 + 1$;
(c) $20 = 4 + 4 + 4 + 4 + 2 + 2$;
(d) $21 = 6 + 5 + 4 + 3 + 2 + 1$;
(e) $29 = 6 + 6 + 5 + 4 + 3 + 3 + 1 + 1$.

27. For n odd the parts are $(n+1)/2$ (one copy) and 1 ($(n-1)/2$ copies). For n even the parts are $n/2$ (one copy), 2 (one copy), and 1 ($n/2 - 2$ copies).

28. Let us view the Ferrers diagram for λ and μ as contained in a $n \times n$ box and justified to the North-West. Consider the $2n - 1$ NW-SE diagonals in this box. One checks that the following are equivalent:

(i) λ is majorized by μ;

(ii) for each NW-SE diagonal the number of dots in μ that lie on or above the diagonal is at least the number of dots in λ that lie on or above the diagonal.

The result follows from this equivalence.

29. We list the partitions of n into parts each at most 2. For even $n = 2r$ they are

$$1^n, 2^1 1^{n-2}, 2^2 1^{n-4}, \ldots, 2^r 1^0$$

for a total of $r+1$ partitions. For odd $n = 2r+1$ they are

$$1^n, 2^1 1^{n-2}, 2^2 1^{n-4}, \ldots, 2^r 1^1$$

for a total of $r+1$ partitions. In either case the total comes to $\lfloor n/2 \rfloor + 1$.

30. Let P_n denote the set of partitions of n. Given a partition in P_{n-1}, we can add 1 to the first part to get a partition in P_n. This procedure gives an injection $P_{n-1} \to P_n$. The injection is not surjective, because it sends no partition in P_{n-1} to the partition in P_n all of whose parts are 1. It follows that $p_{n-1} < p_n$.

Appendix. The entries of S are Stirling numbers of the second kind:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & 0 & 0 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & 0 & 0 \\ 0 & 1 & 63 & 301 & 350 & 140 & 21 & 1 & 0 \\ 0 & 1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1 \end{pmatrix}$$

The absolute values of the entries of S^{-1} are Stirling numbers of the first kind:

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 11 & -6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 24 & -50 & 35 & -10 & 1 & 0 & 0 & 0 \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 & 0 & 0 \\ 0 & 720 & -1764 & 1624 & -735 & 175 & -21 & 1 & 0 \\ 0 & -5040 & 13068 & -13132 & 6769 & -1960 & 322 & -28 & 1 \end{pmatrix}$$