Spring 2007 Math 510 HW11 Solutions

Section 12.3

4. Give an example of a digraph that does not have a closed Eulerian directed trail but whose underlying graph has a closed Eulerian trail.

Solution. Take the graph G as the complete graph with three vertices (thus G is a triangle). Thus G as a closed Eulerian trial. Now give the orientation so that there is one vertex which is the tail of the both arrows connected to it. With this orientation, this digraph does not have a closed Eulerian directed trail.

6. Prove that a digraph is strongly connected if and only if there is a closed directed walk which contains each vertex at least once.

Solution. If a digraph G has a closed direct walk containing each vertex at least once, then for any two vertices v_1 and v_2, there is a segment of the directed walk starts at v_1 and ends at v_2. On this walk, whenever there is vertex that is repeated, the segment walk between the repeated vertex is a closed walk and removing the edges of the closed walk between the repeated vertex would result a short directed walk. The shortest walk resulted by repeatedly removing edges from closed walks must be a path. Thus there is a directed path from v_1 and v_2. Hence the digraph is strongly linked.

Conversely, suppose that the digraph G is strongly connected. We construct a closed walk containing all vertices of G adding more vertices to a closed walk till all vertices are included in the closed walk.

Since G is strongly connected, for any two vertices v and u there is a closed walk containing both u and v. This walk can be contructed connecting a path (or walk) from v to u and a path from u to v. Now assume that

$$\gamma = (v_1 \to v_2 \to \cdots \to v_m^{\to} v_1)$$

is a closed walk. If there is a vertex z of G which is not contained in the closed walk γ, then there is a closed walk

$$(v_m v_{m+1} \to \cdots v_{m+k} \to z \to z_1 \to \cdots , \to z_l \to v_m).$$

Then glueing them at v_m together to get a walk

$$(v_1 \to v_2 \to \cdots \to v_m^{\to} v_{m+1} \to \cdots \to v_{m+k} \to z \to z_1 \to \cdots \to z_l \to v_m \to v_1)$$

is a closed walk containing all vertices of γ and z. Replacing γ by this closed walk and repeate the process if there is a vertex not yet contained in γ. Since G has only finitely many vertices, we must stop with all vertices being contained in γ.

9. Prove that a tournament is strongly connected if and only if it has a directed Hamilton cycle.

Solution. Using the same (simpler) proof as in problem 6, we have that, if G has a directed Hamilton cycle (thus a directed closed walk containing every vertex at least once), then G is strongly connected. We now prove the converse that G has direct Hamilton cycle assuming that G is strongly connected. Since G is a tournament, Theorem 12.1.5 implies tha there is an open Hamilton path there is

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$$2(\text{n-1})= d_1 + d_2 + \cdots + d_{n-k} + d_{n-k+1} + \cdots + d_n = d_1 + (d_2 + \cdots + d_{n-k}) + k$$
$$\geq d_1 + 2(n - k - 1) + k = d_1 + 2(n - 1) - k.$$

From $2(n - 1) \geq d_1 + 2(n - 1) - k$ we have $0 \geq d_1 - k$ or $k \geq d_1$. Now for any vertex v with $p = \deg(v)$ we have $k \geq d_1 \geq p$, i.e., there are at least p pendent vertices.