算法基础
Foundation of Algorithms

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Chapter 19 Binomial Heap (二项堆, in v2)

19.1 Priority Queue and Union op

19.2 Binomial Trees and Binomial Heap

19.3 Operations on a Binomial Heap

The slides are from Prof. LeongHonWai@NUS and taoli@FIU

19.1 Priority Queue and Union op

- Priority queue
- Various implementations
- Comparison of efficiency
- Union operation

Priority Queue

- *Priority Queue* is an ADT （抽象数据类型） for maintaining a set S of elements, each with a *key* value and supports the following operations:

  - INSERT(S, x) — *inserts* element x into S (also write as $S \leftarrow S \cup \{x\}$)
  - MINIMUM(S) — returns element in S with *min* key
  - EXTRACT-MIN(S) — removes and returns element in S with *min* key
  - DECREASE-KEY(S, x, k) — *decreases* the value of element x's key to a *new value* k

PQ Implementations…

- Many data structures proposed for PQ:

1964	Binary Heap	*J. W. J. Williams*
1972	Leftist Heap	*C. A. Crane*
1978	Binomial Heap	*J. Vuillemin*
1984	Fibonacci Heap	*M. L. Fredman, R. E. Tarjan*
1985	Skew Heap	*D. D. Sleator R. E. Tarjan*
1988	Relaxed Heap	*Driscoll, Gabow Shrairman, Tarjan*

Binary Min-Heap (as in Heapsort)

- *Binary min-heap* is an *array* $A[1..n]$ that can be viewed as a nearly complete *binary tree.*
- Number the nodes using level order traversal.
  - LEFT(i) = $2i$ and RIGHT(i) = $2i+1$ and
  - PARENT(i) = $\lfloor i/2 \rfloor$
  - Height of tree $\approx \log n$
- Heap Property: (Each node ≥ its parent node)
  - $A[\text{PARENT}(i)] \leq A[i]$

PQ Implementations…

- Time Bounds for different PQ implementations.
  - n is the number of items in the PQ.

Data Str	INSERT	MIN	Extract -MIN	D-KEY	DELETE	Union
Binary H	$O(\lg n)$	$O(1)$	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$	O(n)
Binomial H	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$
Fibonacci	$O(1)$	$O(1)$	$O(\lg n)$	$O(1)$	$O(\lg n)$	$O(1)$

Comparison of Efficiency

Procedure	Binary (worst-case)	Binomial (worst-case)	Fibonacci (amortized)
Make-Heap	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(\lg n)$	$O(\lg n)$	$\Theta(1)$
Minimum	$\Theta(1)$	$O(\lg n)$	$\Theta(1)$
Extract-Min	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$
Union	$\Theta(n)$	$O(\lg n)$	$\Theta(1)$
Decrease-Key	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$
Delete	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$

Union Operation

- A *mergeable heap* *（可合并堆）* is any data structure that supports the basic heap operation *plus union*.
- Union (H1, H2) creates and returns a new heap.

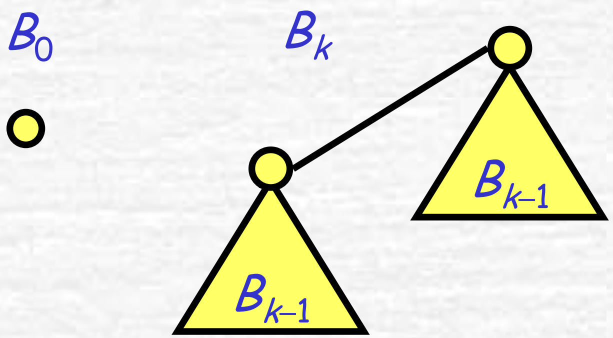
Chapter 19 Binomial Heap (二项堆, in v2)

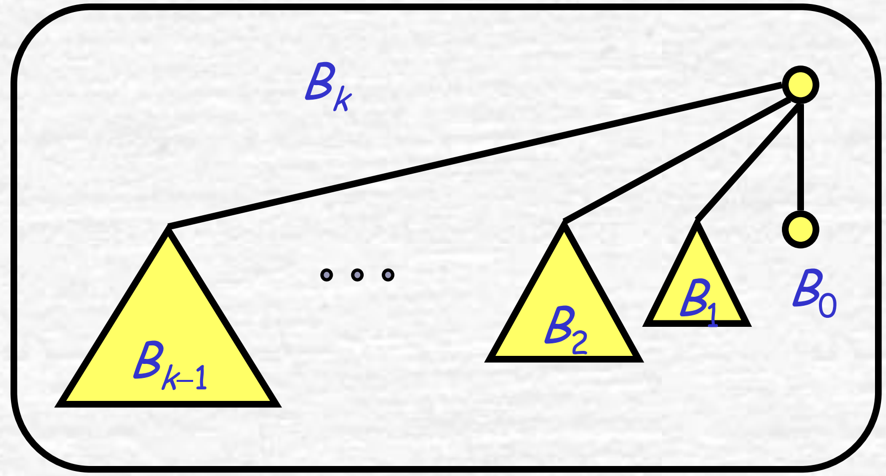
19.2 Binomial Trees and Binomial Heap

- Binomial trees （二项树）
- Properties of binomial trees
- Binomial heaps
- Representing binomial heaps

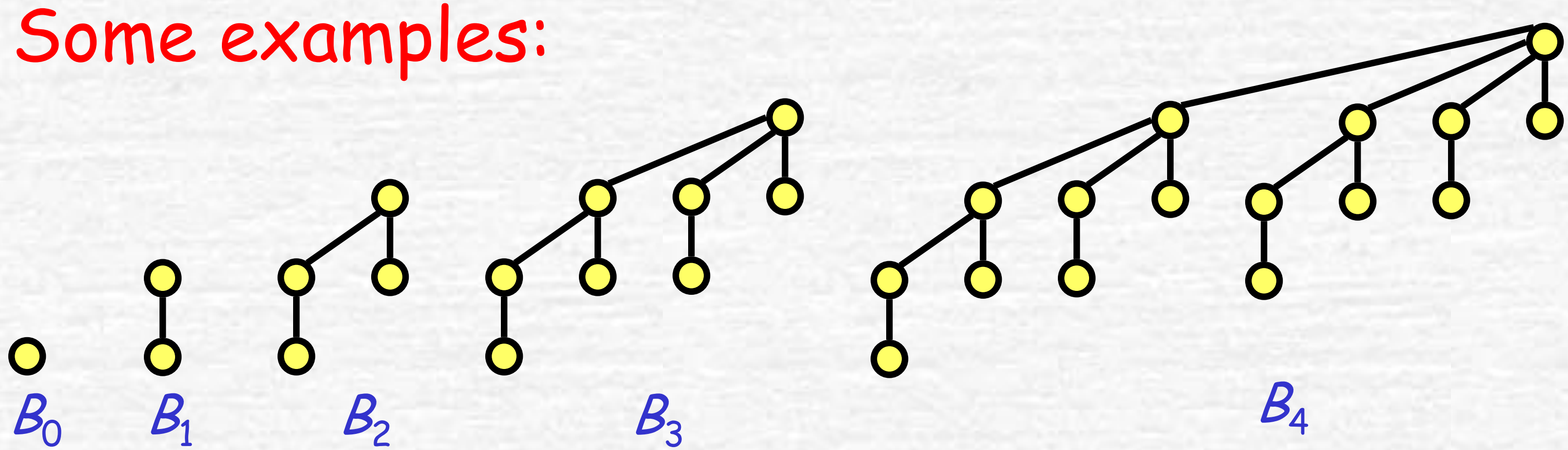
Binomial Trees

Recursive definition:





Some examples:

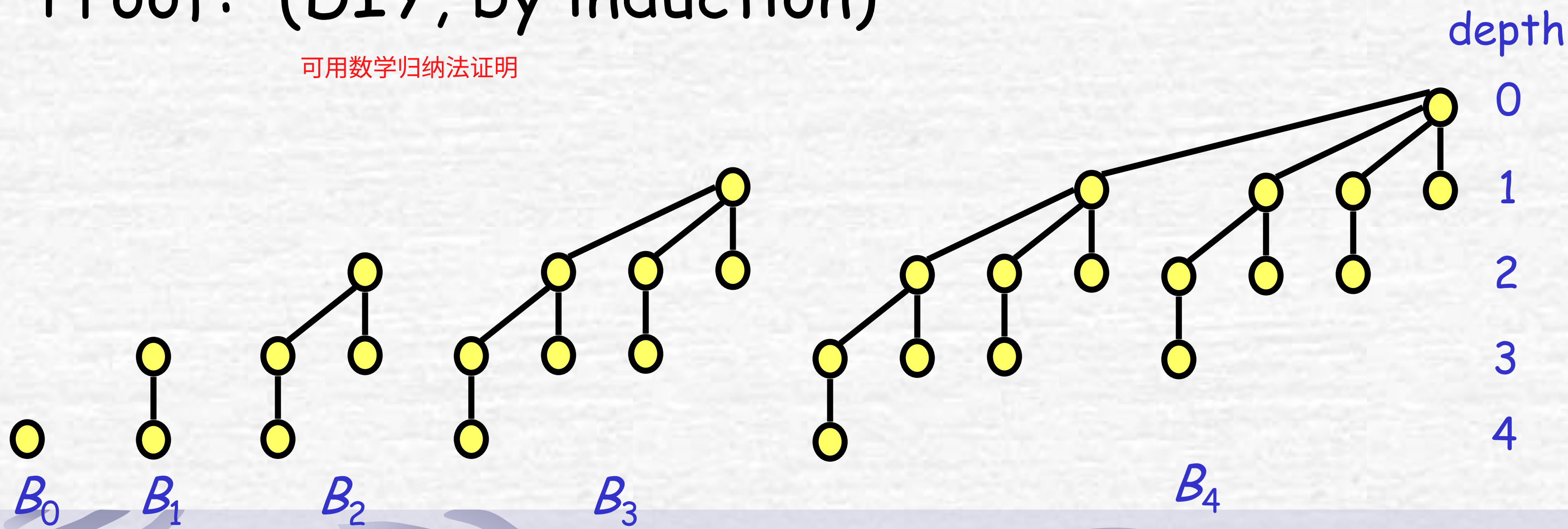


Properties of Binomial Trees

- For a Binomial Tree B_k (of order k)
  1. there are 2^k nodes,
  2. the height of the tree is k, 从0开始记
  3. root has degree k and
  4. deleting the root gives binomial trees $B_0, B_1, \ldots, B_{k-1}$.

Proof: (DIY, by induction)

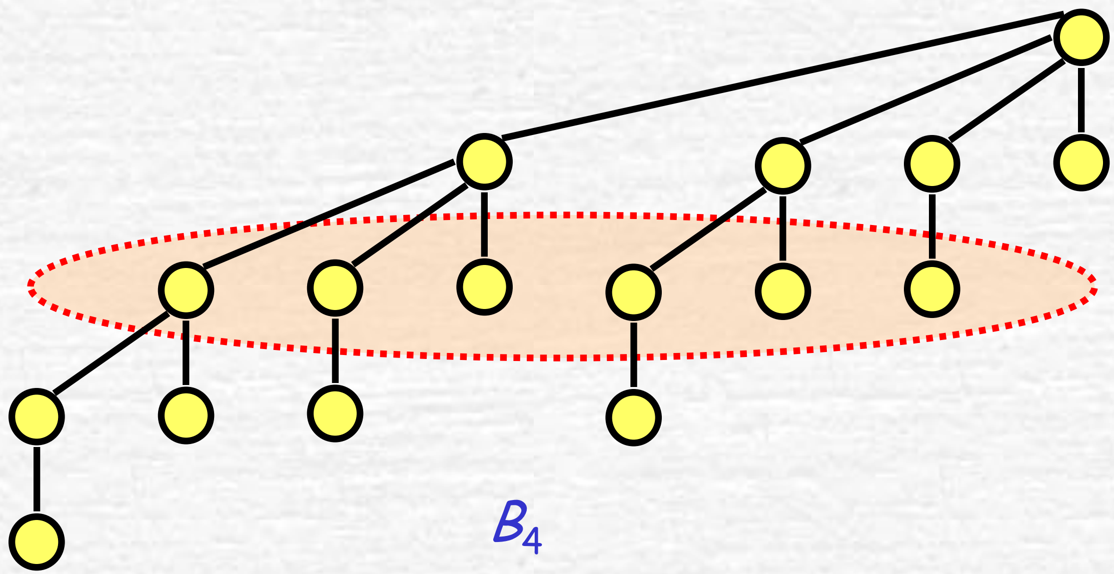
可用数学归纳法证明



Defining Property of Binomial Trees

There are exactly $\binom{k}{i}$ nodes at depth i, for B_k 组合数

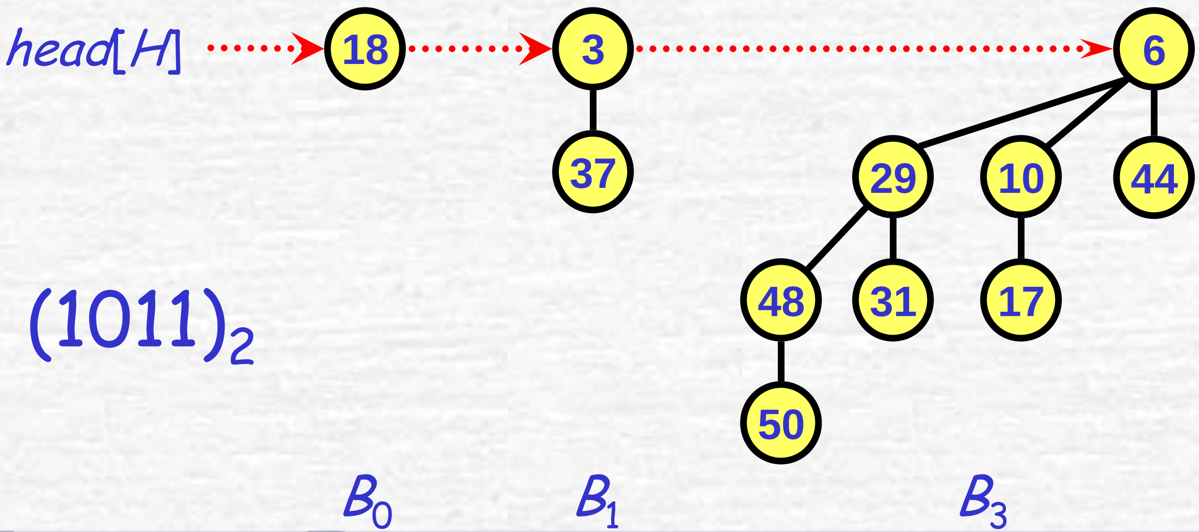
$$(0 \le i \le k)$$



depth	Nodes
0	1
1	4
2	6
3	4
4	1

Binomial Heap (Vuillemin, 1978)

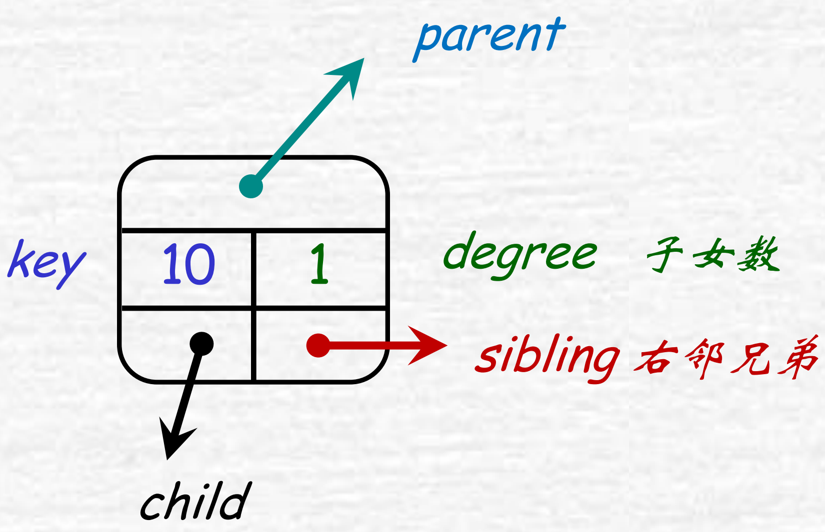
- A sequence of binomial trees that satisfy
  - binomial heap property (each tree B_k is a min-heap)
  - 0 or 1 binomial tree B_k of order k,
- There are at most $\lfloor \log n \rfloor + 1$ binomial trees.
- Eg: A binomial heap H with $n = 11$ nodes.



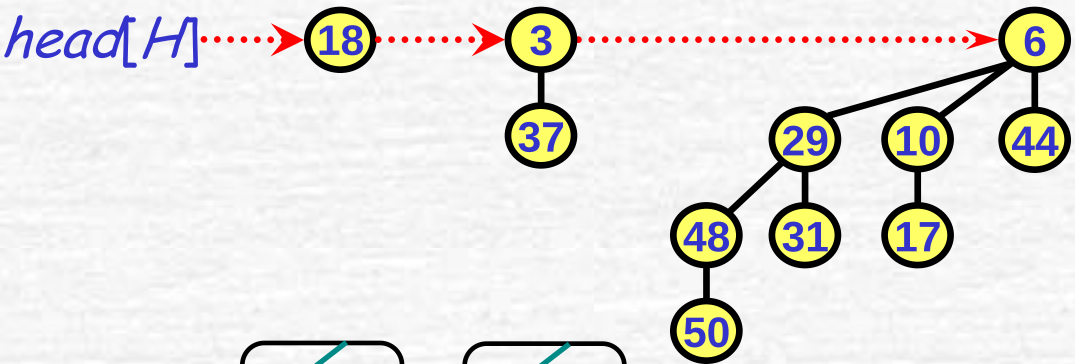
$11 = (1011)_2$

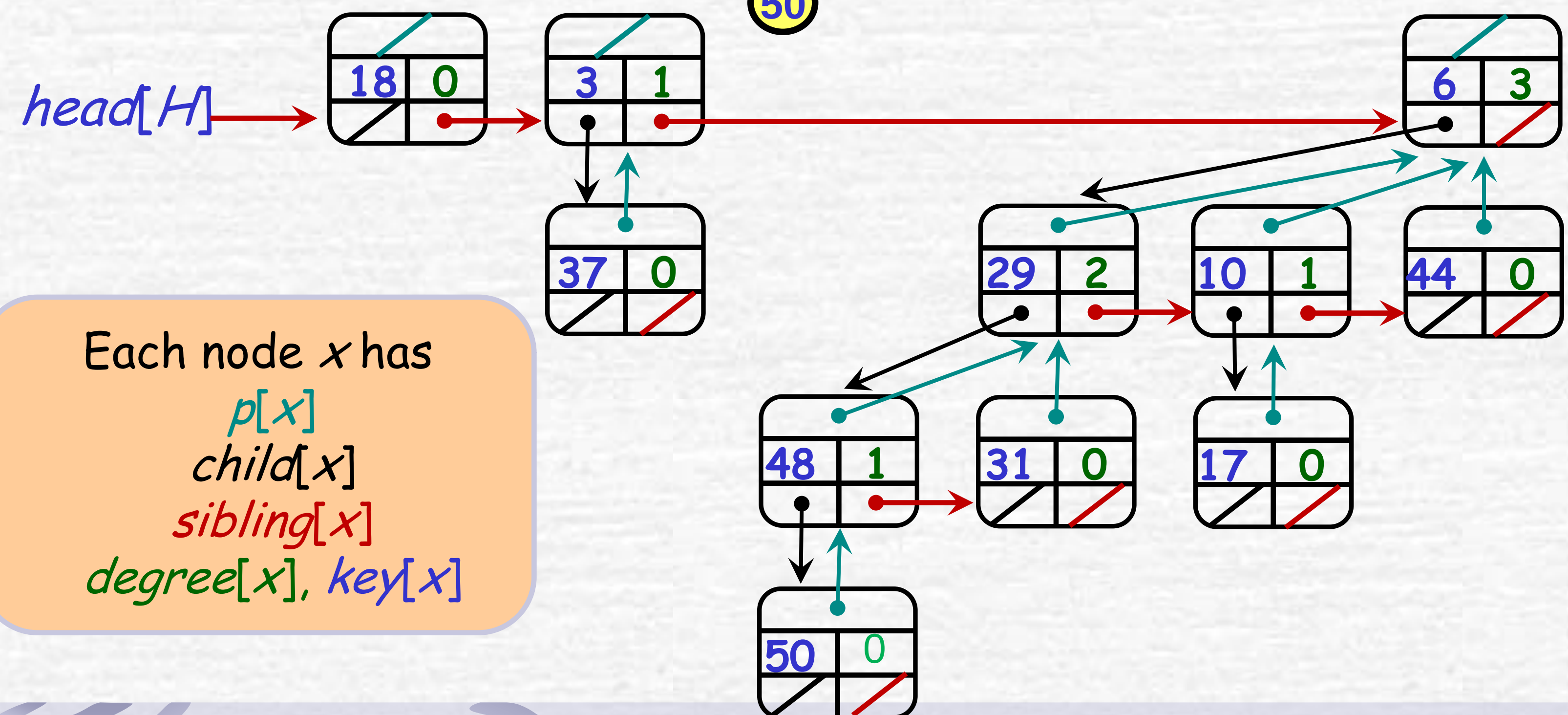
Representing Binomial Heaps (1)

- Each node *x* stores
  - *key*[*x*]
  - *degree*[*x*]
  - *p*[*x*]
  - *child*[*x*]
  - *sibling*[*x*]
- (3 pointers per node)



Representing Binomial Heaps (2)





Each node *x* has
p[*x*]
child[*x*]
sibling[*x*]
degree[*x*], *key*[*x*]

Chapter 19 Binomial Heap (二项堆, in v2)

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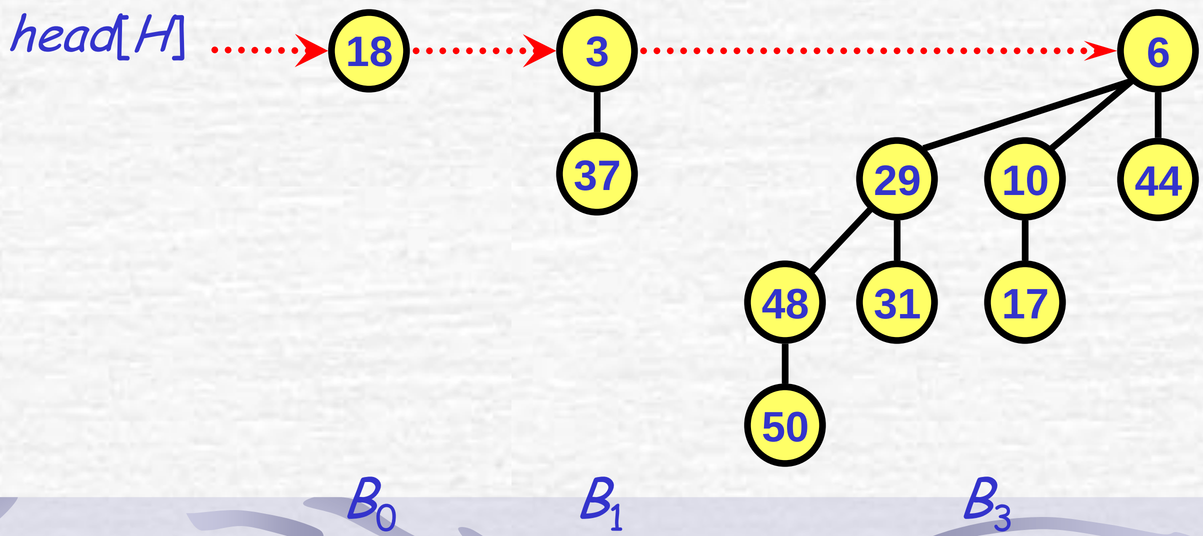
19.3 Operations on a Binomial Heap

- MAKE and MINIMUM
- Linking Step: Fundamental Op
- Binomial Heap Union
- More Operations
- Summary

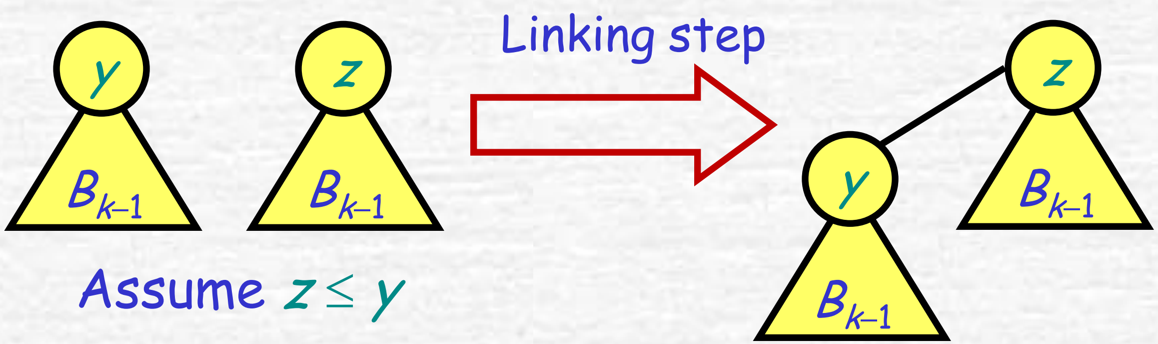
MAKE and MINIMUM

- MAKE-BINOMIAL-HEAP(H)
  - Allocate object H, make $head[H]$ = NIL. $\Theta(1)$.
- BINOMIAL-HEAP-MINIMUM(H)
  - Search the root list for minimum. $O(\log n)$.



Linking Step: Fundamental Op

- BINOMIAL-LINK (y, z)



BINOMIAL-LINK (y, z) ▹ *Assume* $z \le y$

$p[y] \leftarrow z$

$sibling[y] \leftarrow child[z]$

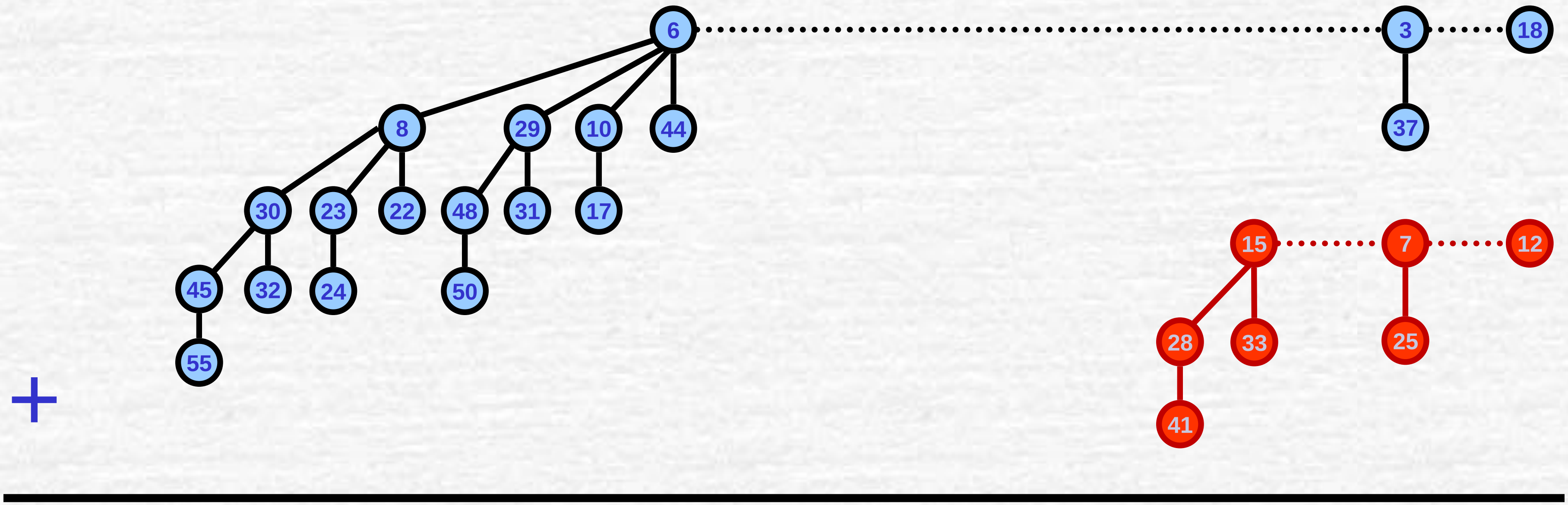
$child[z] \leftarrow y$

$degree[z] \leftarrow degree[z] + 1$

Constant time $O(1)$

Binomial Heap Union（1）

Let us look at the procedure of an example:



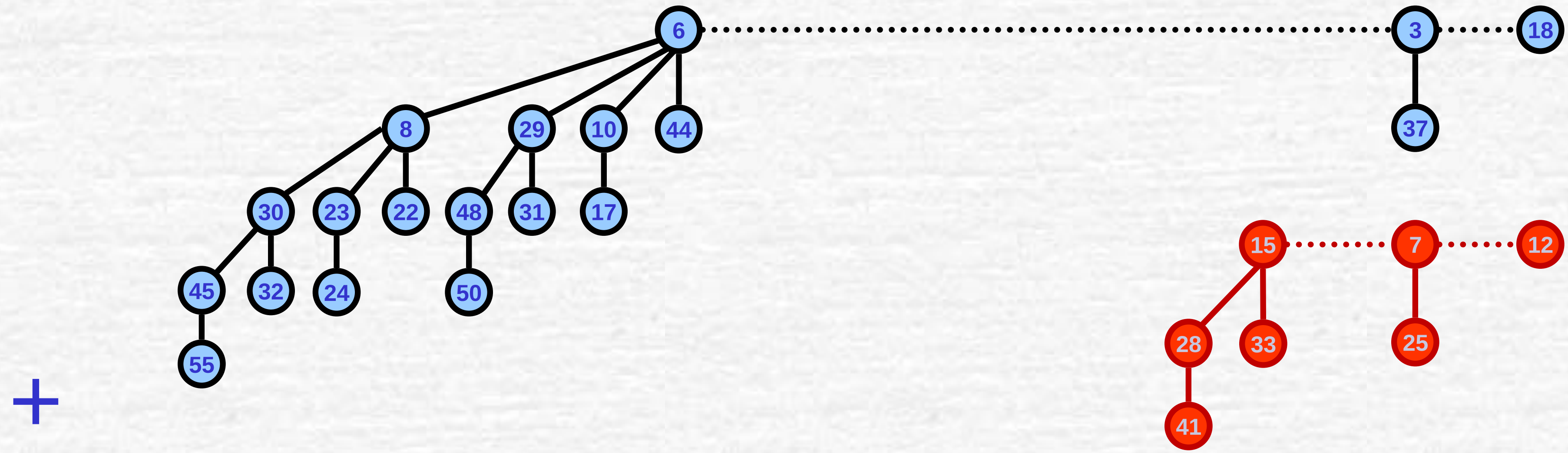
19 + 7 = 26

		1	*1*	*1*	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

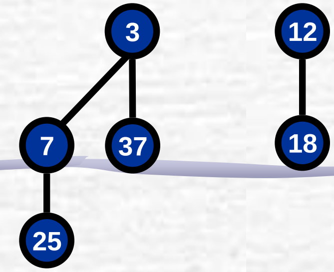
The binomial trees in the Binomial Heap at last: B_1,B_3,B_4

Binomial Heap Union

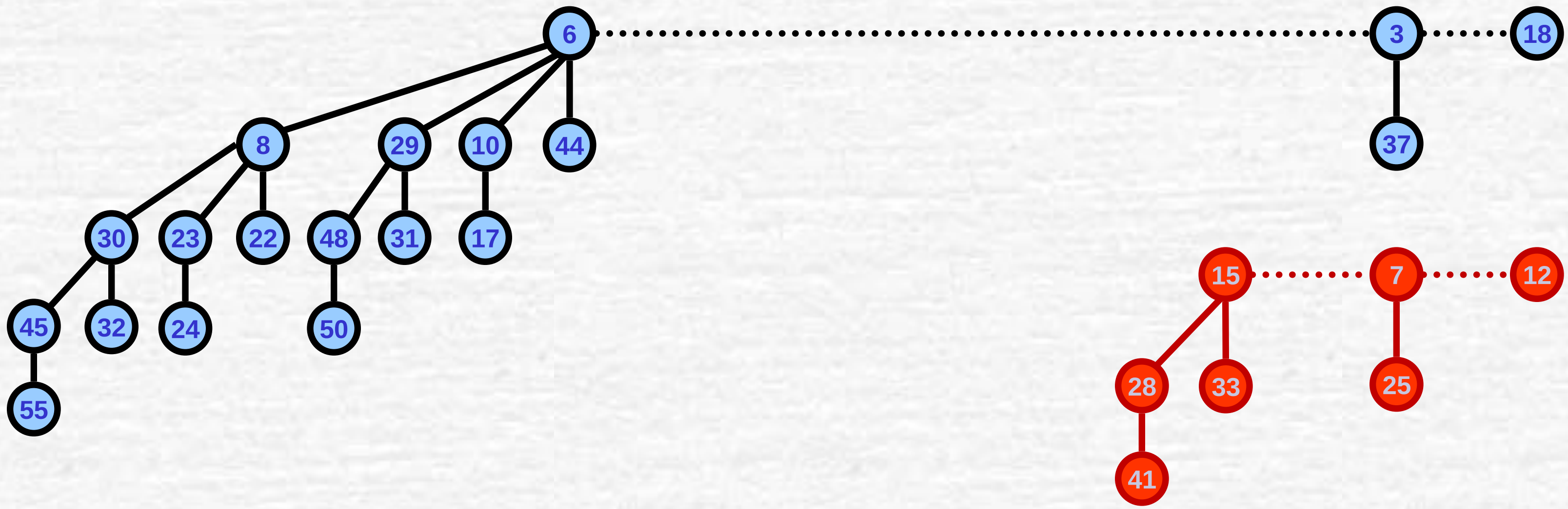
Temporary area:



Stable area:



Temporary area:



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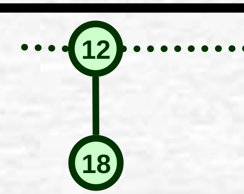
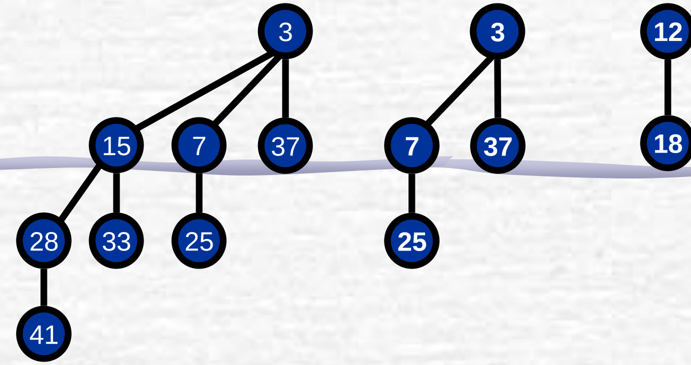
Stable area:





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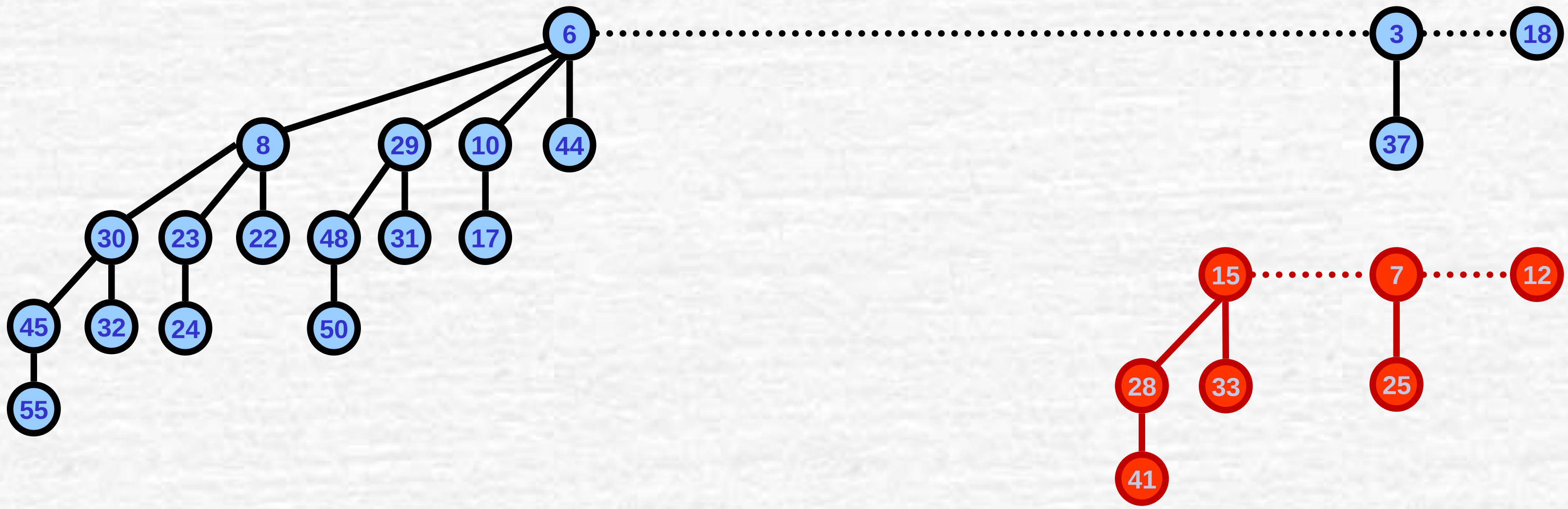

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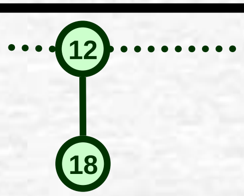
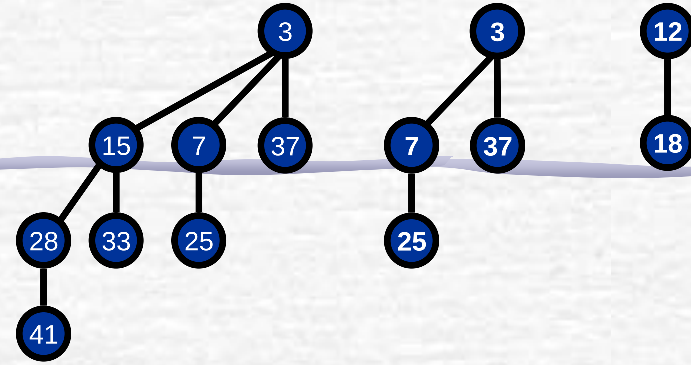
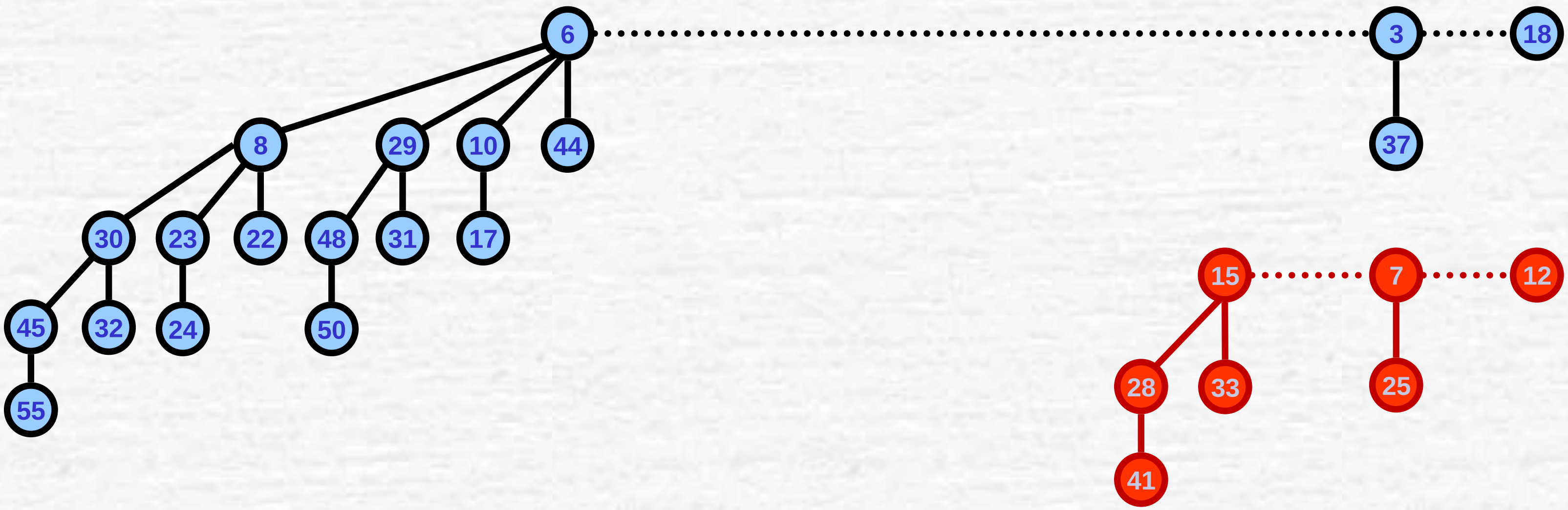
Stable area:





Temporary area:



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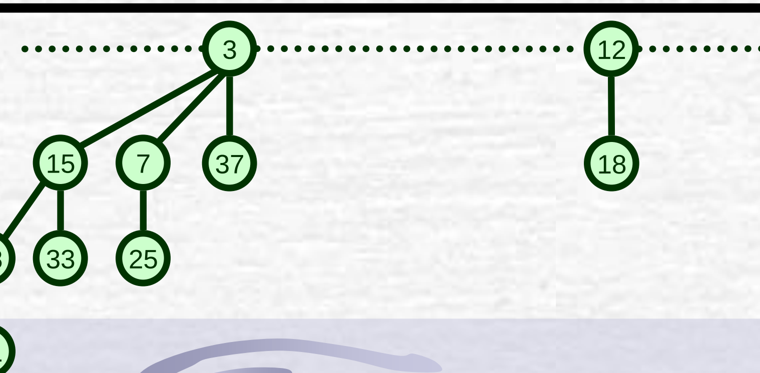
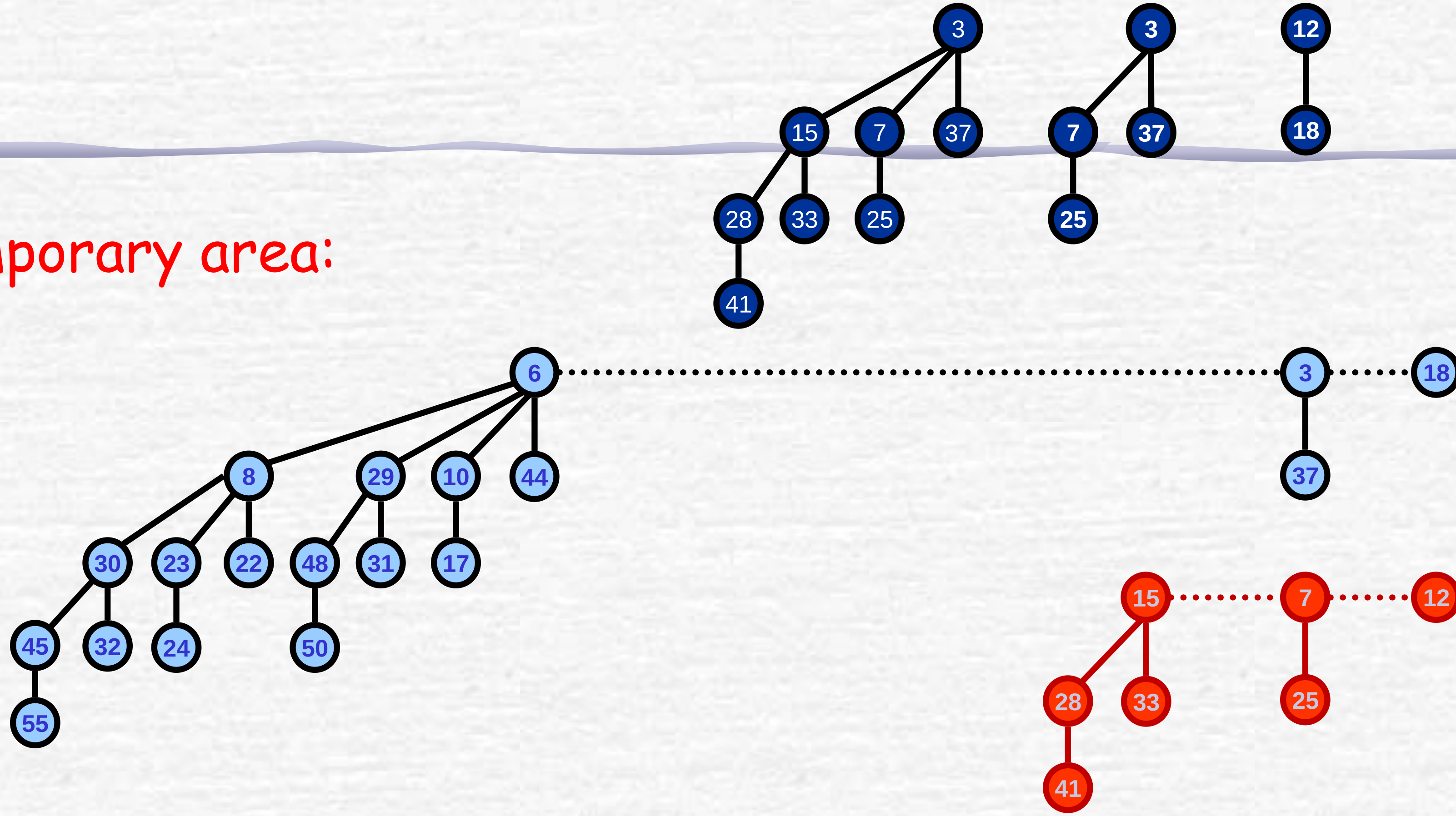
Stable area:



Temporary area:



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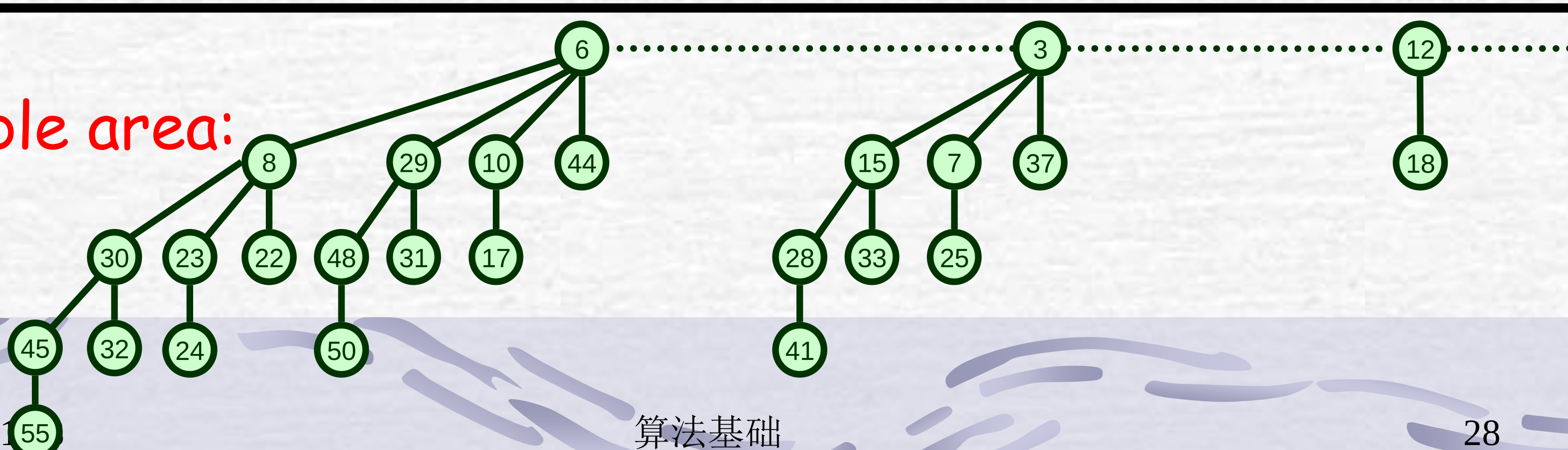
Stable area:



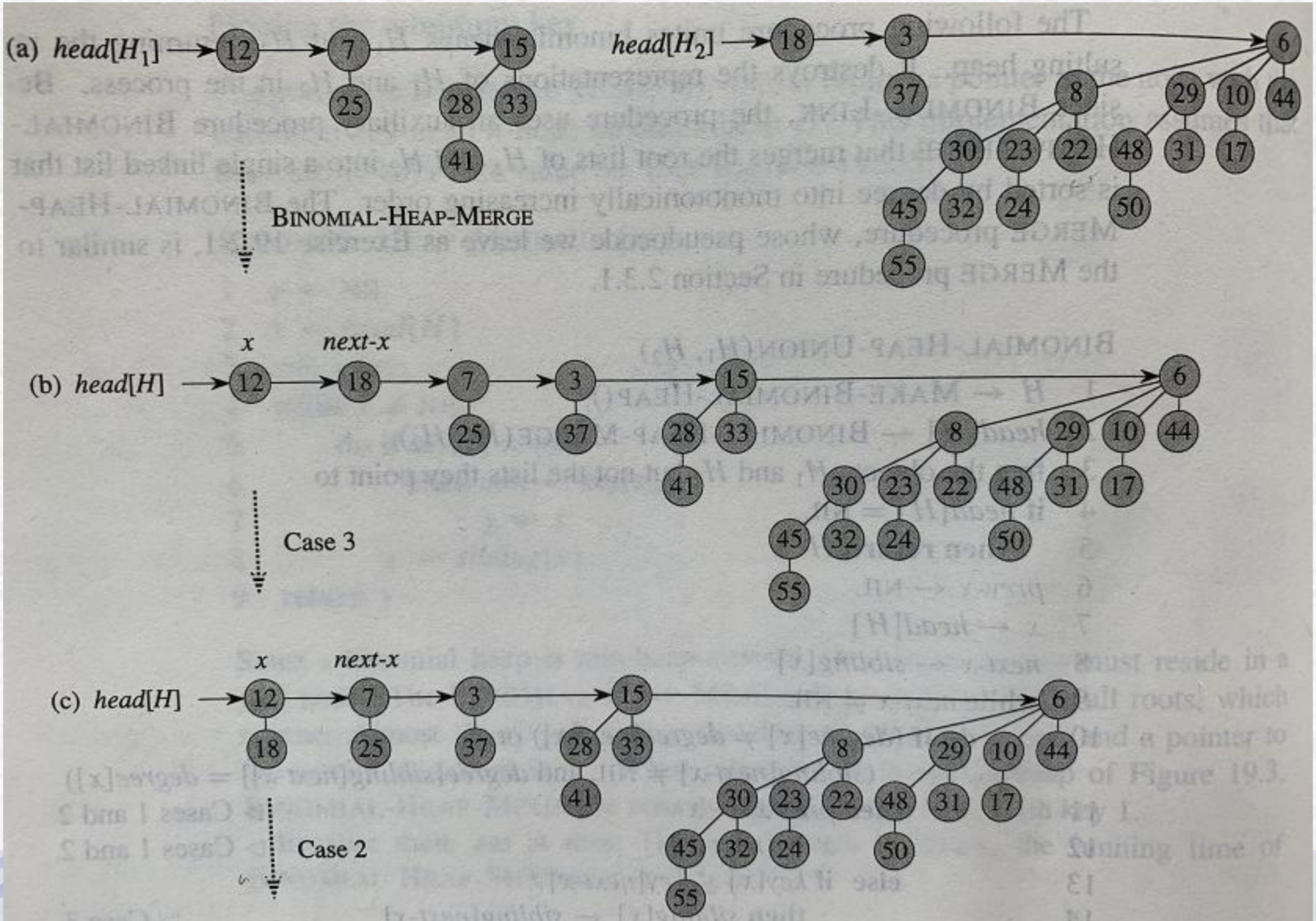
```
BINOMIAL-HEAP-UNION(H1, H2)
H ← MAKE-BINOMIAL-HEAP()
head[H] ← BINOMIAL-HEAP-MERGE(H1, H2)
free the objects H1 and H2 but not the lists they point to
if head[H] = NIL
   then return H
prev-x ← NIL
x ← head[H]
next-x ← sibling[x]
while next-x ≠ NIL
      do if (degree[x] ≠ degree[next-x]) or
               (sibling[next-x] ≠ NIL and degree[sibling[next-x]] = degree[x])
            then prev-x ← x                                   ▷ Cases 1 and 2
                 x ← next-x                                   ▷ Cases 1 and 2
            else if key[x] ≤ key[next-x]
                    then sibling[x] ← sibling[next-x]         ▷ Case 3
                         BINOMIAL-LINK(next-x, x)             ▷ Case 3
            else if prev-x = NIL                              ▷ Case 4
                    then head[H] ← next-x                     ▷ Case 4
                    else sibling[prev-x] ← next-x             ▷ Case 4
                 BINOMIAL-LINK(x, next-x)                     ▷ Case 4
                 x ← next-x                                   ▷ Case 4
         next-x ← sibling[x]
return H
```

Binomial Heap Union（3）

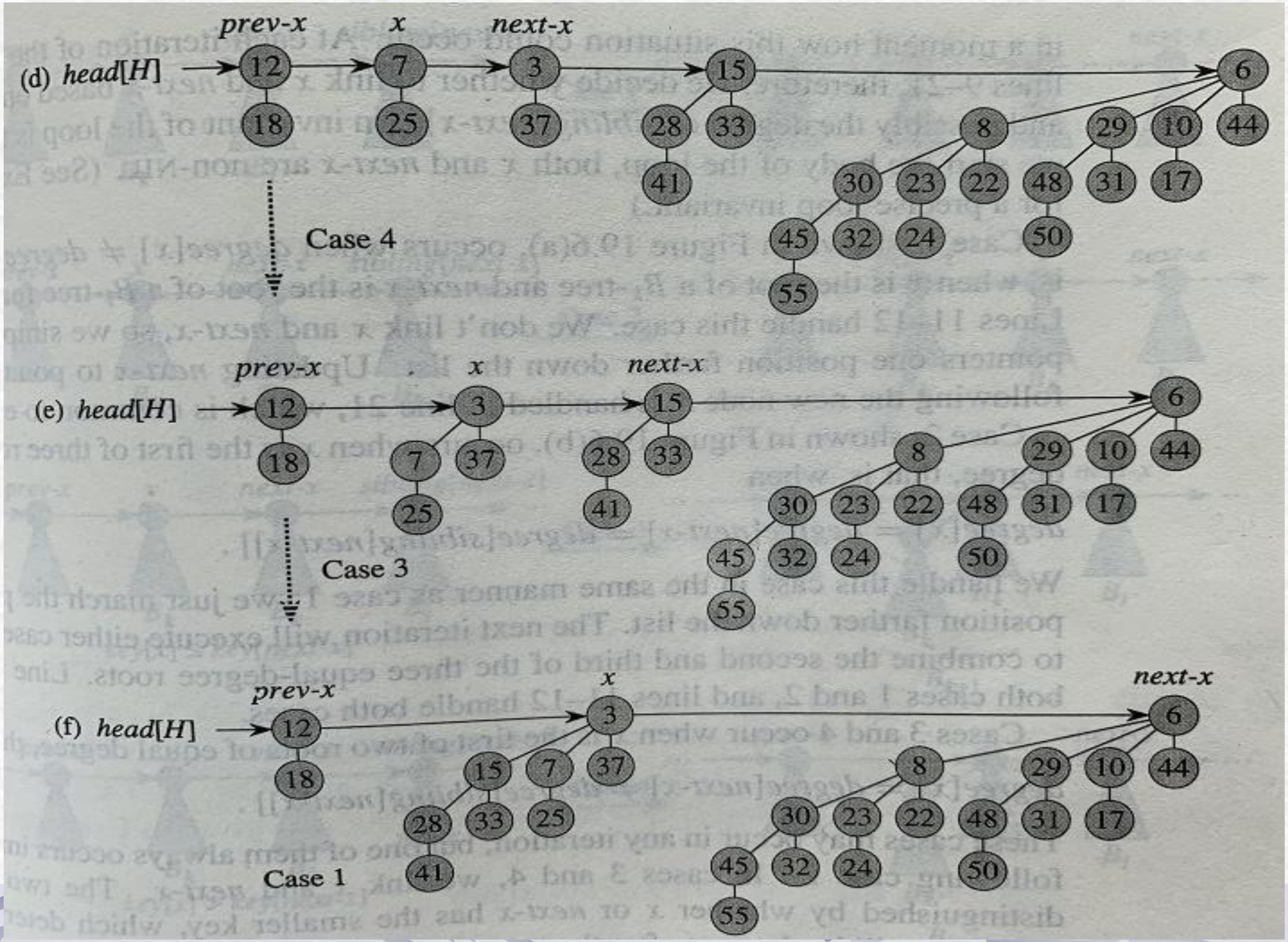
Case classification：

$$
degree[x]\ \begin{array}{l}
\neq degree[\text{next-}x] \quad case\ 1 \\
= degree[\text{next-}x]\ \begin{array}{l}
= degree[sibling[\text{next-}x]] \quad case\ 2 \\
\neq degree[sibling[\text{next-}x]]\ and\ \begin{array}{l}
key[x] \leq key[\text{next-}x] \quad case\ 3 \\
key[x] > key[\text{next-}x] \quad case\ 4
\end{array}
\end{array}
\end{array}
$$

Binomial Heap Union（4）



Binomial Heap Union（5）



Binomial Heap Union（6）

- MAKE-BINOMIAL-HEAP-UNION (H_1, H_2) :
  - Create a heap H that is the union of two heaps H_1 and H_2
  - Analogous to binary addition of n_1 and n_2
- Running time.: $O(\log n)$ $[n = n_1 + n_2]$

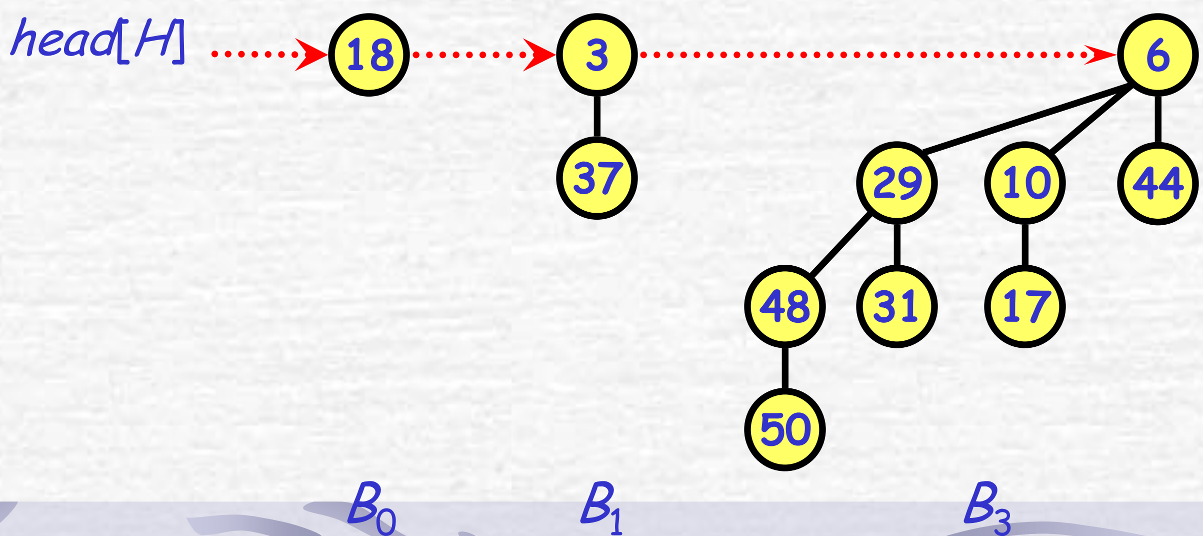
19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

More Operations (1)

- BINOMIAL-HEAP-INSERT(H, x)
  - Create a one-item (x) binomial heap H_1 and then union H and H_1. $O(\lg n)$.
- BINOMIAL-HEAP-EXTRACT-MIN(H)
  - Find minimum, remove root, then union. $O(\lg n)$.

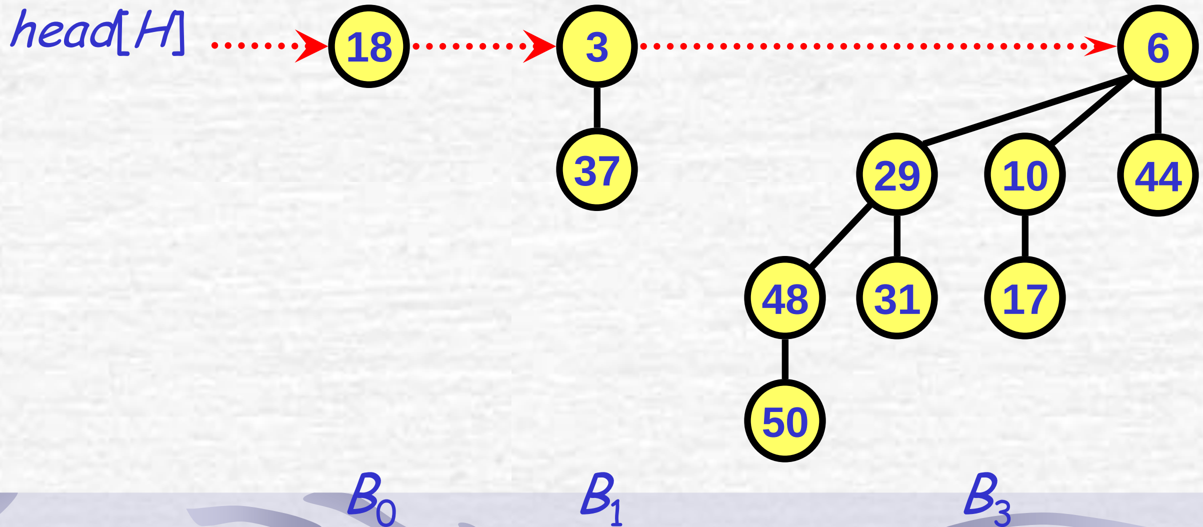
删除最小结点，形成一个新的二项堆，union两个二项堆



More Operations (2)

- BINOMIAL-HEAP-DECREASE (H, x, k) 与整堆操作相似
- BINOMIAL-HEAP-DELETE (H, x)

首先将x置为负无穷，整堆后，再删除最小结点（那个负无穷结点）



Summary

- MINIMUM(H) $O(\lg n)$
- UNION(H_1, H_2) $O(\lg n)$
- INSERT(H, x) $O(\lg n)$
- EXTRACT-MIN(H) $O(\lg n)$
- DECREASE-KEY (H, x, k) $O(\lg n)$
- DELETE (H, x) $O(\lg n)$

End of Ch19