Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 3

1. For $1 \leq k \leq 22$ we show that there exists a succession of consecutive days during which the grandmaster plays exactly k games. For $1 \leq i \leq 77$ let b_i denote the number of games played on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + k\}_{i=0}^{76} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{77}$. There are 154 numbers in the list, all among $1, 2, \ldots, 153$. Therefore the numbers $\{b_1 + b_2 + \cdots + b_i + k\}_{i=0}^{76} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{77}$. are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 77)$ such that $b_{i+1} + \cdots + b_j = k$. During the days $i + 1, \ldots, j$ the grandmaster plays exactly k games.

2. Let S denote a set of 100 integers chosen from $1, 2, \ldots, 200$ such that i does not divide j for all distinct $i, j \in S$. We show that $i \notin S$ for $1 \leq i \leq 15$. Certainly $1 \notin S$ since 1 divides every integer. By construction the odd parts of the elements in S are mutually distinct and at most 199. There are 100 numbers in the list $1, 3, 5, \ldots, 199$. Therefore each of $1, 3, 5, \ldots, 199$ is the odd part of an element of S. We have $3 \times 5 \times 13 = 195 \in S$. Therefore none of $3, 5, 13, 15$ are in S. We have $3^3 \times 7 = 189 \in S$. Therefore neither of $7, 9$ is in S. We have $11 \times 17 = 187 \in S$. Therefore $11 \notin S$. We have shown that none of $1, 3, 5, 7, 9, 11, 13, 15$ is in S. We show neither of $6, 14$ is in S. Recall $3^3 \times 7 = 189 \in S$. Therefore $3^2 \times 7 = 63 \notin S$. Therefore $2 \times 3^2 \times 7 = 126 \in S$. Therefore $2 \times 3 = 6 \notin S$ and $2 \times 7 = 14 \notin S$. We show $10 \notin S$. Recall $3 \times 5 \times 13 = 195 \in S$. Therefore $5 \times 13 = 65 \notin S$. Therefore $2 \times 5 \times 13 = 130 \in S$. Therefore $2 \times 5 = 10 \notin S$. We now show that none of $2, 4, 8, 12$ are in S. Below we list the integers of the form $2^r 3^s$ that are at most 200:

$$
\begin{array}{llllllll}
1, & 2, & 4, & 8, & 16, & 32, & 64, & 128, \\
3, & 6, & 12, & 24, & 48, & 96, & 192, & \\
9, & 18, & 36, & 72, & 144, & & & \\
27, & 54, & 108, & & & & & \\
81, & 162. & & & & & &
\end{array}
$$

In the above array each element divides everything that lies to the southeast. Also, each row contains exactly one element of S. For $1 \leq i \leq 5$ let r_i denote the element of row i that is contained in S, and let c_i denote the number of the column that contains r_i. We must have $c_i < c_{i-1}$ for $2 \leq i \leq 5$. Therefore $c_i \geq 6 - i$ for $1 \leq i \leq 5$. In particular $c_1 \geq 5$ so $r_1 \geq 16$, and $c_2 \geq 4$ so $r_2 \geq 24$. We have shown that none of $2, 4, 8, 12$ is in S. By the above comments $i \notin S$ for $1 \leq i \leq 15$.

3. See the course notes.

4, 5, 6. Given integers $n \geq 1$ and $k \geq 2$ suppose that $n + 1$ distinct elements are chosen from $\{1, 2, \ldots, kn\}$. We show that there exist two that differ by less than k. Partition $\{1, 2, \ldots, nk\} = \cup_{i=1}^{n} S_i$ where $S_i = \{ki, ki - 1, ki - 2, \ldots, ki - k + 1\}$. Among our $n + 1$ chosen elements, there exist two in the same S_i. These two differ by less than k.

7. Partition the set $\{0, 1, \ldots, 99\} = \cup_{i=0}^{50} S_i$ where $S_0 = \{0\}$, $S_i = \{i, 100 - i\}$ for $1 \leq i \leq 49$, $S_{50} = \{50\}$. For each of the given 52 integers, divide by 100 and consider the remainder. The remainder is contained in S_i for a unique i. By the pigeonhole principle, there exist two of the 52 integers for which these remainders lie in the same S_i. For these two integers the sum or difference is divisible by 100.

8. For positive integers m, n we consider the rational number m/n. For $0 \leq i \leq n$ divide the integer $10^i m$ by n, and call the remainder r_i. By construction $0 \leq r_i \leq n - 1$. By the pigeonhole principle there exist integers i, j $(0 \leq i < j \leq n)$ such that $r_i = r_j$. The integer n divides $10^j m - 10^i m$. For notational convenience define $\ell = j - i$. Then there exists a positive integer q such that $nq = 10^i(10^\ell - 1)m$. Divide q by $10^\ell - 1$ and call the remainder r. So $0 \leq r \leq 10^\ell - 2$. By construction there exists an integer $b \geq 0$ such that $q = (10^\ell - 1)b + r$. Writing $\theta = m/n$ we have

$$\begin{aligned} 10^i \theta &= b + \frac{r}{10^\ell - 1} \\ &= b + \frac{r}{10^\ell} + \frac{r}{10^{2\ell}} + \frac{r}{10^{3\ell}} + \cdots \end{aligned}$$

Since the integer r is in the range $0 \leq r \leq 10^\ell - 2$ this yields a repeating decimal expansion for θ.

9. Consider the set of 10 people. The number of subsets is $2^{10} = 1024$. For each subset consider the sum of the ages of its members. This sum is among $0, 1, \ldots, 600$. By the pigeonhole principle the 1024 sums are not distinct. The result follows. Now suppose we consider at set of 9 people. Then the number of subsets is $2^9 = 512 < 600$. Therefore we cannot invoke the pigeonhole principle.

10. For $1 \leq i \leq 49$ let b_i denote the number of hours the child watches TV on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. There are 98 numbers in the list, all among $1, 2, \ldots, 96$. By the pigeonhole principle the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 49)$ such that $b_{i+1} + \cdots + b_j = 20$. During the days $i + 1, \ldots, j$ the child watches TV for exactly 20 hours.

11. For $1 \leq i \leq 37$ let b_i denote the number of hours the student studies on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + 13\}_{i=0}^{36} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{37}$. There are 74 numbers in the list, all among $1, 2, \ldots, 72$. By the pigeonhole principle the numbers $\{b_1 + b_2 + \cdots + b_i + 13\}_{i=0}^{36} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{37}$ are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 37)$ such that $b_{i+1} + \cdots + b_j = 13$. During the days $i + 1, \ldots, j$ the student will have studied exactly 13 hours.

12. Take $m = 4$ and $n = 6$. Pick a among $0, 1, 2, 3$ and b among $0, 1, 2, 3, 4, 5$ such that $a + b$ is odd. Suppose that there exists a positive integer x that yields a remainder of a (resp. b) when divided by 4 (resp. by 6). Then there exist integers r, s such that $x = 4r + a$ and $x = 6s + b$. Combining these equations we obtain $2x - 4r - 6s = a + b$. In this equation the

left-hand side is even and the right-hand side is odd, for a contradiction. Therefore x does not exist.

13. Since $r(3,3) = 6$ there exists a K_3 subgraph of K_6 that is red or blue. We assume that this K_3 subgraph is unique, and get a contradiction. Without loss we may assume that the above K_3 subgraph is red. Let x denote one of the vertices of this K_3 subgraph, and let $\{x_i\}_{i=1}^5$ denote the remaining five vertices of K_6. Consider the K_5 subgraph with vertices $\{x_i\}_{i=1}^5$. By assumption this subgraph has no K_3 subgraph that is red or blue. The only edge coloring of K_5 with this feature is shown in figure 3.2 of the text. Therefore we may assume that the vertices $\{x_i\}_{i=1}^5$ are labelled such that for distinct i, j $(1 \leq i, j \leq 5)$ the edge connecting x_i, x_j is red (resp. blue) if $i - j = \pm 1$ modulo 5 (resp. $i - j = \pm 2$ modulo 5). By construction and without loss of generality, we may assume that each of x_1, x_2 is connected to x by a red edge. Thus the vertices x, x_1, x_2 give a red K_3 subgraph. Now the edge connecting x and x_3 is blue; otherwise the vertices x, x_2, x_3 give a second red K_3 subgraph. Similarly the edge connecting x and x_5 is blue; otherwise the vertices x, x_1, x_5 give a second red K_3 subgraph. Now the vertices x, x_3, x_5 give a blue K_3 subgraph.

14. After n minutes we have removed n pieces of fruit from the bag. Suppose that among the removed fruit there are at most 11 pieces for each of the four kinds. Then our total n must be at most $4 \times 11 = 44$. After $n = 45$ minutes we will have picked at least a dozen pieces of fruit of the same kind.

15. For $1 \leq i \leq n+1$ divide a_i by n and call the reminder r_i. By construction $0 \leq r_i \leq n-1$. By the pigeonhole principle there exist distinct integers i, j among $1, 2, \ldots, n+1$ such that $r_i = r_j$. Now n divides $a_i - a_j$.

16. Label the people $1, 2, \ldots, n$. For $1 \leq i \leq n$ let a_i denote the number of people aquainted with person i. By construction $0 \leq a_i \leq n-1$. Suppose the numbers $\{a_i\}_{i=1}^n$ are mutually distinct. Then for $0 \leq j \leq n-1$ there exists a unique integer i $(1 \leq i \leq n)$ such that $a_i = j$. Taking $j = 0$ and $j = n-1$, we see that there exists a person aquainted with nobody else, and a person aquainted with everybody else. These people are distinct since $n \geq 2$. These two people know each other and do not know each other, for a contradiction. Therefore the numbers $\{a_i\}_{i=1}^n$ are not mutually distinct.

17. We assume that the conclusion is false and get a contradiction. Label the people $1, 2, \ldots, 100$. For $1 \leq i \leq 100$ let a_i denote the number of people aquainted with person i. By construction $0 \leq a_i \leq 99$. By assumption a_i is even. Therefore a_i is among $0, 2, 4, \ldots, 98$. In this list there are 50 numbers. Now by our initial assumption, for each even integer j $(0 \leq j \leq 98)$ there exists a unique pair of integers (r, s) $(1 \leq r < s \leq 100)$ such that $a_r = j$ and $a_s = j$. Taking $j = 0$ and $j = 98$, we see that there exist two people who know nobody else, and two people who know everybody else except one. This is a contradiction.

18. Divide the 2×2 square into four 1×1 squares. By the pigeonhole principle there exists a 1×1 square that contains at least two of the five points. For these two points the distance apart is at most $\sqrt{2}$.

19. Divide the equilateral triangle into a grid, with each piece an equilateral triangle of side length $1/n$. In this grid there are $1+3+5+\cdots+2n-1=n^2$ pieces. Suppose we place $m_n=n^2+1$ points within the equilateral triangle. Then by the pigeonhole principle there exists a piece that contains two or more points. For these two points the distance apart is at most $1/n$.

20. Color the edges of K_{17} red or blue or green. We show that there exists a K_3 subgraph of K_{17} that is red or blue or green. Pick a vertex x of K_{17}. In K_{17} there are 16 edges that contain x. By the pigeonhole principle, at least 6 of these are the same color (let us say red). Pick distinct vertices $\{x_i\}_{i=1}^6$ of K_{17} that are connected to x via a red edge. Consider the K_6 subgraph with vertices $\{x_i\}_{i=1}^6$. If this K_6 subgraph contains a red edge, then the two vertices involved together with x form the vertex set of a red K_3 subgraph. On the other hand, if the K_6 subgraph does not contain a red edge, then since $r(3,3)=6$, it contains a K_3 subgraph that is blue or green. We have shown that K_{17} has a K_3 subgraph that is red or blue or green.

21. Let X denote the set of sequences (a_1,a_2,a_3,a_4,a_5) such that $a_i\in\{1,-1\}$ for $1\le i\le 5$ and $a_1a_2a_3a_4a_5=1$. Note that $|X|=16$. Consider the complete graph K_{16} with vertex set X. We display an edge coloring of K_{16} with colors red, blue, green such that no K_3 subgraph is red or blue or green. For distinct x,y in X consider the edge connecting x and y. Color this edge red (resp. blue) (resp. green) whenever the sequences x,y differ in exactly 4 coordinates (resp. differ in exactly 2 coordinates i,j with $i-j=\pm1$ modulo 5) (resp. differ in exactly 2 coordinates i,j with $i-j=\pm2$ modulo 5). Each edge of K_{16} is now colored red or blue or green. For this edge coloring of K_{16} there is no K_3 subgraph that is red or blue or green.

22. For an integer $k\ge 2$ abbreviate $r_k=r(3,3,\ldots,3)$ (k 3's). We show that $r_{k+1}\le(k+1)(r_k-1)+2$. Define $n=r_k$ and $m=(k+1)(n-1)+2$. Color the edges of K_m with $k+1$ colors $C_1,C_2,\ldots,C_{k+1}$. We show that there exists a K_3 subgraph with all edges the same color. Pick a vertex x of K_m. In K_m there are $m-1$ edges that contain x. By the pigeonhole principle, at least n of these are the same color (which we may assume is C_1). Pick distinct vertices $\{x_i\}_{i=1}^n$ of K_m that are connected to x by an edge colored C_1. Consider the K_n subgraph with vertices $\{x_i\}_{i=1}^n$. If this K_n subgraph contains an edge colored C_1, then the two vertices involved together with x give a K_3 subgraph that is colored C_1. On the other hand, if the K_n subgraph does not contain an edge colored C_1, then since $r_k=n$, it contains a K_3 subgraph that is colored C_i for some i $(2\le i\le k+1)$. In all cases K_m has a K_3 subgraph that is colored C_i for some i $(1\le i\le k+1)$. Therefore $r_k\le m$.

23. We proved earlier that

$$r(m,n)\le\binom{m+n-2}{n-1}.$$

Applying this result with $m=3$ and $n=4$ we obtain $r(3,4)\le 10$.

24. We show that $r_t(t,t,q_3)=q_3$. By construction $r_t(t,t,q_3)\ge q_3$. To show the reverse inequality, consider the complete graph with q_3 vertices. Let X denote the vertex set of this

graph. Color the t-element subsets of X red or blue or green. Then either (i) there exists a t-element subset of X that is red, or (ii) there exists a t-element subset of X that is blue, or (iii) every t-element subset of X is green. Therefore $r_t(t, t, q_3) \leq q_3$ so $r_t(t, t, q_3) = q_3$.

25. Abbreviate $N = r_t(m, m, \ldots, m)$ (k m's). We show $r_t(q_1, q_2, \ldots, q_k) \leq N$. Consider the complete graph K_N with vertex set X. Color each t-element subset of X with k colors $C_1, C_2, \ldots, C_k$. By definition there exists a K_m subgraph all of whose t-element subsets are colored C_i for some i ($1 \leq i \leq k$). Since $q_i \leq m$ there exists a subgraph of that K_m with q_i vertices. For this subgraph every t-element subset is colored C_i.

26. In the $m \times n$ array assume the rows (resp. columns) are indexed in increasing order from front to back (resp. left to right). Consider two adjacent columns $j-1$ and j. A person in column $j-1$ and a person in column j are called *matched* if they occupy the same row of the original formation. Thus a person in column j is taller than their match in column $j-1$. Now consider the adjusted formation. Let L and R denote adjacent people in some row i, with L in column $j-1$ and R in column j. We show that R is taller than L. We assume that L is at least as tall as R, and get a contradiction. In column $j-1$, the people in rows $i, i+1, \ldots, m$ are at least as tall as L. In column j, the people in rows $1, 2, \ldots, i$ are at most as tall as R. Therefore everyone in rows $i, i+1, \ldots, m$ of column $j-1$ is at least as tall as anyone in rows $1, 2, \ldots, i$ of column j. Now for the people in rows $1, 2, \ldots, i$ of column j their match stands among rows $1, 2, \ldots, i-1$ of column $j-1$. This contradicts the pigeonhole principle, so L is shorter than R.

27. Let $s_1, s_2, \ldots, s_k$ denote the subsets in the collection. By assumption these subsets are mutually distinct. Consider their complements $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$. These complements are mutually distinct. Also, none of these complements are in the collection. Therefore $s_1, s_2, \ldots, s_k$, $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$ are mutually distinct. Therefore $2k \leq 2^n$ so $k \leq 2^{n-1}$. There are at most 2^{n-1} subsets in the collection.

28. The answer is 1620. Note that $1620 = 81 \times 20$. First assume that $\sum_{i=1}^{100} a_i < 1620$. We show that no matter how the dance lists are selected, there exists a group of 20 men that cannot be paired with the 20 women. Let the dance lists be given. Label the women $1, 2, \ldots, 20$. For $1 \leq j \leq 20$ let b_j denote the number of men among the 100 that listed woman j. Note that $\sum_{j=1}^{20} b_j = \sum_{i=1}^{100} a_i$ so $(\sum_{j=1}^{20} b_j)/20 < 81$. By the pigeonhole principle there exists an integer j ($1 \leq j \leq 20$) such that $b_j \leq 80$. We have $100 - b_j \geq 20$. Therefore there exist at least 20 men that did not list woman j. This group of 20 men cannot be paired with the 20 women.
Consider the following selection of dance lists. For $1 \leq i \leq 20$ man i lists woman i and no one else. For $21 \leq i \leq 100$ man i lists all 20 women. Thus $a_i = 1$ for $1 \leq i \leq 20$ and $a_i = 20$ for $21 \leq i \leq 100$. Note that $\sum_{i=1}^{100} a_i = 20 + 80 \times 20 = 1620$. Note also that every group of 20 men can be paired with the 20 women.

29. Without loss we may assume $|B_1| \leq |B_2| \leq \cdots \leq |B_n|$ and $|B_1^*| \leq |B_2^*| \leq \cdots \leq |B_{n+1}^*|$. By assumption $|B_1^*|$ is positive. Let N denote the total number of objects. Thus $N = \sum_{i=1}^{n} |B_i|$ and $N = \sum_{i=1}^{n+1} |B_i^*|$. For $0 \leq i \leq n$ define

$$\Delta_i = |B_1^*| + |B_2^*| + \cdots + |B_{i+1}^*| - |B_1| - |B_2| - \cdots - |B_i|.$$

We have $\Delta_0 = |B_1^*| > 0$ and $\Delta_n = N - N = 0$. Therefore there exists an integer r $(1 \leq r \leq n)$ such that $\Delta_{r-1} > 0$ and $\Delta_r \leq 0$. Now

$$0 < \Delta_{r-1} - \Delta_r = |B_r| - |B_{r+1}^*|$$

so $|B_{r+1}^*| < |B_r|$. So far we have

$$|B_1^*| \leq |B_2^*| \leq \cdots \leq |B_{r+1}^*| < |B_r| \leq |B_{r+1}| \leq \cdots \leq |B_n|.$$

Thus $|B_i^*| < |B_j|$ for $1 \leq i \leq r+1$ and $r \leq j \leq n$. Define

$$\theta = |(B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*) \cap (B_r \cup B_{r+1} \cup \cdots \cup B_n)|.$$

We show $\theta \geq 2$. Using $\Delta_{r-1} > 0$ we have

$$\begin{aligned}
|B_1^*| + |B_2^*| + \cdots + |B_r^*| &> |B_1| + |B_2| + \cdots + |B_{r-1}| \\
&= |B_1 \cup B_2 \cup \cdots \cup B_{r-1}| \\
&\geq |(B_1 \cup B_2 \cup \cdots \cup B_{r-1}) \cap (B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*)| \\
&= |B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*| - \theta \\
&= |B_1^*| + |B_2^*| + \cdots + |B_{r+1}^*| - \theta \\
&\geq |B_1^*| + |B_2^*| + \cdots + |B_r^*| + 1 - \theta.
\end{aligned}$$

Therefore $\theta > 1$ so $\theta \geq 2$.