Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 6

1. Define the set $S = \{1, 2, \ldots, 10^4\}$. Let A (resp. B) (resp. C) denote the set of integers in S that are divisible by 4 (resp. 5) (resp. 6). We seek $|\overline{A} \cap \overline{B} \cap \overline{C}|$. We have

set	size	justification
S	10^4	
A	2500	$10^4 = 2500 \times 4$
B	2000	$10^4 = 2000 \times 5$
C	1666	$10^4 = 1666 \times 6 + 4$
$A \cap B$	500	$10^4 = 500 \times 20$
$B \cap C$	333	$10^4 = 333 \times 30 + 10$
$A \cap C$	833	$10^4 = 833 \times 12 + 4$
$A \cap B \cap C$	166	$10^4 = 166 \times 60 + 40$

By inclusion/exclusion

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| &= 10^4 - 2500 - 2000 - 1666 + 500 + 333 + 833 - 166 \\ &= 5334. \end{aligned}$$

2. Define the set $S = \{1, 2, \ldots, 10^4\}$. Let A (resp. B) (resp. C) (resp. D) denote the set of integers in S that are divisible by 4 (resp. 6) (resp. 7) (resp. 10). We seek $|\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}|$. We have

set	size	justification
S	10^4	
A	2500	$10^4 = 2500 \times 4$
B	1666	$10^6 = 1666 \times 6 + 4$
C	1428	$10^4 = 1428 \times 7 + 4$
D	1000	$10^4 = 1000 \times 10$
$A \cap B$	833	$10^4 = 833 \times 12 + 4$
$A \cap C$	357	$10^4 = 357 \times 28 + 4$
$A \cap D$	500	$10^4 = 500 \times 20$
$B \cap C$	238	$10^4 = 238 \times 42 + 4$
$B \cap D$	333	$10^4 = 333 \times 30 + 10$
$C \cap D$	142	$10^4 = 142 \times 70 + 60$
$A \cap B \cap C$	119	$10^4 = 119 \times 84 + 4$
$A \cap B \cap D$	166	$10^4 = 166 \times 60 + 40$
$A \cap C \cap D$	71	$10^4 = 71 \times 140 + 60$
$B \cap C \cap D$	47	$10^4 = 47 \times 210 + 130$
$A \cap B \cap C \cap D$	23	$10^4 = 23 \times 420 + 340$

By inclusion/exclusion

$$\begin{aligned}|\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| &= 10^4 - 2500 - 1666 - 1428 - 1000 \\ &\quad + 833 + 357 + 500 + 238 + 333 + 142 \\ &\quad - 119 - 166 - 71 - 47 + 23 \\ &= 5429.\end{aligned}$$

3. Define a set $S = \{1, 2, \ldots, 10^4\}$. Let A (resp. B) denote the set of integers in S that are perfect squares (resp. perfect cubes). We seek $|\overline{A} \cap \overline{B}|$. We have

set	size	justification
S	10^4	
A	100	$100^2 = 10^4$
B	21	$21^3 = 9261$ and $22^3 = 10648$
$A \cap B$	4	$4^6 = 4096$ and $5^6 = 15625$

By inclusion/exclusion

$$|\overline{A} \cap \overline{B}| = 10^4 - 100 - 21 + 4 = 9883.$$

4. These 12-combinations correspond to the integral solutions for

$$x + y + z + w = 12, \qquad 0 \le x \le 4, \quad 0 \le y \le 3, \quad 0 \le z \le 4, \quad 0 \le w \le 5.$$

Let S denote the set of nonnegative integral solutions to $x + y + z + w = 12$. Let X (resp. Y) (resp. Z) (resp. W) denote the set of elements in S such that $x \ge 5$ (resp. $y \ge 4$) (resp. $z \ge 5$) (resp. $w \ge 6$). We seek $|\overline{X} \cap \overline{Y} \cap \overline{Z} \cap \overline{W}|$. We have

set	size	justification
S	$\binom{15}{3}$	$15 = 12 + 4 - 1$
X	$\binom{10}{3}$	$15 - 5 = 10$
Y	$\binom{11}{3}$	$15 - 4 = 11$
Z	$\binom{10}{3}$	$15 - 5 = 10$
W	$\binom{9}{3}$	$15 - 6 = 9$
$X \cap Y$	$\binom{6}{3}$	$15 - 5 - 4 = 6$
$X \cap Z$	$\binom{5}{3}$	$15 - 5 - 5 = 5$
$X \cap W$	$\binom{4}{3}$	$15 - 5 - 6 = 4$
$Y \cap Z$	$\binom{6}{3}$	$15 - 4 - 5 = 6$
$Y \cap W$	$\binom{5}{3}$	$15 - 4 - 6 = 5$
$Z \cap W$	$\binom{4}{3}$	$15 - 5 - 6 = 4$
$X \cap Y \cap Z$	0	$15 - 5 - 4 - 5 = 1 < 3$
$X \cap Y \cap W$	0	$15 - 5 - 4 - 6 = 0 < 3$
$X \cap Z \cap W$	0	$15 - 5 - 5 - 6 = -1 < 3$
$Y \cap Z \cap W$	0	$15 - 4 - 5 - 6 = 0 < 3$
$X \cap Y \cap Z \cap W$	0	$15 - 5 - 4 - 5 - 6 = -5 < 3$

By inclusion/exclusion

$$
\begin{aligned}
|\overline{X} \cap \overline{Y} \cap \overline{Z} \cap \overline{W}| &= \binom{15}{3} - \binom{10}{3} - \binom{11}{3} - \binom{10}{3} - \binom{9}{3} \\
&\quad + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} \\
&= 34.
\end{aligned}
$$

5. These 10-combinations correspond to the integral solutions for

$$x + y + z + w = 10, \qquad 0 \le x, \quad 0 \le y \le 4, \quad 0 \le z \le 5, \quad 0 \le w \le 7.$$

Let S denote the set of nonnegative integral solutions to $x + y + z + w = 10$. Let Y (resp. Z) (resp. W) denote the set of elements in S such that $y \ge 5$ (resp. $z \ge 6$) (resp. $w \ge 8$). We seek $|\overline{Y} \cap \overline{Z} \cap \overline{W}|$. We have

set	size	justification
S	$\binom{13}{3}$	$13 = 10 + 4 - 1$
Y	$\binom{8}{3}$	$13 - 5 = 8$
Z	$\binom{7}{3}$	$13 - 6 = 7$
W	$\binom{5}{3}$	$13 - 8 = 5$
$Y \cap Z$	0	$13 - 5 - 6 = 2 < 3$
$Y \cap W$	0	$13 - 5 - 8 = 0 < 3$
$Z \cap W$	0	$13 - 6 - 8 = -1 < 3$
$Y \cap Z \cap W$	0	$13 - 5 - 6 - 8 = -6 < 3$

By inclusion/exclusion

$$|\overline{Y} \cap \overline{Z} \cap \overline{W}| = \binom{13}{3} - \binom{8}{3} - \binom{7}{3} - \binom{5}{3} = 185.$$

6. We seek the number of integral solutions for

$$x + y + z = 12, \qquad 0 \le x \le 6, \quad 0 \le y \le 6, \quad 0 \le z \le 3.$$

Let S denote the set of nonnegative integral solutions to $x + y + z = 12$. Let X (resp. Y) (resp. Z) denote the set of elements in S such that $x \ge 7$ (resp. $y \ge 7$) (resp. $z \ge 4$). We seek $|\overline{X} \cap \overline{Y} \cap \overline{Z}|$. We have

set	size	justification
S	$\binom{14}{2}$	$14 = 12 + 3 - 1$
X	$\binom{7}{2}$	$14 - 7 = 7$
Y	$\binom{7}{2}$	$14 - 7 = 7$
Z	$\binom{10}{2}$	$14 - 4 = 10$
$X \cap Y$	0	$14 - 7 - 7 = 0 < 2$
$X \cap Z$	$\binom{3}{2}$	$14 - 7 - 4 = 3$
$Y \cap Z$	$\binom{3}{2}$	$14 - 7 - 4 = 3$
$X \cap Y \cap Z$	0	$14 - 7 - 7 - 4 = -4 < 2$

By inclusion/exclusion

$$|\overline{X} \cap \overline{Y} \cap \overline{Z}| = \binom{14}{2} - \binom{7}{2} - \binom{7}{2} - \binom{10}{2} + \binom{3}{2} + \binom{3}{2} = 10.$$

7. Let S denote the set of nonnegative integral solutions for $x_1 + x_2 + x_3 + x_4 = 14$. For $1 \leq i \leq 4$ let A_i denote the set of elements in S with $x_i \geq 9$. We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4|$. We have

set	size	justification
S	$\binom{17}{3}$	$17 = 14 + 4 - 1$
A_i	$\binom{8}{3}$	$17 - 9 = 8$
$A_i \cap A_j$	0	$17 - 9 - 9 = -1 < 3$

By inclusion/exclusion

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4| = \binom{17}{3} - 4\binom{8}{3} = 456.$$

8. Let S denote the set of positive integral solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 14$. For $1 \leq i \leq 5$ let A_i denote the set of elements in S with $x_i \geq 6$. We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4 \cap \overline{A}_5|$. We have

set	size	justification
S	$\binom{13}{4}$	$13 = 14 - 5 + 5 - 1$
A_i	$\binom{8}{4}$	$13 - 5 = 8$
$A_i \cap A_j$	0	$13 - 5 - 5 = 3 < 4$

By inclusion/exclusion

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4 \cap \overline{A}_5| = \binom{13}{4} - 5\binom{8}{4} = 365.$$

9. We make a change of variables

$$y_1 = x_1 - 1, \qquad y_2 = x_2, \qquad y_3 = x_3 - 4, \qquad y_4 = x_4 - 2.$$

We seek the number of integral solutions to

$$y_1 + y_2 + y_3 + y_4 = 13, \qquad 0 \leq y_1 \leq 5, \quad 0 \leq y_2 \leq 7, \quad 0 \leq y_3 \leq 4, \quad 0 \leq y_4 \leq 4.$$

Let S denote the set of nonnegative integral solutions to $y_1 + y_2 + y_3 + y_4 = 13$. Let A_1 (resp. A_2) (resp. A_3) (resp. A_4) denote the set of elements in S such that $y_1 \geq 6$ (resp. $y_2 \geq 8$) (resp. $y_3 \geq 5$) (resp. $y_4 \geq 5$). We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4|$. We have

set	size	justification
S	$\binom{16}{3}$	$16 = 13 + 4 - 1$
A_1	$\binom{10}{3}$	$16 - 6 = 10$
A_2	$\binom{8}{3}$	$16 - 8 = 8$
A_3	$\binom{11}{3}$	$16 - 5 = 11$
A_4	$\binom{11}{3}$	$16 - 5 = 11$
$A_1 \cap A_2$	0	$16 - 6 - 8 = 2 < 3$
$A_1 \cap A_3$	$\binom{5}{3}$	$16 - 6 - 5 = 5$
$A_1 \cap A_4$	$\binom{5}{3}$	$16 - 6 - 5 = 5$
$A_2 \cap A_3$	$\binom{3}{3}$	$16 - 8 - 5 = 3$
$A_2 \cap A_4$	$\binom{3}{3}$	$16 - 8 - 5 = 3$
$A_3 \cap A_4$	$\binom{6}{3}$	$16 - 5 - 5 = 6$
$A_i \cap A_j \cap A_k$	0	$16 - 6 - 5 - 5 = 0 < 3$

By inclusion/exclusion

$$\begin{aligned}|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4| &= \binom{16}{3} - \binom{10}{3} - \binom{8}{3} - \binom{11}{3} - \binom{11}{3} \\ &\quad + \binom{5}{3} + \binom{5}{3} + \binom{3}{3} + \binom{3}{3} + \binom{6}{3} \\ &= 96.\end{aligned}$$

10. The r-combinations of S correspond to the integral solutions for

$$\sum_{i=1}^{k} x_i = r, \qquad 0 \le x_i \le n_i \quad (1 \le i \le k).$$

We assume that there exists at least one solution, so $r \le \sum_{i=1}^k n_i$. Let U denote the set of nonnegative integral solutions to $\sum_{i=1}^k x_i = r$. For $1 \le i \le k$ let A_i denote the set of elements in U such that $x_i > n_i$. To get our answer, we would apply inclusion/exclusion to the sets $\{A_i\}_{i=1}^k$. Note that $A_1 \cap A_2 \cap \cdots \cap A_k$ consists of the elements in U such that $x_i > n_i$ for $1 \le i \le k$. For such an element $r = \sum_{i=1}^k x_i > \sum_{i=1}^k n_i \ge r$, a contradiction. Hence $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$.

11. Let the set S consist of the permutations of $\{1, 2, \ldots, 8\}$. For $i \in \{2, 4, 6, 8\}$ let A_i denote the set of permutations in S for which i is in its natural position. We seek $|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8|$. We have

set	size
S	$8!$
A_i	$7!$
$A_i \cap A_j$	$6!$
$A_i \cap A_j \cap A_k$	$5!$
$A_i \cap A_j \cap A_k \cap A_\ell$	$4!$

By inclusion/exclusion

$$|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8| = 8! - 4 \times 7! + 6 \times 6! - 4 \times 5! + 4! = 24024.$$

12. Let X denote the set of permutations of $\{1, 2, \ldots, 8\}$ for which exactly four integers are in their natural position. We compute $|X|$. To do this we construct an element of X in stages:

stage	to do	# choices
1	select the four fixed integers	$\binom{8}{4}$
2	select a derangement of the remaining four integers	D_4

Therefore $|X| = \binom{8}{4} D_4$. We have $\binom{8}{4} = 70$ and

$$D_4 = 4! - 4 \times 3! + 6 \times 2! - 4 \times 1! + 1 = 9$$

so $|X| = 70 \times 9 = 630$.

13. Let the set S consist of the permutations of $\{1, 2, \ldots, 9\}$. For $i \in \{1, 3, 5, 7, 9\}$ let A_i denote the set of permutations in S for which i is in its natural position. We seek $|S| - |\overline{A}_1 \cap \overline{A}_3 \cap \overline{A}_5 \cap \overline{A}_7 \cap \overline{A}_9|$. We have

set	size
A_i	$8!$
$A_i \cap A_j$	$7!$
$A_i \cap A_j \cap A_k$	$6!$
$A_i \cap A_j \cap A_k \cap A_\ell$	$5!$
$A_i \cap A_j \cap A_k \cap A_\ell \cap A_m$	$4!$

By inclusion/exclusion

$$|S| - |\overline{A}_1 \cap \overline{A}_3 \cap \overline{A}_5 \cap \overline{A}_7 \cap \overline{A}_9| = 5 \times 8! - 10 \times 7! + 10 \times 6! - 5 \times 5! + 1 \times 4!.$$

14. Let X denote the set of permutations of $\{1, 2, \ldots, n\}$ for which exactly k integers are in their natural position. We compute $|X|$. To do this we construct an element of X in stages:

stage	to do	# choices
1	select the k fixed integers	$\binom{n}{k}$
2	select a derangement of the remaining $n-k$ integers	D_{n-k}

Therefore $|X| = \binom{n}{k} D_{n-k}$.

15. (a) D_7; (b) $7! - D_7$; (c) $7! - 7D_6 - D_7$.

16. We use combinatorial reasoning to show

$$n! = \sum_{i=0}^{n} \binom{n}{i} D_{n-i}.$$

Let S denote the set of permutations of $\{1, 2, \ldots, n\}$. For $0 \le i \le n$ let S_i denote the set of permutations in S for which exactly i integers are in their natural position. The sets $\{S_i\}_{i=0}^{n}$ partition S, so $|S| = \sum_{i=0}^{n} |S_i|$. We have $|S| = n!$ and by Problem 14 $|S_i| = \binom{n}{i} D_{n-i}$ for $0 \le i \le n$. The result follows.

17. Let X denote the set of permutations of S. Let A (resp. B) (resp. C) denote the set of elements in X such that aaa (resp. $bbbb$) (resp. cc) are consecutive. We seek $|\overline{A} \cap \overline{B} \cap \overline{C}|$. Using Theorem 2.4.2,

set	size
X	$\binom{9}{3\,4\,2}$
A	$\binom{7}{1\,4\,2}$
B	$\binom{6}{3\,1\,2}$
C	$\binom{8}{3\,4\,1}$
$A \cap B$	$\binom{4}{1\,1\,2}$
$A \cap C$	$\binom{6}{1\,4\,1}$
$B \cap C$	$\binom{5}{3\,1\,1}$
$A \cap B \cap C$	$\binom{3}{1\,1\,1}$

By inclusion/exclusion

$$\begin{aligned}
|\overline{A}\cap\overline{B}\cap\overline{C}| &= \binom{9}{3\;4\;2} - \binom{7}{1\;4\;2} - \binom{6}{3\;1\;2} - \binom{8}{3\;4\;1} \\
&\quad + \binom{4}{1\;1\;2} + \binom{6}{1\;4\;1} + \binom{5}{3\;1\;1} - \binom{3}{1\;1\;1} \\
&= 871.
\end{aligned}$$

18. View $(n-1)! = (n-1)\times(n-2)!$.

19. Using Problem 18 and Theorem 6.3.1,

$$\begin{aligned}
&D_n - (n-1)(D_{n-2}+D_{n-1}) \\
&= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} - (n-1)(n-2)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} - (n-1)(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} \\
&= n!\frac{(-1)^{n-1}}{(n-1)!} + n!\frac{(-1)^n}{n!} - (n-1)(n-1)!\frac{(-1)^{n-1}}{(n-1)!} \\
&= n(-1)^{n-1} + (-1)^n - (n-1)(-1)^{n-1} \\
&= (-1)^{n-1}(n-1-(n-1)) \\
&= 0.
\end{aligned}$$

20. We show

$$D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} \qquad\qquad n = 1,2,\ldots$$

We use induction on n. The above equation holds for $n=1$ since each side is zero. Next assume $n \geq 2$. By induction

$$D_{n-1} = (n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}.$$

Therefore

$$\begin{aligned}
D_n &= nD_{n-1} + (-1)^n \\
&= (-1)^n + n(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}. \\
&= (-1)^n + n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}. \\
&= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}.
\end{aligned}$$

21. We show that D_n is even if and only if n is odd. We use induction on n. First assume $n = 1$. In this case the result holds since $D_1 = 0$ is even and 1 is odd. Next assume $n \geq 2$. Recall $D_n = nD_{n-1} + (-1)^n$. If n is even then $n-1$ is odd, so by induction D_{n-1} is even. Therefore nD_{n-1} is even so $D_n = nD_{n-1} + 1$ is odd. If n is odd then $n-1$ is even, so by induction D_{n-1} is odd. Therefore nD_{n-1} is odd so $D_n = nD_{n-1} - 1$ is even.

22. Using Theorem 6.5.1,

$$\begin{aligned}
Q_n &= \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i)! \\
&= \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} (-1)^i (n-i)! \\
&= (n-1)! \sum_{i=0}^{n-1} \frac{(-1)^i (n-i)}{i!}
\end{aligned}$$

23. Using Problem 22 and a change of variables $j = i - 1$,

$$\begin{aligned}
Q_n &= (n-1)! \sum_{i=0}^{n-1} \frac{(-1)^i (n-i)}{i!} \\
&= (n-1)! \sum_{i=0}^{n} \frac{(-1)^i (n-i)}{i!} \\
&= (n-1)! \sum_{i=0}^{n} \frac{(-1)^i n}{i!} - (n-1)! \sum_{i=0}^{n} \frac{(-1)^i i}{i!}. \\
&= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} - (n-1)! \sum_{i=1}^{n} \frac{(-1)^i i}{i!} \\
&= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} + (n-1)! \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!} \\
&= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} + (n-1)! \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \\
&= D_n + D_{n-1}.
\end{aligned}$$

24. For $0 \leq k \leq 6$ let r_k denote the number of ways to place k nonattacking rooks in the forbidden positions. Consider the number of ways to place six nonattacking rooks on the chessboard such that no rook is in a forbidden position. By Theorem 6.4.1, this number is

$$\sum_{k=0}^{6} r_k (-1)^k (6-k)!.$$

For case (a),

k	0	1	2	3	4	5	6
r_k	1	6	12	8	0	0	0

For case (b),

k	0	1	2	3	4	5	6
r_k	1	12	54	112	108	48	8

For case (c),

k	0	1	2	3	4	5	6
r_k	1	8	22	24	9	1	0

25. We interpret this problem in terms of placing six nonattacking rooks on a 6×6 chessboard. The answer is $\sum_{k=0}^{6} r_k(-1)^k(6-k)!$ where

k	0	1	2	3	4	5	6
r_k	1	8	20	20	7	0	0

26. We interpret this problem in terms of placing six nonattacking rooks on a 6×6 chessboard. The answer is $\sum_{k=0}^{6} r_k(-1)^k(6-k)!$ where

k	0	1	2	3	4	5	6
r_k	1	9	26	26	8	0	0

27. Choose a circular labelling $1, 2, \ldots, 8$ of the seats in order around the carousel, with seat i facing seat $i+1$ for $1 \leq i \leq 7$ and seat 8 facing seat 1. For $1 \leq i \leq 8$ the girl in seat i moves to a new seat, labelled s_i. Then $s_1 s_2 \cdots s_8$ is a permutation of $\{1, 2, \ldots, 8\}$ such that seat s_i does not face seat s_{i+1} for $1 \leq i \leq 7$ and seat s_8 does not face seat s_1. We compute the number of such permutations. Let P denote the set of permutations of $\{1, 2, \ldots, 8\}$. For $1 \leq i \leq 7$ let A_i denote the set of permutations $s_1 s_2 \cdots s_8$ in P such that seat s_i faces seat s_{i+1}. Let A_8 denote the set of permutations $s_1 s_2 \cdots s_8$ in P such that s_8 faces s_1. We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_8|$. For each subset $s \subseteq \{1, 2, \ldots, 8\}$ define $A_s = \cap_{i \in s} A_i$. We routinely find $|A_s| = 8(7 - |s|)!$ if $|s| \leq 7$. Moreover $|A_1 \cap A_2 \cap \cdots \cap A_8| = 8$. By inclusion/exclusion

$$
\begin{aligned}
|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_8| &= \sum_{s \subseteq \{1,2,\ldots,8\}} |A_s|(-1)^{|s|} \\
&= \sum_{k=0}^{7} \binom{8}{k} 8(7-k)!(-1)^k + 8 \\
&= 13000.
\end{aligned}
$$

Now assume that the seats are indistinguishable. In this case, any two of the original seating arrangements become indistinguishable whenever one is obtained from the other by a circular permutation. Under the new assumption the answer is $13000/8 = 1625$.

28. Choose a circular labelling $1, 2, \ldots, 8$ of the seats in order around the carousel. Note that seats labelled $i, i+4$ are opposite for $1 \le i \le 4$. For $1 \le i \le 8$ the boy in seat i moves to a new seat, labelled s_i. Then $s_1 s_2 \cdots s_8$ is a permutation of $\{1, 2, \ldots, 8\}$ such that seats s_i, s_{i+4} are not opposite for $1 \le i \le 4$. We compute the number of such permutations. Let P denote the set of permutations of $\{1, 2, \ldots, 8\}$. For $1 \le i \le 4$ let A_i denote the set of permutations $s_1 s_2 \cdots s_8$ in P such that seats s_i, s_{i+4} are opposite. We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4|$. For each subset $s \subseteq \{1, 2, 3, 4\}$ define $A_s = \cap_{i \in s} A_i$. We routinely find

$\lvert s\rvert$	0	1	2	3	4
$\lvert A_s\rvert$	$8!$	$8 \times 6!$	$8 \times 6 \times 4!$	$8 \times 6 \times 4 \times 2!$	$8 \times 6 \times 4 \times 2 \times 0!$

By inclusion/exclusion

$$
\begin{aligned}
&|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4| \\
&\quad = \sum_{s \subseteq \{1,2,\ldots,4\}} |A_s| (-1)^{|s|} \\
&\quad = 8! - 4 \times 8 \times 6! + 6 \times 8 \times 6 \times 4! - 4 \times 8 \times 6 \times 4 \times 2! + 8 \times 6 \times 4 \times 2 \times 0! \\
= \;& 23040.
\end{aligned}
$$

Now assume that the seats are indistinguisable. In this case, any two of the original seating arrangements become indistinguishable whenever one is obtained from the other by a circular permutation. Under the new assumption the answer is $23040/8 = 2880$.

29. Label the people on the platform $1, 2, \ldots, 10$. For $1 \le i \le 10$ let s_i denote the stop where person i exits the subway. Thus $1 \le s_i \le 6$. Let S denote the set of sequences of integers $s_1 s_2 \cdots s_{10}$ such that $1 \le s_i \le 6$ for $1 \le i \le 10$. For $1 \le j \le 6$ let A_j denote the set of sequences $s_1 s_2 \cdots s_{10}$ in S such that $s_i \ne j$ for $1 \le i \le 10$. We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_6|$. For each subset $s \subseteq \{1, 2, \ldots, 6\}$ define $A_s = \cap_{i \in s} A_i$. We routinely find $|A_s| = (6 - |s|)^{10}$. By inclusion/exclusion

$$
\begin{aligned}
|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_6| &= \sum_{s \subseteq \{1,2,\ldots,6\}} |A_s| (-1)^{|s|} \\
&= \sum_{k=0}^{6} \binom{6}{k} (6-k)^{10} (-1)^k.
\end{aligned}
$$

30. (Problem statement is suspect) If we accept the problem statement verbatim, then the answer is 0. Reason: Since d appears in the multiset with multiplicity one, it is vacuously true that for any circular permutation of the multiset, all occurrences of d will appear consecutively. We now adjust the problem statement to read ..for each type of letter except d, all letters of that type do not appear consecutively.. Let S denote the set of circular permutations of

$$\{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\}.$$

Let A (resp. B) (resp. C) denote the set of elements in S such that all occurrences of a (resp. b) (resp. c) are consecutive. We seek $|\overline{A} \cap \overline{B} \cap \overline{C}|$. We have

set X	X contains CP of	$\lvert X\rvert$
S	$\{3\cdot a, 4\cdot b, 2\cdot c, 1\cdot d\}$	$\frac{9!}{3!4!2!1!}$
A	$\{1\cdot aaa, 4\cdot b, 2\cdot c, 1\cdot d\}$	$\frac{7!}{1!4!2!1!}$
B	$\{3\cdot a, 1\cdot bbbb, 2\cdot c, 1\cdot d\}$	$\frac{6!}{3!1!2!1!}$
C	$\{3\cdot a, 4\cdot b, 1\cdot cc, 1\cdot d\}$	$\frac{8!}{3!4!1!1!}$
$A\cap B$	$\{1\cdot aaa, 1\cdot bbbb, 2\cdot c, 1\cdot d\}$	$\frac{4!}{1!1!2!1!}$
$A\cap C$	$\{1\cdot aaa, 4\cdot b, 1\cdot cc, 1\cdot d\}$	$\frac{6!}{1!4!1!1!}$
$B\cap C$	$\{3\cdot a, 1\cdot bbbb, 1\cdot cc, 1\cdot d\}$	$\frac{5!}{3!1!1!1!}$
$A\cap B\cap C$	$\{1\cdot aaa, 1\cdot bbbb, 1\cdot cc, 1\cdot d\}$	$\frac{3!}{1!1!1!1!}$

By inclusion/exclusion

$$|\overline{A}\cap\overline{B}\cap\overline{C}| = \frac{9!}{3!4!2!1!} - \frac{7!}{1!4!2!1!} - \frac{6!}{3!1!2!1!} - \frac{8!}{3!4!1!1!} + \frac{4!}{1!1!2!1!} + \frac{6!}{1!4!1!1!} + \frac{5!}{3!1!1!1!} - \frac{3!}{1!1!1!1!}.$$

31. Let S denote the set of circular permutations of

$$\{2\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\}.$$

Let A (resp. B) (resp. C) (resp. D) denote the set of elements in S such that all occurrences of a (resp. b) (resp. c) (resp. d) are consecutive. We seek $|\overline{A}\cap\overline{B}\cap\overline{C}\cap\overline{D}|$. We have

set X	X contains CP of	$\lvert X\rvert$
S	$\{2\cdot a, 3\cdot b, 4\cdot c, 5\cdot d\}$	$\frac{13!}{2!3!4!5!}$
A	$\{1\cdot aa, 3\cdot b, 4\cdot c, 5\cdot d\}$	$\frac{12!}{1!3!4!5!}$
B	$\{2\cdot a, 1\cdot bbb, 4\cdot c, 5\cdot d\}$	$\frac{11!}{2!1!4!5!}$
C	$\{2\cdot a, 3\cdot b, 1\cdot cccc, 5\cdot d\}$	$\frac{10!}{2!3!1!5!}$
D	$\{2\cdot a, 3\cdot b, 4\cdot c, 1\cdot ddddd\}$	$\frac{9!}{2!3!4!1!}$
$A\cap B$	$\{1\cdot aa, 1\cdot bbb, 4\cdot c, 5\cdot d\}$	$\frac{10!}{1!1!4!5!}$
$A\cap C$	$\{1\cdot aa, 3\cdot b, 1\cdot cccc, 5\cdot d\}$	$\frac{9!}{1!3!1!5!}$
$A\cap D$	$\{1\cdot aa, 3\cdot b, 4\cdot c, 1\cdot ddddd\}$	$\frac{8!}{1!3!4!1!}$
$B\cap C$	$\{2\cdot a, 1\cdot bbb, 1\cdot cccc, 5\cdot d\}$	$\frac{8!}{2!1!1!5!}$
$B\cap D$	$\{2\cdot a, 1\cdot bbb, 4\cdot c, 1\cdot ddddd\}$	$\frac{7!}{2!1!4!1!}$
$C\cap D$	$\{2\cdot a, 3\cdot b, 1\cdot cccc, 1\cdot ddddd\}$	$\frac{6!}{2!3!1!1!}$
$A\cap B\cap C$	$\{1\cdot aa, 1\cdot bbb, 1\cdot cccc, 5\cdot d\}$	$\frac{7!}{1!1!1!5!}$
$A\cap B\cap D$	$\{1\cdot aa, 1\cdot bbb, 4\cdot c, 1\cdot ddddd\}$	$\frac{6!}{1!1!4!1!}$
$A\cap C\cap D$	$\{1\cdot aa, 3\cdot b, 1\cdot cccc, 1\cdot ddddd\}$	$\frac{5!}{1!3!1!1!}$
$B\cap C\cap D$	$\{2\cdot a, 1\cdot bbb, 1\cdot cccc, 1\cdot ddddd\}$	$\frac{4!}{2!1!1!1!}$
$A\cap B\cap C\cap D$	$\{1\cdot aa, 1\cdot bbb, 1\cdot cccc, 1\cdot ddddd\}$	$\frac{3!}{1!1!1!1!}$

By inclusion/exclusion

$$\begin{aligned}|\overline{A}\cap\overline{B}\cap\overline{C}\cap\overline{D}| = & \frac{13!}{2!3!4!5!} - \frac{12!}{1!3!4!5!} - \frac{11!}{2!1!4!5!} - \frac{10!}{2!3!1!5!} - \frac{9!}{2!3!4!1!} \\ & + \frac{10!}{1!1!4!5!} + \frac{9!}{1!3!1!5!} + \frac{8!}{1!3!4!1!} + \frac{8!}{2!1!1!5!} + \frac{7!}{2!1!4!1!} + \frac{6!}{2!3!1!1!} \\ & - \frac{7!}{1!1!1!5!} - \frac{6!}{1!1!4!1!} - \frac{5!}{1!3!1!1!} - \frac{4!}{2!1!1!1!} + \frac{3!}{1!1!1!1!}.\end{aligned}$$

32. Let S denote the set $\{1, 2, \ldots, n\}$. For $1 \le i \le k$ let A_i denote the set of integers in S that are divisible by p_i. Note that $\phi(n) = |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_k|$. To find $\phi(n)$ we use inclusion/exclusion. For each subset $s \subseteq \{1, 2, \ldots, k\}$ define $A_s = \cap_{i \in s} A_i$. For notational convenience define $p_s = \prod_{i \in s} p_i$. Note that $|A_s| = np_s^{-1}$. By inclusion/exclusion

$$\begin{aligned}\phi(n) &= \sum_{s \subseteq \{1,2,\ldots,k\}} |A_s|(-1)^{|s|} \\ &= n \sum_{s \subseteq \{1,2,\ldots,k\}} p_s^{-1}(-1)^{|s|} \\ &= n \prod_{i=1}^{k} (1 - p_i^{-1}).\end{aligned}$$

33. (Problem statement contains typo) For an $n \times n$ chessboard define a set F of "forbidden" locations, consisting of $(1,1), (2,2), \ldots, (n,n)$ and $(1,2), (2,3), \ldots, (n-1,n), (n,1)$. For $0 \le k \le n$ let $a(n,k)$ denote the number of ways to place k nonattacking rooks on the chessboard, such that each rook is contained in F. We show that

$$a(n,k) = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

Order the elements of F in a circular fashion

n	1	2	3	4	$\cdots$	$2n-2$	$2n-1$	$2n$
nth location	$(1,1)$	$(1,2)$	$(2,2)$	$(2,3)$	$\cdots$	$(n-1,n)$	(n,n)	$(n,1)$

with the first location following the last one. Given two locations in F, they are adjacent in the circular ordering if and only if they are in the same row or column of the chessboard. Thus $a(n,k)$ represents the number of k-subsets of the set F, such that no two elements of the subset are adjacent in the circular ordering. Denote such a k-subset by $\{a_i\}_{i=1}^k$ with $1 \le a_1 < a_2 < \cdots < a_k \le 2n$. Define

$$x_1 = a_1 - 1, \quad x_2 = a_2 - a_1 - 2, \quad \ldots, \quad x_k = a_k - a_{k-1} - 2, \quad x_{k+1} = 2n - a_k.$$

Then

$$x_i \ge 0 \quad (1 \le i \le k+1), \qquad \sum_{i=1}^{k+1} x_i = 2n - 2k + 1, \tag{1}$$

with x_1, x_{k+1} not both zero. Initially ignoring the constraint on the pair (x_1, x_{k+1}), the number of integral solutions to (1) is $\binom{2n-k+1}{k}$. Now consider the number of integral solutions to (1) such that $x_1 = 0$ and $x_{k+1} = 0$. This number is $\binom{2n-k-1}{k-2}$. Therefore

$$\begin{aligned} a(n,k) &= \binom{2n-k+1}{k} - \binom{2n-k-1}{k-2} \\ &= \frac{2n}{2n-k}\binom{2n-k}{k}. \end{aligned}$$

We now compute the number of ways to place n nonattacking rooks on the chessboard, such that no rook is contained in F. By Theorem 6.4.1 this number is

$$\sum_{k=0}^{n}(-1)^k (n-k)!\, a(n,k) = \sum_{k=0}^{n} \frac{(-1)^k (n-k)!\, 2n}{2n-k}\binom{2n-k}{k}. \tag{2}$$

We evaluate the sum (2) for small values of n:

n	2	3	4	5	6	7	8	9
sum	0	1	2	13	$2^4 \times 5$	3×193	$2 \times 23 \times 103$	43×1009

The large prime factors in the above table suggest that the sum (2) cannot be easily expressed in closed form.

34. The convolution product is matrix multiplication in disguise.

35. $F(1) = G(1)$ and $F(m) = G(m) - G(m-1)$ for $2 \le m \le n$.

36. The answer is 6. We suppress the details of the calculation.

37. The inverse f^{-1} is defined just like f, with the sequence $1, 2, 1, -1$ replaced by $1, -2, 7, -35$.

38. Given a partition in Π_n, define its *type* to be the sequence consisting of the cardinalities of the sets that make up the partition, listed in nonincreasing order. For $n = 3$ the possible types are

$$111, \qquad 21, \qquad 3.$$

In Π_3 there is one element of type 111, three elements of type 21, and one element of type 3. Consider the Mobius funtion μ for Π_3. Given elements A, B of Π_3 we have $\mu(A, B) = 1$ if $A = B$, $\mu(A, B) = -1$ if B covers A, and $\mu(A, B) = 2$ A has type 111 and B has type 3. We have $\mu(A, B) = 0$ for all other A, B.
For $n = 4$ the possible types are

$$1111, \qquad 211, \qquad 22, \qquad 31, \qquad 4.$$

In Π_4 there is one element of type 1111, six elements of type 211, three elements of type 22, four elements of type 31, and one element of type 4. Consider the Mobius funtion μ for Π_4. Given elements A, B in Π_4 we have

case	$\mu(A, B)$
$A = B$	1
B covers A	-1
B has type 4 and A has type 211	2
B has type 4 and A has type 1111	-6
B has type 22 and A has type 1111	1
B has type 31 and A has type 1111	2
other	0

39. Let $[a, b]$ denote the set of elements in the poset which are both divisible by a and divide b. View $[a, b]$ as a poset with the inherited partial order. Then the map $[a, b] \to [1, b/a]$, $x \mapsto x/a$ is a bijection which preserves the partial order. In other words the poset $[a, b]$ is the poset $[1, b/a]$ in disguise. It follows that $\mu(a, b) = \mu(1, b/a)$.

40. Let $(P, \le)$ denote the poset described in the problem. Pick k distinct primes $\{p_i\}_{i=1}^k$ and define $n = \prod_{i=1}^k p_i^{n_i}$. Let $(Q, |)$ denote the poset consisting of the positive integer divisors of n, with partial order given by divisibility. Then the map $P \to Q$, $\{m_1 \cdot a_1, \ldots, m_k \cdot a_k\} \mapsto p_1^{m_1} \cdots p_k^{m_k}$ is a bijection which preserves the partial order. In other words the poset $(P, \le)$ is the poset $(Q, |)$ in disguise. The Mobius function for $(Q, |)$ is the classical one given above Theorem 6.6.4, so the Mobius function for $(P, \le)$ is the same.