LECTURE 7:

Before we begin I'd like to explain this magical "~" sign I've be using to simplify complex arguements and to make physical reasoning mathematical. It is by far the weirdest, craziest, simplest, and most productive math an engineer can know. Mathematicians use it in Number Theory, Physicist use it in Cosmology, we use it all the time and its called...

Scaling

* some sick 80's hair metal plays in the background

Let's start by looking at this simple algebraic statement.

$$4.000001 = 3 + 1 + \underbrace{0.000001}$$

Do we really need to keep this number?

Now say this equation changes over time

$$4.0002 = 3.5 + 0.5 + 0.0002 \qquad t = 1s$$

$$4.00001 = 3.1 + 0.9 + 0.0001 \qquad t = 2s$$

$$\vdots$$

$$4.0068 = 2.1 + 1.9 + \underbrace{0.0068} \qquad t = 8s$$

Do we really need to keep this term around?!

LECTURE 7

Those were numbers but lets say they were rally derivatives evaluated at a certain time instant.

at $t=1$:=> $\frac{d^2f}{dt^2}(1) = 4.0002$, $\frac{df}{dt}(1) = 3.5$, $f(1) = 0.5$, $a \cdot \sqrt{f}'(1) = 0.0002$

So really I was describing a diffeential equation but look what the benefit of throwing away the last term does.

$$\frac{d^2f}{dt^2} = \frac{df}{dt} + f + a \cdot \sqrt{f}$$ (non-linear!) NOT-SOLVABLE

$$\frac{d^2f}{dt^2} = \frac{df}{dt} + f$$ linear! SOLVABLE

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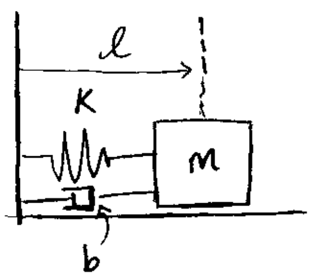
Parameters vs. Variables

That's the goal of scaling THROW OUT INSIGNIFICANT STUFF! But in the little number toy example we cheated! I told you what each term evaluated to, we can't do this given any diff eq... Well you could numerically solve it and then see what the terms evaluate to, but trust me this is way faster. Take a mass-damper-spring system.



l := rest length oscillations occur around this value.

Equation of motion is ...

$$m\ddot{x} + b\dot{x} + Kx = 0$$

Variables : Dependent x Independent t

Parameters : m, b, k, l

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Scaling and Non-Dimensionalization are intertwined. One informs the other. So let's combine our parameters (m, b, k, ℓ) to make our equation non-dimensional.

X oscillates around ℓ so...

I use a star to indicate a variable is non-dimensional $\therefore$

$$X^* = \frac{X}{\ell}$$

t is about ... um

Scaling let's us solve for a characteristic frequency or oscillation time. Let's go along as if we knew a time-scale τ.

$$t^* = \frac{t}{\tau}$$

Now sub-everything into our equation.

$$m \frac{d^2 x}{d t^2} + b \frac{dx}{dt} + k\, x = 0$$

$$m \frac{d^2(\ell x^*)}{d(\tau t^*)^2} + b \frac{d(\ell x^*)}{d(\tau t^*)} + k(\ell x^*) = 0$$

(τ, ℓ) are constant values so they can come out of $\frac{d}{dt}$ terms.

$$\frac{m\cancel{\ell}}{\tau^2} \frac{d^2 x^*}{d t^{*2}} + \frac{b\cancel{\ell}}{\tau} \frac{d x^*}{d t^*} + k\cancel{\ell} x^* = 0$$

$$\frac{m}{\tau^2} \ddot{X}^* + \frac{b}{\tau} \dot{X}^* + k X^* = 0$$

This has all just been Non-dimensionalization, now we shall use scaling.

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Let's skip the reasoning but importantly once you get the derivative terms or Variables non-dimensional you turn the differential equation for the variables into an algebraic equation for parameters.

$$\frac{m}{\tau^2}\ddot{x}^* + \frac{b}{\tau}\dot{x}^* = -Kx^*$$

$$\overset{\mathrm{I}}{\frac{m}{\tau^2}} , \quad \overset{\mathrm{II}}{\frac{b}{\tau}} \quad \sim \quad \overset{\mathrm{III}}{K}$$

It is a rule that nothing can scale to zero so always move a term over if it equals zero

use commas instead of +/- because ~ doesn't care about sign

Now we can compare terms. Let's say we know b is super small. So we can neglect it. Then

$$\mathrm{I} \sim \mathrm{III} \quad \Rightarrow \quad \frac{m}{\tau^2} \sim K \quad \Rightarrow \quad \underbrace{\tau_1 \sim \sqrt{\frac{m}{K}}}_{\text{Recognize this guy!?}}$$

Say we know that k is just a pathetically tiny spring constant. then

$$\mathrm{I} \sim \mathrm{II} \quad \Rightarrow \quad \frac{m}{\tau^2} \sim \frac{b}{\tau} \quad \Rightarrow \quad \tau_2 \sim \frac{m}{b}$$

What if it is very light?

$$\mathrm{II} \sim \mathrm{III} \quad \Rightarrow \quad \frac{b}{\tau} \sim K \quad \Rightarrow \quad \tau_3 \sim \frac{b}{K}$$

* Now you try it! It's so fun!

LECTURE 7

Extra Credit

Solve for a velocity scale for the differential equation

$$\eta \dot{V}^2 - \beta \ddot{V} V^{1/3} = 0$$

Given known (η, β, τ)

Extra Credit

Solve for the velocity scale in the 2D continuity equation. How fast is the y-velocity? V?

$$\nabla \cdot \vec{V} = 0$$

Given $x \sim L$ and $y \sim \delta$ and $u \sim U_\infty$

Extra Credit

Find all time-scales by comparing 2 terms in the Navier Stokes. Rank them given fluid properties you've looked up online

$$\rho \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right\} = -\frac{dP}{dx} + \mu \frac{\partial^2 u}{\partial x^2}$$

Given $x \sim L$, $u \sim U_\infty$, $P \sim P_0$

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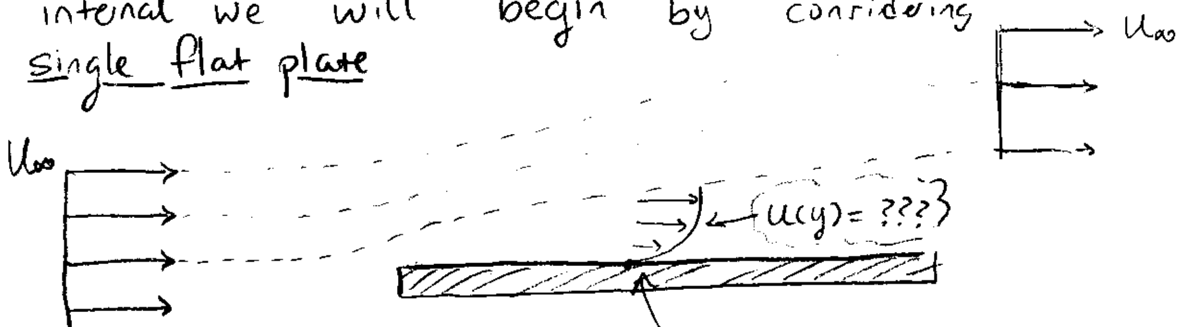
Now let's use these tools to "zoom" in on the very surface of a plate sitting in an external flow field

* Just as we first considered a single straight pipe for internal we will begin by considering a single flat plate



We know beyond the shadow of a doubt $u = 0$ here

No-SLIP

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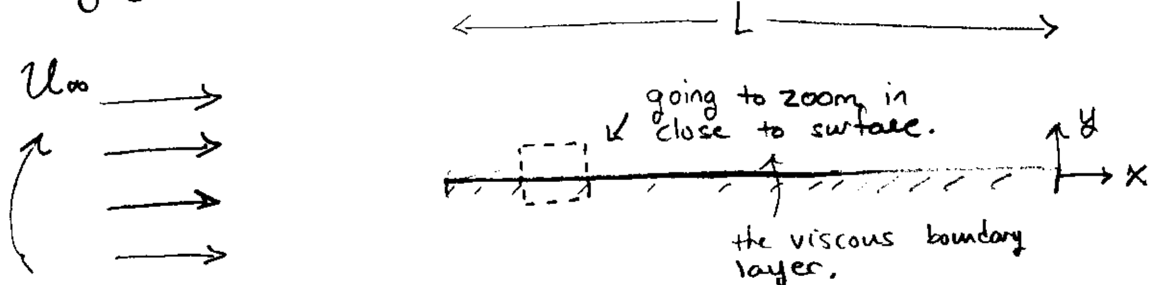
So with this concept of <u>scaling</u>. let's write out the navier-stokes equations, turn our physical dimensions into <u>math</u> (scaling), and derie the Boundary Layer Equations. So in <u>vector</u> <u>form</u> the NAVER STOKES Equations for <u>incompressible</u> <u>viscous</u> flow are. $\tilde{u} \in \mathbb{R}^3$ (a vector)

Newtonian, too!

(1) $$\rho \frac{d\tilde{u}}{dt} + \rho(\tilde{u}\cdot\nabla)\tilde{u} = -\nabla P + \mu \nabla^2 \tilde{u} \qquad \text{(N.V.S)}$$

(2) $$\nabla \cdot \tilde{u} = 0 \qquad \text{(Continuity)}$$

You should have in your head now the picture we're trying to describe. A flat plate sitting in <u>2D</u> flow.



In this set-up we will say the z-direction velocity is basically zero $w \approx 0$

We scale our dimensions to make the equations non-dimensional.

$x \sim L$ (plate length)

$y \sim \delta$ (boundary layer height)

$u \sim U_\infty$ (free-stream velocity)

You see even steady-state can be made into a scaling arguement. $\longrightarrow$ $t \sim \infty$ (steady-state)

$w \sim \varepsilon \approx 0$ (z - velocity)

LECTURE 7:

Start with continuity (2).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\underbrace{\frac{U_\infty}{L} \frac{\partial u^*}{\partial x^*} + \frac{V}{\delta} \frac{\partial v^*}{\partial y^*}}_{\text{these are basically 1 now.}} + \underbrace{\cancel{\frac{W}{Z} \frac{\partial w^*}{\partial z^*}}}_{W \text{ small}} = 0$$

Now we can determine the y-direction velocity by solving the algebraic scale equation.

$$\frac{U_\infty}{L} \sim \frac{V}{\delta}$$

$$\therefore \boxed{V \sim \frac{\delta}{L} U_\infty}$$

Appreciate how this agrees with intuition because δ is our zoom in height. It's a small number. L is the length of the plate, it's usually pretty large. $\delta << L$. This means..

$$V \sim \{\text{supers small number}\}\{\text{air-velocity}\}.$$

$$\sim \text{tiny velocity upwards!}$$

Continuity helped us determine a scale for V. Now lets use it in the momentum equation (Navier-Stokes.)

$$\tilde{U} \sim \langle U_\infty u^*, \frac{\delta}{L} U_\infty v^*, 0 \rangle$$

$$\nabla \sim \langle \frac{1}{L}\frac{\partial}{\partial x^*}, \frac{1}{\delta}\frac{\partial}{\partial y^*}, \frac{1}{Z}\frac{\partial}{\partial z^*} \rangle$$

$$P \sim \rho U_\infty^2$$ (remember we always Non-Dim pressure by the Dynamic Pressure!)

LECTURE 7:

$$\frac{\rho U_\infty^2}{L}\left\{ u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right\} = -\frac{\rho U_\infty^2}{L}\frac{\partial P^*}{\partial x^*} + \frac{\mu U_\infty}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\mu U_\infty}{L\delta}\frac{\partial^2 u^*}{\partial y^{*2}} \quad \text{X-MOM}$$

Extra credit on $\delta << L$: write out the y-mom equation and based conclude "something", Compare the scale of the entire y-equation to the x-momentum.

Now we divide by $\rho U_\infty^2/L$ because we want to compare the scale of the left-hand side (inertia) to the right hand side (viscous + pressure forces).

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \underbrace{\frac{\mu}{\rho U_\infty L}}\frac{\partial^2 u^*}{\partial x^{*2}} + \underbrace{\frac{\mu}{\rho U_\infty \delta}}\cdot\frac{\partial^2 u^*}{\partial y^{*2}} \qquad (3)$$

* Since $\delta << L$ the second term is much bigger than the first

Remember in last lecture we derived a scale for δ based on a Reynold's!

$$\delta \sim \frac{L}{\sqrt{Re_L}}$$

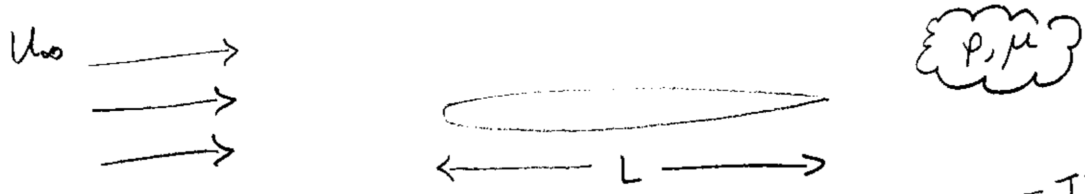
This fixes this mis-match of TWO Reynolds # that appear in (3). Sub it in and find another conclusion.

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{1}{Re^{1/2}}\frac{\partial^2 u^*}{\partial y^{*2}}$$

* This difference in power is critical

LECTURE 7

In boundary layer analysis we require large Re values for the free-stream velocity. Not turbulent, just not really-slow. So for



$$Re = \frac{\rho U_\infty L}{\mu} \gg 1$$

This is also the only Reynolds I have to calculate to conclude stuff

Say then $Re = 1300$, still laminar but now look at.

$$\frac{1}{Re} = \frac{1}{1300} \ll \frac{1}{36} = \frac{1}{Re^{1/2}} \; !$$

So the second term matters more! $\frac{\partial^2 u}{\partial y^2}$ is more important

This is why the Boundary Layer Equations we study are. Non-dimensional

The Boundary Layer Equations!

$$\left\{ \begin{aligned}
& u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dP^*}{dx^*} + \frac{1}{Re^{1/2}} \frac{\partial^2 u^*}{\partial y^{*2}} \\
& \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \\
& \left. \begin{aligned} u^*(x^*, 0) &= 0 \\ v^*(x^*, 0) &= 0 \end{aligned} \right\} \text{"no-slip at } y=0\text{"} \\
& \lim_{y^* \to 1} u^*(x^*, y^*) = U_\infty \; \} \text{ "stream-lines approach inviscid parallel constant flow } U_\infty \text{ after the boundary layer"}
\end{aligned} \right.$$

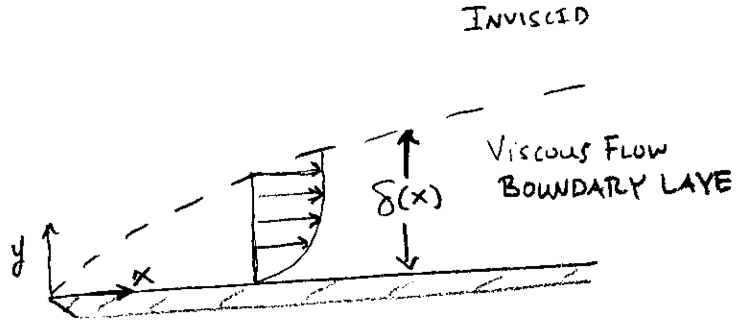
Extra-Credit "Dimensionalize" these equations

LECTURE 7:

Literally EVERYTHING WE KNOW ABOUT THE BOUNDARY LAYER, COMES FROM THOSE EQUATIONS!

* These equations are differential equations whos solution yields $u(x,y)$ & $v(x,y)$... Sound familiar... In advanced class yes you turn this into a stream function problem and get the solution for a flat-plate. But we don't need to go that far for this class. We will just list the useful equations.

Blasius Flat Plate Problem



(EXACT SOLUTIONS FOR THE MOST IDEAL CASE THAT NEVER HAPPENS...)

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \quad \text{(Boundary Layer)}$$

$$\frac{\delta^*}{x} = 1.721\, Re_x^{-1/2} \quad \text{(Displacement Thickness)}$$

I'LL EXPLAIN!

$$\frac{\Theta}{x} = 0.664\, Re_x^{-1/2} \quad \text{(Momentum Thickness)}$$

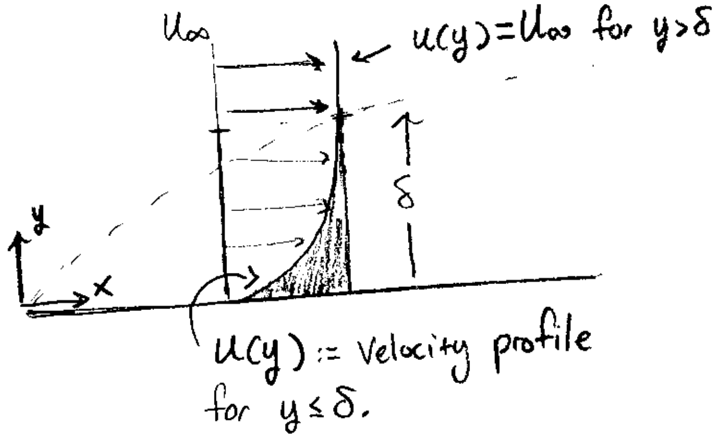
$$C_D = 1.328\, Re_L^{-1/2} \quad \text{(Drag Coefficient)}$$

This is what we're after! To see why refer to pg. 7 lecture 6.

LECTURE 7:

I now want to describe these δ^* and Θ. They are helpful quantities similar to the $\bar{V}$ and L_{eq} for internal pipe flow. Let's draw that picture ... again

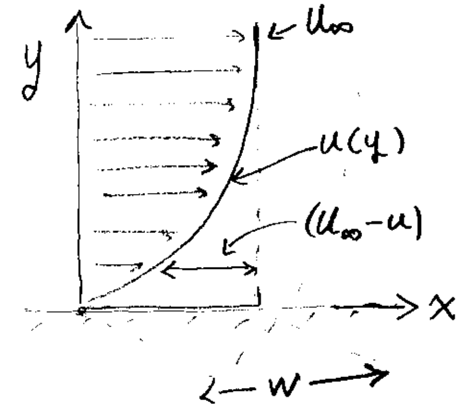
Displacement Thickness δ^*



* How much fluid is not flowing due to this no-slip condition which produces a boundary-layer?

$$Q = \int_0^\delta (\text{not flowing}) \cdot (\text{width}) \cdot (\text{little thickness})$$

$$= \int_0^\delta (U_\infty - u) \, W \, dy$$



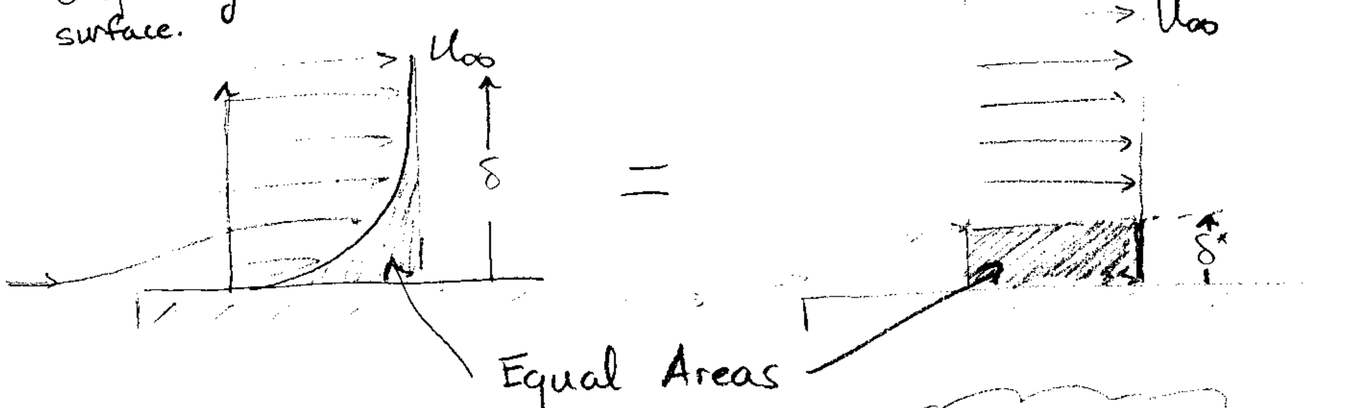
Just like we want numbers not integrals we equate this to a simple scalar expression.

multiplication of 2 scalar numbers! (easy) $\rightarrow$ $$U_\infty \cdot A = W \int_0^\delta (U_\infty - u) \, dy$$ $\leftarrow$ requires knowledge of calculus

$$\cancel{U_\infty}(\cancel{W}\delta^*) = \cancel{U_\infty W} \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\therefore \quad \delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

Graphically δ^* means, how far a streamline is displaced from the surface.



Reality

$u=0$ @ $y=0$

Idealization
with all invisicd flow even at surface

$u \neq 0$ @ $y=0$

* Important to notice that δ^* is a point wise property. It is not constant as we traverse the plate.

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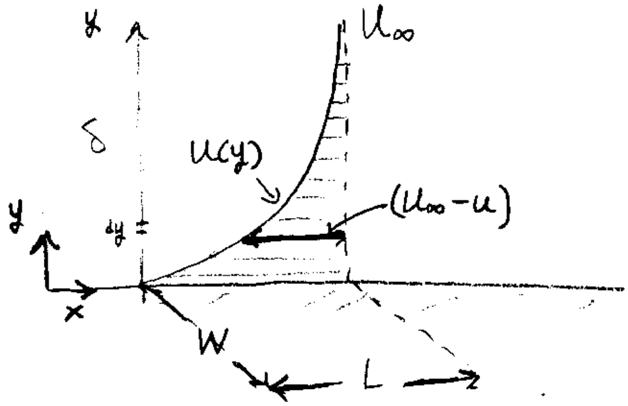
Momentum Thickness Θ

* What is the Kinetic energy of boundary layer for the fluid NOT flowing.



$$\int_0^\infty \frac{1}{2} m u^2 \, dy + \int_0^\infty \frac{1}{2} m (U_\infty - u)^2 \, dy = \underbrace{K_{TOT}}_{\equiv 0}$$

$$\rho WL \int_0^\infty (u^2 - uU_\infty)\, dy + \frac{\rho WL}{2} \int_0^\infty U_\infty^2 \, dy = 0$$ ← No Kinetic energy cause it's not moving.

factor out a U_∞^2.

$$\therefore \rho \frac{WL}{2} \int_0^\infty U_\infty^2 dy = \rho WL \Theta U_\infty^2 = \rho WL \int_0^\infty (uU_\infty - u^2)\, dy$$

CONT

LECTURE 7

This is weird I know, but again just like $\bar{V}$ and L_{eq} we're using math to equate things to turn calculations involving <u>calculus</u> $\Rightarrow$ into calculations involving <u>algebra</u>. Our little arguement left us at.

$$\cancel{\rho W L}\, \Theta\, \cancel{U_\infty^2} = \cancel{\rho W L U_\infty^2} \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

I want to know this <u>number</u>

$$\therefore \quad \Theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Extra Credit: <u>show</u> why we <u>don't</u> have to integrate to $y \to \infty$ but only δ the boundary layer thickness. Prove the equivalent def is actually.

$$\Theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

So the <u>3</u> <u>fundamental</u> qualtities of B.L. analysis are just that

$$\{\, \delta\ ,\ \delta^*\ ,\ \Theta \,\}$$

{ Boundary Layer Thickness, Displacement Thickness, Momentum Thickness }

LECTURE 7:

IT TURNS OUT THAT THESE QUANTITIES MUCH LIKE THE FRICTION FACTORS ARE FUNCTIONS OF REYNOLDS!

$$\frac{\delta}{x} = \frac{C_1}{\sqrt{Re_x}}$$

δ^* helps to make a computer plot streamlines $\longrightarrow$
$$\frac{\delta^*}{x} = \frac{C_2}{\sqrt{Re_x}}$$

Θ helps to calculate τ_w $\longrightarrow$
$$\frac{\Theta}{x} = \frac{C_3}{\sqrt{Re_x}}$$

These constants depend on geometry of the surface or approximations of $u(y)$ in the boundary layer

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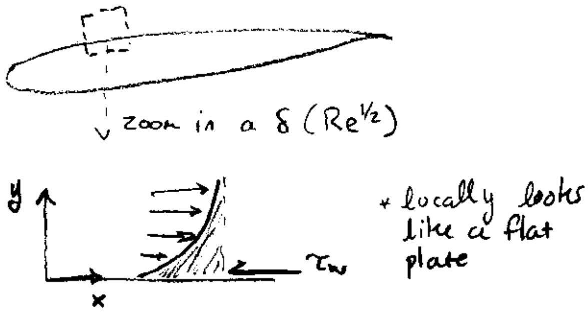
Shear Stress at surface walls?

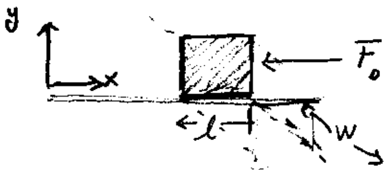
* Let's consider the drag force F_D.



$$\underbrace{\Delta KE} = W = F_D \cdot l$$

$$\rho W \cancel{l} \Theta U_\infty^2 = F_D \cancel{l}$$

$$\therefore \quad F_{DRAG} = \rho W U_\infty^2 \Theta$$



* dF/dx tells us how drag changes along the plate.

$$\frac{dF_D}{dx} = \rho W U_\infty^2 \frac{d\Theta}{dx}$$

* $dF_D = \tau_w W dx$

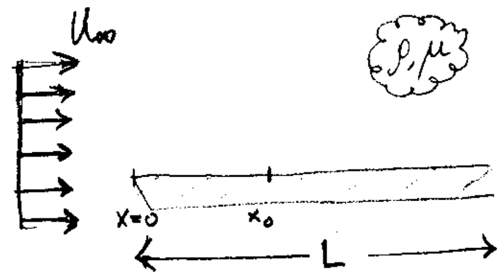
$$\therefore \quad \boxed{\tau_w = \rho U_\infty^2 \frac{d\Theta}{dx}}$$

* We can guess a $u(y)$ and compute the right hand side!

LECTURE 7

Let's now start from ground zero and see all this work together to calculate stuff. Say we have a plate in a flow that

1) What is δ at x_o
2) τ_w at x_o
3) plot a streamline flowing at U_∞ at x_o
4) Drag force on top plate



0) Calculate $Re_{x_o} = \dfrac{\rho U_\infty x_o}{\mu}$

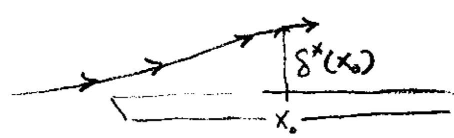
i) Is $Re < 5\times10^5$? ← This is the turbulent critical value.

ii) If yes continue, if no wait for next lecture.

1) $$\delta = \frac{x_o \cdot 5}{\sqrt{Re_{x_o}}}$$ plug in-numbers.

2) $$\tau_w = 0.332\, U_\infty^{3/2} \sqrt{\frac{\rho\mu}{x_o}}$$

3) $$\delta^* = \frac{1.721\, x_o}{\sqrt{Re_{x_o}}},$$



4) $$F_D = \rho W U_\infty^2 \Theta, \quad \Theta = \frac{0.664 \cdot x_o}{\sqrt{Re_{x_o}}}$$

Extra Credit

Make excel give you all values based on $U_\infty, x_o, \rho, \mu, L$. Make sure it has a check for $Re_{x_o} < 5\times10^5$