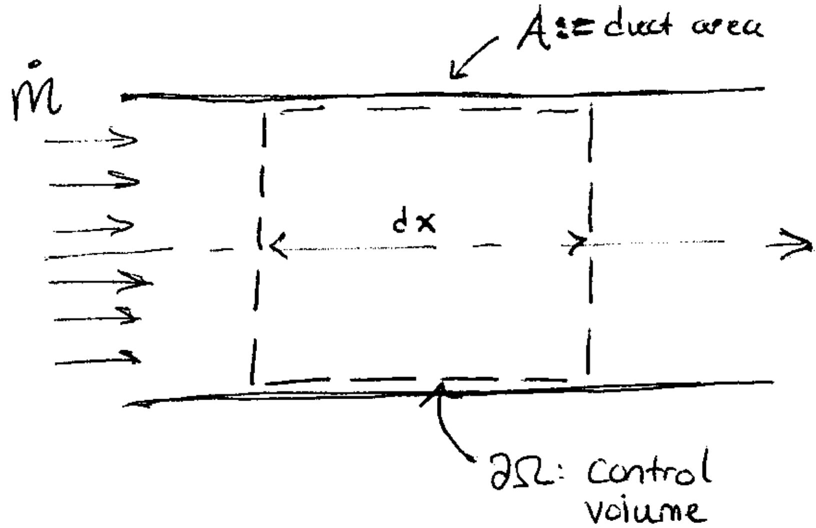
LECTURE 16 : What if the walls are rough?

Now we want to look at the rough wall simple problem. We will use a single straight section of pipe as well.



$\frac{dA}{dx} = 0$ (straight)

$q = 0$ (adiabatic)

$\mu, \varepsilon, \Delta S > 0$ (frictional losses)

* If we are going to consider frictional losses we must consider shear at the wall. τ_w

* We also wouldn't you believe it need a FANNING FRICTION FACTOR "$\mathcal{F}$" which we will define as.

$$\mathcal{F} = \frac{\tau_w}{\frac{1}{2}\rho V^2} \qquad (1)$$

Since we are working usually with NON-CIRCULAR DUCTS all diameters will be hydraulic

$$D_H = \frac{4A}{P} \qquad (2)$$

LECTURE 16 :

For our mass balance of the control volume we can just use the previous result.

$$\cancel{\frac{dA}{A}} + \frac{d\rho}{\rho} + \frac{dV}{V} = 0$$

$$\therefore \boxed{\frac{d\rho}{\rho} + \frac{dV}{V} = 0}$$

Like wise we will assume an ideal gas (as always).

$$\boxed{P = \rho R T}$$

The energy equation for a steady state gas problem is phrased in terms of enthalpy.

$$\boxed{h + \frac{V^2}{2} = C_o} \quad \text{(for all states)}$$

All we need is a momentum relationship to tell us how pressure (forces) prescribe motion (velocity).

$$\Sigma F_{ext} = \int_{\partial\Omega} \rho A V(\vec{V}\cdot\hat{n})\,dA$$

$$AP - A(P+dP) - \tau_w P\,dx = \rho A V(V+dV-V)$$

$$\therefore \boxed{dp + \frac{4\mathcal{F}}{D_H}\frac{\rho V^2}{2}dx + \frac{\rho V^2}{2}\frac{dV^2}{V^2} = 0}$$

LECTURE 16

Essentially all we do now is use all our equations to sub into our momentum relationship. But let's write them all out to see how one would come to this solution.

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \qquad \text{"continuity"} \qquad (1)$$

$$P - \rho RT = 0 \qquad \text{"equation of state"} \qquad (2)$$

$$dh + VdV = 0 \qquad \text{"energy"} \qquad (3)$$

$$Ma^2 - \frac{V^2}{\gamma RT} = 0 \qquad \text{"Mach Definition"} \qquad (4)$$

$$ds \geq 0 \qquad \text{"Irreversibility"} \qquad (5)$$

$$dP + \frac{4\mathcal{F}}{D_H}\frac{\rho V^2}{2}dx + \frac{\rho V^2}{2}\frac{dV^2}{V^2} = 0 \qquad \text{"Momentum"} \qquad (6)$$

We have count'em 7 unknowns and 6 equations!

$$(Ma, V, P, \rho, T, s, \mathcal{F}) \; ????$$

This means we have to set a variable to get a solution.

What variable should we choose to set then? Well again we are modeling the question...

How Does Friction Effect Flow?

So it's reasonable then to say we should set the friction term as an indepedent variable to probe it's effects on flow variables (V, P, ρ, T, s, Ma)!

Now this get's a little rough but let's hold onto a goal. In the end we are trying to relate everything to (Ma, γ).

$$\frac{dMa}{Ma} = f_1(\mathcal{F}) \qquad \frac{dP}{P} = f_3(\mathcal{F}) \qquad ds = f_5(\mathcal{F})$$

$$\frac{dV}{V} = f_2(\mathcal{F}) \qquad \frac{dT}{T} = f_4(\mathcal{F})$$

We are trying to determe these functions, So we are going to combine all our equations by subbing and resubbing until we relate

$$\frac{dMa}{Ma} \sim \frac{4f\,dx}{D_H}$$ → then back sub for the rest!

You might as well see solving this many equation at least <u>once</u> in your undergrad. I will highlight common tricks and rational. We are going to use all (1)-(6) to arrive at $\frac{4\mathcal{f}dx}{D_H} = C(M,\gamma)\frac{dM}{M}$.

Start with momentum. Why!? Because it's the only one with the $\mathcal{f}$ term!

$$\frac{dP}{P} + \frac{4\mathcal{f}}{D_H}\frac{\rho V^2}{2P}dx + \frac{\rho V^2}{2P}\frac{dV^2}{V^2} = 0$$

$$\therefore \quad \frac{\rho V^2}{2P}\cdot\frac{4\mathcal{f}}{D_H}dx = \underbrace{-\frac{dP}{P} - \frac{\rho V^2}{2P}\frac{dV^2}{V^2}}_{\text{use (1)-(6) to get in terms of Ma!}}$$

We use (2) to get rid of dP/P. Take (d) of (2)

$$dP = d\rho RT + dT\rho R \quad / \tfrac{1}{P}$$

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Now we can use (1) to replace $d\rho/\rho$!

$$\frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V}$$

Where are we now then...

$$\frac{\rho V^2}{2P}\cdot\frac{4\mathcal{f}}{D_H}dx = \frac{dV}{V} - \frac{dT}{T} - \frac{\rho V^2}{2P}\frac{dV^2}{V^2}$$

Now we must get $\frac{dV}{V}$, $\frac{dT}{T}$, $\frac{\rho V^2}{2P}$, & $\frac{dV^2}{V^2}$ terms into Ma & γ expressions.

Lecture 16

We use (3) and the definition of c_p (This is our first trick but it makes sense as we want to go from $dh \rightarrow dT$ which is what c_p does!)

$$\underbrace{dh = -VdV = -\tfrac{1}{2}\,dV^2}_{\substack{d(\,)\text{ basically}\\ \text{behaves like}\\ \text{the derivative}\ldots}} = \underbrace{c_p dT}_{\substack{\text{since } c_p = \frac{dh}{dT}\\ \text{FOR IDEAL GAS!!!}}}$$

{divide by $c_p T$}

$$\frac{dT}{T} = -\frac{1}{2}\frac{dV^2}{c_p T} = \frac{-dV^2}{2 c_p T}$$

We need to get to Ma relations so c_p & T are problematic.

$$c_p - c_v = R \Rightarrow c_p = R + c_v = R + (c_p - R)$$

$$c_p = \gamma R / (\gamma - 1) \qquad \text{(a dirty trick...)}$$

$$\frac{dT}{T} = \frac{-dV^2}{2}\frac{(\gamma-1)}{\underbrace{\gamma RT}_{\text{use (4)}}} = -\frac{dV^2}{V^2} Ma^2 \frac{\gamma-1}{2} \checkmark$$

Revaluate now...

$$\frac{\rho V^2}{2P}\cdot\frac{4f}{D_H}\,dx = \frac{dV}{V} + \frac{\gamma-1}{2} Ma^2 \frac{dV^2}{V^2} - \frac{\rho V^2}{2P}\frac{dV^2}{V^2}$$

Lecture 16

We now have to deal with this $\rho V^2/2P$ term...

$$\rho V^2 = \rho \frac{V^2}{\gamma R T} \gamma R T = \gamma Ma^2 P \quad \text{(dirty trick...)}$$

Revaluate...

$$\frac{4 f}{D_H} dx = \frac{dV}{V} + \frac{\gamma-1}{2} Ma^2 \frac{dV^2}{V^2} - \frac{\gamma Ma^2 \cancel{P}}{2 \cancel{P}} \frac{dV^2}{V^2}$$

eliminated pressure!!

Now we need to get dV^2/V^2 into Ma expressions as well. Let's do a d( ) to (4) to build up dMa^2/Ma from scratch.

$$d Ma^2 = d\left(\frac{V^2}{\gamma RT}\right) = \frac{dV^2 \gamma RT - \gamma R dT V^2}{(\gamma RT)^2} \quad \Big/ \frac{1}{Ma^2}$$

$$\frac{dMa^2}{Ma^2} = \frac{dV^2}{\gamma RT\, Ma^2} - \frac{\gamma R V^2 dT}{(\gamma RT)^2 Ma^2}$$

$$= \frac{dV^2}{V^2} - \frac{dT}{T}$$ because $Ma^2 = V^2/\gamma RT$

We also have an expression for dT/T though!

$$\frac{dMa^2}{Ma^2} = \frac{dV^2}{V^2} + \frac{\gamma-1}{2} Ma^2 \frac{dV^2}{V^2} = \left(1 + \frac{\gamma-1}{2} Ma^2\right) \frac{dV^2}{V^2}$$ Boom!

Revaluate!

$$\frac{\rho V^2}{2P} \frac{4 f}{D_H} dx = \frac{dV}{V} + \left(\frac{\gamma-1}{2} Ma^2 - \frac{\gamma Ma^2}{2}\right) \frac{dV^2}{V^2}$$

Lecture 16

Almost done! Just need to take care of dV/V.

$$\frac{\rho V^2}{2P}\frac{4\mathcal{F}}{D_H}dx = \underbrace{\frac{dV}{V}}_{\text{Last son of a bitch!}} + \left(\frac{\cancel{\gamma}-1}{\cancel{2}}\cancel{Ma^2} - \frac{\gamma \cancel{Ma^2}}{\cancel{2}}\right)\left(\frac{\gamma-1}{2}Ma^2 + 1\right)^{-1}\frac{dMa^2}{Ma}$$

! We can't just square root our dV^2/V^2 term !
! Because its $d(V^2)/V^2$ not $(dV)^2/V^2$!!!!!!

We can use (1) since we haven't touched it yet.

$$\frac{dV}{V} = -\frac{d\rho}{\rho}$$ or divide now by $\rho V^2/2P$.

$$\frac{4\mathcal{F}}{D_H} = \frac{2P}{\rho V^2}\left\{\frac{dV}{V} + \frac{-Ma^2/2}{\left(\frac{\gamma+1}{2}Ma^2+1\right)}\frac{dMa^2}{Ma^2}\right\}$$

$$= \frac{2\cancel{P}}{\gamma Ma^2 \cancel{P}}\left\{\frac{dV}{V} + \dots\right\}$$

$$= \frac{2}{\gamma Ma^2}\left\{\frac{1}{2}\frac{dV^2}{V^2} + \dots\right\}$$

Just sub in for dV^2/V^2 our dMa^2/Ma^2 rearrange to get

(**)

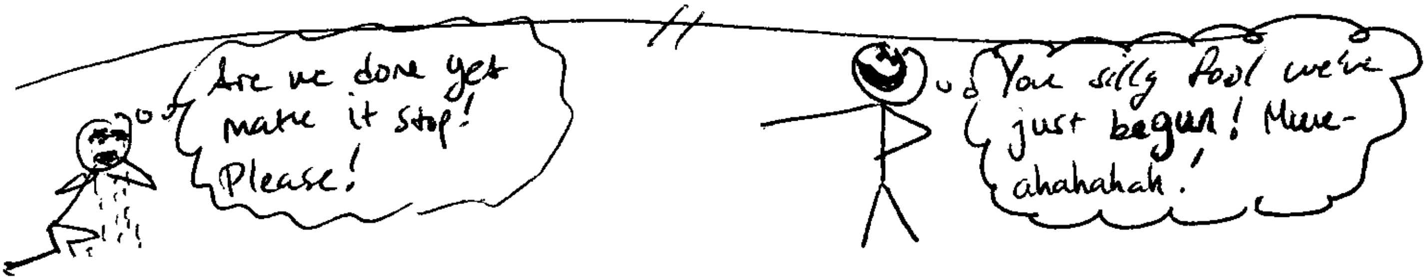
$$\frac{4\mathcal{F}}{D_H}dx = \frac{2(1-Ma^2)}{\gamma Ma^3\left(1+\frac{\gamma-1}{2}Ma^2\right)}\frac{dMa}{Ma}$$

This ode for $Ma(x)$ tells us how losses effect speed as we traverse the rough duct. The ODE is found solving for dMa/dx!

$$\frac{\cancel{4}\mathcal{F}\,Ma}{D_H}\,\frac{\gamma Ma^3\left(1+\frac{\gamma-1}{2}Ma^2\right)}{\cancel{2}(1-Ma^2)} = \frac{dMa}{dx}$$

$$\therefore\quad \frac{dMa}{dx} = \frac{2\mathcal{F}\gamma\, Ma^4\left(1+\frac{\gamma-1}{2}Ma^2\right)}{(1-Ma^2)}$$

- Non-linear
- First Order
- Autonomous
- singular



We can use (**) to just back sub now for dp/p, dT/T, $d\rho/\rho$, and dV/V!

$$\frac{dV}{V} = \frac{\gamma Ma^2}{2(1-Ma^2)}\,\frac{4\mathcal{F}dx}{D_H} \implies \text{ODE for } V(x)$$

$$\frac{dP}{P} = -\frac{\gamma Ma^2\left(1+[\gamma-1]Ma^2\right)}{2(1-Ma^2)}\,\frac{4\mathcal{F}dx}{D_H} \implies \text{ODE for } P(x)$$

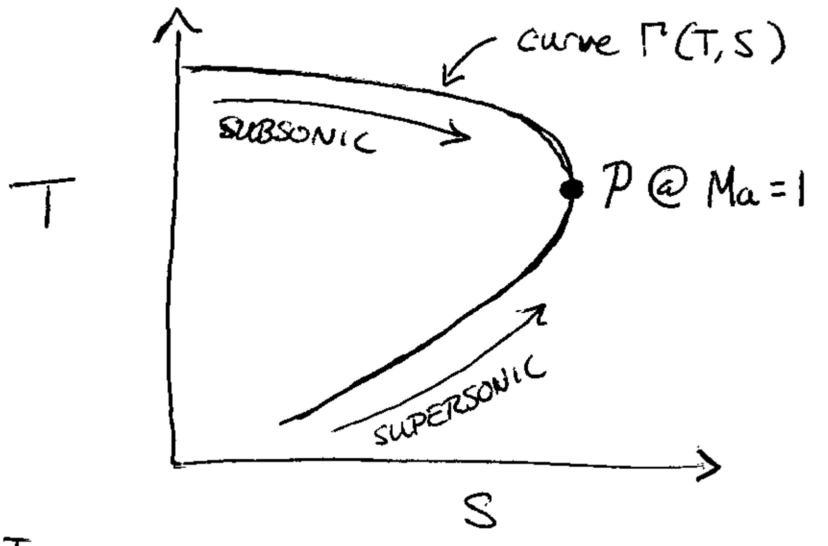
$$\frac{dT}{T} = -\frac{\gamma(\gamma-1)Ma^4}{2(1-Ma^2)}\,\frac{4\mathcal{F}dx}{D_H} \implies \text{ODE for } T(x)$$

Lecture 16

More often we actually use "Fanno" Lines to analyze systems though. I'll spare you their derivations after that last round and just explain them.

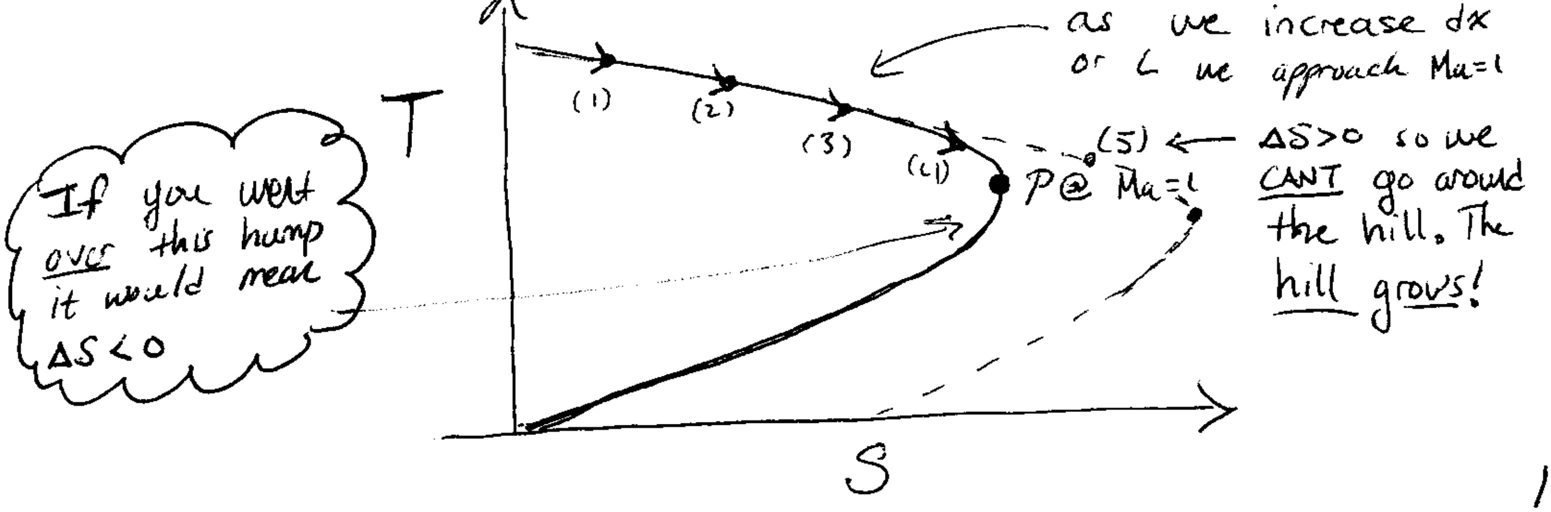
- The curve Γ represents all the states possible for Fanno Flow with a FIXED $\dot{m}$ and h.

- For a given state at a given mass flowrate states MOVE TO THE RIGHT.



I honestly believe this is the first time you have to apply the 2nd law to rationalize something. Usually its all 1st Law.

a) What happens if flow starts Ma < 1 and length is continually added to the section of pipe?



Lecture 16

As we approach $Ma = 1$ the flow becomes <u>choked</u> by friction. A <u>decrease</u> in $\dot{m}$ results

THIS MEANS FOR A GIVEN INLET Ma CONDITION THERE EXIST A L_{MAX} PAST WHICH $\dot{m}$ DECREASES

* It is surprising that this compressible phenomena of "choking" or maxing out $\dot{m}$ occurs even for a straight pipe due to frictional effects

//

The other game for Fanno Flow is the use of $\boxed{\text{Tables}}$ that tell the user about choked flow

Ma	T/T_*	P/P_*	P_t/P_{t*}	V/V_*	$\bar{f}L_{max}/D$
.	⋮	⋮	⋮	⋮	⋮

Where $(\,)_*$ is a property at <u>$Ma = 1$</u>.

Lecture 16

The values in these tables come from Numerical Integration of our awful, disgusting vile, reprehensible equations we derived earlier.

For example...

$$\int_0^{L_{max}} \frac{4 \mathcal{F} dx}{D_H} = \int_{M = Ma \in \{0.01, 0.02, \ldots, 4.99, 5\}}^{M=1} f(Ma)\, dMa$$

upper limit we know is for Ma=1

computer integrates for a discrete set of Ma values.

start at any arbitrary Ma value

$$\therefore \quad \frac{\mathcal{F} L_{max}}{D_H} = \left\{ \text{Make compooter go brrrrr.} \right\}$$

We do the exact same for the other relationships.

$$\int_P^{P^*} \frac{dP}{P} = \int_M^1 -\frac{\gamma Ma^2 (1-(\gamma-1)Ma^2)}{2(1-Ma^2)} \frac{4\mathcal{F} dx}{D_H}$$

but we can express this in terms of Ma number! (**)

$$= \int_1^M \left[\frac{1+(\gamma-1)Ma^2}{1+\frac{\gamma-1}{2}Ma^2} \right] \frac{dM}{M} = \ldots$$

$$= \left\{ \text{compooter calculates for a bunch of Ma values} \right\}$$

Lecture 16

Make sure you realize that you have to exponentiate the right-hand side after the compooter is done integrating, because say you get a value of α for integration.

$$\int_P^{P^*} \frac{dP}{P} = \ln(P)\Big|_P^{P^*} = \ln\left(\frac{P_*}{P}\right) = \alpha \Rightarrow \boxed{\frac{P_*}{P} = e^{\alpha}}$$

SO WHEN CONNECTING 2 STATES OF GAS FLOW THAT FALL IN THE FANNO ASSUMPTIONS YOU MUST CONSULT A TABLE/DATABASE AND RELATE A STATE TO A CHOKED/MAXIMUM FLOW

Some qualitative properties that can be drawn from Fano Lines.

If $Ma < 1 \Rightarrow T \downarrow$ as $Ma \uparrow$

If $Ma > 1 \Rightarrow T \uparrow$ as $Ma \downarrow$

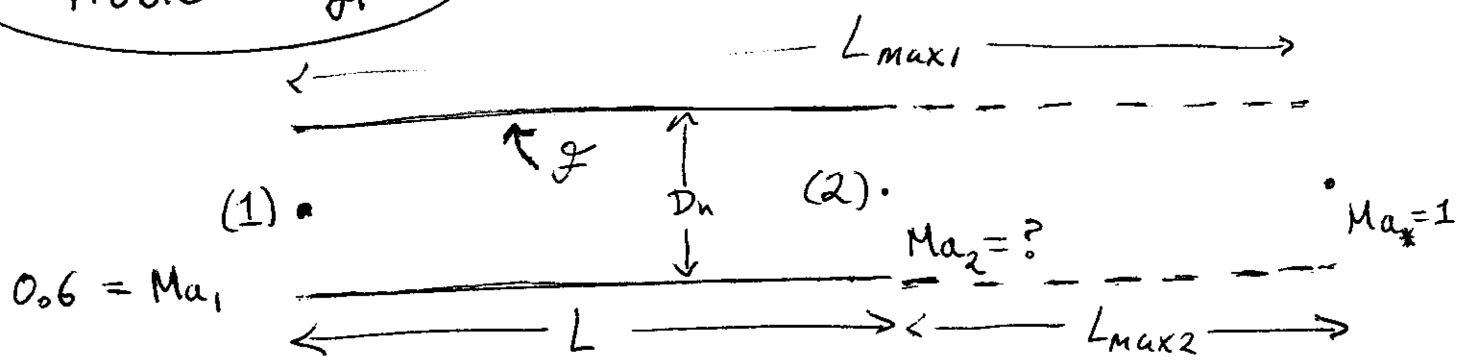
Some tricks using tables..., say you want to get from state 1 to 2 but don't have to consider $(\,)_*$ properties in your problem?

$$\frac{P_2/P_*}{P_1/P_*} = \frac{P_2}{P_1}$$ dirty trick right...

Lecture 16

More on the "flow" of using tables is that you must specify the Ma value or the friction term $4\bar{f}L_{max}/D_h$ to start solving anything. This is because this will tell you which row you're on.

Problem Types



Determine exit properties at (2)?

$$\alpha = \frac{\bar{f}L_{max_1}}{D_H} = \frac{\bar{f}L_{max2}}{D_H} + \frac{\bar{f}L}{D_H}$$

* you can get this number from a table because $Ma_1 = 0.6$ which specifies a row to look up the friction term.

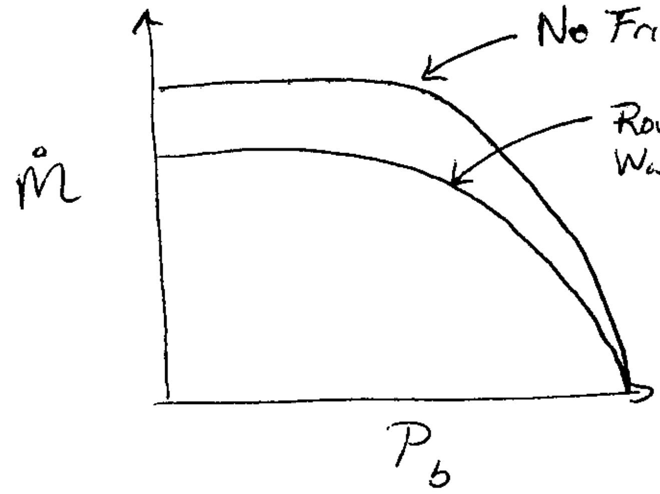
$$\therefore \quad \frac{\bar{f}L_{max_2}}{D_H} = \frac{\bar{f}L_{max_1}}{D_H} - \alpha$$

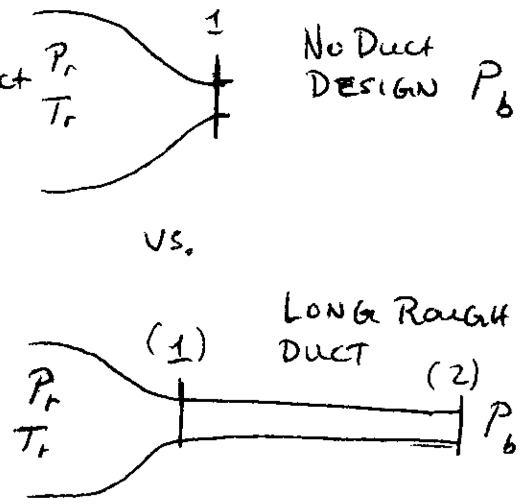
* Since this specifies another row both states are now completely specified and both states can be related by

$$\frac{(\ )_2}{(\ )_1} = \frac{(\ )_2/(\ )_*}{(\ )_1/(\ )_*} \quad !$$

LECTURE 16

If we recall our $\dot{m}$ graphs from last time this frictional loss is reducing $\dot{m}_{max}$ from the ideal case.





However... would you believe that people actually figured out all those integrals? Thru crazy algebra you can show... i.e. (partial fractions)

$$\frac{4L_{max}}{D_H}f = \frac{1-Ma^2}{\gamma Ma^2} + \frac{\gamma+1}{2\gamma}\ln\left(\frac{(\gamma+1)Ma^2}{2\left(1+\frac{\gamma-1}{2}Ma^2\right)}\right) \qquad (1)$$

$$\frac{V}{V_*} = Ma\left\{\frac{\gamma+1}{2+(\gamma-1)Ma^2}\right\}^{1/2} \qquad (2)$$

$$\frac{P}{P_*} = \left\{\frac{\gamma+1}{Ma^2(2+(\gamma-1)Ma^2)}\right\}^{1/2} \qquad (3)$$

Extra credit make your own Fanno Tables in excel

$$\frac{T}{T_*} = \frac{\gamma+1}{2+(\gamma-1)Ma^2} \qquad (4)$$

$$\frac{\rho}{\rho_*} = \frac{V^*}{V} \quad \text{and} \quad \frac{s-s_*}{c_p} = \ln\left\{Ma^2\right\}\left\{\frac{\gamma+1}{Ma^2[2+(\gamma-1)Ma^2]}\right\}^{\frac{\gamma+1}{2\gamma}} \qquad (5)$$