Lecture 15 : The Most <u>Ideal</u> Case of This Difficult Problem.

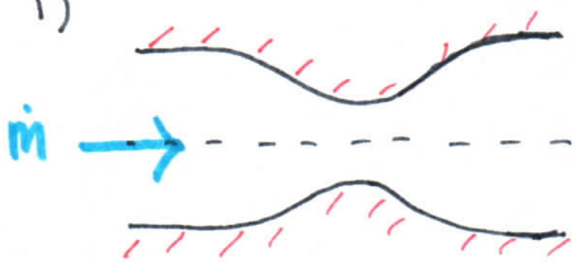
We now would like to start making predictions about the flow part of compressible flow.

compressible — a thermodynamic problem

flow — this is simply a "process" that thermo speaks of.

Of the simple types of flows we could consider well let's draw some simple pictures.

1)

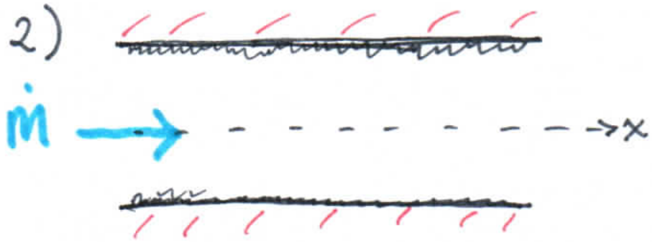


$\frac{dA}{dx} \neq 0$ — changing area

(adiabatic) $q \approx 0$ — negligable heat transfer thru walls

(isentropic) $\mu, \varepsilon \approx 0$ — smooth walls

2)

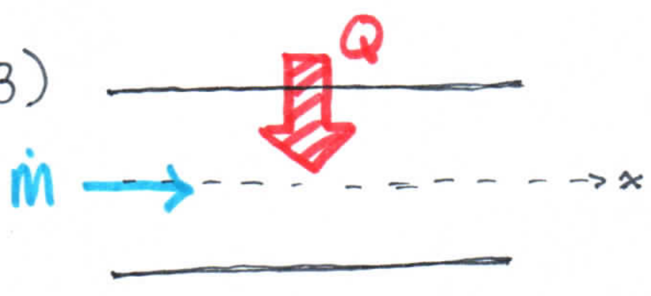


$\frac{dA}{dx} \cong 0$ — constant duct area

$q \approx 0$ — no heat transfer

$\mu, \varepsilon > 0$ — frictional effects

3)



$\frac{dA}{dx} = 0$ — constant duct area

$q \gg 0$ — considerable heat flux

$\mu, \varepsilon \approx 0$ — smooth walls

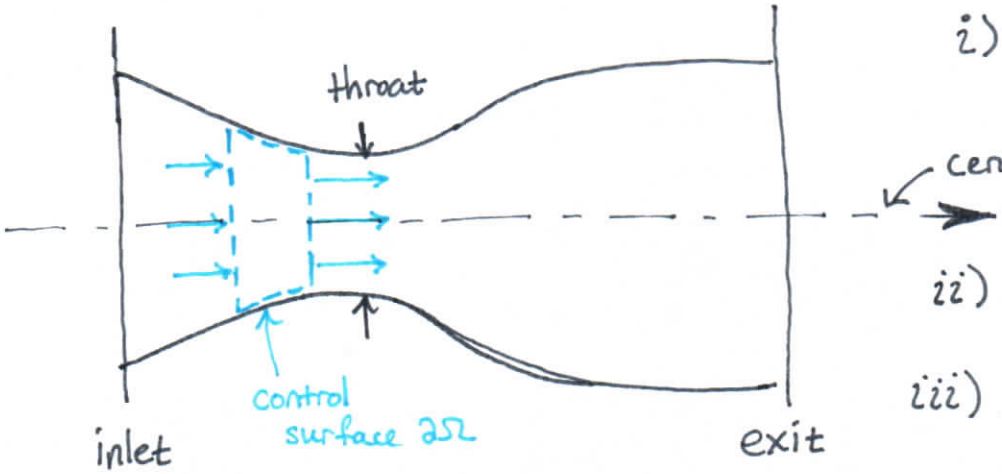
In reality these cases depend on the relative magnitude of the three effects.

These 3 simple problems allow for simple analysis, but also is the case for all analysis really. I like to think of all these cases as a well ordering of flow effects.

Ordering		Name
$\lvert q\rvert \ll \left\lvert \frac{dA}{dx} \right\rvert$ $\lvert\mu\rvert, \lvert\varepsilon\rvert \ll \left\lvert \frac{dA}{dx} \right\rvert$	**EASY**	"Isentropic Flow"
$\left\lvert \frac{dA}{dx} \right\rvert \ll \lvert\varepsilon\rvert$ $\lvert q\rvert \ll \lvert\varepsilon\rvert$		"Fanno Flow"
$\left\lvert \frac{dA}{dx} \right\rvert \ll \lvert q\rvert$ $\lvert\varepsilon\rvert, \lvert\mu\rvert \ll \lvert q\rvert$		"Rayleigh Flow"

It turns out that isentropic flow just produces basically the simplest analysis. There would no way to know this without just going down each rabbit hole back in the day and quitting the more difficult cases after considerable hair loss.

Let's draw some pictures and control volumes!!!



i) throat $:\Rightarrow \left.\frac{dA}{dx}\right|_{x_t} = 0$

ii) $\frac{dA}{dx} > 0 \Rightarrow$ "Diffuser"

iii) $\frac{dA}{dx} < 0 \Rightarrow$ "Nozzle"

I. Conserve Mass (Steady Flow)

$$\{Out\} - \{In\} = 0$$

$$(\cancel{\rho}+d\rho)(\cancel{A}+dA)(\cancel{V}+dV) - \cancel{\rho A V} = 0$$

a) The first triple product will cancel out

b) Ignore all dxdy terms as they are negligable.

$$\boxed{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0}$$

II. Use God Damn Bernoulli's Again (My own way...)

Since we assumed frictionless flow we can use the Euler Equations $\Rightarrow$ Bernoulli Time! Consider the velocity along the center line which makes $dz \equiv 0$.

One Way is to start from the 1D Euler Equation...

$$\underbrace{\cancel{\rho \frac{du}{dt}}}_{\text{steady}} + \rho \frac{dV}{\cancel{dx}} \cdot V = \frac{dP}{\cancel{dx}}$$

if dx/dt is gone these "cancel"... and also boil mathematician blood!

$$\therefore \quad \frac{1}{\rho} = -V\frac{dV}{dP}$$

LECTURE 13

* Another way would be to use bernoullir along the center line

cancel.

$$\left(\cancel{P} + \frac{1}{2}\rho V^2 + \cancel{z}\right) = \left((P+dP) + \frac{1}{2}(\rho+d\rho)(V+dV)^2 + \cancel{z}\right)$$

along the center line.

$$\frac{1}{2}\rho V^2 = dP + \frac{1}{2}(\rho+d\rho)(V+dV)^2$$

$$+\frac{1}{2}(\rho+d\rho)(V^2 + 2VdV + \cancel{dV^2})$$

always ignore $d(\ )^2$ terms

$$\cancel{\frac{1}{2}\rho V^2} = \cancel{\frac{1}{2}\rho V^2} + \underbrace{\rho V dV + \frac{1}{2}d\rho V^2} + \cancel{\rho V d\rho dV} + dP$$

• Which of these terms matters <u>most</u>

Extra Credit reason why $\rho V dV \gg \frac{1}{2}d\rho V^2$

$$\therefore \quad \boxed{\frac{1}{\rho} = -V\frac{dV}{dP}} \quad \text{again!}$$

So now we sub our $1/\rho$ relation into I.

$$-V\frac{dV}{dP}d\rho + \frac{dA}{A} + \frac{dV}{V} = 0$$

<u>REMARK!</u> In incompressible flow continuity was usually used to find substitutions for velocity. For <u>compressible</u> flow we use it for density subs.

LECTURE 15

Now let's really piss off mathematicians...

$$\underbrace{-V^2 \frac{d\rho}{dP}\frac{dV}{V}} + \frac{dA}{A} + \frac{dV}{V} = 0$$

I swapped $d\rho$ and dV and factored out a V... take that rigor police!

All this because recall.

$$c^2 = \left.\frac{dP}{d\rho}\right|_{s=s_0} \quad \text{for an } \underline{\text{isentropic}} \text{ process!}$$

yeah that's right I just flipped the derivative like a fraction! Come at me bro!?

$$-\frac{V^2}{c^2}\frac{dV}{V} + \frac{dV}{V} = -\frac{dA}{A}$$

$$\frac{V^2}{c^2}\frac{dV}{V} - \frac{dV}{V}$$

$$(Ma^2 - 1)\frac{dV}{V} = \frac{dA}{A}$$

Which is more commonly presented as...

$$\boxed{\frac{dV}{V} = \frac{1}{Ma^2 - 1}\frac{dA}{A}} \qquad (*)$$

Which I like to think of as a non-linear ODE. Which determines a $V(A)$ function.

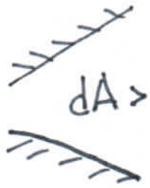
$$\frac{dV}{dA} = \Gamma(V)\frac{V}{A} \quad \text{for } \Gamma(V) = \frac{1}{(V/c)^2 - 1}$$

- Nonlinear ODE
- Non-autonomous
- First Order
- Singularities at $V=c$ & $A=0$

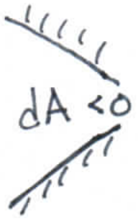
LECTURE 15 :

We need to STAHP and appreciate the non-intuitive equation we just wrote down. Because of the $\Gamma(Ma)$ term flow behaves differently if $Ma > 1$ or $Ma < 1$!

$Ma < 1$ (Subsonic Flow)



$$\Rightarrow \quad dV < 0 \,,\; dP > 0 \,,\; d\rho > 0$$



$$\Rightarrow \quad dV > 0 \,,\; dP < 0 \,,\; d\rho < 0$$

$Ma > 1$ (SuperSonic Flow)

$$dA > 0 \quad \Rightarrow \quad dV > 0 \,,\; \{dP \,,\; d\rho\} < 0$$

$$dA < 0 \quad \Rightarrow \quad dV < 0 \,,\; \{dP \,,\; d\rho\} > 0$$

This is weird right?! It's doing the opposite of what we're used to! An increase in area causes an increase in velocity?

The dP and dρ relation can be dertimined from just rewriting (*) interms of dP or dρ.

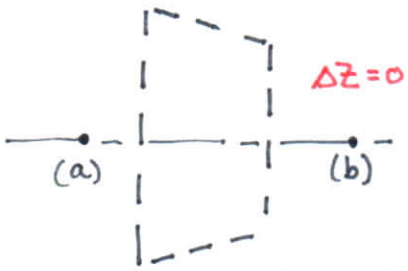
$$\boxed{\frac{dP}{\rho V^2} = \frac{1}{1 - Ma^2}\frac{dA}{A}} \quad \& \quad \boxed{\frac{d\rho}{\rho} = \frac{Ma^2}{1 - Ma^2}\frac{dA}{A} = -Ma^2\frac{dV}{V}}$$

* because

$$\frac{1}{\rho} = -V\frac{dV}{dP}$$

LECTURE 15 :

Has anyone realized we have not yet invoked any thing about an ideal gas. We haven't used our equation-of-state! Recall equations-of-state relate $\rho(P,T)=0$. So we can bring in temperature into our relationships. Let's look at the centerline again and energy.



$$\left(\frac{P}{\rho} + \frac{V^2}{2} + \cancel{gz}\right)_a = \left(\frac{P}{\rho} + \frac{V^2}{2} + \cancel{gz}\right)_b$$

$$\frac{P_a}{\rho_b} + \frac{V_a^2}{2} = \frac{P_b}{\rho_b} + \frac{V_b^2}{2}$$

$P = \rho RT$

$\therefore \boxed{\frac{P}{\rho} = RT}$!

a) We can use the ideal gas law to make this temperature!

b) We will also consider stagnation for a reference. Or $V_b \equiv 0$.

$$RT_* + \frac{V_*^2}{2} = RT_0$$ ← Stagnation temp!

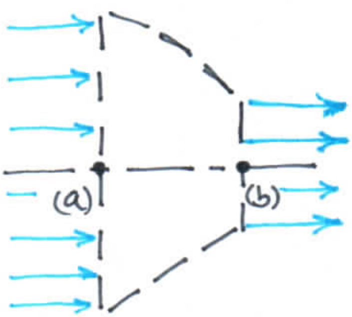
$$\therefore \quad \frac{T_0}{T} = 1 + \frac{V^2}{2RT}$$

$$= 1 + \frac{V^2}{2(c_p - c_v)T}$$

This is wrong, let's see why!

Lecture 15

I did the wrong thing to highlight a subtle annoying part of compressible flow. ENTHALPY! ENTHALPY! ENTHALPY!



* Our first balance neglected internal energy!

$\Delta u \neq 0$ for compressible flow!

$$\left(\underbrace{u_a + \frac{P_a}{\rho_a}}_{h_a\text{: this is enthalpy}} + \frac{V_a^2}{2} + \cancel{g z_a}\right) = \left(\underbrace{u_b + \frac{P_b}{\rho_b}}_{h_b} + \frac{V_b^2}{2} + \cancel{g z_b}\right)$$

But if we write it interms of stagnation ENTHALPY! we get the correct relation.

(STAGNATION) $\qquad h_o = h + \dfrac{V^2}{2} \qquad$ (REFERENCE)

$$h_o - h = \frac{V^2}{2}$$

For an ideal gas recall $\dfrac{\Delta h}{\Delta T} = C_p \equiv \text{CONSTANT}$

$$C_p (T_o - T) = \frac{V^2}{2}$$

$$\therefore \quad \boxed{\frac{T_o}{T} = 1 + \frac{V^2}{2 C_p T}}$$

Extra Credit show a % Error calculation if you used the false relation from last page

Now let's put this in terms of Ma.

$$Ma^2 = \frac{V^2}{c^2} = \frac{V^2}{\gamma R T}$$

This step also invokes the ideal gas law.

We're almost there! The whole point of this is to derive relationships for T_o/T, P_o/P, ρ_o/ρ, c_o/c as functions of (Ma, γ) of f(speed, medium).

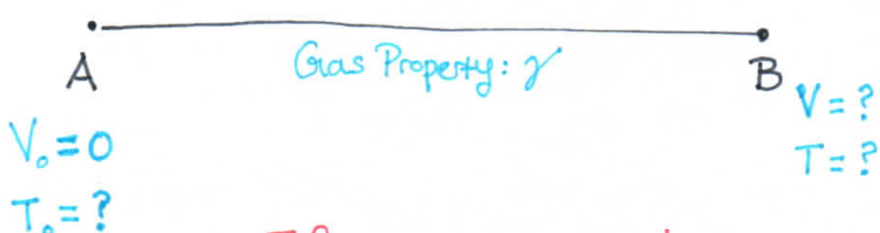
$$\frac{T_o}{T} = 1 + \frac{Ma^2 \gamma R \cancel{T}}{2 C_p \cancel{T}}$$

$$= 1 + \frac{Ma^2 (\cancel{C_p}/C_v)(C_p - C_v)}{2 \cancel{C_p}}$$

$$\boxed{\frac{T_o}{T} = 1 + \frac{Ma^2 (\gamma - 1)}{2}}$$

Amazing! This is so simple, and can tell you so much!!!

A Problem of Gas From A → B



If you can provide 2 of the variables $\{V, T, T_o\}$ then you can calculate the third.

LECTURE 15:

Let's say we have an unknown gas $\gamma = ?$ But measured $\{T, T_0, V\}$.

i) Calculate $Ma = \frac{V}{c}$.

ii) $\gamma = \frac{1}{Ma^2}\left\{\frac{2(T_0 - T)}{T} + Ma^2\right\}$

iii) Use a table to approximate what gas it is based on γ value.

//

We can go deeper if we invoke the isentropic Gibbs Relations For An Ideal Gas!

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/\gamma-1}$$

Simple substitution yields the ISENTROPIC GAS RELATIONSHIPS AT STAGNATION REFERENCE.

$$\frac{T}{T_0} = 1 + \frac{\gamma-1}{2} Ma^2$$

$$\frac{c_0}{c} = \left\{1 + \frac{\gamma-1}{2} Ma^2\right\}^{1/2}$$

since $c = \sqrt{\gamma R T}$

$$\frac{c_0}{c} = \sqrt{\frac{T_0}{T}}.$$

$$\frac{P_0}{P} = \left\{1 + \frac{\gamma-1}{2} Ma^2\right\}^{\gamma/(\gamma-1)}$$

$$\frac{\rho_0}{\rho} = \left\{1 + \frac{\gamma-1}{2} Ma^2\right\}^{1/\gamma-1}$$

LECTURE 15:

Stagnation is just an easy reference since $V=0$ another easy reference is if $V=c$. This is when $Ma=1$. Properties that use $V=c$ as a reference are called CRITICAL PROPERTIES.

They are usually denoted with a $*$ or c subscript.

$$\frac{T_*}{T_o} = \frac{2}{\gamma+1} \qquad \frac{P_*}{P_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1}$$

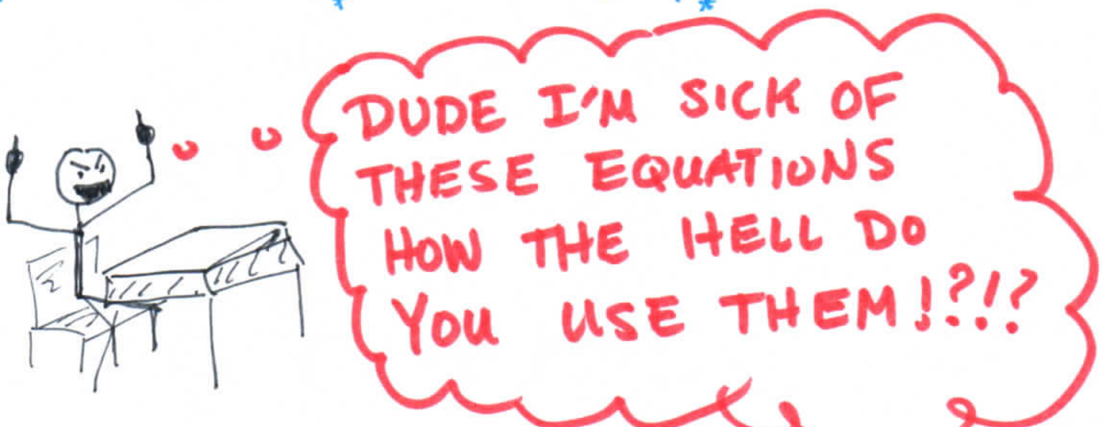
$$\frac{\rho_*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1} \qquad \frac{C_*}{C_o} = \left(\frac{2}{\gamma+1}\right)^{1/2}$$

Now we can use these values to derive critical references or X/X_* ratios.

This get's pretty large so notationally let me define a value we will just raise to powers.

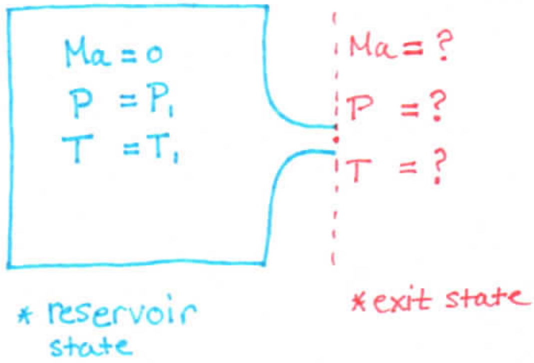
For $\Pi = (\gamma+1)/2\{1+\frac{\gamma-1}{2}Ma^2\}$ critical ratios are...

$$\frac{T}{T_*} = \Pi \;;\; \frac{P}{P_*} = \Pi^{\gamma/\gamma-1} \;;\; \frac{\rho}{\rho_*} = \Pi^{1/\gamma-1} \;;\; \frac{c}{c_*} = \Pi^{1/2}$$



LECTURE 15 :

The stagnation reference equations always help solve the "exploding" tank type problem.

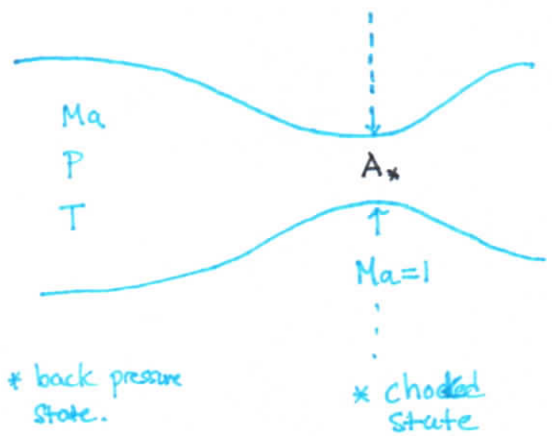


* We use the stagnation reference equations to solve various types of problems like this

* Provide 5 quantities solve for 6th

These types of problems usually involve knowing properties inside the reseroir state and solving for exit conditions but it doesn't have to be. You could solve for tank properties if you set exit conditions.

The critical state equation involve choked flow problems.



* The critical equations are used for <u>sizing</u> a nozzle

* Help determine A_* = "critical area"

* Always must set Ma=1 at the necked region or else we can't use the critical equations to relate 2 states.

Incompressible Summary

i) Get a <u>liquid</u> from state A to state B. Define vars at A and B.

ii) Use Bernoullis to figure stuff out. Solve for unknowns in state A or B.

Compressible Summary

i) Get a <u>Gas</u> from State A to State B. A or B must be stagnate or at Ma=1.

ii) Use fancier isentropic equations to figure stuff out. Solve for unknowns in State A or B.

LECTURE 13 :

Let's talk about flowrate before we conclude. We have to use mass-flowrate for compressible flow because.

$$\dot{m} = \rho A V$$

this can CHANGE now!

Before it was just these that could change.

So for an ideal gas,

$$\dot{m} = \frac{P}{RT} \cdot AV$$

$$= \left(\frac{P}{RT}\right) A \left(\frac{V}{c}\right) c$$

a dirty trick to get an expression in terms of Ma! I've just multiplied by 1!

$$= \left(\frac{P}{RT}\right) Ma \cdot A \sqrt{\gamma R T}$$

← for an ideal gas!

$$= A \cdot Ma \cdot P \cdot \sqrt{\frac{\gamma}{RT}}$$

The name of the game for Comp flow is to relate to stagnation / critical ratios. Recall our equations had the form $P_o/p = (f(Ma,\gamma))^{\gamma/\gamma-1}$ or $T_*/T = g(Ma,\gamma)$ or $\rho_*/\rho = (g(Ma,\gamma))^{1/\gamma-1}$

This means that we could sub in a Ma expression for P and T. It's really ugly though...

$$\dot{m} = \frac{P_o}{\sqrt{T_o}} \left(\sqrt{\frac{\gamma}{R}}\right) \left\{ Ma \Big/ \left(1 + \frac{\gamma-1}{2} Ma^2\right)^{\frac{\gamma+1}{2(\gamma-1)}} \right\} A$$

Critical at Ma=1 then.

$$\dot{m} = \frac{P_o}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \, A_*$$

~ a critical area usually use to size a choked region.

LECTURE 13:

We solve for a A/A_* to get the critical area ratio.

$$\frac{A}{A_*} = \frac{1}{Ma}\left\{\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}Ma^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}$$

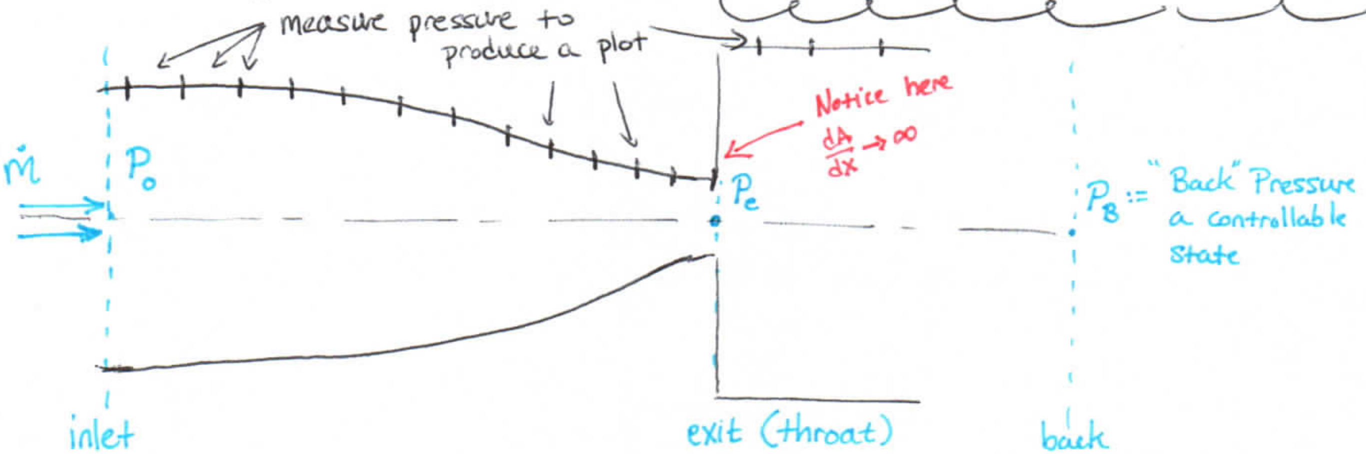
this is some Ma at another state, clearly not Ma=1.

Now we can use this to size a necked region in a converging nozzle to achieve given requirements.

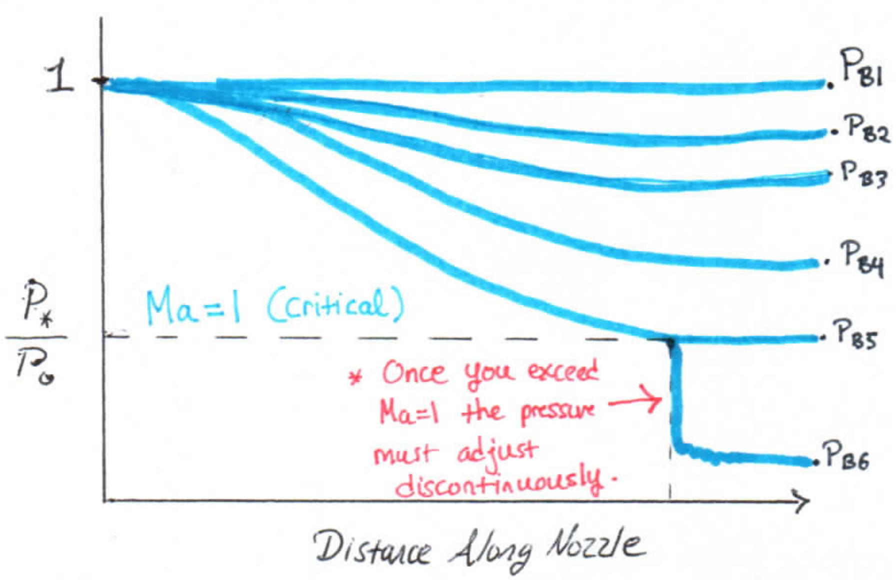
* Surprisingly ONLY now are we equiped with machinery to talk about the physical implications of NOZZLES.

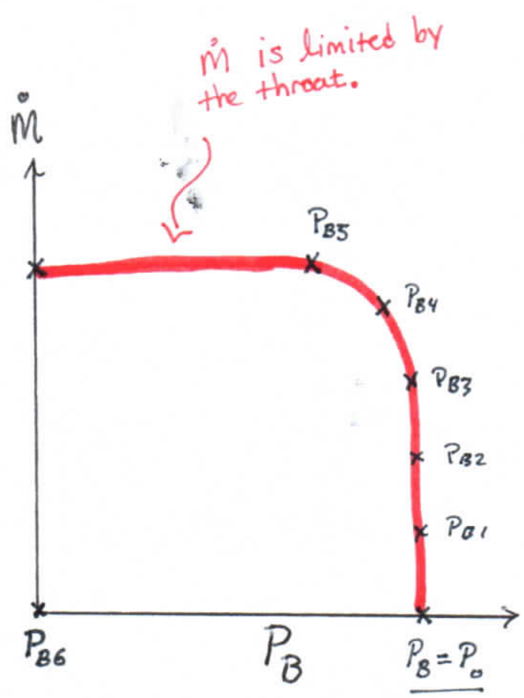
Convergent NOZZLES

Imagine we slowly decrease P_B?

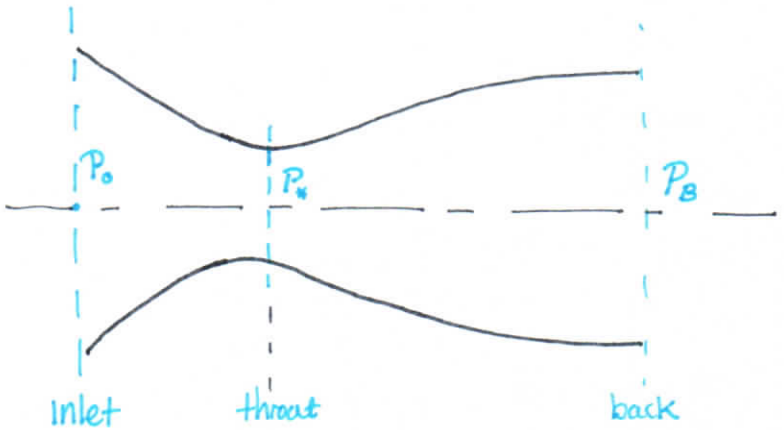


Results $\{P_{B1} > \ldots > P_{B6}\}$



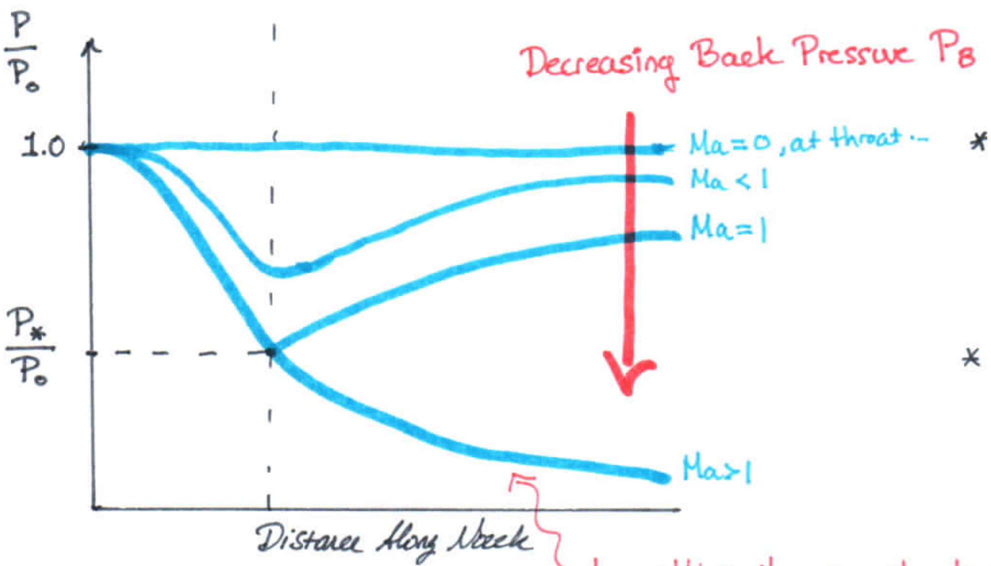


Convergent – Divergent Nozzles



* We can preform the same experiment only now we will arrive at an amazing pressure plot!

* This is a rocket nozzle btw...



* Once you have supersonic flow the expansion causes an INCREASE in velocity <=> DECREASE in pressure!?

* The throat still limits the mass flow rate. i.e. there exists a $\dot{m}_{max}$ for all gas problems.

by adding the smooth diverging area section we allow a continuous decrease in pressure as opposed to the previous plot (discontinuous)

Extra Credit

Using expressions for $\frac{dV}{dA} = ?$ and $\frac{d\rho}{dA} = ?$ explain why gas accelerates at supersonic speeds in divergent chanels.

Extra Credit

Explain this implications of a $\dot{m}_{max}$ when it comes to selecting fuels for rockets