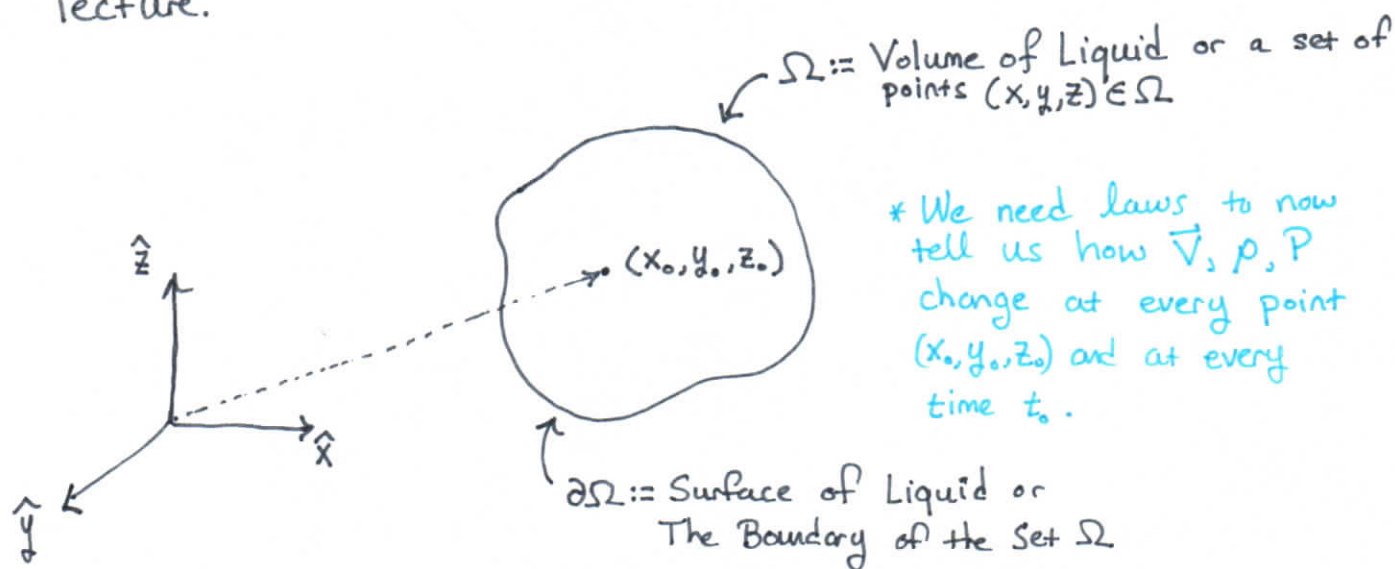# LECTURE 14 : WHAT IF IT'S GAS FLOWING INSTEAD OF A LIQUID?

Up until now we have been under the assumption that of __INCOMPRESSIBLE FLUID__. It is natural that we round out this intro to applied fluids by introducing the applications of compressible flow. How do you tell your problem is compressible? Here's just a list of a few.

1) Fluid is a __gas__

2) VERY HIGH Temperature

3) Enormous Pressures

4) $Ma \sim \mathcal{O}(1)$

Mathematically __COMPRESSIBLE__ is defined for flow with non-constant density field. Recall our very first lecture.

$\Omega :=$ Volume of Liquid or a set of points $(x,y,z) \in \Omega$

(x_0, y_0, z_0)

* We need laws to now tell us how $\vec{V}, \rho, P$ change at every point (x_0, y_0, z_0) and at every time t_0.

$\partial\Omega :=$ Surface of Liquid or The Boundary of the Set Ω

$$(1) \qquad \frac{D\vec{V}}{Dt} = \frac{1}{\rho} \left\{ Forces \right\}$$

$$(2) \qquad \frac{D\rho}{Dt} = \left\{ Prescribe\ how\ Density\ Changes\ in\ \Omega \right\}$$

$$(3) \qquad \frac{DP}{Dt} = \left\{ Prescribe\ how\ P\ is\ coupled\ with\ \vec{V}\ \&\ \rho \right\}$$

LECTURE 14:

The only thing NEW is (3) which a thermodynamical Equation of State. We know that (2) is still conservation of mass only this time $\rho \neq$ constant so it won't simplify to $\nabla \cdot \vec{V} = 0$. Intuitavely then, you should understand that __Density__ and __Velocity__ will be __coupled__. To compare both situations let's write out the governing equations. $\vec{V} = \langle u(x,y,z,t), v(x,y,z,t), w(x,y,z,t) \rangle \in \mathbb{R}^3$, $\rho = \rho(x,y,z,t) \in \mathbb{R}$ and $P = P(x,y,z,t) \in \mathbb{R}$.

INCOMPRESSIBLE $\rho = \rho_0$

(1a) $$\nabla \cdot \vec{V} = 0$$

(2a) $$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu \nabla^2 \vec{V} + \rho \vec{F}_b$$

4 equations, (2a) is 3 equations
1 for each dimension (x,y,z),
and 4 unknowns.

COMPRESSIBLE $\rho \neq$ constant

(1b) $$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{V})$$

(2b) $$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu \nabla^2 \vec{V} + \lambda \nabla(\nabla \cdot \vec{V}) + \rho \vec{F}_b$$

(3) $$\frac{DP}{Dt} = \frac{B}{A} \rho (\nabla \cdot \vec{V}) + \frac{C}{A}$$

5 equations and 5 unknowns

__Extra Credit__
List the __dependent functions__ in each case,
Show the 5th equation is technically still there for incompressible
Show compressible reduces to incompressible if something is zero

Equation 3 is a __general__ form of an equation of State. Commonly we just use a __very__ simple equation of state for __air__.

$$\rho = \frac{P}{RT} \qquad \text{(ideal gas law)}$$

2

LECTURE 14

The biggest take away is compressible flow adds terms and equations to the governing equation of motion. The addition of these terms creates a closure problem, (to many unknowns, not enough equations) to close this system of equations we must use THERMODYNAMICS in order to relate ...

density ~ $f(P,T)$ "Some function of pressure and Temperature"

Crash Course In Thermo · Sam Style...

0th Law : "Allows use to <u>define</u> temperature as a property of a system."

2 systems that are thermal equilibrium are at the same temperature.

1st Law : "Allows us to introduce the property energy to a system."

The sum of infinitesimal amounts of heat and work equal the change in energy to the system.

$$\delta Q + \delta W = dE$$

This means "variation" and means that the quantity is Path dependent.

Neither of these things are properties of a system yet their algebraic sum IS a property of a system.

3

LECTURE 14

2nd Law: "Allows us to introduce an absolute scale for temperature and define a property called entropy."

The change in entropy is bounded below by the positive ratio of heat exchange and temperature.

$$dS \geq \frac{\delta Q}{T}$$

Equality holds for ideal reversible processes.

Let's combine the first and second law and start to build new relations. First we will define Enthalpy.

$$H = U + P\forall \qquad \{\forall := \text{volume}\}$$

Per unit mass $\rightarrow h = u + \frac{P}{\rho}$

Now using (2) rewrite δQ and we get. if we neglect gravity, motion, electrical, capillary, magnetic effects.

$$T dS - P d\forall = dU$$

But look, what if we say internal energy is a function of properties.

$$U = U(S, \forall)$$

$$dU = \left(\frac{\partial U}{\partial S}\right) dS + \left(\frac{\partial U}{\partial \forall}\right) d\forall$$

Cool right!

4

LECTURE 14

Now we have a new interpretation of T & P.

$$T = \left(\frac{\partial U}{\partial S}\right) = \frac{\text{change in internal energy}}{\text{change in system entropy}}$$

$$P = -\left(\frac{\partial U}{\partial \forall}\right) = \frac{\text{negative change in internal energy}}{\text{change in system volume.}}$$

Let's keep going. More derivatives!!!

$$\frac{\partial T}{\partial \forall} = \frac{\partial^2 U}{\partial \forall \, \partial S}$$

$$\frac{\partial P}{\partial S} = -\frac{\partial^2 U}{\partial \forall \, \partial S}$$

order of $\partial \forall \, \partial S$ doesn't matter.

$$\therefore \left(\frac{\partial S}{\partial P}\right) = -\left(\frac{\partial \forall}{\partial T}\right)$$

Maxwell's Relation

Now let's look into our density issue. What do changes in $\rho(P,T)$ look like.

are you seeing a "theme" in thermo?

$$d\rho = \left(\frac{\partial \rho}{\partial T}\right)dT + \left(\frac{\partial \rho}{\partial P}\right)dP$$

(1)
$$\frac{d\rho}{\rho} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)dT + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)dP$$

Same expression but in terms of volume.

(2)
$$\frac{d\forall}{\forall} = \frac{1}{\forall}\left(\frac{\partial \forall}{\partial T}\right)dT + \frac{1}{\forall}\left(\frac{\partial \forall}{\partial P}\right)dP$$

5

Let's go back to densities definition.

$$\rho(T,P) = \frac{m}{\Psi(T,P)}$$

So in order to reconstruct (1) and (2)

$$d\rho = d\left(\frac{m}{\Psi}\right) \quad \& \quad d\Psi = d\left(\frac{m}{\rho}\right)$$

Then using the quotient rule.

$$d\rho = \frac{\overset{\text{conservation of mass (=0)}}{\cancel{dm}\Psi} - md\Psi}{\Psi^2} = -\frac{m}{\Psi}\frac{d\Psi}{\Psi}$$

$$d\Psi = \frac{\overset{=0}{\cancel{dm}\rho} - \cancel{m}d\rho}{\rho^2} = -\frac{m}{\rho}\frac{d\rho}{\rho}$$

Divide both by ρ and Ψ respectively.

$$\frac{d\rho}{\rho} = \underset{=1}{-\frac{m}{\cancel{\rho\Psi}}}\frac{d\Psi}{\Psi}$$

$$\frac{d\Psi}{\Psi} = \underset{=1}{\frac{-m}{\cancel{\Psi\rho}}}\frac{d\rho}{\rho}$$

Now by adding (1) and (2) we can derive 2 more properties of fluids and systems.

$$\frac{d\rho}{\rho} + \cancel{\frac{d\Psi}{\Psi}} = \left\{\frac{1}{\rho}\frac{\partial\rho}{\partial T} + \frac{1}{\Psi}\frac{\partial\Psi}{\partial T}\right\}dT + \left\{\frac{1}{\rho}\frac{\partial\rho}{\partial P} + \frac{1}{\Psi}\frac{\partial\Psi}{\partial P}\right\}dP$$

=0 by derivation
and true symmetrically

* These coefficients must be
identically zero since dT, dP are arbitrary.

6

Coefficient of Volume Expansion

$$\beta \equiv \frac{1}{\forall}\left(\frac{\partial \forall}{\partial T}\right) \equiv -\frac{1}{P}\left(\frac{\partial P}{\partial T}\right)$$

$$K \equiv -\frac{1}{\forall}\left(\frac{\partial \forall}{\partial P}\right) \equiv \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)$$

Coefficient of Compressibility

We can combine these more to derive a definition for the.

The Bulk Modulus of Elasticity

$$E = \frac{1}{K - \beta\frac{dT}{dP}}$$

~ for liquids and solids this term is super <u>small</u> so $E \cong 1/K$.

* For a gas $\frac{dT}{dP} \gg 1$ so and *
* therefore E is <u>not</u> a property. *

* Compressibility depends on the *
* <u>process</u> itself not the <u>fluid</u> *

* LET'S <u>USE</u> THE IDEAL GAS LAW *
* FINALLY TO DETERMINE SOME ×
  COMPRESSIBLE FLOW STUFF!

The equation of state for an ideal perfect gas.

$$P\forall = mRT \qquad (*)$$

Or in terms of density.

$$\rho(T,P) = \frac{P}{RT} \qquad (E.S.)$$

This allows us to <u>take</u> derivatives and go further go deeper. So from a <u>perfect</u> gas we go deeper with 2nd law!

$$TdS = dU + Pd\forall$$

$$T\left(\frac{\partial S}{\partial \forall}\right) = \left(\frac{\partial U}{\partial \forall}\right) + P$$

From here we can show that $U(T)$ and <u>not</u> $U(T,\forall)$. This is important because for a single var function we can taylor expand for representations. If we consider internal energy per unit mass $u(T)$

Lower case always mean per unit mass

$$u(T) \cong u_0 + \left(\boxed{\frac{\partial u}{\partial T}}\right)\delta T + \frac{\partial^2 u}{\partial T^2}(\delta T)^2$$

assume constant   neglect

Same for <u>enthalpy</u> since it's related to internal energy.

$$h(T) = u(T) + \frac{P}{\rho}$$

$$h(T) = h_0 + \left(\boxed{\frac{\partial h}{\partial T}}\right)\delta T + \frac{\partial^2 h}{\partial T^2}(\delta T)^2$$

assume constant value for simplisity   neglect

We have _arrived_ at the specific heat variables!

$$C_v = \left(\frac{\partial u}{\partial T}\right)_{V=V_0}$$

constant volume specific-heat

$$C_p = \left(\frac{\partial h}{\partial T}\right)_{P=P_0}$$

constant pressure specific-heat.

For ideal gases as well. By definition,

$$C_p - C_v = R$$

Extra Credit show this from C_p & C_v defs

A great tool for state analysis is the ratio

$$\gamma = \frac{C_p}{C_v}$$

For air under moderate pressure & temperatures.

$$C_p = 1.0035 \left[\frac{KJ}{kg\ K}\right]$$

$$C_v = 0.7165 \left[\frac{KJ}{kg\ K}\right]$$

$$\gamma \cong 1.4$$

We return to the entropy version of the 2nd law.

$$dS = \frac{dU}{T} + \frac{P}{T}dV$$

LECTURE 14

$$dS = \frac{dh}{T} - \frac{dP}{\rho T} \qquad \text{(Enthalpy Form)}$$

For an ideal gas though if C_p is constant.

$$C_p = \frac{dh}{dT}$$

$$C_p dT = dh$$

$$\therefore C_p \frac{dT}{T} = \frac{dh}{T}$$

Similarly since $\rho = P/RT$

$$\rho T = \frac{P}{R}$$

$$\therefore \frac{dP}{\rho T} = R \frac{dP}{P}$$

Put it all together and get.

$$dS = C_p \frac{dT}{T} - R \frac{dP}{P}$$

Integrate equation to get relationship based on state $1 \to 2$.

$$\Delta S = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right)$$

LECTURE 14

So for _any_ process (even let's say flow) an ideal gas must obey, from state $1 \to 2$

Now you know C_p & C_v are actually values of _entropy_ $\longrightarrow$

$$\Delta S = C_p \ln\left(\frac{T_2}{T_1}\right) - (C_p - C_v) \ln\left(\frac{P_2}{P_1}\right)$$

$$\frac{\Delta S}{C_v} = \gamma \ln\left(\frac{T_2}{T_1}\right) - \left(\frac{C_p - C_v}{C_v}\right) \ln\left(\frac{P_2}{P_1}\right)$$

normalized entropy w/respect to gas

$$\Delta \tilde{S} = \gamma \ln\left(\frac{T_2}{T_1}\right) - (\gamma - 1) \ln\left(\frac{P_2}{P_1}\right)$$

Now if we assume the process is _Isentropic_ $\Rightarrow \Delta S = 0$. This means _no_ change in entropy! But this is also called a reversible case process. Recall.

$$dS \geq \frac{\delta Q}{T}$$

Extra credit how can we use the second law to show something is almost isentropic, you need 2-things

If $dS \approx 0$ we get the isentropic relationships for a ideal-gas (air).

If you didn't know 2 states you could write both down but you'll need to provide more state information

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{(\gamma-1)}{\gamma}}$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

You have to know 3 vars to "use" _one_ of these equations.

The Gibbs Equations

LECTURE 14

The last thing is we will derive the "speed of sound" based on "thermodynamical" properties! This is important to do because we need an _upper limit_ of velocity that is based on something that sits outside Kinematics or Classical Mechanics. We need our velocity that normalizes all velocity values to be from a seperate theory.
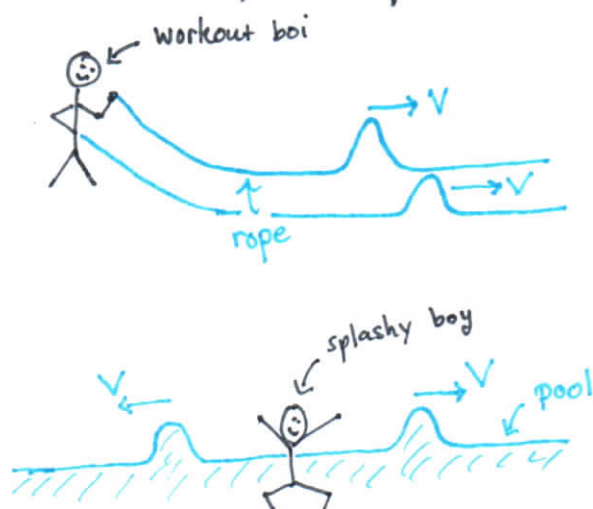
//

The Speed of Sound "c"

* This speed is simple the velocity of a "pressure wave" traveling thru a medium. By pressure wave we mean the compression and expansion of air molecules.
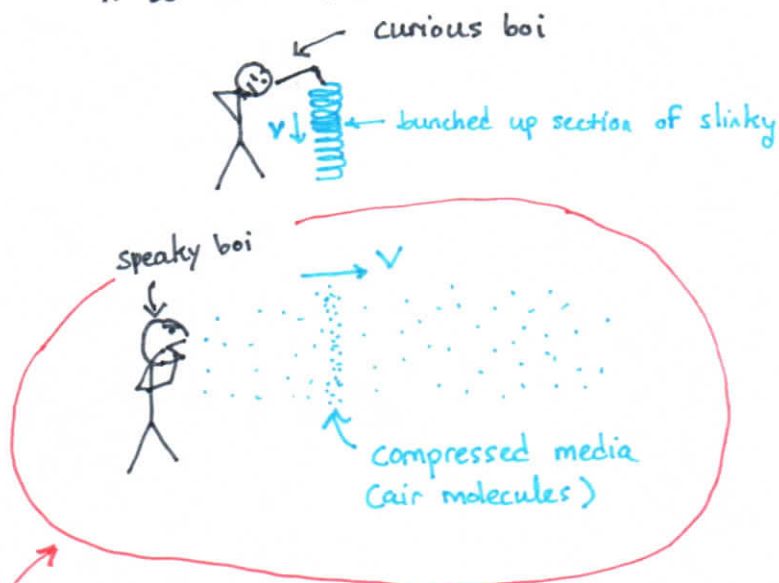
Transverse Wave Speed

* flicking a rope and watching the bump move away from you

* The top of liquid surface

workout boi

→V

↑ rope

→V

splashy boy

V← →V pool

Longitudinal Wave Speed

* Sending a "pulse" down a slinky

* The air pulse from one of those air guns (really a vortex)

* a sound!

curious boi

v↓ ← bunched up section of slinky
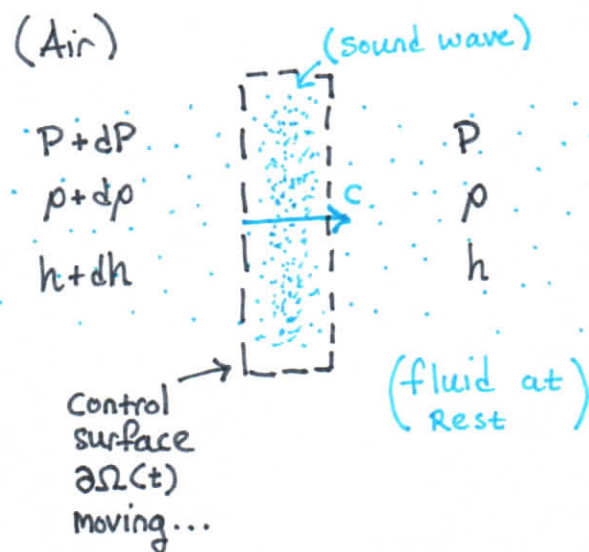
speaky boi

→V

compressed media (air molecules)

these are the physical waves we're talking about all class.

LECTURE 14

Well let's look at that section of bunched up air molecules.

* Since our surface $\partial \Omega(t)$ is moving with the wave we must switch to a relative view!

(Air)   (sound wave)

$$P + dP$$
$$\rho + d\rho$$
$$h + dh$$

Control surface $\partial \Omega(t)$ moving...

$$P$$
$$\rho$$
$$h$$

(fluid at Rest)

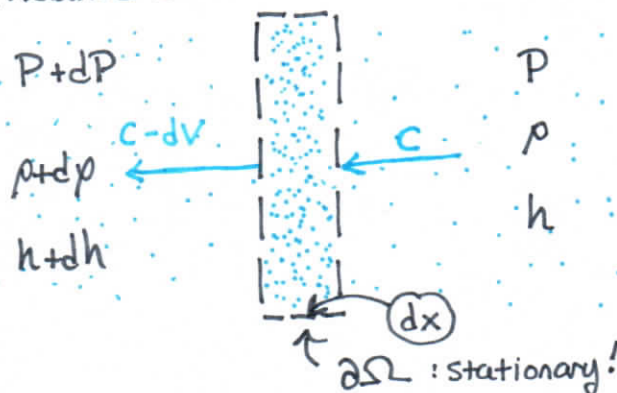Conserve Mass in $\partial \Omega$

$$Q_{in} = Q_{out}$$

$$\rho A c = (\rho + d\rho) A (c - dV)$$

$$= (\rho + d\rho)(Ac - AdV)$$

$$\cancel{\rho A c} = \cancel{\rho A c} - \rho A dV + Ad\rho c - \cancel{Ad\rho dV}$$

* The multiplication of $d\rho dV$ is negligible!
(0.0001)(0.0001) << (0.0001)

$$\boxed{\rho dV = c \, d\rho}$$

(Relative Control Volume)

$$P + dP$$
$$\rho + d\rho$$
$$h + dh$$

$$c - dV$$
$$c$$

$$P$$
$$\rho$$
$$h$$

dx

$\partial \Omega$: stationary!

Conserve Momentum in $\partial \Omega$ (this will invoke pressure P)

$$Work = \Delta KE \quad \text{(I don't know why but I can only remember the Work Energy Thm)}$$

$$\left(\sum F_{ext}\right) dx = \{PA - (P + dP)A\}\cancel{dx} = \left[\rho A c \{(c - dV) - c\}\right]\cancel{dx}$$

$$\boxed{dP = \rho c \, dV} \quad \text{after neglecting } d(\,)d(\,) \text{ terms.}$$

Solving for c we finally arrive at our velocity.

$$dV = \frac{dP}{\rho c} \implies \cancel{\rho}\frac{dP}{\rho c} = c \, d\rho \implies \qquad C^2 = \frac{dP}{d\rho} \quad \begin{array}{l}\text{* always}\\ \text{positive}\\ \text{quantity}\end{array}$$

13

Since the right hand side is a derivative of some __EQUATION OF STATE__, thermodynamical quantities govern this velocity limit.

$$c = \sqrt{\frac{\partial P}{\partial \rho}}$$

Now for a ideal gas we had the realtionship

$$P = \rho RT \quad \leftsquigarrow \text{ you will be wrong if you just this relationship.}$$

But recall we also showed that for a __isentropic__ process

$$P = C\rho^\gamma$$

Either way we will get the speed of sound for an isentropic process of a ideal gas.

$$\frac{\partial P}{\partial \rho} = C\gamma\rho^{\gamma-1} = \gamma C\left(\frac{\rho^\gamma}{\rho}\right)$$

$$= \gamma C\left(\frac{P}{C\rho}\right)$$

$$= \gamma\left(\frac{P}{\rho}\right)$$

$$= \gamma RT$$

$$\therefore \quad c = \sqrt{\gamma RT} \sim 50\sqrt{T}\ [ft/s]$$

$$\sim 340\ [m/s] \quad @T = 15°C$$

$$\sim 20\sqrt{T}\ [m/s]$$

14

* This relationship is true for __all__ media to some degree.

$$C = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_{S=S_o}} \qquad \text{(most general)}$$

Recall we had derived K & β, true for __all__ media.

$$K = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)$$

← for solids and liquids this derivative is too small though!!!

$$C = \sqrt{\frac{1}{\rho K}}$$

Moreover recall we made a derived material prop E.

$$E = \frac{1}{K - \beta\frac{dT}{dP}}$$

← We needed to do this because $\partial\rho/\partial P \to 0$ and would have predicted $C \to \infty$.

For liquids & solids $dT/dP \approx 0$. So speed in solid and liquids is then.

$$C = \sqrt{\frac{E}{\rho}} \quad \sim \quad 1400 \text{ M/s} \quad \text{Water}$$

$$\sim \quad 6000 \text{ M/s} \quad \text{Steel}$$

The point is we can put __all values of__ Velocity on an __absolute scale__.

$$Ma = \frac{U_\infty}{C}$$

← scale "problem" velocity. This is the velocities our problem will be around.

↖ an "upperbound" to problem velocity. If $Ma>1$ you're travelling waves in the medial!?!?

15