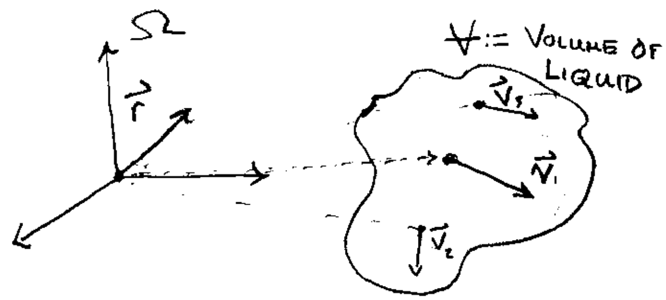
LECTURE 1 : LITTLE BITS OF FLUID.



Ω := OUR COORDINATE SYSTEM

$\forall$:= SOME VOLUME OF LIQUID

* EVERY POINT INSIDE $\forall$ CAN BE THOUGHT OF AS SOME "POINT" OF LIQUID. YOU CAN REASON IT AS US "TRACKING" EACH MOLECULE OF LIQUID. THIS LET'S US CONSIDER GOOD OLE NEWTON!

$\forall$ IS SOME UNGULATING SLOSHING FIELD OF LIQUID SO IT REASONS TO DESCRIBE IT WITH A VECTOR FIELD!

$$\underbrace{\vec{V}}_{\text{THE VELOCITY OF A POINT OF LIQUID}} = \vec{V}\underbrace{(x, y, z, t)}_{\text{YOU ASSIGN A CARTESIAN POINT IN THE VOLUME AND A VALUE OF TIME}} \qquad \underbrace{\forall\, x, y, z \in \Omega}_{\text{"FOR ALL } x, y, \text{ and } z \text{ IN THE REGION } \Omega\text{."}}$$

NEWTON SAYS WHAT...

$$\sum \vec{F} = m\,\vec{a}$$

THIS SIDE IS ACTUALLY THE HARDEST TO GET, WE WILL DO IT LAST.

LETS TACKLE THIS TERM FIRST.

THIS IS JUST MASS, THIS ISN'T TOO DIFFICULT IF WE DEFINE DENSITY $\rho = \frac{m}{\forall} = \frac{\text{mass}}{\text{volume}}$ ...

$\vec{a}$? THIS IS WHY WE TAKE VECTOR CALC.

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{V}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{V}}{\partial z}\frac{\partial z}{\partial t}$$

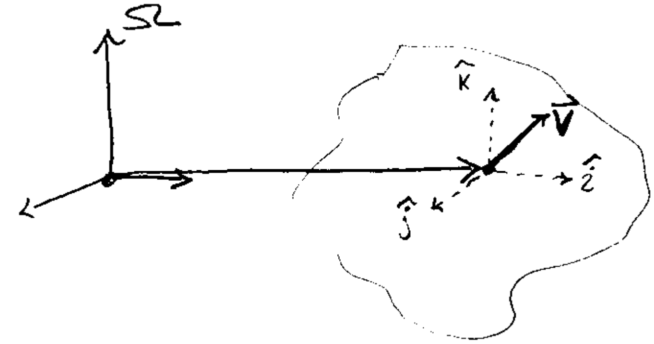
I WILL EXPLAIN WHY I USED CAPITAL LETTERS LATER.

Velocity in x, call it u

Velocity in y, call it v

Velocity in z call it w

LECTURE 1: LITTLE BITS OF FLUID



* VECTOR CAN BE SPLIT UP INTO THEIR X, y, z COMPONENTS

$$\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$$

WHERE $u(x,y,z,t)$, $v(x,y,z,t)$, & $w(x,y,z,t)$
THIS IS NICE ONLY BECAUSE THE FUNCTIONS u, v, w ARE SCALAR VALUED NOT VECTOR VALUED. SO $\vec{a}$ HAS 3 COMPONENTS

$$(1) \quad \vec{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{matrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{matrix}$$

EXTRA CREDIT
DO THIS CALCULATION TO SHOW THIS

NOTATION IS GREAT CAUSE MY HAND ALREADY HURTS! LET'S DEFINE A VECTOR OPERATION. THE "MATERIAL" DERIVATIVE...

$$\frac{D(\;)}{Dt} := \frac{\partial(\;)}{\partial t} + (\vec{V}\cdot\nabla)(\;)$$

VECTOR DOT PRODUCT.

WHERE,

$$\nabla := \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{or} \quad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right].$$

EXTRA CREDIT
VERIFY $\frac{D\vec{V}}{Dt}$ EQUALS (1)

THIS ALLOWS US TO WRITE ACCELERATION JUST LIKE WE'RE USED TO!

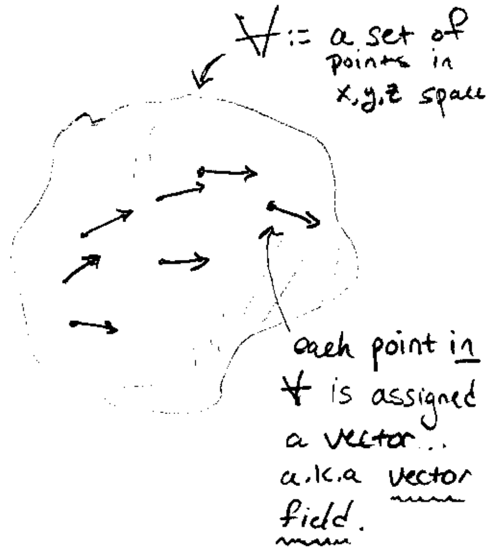
$$\vec{a} = \frac{D\vec{V}}{Dt}$$

IN PHYSICS CLASS WE WROTE
$a = \frac{dv}{dt}$

$$\begin{Bmatrix}\text{VECTOR} \\ \text{FIELD}\end{Bmatrix} = \begin{Bmatrix}\text{MATERIAL} \\ \text{DERIVATIVE}\end{Bmatrix} \cdot \begin{Bmatrix}\text{VECTOR} \\ \text{FIELD}\end{Bmatrix}$$

LECTURE 1 : LITTLE BITS OF FLUID

JUST FROM OUR MATERIAL DERIVATIVE WE CAN ALREADY GLEAN SOME FLUID MOTION INTUITION.



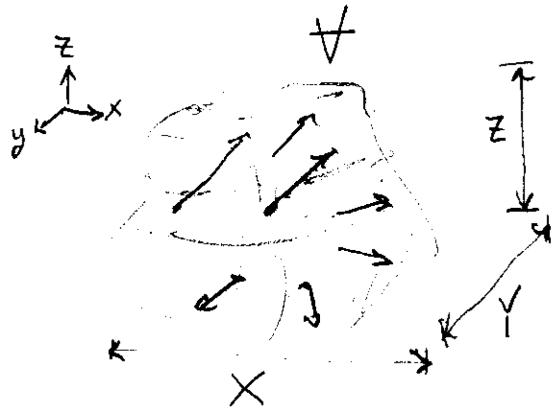
$\vec{V}$:= A VECTOR FIELD THAT DESCRIBES THE VELOCITY AT EACH POINT IN ∀.

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \underbrace{(\vec{V}\cdot\nabla)}$$

THIS IS AN EXTRA TERM THAT DEPENDS ON $\vec{V}$ ITSELF.
NON-LINEARITY! ☹

WHICH TERM SHOULD I CARE ABOUT? JUST BARE WITH ME ON THIS FOR A BIT, IT'S CALLED "SCALING" BUT IT'S SOO POWERFUL! LET'S JUST THROW OUT ALL THE RULES OF VECTORS AND TREAT THEM LIKE SCALARS.

$$\frac{D}{Dt} \sim \frac{1}{T} + \frac{u}{X} + \frac{v}{Y} + \frac{w}{Z}$$



1) IF u, v, w ARE SUPER SMALL

$$\frac{D}{Dt} \sim \frac{1}{T} \sim \frac{\partial}{\partial t}$$

2) IF WE ARE TRYING TO INVESTIGATE VERY FAST DYNAMICS THEN $T \ll 1$, THEN

$$\frac{D}{Dt} \sim \frac{1}{T}$$

3) IF u, v, w ARE SUPER BIG THEN

$$\frac{D}{Dt} \sim (\vec{V}\cdot\nabla)$$

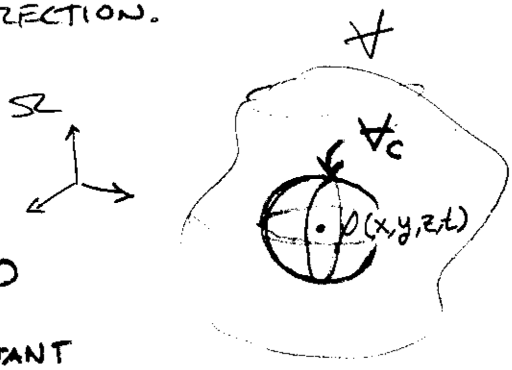
4. ...

EXTRA CREDIT! WRITE OUT MORE CASES THAT SIMPLIFY THIS OPERATOR

LECTURE 1: LITTLE BITS OF FLUID.

WITH OUR NEW MATERIAL DERIVATIVE WE CAN TALK ABOUT MASS OF A BLOB OF FLUID DYNAMICALLY. WE JUST NEED TO CONSIDER A DIFFERENT FUNCTION TO OPERATE ON. BEFORE WE ASKED QUESTION ABOUT VELOCITY BECAUSE WE WANTED TO DETERMINE THE ACCELERATION AT EVERY POINT IN $\forall$.

NOW LETS CONSIDER DENSITY $\rho(x,y,z,t)$. DENSITY IS NOT A VECTOR. IT IS A SCALAR. SOMETHING JUST WEIGHS 100kg, IT DOESN'T HAVE A DIRECTION.



CONSERVATION OF MASS REQUIRES

$$\frac{D\rho}{Dt} \equiv 0 \Longleftrightarrow \frac{DM}{Dt} \equiv \frac{D\rho\forall_c}{Dt} \equiv 0$$

CONSTANT CONTROL VOLUME

BUT LETS SEE MORE TERMS

$$\frac{\partial\rho}{\partial t} + \underbrace{(\vec{V}\cdot\nabla)(\rho)} = 0$$

- THERE IS AN ISSUE HERE WE NEED TO FIX THIS PRODUCT PRODUCES A VECTOR BUT $\frac{\partial\rho}{\partial t}$ IS A SCALAR. SO THIS IS ADDING APPLE AND ORANGES.

READ THE BOOK FOR WHY BUT HERE'S THE FIX.

$$\frac{\partial\rho}{\partial t} + (\nabla\cdot\vec{V})\rho = 0$$

WE JUST FLIPPED THE ORDER.

INCOMPRESSIBLE ASSUMPTION IS $\rho =$ CONSTANT.

$$\therefore \quad (\nabla\cdot\vec{V}) \equiv 0 \quad \leftarrow$$

VERY IMPORTANT EQUATION.

LECTURE 1: LITTLE BITS OF FLUID

ONE SIMPLIFICATION FOR INCOMPRESSIBLE FLOW IS TO SAY FLOW IS **2D**. THAT MEANS ONE DIMENSION HAS NO CHANGE TO IT.

$$\nabla \cdot \vec{V} = 0$$

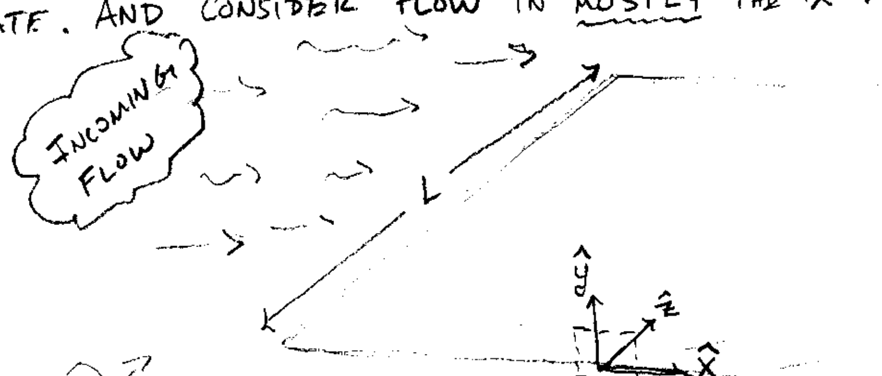
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

• Let's say it doesn't vary in the z direction

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

HOW CAN YOU SAY FLOW IS 2D THOUGH... MAGIC SCALES. CONSIDER A LONG FLAT PLATE. AND CONSIDER FLOW IN MOSTLY THE $\hat{x}$ DIRECTION.



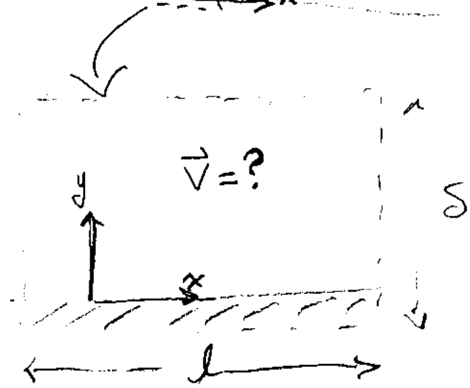
$L = 10\,m$

$\delta = 1\,cm$

$l = 50\,cm$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \sim \frac{u}{l} + \frac{V}{\delta} + \frac{W}{L}$$

L IS REALLY BIG SO THIS NUMBER IS REALLY SMALL



$$\sim \frac{u}{l} + \frac{V}{\delta}$$

WE SAID MOSTLY IN THE X DIRECTION BUT REMEMBER δ IS REALLY SMALL SO WE MUST CONSIDER THIS TERM.

LECTURE 1: LITTLE BITS OF FLUID.

SO SAY WE HAVE 2D FLOW. THERE IS A LITTLE MATH TRICK WE CAN PULL TO MAKE OUR LIVES EASIER. WE SUPPOSE THAT u & v ARE THE RESULT OF DIFFERENTIATING A SPECIAL FUNCTION CALLED A

STREAM-FUNCTION

$$\psi(x, y)$$

SUCH THAT.

IF NOTHING ELSE REMEMBER THIS AND HOW TO CALCULATE IT

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad (*)$$

ANY FUNCTION ψ THAT SATISFYS $(*)$ ALSO OBEYS CONSERVATON OF MASS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0 \checkmark$$

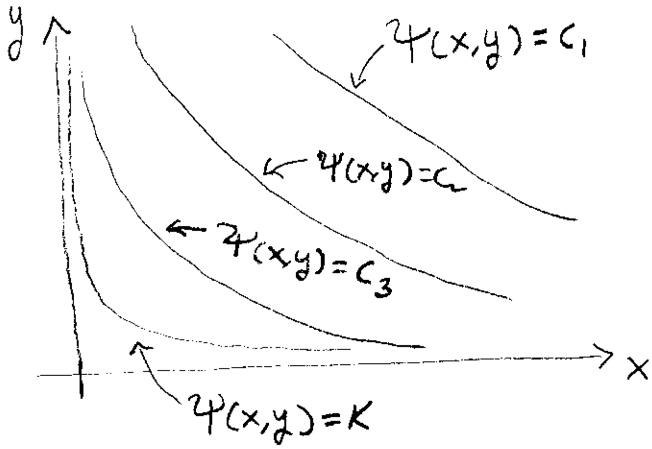
$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ ORDER DOESN'T MATTER

THIS TRICK OF SAYING THE VELOCITY COMPONENTS COME FROM A SPECIAL FUNCTION LAYS THE FIELD OF <u>POTENTIAL FLOWS</u>. THE STREAM FUNCTION ALSO TELLS US ABOUT <u>FLOW RATE</u> AND <u>FLOW VISUALIZATION</u>.

THE WHOLE POINT IS NOW WE JUST NEED TO DETERMINE $\psi(x,y)$ INSTEAD OF $u(x,y)$ & $v(x,y)$

LECTURE I: LITTLE BITS OF FLUID

* LINES OF CONSTANT VALUES OF ψ ARE CALLED **STREAMLINES.**



SAY WE CONSIDER ONLY THE X,Y POINTS SUCH THAT

$$\psi(x,y) = K$$

THIS IS A CONTOUR PLOT OF A SURFACE IF YOU REMEMBER.

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

Theorem from Calc IV.

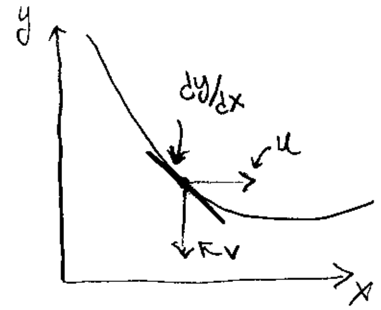
Along a streamline there is no change

$$0 =$$

$$= -v\,dx + u\,dy$$

Definition of ψ

$$\therefore \underbrace{\frac{dy}{dx}}_{\text{THE SLOPE ALONG ANY STREAMLINE}} = \frac{v}{u} = \frac{\text{y-velocity value}}{\text{x-velocity value}}$$



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FLOWRATE

$$q = q_x + q_y$$

$$dq = u\,dy - v\,dx = d\psi$$

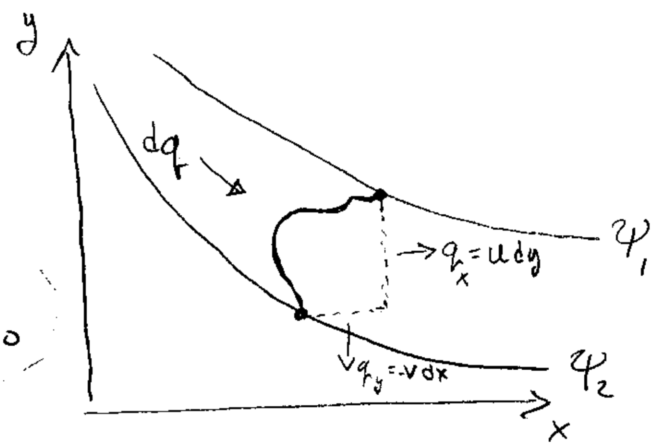
$$\therefore\ dq = d\psi$$

THERE IS NO FLOWRATE THRU A STREAMLINE $d\psi = 0$

BUT BETTER!

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

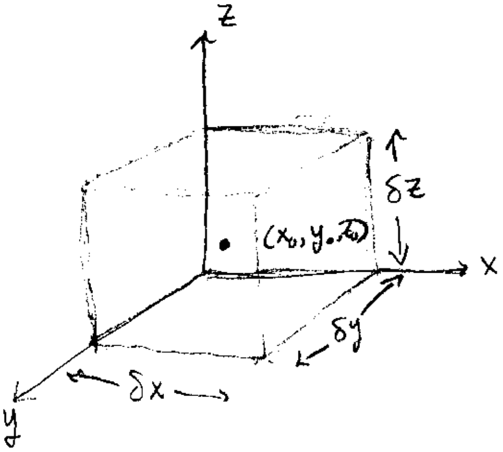
YOU KNOW HOW SIMPLE THAT IS TO CALCULATE



LECTURE 1 : LITTLE BITS OF FLUID

THE HARDEST TERM WE NEVER DEALT WITH WAS THE FORCES $\Sigma \vec{F}$ TERM OF $\vec{F} = m\vec{a}$. THIS IS THE MOST DIFFICULT TERM TO MATH OUT. WE NEED TO LOOK AT A LITTLE CUBE OF FLUID. (Ref 6.3)

1) CUBES ARE BETTER THAN BLOBS SO WE CAN CALCULATE AREAS MORE CONVIENIEANTLY.



2) CUBES HAVE 9 SIDES SO WE HAVE TO DESCRIBE THE FORCES ON EACH FACE ☹ (I TOLD YOU THIS PART IS HARD...)

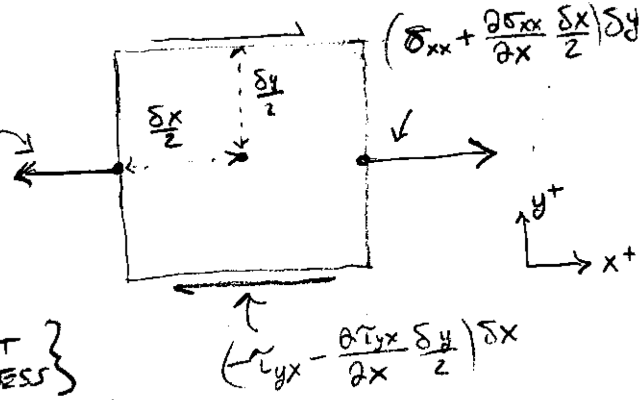
3) CONSIDER THE CENTER OF THE CUBE (x_0, y_0, z_0) AND TRY TO ESTIMATE FORCES ON THE FACES FROM THIS POINT.

4) SINCE WERE ENGINEERS ANY FUNCTION CAN BE "TAYLOR EXPANDED"

$$\underbrace{f(x)}_{\text{any function}} = \underbrace{f(x_0)}_{\text{the function evaluated at some point -- like }(x_0, y_0, z_0)} + \underbrace{\frac{\partial f}{\partial x}}_{\text{The derivative evaluated at that point}} \delta x$$

δx — some tiny distance away from (x_0, y_0, z_0)

5) I'LL DO ONE FACE JUST TO SEE HOW THIS GOES... THE NET FORCE ON THE POINT IN THE X-DIRECTION.



$$\left(\cancel{\sigma_{xx}} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right)\delta y - \left(\cancel{\sigma_{xx}} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right)\delta y$$

$$\therefore \left(\frac{\partial \sigma_{xx}}{\partial x}\right)\delta x \delta y + \left\{\begin{matrix}\text{SHEAR TERMS THAT} \\ \text{FOLLOW SAME PROCESS}\end{matrix}\right\}$$

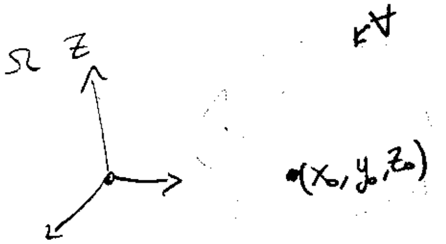
EXTRA CREDIT
DO THE SHEAR TERMS

LECTURE 1: LITTLE BITS OF FLUID

WHAT ARE THESE σ_{xx}, τ_{xy}, etc TERMS? THEY ARE COMPONENTS OF SOMETHING CALLED A CAUCHY STRESS-TENSOR

DONT WORRY... ALL IT DOES IS SPIT OUT A MATRIX IF YOU GIVE IT A POINT IN SPACE (x, y, z) IN $\forall$

$$(x_0, y_0, z_0) \longrightarrow \boxed{\text{CAUCHY STRESS TENSOR}} \longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



- LIKE MY MIND ALREADY BROKE TRYING TO UNDERSTAND A VECTOR ASSIGNED FOR EVERY POINT... JEEZ...

DIFFERENT ASSUMPTIONS ABOUT THIS MONSTER PRODUCE DIFFERENT FLOW THEORIES.

$\tau_{xy} = \tau_{xz} = \dots \tau_{zy} = 0$ $\Rightarrow$ INVISCID FRICTIONLESS "DRY" FLUID $\Rightarrow$ EULER'S EQUATIONS OF MOTION!

$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$

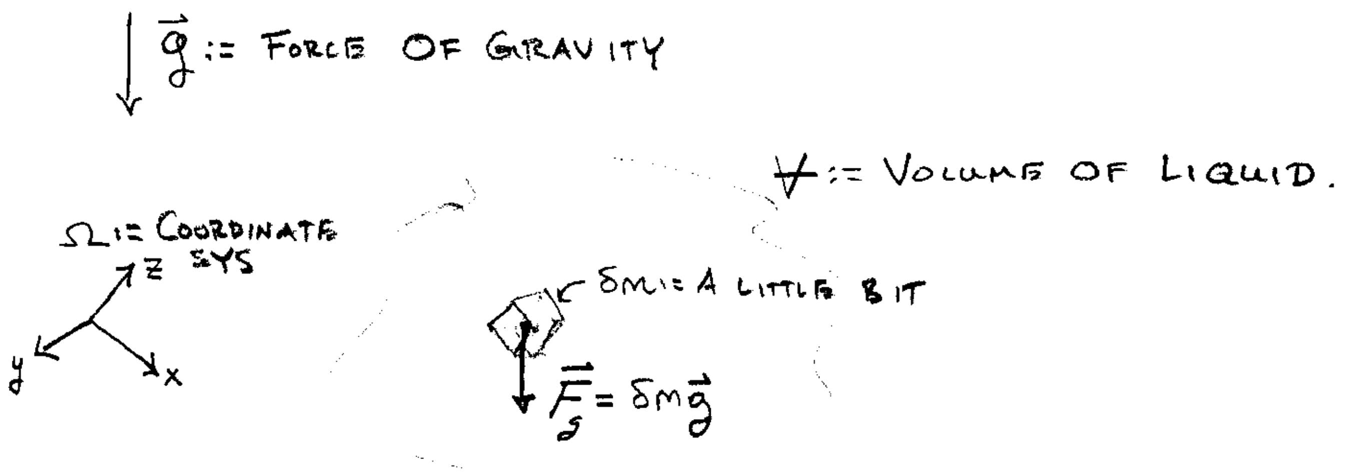
$$\sigma_{xx} = -P + 2\mu \frac{\partial u}{\partial x}$$
$$\vdots$$
$$\sigma_{zz} = -P + 2\mu \frac{\partial w}{\partial z}$$

$\Rightarrow$ VISCOUS FLOW 'WET' FLUID NEWTONIAN FLUID $\Rightarrow$ NAVIER-STOKES EQUATIONS!

IT IS AN OPEN PROBLEM IN MATHEMATICS THAT IN THE LIMIT AS $\mu \to 0$ FOR NAVIER-STOKES THE SOLUTION IS THE SAME AS A SOLUTION TO EULER'S

LECTURE 1 : LITTLE BITS OF FLUID

THERE ONE EASY BODY FORCE ON A LITTLE BIT OF FLUID (AT LEAST ON EARTH)... GRAVITY.



WHAT EVER Ω WE ARE IN WE CAN DECOMPOSE $\vec{g}$ INTO THE COMPONENTS. WE PROJECT $\vec{g}$ ONTO EACH AXIS.

$$\left.\begin{aligned} F_{gx} &= \delta m\, g_x \\ F_{gy} &= \delta m\, g_y \\ F_{g,z} &= \delta m\, g_z \end{aligned}\right\} \text{COMPONENTS OF VECTORS } \vec{F}_g \text{ \& } \vec{g}.$$

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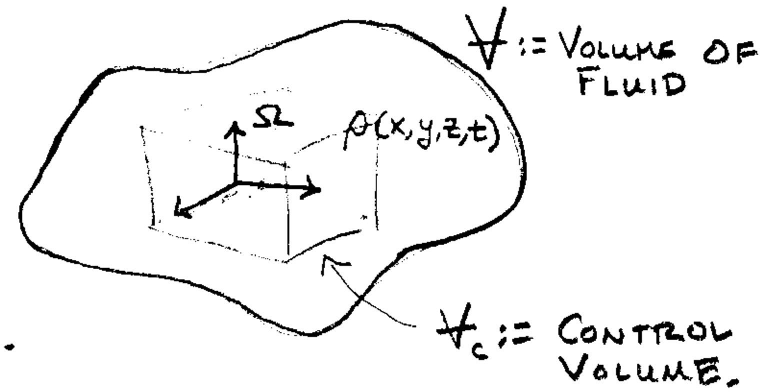
ONE NOTE : ANOTHER WAY YOU CAN DERIVE ALL OF THIS STUFF IS WITH "CONTROL VOLUMES". THESE ARE <u>INTEGRAL</u> ARGUMENTS INSTEAD OF <u>DIFFERENTIAL</u>. YOUR BOOK USES CONTROL VOLUMES.

LET'S LOOK AT THE DIFFERENCE OF THESE APPROACHES USING CONSERVATION OF MASS.

* A CONTROL VOLUME IS A VOLUME YOU HAVE .. CONTROL OVER. LET'S SAY $\forall$ COULD BE SLOSHING AND UNGULATING AROUND.



YOU CAN'T STOP ME FROM CONSIDERING, IMAGINING, OR DRAWING THIS CONTROL BOUNDARY.

DIFFERENTIAL VS. **INTEGRAL**

Differential:

$$\frac{DM}{Dt} = 0$$

$$\frac{D}{Dt}(\rho \forall_c) = 0$$

$$\forall_c \frac{D\rho}{Dt} = 0$$

I DREW $\forall_c$ SO I CAN KEEP THE DIMENSIONS CONSTANT!

$$\therefore \quad \frac{D\rho}{Dt} = 0$$

Integral:

$$\left\{\begin{matrix}\text{CHANGE}\\ \text{INSIDE } \forall_c\end{matrix}\right\} + \left\{\begin{matrix}\text{FLUX OF MASS}\\ \text{THROUGH } \forall_c\end{matrix}\right\} = 0$$

$$\frac{d}{dt}\int \rho \, d\forall + \int \rho \, V \cdot \hat{n} \, dA = 0$$

$$\underbrace{\int_{\forall} \frac{\partial \rho}{\partial t} d\forall}_{\text{VOLUME INT}} + \underbrace{\int_{\partial \forall} \rho \, V \cdot \hat{n} \, dA}_{\text{SURFACE INTEGRAL}} = 0$$

MATH TRICK THE DIVERGENCE THM CONVERTS SURFACE INTEGRALS INTO VOLUME INTEGRALS

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho V \right\} dV = 0$$

non-zero quantity!

$$\therefore \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

EXAMPLE CALCULATIONS (STREAMFUNCTIONS)

GIVEN $\psi(x,y)$ FIND u & v. (EASY)

$$\psi(x,y) = x y^2 \cos(\ln(x))$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left\{ x y^2 \cos(\ln(x)) \right\}$$

$$= 2yx \cos(\ln(x)) \checkmark$$

you're home... use software or google it!

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left\{ x y^2 \cos(\ln(x)) \right\}$$

$$= -\left\{ y^2 \cos(\ln(x)) + x y^2 \frac{\partial}{\partial x}\cos(\ln(x)) \right\}$$

$$= -\left\{ y^2 \cos(\ln(x)) - \cancel{x} y^2 \frac{\sin(\ln(x))}{\cancel{x}} \right\}$$

$$\therefore v = y^2 \sin(\ln(x)) - y^2 \cos(\ln(x)) \checkmark$$

WHATEVER THE QUESTION IT INVOLVES

i) TAKING PARTIAL DERIVATIVES (u, v)

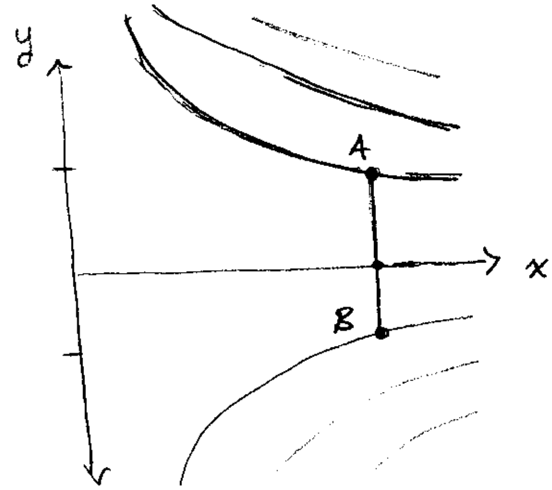
ii) SETTING THESE EXPRESSION EQUAL TO SOMETHING

$\psi(x,y) = K$ or $\omega_z \equiv 0$

Vorticity.

EXAMPLE (STREAM FUNCTIONS)

GIVEN $\psi(x,y)$ CALCULATE FLOWRATE Q.



$$\psi(x,y) = xy^2 \cos(\ln(x))$$

* USE A COMPUTER TO MAKE A CONTOUR PLOT. CONTOURS $\Leftrightarrow$ STREAMLINES

$A := (x=2,\ y=4)$
$B := (x=2,\ y=-4)$

* $q = \psi(B) - \psi(A)$

$= \psi(2,-4) - \psi(2,4)$

$= 2(-4)^2\cos(\ln(2)) - 2(4)^2\cos(\ln(2)) = 0$ No Flowrate...

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GIVEN $u(x,y)$ & $v(x,y)$ DETERMINE $\psi(x,y)$ (HARD)

$u(x,y) = x + x^2 - 2xy - 3y^2$ & $v(x,y) = -y - x^2 + y^2 - 3x^2$

* REMEMBER THE DEF OF STREAMFUNCTIONS!

$\frac{\partial \psi}{\partial y} = u \Rightarrow \psi(x,y) = \int u\, dy = xy + x^2y - xy^2 - y^3 + f(x)$

$-\frac{\partial \psi}{\partial x} = v \Rightarrow \psi(x,y) = -\int v\, dx = xy + x^2y - xy^2 + x^3 + g(y)$

* LOOK FOR NOW $f(x)$ & $g(y)$. BUT WE SEE BY INSPECTION

$f(x) \equiv x^3$ & $g(y) \equiv -y^3$

$\therefore \quad \psi(x,y) = xy + x^2y - xy^2 + x^3 - y^3$ ✓