LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE

LET'S TRY TO FIGURE OUT THE VELOCITY DISTRIBUTION OF LIQUID FLOWING IN A STRAIGHT PIPE. WE ALSO WILL MAKE SOME ASSUMPTIONS TO MAKE OUR LIVES EASIER.

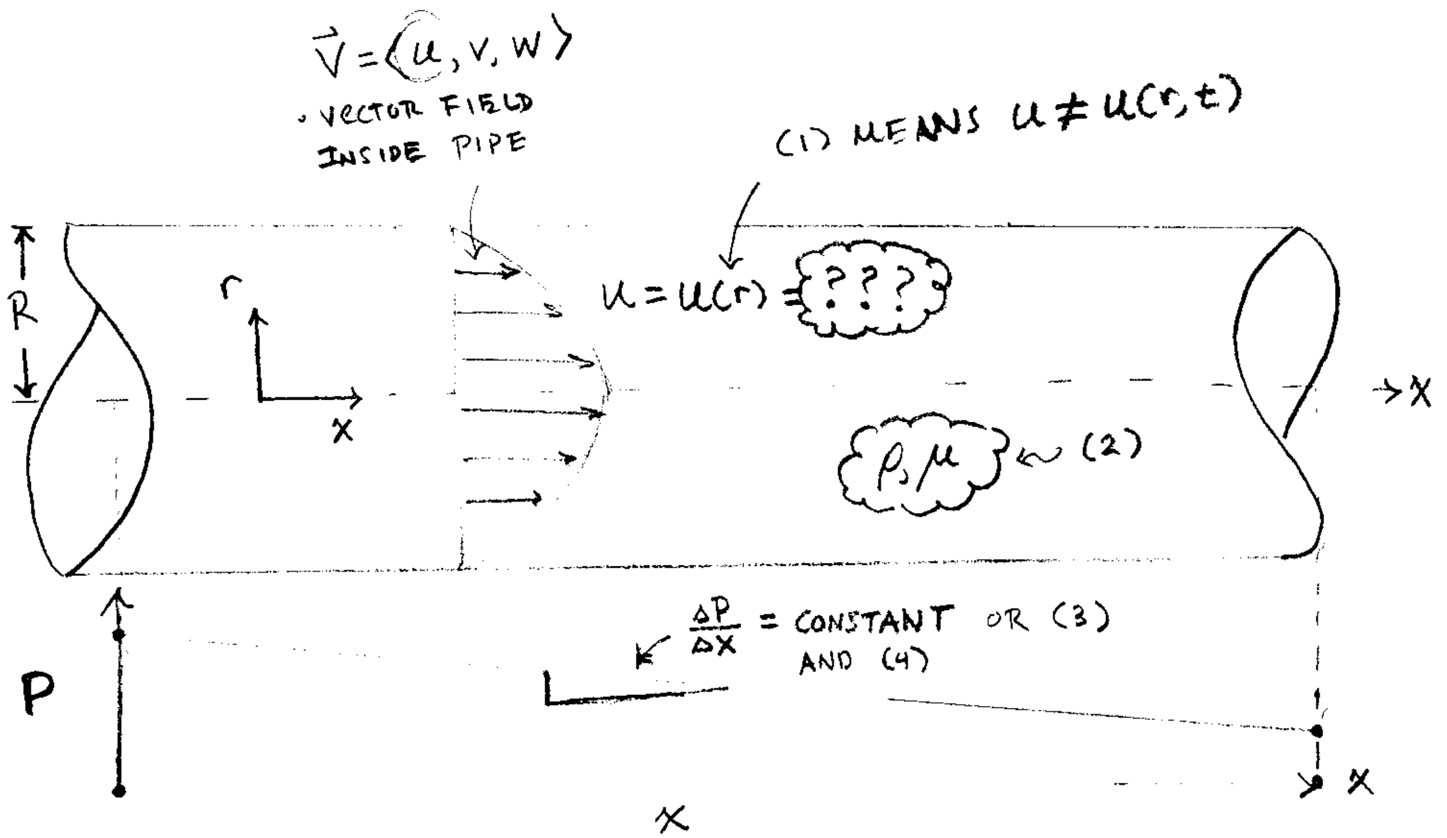
1) STEADY-FLOW $\partial_t \equiv 0$

2) INCOMPRESSIBLE (ALWAYS IN THIS CLASS)

3) FULLY-DEVELOPED

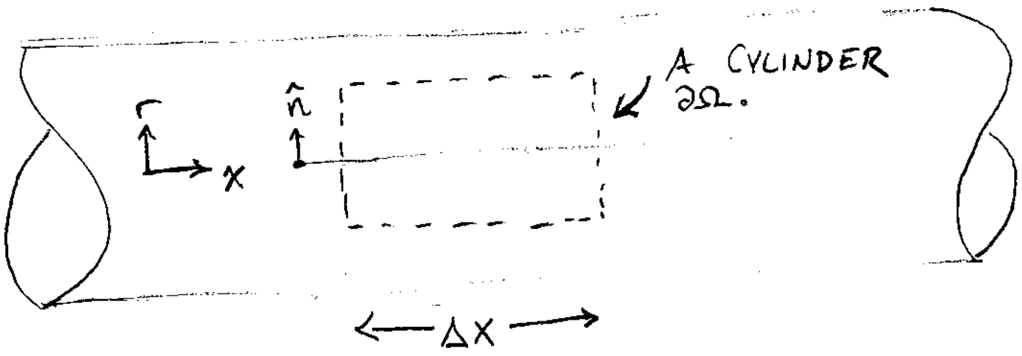
4) PRESSURE ONLY A FUNCTION OF AXIAL POSITION.

LET'S DRAW A PICTURE TO POINT THESE OUT.



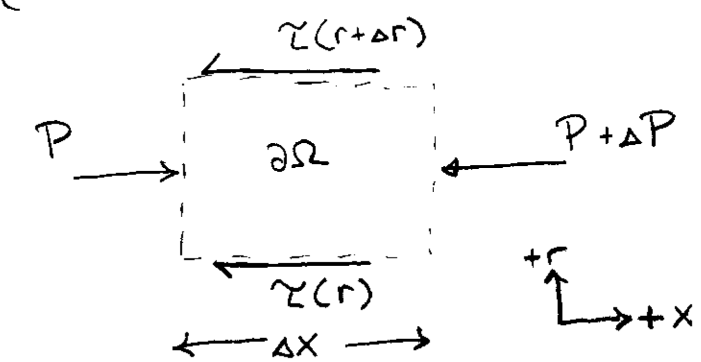
LECTURE 3: INSIDE A SINGLE STRAIGHT PIPE.

NOW LET'S DETERMINE THIS $U(r)$ USING A CONTROL VOLUME APPROACH, THAT MEANS WE WILL DRAW AN IMAGINARY SURFACE IN THE FLOW AND SAY TO OURSELVES "WHAT COMES IN MUST COME OUT"

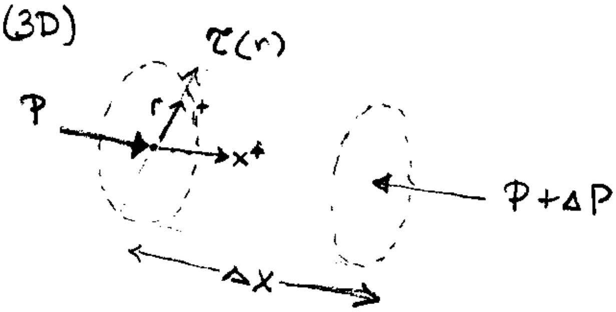


WHAT ARE THE FORCES ON OUR $\partial\Omega$ CONTROL VOLUME?

(2D)



(3D)



NEWTON IN THE X-DIRECTION.

$$\sum F_x = \frac{dp}{dt} \quad \text{In terms of momentum } p = mv.$$

$$= \underbrace{\cancel{\frac{\partial}{\partial t}\int_{\Omega} V \rho \, d\forall}}_{\text{VARIATION INSIDE BODY}}^{\to 0,\ \text{steady}} + \underbrace{\int_{\partial\Omega} \rho V_x (V\cdot\hat{n})\, dA}_{\text{FLUX THRU BODY SURFACE INTEGRAL}}$$

THIS DOT PRODUCT MEANS THE FRONT AND BACK OF THE CYLINDER DO NOT CONTRIBUTE

$$= \int_{\partial\Omega} \rho V_x (V\cdot\hat{n})\, dA$$

LECTURE 3: INSIDE A SINGLE STRAIGHT PIPE

WE ALREADY MADE THE ASSUMPTION THAT

$$V = \langle u(r), 0, 0 \rangle$$

SO OUR SURFACE INTEGRAL BECOMES.

$$\Sigma F_x = \int_0^r \rho\, u\, (u \cdot \hat{n}) \underbrace{2\pi r dr}_{\text{just } dA} + \int_0^{-r} \rho u (u \cdot \hat{n}) 2\pi r dr$$

$$= \int_0^r \rho\, u^2\, 2\pi r dr - \int_0^r \rho u^2 2\pi r dr$$

flipped integral.

$$= \phi\ !$$

THIS IS FUNDAMENTAL TO FULLY DEVELOPED FLOW! MOMENTUM AT X AND ΔX ARE THE SAME. LET'S NOW CALCULATE FORCES IN THE X-DIRECTION. REFER TO THE 2D DRAWING.

$$\Sigma F_x = 0$$

$$\underbrace{P\pi r^2}_{F = P \cdot A} - (P - \Delta P)\pi r^2 - \underbrace{\tau 2\pi r \Delta X}_{\{\text{SHEAR}\}\cdot\{\text{SURFACE AREA OF CYLINDER}\}.} = 0$$

(−) BECAUSE OUR ARROW IS POINTED IN

* SOLVE FOR $\Delta P / \Delta X$.

$$\underbrace{\frac{\Delta P}{\Delta X} = \frac{2\tau}{r}}_{\text{TRUE FOR LAMINAR AND TURBULENT.}} \qquad (*)$$

LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE.

REMEMBER FOR FULLY DEVELOPED FLOWS THOUGH!

$$\frac{\Delta P}{\Delta X} = K$$

USING (*) WE SEE SOMETHING SURPRISING.

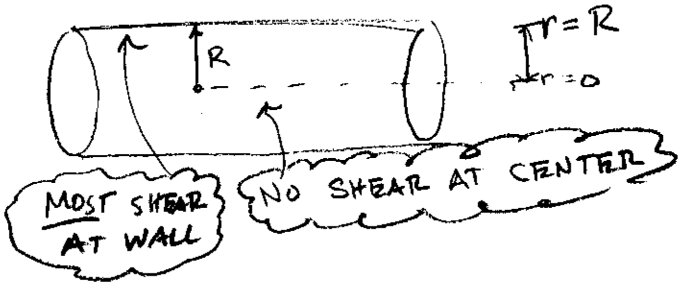
$$\tau = \frac{K}{2} r$$

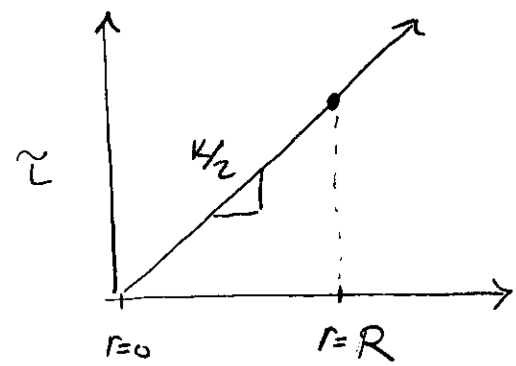
{SHEAR} = {LINEARLY PROPORTIONAL TO RADIUS}.

WHAT DOES THIS MEAN. WELL...

$$\tau_{max} = \frac{K}{2} R$$

RADIUS OF PIPE





WE NOW INTRODUCE NEWTONS LAW OF VISCOSITY

$$(**) \qquad \tau = -\mu \frac{du}{dr}$$

FOR NEWTONIAN FLUIDS μ IS THE SLOPE OF A LINE.

{SHEAR} = {LINEARLY PROPORTIONAL TO A CHANGE IN VELOCITY}

* THIS IS WHERE CONSTRAINT OF LAMINAR IS INTRODUCED.

SUB (**) INTO (*).

$$\frac{dP}{dX} = -\frac{2\mu}{r}\frac{du}{dr}$$

"SEPERATE AND INTEGRATE"

$$du = -\left(\frac{1}{2\mu}\frac{\Delta P}{\Delta X}\right) r\,dr$$

$$\int du = -\left(\frac{1}{2\mu}\frac{\Delta P}{\Delta X}\right)\int r\,dr$$

INTEGRATON CONSTANT

$$\therefore \quad u(r) = -\left(\frac{1}{2\mu}\frac{\Delta P}{\Delta X}\right)\frac{r^2}{2} + C$$

TO SOLVE FOR C WE NEED A BOUNDARY CONDITION.

$$u(r=R) = 0 \qquad \text{(NO SLIP CONDITION)}$$

$$\therefore \quad C = \frac{1}{2\mu}\frac{\Delta P}{\Delta X}\frac{R^2}{2}$$

EXTRA CREDIT. SHOW THE SAME BUT USING $\frac{du}{dr}(r=0) = 0$. EXPLAIN WHAT THIS CONDITION MEANS

FINALLY ...

$$u(r) = \left(\frac{1}{4\mu}\frac{\Delta P}{\Delta X}\right)(R^2 - r^2)$$

$$\boxed{u(r) = \frac{D^2\,\Delta P}{16\mu\,\Delta X}\left(1 - \left(\frac{r}{R}\right)^2\right)}$$

- D IS EASIER TO MEASURE
- (r/R) IS NON-DIMENSIONAL

LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE.

Q: WHAT IS THE MAXIMUM VELOCITY?

$$u(r) = \frac{D^2}{16\mu}\frac{\Delta P}{\Delta x}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \text{MAX WHEN } r = 0$$

EXTRA CREDIT. SHOW THIS WITH CALCULUS.

$$\therefore \quad \boxed{V_{max} = \frac{D^2}{16\mu}\cdot\frac{\Delta P}{\Delta x}}$$

Q: WHAT IS THE FLOWRATE IN THE PIPE?

$$Q = \int u(r)\, dA$$

$$= \int_0^R u(r)\, 2\pi r\, dr$$

EXTRA CREDIT DO THIS INTEGRATION

Q: I LIKE TO CALCULATE Q WITHOUT INTEGRATING! I WANT TO JUST MULTIPLY A VELOCITY AND AN AREA. $Q = \bar{V}A$. WHAT VALUE IS THIS $\bar{V}$ I'M LOOKING FOR.

$$Q = \bar{V} \cdot A$$

$$\bar{V} = \frac{1}{A} Q$$

$$= \frac{1}{A}\int_0^R u(r)\, 2\pi r\, dr$$

$$\vdots$$

EXTRA CREDIT COMPLETE STEPS TO SHOW THIS

$$\boxed{\bar{V} = \frac{V_{max}}{2}}$$

LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE.

THE FLOWRATE Q IN THE PIPE IS SHOWN AS

$$\boxed{Q = \frac{\pi D^4}{128\,\mu}\,\frac{\Delta P}{\Delta X}}$$

NOTICE HOW SENSITIVE IT IS TO THE DIAMETER! D^4!

WHAT HAVE WE LEARNED?

(1) $\frac{\Delta P}{\Delta X} = \frac{2\tau}{r}$ or $\tau = \frac{K}{2} r$

- SHEAR IS LINEARLY PROPORTION TO RADIUS

(2) $u(r) = \frac{D^2}{16\mu}\frac{\Delta P}{\Delta X}\left(1-\left(\frac{r}{R}\right)^2\right)$ or $V_{max}\left(1-\left(\frac{r}{R}\right)^2\right)$

- $u\uparrow$ if $\frac{\Delta P}{\Delta X}\uparrow$, $D\uparrow$, or $\mu\downarrow$

(3) $V_{max} = \frac{D^2}{16\mu}\frac{\Delta P}{\Delta X}$

- THE FASTEST FLOW IS IN THE CENTER OF THE PIPE.

(4) $\bar{V} = \frac{V_{max}}{2}$ for $Q = \bar{V}\cdot A$

- A CONVINENT CALCULATION, WE CAN MEASURE V_{max}...

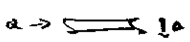
(5) $Q = \frac{\pi D^4}{128\,\mu}\frac{\Delta P}{\Delta X}$

- FLOWRATE IS EXTREMELY SENSITIVE TO PIPE DIMENSIONS

NON-DIMENSONAL FORM

* A NON-DIMENSIONAL EXPRESSION IS ∞ MORE USEFUL THAN A DIMENSIONAL ONE

* IT REMAINS TRUE FOR MANY SCALES OF FLOW!



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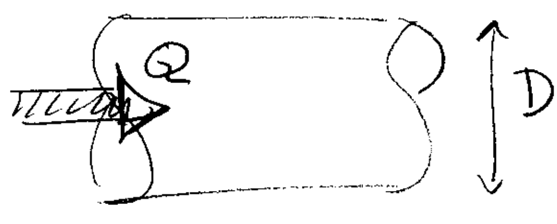


LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE.

LETS USE THAT $\bar{V}$ OR AVERAGE VELOCITY EXPRESSION

$$\underbrace{V_{max}} = \frac{D^2}{16\mu}\frac{\Delta P}{\Delta X}$$

$$\bar{V} = \frac{V_{max}}{2}$$

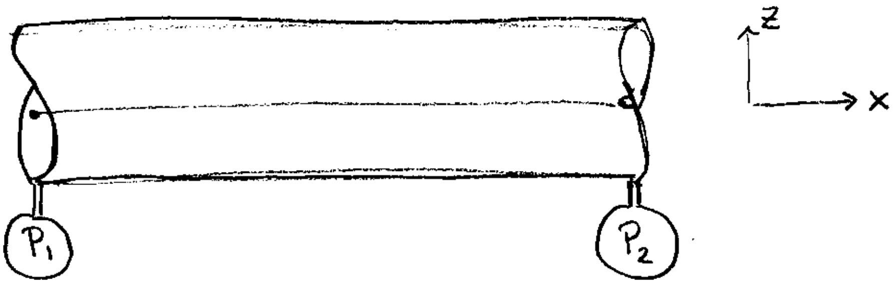
$$\bar{V} = \frac{D^2}{32\mu}\frac{\Delta P}{\Delta X}$$

WHAT IS THE CHANGE IN PRESSURE ΔP ACROSS THE PIPE?

$$\Delta P = \bar{V} \cdot \frac{32\mu\,\Delta X}{D^2}$$

DIMENSIONAL

WE WANT TO DIVIDE BY A PRESSURE TO MAKE THIS NON-DIMENSIONAL. BUT WHAT PRESSURE SHOULD WE CHOOSE? LETS LOOK AT BERNOULLI'S EQUATION.



$$\rho\frac{V^2}{2} + \cancel{gz} + P = C$$

"STRAIGHT PIPE"

$$\underbrace{\frac{\rho V^2}{2}} + P = C$$

IN FLUIDS WE USUALLY CHOOSE THIS DYNAMIC PRESSURE TO NON-DIMENSIONLIZE PRESSURE TERMS

LECTURE 3: INSIDE A SINGLE STRAIGHT PIPE

$$\frac{\Delta P}{\frac{1}{2}\rho \bar{V}^2} = \frac{32\mu}{D}\,\frac{\Delta X}{D}\,\frac{2}{\rho \bar{V}^{\cancel{2}}}\,\cancel{\bar{V}}$$

WELL I'LL BE DAMNED THAT'S A REYNOLD'S #

$$\Delta P^* = 64\left(\frac{\mu}{\rho \bar{V} D}\right)\left(\frac{\Delta X}{D}\right)$$

$$\underbrace{\Delta P^* \frac{D}{\Delta X}} = \frac{64}{Re}$$

WE CALL THIS THE DARCY-FRICTION FACTOR!

$$f_{laminar} = \frac{64}{Re}$$

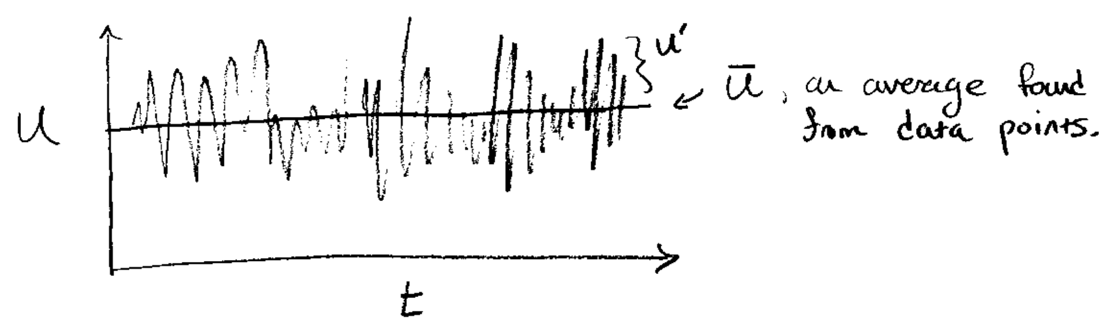
WHAT IS ΔX IN THIS EXPRESSION THOUGH? IT IS SOME AXIAL DISTANCE THAT MUST CHARACTERIZE OUR PIPE... UM I'D THE <u>LENGTH OF THE PIPE DOES A GREAT JOB AT THAT</u>!

WHAT ABOUT $f_{turbulent}$?!

//

REMEMBER THAT IN TURBULENT FLOWS WE MODEL VARIATIONS

$$u = \bar{u} + u'$$

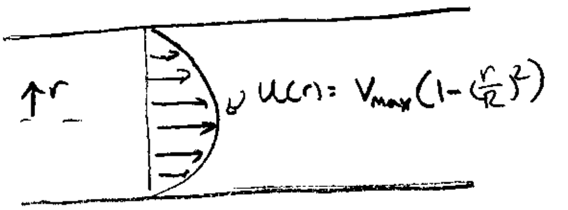


LECTURE 3: INSIDE A SINGLE STRAIGHT PIPE

* WE CANNOT DETERMINE $\bar{u}(r)$ FROM THE GOVERNING EQUATIONS OR A FORCE BALANCE.

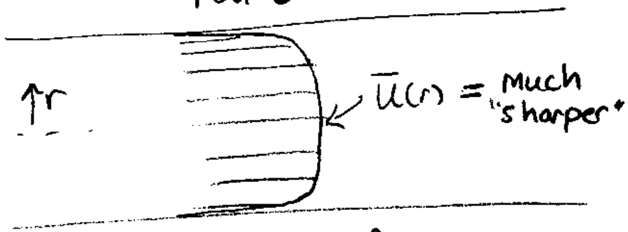
* WE DO KNOW WHAT TURBULENT FLOW LOOKS LIKE FROM EXPERIMENTS

LAMINAR



vs.

TURBULENT



- quadratic profile
- Almost looks like a plug flow

* WE NEED SOME $\bar{u}(r)$ FUNCTION TO CALCULATE ANYTHING THOUGH!

PEOPLE FIGURED OUT FROM EXPERIMENTAL DATA THAT

$$u^+ = \frac{1}{\kappa}\ln(y^+) + B$$

AND

$$\kappa = 0.41$$

$$B = 5$$

WHERE $u^+ = u/u^*$ OR NON-DIMENSIONAL VELOCITY. AND $y^+ = y/y^*$

$$u^* = \sqrt{\frac{\tau_w}{\rho}} := \text{"THE FRICTION VELOCITY"}$$

$$y^* = \frac{\nu}{u^*} := \text{"FRICTION HEIGHT"}$$

$\nu = \frac{\mu}{\rho}$

LECTURE 3 : INSIDE A SINGLE STRAIGHT PIPE

SO WE WILL JUST ACCEPT THIS $u^+(y^+)$ RELATION
NOW WE CAN DETERMINE $f_{turbulent}$!

1) Express $u^+(y^+)$ IN RADIAL FORM.

$$u^+ = \frac{u(y)}{u^*} = \frac{u(R-r)}{u^*}$$

$$\boxed{\frac{u(r)}{u^*} = \frac{1}{\kappa} \ln\left(\frac{u^*(R-r)}{\nu}\right) + B} \qquad (*)$$

2) AVERAGE VELOCITY $\bar{V}$. JUST AS BEFORE
$Q = \bar{V} \cdot A \Rightarrow \bar{V} = \frac{1}{A} Q$:

$$\bar{V} = \frac{1}{\pi R^2} \int_0^r u(r)\, 2\pi r\, dr$$

EXTRA CREDIT
SUB IN (*)
AND DO INTEGRATION

$$\frac{\bar{V}}{u^*} = 2.44 \ln\left(\frac{R u^*}{\nu}\right) + 1.34 \qquad (\Delta\Delta\Delta)$$

3) Manipulate!

$$\underbrace{\bar{V}\sqrt{\frac{\rho}{\tau_w}}}_{\text{Def of } u^*} = \sqrt{\frac{\rho \bar{V}^2}{\tau_w}}$$

4) DEFINE OUR FRICTION FACTOR

$$f \equiv \frac{\Delta P}{\frac{1}{2}\rho \bar{V}^2} \frac{D}{\Delta X} \qquad (\Delta)$$

LECTURE 3. INSIDE A SINGLE STRAIGHT PIPE.

* REMEMBER THAT SHEAR RELATIONSHIP WE FOUND.

(ΔΔ) $$\frac{\Delta P}{\Delta X} = \frac{2\tau}{R}$$ (STILL TRUE FOR TURBULENT FLOW)

$$f \equiv \frac{8\tau}{\rho \bar{V}^2}$$

~ SUBBED (ΔΔ) INTO (Δ).

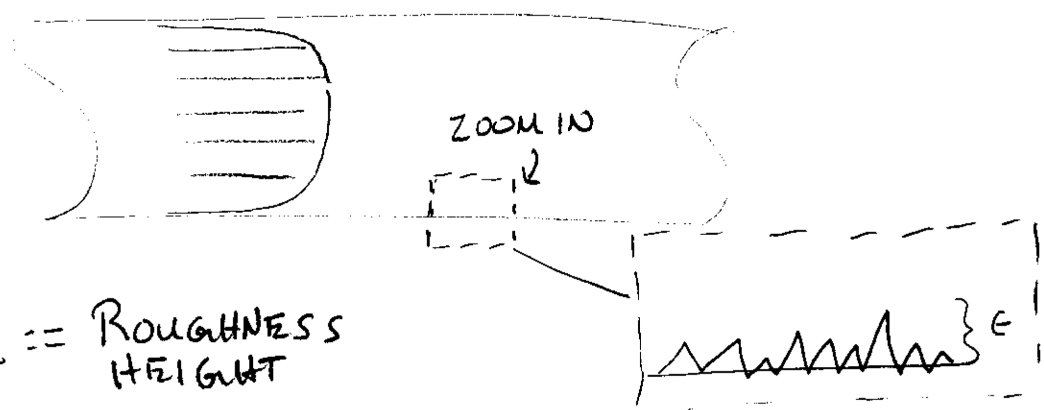
SO WHAT IS $f_{turbulent}$!? COMBINE (Δ), (ΔΔ), AND (ΔΔΔ) TO GET!

$$\boxed{\frac{1}{\sqrt{f}} = 2.00 \log_{10}\left(Re\sqrt{f}\right) - 0.8}$$

SO f_{turb} MUST BE SOLVED WITH A COMPUTER! WE CAN'T WRITE OUT $f(Re)$ LIKE WE DID WITH LAMINAR FLOW.

//

WHAT ABOUT TURBULENT ROUGH PIPES!?



ϵ := ROUGHNESS HEIGHT

LECTURE 3:

I'LL SPARE THE DETAILS BUT WE GET FROM THE <u>EXACT</u> <u>SAME</u> <u>PROCESS</u> BUT A DIFFERENT $u^+(y^+)$.

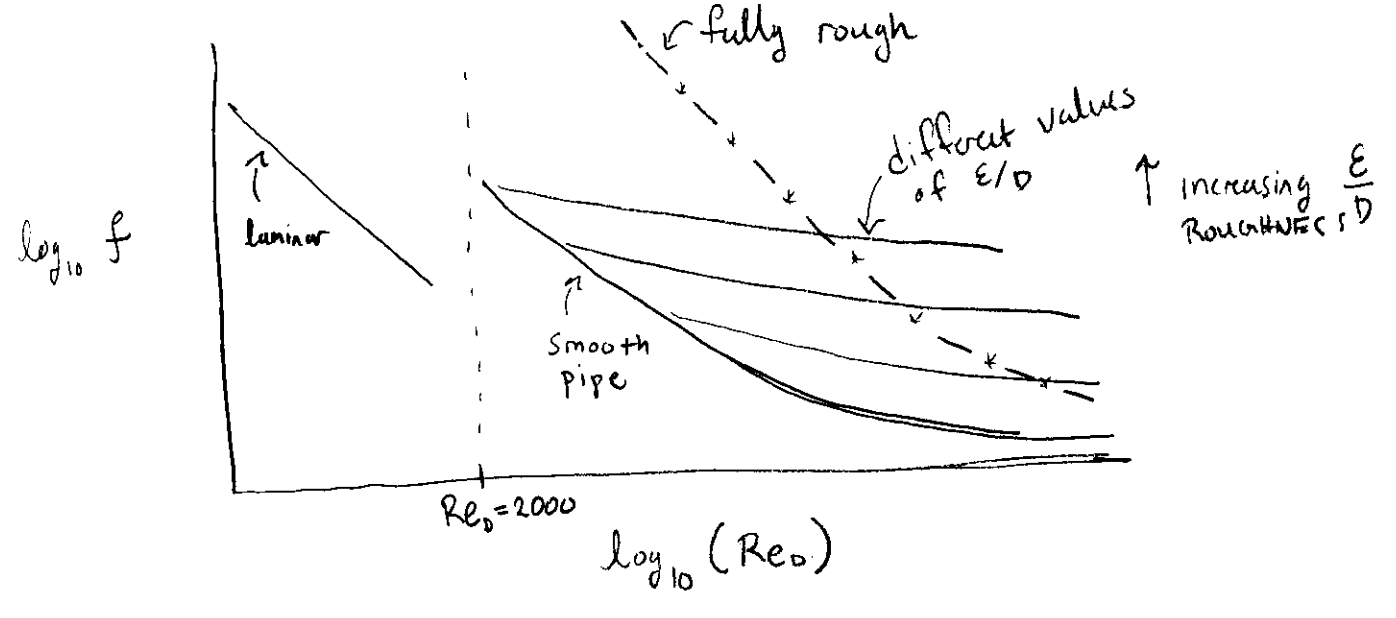
$$\frac{1}{\sqrt{f}} = -2.0 \log_{10}\left[\frac{\epsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right] \qquad (1)$$

OR

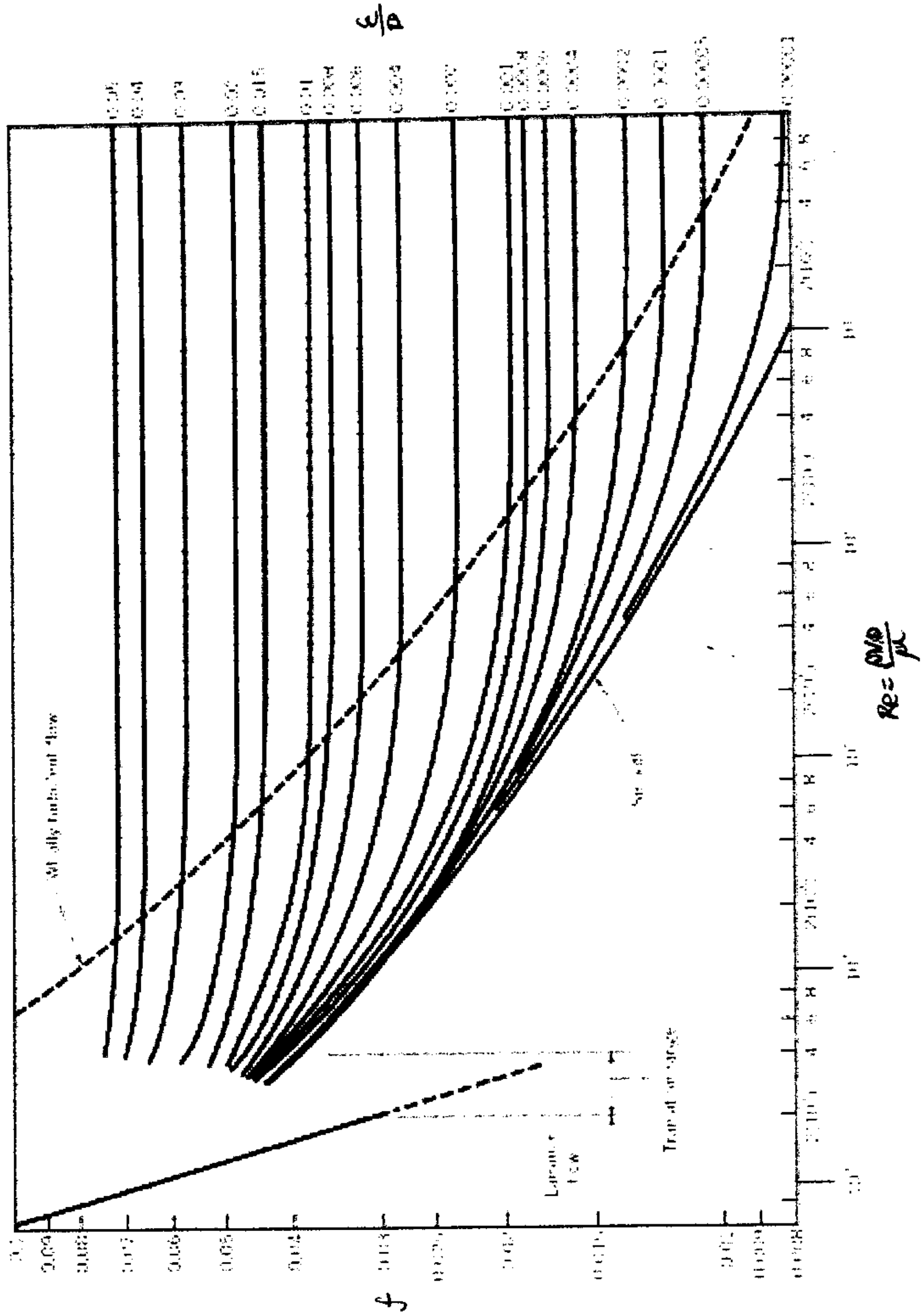
$$\frac{1}{\sqrt{f}} \approx -1.8 \log_{10}\left[\frac{6.9}{Re_D} + \left(\frac{\epsilon/D}{3.7}\right)^{1.11}\right] \qquad (2)$$

THE BENEFIT OF (2) IS IT IS NOT AN IMPLICIT EQUATION. YOU CAN PLUG IT INTO A CALCULATOR. WITH (1) YOU MUST USE A ITERATIVE SOLVER.

* THE MOODY CHART IS A WAY TO DETERMINE A FRICTION FACTOR <u>GRAPHICALLY</u>.



* RESOLUTION MAY BE BAD USE GRAPH IN BOOK OR GOOGLE "MOODY CHART"



Lecture 3: Inside a Single Straight Pipe

- That's it for a single straight pipe!
- Our job is to find friction factors!
- Easy if laminar, slightly more complicated if turbulent.
- We will see how to use f next time. But for now we should get good at finding an f given a pipe problem
- Process...