Shahjalal University of Science &Technology, Sylhet

Department of Computer Science and Engineering

Course No: Math 204 D

Assignment No: 02

Complex Variables

Submitted To

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Session : 2018-19

Submission date: 24/11/2020

(a)

Problem Statement:

Find all the values of z for which $z^7 = -7$ and locate these values in the complex plane.

Solution:

$$z^7 = -7$$
$$= 7\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\} \quad [k = 0, \pm 1, \pm 2, \cdots] \quad \text{--- (I)}$$

Let, $z = r(\cos\theta + i\sin\theta)$.

Using De Moivre's theorem,

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$
$$\therefore z^7 = r^7(\cos 7\theta + i\sin 7\theta) \quad \text{--- (II)}$$

Comparing equations (I) and (II) we get,

$$r^7 = 7$$
$$\Rightarrow r = 7^{\frac{1}{7}}$$

and

$$7\theta = \pi + 2k\pi$$
$$\Rightarrow \theta = \frac{\pi + 2k\pi}{7}$$

So,

$$z = r(\cos\theta + i\sin\theta)$$

$$= 7^{\frac{1}{7}}\left\{\cos\left(\frac{\pi+2K\pi}{7}\right) + i\sin\left(\frac{\pi+2K\pi}{7}\right)\right\}$$

If, $K=0$,

$$z = z_1 = 7^{\frac{1}{7}}\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$$

If, $K=1$,

$$z = z_2 = 7^{\frac{1}{7}}\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)$$

If, $K=2$,

$$z = z_3 = 7^{\frac{1}{7}}\left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$$

If, $K=3$,

$$z = z_4 = 7^{\frac{1}{7}}\left(\cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7}\right) = -7^{\frac{1}{7}}$$

If, $K=4$,

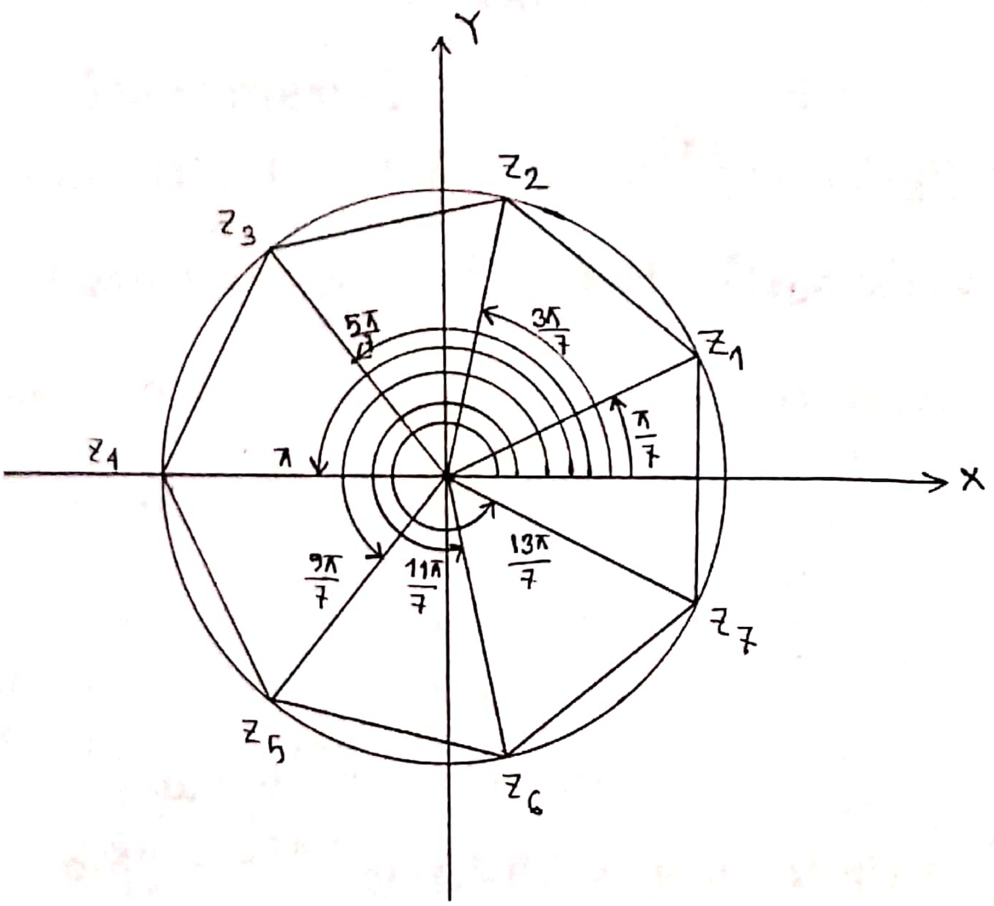
$$z = z_5 = 7^{\frac{1}{7}}\left(\cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}\right)$$

If, $K=5$,

$$z = z_6 = 7^{\frac{1}{7}}\left(\cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}\right)$$

If, $K=6$,

$$z = z_7 = 7^{\frac{1}{7}}\left(\cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}\right)$$



The values of z are located in the above complex plane.

(b)

Problem Statement:

Prove that the function $u = e^{-2xy}\sin(x^2-y^2)$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Is v harmonic? Justify your answer. Express $f(z)$ in terms of z.

Solution:

Given,

$$u = e^{-2xy}\sin(x^2-y^2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}\cdot\cos(x^2-y^2).2x + \sin(x^2-y^2)\,e^{-2xy}\cdot(-2y)$$

$$= 2x e^{-2xy}\cos(x^2-y^2) - 2y e^{-2xy}\sin(x^2-y^2)$$

$$= 2e^{-2xy}\{x\cos(x^2-y^2) - y\sin(x^2-y^2)\} \quad \text{--- (1)}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 2\left[e^{-2xy}\frac{\partial}{\partial x}\{x\cos(x^2-y^2) - y\sin(x^2-y^2)\} + \{x\cos(x^2-y^2) - y\sin(x^2-y^2)\}\cdot e^{-2xy}\cdot(-2y)\right]$$

$$= 2e^{-2xy}\left[x\{-\sin(x^2-y^2)\}2x + \cos(x^2-y^2) - y\cos(x^2-y^2).2x\right] - 4ye^{-2xy}\{x\cos(x^2-y^2) - y\sin(x^2-y^2)\}$$

$$= 2e^{-2xy}\left[-2x^2\sin(x^2-y^2) + \cos(x^2-y^2) - 2xy\cos(x^2-y^2)\right] - 4xye^{-2xy}\cos(x^2-y^2) - 2xy\cos(x^2-y^2)]$$

$$= -4x^2 e^{-2xy}\sin(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2) - 4xye^{-2xy}\cos(x^2-y^2)$$
$$- 4xye^{-2xy}\cos(x^2-y^2) + 4y^2e^{-2xy}\sin(x^2-y^2)$$

$$= -4x^2 e^{-2xy}\sin(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2) - 8xye^{-2xy}\cos(x^2-y^2)$$
$$+ 4y^2 e^{-2xy}\sin(x^2-y^2) \quad \text{——— (2)}$$

Again,

$$u = e^{-2xy}\sin(x^2-y^2)$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{-2xy}\cos(x^2-y^2)\cdot(-2y) + \sin(x^2-y^2)\, e^{-2xy}(-2x)$$

$$= -2ye^{-2xy}\cos(x^2-y^2) - 2xe^{-2xy}\sin(x^2-y^2) \quad \text{——— (3)}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -2\Big[e^{-2xy}\frac{\partial}{\partial y}\{y\cos(x^2-y^2) + x\sin(x^2-y^2)\} + \{y\cos(x^2-y^2) +$$
$$x\sin(x^2-y^2)\}e^{-2xy}(-2x)\Big]$$

$$= -2e^{-2xy}\big[y\{-\sin(x^2-y^2)\}(-2y) + \cos(x^2-y^2) + x\cos(x^2-y^2)$$
$$\cdot(-2y)\big]$$
$$+ 4xe^{-2xy}\{y\cos(x^2-y^2) + x\sin(x^2-y^2)\}$$

$$= -4y^2e^{-2xy}\sin(x^2-y^2) - 2e^{-2xy}\cos(x^2-y^2) + 4xye^{-2xy}$$
$$+\cos(x^2-y^2) + 4xye^{-2xy}\cos(x^2-y^2) + 4x^2e^{-2xy}\sin(x^2-y^2)$$

$$= -4y^2 e^{-2xy}\sin(x^2-y^2) - 2e^{-2xy}\cos(x^2-y^2) + 8xye^{-2xy}\cos(x^2-y^2)$$
$$+ 4x^2 e^{-2xy}\sin(x^2-y^2) \text{ ——— } (4)$$

From (2) + (4) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So, u is harmonic.

Now,

$$\frac{\partial u}{\partial x} = 2e^{-2xy}\{x\cos(x^2-y^2) - y\sin(x^2-y^2)\} \text{ ——— } (1)$$

$$\frac{\partial u}{\partial y} = -2e^{-2xy}\{y\cos(x^2-y^2) + x\sin(x^2-y^2)\} \text{ ——— } (3)$$

We know,

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(x+iy) = u_1(x,y) - iu_2(x,y)$$

Putting $y=0$

$$f'(x) = u_1(x,0) - iu_2(x,0)$$

Replacing x by z,

$$f'(z) = u_1(z,0) - iu_2(z,0) \text{ ——— } (5)$$

$u_1(z,0) = 2z\cos z^2$

$u_2(z,0) = -2z\sin z^2$

From eq. 5

$$f'(z) = 2z\cos z^2 + 2iz\sin z^2$$

Integrating with respect to z,

$$f(z) = \int 2z\cos z^2\,dz + i\int 2z\sin z^2\,dz \qquad \begin{array}{|l} z^2 = p \\ \Rightarrow 2z\,dz = dp \end{array}$$

$$= \int \cos p\,dp + i\int \sin p\,dp$$

$$= \sin p + i(-\cos p) + c$$

$\therefore f(z) = \sin z^2 - \cos z^2$ [Ignoring constant]

Now, From Cauchey-Reimann equation,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2xe^{-2xy}\cos(x^2-y^2) - 2ye^{-2xy}\sin(x^2-y^2) \quad (7)$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} = 2ye^{-2xy}\cos(x^2-y^2) + 2xe^{-2xy}\sin(x^2-y^2) \quad (8)$$

Integrating ① w.r.t y we get,

$$v = \int 2x e^{-2xy} \cos(x^2-y^2)\, dy - \int 2y e^{-2xy} \sin(x^2-y^2)\, dy$$

$$= \int 2x e^{-2xy} \cos(x^2-y^2) dy - e^{-2xy} \int 2y \sin(x^2-y^2)\, dy + \int \{ (-2x) e^{-2xy} \int 2y \sin(x^2-y^2)\, dy \} dy$$

$$= \int 2x e^{-2xy} \cos(x^2-y^2)\, dy - e^{-2xy} \int dp - \int \{ 2x e^{-2xy} \int dp \} dy$$

Let,

$\cos(x^2-y^2) = p$

$\Rightarrow -\sin(x^2-y^2).(-2y) dy = dp$

$\Rightarrow 2y \sin(x^2-y^2) dy = dp$

$$= \int 2x e^{-2xy} \cos(x^2-y^2)\, dy - e^{-2xy} . p - \int 2x e^{-2xy} p\, dy$$

$$= \int 2x e^{-2xy} \cos(x^2-y^2)\, dy - e^{-2xy} \cos(x^2-y^2) - \int 2x e^{-2xy} \cos(x^2-y^2)\, dy$$

$$= -e^{-2xy} \cos(x^2-y^2) + F(x) \quad \text{—— ⑨}$$

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Differentiating (9) partially with respect to x,

$$\frac{\partial v}{\partial x} = -\left[e^{-2xy}\{-\sin(x^2-y^2)\}\cdot 2x + \cos(x^2-y^2)\cdot e^{-2xy}\cdot(-2y)\right] + F'(x)$$

$$= 2x\,e^{-2xy}\sin(x^2-y^2) + 2y\,e^{-2xy}\cos(x^2-y^2) + F'(x) \quad (10)$$

Comparing eq. (8) and (10) we get,

$F'(x) = 0$. So, $F(x) = c$, a constant.

From, (9) we get,

$$v = -e^{-2xy}\cos(x^2-y^2) + c.$$

which is the required conjugate function of u.

and v is harmonic.

So, we can say that, $f(z)$ is analytic.

(C)

Problem Statement:

Compute $\oint_C \frac{dz}{(z-1)(z+1)(z-i)}$, where C is the simple closed path in the following three cases;

i. The point 1 is in the interior of C and the points $\pm i$ are in the exterior of C.

ii. The point 1 and i are in the interior of C and point $-i$ is in the exterior of C.

iii. All three points are in the interior of C.

Solution:

i)

$$\oint_C \frac{dz}{(z-1)(z+i)(z-i)}$$

$$= \oint_C \frac{dz}{(z-1)(z^2+1)}$$

$$= \oint_C f(z)\, dz$$

Point 1 is in the interior of C. So, $f(z)$ has a pole of order 1 at $z = 1$

Residue at $z = 1$ is, $[z_0 = 1, m = 1]$

$$a_{-1} = \underset{z \to z_0}{Lt} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m \cdot f(z) \right\}$$

$$= \underset{z \to z_0}{Lt} \frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z-1)^1 \cdot \frac{1}{(z-1)(z^2+1)} \right\}$$

$$= \underset{z \to z_0}{Lt} \frac{1}{z^2+1}$$

$$= \underset{z \to 1}{Lt} \frac{1}{z^2+1}$$

$$= \frac{1}{2} \qquad \text{———— (1)}$$

By Residue Theorem,

$$\oint_C \frac{dz}{(z-1)(z+i)(z-i)} = 2\pi i \cdot a_{-1}$$

$$= 2\pi i \cdot \frac{1}{2}$$

$$= \pi i$$

ii)

The points 1 and i are in the interior of C.

So, f(z) has two poles of order 1 at $z = 1$ and $z = i$

Residue at $z = i$ is,

$$b_{-1} = \underset{z \to i}{Lt} \ (z-i) \cdot \frac{1}{(z-1)(z+i)(z-i)}$$

$$= \underset{z \to i}{Lt} \ \frac{1}{(z-1)(z+i)}$$

$$= \frac{1}{2i(i-1)}$$

$$= \frac{1}{2(i^2-i)}$$

$$= \frac{-1}{2(i+1)} \quad \text{———} \ (2)$$

$$= \frac{i-1}{4}$$

By Residue theorem,

$$\oint_c \frac{dz}{(z-1)(z+i)(z-i)} = 2\pi i \left(a_{-1} + b_{-1}\right)$$

[using equation (1) and (2)]

$$= 2\pi i \left(\frac{1}{2} - \frac{1}{i+1}\right)$$

$$= \pi i \left(1 - \frac{1}{i+1}\right)$$

$$= \pi i \ \frac{i}{i+1}$$

$$= \frac{-\pi}{i+1}$$

$$= \frac{-\pi(1-i)}{(1+i)(1-i)}$$

$$= \frac{\pi(i-1)}{2}$$

iii)

All points $1, \pm i$ are in the interior C. So, $f(z)$ has three poles, all of order 1 at $z=1$, $z=i$ and $z=-i$

So, Residue at $z=-i$ is,

$$c_{-1} = \underset{z \to -i}{Lt} (z+i) \frac{1}{(z-1)(z+i)(z-i)}$$

$$= \underset{z \to -i}{Lt} \frac{1}{(z-1)(z-i)}$$

$$= \frac{1}{(-i-1)(-2i)}$$

$$= \frac{1}{2i(i+1)}$$

$$= \frac{1}{2(i^2+i)}$$

$$= \frac{1}{2(i-1)}$$

$$= \frac{(i+1)}{2(i-1)(i+1)}$$

$$= \frac{i+1}{2(i^2-1)}$$

$$= \frac{i+1}{-4}$$

$$= \frac{-1}{4}(i+1)$$

By Residue theorem,

$$\oint_C \frac{dz}{(z-1)(z+i)(z-i)} = 2\pi i (a_{-1} + b_{-1} + c_{-1})$$

$$= 2\pi i \left[\frac{1}{2} + \frac{i-1}{4} - \frac{i+1}{4} \right]$$

$$= 0$$

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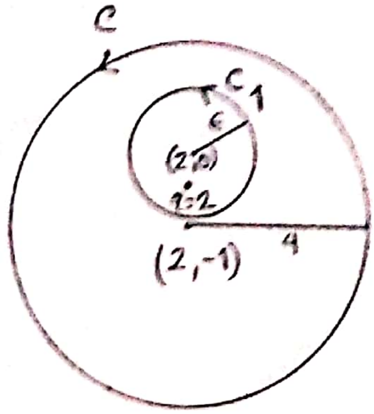
(d)

Problem Statement:

Evaluate $\oint_C \frac{1}{z-2}\, dz$ where $C: |z-2+i| = 4$

Solution:



Here,

$$|z-2+i| = 4$$
$$\Rightarrow |x+iy-2+i| = 4$$
$$\Rightarrow |(x-2)+i(y+1)| = 4$$
$$\Rightarrow \sqrt{(x-2)^2+(y+1)^2} = 4$$
$$\Rightarrow (x-2)^2+(y+1)^2 = 4^2$$

It is an equation of a circle whose centre is at $(2,-1)$ and radius.

So, $z=2$ is inside C and C_1 be a circle at $z=2$ with radius ϵ. So that C_1 is inside C.

We can write,

$$\oint_C \frac{1}{z-2}\, dz = \oint_{C_1} \frac{dz}{z-2} \quad \text{——— (1)}$$

Now on C_1, $|z-2| = \epsilon$ ——— ②

Let, $z-2 = \epsilon e^{i\theta}$

$\Rightarrow |z-2| = |\epsilon e^{i\theta}|$

$\Rightarrow |z-2| = \epsilon |\cos\theta + i\sin\theta|$

$= \epsilon \sqrt{\cos^2\theta + \sin^2\theta}$

$= \epsilon$

So, $|z-2| = |\epsilon e^{i\theta}| = \epsilon$

So,

$\Rightarrow |z-2| = |\epsilon e^{i\theta}|$

$\Rightarrow z-2 = \epsilon e^{i\theta}$

From, equation ②

$$|z-2| = \epsilon$$

$$\Rightarrow z-2 = \epsilon e^{i\theta}$$

$$\therefore dz = \epsilon i e^{i\theta} d\theta \quad ; \quad 0 \leq \theta \leq 2\pi$$

The right hand side of eq. ① becomes

$$\int_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}}$$

$$= i \int_0^{2\pi} d\theta$$

$$= i [\theta]_0^{2\pi}$$

$$= 2\pi i$$

So, $\oint_C \frac{dz}{z-2} = 2\pi i$ [using eq. ①]

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(e)

Problem Statement:

Find the residue of $f(z) = \frac{z^4}{(z^2+3)^3}$

Solution:

Here, $f(z) = \frac{z^4}{(z^2+3)^3}$

$$= \frac{z^4}{(z+\sqrt{3}i)^3 (z-\sqrt{3}i)^3}$$

$f(z)$ has two pole of order 3 at $z=\sqrt{3}i$ and $z=-\sqrt{3}i$

$\therefore$ Residue at, $z=\sqrt{3}i$ is,

$$a_{-1} = \underset{z\to\sqrt{3}i}{Lt} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-\sqrt{3}i)^m \cdot f(z) \right\}$$

$$= \underset{z\to\sqrt{3}i}{Lt} \cdot \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} \left\{ (z-\sqrt{3}i)^3 \cdot \frac{z^4}{(z+\sqrt{3}i)^3 (z-\sqrt{3}i)^3} \right\}$$

$$= \underset{z\to\sqrt{3}i}{Lt} \frac{1}{2} \frac{d^2}{dz^2} \left\{ \frac{z^4}{(z+\sqrt{3}i)^3} \right\}$$

$$= \underset{z\to\sqrt{3}i}{Lt} \cdot \frac{1}{2} \frac{d}{dz} \left\{ \frac{(z+\sqrt{3}i)^3 \cdot 4z^3 - z^4 \cdot 3(z+\sqrt{3}i)^2}{(z+\sqrt{3}i)^6} \right\}$$

$$= \underset{z\to\sqrt{3}i}{Lt} \frac{1}{2} \frac{d}{dz} \left\{ \frac{4z^3}{(z+\sqrt{3}i)^3} - \frac{3z^4}{(z+\sqrt{3}i)^4} \right\}$$

$$= \underset{z \to \sqrt{3}i}{Lt} \frac{1}{2}\left[4 \cdot \frac{(z+\sqrt{3}i)^3 \cdot 3z^2 - z^3 \cdot 3(z+\sqrt{3}i)^2}{(z+\sqrt{3}i)^6} - 3 \cdot \frac{(z+\sqrt{3}i)^4 \cdot 4z^3 - z^4 \cdot 4(z+\sqrt{3}i)^3}{(z+\sqrt{3}i)^8}\right]$$

$$= \underset{z \to \sqrt{3}i}{Lt} \frac{1}{2}\left[4 \cdot \frac{3z^2(z+\sqrt{3}i)^2(z+\sqrt{3}i-z)}{(z+\sqrt{3}i)^6} - 3 \cdot \frac{4z^3(z+\sqrt{3}i)^3(z+\sqrt{3}i-z)}{(z+\sqrt{3}i)^8}\right]$$

$$= \underset{z \to \sqrt{3}i}{Lt} \cdot \frac{1}{2}\left[\frac{12z^2 \cdot \sqrt{3}i}{(z+\sqrt{3}i)^4} - \frac{12z^3 \cdot \sqrt{3}i}{(z+\sqrt{3}i)^5}\right]$$

$$= \underset{z \to \sqrt{3}i}{Lt} \cdot \frac{1}{2}\left[\frac{12\sqrt{3}i\,z^2}{(z+\sqrt{3}i)^4}\left(1 - \frac{z}{z+\sqrt{3}i}\right)\right]$$

$$= \underset{z \to \sqrt{3}i}{Lt} \cdot \frac{1}{2}\left[\frac{12\sqrt{3}i\,z^2}{(z+\sqrt{3}i)^4} \cdot \frac{\sqrt{3}i}{z+\sqrt{3}i}\right]$$

$$= \underset{z \to \sqrt{3}i}{Lt} \frac{1}{2}\left[\frac{-36z^2}{(z+\sqrt{3}i)^5}\right]$$

$$= \frac{-18(\sqrt{3}i)^2}{(2\sqrt{3}i)^5}$$

$$= \frac{\overset{9}{\cancel{18}} \times 3}{\underset{16}{\cancel{32}} \times 3^{\frac{5}{2}} \times i}$$

$$= \frac{3^3\, i}{16 \times 3^{5/2} \cdot i^2}$$

$$= \frac{-\sqrt{3}i}{16}$$

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Again, Residue at $z=-\sqrt{3}i$ is,

$$\lim_{z\to -\sqrt{3}i} \cdot \frac{1}{2} \cdot \frac{d^2}{dz^2} \left\{ \frac{z^4}{(z-\sqrt{3}i)^3} \right\}$$

$$= \frac{-(-1)\sqrt{3}}{16}$$

$$= \frac{\sqrt{3}i}{16}$$

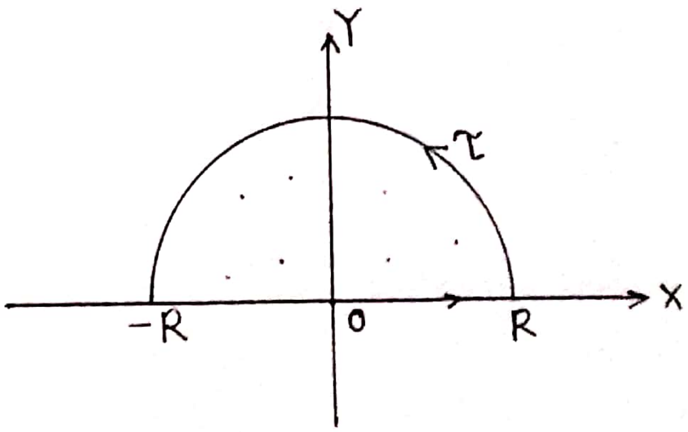
(f)

Problem Statement:

Evaluate,

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dx$$

Solution:



The poles of $\frac{z^2}{1+z^2}$ enclosed by the contour c is $z=i$ of order 1

Residue at $z = i$ is,

$$\underset{z \to i}{Lt} \; (z-i) \frac{z^2}{(z+i)(z-i)}$$

$$= \underset{z \to i}{Lt} \frac{(z)^2}{z+i} \Rightarrow \underset{z \to i}{Lt} \frac{z^2}{z+i}$$

$$= \frac{-1}{2i}$$

By Residue theorem,

$$\oint_C \frac{z^2}{z^2+1} \, dz = 2\pi i \cdot \frac{-1}{2i}$$

$$= -\pi$$

$$\Rightarrow \int_{-R}^{R} \frac{x^2}{1+x^2} \, dx + \int_{\Gamma} \frac{z^2}{1+z^2} \, dz = -\pi$$

Taking the limit as $R \to \infty$ and ~~no~~ noting that the second integral approaches to zero,

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \, dx = -\pi$$