

Problem Set 2

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Optimization methods

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Exercise 1

Check, if:

a) $\forall_{i \in I} Z_i - \text{convex set} \implies \bigcap_{i \in I} Z_i - \text{convex set}$

Proof.

Let $a, b \in \bigcap_{i \in I} Z_i$

We have to check, if $\forall_{\lambda \in [0,1]} \lambda a + (1 - \lambda)b \in \bigcap_{i \in I} Z_i$

But $\forall_{i \in I} a, b \in Z_i$ and $Z_i - \text{convex}$

Which means $\forall_{i \in I} \forall_{\lambda \in [0,1]} \lambda a + (1 - \lambda)b \in Z_i$

Which implies $\forall_{\lambda \in [0,1]} \lambda a + (1 - \lambda)b \in \bigcap_{i \in I} Z_i$ ■

b) $Z - \text{convex set} \implies \text{conv}(Z) = Z$

Proof.

Suppose, that $Z - \text{convex set} \wedge \text{conv}(Z) \neq Z$

From definition of convex hull $Z \subset \text{conv}(Z)$, so the only possibility to $\text{conv}(Z) \neq Z$ is $\text{conv}(Z) \not\subseteq Z$

From that $\exists_{a,b \in Z} \exists_{\lambda \in [0,1]} : \lambda a + (1 - \lambda)b \in \text{conv}(Z) \wedge \lambda a + (1 - \lambda)b \notin Z$

But that implies, that $Z - \text{not convex set}$, which goes against the assumption. ■

c) $Z_1, Z_2 - \text{convex sets} \implies Z_1 + Z_2 - \text{convex set}$

Proof.

Let $x, y \in Z_1 + Z_2$, that means $\exists_{\substack{x_1, y_1 \in Z_1 \\ x_2, y_2 \in Z_2}} x = x_1 + x_2, y = y_1 + y_2$

We have to check, if $\forall_{\lambda \in [0,1]} \lambda x + (1 - \lambda)y \in Z_1 + Z_2$

Let's establish the specific $\lambda \in [0, 1]$

Consider points:

$$\lambda x_1 + (1 - \lambda)y_1 \in Z_1 \text{ (because } Z_1 - \text{convex set)}$$

$$\lambda x_2 + (1 - \lambda)y_2 \in Z_2 \text{ (because } Z_2 - \text{convex set)}$$

Sum of that points: $\lambda x_1 + (1 - \lambda)y_1 + \lambda x_2 + (1 - \lambda)y_2$

is producing $\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2)$,

and that's $\lambda x + (1 - \lambda)y$ ■

d) $Z_1 \subset \mathbb{R}^n, Z_2 \subset \mathbb{R}^m$ -convex sets $\implies Z_1 \times Z_2$ -convex set in $\mathbb{R}^n \times \mathbb{R}^m$

Proof.

Let $x, y \in Z_1 \times Z_2$,

$$x = (\overbrace{a_1^1, a_2^1, \dots, a_n^1}^{a_1}, \overbrace{b_1^1, b_2^1, \dots, b_m^1}^{b_1}), y = (\overbrace{a_1^2, a_2^2, \dots, a_n^2}^{a_2}, \overbrace{b_1^2, b_2^2, \dots, b_m^2}^{b_2})$$

$a_1, a_2 \in Z_1, b_1, b_2 \in Z_2$

We have to check, if $\forall_{\lambda \in [0,1]} \lambda x + (1 - \lambda)y \in Z_1 \times Z_2$

We can rewrite it as $\forall_{\lambda \in [0,1]}$

$$(\lambda a_1^1 + (1 - \lambda)a_1^2, \lambda a_2^1 + (1 - \lambda)a_2^2, \dots, \lambda a_n^1 + (1 - \lambda)a_n^2, \lambda b_1^1 + (1 - \lambda)b_1^2, \lambda b_2^1 + (1 - \lambda)b_2^2, \dots, \lambda b_m^1 + (1 - \lambda)b_m^2)$$

Let's establish the specific $\lambda \in [0, 1]$

Consider points:

$$\begin{aligned} \lambda a_1 + (1 - \lambda)a_2 &\in Z_1 \text{ (because } Z_1 \text{ -convex set)} \\ \lambda b_1 + (1 - \lambda)b_2 &\in Z_2 \text{ (because } Z_2 \text{ -convex set)} \end{aligned}$$

In Cartesian product $Z_1 \times Z_2$ those points will give us:

$$(\lambda a_1^1 + (1 - \lambda)a_1^2, \lambda a_2^1 + (1 - \lambda)a_2^2, \dots, \lambda a_n^1 + (1 - \lambda)a_n^2, \lambda b_1^1 + (1 - \lambda)b_1^2, \lambda b_2^1 + (1 - \lambda)b_2^2, \dots, \lambda b_m^1 + (1 - \lambda)b_m^2)$$

and that's $\lambda x + (1 - \lambda)y \in Z_1 \times Z_2$ ■

e) Z -convex set $\implies \forall_{\alpha \in \mathbb{R}} \alpha Z$ -convex set

Proof.

Let's establish specific $\alpha \in \mathbb{R}$

Let $x, y \in \alpha Z$

We have to check, if $\forall_{\lambda \in [0,1]} \lambda x + (1 - \lambda)y \in \alpha Z$

If $x, y \in \alpha Z$, then $\exists_{x_1, y_1 \in Z} x = \alpha x_1, y = \alpha y_1$

Let's establish the specific $\lambda \in [0, 1]$

And consider point $\lambda x + (1 - \lambda)y$, equals $\lambda \alpha x_1 + (1 - \lambda)\alpha y_1$

Let's transform to $\alpha(\lambda x_1 + (1 - \lambda)y_1)$

Point $\lambda x_1 + (1 - \lambda)y_1 \in Z$ ■

Exercise 2

Check, if sets are convex:

a) $H = \{x \in \mathbb{R}^n : \langle x, a \rangle = \gamma\}, \quad a \in \mathbb{R}^n, \gamma \in \mathbb{R}$

Set H is the convex set

Proof.

Let $x, y \in H$

We have to consider, if $\forall \lambda \in [0,1] \lambda x + (1-\lambda)y \in H$, or equivalently $\forall \lambda \in [0,1] \langle \lambda x + (1-\lambda)y, a \rangle = \gamma$
 $\langle \lambda x + (1-\lambda)y, a \rangle = \lambda \langle x, a \rangle + (1-\lambda) \langle y, a \rangle = \lambda \gamma + (1-\lambda)\gamma = \gamma$

But we know (definition x, y), that $\langle x, a \rangle = \gamma, \quad \langle y, a \rangle = \gamma$

Finally, we have $\lambda \gamma + (1-\lambda)\gamma = \gamma \implies \lambda x + (1-\lambda)y \in H$ ■

b) $\mathbb{R}^n, \mathbb{R}_+^n$

Sets \mathbb{R}^n and \mathbb{R}_+^n are convex

Proof.

Let $x, y \in \mathbb{R}^n, \quad x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n), \quad \forall i \in \{1, \dots, n\} x_i, y_i \in \mathbb{R}$

Let's establish the specific λ

Then, $\lambda x + (1-\lambda)y = (\overbrace{\lambda x_1 + (1-\lambda)y_1}^{\mathbb{R}}, \overbrace{\lambda x_2 + (1-\lambda)y_2}^{\mathbb{R}}, \dots, \overbrace{\lambda x_n + (1-\lambda)y_n}^{\mathbb{R}}) \in \mathbb{R}^n$

Let $x, y \in \mathbb{R}_+^n, \quad x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n), \quad \forall i \in \{1, \dots, n\} x_i, y_i \in \mathbb{R}_+$

Let's establish the specific $\lambda \in [0, 1]$

Then, $\lambda x + (1-\lambda)y = (\overbrace{\lambda x_1 + (1-\lambda)y_1}^{\mathbb{R}_+}, \overbrace{\lambda x_2 + (1-\lambda)y_2}^{\mathbb{R}_+}, \dots, \overbrace{\lambda x_n + (1-\lambda)y_n}^{\mathbb{R}_+}) \in \mathbb{R}_+^n$ ■

c) Open/closed ball, sphere

Open/closed ball is a convex set, sphere is not

Proof.

We can assume, that center of sphere/ball is $(0, 0, 0)$ and radius is equal r (We'll call it B)

Let $x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \quad x_1^2 + x_2^2 + x_3^2 \leq r^2, \quad y_1^2 + y_2^2 + y_3^2 \leq r^2$

Let's establish the specific $\lambda \in [0, 1]$

Then $\lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \lambda x_3 + (1-\lambda)y_3)$

We want to prove, that $(\lambda x_1 + (1-\lambda)y_1)^2 + (\lambda x_2 + (1-\lambda)y_2)^2 + (\lambda x_3 + (1-\lambda)y_3)^2 \leq r^2$

Let's transform left side of inequality to:

$$\begin{aligned} & \lambda^2 \overbrace{(x_1^2 + x_2^2 + x_3^2)}^{\leq r^2} + 2\lambda(1-\lambda) \overbrace{(x_1 y_1 + x_2 y_2 + x_3 y_3)}^{\text{We'll prove, that } \leq r^2} + (1-\lambda)^2 \overbrace{(y_1^2 + y_2^2 + y_3^2)}^{\leq r^2} \leq \\ & \leq r^2(\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2) = r^2(\lambda + (1-\lambda))^2 = r^2 \end{aligned}$$

So, if only $x_1 y_1 + x_2 y_2 + x_3 y_3 \leq r^2$, $\lambda x + (1-\lambda)y \in B$

But, we can easily see, that:

$$x_1y_1 + x_2y_2 + x_3y_3 = \langle x, y \rangle = \overbrace{|x|}^{\leq r} \overbrace{|y|}^{\leq r} \overbrace{\cos \alpha}^{\leq 1} \leq r^2 \quad \alpha \in [0, \pi]$$

We can apply the same reasoning to open ball

To sphere, let's take sphere with center in $(0, 0, 0)$ and radius 1. We can take segment $S = ((1, 0, 0), (-1, 0, 0))$. Then $(0, 0, 0) \in S$, but $(0, 0, 0)$ is not in sphere. ■

d) $E = \{(x_1, x_2) : 3x_1^2 + x_2^2 \geq 6\}$

E is not a convex set

Proof.

Let's take points $A = (-4, 0)$, $B = (4, 0)$ and segment AB

We can easily see, that $(0, 0) \in AB$ and $A, B \in E$, but $(0, 0) \notin E$ ■

e) $F = \{(x_1, x_2) : x_1^2 x_2 \leq 0 \wedge x_1, x_2 \geq 0\}$

F is not a convex set

Proof.

Let's take points $A = (0, 2)$, $B = (2, 0)$ and segment AB

We can easily see, that $(1, 1) \in AB$ and $A, B \in F$, but $(0, 0) \notin F$ ■

f) $Z = Z_1 \cup Z_2, Z_1, Z_2 - \text{convex sets}$

Z is not a convex set

Proof.

Let's take sets $Z_1 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_1, x_2 \in \mathbb{R}\}$, $Z_2 = \{(3, 3)\}$

Then, $Z = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \cup \{(2, 2)\}$

Because $Z - \text{convex set} \implies \forall_{x, y \in Z} \forall_{\lambda \in [0, 1]} \lambda x + (1 - \lambda)y \in Z$

Let's take points $A = (0, 0)$, $B = (3, 3)$ and segment AB

We can easily see, that $(2, 2) \in AB$ and $A, B \in Z$, but $(2, 2) \notin Z$ ■