Problem Set 3

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Optimization methods

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Exercise 1

Check, if set is a cone:

a) $A = \{x \in \mathbb{R}^n : x_i \geq 0\}$

The set is a cone.

Proof. We have to check, if $\forall_{x \in A} \forall_{\alpha \in \mathbb{R}, \alpha \geq 0} \alpha x \in A$
Let $x \in A, x = (x_1, x_2, \ldots, x_n), \alpha \in \mathbb{R}, \alpha \geq 0$
Then, $\alpha x = (\overbrace{\alpha x_1}^{R_+}, \overbrace{\alpha x_2}^{R_+}, \ldots, \overbrace{\alpha x_n}^{R_+}) \in A$ ∎

c) $C = \{x \in \mathbb{R}^n :< x, a > \leq 0\}, a \in \mathbb{R}^n$

The set is a cone.

Proof. Let $x \in C, \alpha \in \mathbb{R}, \alpha \geq 0$
Then $< \alpha x, a >= \alpha < x, a > \leq 0 \implies \alpha x \in C$ ∎

d) $D = (a) := \{x \in \mathbb{R}^n : x = \alpha a, \alpha \geq 0\}, a \in \mathbb{R}^n$

The set is a cone.

Proof. Let $x \in D, \alpha \in \mathbb{R}, \alpha \geq 0$
Then $\alpha x \in D$ (definition of D) ∎

e) $E = \{x \in \mathbb{R}^n : x = \alpha a, \alpha > 0\}, a \in \mathbb{R}^n$

The set is a cone for $a = 0$.
If $\alpha = 0 \wedge a \neq 0 \implies \alpha x \notin E \implies E$ - not a cone.

f) $F = \{x \in \mathbb{R}^2 : x_1 = 2\}$

The set is a cone.
If $\alpha = 2 \wedge x = (2, 4) \implies \alpha x = (4, 8) \notin F \implies F$ - not a cone.

Exercise 2

Prove, that:

a) $C_1, C_2 - cones \implies C_1 \cap C_2 - cone$

Proof. Let $x \in C_1 \cap C_2$
Then $x \in C_1 \wedge x \in C_2$
But $C_1, C_2 - cones \implies \forall_{\alpha \geq 0}(x\alpha \in C_1 \wedge x\alpha \in C_2) \implies$
$\implies \forall_{\alpha \geq 0} x\alpha \in C_1 \cap C_2 \implies C_1 \cap C_2 - cone$ ■

If sets are convex, then intersection is a convex set.

b) $S_1, S_2 - cones \implies S_1 \cup S_2 - cone$

Proof. Let $x \in S_1 \cup S_2$
Then $x \in S_1 \vee x \in S_2$
If $x \in S_1 \implies \forall_{\alpha \geq 0} \alpha x \in S_1 \implies S_1 \cup S_2$
If $x \in S_2 \implies \forall_{\alpha \geq 0} \alpha x \in S_2 \implies S_1 \cup S_2$
So $S_1 \cup S_2 - cone$ ■

Let $S_1 = \{x : \alpha[1,1], \alpha \geq 0\}, S_2 = \{x : \alpha[-1,1], \alpha \geq 0\}$
Then S_1, S_2 - convex cones, but point $(0,1)$ is in segment $((-1,1),(1,1))$ and $(0,1) \notin S_1 \cup S_2 \implies S_1 \cup S_2 - notconvexset$

c) $S_1, S_2 - cones \implies S_1 + S_2 - cone$

Proof. Let $x \in S_1 + S_2, x = x_1 + x_2, x_1 \in S_1, x_2 \in S_2$
Then $\forall_{\alpha \geq 0} \alpha x_1 \in S_1, \alpha x_2 \in S_2$
And $\forall_{\alpha \geq 0} \alpha x_1 + \alpha x_2 \iff \forall_{\alpha \geq 0} \alpha x$
And all x could be presented as $x_1 + x_2, x_1 \in S_1, x_2 \in S_2$ ■

For all convex sets $S_1, S_2, S_1 + S_2 - convex$

d) $S - cone \implies -S - cone$

Proof. Let $x \in -S$
Then $\forall_{\alpha \geq 0} \alpha(-x) = -\alpha x, \alpha x \in S \implies -S - cone$ ■

e) $S \subset \mathbb{R}^n - cone, f : \mathbb{R}^n \mapsto \mathbb{R}^m - linear \implies f(S) - cone$

Proof. Let $x \in f(S), \alpha \geq 0, x = f(x_1), x_1 \in S$
Because any linear tramsformation could be represented as a matrix $f = A, A \in M_{m \times n}$
Then $\alpha x = \alpha f(x_1) = \alpha A x_1 = A \alpha x_1 = f(\alpha x_1)$
and $\alpha x \in S$, because $S - cone \implies f(S) - cone$ ■

Exercise 3

Give geometric interpretation of cone and conjugated cone:

c) $D = ([2, -1, 6]^T)$

It's half line set down by $[2, -1.6]$, conjugated cone is $D^* = \{(x, y, z) : 2x - y + 6z \leq 0\}$

d) $E = \{x \in \mathbb{R}^3 : x_i \geq 0\}$

Conjugated cone to E is $E^* = \{x \in \mathbb{R}^3 : x_i \leq 0\}$

e) $F = \{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} x \geq 0\}$

$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq 0$
$y \geq \frac{-x}{2} \wedge y \leq 2x$

$F = \{x \in \mathbb{R}^2 : x_1 \in R \wedge \frac{-x_1}{2} \leq x_2 \leq 2x_1\}$
$F^* = \{x \in \mathbb{R}^2 : x_1 \in R \wedge \frac{-x_1}{2} \geq x_2 \geq 2x_1\}$