Problem Set 2

Jakub Gadawski
Optimization methods

March 16, 2020

Exercise 1

Check, if:

a) $\forall_{i\in I} Z_i - convex\, set \implies \bigcap_{i\in I} Z_i - convex\, set$

Proof.
Let $a, b \in \bigcap_{i\in I} Z_i$
We have to check, if $\forall_{\lambda\in[0,1]} \lambda a + (1-\lambda)b \in \bigcap_{i\in I} Z_i$
But $\forall_{i\in I} a, b \in Z_i$ and $Z_i - convex$
Which means $\forall_{i\in I}\forall_{\lambda\in[0,1]} \lambda a + (1-\lambda)b \in Z_i$
Which implies $\forall_{\lambda\in[0,1]} \lambda a + (1-\lambda)b \in \bigcap_{i\in I} Z_i$ ■

b) $Z - convex\, set \implies conv(Z) = Z$

Proof.
Suppose, that $Z - convex\, set \wedge conv(Z) \neq Z$
From definition of convex hull $Z \subset conv(Z)$, so the only possibility to $conv(Z) \neq Z$ is $conv(Z) \nsubseteq Z$
From that $\exists_{a,b\in Z}\exists_{\lambda\in[0,1]} : \lambda a + (1-\lambda)b \in conv(Z) \wedge \lambda a + (1-\lambda)b \notin Z$
But that implies, that $Z - not\, convex\, set$, which goes against the assumption. ■

c) $Z_1, Z_2 - convex\, sets \implies Z_1 + Z_2 - convex\, set$

Proof.
Let $x, y \in Z_1 + Z_2$, that means $\exists_{\substack{x_1,y_1\in Z_1\\ x_2,y_2\in Z_2}} \quad x = x_1 + x_2,\, y = y_1 + y_2$
We have to check, if $\forall_{\lambda\in[0,1]} \lambda x + (1-\lambda)y \in Z_1 + Z_2$
Let's establish the specific $\lambda \in [0, 1]$
Consider points:

$$\begin{array}{c} \lambda x_1 + (1-\lambda)y_1 \ \in Z_1 \text{ (because } Z_1 - convex\, set) \\ \lambda x_2 + (1-\lambda)y_2 \in Z_2 \text{ (because } Z_2 - convex\, set) \end{array}$$

Sum of that points: $\lambda x_1 + (1-\lambda)y_1 + \lambda x_2 + (1-\lambda)y_2$
is producing $\lambda(x_1 + x_2) + (1-\lambda)(y_1 + y_2)$,
and that's $\lambda x + (1-\lambda)y$ ■

d) $Z_1 \subset \mathbb{R}^n, Z_2 \subset \mathbb{R}^m$ *−convex sets* $\implies Z_1 \times Z_2$ *− convex set in* $\mathbb{R}^n \times \mathbb{R}^m$

Proof.
Let $x, y \in Z_1 \times Z_2$,
$x = (\overbrace{a_1^1, a_2^1, \ldots, a_n^1}^{a_1}, \overbrace{b_1^1, b_2^1, \ldots, b_m^1}^{b_1}),\ y = (\overbrace{a_1^2, a_2^2, \ldots, a_n^2}^{a_2}, \overbrace{b_1^2, b_2^2, \ldots, b_m^2}^{b_2})$
$a_1, a_2 \in Z_1,\ b_1, b_2 \in Z_2$
We have to check, if $\forall_{\lambda \in [0,1]} \lambda x + (1-\lambda)y \in Z_1 \times Z_2$
We can rewrite it as $\forall_{\lambda \in [0,1]}$
$(\lambda a_1^1 + (1-\lambda)a_1^2, \lambda a_2^1 + (1-\lambda)a_2^2, \ldots, \lambda a_n^1 + (1-\lambda)a_n^2, \lambda b_1^1 + (1-\lambda)b_1^2, \lambda b_2^1 + (1-\lambda)b_2^2, \ldots, \lambda b_m^1 + (1-\lambda)b_m^2)$
Let's establish the specific $\lambda \in [0,1]$
Consider points:

$$\lambda a_1 + (1-\lambda)a_2 \in Z_1 \text{ (because } Z_1 - \textit{convex set})$$
$$\lambda b_1 + (1-\lambda)b_2 \in Z_2 \text{ (because } Z_2 - \textit{convex set})$$

In Cartesian product $Z_1 \times Z_2$ those points will give us:
$(\lambda a_1^1 + (1-\lambda)a_1^2, \lambda a_2^1 + (1-\lambda)a_2^2, \ldots, \lambda a_n^1 + (1-\lambda)a_n^2, \lambda b_1^1 + (1-\lambda)b_1^2, \lambda b_2^1 + (1-\lambda)b_2^2, \ldots, \lambda b_m^1 + (1-\lambda)b_m^2)$
and that's $\lambda x + (1-\lambda)y \in Z_1 \times Z_2$ ■

e) Z *− convex set* $\implies \forall_{\alpha \in \mathbb{R}} \alpha Z$ *− convex set*

Proof.
Let's establish specific $\alpha \in \mathbb{R}$
Let $x, y \in \alpha Z$
We have to check, if $\forall_{\lambda \in [0,1]} \lambda x + (1-\lambda)y \in \alpha Z$
If $x, y \in \alpha Z$, then $\exists_{x_1, y_1 \in Z} x = \alpha x_1, \quad y = \alpha y_1$
Let's establish the specific $\lambda \in [0,1]$
And consider point $\lambda x + (1-\lambda)y$, equals $\lambda \alpha x_1 + (1-\lambda)\alpha y_1$
Let's transform to $\alpha(\lambda x_1 + (1-\lambda)y_1)$
Point $\lambda x_1 + (1-\lambda)y_1 \in Z$ ■

Exercise 2

Check, if sets are convex:

a) $H = \{x \in \mathbb{R}^n :< x, a >= \gamma\}, \quad a \in \mathbb{R}^n, \gamma \in \mathbb{R}$

Set H is the convex set

Proof.
Let $x, y \in H$
We have to consider, if $\forall_{\lambda \in [0,1]} \lambda x + (1-\lambda)y \in H$, or equivalently $\forall_{\lambda \in [0,1]} < \lambda x + (1-\lambda)y, a >= \gamma$
$< \lambda x + (1-\lambda)y, a >=< \lambda x, a > + < (1-\lambda)y, a >= \lambda < x, a > + (1-\lambda) < y, a >$
But we know (definition x, y), that $< x, a >= \gamma, \quad < x, a >= \gamma$
Finally, we have $\lambda\gamma + (1-\lambda)\gamma = \gamma \implies \lambda x + (1-\lambda)y \in H$ ■

b) $\mathbb{R}^n, \mathbb{R}^n_+$

Sets $\mathbb{R}^n$ and $\mathbb{R}^n_+$ are convex

Proof.
Let $x, y \in \mathbb{R}^n, \quad x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n), \quad \forall_{i \in \{1,\ldots,n\}} x_i, y_i \in \mathbb{R}$
Let's establish the specific λ

Then, $\lambda x + (1-\lambda)y = (\overbrace{\lambda x_1 + (1-\lambda)y_1}^{\mathbb{R}}, \overbrace{\lambda x_2 + (1-\lambda)y_2}^{\mathbb{R}}, \ldots, \overbrace{\lambda x_n + (1-\lambda)y_n}^{\mathbb{R}}) \in \mathbb{R}^n$

Let $x, y \in \mathbb{R}^n_+, \quad x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n), \quad \forall_{i \in \{1,\ldots,n\}} x_i, y_i \in \mathbb{R}_+$
Let's establish the specific $\lambda \in [0, 1]$

Then, $\lambda x + (1-\lambda)y = (\overbrace{\lambda x_1 + (1-\lambda)y_1}^{\mathbb{R}_+}, \overbrace{\lambda x_2 + (1-\lambda)y_2}^{\mathbb{R}_+}, \ldots, \overbrace{\lambda x_n + (1-\lambda)y_n}^{\mathbb{R}_+}) \in \mathbb{R}^n_+$

■

c) Open/closed ball, sphere

Open/closed ball is a convex set, sphere is not

Proof.
We can assume, that center of sphere/ball is $(0, 0, 0)$ and radius is equal r (We'll call it B)
Let $x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \quad x_1^2 + x_2^2 + x_3^2 \le r^2, \quad y_1^2 + y_2^2 + y_3^2 \le r^2$
Let's establish the specific $\lambda \in [0, 1]$
Then $\lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \lambda x_3 + (1-\lambda)y_3)$
We want to prove, that $(\lambda x_1 + (1-\lambda)y_1)^2 + (\lambda x_2 + (1-\lambda)y_2)^2 + (\lambda x_3 + (1-\lambda)y_3)^2 \le r^2$
Let's transform left side of inequality to:
$\lambda^2(\overbrace{x_1^2 + x_2^2 + x_3^2}^{\le r^2}) + 2\lambda(1-\lambda)(\overbrace{x_1y_1 + x_2y_2 + x_3y_3}^{We'll\, prove,\, that \le r^2}) + (1-\lambda)^2(\overbrace{y_1^2 + y_2^2 + y_3^2}^{\le r^2}) \le$
$\le r^2(\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2) = r^2(\lambda + (1-\lambda))^2 = r^2$
So, if only $x_1y_1 + x_2y_2 + x_3y_3 \le r^2$, $\lambda x + (1-\lambda)y \in B$

But, we can easily see, that:

$$x_1y_1 + x_2y_2 + x_3y_3 =< x, y >= \overbrace{|x|}^{\leq r}\overbrace{|y|}^{\leq r}\overbrace{\cos\alpha}^{\leq 1} \leq r^2 \quad \alpha \in [0, \pi]$$

We can apply the same reasoning to open ball

To sphere, let's take sphere with center in $(0, 0, 0)$ and radius 1. We can take segment $S = ((1, 0, 0), (-1, 0, 0))$. Then $(0, 0, 0) \in S$, but $(0, 0, 0)$ is not in sphere. ■

d) $E = \{(x_1, x_2) : 3x_1^2 + x_2^2 \geq 6\}$

E is not a convex set

Proof.
Let's take points $A = (-4, 0)$, $B = (4, 0)$ and segment AB
We can easily see, that $(0, 0) \in AB$ and $A, B \in E$, but $(0, 0) \notin E$ ■

e) $F = \{(x_1, x_2) : x^1x_2 \leq 0 \wedge x_1, x_2 \geq 0\}$

F is not a convex set

Proof.
Let's take points $A = (0, 2)$, $B = (2, 0)$ and segment AB
We can easily see, that $(1, 1) \in AB$ and $A, B \in F$, but $(0, 0) \notin F$ ■

f) $Z = Z_1 \cup Z_2, Z_1, Z_2 - convex\ sets$

Z is not a convex set

Proof.
Let's take sets $Z_1 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_1, x_2 \in \mathbb{R}\}, Z_2 = \{(3, 3)\}$
Then, $Z = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \cup \{(2, 2)\}$
Because $Z - convex\ set \implies \forall_{x,y\in Z}\forall_{\lambda\in[0,1]}\lambda x + (1 - \lambda)y \in Z$
Let's take points $A = (0, 0)$, $B = (3, 3)$ and segment AB
We can easily see, that $(2, 2) \in AB$ and $A, B \in Z$, but $(2, 2) \notin Z$ ■

Exercise 3

Find convex hull of sets:

a) $Z = \{x, y\}, x \neq y, x, y \in \mathbb{R}^n$

For only two points, we have to take all points in segment xy
Then $conv(Z) = \{a : a = \lambda x + (1 - \lambda)y, \lambda \in [0, 1]\}$

b) $Z = \{x, y, z\}, x, y, z \in \mathbb{R}^n \ x, y, z - linearly\, intependent$

For three points, we have to take all points beetwen segments xy, yz, xz
Then $conv(Z) = \{a : a \in \triangle(xyz)\}$

c) $Z = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1(x_1 - 1) = 0 \wedge x_2(x_2 - 1) = 0\}$

We can consider set Z as set of four segments:
$((0, 0), (0, 1)), ((0, 1), (1, 1)), ((1, 1), (1, 0)), ((1, 0), (0, 0))$
Then $conv(Z) = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1(x_1 - 1) \leq 0 \wedge x_2(x_2 - 1) \leq 0\}$