Problem Set 4

Jakub Gadawski
Optimization methods

April 3, 2020

Exercise 1

Let $f : X \to \mathbb{R}, X \in \mathbb{R}^n - convex$ **Prove, that:**

a) $f - convex \iff epif - convex$

Proof. " $\implies$ "
Let $f - convex \wedge epif - not\, convex \implies$
$\exists_{(x,z_1),(y,z_2)\in epif} \exists_{\lambda\in[0,1]} \lambda x + (1-\lambda)y \notin X \vee f(\lambda x + (1-\lambda)y) > \lambda z_1 + (1-\lambda)x_2$
$X - convex \implies \lambda x + (1-\lambda)y \in X$
We have to consider, if $f(\lambda x + (1-\lambda)y) > \lambda z_1 + (1-\lambda)z_2$ is possible.
$f - convex \implies \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$
On the other hand, from definition of $z_1, z_2, \lambda z_1 + (1-\lambda)z_2 \geq \lambda f(x) + (1-\lambda)f(y)$
So, from the first inequality $\lambda f(x) + (1-\lambda)f(y) > \lambda f(x) + (1-\lambda)f(y)$
Which is impossible.

" $\impliedby$ "
We know, that $epif - convex \iff$
$\forall_{(x,z_1),(y,z_2)\in epif} \forall_{\lambda\in[0,1]} \lambda x + (1-\lambda)y \in X \wedge f(\lambda x + (1-\lambda)y) \leq \lambda z_1 + (1-\lambda)z_2 \implies f - convex$
Especially for $z_1 = f(x), z_2 = f(y)$
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ ∎

b) $f - convex \implies \forall_{\alpha\in\mathbb{R}} Z_\alpha(f) - convex$

Proof. Let $f - convex \wedge \exists_{\alpha\in\mathbb{R}} Z_\alpha(f) - not\, convex \implies \exists_{\lambda\in[0,1]} \exists_{x,y\in Z_\alpha(f)} f(\lambda x + (1-\lambda)y) > \alpha$
But, $f - convex \implies f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
Which means, that $\lambda f(x) + (1-\lambda)f(y) < \lambda \implies x \notin Z_\alpha(f) \vee y \notin Z_\alpha(f)$
Which is impossible. ∎

Left handed implication is impossible. Let $\alpha = 2 \wedge f(x) = sin(x)$
Then $Z_2(sin(x)) = \mathbb{R} - convex$, but $sin(x) - not\, convex$

c) $f - quasiconvex \iff \forall_{\alpha\in\mathbb{R}} Z_\alpha(f) - convex$

Proof. ∎