Basics

Hubble's Law: *"Close Galaxies"*

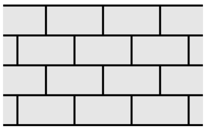
$$v = H_0 d$$

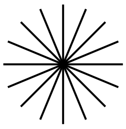
$\implies$ *The universe is expanding...*

Cosmological Principle:

On large scales, the universe looks the same for any observer...

Homogenous: Isotropic:





Redshift:

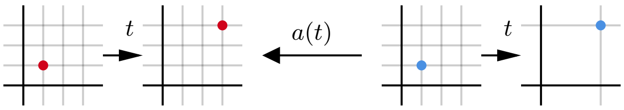
$$\frac{v}{c} + 1 = \frac{\lambda_{\text{obs}}}{\lambda_{\text{rest}}} \qquad z + 1 = \frac{\lambda_{\text{obs}}}{\lambda_{\text{rest}}}$$

"The higher the redshift, the further and earlier in the universe we are observing"

The Scale Factor:

$$\underline{\mathbf{r}} = a(t)\underline{\mathbf{x}}$$

Physical: $\underline{\mathbf{r}}$ Comoving: $\underline{\mathbf{x}}$

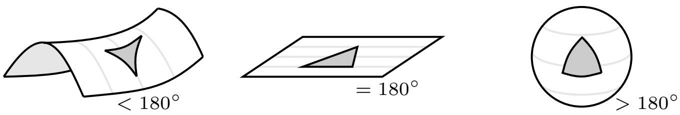


The Friedmann–Lemaître–Robertson–Walker Metric:

For an expanding universe with curvature, the metric is...

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2\right]$$

Open: $k < 0$ Flat: $k = 0$ Closed: $k > 0$



Newtonian Gravity

Friedmann Equation: *"Expansion of the universe"*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

"The above form is in terms of energy density, with $[k] = \mathrm{s}^{-2}$*"*

Hubble's Parameter:

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$$

The Fluid Equation: *"Evolution of a perfect fluid"*

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

Equation of State:

$$P = wc^2\rho$$

Matter: $w = 0$ Radiation: $w = 1/3$ Dark Energy: $w = -1$

Baryonic and Dark

The Acceleration Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

Simple Cosmological Models

Expansion and Redshift:

$$a = \frac{1}{1+z}$$

Matter Domination:

Matter is any non-relativistic massive substance with negligible pressure. It includes baryons on large scales and dark matter...

$$P_m = 0 \implies \rho_m(a) = \frac{\rho_m^0}{a^3}$$

Using an ansatz of $a = Ct^q$ *in the Friedmann equation...*

$$a = \left(\frac{t}{t_0}\right)^{2/3} \qquad H(t) = \frac{2}{3t}$$

Radiation Domination:

Radiation is any relativistic substance, including photons and early universe neutrinos...

$$P_r = \frac{1}{3}c^2\rho_r \implies \rho_r(a) = \frac{\rho_r^0}{a^4}$$

"Extra factor of a due to expansion increasing wavelength"

$$a = \left(\frac{t}{t_0}\right)^{1/2} \qquad H(t) = \frac{1}{2t}$$

Matter Vs Radiation:

"Matter-dominated universe expands faster than a radiation-dominated universe"

Mixtures:

In a flat universe with matter and radiation...

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_m^0}{a^3} + \frac{\rho_r^0}{a^4}\right)$$

This must be solved numerically, or, a dominant component can be assumed to allow for analytic solutions...

"As radiation density ($\rho_r \sim t^{-2}$) *falls off faster than matter density* ($\rho_m \sim t^{-1.5}$)*, matter will always eventually dominate"*

Evolution with Curvature:

Open Universe: $k < 0$

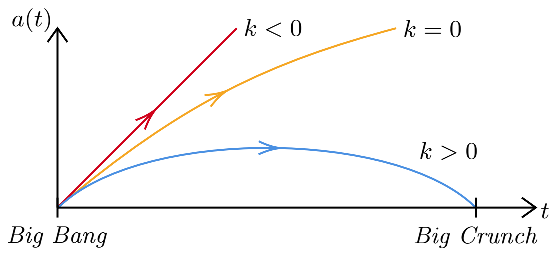
$$\frac{8\pi G}{3}\rho - \frac{k}{a^2} > 0 \implies \left(\frac{\dot{a}}{a}\right)^2 > 0 \qquad a(t) \approx \sqrt{-k}\,t$$

"This universe expands forever"

Closed Universe: $k > 0$

As this universe evolves, $\dot{a}$ *can become zero as* $k > 0$*...*

$$\frac{8\pi G\rho}{3} = \frac{k}{a^2} \implies \left(\frac{\dot{a}}{a}\right)^2 = 0 \qquad \underbrace{\dot{a} \approx \sqrt{-k} = \mathrm{i}\sqrt{k}}_{\textit{Sinusoidal Sol.}}$$



Observational Parameters

"A cosmological parameter is a parameter measured by observations of the universe"

Hubble's Parameter: *"Expansion Rate"*

$$H_0 \approx 70 \text{ km/s/Mpc} \qquad H_0 = 100h \text{ km/s/Mpc}$$

Density Parameters:

At a given $H(t)$, there exists a particular density at a given time that would correspond to a flat universe...

$$\rho_{crit}(t) = \frac{3H^2(t)}{8\pi G} \qquad \rho^0_{crit} = \frac{3H_0^2}{8\pi G}$$

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\text{crit}}(t)}$$

By introducing a curvature density parameter, we can simplify the Friedmann equation...

$$\Omega_k(a) \equiv -\frac{k}{H^2(a)a^2} \implies \Omega_m(a) + \Omega_r(a) + \Omega_k(a) = 1$$

Deceleration parameter:

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2}$$

"We expect a positive deceleration as all our models so far suggest this... However, from Type Ia Supernovae, we measure an accelerating expansion"

The Cosmological Constant

Introducing Λ:

From GR, we reintroduce Λ to explain a negative q_0...

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

The "Fluid" Description of Λ: *"Dark Energy"*

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \implies P_\Lambda = -c^2\rho_\Lambda$$

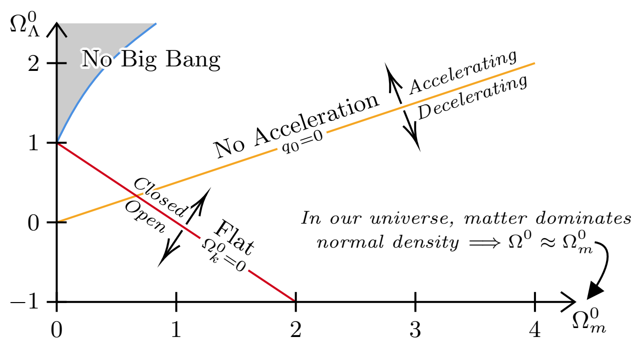
This could be the zero-point energy density of free space... But particle physics puts this energy orders of magnitude lower. This is the ***cosmological constant problem****.*

"We in fact get accelerated expansion when $w < 1/3$"

Cosmological models with Λ: *ΛCDM*

We can infer the fate of the universe with Λ involved...

$$\Omega^0 + \Omega^0_k + \Omega^0_\Lambda = 1 \qquad \underbrace{q_0 = \frac{\Omega^0}{2} - \Omega^0_\Lambda}_{\text{"Flat Universe"}}$$



Constraining the Cosmological Constant:

If the universe is flat, matter-dominated and accelerating...

$$\Omega^0_\Lambda > \frac{1}{3}$$

"We measure $\Omega^0_\Lambda \approx 0.7$ and $\Omega^0_m \approx 0.3$"

The Cosmic Microwave Background

"Redshifted light from the Big Bang"

Thermal Physics:

$$\varepsilon_r = \alpha T^4 \qquad \rho_r = \frac{\varepsilon_r}{c^2} = \frac{1}{c^2}\alpha T^4$$

"In thermal equilibrium, photons follow a blackbody"

Properties of CMB:

$$T_0 = 2.725\text{K} \qquad \Omega^0_r = 2.47 \times 10^{-5}\ h^{-2}$$

Fractional energy density today of Radiation

Expansion and Temperature:

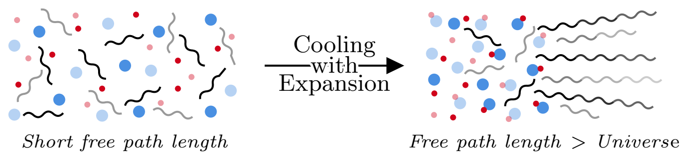
$$T \propto \frac{1}{a}$$

The Origin of CMB:

When the early universe was a millionth of its current volume $a \sim 10^{-6}$...

$$T \sim 3k_B \times 10^6 \text{ K} \gg 13.6 \text{ eV} \implies \textit{No atoms...}$$

As a result, there only exists photons with a short mean free path length in an ionised plasma of particles...



"Decoupling was when photons had a free path length greater than the radius of the universe"

Temperature of Decoupling I:

Assuming decoupling occurs when the mean photon energy is equal to the ionisation energy of Hydrogen...

$$E_{mean} = E_{ion} \ :\ 3k_BT = 13.6 \text{ eV} \implies T \approx 5.2 \times 10^4 \text{ K}$$

But, we must account for the Photon to Baryon ratio...

The Photon to Baryon Ratio:

By considering $\varepsilon^0_r = \alpha T_0^4$ and $\bar{\varepsilon}^0_r = 3k_BT_0$, the number density of photons today is...

$$n^0_r = \frac{\varepsilon^0_r}{\bar{\varepsilon}^0_r} = 3.7 \times 10^8 \ \frac{\text{photons}}{\text{m}^3}$$

Similarly, by considering $\varepsilon^0_B = \rho^0_B c^2 = \rho^0_c\Omega^0_B c^2$ with the observed $\Omega \approx 0.023\ h^{-2}$ and $\bar{\varepsilon}^0_B \approx m_pc^2$...

$$n^0_B = \frac{\varepsilon^0_B}{\bar{\varepsilon}^0_B} = 0.26 \ \frac{\text{baryons}}{\text{m}^3}$$

$$\frac{n_B}{n_r} \sim 10^9$$

"There are many more photons than baryons"

Recombination:

"Recombination was when electrons and protons first formed neutral hydrogen."

The fraction of free baryons χ is determined by the Saha equation...

$$\frac{1-\chi}{\chi^2} \approx 3.8 \; \frac{n_B}{n_r} \; \left(\frac{k_B T}{m_e c^2}\right)^{3/2} \exp\left[\frac{13.6 \text{ eV}}{k_B T}\right]$$

We conventionally define recombination to occur when ***90% of baryons*** *are recombined...*

$$\chi_{\text{rec}} = 0.1 \implies T_{\text{rec}} \approx 3600 \text{ K}$$

Temperature of Decoupling II:

As decoupling doesn't occur instantaneously with recombination by pushing $\chi_{rec} \to 0$...

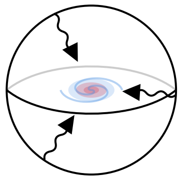
$$T_{\text{dec}} \approx 3000 \text{ K}$$

Thus, the time of decoupling is...

$$T \propto \frac{1}{a} \implies a_{dec} \approx 10^{-3}$$

Surface of Last Scattering:

"The surface of last scattering is a shell at the distance where photons from decoupling are arriving to us now"



A Thermal History of the Early Universe

Neutrinos:

Neutrinos are expected to have a very very low mass and were relativistic in the early universe...

$$\Omega_\nu = 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \Omega_r = 0.68 \; \Omega_r$$

Three neutrino species — *Fermion factor* — *photon to neutrino*

$$\Omega_\nu^0 = 1.68 \times 10^{-5} \; h^{-2}$$

Total Relativistic Density:

$$\Omega_{\text{rel}}^0 = \Omega_r^0 + \Omega_\nu^0 = 4.15 \times 10^{-5} \; h^{-2}$$

Matter-Radiation at Decoupling:

Consider the matter-radiation energy density ratio...

$$\frac{\Omega_{rel}}{\Omega_m} \equiv \frac{\rho_{rel}}{\rho_m} = \frac{\rho_{rel}^0}{\rho_m^0}\frac{1}{a} \implies \frac{\Omega_{rel}}{\Omega_m} = \frac{\Omega_{rel}^0}{\Omega_m^0}\frac{1}{a} = \frac{4.15 \times 10^{-5}}{\Omega_m^0 h^2}\frac{1}{a}$$

$$a_{\text{dec}} = 10^{-3} \implies \frac{\Omega_{\text{rel}}}{\Omega_m} \approx 0.3$$

"At decoupling, there was more relativistic density than today"

Matter-Radiation Equality:

$$\frac{\Omega_{\text{rel}}}{\Omega_m} = 1 \implies a_{\text{eq}} \approx 3 \times 10^{-4}$$

Temperature and Time: $T(t)$

We assume the early universe has $k = 0$ and $\Lambda = 0$ with an instantaneous transition between radiation & matter domination...

Matter Domination:

Matter domination occurs after equality...

$$\frac{T}{T_0} = \frac{a_0}{a} = \left(\frac{t_0}{t}\right)^{2/3} \qquad t_0 \approx 12 \text{ billion years}$$

"We fuck up t_0 to account for Λ domination in the late universe"

$$\frac{T}{2.725 \text{ K}} = \left(\frac{4 \times 10^{16} \text{ s}}{t}\right)^{2/3}$$

$$\implies t_{\text{eq}} \approx 60,000 \text{ years} \qquad t_{\text{dec}} \approx 350,000 \text{ years}$$

Radiation Domination:

Radiation domination occurs before equality...

$$\frac{T}{T_{eq}} = \frac{a_{eq}}{a} = \left(\frac{t_{eq}}{t}\right)^{1/2}$$

$$\frac{T}{9.6 \times 10^3 \text{ K}} = \left(\frac{1.9 \times 10^{12} \text{ s}}{t}\right)^{1/2}$$

Another option during radiation domination is to directly use ideas from thermal physics...

$$\rho_{rad} = \frac{1}{c^2}\alpha T^4 \times (1.68)$$

Photons + Neutrinos

Combining this with the Friedmann equation gives...

$$T(t) = \left[\frac{3c^2}{8\pi G\alpha(1.68)}\right]^{1/4} \left(\frac{1}{2t}\right)^{1/2}$$

Big Bang Nucleosynthesis

Composition of the Early Universe:

Historically, we believe only hydrogen existed in the early universe after recombination, with heavier elements formed later through stellar evolution. But, we observed other light elements in early universe stars...

Freeze Out:

The ratio of neutrons to protons is...

$$\frac{N_n}{N_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[-\frac{(m_n - m_p)c^2}{k_B T}\right] \approx \exp\left[-\frac{1.3 \text{ Mev}}{k_B T}\right]$$

At high temperatures, protons and neutrons interconvert with a high reaction rate in thermal equilibrium...

$$k_B T > 0.8 \text{ MeV} \implies \frac{N_n}{N_p} = 1$$

Once the universe cools to below 0.8 MeV, the ratio freezes out except for ***free neutron decay****...*

$$\frac{N_n}{N_p} = \exp\left[-\frac{1.3 \text{ Mev}}{0.8 \text{ MeV}}\right] \approx \frac{1}{5}$$

"5 protons for every 1 neutron"

Light nuclei can now attempt to form but are broken apart by high energy photons until $k_B T \sim 0.06 MeV$...

Free Neutron Decay:

How does the ratio change before nucleosynthesis begins? Using the radiation-dominated thermal history...

$$t_{nuc} \approx 340\text{ s} \qquad t_{fo} \approx 1\text{ s} \qquad t_{1/2} = 614\text{ s}$$

$$\implies \frac{N_n}{N_p} = \frac{1}{5}\exp\left[-\frac{339\text{ s}}{614\text{ s}}\ln(2)\right] \approx \frac{1}{7.3}$$

"7 protons for every 1 neutron"

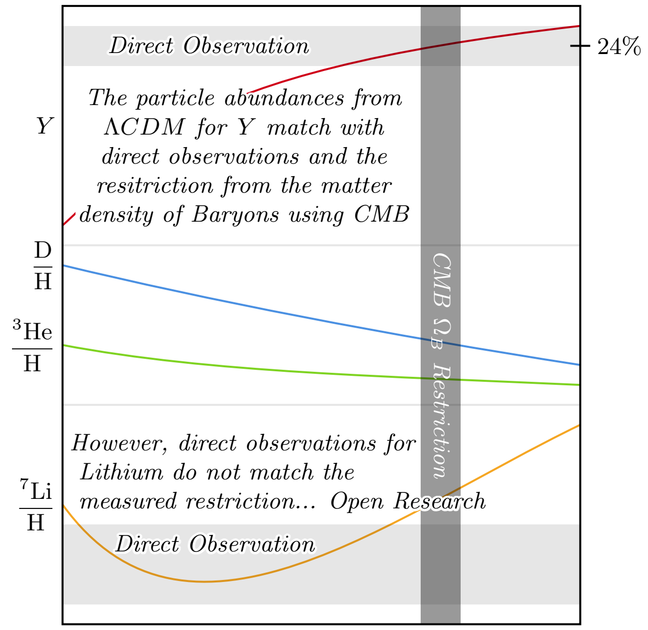
Nucleosynthesis of Light Nuclei:

Assuming all neutrons contribute to Helium and the remaining protons are Hydrogen, the mass fractions are...

Hydrogen:	Helium:	Metals:
$X \approx \frac{12}{16} \approx 76\%$	$Y \approx \frac{4}{16} \approx 24\%$	$Z = 0$

"More detailed analysis produces other light elements such as Lithium, which are detected in early universe stars"

Comparing with Observations:



The Inflationary Universe

Problems with the Hot Big Bang:

The Flatness Problem:

We observe, conservatively, that $0.5 < \Omega^0_\Lambda + \Omega^0_m < 1.5$*...*

$$|\Omega_{tot}(t_0) - 1| = |\Omega_k(t_0)| \equiv \left|\frac{k}{a^2H^2}\right|_{t_0} < 0.5$$

As $|\Omega_k|$ *grows with time, it must've been smaller earlier...*

"Of all the possibilities, why is the universe so close to flat?"

The Horizon Problem:

We observe all CMB radiation at the same temperature... How close must two photons be to thermalise?

$$d_{lss} \approx ct_0 = 1.3 \times 10^{26}\text{ m} \qquad d_{exp} = \frac{ct_{dec}}{a_{dec}} \approx 3.3 \times 10^{24}\text{ m}$$

$$\cdots \implies \theta \approx 1.4^\circ$$

"The smallest angular distance on our sky for photons to be in uniform thermal equilibrium is $\sim 1.4^\circ$*"*

Inflationary Expansion:

"Introduce accelerating expansion in the very early universe"

In the very early universe, the cosmological constant can dominate...

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \approx \frac{\Lambda}{3} \implies a(t) \propto \exp\left[\sqrt{\frac{\Lambda}{3}}\,t\right]$$

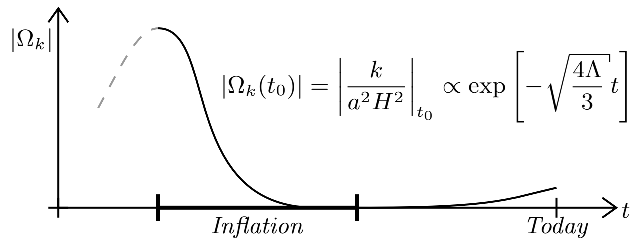
Radiation:	Matter:	Curvature:	Inflation:
$a \propto t^{1/2}$	$a \propto t^{2/3}$	$a \propto t$	$a \propto \exp(\sqrt{\Lambda/3}\,t)$

Grows much faster

Solving the Big Bang Problems:

Solving The Flatness Problem:

During inflation $|\Omega_k|$ *collapses to zero exponentially...*



Solving The Horizon Problem:

"Pre-inflation regions thermalise before inflated expansion occurs faster than the speed of light"

Quantifying Inflation:

How much inflation solves the flatness problem?

Assuming that inflation ends at $t = 10^{-34}$ *s and we observe that* $|\Omega_k(t)| < 0.1(t/t_0)$*...*

$$|\Omega_k(10^{-34}\text{ s})| < 0.1\left(\frac{10^{-34}\text{ s}}{4 \times 10^{17}\text{ s}}\right) \approx 3 \times 10^{-53}$$

Using the exponential inflation model...

$$|\Omega_k(t)| \propto \frac{1}{|a^2|} \implies \frac{1}{|a^2|} \sim 10^{54} \implies a \text{ increases by } 10^{27}$$

This exponential model of inflation has 27 ten-folds...

Inflation and Particle Physics:

"There could be exotic physics that drove inflation, such as high energy particles or scalar fields, which decay away after a universal phase transition"

Distances in Classic Cosmology

Luminosity Distance: *"Standard Candle"*

$$d_{\text{lum}} \equiv \sqrt{\frac{L}{4\pi F}}$$

In a static universe, $d_{lum} = d_{phys}$*, but more generally...*

$$d_{\text{lum}} = d_{\text{phys}}(1+z) = a_0\, d_{\text{com}}(1+z)$$

Distances can be extended into our cosmological model...

$$d_{\text{com}}(z) = \int_0^z \frac{c\,\mathrm{d}z'}{H(z')} \implies d_{\text{lum}}(z) = c(1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')}$$

"We can test and constrain our cosmological model using Type Ia Supernovae"

Angular Diameter Distance: "Standard Ruler"

$$d_{\mathrm{diam}} \equiv \frac{\ell}{\theta}$$

In a static universe, $d_{diam} = d_{phys}$, but more generally...

$$d_{\mathrm{diam}} = \frac{d_{\mathrm{phys}}}{1+z} = \frac{d_{\mathrm{lum}}}{(1+z)^2}$$

Extending this into our cosmological model...

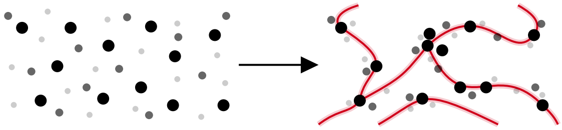
$$d_{\mathrm{diam}} = \frac{c}{(1+z)} \int_0^z \frac{\mathrm{d}z'}{H(z')}$$

"We can test our cosmological model using baryon acoustic oscillations and anisotropies in CMB"

Structures in The Universe

Observed Structures:

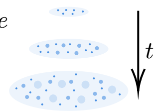
Inhomogeneity is universal structure in position, whereas anisotropy is structure in orientation...



"We observe clustered galaxies and CMB anisotropies"

Origin of Structure:

"Quantum fluctuations of elementary fields are magnified during inflation over many scales. This seeds early overdense regions of matter, which gravitationally attracts more matter over time, later causing clustering and large structure."

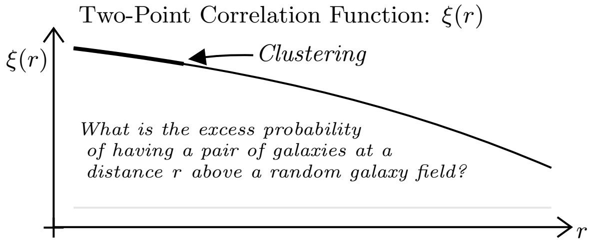


Clustering of Galaxies:

$$\rho(\underline{\mathbf{r}}) = \overline{\rho} + \Delta\rho(\underline{\mathbf{r}}) = \overline{\rho}\left(1 + \frac{\Delta\rho(\underline{\mathbf{r}})}{\overline{\rho}}\right) = \overline{\rho}\big(1 + \delta(\underline{\mathbf{r}})\big)$$

*Line-of-site measurements of galaxy clustering constrains cosmological parameters by modelling the distribution of galaxies using **linear perturbation theory** for small inhomogeneities ($\delta(\underline{\mathbf{r}}) \ll 1$). Summary statistics are used for analysis...*

Two-Point Correlation Function: $\xi(r)$



Power Spectrum: $P(k)$

"Decomposes the pattern into Fourier k-space"

Cosmic Microwave Background Anisotropies:

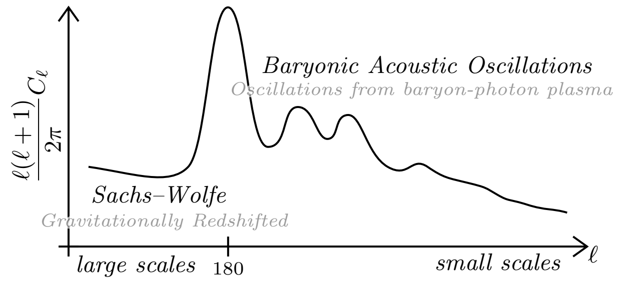
$$T(\theta,\phi) = \overline{T} + \Delta T(\theta,\phi) = \overline{T}\left(1 + \frac{\Delta T}{\overline{T}}(\theta,\phi)\right) \qquad \frac{\Delta T}{\overline{T}} \sim 10^{-5}$$

CMB temperature anisotropies across the sky occur in the early universe, meaning linear perturbation theory is valid. Suppose we expand in spherical harmonics...

$$\frac{\Delta T}{\overline{T}}(\theta,\phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_m^{\ell}(\theta,\phi)$$

$a_{\ell m}$: *Size of anisotropies on different scales*

Angular Power Spectrum: $C_\ell = \langle |a_{\ell m}|^2 \rangle$



"C_ℓ fluctuates roughly on a scale $\theta \sim \frac{180°}{\ell}$"

Timeline of the Early Universe

Inflation	$T_{\mathrm{inf}} \sim 10^{28}$ K $\quad t_{\mathrm{inf}} \sim 10^{-34}$ s *Faster than light expansion in the early universe*
Quarks	$T_{\mathrm{qua}} \sim 10^{14}$ K $\quad t_{\mathrm{qua}} \sim 10^{-12}$ s *Quark-gluon plasma with free electons*
Hadrons	$T_{\mathrm{had}} \sim 10^{12}$ K $\quad t_{\mathrm{had}} \sim 10^{-6}$ s *Quarks form free protons and neutrons*
Nucleosynthesis	$T_{\mathrm{nuc}} \sim 10^{10}$ K $\quad t_{\mathrm{nuc}} \sim 1 - 400$ s *Protons and neutrons form light nuclei*
Equality	$T_{\mathrm{eq}} \sim 10^{4}$ K $\quad t_{\mathrm{eq}} \sim 60,000$ years *Matter starts dominating over radiation*
Recombination	$T_{\mathrm{rec}} \sim 3600$ K $\quad t_{\mathrm{rec}} \sim 300,000$ years *Electrons combine with nuclei to form neutral atoms*
Decoupling	$T_{\mathrm{dec}} \sim 3000$ K $\quad t_{\mathrm{dec}} \sim 360,000$ years *Transparent universe allows free streaming photons*