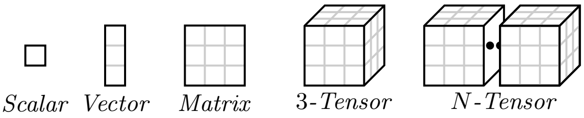
Mathematical Background

Scalars, Vectors, Matrices and Tensors:

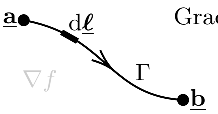


The Wave Equation:

> *The 3D wave equation is...*
> $$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Gradient of a Scalar Field:

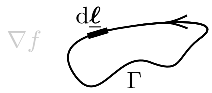
$$\nabla f(\underline{\mathbf{r}}) = \frac{\partial f}{\partial x}\,\hat{\underline{\mathbf{i}}} + \frac{\partial f}{\partial y}\,\hat{\underline{\mathbf{j}}} + \frac{\partial f}{\partial z}\,\hat{\underline{\mathbf{k}}}$$



Gradient Theorem:

$$\int_\Gamma (\nabla f)\cdot \mathrm{d}\underline{\boldsymbol{\ell}} = f(\underline{\mathbf{a}}) - f(\underline{\mathbf{b}})$$

"This line integral is path independent"

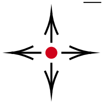


$$\Longrightarrow \oint_\Gamma (\nabla f)\cdot \mathrm{d}\underline{\boldsymbol{\ell}} = 0$$

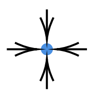
Divergence of Vector Field:

$$\nabla\cdot\underline{\mathbf{F}} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Source: $\nabla\cdot\underline{\mathbf{F}} > 0$

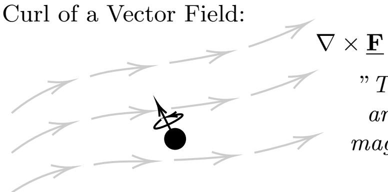


Sink: $\nabla\cdot\underline{\mathbf{F}} < 0$



"If you feel negative, the world is closing in"

Curl of a Vector Field:

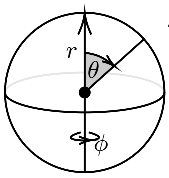


"The direction of the curl is an axis of rotation and the magnitude is that of rotation"

The Laplacian Operator:

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Spherical Polar Coordinates:



$r\in[0,\infty)\quad \theta\in[0,\pi]\quad \phi\in[0,2\pi)$

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

Line Element:

$$\mathrm{d}\underline{\boldsymbol{\ell}} = \mathrm{d}r\,\hat{\underline{\mathbf{r}}} + r\,\mathrm{d}\theta\,\hat{\underline{\boldsymbol{\theta}}} + r\sin(\theta)\,\mathrm{d}\phi\,\hat{\underline{\boldsymbol{\phi}}}$$

Volume Element:

$$\mathrm{d}V = r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Spherically Symmetric:

$$\nabla f^{\mathrm{ss}} = \frac{\partial f}{\partial r}\,\hat{\underline{\mathbf{r}}} \qquad \nabla\cdot\underline{\mathbf{A}}^{\mathrm{ss}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) \qquad \nabla\times\underline{\mathbf{A}}^{\mathrm{ss}} = 0$$

$$\nabla^2 f^{\mathrm{ss}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf)$$

Gauss's Theorem:

$$\oiint_S \underline{\mathbf{F}}\cdot\hat{\underline{\mathbf{n}}}\,\mathrm{d}S = \iiint_V \nabla\cdot\underline{\mathbf{F}}\,\mathrm{d}V$$

The Normalised Gaussian:

$$g(\underline{\mathbf{r}}) = \frac{\alpha^3}{\pi^{3/2}}\mathrm{e}^{-\alpha^2|\underline{\mathbf{r}}|^2}$$

The Error Function:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{erf}(\alpha z)\right] = \frac{2\alpha}{\sqrt{\pi}}\mathrm{e}^{-\alpha^2 z^2}$$

The Dirac Delta Function:

$$\delta(\underline{\mathbf{r}}) = \lim_{\alpha\to\infty} g(\underline{\mathbf{r}}) = \lim_{\alpha\to\infty}\frac{\alpha^3}{\pi^{3/2}}\mathrm{e}^{-\alpha^2|\underline{\mathbf{r}}|^2}$$

Sifting Property:

$$f(\underline{\mathbf{r}}) = \int_{\mathbb{R}^n} f(\underline{\mathbf{r}}')\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\,\mathrm{d}^n\underline{\mathbf{r}}'$$

Fourier Exponential Form:

$$\delta(t) = \int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{\mathrm{d}\omega}{2\pi}$$

Poisson Kernel:

> *Consider the Laplacian of $\mathcal{V} = \mathrm{erf}(\alpha r)/r$ in ss spc...*
> $$\nabla^2\mathcal{V} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\mathcal{V}) = \cdots = -4\pi\frac{\alpha^3}{\pi^{3/2}}\mathrm{e}^{-\alpha^3 r^2} = -4\pi g(r)$$
> *By taking the limit as $\alpha\to\infty$...*
> $$\nabla^2\left[\frac{\mathrm{erf}(\alpha r)}{r}\right] = -4\pi g(r) \overset{\lim_{\alpha\to\infty}}{\Longrightarrow} \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$$

$$\nabla^2\left(\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\right) = -4\pi\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')$$

Stokes' Theorem:

$$\oint_\Gamma \underline{\mathbf{F}}\cdot\mathrm{d}\underline{\boldsymbol{\ell}} = \iint_S (\nabla\times\underline{\mathbf{F}})\cdot\hat{\underline{\mathbf{n}}}\,\mathrm{d}S$$

Free Fields:

Curl-free:

$$\nabla\times(\nabla\phi) = 0$$

Divergence-free:

$$\nabla\cdot(\nabla\times\underline{\mathbf{C}}) = 0$$

The Fundamental Theorem of Vector Calculus:

Let $\underline{\mathbf{F}}$ be a twice continuously differentiable (smooth C^2) vector field in the domain V bounded by the surface S. Then, $\underline{\mathbf{F}}$ can be decomposed as...

$$\underline{\mathbf{F}} = \underbrace{-\nabla\phi}_{\textit{Curl-free}} + \underbrace{\nabla\times\underline{\mathbf{A}}}_{\textit{Divergence-free}}$$

> *The full scalar and vector potential can be derived using the Dirac Delta with the kernel and vector identities...*
> $$\phi(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}' - \frac{1}{4\pi}\int_S \hat{\underline{\mathbf{n}}}\cdot\frac{\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^2\underline{\mathbf{r}}'$$
> $$\underline{\mathbf{A}}(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int_V \frac{\nabla'\times\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}' - \frac{1}{4\pi}\int_S \hat{\underline{\mathbf{n}}}\times\frac{\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^2\underline{\mathbf{r}}'$$

If V is taken to infinity and $\underline{\mathbf{F}}$ decays faster than $1/r$ as $r\to\infty$, then the surface integrals vanish...

$$\phi(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}' \qquad \underline{\mathbf{A}}(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int_V \frac{\nabla'\times\underline{\mathbf{F}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}'$$

"Describing the field $\underline{\mathbf{F}}$ requires only its curl and divergence"

Electrostatics I

Consider a continuous charge density $\rho(\underline{\mathbf{r}})$...

$$\rho(\underline{\mathbf{r}}) = \int_V \rho(\underline{\mathbf{r}}')\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}' \equiv \int_V \frac{\rho(\underline{\mathbf{r}}')}{-4\pi}\nabla^2\left(\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\right)\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\implies 4\pi\rho = -\nabla^2 \underbrace{\int_V \frac{\rho(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}'}_{\textit{Scalar Field: } \phi(\underline{\mathbf{r}})} = -\nabla^2\phi(\underline{\mathbf{r}})$$

By choosing, say, $\underline{\mathbf{E}} = -\nabla\phi$...

$$4\pi\rho = -\nabla^2\phi = \nabla\cdot(-\nabla\phi) = \nabla\cdot\underline{\mathbf{E}}$$

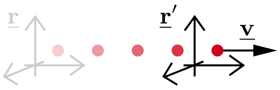
$$\nabla\cdot\underline{\mathbf{E}} = 4\pi\rho(\underline{\mathbf{r}}) \qquad \nabla\times\underline{\mathbf{E}} = 0$$

$$\frac{\partial\rho(\underline{\mathbf{r}})}{\partial t} = \frac{\partial\underline{\mathbf{E}}}{\partial t} = 0$$

Inference of Maxwell's Equations

Galilean Transformations:

$$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \underline{\mathbf{v}}t \qquad t' = t$$



Consider a 1D Galilean transform...

$$\frac{\partial}{\partial t'}\psi(x,t) = \frac{\partial t}{\partial t'}\frac{\partial\psi}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial\psi}{\partial x} \equiv \frac{\partial\psi}{\partial t} + v_x\frac{\partial\psi}{\partial x}$$

This generalises to 3D...

$$\frac{\partial}{\partial t'}\psi(\underline{\mathbf{r}},t) = \frac{\partial\psi}{\partial t} + v_x\frac{\partial\psi}{\partial x} + v_y\frac{\partial\psi}{\partial y} + v_z\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial t} + \underline{\mathbf{v}}\cdot\nabla\psi$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \underline{\mathbf{v}}\cdot\nabla$$

Vector Identities:

$$\nabla\times(\underline{\mathbf{A}}\times\underline{\mathbf{B}}) = \underline{\mathbf{A}}(\nabla\cdot\underline{\mathbf{B}}) - \underline{\mathbf{B}}(\nabla\cdot\underline{\mathbf{A}}) + (\underline{\mathbf{B}}\cdot\nabla)\underline{\mathbf{A}} - (\underline{\mathbf{A}}\cdot\nabla)\underline{\mathbf{B}}$$

$$\nabla\cdot(\underline{\mathbf{A}}\times\underline{\mathbf{B}}) = \underline{\mathbf{B}}\cdot(\nabla\times\underline{\mathbf{A}}) - \underline{\mathbf{A}}\cdot(\nabla\times\underline{\mathbf{B}}) \qquad \nabla\times(\nabla\times\underline{\mathbf{A}}) = \nabla(\nabla\cdot\underline{\mathbf{A}}) - \nabla^2\underline{\mathbf{A}}$$

$$\nabla\times(\phi\underline{\mathbf{A}}) = \phi(\nabla\times\underline{\mathbf{A}}) + (\nabla\phi)\times\underline{\mathbf{A}}$$

The Continuity Equation:

Consider the rest frame of a charge arrangement moving with uniform velocity $\underline{\mathbf{v}}$...

$$\frac{\partial\rho}{\partial t} = 0 \to \frac{\partial\rho}{\partial t} + \underline{\mathbf{v}}\cdot\nabla\rho = 0 \implies \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{\mathbf{v}}) = 0$$

Recognising the electric current density as $\underline{\mathbf{J}} = \rho\underline{\mathbf{v}}$...

$$\frac{\partial}{\partial t}\rho(\underline{\mathbf{r}},t) + \nabla\cdot\underline{\mathbf{J}}(\underline{\mathbf{r}},t) = 0$$

Ampère's Law with Maxwell's Addition:

By considering the transformation of the electric field $\underline{\mathbf{E}}$...

$$\frac{\partial\underline{\mathbf{E}}}{\partial t} + (\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{E}} = 0$$

Using the $\nabla\times(\underline{\mathbf{A}}\times\underline{\mathbf{B}})$ vector identity...

$$\nabla\times(\underline{\mathbf{v}}\times\underline{\mathbf{E}}) = \underbrace{\underline{\mathbf{v}}(\nabla\cdot\underline{\mathbf{E}})}_{\textit{Statics: } 4\pi\underline{\mathbf{J}}} - \cancel{\underline{\mathbf{E}}(\nabla\cdot\underline{\mathbf{v}})} + \cancel{(\underline{\mathbf{E}}\cdot\nabla)\underline{\mathbf{v}}} - (\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{E}}$$

$$\implies \nabla\times(\underline{\mathbf{v}}\times\underline{\mathbf{E}}) = 4\pi\underline{\mathbf{J}} - (\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{E}} \equiv 4\pi\underline{\mathbf{J}} + \frac{\partial\underline{\mathbf{E}}}{\partial t}$$

Now, let $\underline{\mathbf{v}}\times\underline{\mathbf{E}} = c\underline{\mathbf{B}}$...

$$\nabla\times\underline{\mathbf{B}} = \frac{4\pi}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}},t) + \frac{1}{c}\frac{\partial\underline{\mathbf{E}}}{\partial t}$$

"Moving charges give rise to magnetic fields"

Gauss's law for Magnetism:

How about the divergence of $\underline{\mathbf{B}}$?

$$\underline{\mathbf{v}}\times\underline{\mathbf{E}} = c\underline{\mathbf{B}} \implies \nabla\cdot\underline{\mathbf{B}} = \nabla\cdot\left(\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{E}}\right)$$

Using the $\nabla\cdot(\underline{\mathbf{A}}\times\underline{\mathbf{B}})$ vector identity...

$$\nabla\cdot\left(\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{E}}\right) = \cancel{\underline{\mathbf{E}}\cdot\left(\nabla\cdot\frac{\underline{\mathbf{v}}}{c}\right)} - \frac{\underline{\mathbf{v}}}{c}\cdot\underbrace{(\nabla\times\underline{\mathbf{E}})}_{\textit{Statics: } 0} = 0$$

$$\nabla\cdot\underline{\mathbf{B}} = 0$$

Maxwell–Faraday Equation:

We infer from experimental observations that light is an electromagnetic wave...

$$\nabla^2\underline{\mathbf{E}} = \frac{1}{c^2}\frac{\partial^2\underline{\mathbf{E}}}{\partial t^2}$$

Considering a point in free space far from any charge and using the $\nabla\times(\underline{\mathbf{A}}\times\underline{\mathbf{B}})$ vector identity...

$$\nabla\times(\nabla\times\underline{\mathbf{E}}) = \nabla\underbrace{(\nabla\cdot\underline{\mathbf{E}})}_{\textit{Free: } 0} - \nabla^2\underline{\mathbf{E}} \implies \nabla^2\underline{\mathbf{E}} = -\nabla\times(\nabla\times\underline{\mathbf{E}})$$

Similarly, in free space, the current density vanishes...

$$\nabla\times\underline{\mathbf{B}} = \cancel{\frac{4\pi}{c}\underline{\mathbf{J}}} + \frac{1}{c}\frac{\partial\underline{\mathbf{E}}}{\partial t} \implies \frac{1}{c}\nabla\times\frac{\partial\underline{\mathbf{B}}}{\partial t} = \frac{1}{c^2}\frac{\partial^2\underline{\mathbf{E}}}{\partial^2 t} \equiv \nabla^2\underline{\mathbf{E}}$$

$$\implies \nabla^2\underline{\mathbf{E}} = \frac{1}{c}\nabla\times\frac{\partial\underline{\mathbf{B}}}{\partial t} \equiv -\nabla\times(\nabla\times\underline{\mathbf{E}})$$

We physically accept only this curl relation...

$$\nabla\times\underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial\underline{\mathbf{B}}}{\partial t}$$

Maxwell's Equations:

$$\nabla\cdot\underline{\mathbf{E}} = 4\pi\rho(\underline{\mathbf{r}},t) \qquad \nabla\cdot\underline{\mathbf{B}} = 0$$

$$\nabla\times\underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial\underline{\mathbf{B}}}{\partial t} \qquad \nabla\times\underline{\mathbf{B}} = \frac{4\pi}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}},t) + \frac{1}{c}\frac{\partial\underline{\mathbf{E}}}{\partial t}$$

$$\oiint_S \underline{\mathbf{E}}\cdot\mathrm{d}\underline{\mathbf{S}} = 4\pi\iiint_V \rho(\underline{\mathbf{r}},t)\,\mathrm{d}V \qquad \oiint_S \underline{\mathbf{B}}\cdot\mathrm{d}\underline{\mathbf{S}} = 0$$

$$\oint_\Gamma \underline{\mathbf{E}}\cdot\mathrm{d}\underline{\boldsymbol{\ell}} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}\cdot\mathrm{d}\underline{\mathbf{S}}$$

$$\oint_\Gamma \underline{\mathbf{B}}\cdot\mathrm{d}\underline{\boldsymbol{\ell}} = \frac{4\pi}{c}\iint_S \underline{\mathbf{J}}(\underline{\mathbf{r}},t)\cdot\mathrm{d}\underline{\mathbf{S}} + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{E}}\cdot\mathrm{d}\underline{\mathbf{S}}$$

"The finite speed of light allows light to exist"

Electrostatics II

Poisson's Equation:

$$\nabla\cdot\underline{\mathbf{E}} = 4\pi\rho(\underline{\mathbf{r}}) \qquad \nabla\times\underline{\mathbf{E}} = 0 \qquad \underline{\mathbf{E}} = -\nabla\phi(\underline{\mathbf{r}})$$

From the fundamental theorem of vector calculus...

$$\phi(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\underline{\mathbf{E}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}' = \frac{1}{4\pi}\int_V \frac{4\pi\rho(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}'$$

Consider the Laplacian of the potential...

$$\implies \nabla^2\phi(\underline{\mathbf{r}}) = \int_V \rho(\underline{\mathbf{r}}')\nabla^2\left(\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\right)\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\equiv -4\pi\int_V \rho(\underline{\mathbf{r}}')\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\nabla^2\phi(\underline{\mathbf{r}}) = -4\pi\rho(\underline{\mathbf{r}})$$

Potential *Source*

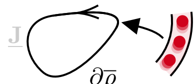
Electrostatic Gauge Transforms:

$$\phi \mapsto \phi - \kappa$$

"The charge density $\rho(\underline{\mathbf{r}})$ is invariant under a constant shift κ in the electric scalar potential $\phi(\underline{\mathbf{r}})$"

Magnetostatics

Realisation:



$$\frac{\partial \rho}{\partial t} = 0$$

"Magnetostatics occur when the charge density is time-independent to a good approximation"

$$\implies \nabla \cdot \underline{\mathbf{B}} = 0 \qquad \nabla \times \underline{\mathbf{B}} = \frac{4\pi}{c} \mathbf{J}(\underline{\mathbf{r}}) \qquad \underline{\mathbf{B}} = \nabla \times \underline{\mathbf{A}}$$

Synchrotron radiation is emitted when the charge density varies in time $\implies$ Wires do not emit radiation.

Magnetostatic Gauge Transforms:

$$\underline{\mathbf{A}} \mapsto \underline{\mathbf{A}} - \nabla\psi$$

"The magnetic field $\underline{\mathbf{B}}(\underline{\mathbf{r}})$ is invariant under a gradient shift of a smooth scalar field ψ in the magnetic vector potential $\underline{\mathbf{A}}(\underline{\mathbf{r}})$"

Coulomb Gauge:

Under the Coulomb gauge we define $\nabla^2\psi \equiv \nabla \cdot \underline{\mathbf{A}}$, which maps...

$$\nabla \cdot \underline{\mathbf{A}} = 0$$

Poisson Equivalent:

Consider the curl of the magnetic field in the Coulomb gauge using the $\nabla \times (\nabla \times \underline{\mathbf{A}})$ vector identity...

$$\nabla \times \underline{\mathbf{B}} = \nabla \times \nabla \times \underline{\mathbf{A}} = \nabla \underbrace{(\nabla \cdot \underline{\mathbf{A}})}_{\textit{Coulomb: } 0} - \nabla^2 \underline{\mathbf{A}}$$

$$\nabla^2 \underline{\mathbf{A}}(\underline{\mathbf{r}}) = -\frac{4\pi}{c} \underline{\mathbf{J}}(\underline{\mathbf{r}})$$

Generalised Biot–Savart Law:

From the fundamental theorem of vector calculus...

$$\underline{\mathbf{A}}(\underline{\mathbf{r}}) = \frac{1}{4\pi} \int_V \frac{\nabla' \times \underline{\mathbf{B}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \, \mathrm{d}^3\underline{\mathbf{r}}' = \frac{1}{4\pi} \int_V \frac{4\pi}{c} \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \, \mathrm{d}^3\underline{\mathbf{r}}'$$

$$\implies \underline{\mathbf{B}}(\underline{\mathbf{r}}) = \nabla \times \underline{\mathbf{A}}(\underline{\mathbf{r}}) = \frac{1}{c} \int_V \nabla \times \left(\frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \right) \mathrm{d}^3\underline{\mathbf{r}}'$$

Using the $\nabla \times (\phi \underline{\mathbf{A}})$ vector identity...

$$\nabla \times \left(\frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \right) = \frac{1}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \underbrace{\nabla \times \underline{\mathbf{J}}(\underline{\mathbf{r}}')}_{\underline{\mathbf{r}} \neq \underline{\mathbf{r}}' : 0} + \nabla \left(\frac{1}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|} \right) \times \underline{\mathbf{J}}(\underline{\mathbf{r}}')$$

$$= -\frac{(\underline{\mathbf{r}} - \underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|^3} \times \underline{\mathbf{J}}(\underline{\mathbf{r}}') = \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times (\underline{\mathbf{r}} - \underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|^3}$$

$$\underline{\mathbf{B}}(\underline{\mathbf{r}}) = \frac{1}{c} \int_V \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times (\underline{\mathbf{r}} - \underline{\mathbf{r}}')}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}'|^3} \, \mathrm{d}^3\underline{\mathbf{r}}'$$

Now let's try to add time back in...

Potential Formulation of Maxwell's Equations

Maxwell's Equations:

$$\nabla \cdot \underline{\mathbf{E}} = 4\pi\rho(\underline{\mathbf{r}}, t) \qquad \nabla \cdot \underline{\mathbf{B}} = 0$$

$$\nabla \times \underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \nabla \times \underline{\mathbf{B}} = \frac{4\pi}{c} \underline{\mathbf{J}}(\underline{\mathbf{r}}, t) + \frac{1}{c}\frac{\partial \underline{\mathbf{E}}}{\partial t}$$

Electric Vector Potential Equation:

Consider the Maxwell–Faraday equation...

$$\nabla \times \underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial \underline{\mathbf{B}}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left[\nabla \times \underline{\mathbf{A}}\right] = \nabla \times \frac{1}{c}\frac{\partial \underline{\mathbf{A}}}{\partial t}$$

$$\implies \underline{\mathbf{E}} = -\nabla\phi - \frac{1}{c}\frac{\partial \underline{\mathbf{A}}}{\partial t}$$

Taking the divergence of the electric field...

$$\nabla \cdot \underline{\mathbf{E}} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \underline{\mathbf{A}} \equiv 4\pi\rho$$

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \underline{\mathbf{A}} = -4\pi\rho(\underline{\mathbf{r}}, t)$$

Magnetic Vector Potential Equation:

Consider Ampère's Law...

$$\nabla \times \underline{\mathbf{B}} = \frac{4\pi}{c}\underline{\mathbf{J}} + \frac{1}{c}\frac{\partial \underline{\mathbf{E}}}{\partial t} \equiv \frac{4\pi}{c}\underline{\mathbf{J}} + \frac{1}{c}\frac{\partial}{\partial t}\left[-\nabla\phi - \frac{1}{c}\frac{\partial \underline{\mathbf{A}}}{\partial t}\right]$$

$$\implies \nabla \times \underline{\mathbf{B}} = \frac{4\pi}{c}\underline{\mathbf{J}} - \frac{1}{c}\nabla\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \underline{\mathbf{A}}}{\partial t^2}$$

As we're not in the Coulomb gauge...

$$\nabla \times \underline{\mathbf{B}} = \nabla \times (\nabla \times \underline{\mathbf{A}}) = \nabla(\nabla \cdot \underline{\mathbf{A}}) - \nabla^2\underline{\mathbf{A}}$$

$$\implies \nabla(\nabla \cdot \underline{\mathbf{A}}) - \nabla^2\underline{\mathbf{A}} = \frac{4\pi}{c}\underline{\mathbf{J}} - \frac{1}{c}\nabla\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \underline{\mathbf{A}}}{\partial t^2}$$

$$\nabla^2\underline{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2\underline{\mathbf{A}}}{\partial t^2} - \nabla\left(\nabla \cdot \underline{\mathbf{A}} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}}, t)$$

Lorenz Gauge:

Under the Lorentz gauge...

$$\nabla \cdot \underline{\mathbf{A}} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

Electromagnetic Vector Potential Equations in the Lorenz Gauge:

$$\nabla^2\phi(\underline{\mathbf{r}}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\underline{\mathbf{r}}, t) = -4\pi\rho(\underline{\mathbf{r}}, t)$$

$$\nabla^2\underline{\mathbf{A}}(\underline{\mathbf{r}}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{\mathbf{A}}(\underline{\mathbf{r}}, t) = -\frac{4\pi}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}}, t)$$

The d'Alembert operator is the Laplacian of 4-vector Minkowski space...

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\Box^2\phi(\underline{\mathbf{r}}, t) = -4\pi\rho(\underline{\mathbf{r}}, t) \qquad \Box^2\underline{\mathbf{A}}(\underline{\mathbf{r}}, t) = -\frac{4\pi}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}}, t)$$

"The sets $(\phi, \underline{\mathbf{A}})$ and $(\rho, \underline{\mathbf{J}})$ are starting to look like 4-vectors"

Maxwell's Wave Equations:

$$\Box^2\underline{\mathbf{E}}(\underline{\mathbf{r}}, t) = 0 \qquad \Box^2\underline{\mathbf{B}}(\underline{\mathbf{r}}, t) = 0$$

Green's Functions

Motivation:

The Green's function $G(x, x')$ of any linear differential operator $\widehat{O}$ is...

$$\widehat{O}\, G(x, x') = \delta(x - x')$$

This can be used to solve differential equations of the form...

$$\widehat{O}\, u(x) = f(x) \implies u(x) = \int_{-\infty}^{\infty} G(x, x') f(x') \, \mathrm{d}x'$$

Electrostatics:

In electrostatic, we had Poisson's equation...

$$\nabla^2\phi(\underline{\mathbf{r}}) = -4\pi\rho(\underline{\mathbf{r}})$$

This differential equation can be solved using a Green's function...

$$\phi(\underline{\mathbf{r}}) = \int_V G(\underline{\mathbf{r}},\underline{\mathbf{r}}')\rho\underline{\mathbf{r}}'\,\mathrm{d}\underline{\mathbf{r}}',$$

where the Green's satisfies $\nabla^2 G(\underline{\mathbf{r}},\underline{\mathbf{r}}') = -4\pi\rho(\underline{\mathbf{r}})$*...*

$$\implies G(\underline{\mathbf{r}},\underline{\mathbf{r}}') \equiv \frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|} \text{ as seen before...}$$

Green's Functions for Vector Potentials in the Lorentz Gauge:

$$\Box^2\phi(\underline{\mathbf{r}},t) = -4\pi\rho(\underline{\mathbf{r}},t) \qquad \Box^2\underline{\mathbf{A}}(\underline{\mathbf{r}},t) = -4\pi\frac{1}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}},t)$$

These differential equations can be solved using Green's functions...

$$\phi(\underline{\mathbf{r}},t) = \int_t\int_V G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t-t')\rho(\underline{\mathbf{r}}',t')\,\mathrm{d}^3\underline{\mathbf{r}}'\,\mathrm{d}t',$$

$$\underline{\mathbf{A}}(\underline{\mathbf{r}},t) = \int_t\int_V G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t-t')\frac{1}{c}\underline{\mathbf{J}}(\underline{\mathbf{r}}',t')\,\mathrm{d}^3\underline{\mathbf{r}}'\,\mathrm{d}t',$$

where the Green's function satisfies...

$$\Box^2 G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t-t') = -4\pi\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\delta(t-t')$$

Using the exponential Fourier form of delta...

$$G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t-t') = \int_{-\infty}^{\infty} G_w(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\mathrm{e}^{-\mathrm{i}\omega(t-t')}\frac{\mathrm{d}\omega}{2\pi}$$

$$\implies \Box^2 G_w\mathrm{e}^{-\mathrm{i}\omega(t-t')} = \ldots = \left[\nabla^2+\frac{\omega^2}{c^2}\right]G_w\mathrm{e}^{-\mathrm{i}\omega(t-t')}$$

Thus...

$$\Box^2 G = \int_{-\infty}^{\infty}\left[\nabla^2+\frac{\omega^2}{c^2}\right]G_w\mathrm{e}^{-\mathrm{i}\omega(t-t')}\frac{\mathrm{d}\omega}{2\pi} = -4\pi\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\delta(t-t')$$

$$\implies \left[\nabla^2+\frac{\omega^2}{c^2}\right]G_w(\underline{\mathbf{r}}-\underline{\mathbf{r}}') = -4\pi\delta(\underline{\mathbf{r}}-\underline{\mathbf{r}}')$$

Solving this in ss spc gives a general solution...

$$G_\omega(\underline{\mathbf{r}}-\underline{\mathbf{r}}') = \frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\mathrm{e}^{\pm\mathrm{i}\omega|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|/c}$$

"In the static case of $\omega\to 0$, $G_0 \to 1/|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|$ *as expected"*

Retarded and Advanced Green Functions:

Thus, using the general solution...

$$G(\underline{\mathbf{r}}-\underline{\mathbf{r}}',t-t') = \int_{-\infty}^{\infty}\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\mathrm{e}^{\mathrm{i}\omega\left[\pm\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|-(t-t')\right]}\frac{\mathrm{d}\omega}{2\pi}$$

$$\equiv \frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\delta\left(\pm\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|-(t-t')\right)$$

This has two non-vanishing solutions, advanced time t_a' *and retarded time* t_r'*...*

$$-: t_a' = t+\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'| \qquad +: t_r' = t-\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|$$

We only accept retarded time as this follows causality...

$$G_r(|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|,t-t') = \frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\delta\left(\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|-(t-t')\right)$$

$$\phi(\underline{\mathbf{r}},t) = \int_t\int_V\frac{\rho(\underline{\mathbf{r}}',t')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\delta\left(\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|-(t-t')\right)\mathrm{d}^3\underline{\mathbf{r}}'\,\mathrm{d}t'$$

$$\underline{\mathbf{A}}(\underline{\mathbf{r}},t) = \int_t\int_V\frac{1}{c}\frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}',t')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\delta\left(\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|-(t-t')\right)\mathrm{d}^3\underline{\mathbf{r}}'\,\mathrm{d}t'$$

Liénard-Wiechert Potentials:

By integrating over t' *we obtain the retarded potentials...*

$$\phi(\underline{\mathbf{r}},t) = \int_V\frac{\rho(\underline{\mathbf{r}}',t_r)}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\mathrm{d}\underline{\mathbf{r}}' \qquad \underline{\mathbf{A}}(\underline{\mathbf{r}},t) = \frac{1}{c}\int_V\frac{\underline{\mathbf{J}}(\underline{\mathbf{r}},t_r)}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\mathrm{d}\underline{\mathbf{r}}'$$

$$t_r = t-\frac{1}{c}|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|$$

Jefimenko's Equations:

Consider the electric and magnetic fields...

$$\underline{\mathbf{E}} = -\nabla\phi-\frac{1}{c}\frac{\partial\underline{\mathbf{A}}}{\partial t} \qquad \underline{\mathbf{B}} = \nabla\times\underline{\mathbf{A}}$$

Using the Liénard-Wiechert potentials...

$$\underline{\mathbf{E}} = \int_V\left[\left(\frac{\rho}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|^3}+\frac{1}{c|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|^2}\frac{\partial\rho}{\partial t}\right)(\underline{\mathbf{r}}-\underline{\mathbf{r}}')-\frac{1}{c^2|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\frac{\partial\underline{\mathbf{J}}}{\partial t}\right]\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\underline{\mathbf{B}} = \frac{1}{c}\int_V\left[\frac{\underline{\mathbf{J}}}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|^3}+\frac{1}{c|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|^2}\frac{\partial\underline{\mathbf{J}}}{\partial t}\right]\times(\underline{\mathbf{r}}-\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}'$$

Lorentz Force

Consider a charge q *moving with velocity* $\underline{\mathbf{v}}$ *through electric and magnetic fields. In the co-moving rest frame, the force on the particle is...*

$$\underline{\mathbf{F}} = q\underline{\mathbf{E}}_{\mathit{eff}}$$

When transforming to this rest frame, the particle is given an addition counter velocity $-\underline{\mathbf{v}}$*. We infer that...*

$$\text{"}\underline{\mathbf{E}} = -\frac{(-\underline{\mathbf{v}})}{c}\times\underline{\mathbf{B}}\text{"} \implies \underline{\mathbf{E}}_{\mathit{eff}} = \underline{\mathbf{E}}+\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{B}}$$

$$\underline{\mathbf{F}} = q\left(\underline{\mathbf{E}}+\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{B}}\right)$$

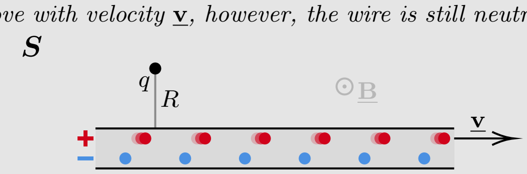
Special Relativity

Infinite Wire:

$$E = \frac{2\lambda}{R} \qquad B = \frac{2I}{cR}$$

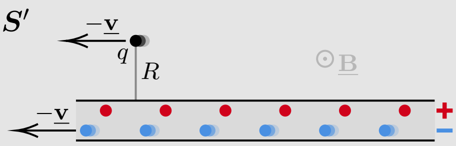
Newtonian Relativity:

Consider a charge q *near an infinite wire with both positive and negative charges. In frame* $\boldsymbol{S}$*, the positive charges move with velocity* $\underline{\mathbf{v}}$*, however, the wire is still neutral...*



$$\lambda = \lambda_+ + \lambda_- = 0 \implies \underline{\mathbf{F}} = q\left(\underline{\mathbf{E}}+\frac{\underline{\mathbf{v}}_q}{c}\times\underline{\mathbf{B}}\right) = 0$$

Now consider the rest frame $\boldsymbol{S'}$ *of the positive charges...*



$$\underline{\mathbf{F}} = q\left(\underline{\mathbf{E}}+\frac{\underline{\mathbf{v}}_q}{c}\times\underline{\mathbf{B}}\right) = -\frac{q}{c}\underline{\mathbf{v}}\times\underline{\mathbf{B}} \neq 0$$

But, both observers must see the same result...

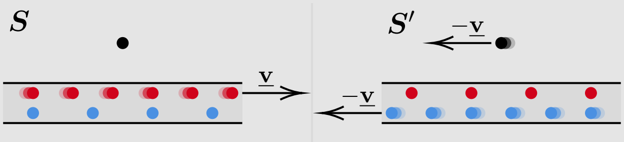
Length Contraction:

In special relativity, moving lengths get shorter...

$$\ell = \frac{\ell_0}{\gamma} = \ell_0\sqrt{1-v^2/c^2}$$

Special Relativity:

Due to length contraction, linear charge density increases for moving charges...



$$\lambda = \lambda_+\gamma + \lambda_- = 0 \qquad\qquad \lambda' = \lambda_+ + \lambda_-\gamma$$

In the frame $\boldsymbol{S}$, the moving charge current is $\lambda_+\gamma$...

$$|\underline{\mathbf{E}}| = \frac{2\lambda}{R} = 0 \qquad\qquad |\underline{\mathbf{B}}| = \frac{2I}{cR} = \frac{2\lambda_+\gamma}{cR}$$

In the other frame $\boldsymbol{S'}$, the moving charge current is $\lambda_-\gamma$...

$$\lambda_- = -\lambda_+\gamma \implies \lambda' = \lambda_+(1-\gamma^2) = -\lambda_+\frac{v^2}{c^2}\gamma^2$$

$$|\underline{\mathbf{E}}| = \frac{2\lambda'}{R} = \frac{2\lambda_+}{R}\left(\frac{v}{c}\right)^2\gamma^2 \qquad |\underline{\mathbf{B}}| = \frac{2\lambda_- v}{cR} = \frac{2\lambda_+}{c}\left(\frac{v}{c}\right)\gamma^2$$

From the Lorentz force, the magnitude of the electric and magnetic components can be found...

$$|\underline{\mathbf{F}}_E| = q|\underline{\mathbf{E}}| = q\frac{2\lambda_+}{R}\left(\frac{v}{c}\right)^2\gamma^2 \qquad |\underline{\mathbf{F}}_B| = q\frac{|\underline{\mathbf{v}}|}{c}|\underline{\mathbf{E}}| = q\frac{2\lambda_+}{R}\left(\frac{v}{c}\right)^2\gamma^2$$

"These forces are equal and opposite, meaning there is no total force as expected... The universe is all good again!"

Lorentz Invariance:

Property	**$\pm$**	**Invariant?**	**Fields**
Mass	$+$	*No*	*Gravitational*
Charge	$\pm$	*Yes*	*Electromagnetic*

We must consider both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ together...

Magnetic Charge

Inference of Maxwell's Equations:

Back when inferring Maxwell's equations, we unknowingly accepted a curl relation to allow electrostatics...

$$\underline{\mathbf{E}} = -\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{B}} \rightarrow \nabla\times\underline{\mathbf{E}} = -\nabla\times\left(\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{B}}\right)$$

However, we could've also done similar earlier to allow for the existence of magnetostatics...

$$\underline{\mathbf{B}} = \frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{E}} \rightarrow \nabla\times\underline{\mathbf{B}} = \nabla\times\left(\frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{E}}\right)$$

Magnetic Charge:

By introducing charge density for both electric ρ_e and magnetic ρ_m fields, Maxwell's equations become...

$$\nabla\cdot\underline{\mathbf{E}} = 4\pi\rho_e(\underline{\mathbf{r}},t) \qquad\qquad \nabla\cdot\underline{\mathbf{B}} = 4\pi\rho_m(\underline{\mathbf{r}},t)$$

$$-\nabla\times\underline{\mathbf{E}} = \frac{4\pi}{c}\underline{\mathbf{J}}_m(\underline{\mathbf{r}},t) + \frac{1}{c}\frac{\partial\underline{\mathbf{B}}}{\partial t} \qquad \nabla\times\underline{\mathbf{B}} = \frac{4\pi}{c}\underline{\mathbf{J}}_e(\underline{\mathbf{r}},t) + \frac{1}{c}\frac{\partial\underline{\mathbf{E}}}{\partial t}$$

Duality Transforms:

These equations are invariant under the transforms...

$$\rho_e \underset{-1}{\overset{1}{\rightleftarrows}} \rho_m \qquad \underline{\mathbf{E}} \underset{-1}{\overset{1}{\rightleftarrows}} \underline{\mathbf{B}} \qquad \underline{\mathbf{J}}_e \underset{-1}{\overset{1}{\rightleftarrows}} \underline{\mathbf{J}}_m$$

Generalised Lorentz Force:

For a theoretical particle carrying both electric charge e and magnetic charge g...

$$\underline{\mathbf{F}} = e\left(\underline{\mathbf{E}} + \frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{B}}\right) + g\left(\underline{\mathbf{B}} - \frac{\underline{\mathbf{v}}}{c}\times\underline{\mathbf{E}}\right)$$

Conservation of Energy

Rate of Energy Transfer:

Using the generalised Lorentz force, consider the rate of energy transfer to particles in an EM field...

$$\underline{\mathbf{F}}\cdot\underline{\mathbf{v}} = e\underline{\mathbf{v}}\cdot\underline{\mathbf{E}} + g\underline{\mathbf{v}}\cdot\underline{\mathbf{B}} = \int_V \rho_e\underline{\mathbf{v}}\cdot\underline{\mathbf{E}}\,\mathrm{d}^3\underline{\mathbf{r}} + \int_V \rho_m\underline{\mathbf{v}}\cdot\underline{\mathbf{B}}\,\mathrm{d}^3\underline{\mathbf{r}}$$

$$\implies \underline{\mathbf{F}}\cdot\underline{\mathbf{v}} = \int_V \underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}}\;\mathrm{d}^3\underline{\mathbf{r}}$$

"We interpret $\underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}}$ as the rate of energy transfer from the field to the particles per unit volume"

Inserting Maxwell's Equations:

Using the generalised Maxwell's equations, the power density $\underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}}$ becomes...

$$\frac{c}{4\pi}\left(\nabla\times\underline{\mathbf{B}} - \frac{1}{c}\frac{\partial\underline{\mathbf{E}}}{\partial t}\right)\cdot\underline{\mathbf{E}} - \frac{c}{4\pi}\left(\nabla\times\underline{\mathbf{E}} + \frac{1}{c}\frac{\partial\underline{\mathbf{B}}}{\partial t}\right)\cdot\underline{\mathbf{B}}$$

Using the $\nabla\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})$ vector identity...

$$\underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}} = -\frac{\partial}{\partial t}\left(\frac{\underline{\mathbf{E}}^2+\underline{\mathbf{B}}^2}{8\pi}\right) - \nabla\cdot\left(\frac{c}{4\pi}\underline{\mathbf{E}}\times\underline{\mathbf{B}}\right)$$

Energy Density and Poynting Vector:

Consider when $\underline{\mathbf{J}}_m = \underline{\mathbf{J}}_e = 0$...

$$\frac{\partial}{\partial t}\left(\frac{\underline{\mathbf{E}}^2+\underline{\mathbf{B}}^2}{8\pi}\right) + \nabla\cdot\left(\frac{c}{4\pi}\underline{\mathbf{E}}\times\underline{\mathbf{B}}\right) = 0$$

By introducing the energy density U and Poynting Vector $\underline{\mathbf{S}}$, this simplifies...

$$U = \frac{\underline{\mathbf{E}}^2+\underline{\mathbf{B}}^2}{8\pi} \qquad\qquad \underline{\mathbf{S}} = \frac{c}{4\pi}\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$

Considering now when $\underline{\mathbf{J}}_e \neq 0$ and $\underline{\mathbf{J}}_m \neq 0$...

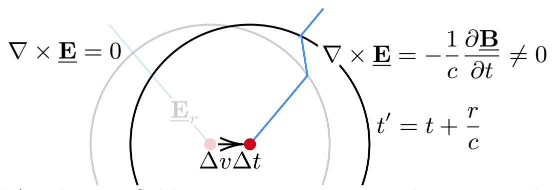
$$\frac{\partial U}{\partial t} + \nabla\cdot\underline{\mathbf{S}} + \underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}} = 0$$

Integrating this over a volume gives energy conservation...

$$\frac{\partial}{\partial t}\int_V U\,\mathrm{d}^3\underline{\mathbf{r}} + \int_S \underline{\mathbf{S}}\cdot\hat{\underline{\mathbf{n}}}\,\mathrm{d}^2\underline{\mathbf{r}} + \int_V \underline{\mathbf{J}}_e\cdot\underline{\mathbf{E}} + \underline{\mathbf{J}}_m\cdot\underline{\mathbf{B}}\,\mathrm{d}^3\underline{\mathbf{r}} = 0$$

Accelerating Charges

Consider the spherical electric field about a static charge, which is then accelerated, displacing the charge by $\Delta v\Delta t$...



"As electric fields are continuous, accelerating a charge produces a time-varying electromagnetic pulse (light)"

To properly analyse this, we must solve the Liénard-Wiechert potentials $\phi(\underline{\mathbf{r}},t)$ and $\underline{\mathbf{A}}(\underline{\mathbf{r}},t)$...

$$\underline{\mathbf{E}} = -\nabla\phi - \frac{1}{c}\frac{\partial\underline{\mathbf{A}}}{\partial t} \qquad\qquad \underline{\mathbf{B}} = \nabla\times\underline{\mathbf{A}}$$

This is mathematically complicated for moving charges...

Multipoles

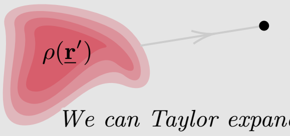
Taylor Expansion:

The 3D Taylor expansion near $f(\underline{\mathbf{r}})$ is...

$$f(\underline{\mathbf{r}}+\underline{\mathbf{a}}) = f(\underline{\mathbf{r}}) + \underline{\mathbf{a}}\cdot\nabla f(\underline{\mathbf{r}}) + \frac{1}{2}\underline{\mathbf{a}}\cdot\nabla\nabla f(\underline{\mathbf{r}})\cdot\underline{\mathbf{a}} + \dots$$

Potential Expansion:

Consider a point $\underline{\mathbf{r}}$ far from a charge distribution $\rho(\underline{\mathbf{r}}')$...



$$\phi(\underline{\mathbf{r}}) = \int_V \frac{\rho(\underline{\mathbf{r}}')}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|}\,\mathrm{d}^3\underline{\mathbf{r}}'$$

We can Taylor expand the Kernel to simplify...

$$\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|} = \frac{1}{|\underline{\mathbf{r}}|} + \frac{\underline{\mathbf{r}}\cdot\underline{\mathbf{r}}'}{|\underline{\mathbf{r}}|^3} + \frac{1}{2}\frac{1}{|\underline{\mathbf{r}}|^5}\underline{\mathbf{r}}\cdot\left(3\underline{\mathbf{r}}'\underline{\mathbf{r}}' - I|\underline{\mathbf{r}}'|^2\right)\cdot\underline{\mathbf{r}} + \dots$$

Thus, the potential becomes...

$$\phi(\underline{\mathbf{r}}) = \frac{e}{r} + \frac{\underline{\mathbf{r}}\cdot\underline{\mathbf{d}}}{r^3} + \frac{1}{2}\frac{\underline{\mathbf{r}}\cdot\underline{\mathbf{q}}\cdot\underline{\mathbf{r}}}{r^5} + \dots$$

"The potential is $\phi(r) = e/r$ if the charge is spherical, with additional terms arising from structure"

$$\text{Total Charge: } e = \int_V \rho(\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\text{Dipole Moment: } \underline{\mathbf{d}} = \int_V \underline{\mathbf{r}}'\rho(\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}'$$

$$\text{Quadrupole Moment: } \underline{\mathbf{q}} = \int_V \left(3\underline{\mathbf{r}}'\underline{\mathbf{r}}' - I|\underline{\mathbf{r}}'|^2\right)\rho(\underline{\mathbf{r}}')\,\mathrm{d}^3\underline{\mathbf{r}}'$$

The interaction energy is...

$$\varepsilon = Q\phi(\underline{\mathbf{r}}) = \frac{Qe}{r} + \frac{Q\underline{\mathbf{r}}\cdot\underline{\mathbf{d}}}{r^3} + \frac{1}{2}Q\frac{\underline{\mathbf{r}}\cdot\underline{\mathbf{q}}\cdot\underline{\mathbf{r}}}{r^5} + \dots$$

This can be interpreted as the interaction energy of the charge's various moments with $\underline{\mathbf{E}}$...

$$\varepsilon = \frac{Qe}{r} - \underline{\mathbf{d}}\cdot\underline{\mathbf{E}} + \frac{1}{6}\nabla\cdot(\underline{\mathbf{q}}\underline{\mathbf{E}}) + \dots \qquad \underline{\mathbf{E}} = -\frac{Q\underline{\mathbf{r}}}{r^3}$$

Dipole-Dipole Interactions:

When two neutral ($e = 0$) charge distributions interact, the dipole moment dominates as it falls off slower than higher-order interactions. The dipole-dipole interaction energy is...

$$\varepsilon = -\underline{\mathbf{d}}_2\cdot\left[-\nabla\frac{\underline{\mathbf{r}}\cdot\underline{\mathbf{d}}_1}{r^3}\right]$$

Expansion in Spherical Harmonics:

A more natural expansion for the potential is through spherical harmonics...

$$\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}'|} = \sum_\ell\sum_m \frac{r'^\ell}{r^{\ell+1}}\frac{4\pi}{2\ell+1}Y^\ell_m(\theta,\phi)Y^{\ell\,*}_m(\theta',\phi')$$

This can be used to simplify the potential again...