

## Introduction

Dimensions:

$$[Mass] = M \quad [Length] = L \quad [Time] = T$$

Coordinate Systems:

Cartesian:

$$\underline{\mathbf{x}} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z = (x, y, z)$$

$$d\underline{\mathbf{l}} = (dx, dy, dz)$$

$$dV = dx dy dz$$

$$dS = dx dy \quad \leftarrow \text{Constant } z$$

Cylindrical Polar:

$$\underline{\mathbf{x}} = r\hat{\mathbf{e}}_r(\phi) + z\hat{\mathbf{e}}_z = (r, \phi, z)$$

$$d\underline{\mathbf{l}} = (dr, r d\phi, dz)$$

$$dV = r dr d\phi dz$$

$$dS = r dr d\phi = r d\phi dz \quad \leftarrow \text{Constant } r$$

$$\hat{\mathbf{e}}_r = \cos(\phi)\hat{\mathbf{e}}_x + \sin(\phi)\hat{\mathbf{e}}_y \quad \hat{\mathbf{e}}_\phi = -\sin(\phi)\hat{\mathbf{e}}_x + \cos(\phi)\hat{\mathbf{e}}_y$$

Density:

$$dm = \rho dV \implies m = \iiint_V \rho(\underline{\mathbf{x}}) dV$$

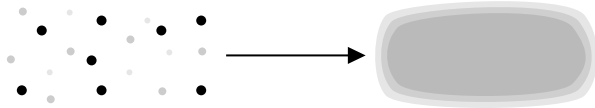
The Continuum Approximation:

We assume the fluid is a continuous distribution of matter whose properties are defined everywhere in space and time...

$$\rho = \rho(\underline{\mathbf{x}}, t) \quad \underline{\mathbf{v}} = \underline{\mathbf{v}}(\underline{\mathbf{x}}, t)$$

Microscopic:

Macroscopic:



"The **Fluid Parcel** is an infinitesimal volume of fluid large enough to be valid under the continuum approximation"

## Kinematics

The Flow Velocity:

$$\underline{\mathbf{v}} = \underline{\mathbf{v}}(\underline{\mathbf{x}}, t)$$

" $\mathbb{R}^n$  flow is a function of  $n$ -spatial and 1-temporal dimensions, with no components in any remaining dimensions."

Uniform Flow:

$$\frac{\partial \underline{\mathbf{v}}}{\partial x} = \frac{\partial \underline{\mathbf{v}}}{\partial y} = \frac{\partial \underline{\mathbf{v}}}{\partial z} = 0$$

Steady Flow:

$$\frac{\partial \underline{\mathbf{v}}}{\partial t} = 0$$

Stagnation Points:

Stagnation points are where the flow stagnates...

$$\underline{\mathbf{v}}(\underline{\mathbf{x}}) = 0$$

Streamlines:

Streamlines are tangential to the instantaneous flow velocity...

$$\frac{d\underline{\mathbf{x}}(s)}{ds} = \underline{\mathbf{v}}(\underline{\mathbf{x}}(s), t)$$

Implicit 3D:

$$\frac{dx}{ds} = v_x \quad \frac{dy}{ds} = v_y \quad \frac{dz}{ds} = v_z$$

Explicit 2D:

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

Pathlines:

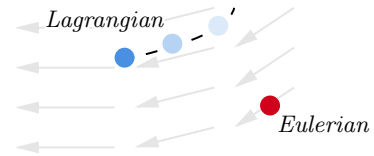
Pathlines are the trajectory of a fluid parcel moving with the flow...

$$\frac{d\underline{\mathbf{x}}(t)}{dt} = \underline{\mathbf{v}}(\underline{\mathbf{x}}(t), t)$$

"If the flow is steady, the streamlines are the pathlines"

Eulerian and Lagrangian Views:

"Eulerian observer's reference point is fixed (streamline), whereas Lagrangian observer's moves with the flow (pathline)"



Material Derivative:

Convective Rate of Change  
Lagrangian

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\underline{\mathbf{v}} \cdot \nabla)f$$

Temporal Rate of Change  
Eulerian

The Fluid Acceleration:

$$\underline{\mathbf{a}} = \frac{D\underline{\mathbf{v}}}{Dt} = \frac{\partial \underline{\mathbf{v}}}{\partial t} + (\underline{\mathbf{v}} \cdot \nabla)\underline{\mathbf{v}}$$

"Always calculate fluid parcel acceleration by component"

Vortices and Vorticity:

A vortex is a rotating flow about some axis, characterised by its vorticity  $\underline{\omega}$ ...

$$\underline{\omega} = \nabla \times \underline{\mathbf{v}}$$

"**Vortex Lines** are tangential to the vorticity"

Circulation:

$$\Gamma = \oint_C \underline{\mathbf{v}} \cdot d\underline{\mathbf{l}} \equiv \iint_S \underline{\omega} \cdot \hat{\mathbf{n}} dS$$

## Static Fluids

Pressure Forces:

$$\underline{\mathbf{F}}_P = - \iint_S P \hat{\mathbf{n}} dS \equiv - \iiint_V \nabla P dV$$

Body Forces:

$$\underline{\mathbf{F}}_B = \iiint_V \rho \underline{\mathbf{f}} dV \quad \leftarrow \text{Body force per unit mass}$$

Conservative:

$$\underline{\mathbf{f}} = -\nabla\phi$$

Gravitational:

$$\underline{\mathbf{f}} = \underline{\mathbf{g}} = -g \hat{\mathbf{e}}_z$$

Fluid Equilibrium:

$$\underline{\mathbf{F}}_P + \underline{\mathbf{F}}_B = 0 \implies \nabla P = \rho \underline{\mathbf{f}}$$

Incompressible Fluids:

$$P_{\text{atm}} \approx 10^5 \text{ Pa}$$

$$\rho = \rho_0 \implies P(z) = P_0 - \rho_0 g z$$

Pascal's Theorem and Transfer of Pressure:

"A change of pressure  $\Delta P$  occurring anywhere in a fluid is transmitted throughout the fluid instantaneously so that  $\Delta P$  is felt everywhere"

Pressure as Weight of a Fluid:

$$\underline{\mathbf{F}}_W = \iiint_V \rho(\underline{\mathbf{x}}) \underline{\mathbf{g}} dV \quad P = \frac{|\underline{\mathbf{F}}_W|}{A}$$

Buoyancy Forces:

A body immersed in a fluid under gravity experiences a buoyancy force  $\underline{\mathbf{F}}_b$ , which is equal and opposite to the weight of the displaced fluid...

$$\underline{\mathbf{F}}_b = -\underline{\mathbf{F}}_W^{\text{fluid}} = m^{\text{fluid}} g \hat{\mathbf{e}}_z$$

$\dots \implies \text{Sink Vs Rise?}$

## Inviscid Fluid under Motion

Mass Flux:

$$\mu = \iint_S \rho \underline{\mathbf{v}} \cdot \hat{\mathbf{n}} dS$$

Conservation of Mass:

Mass cannot be created or destroyed...

$$\frac{dm}{dt} = \text{Total Mass Flux } \mu \text{ into } V$$

"For an incompressible fluid in a steady flow,  $\dot{m} = -\mu = 0$ "

Continuity Equation:

Integral Form:

$$\iiint_V \frac{\partial \rho}{\partial t} dV = - \iint_S \rho \underline{\mathbf{v}} \cdot \hat{\mathbf{n}} dS$$

Differential Form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\mathbf{v}}) = 0$$

Incompressible Flow:

"Incompressible fluids have constant density, whereas in incompressible flows fluid parcels remain at uniform density as they move along pathlines"

$$\text{Incomp. Fluids : } \rho = \rho_0 \implies \text{Incomp. Flow : } \nabla \cdot \underline{\mathbf{v}} = 0$$

The Euler Equation:

For a fluid parcel of density  $\rho$  with a body force per unit mass  $\underline{\mathbf{f}}$  acting on it, the net force is...

$$\text{Net Force on the Fluid Parcel} = -\nabla P dV + \rho \underline{\mathbf{f}} dV$$

$$\underline{\mathbf{F}} = m \underline{\mathbf{a}} \implies \rho dV \underline{\mathbf{a}} = \rho dV \frac{D\underline{\mathbf{v}}}{Dt} = -\nabla P dV + \rho \underline{\mathbf{f}} dV$$

$$\frac{D\underline{\mathbf{v}}}{Dt} = -\frac{1}{\rho} \nabla P + \underline{\mathbf{f}}$$

The Incompressible Euler Equation:

For an incompressible fluid with a conservative body force...

$$\frac{\partial \underline{\mathbf{v}}}{\partial t} + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}} = -\nabla \left( \frac{P}{\rho_0} + \frac{1}{2} |\underline{\mathbf{v}}|^2 + \phi \right) = -\nabla B$$

$$\text{Bernoulli's Function: } B = \frac{P}{\rho_0} + \frac{1}{2} |\underline{\mathbf{v}}|^2 + \phi$$

Vorticity Transport Equation:

$$\frac{D\underline{\boldsymbol{\omega}}}{Dt} = \frac{\partial \underline{\boldsymbol{\omega}}}{\partial t} + (\underline{\mathbf{v}} \cdot \nabla) \underline{\boldsymbol{\omega}} = (\underline{\boldsymbol{\omega}} \cdot \nabla) \underline{\mathbf{v}}$$

Bernoulli's Theorem:

"For a steady flow of an incompressible fluid subject to a conservative body force, the Bernoulli function  $B$  is constant along a **streamline**"

Internal  $\nearrow$

$$B = \frac{P}{\rho_0} + \frac{1}{2} |\underline{\mathbf{v}}|^2 + \phi \equiv \text{Energy of a Fluid Parcel per unit mass}$$

$\nwarrow$   $\nearrow$

Kinetic Potential

Irrotational Bernoulli's Theorem:

"If a steady flow is irrotational, incompressible and subject to a conservative body force, the Bernoulli function  $B$  is constant **everywhere** in the fluid"

The Bernoulli Effect:

"For a constant potential  $\phi$ , under Bernoulli's Theorem, regions of higher flow speed have reduced fluid pressure"

## Potential Flow Theory

"Irrotational and Incompressible Flow"

Velocity Potential:

The velocity potential  $\Phi(\underline{\mathbf{x}})$  is defined such that...

$$\underline{\mathbf{v}} = \nabla \Phi$$

$$\underline{\mathbf{v}} = (v_x, v_y) = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right) \quad \underline{\mathbf{v}} = (v_r, v_\phi) = \left( \frac{\partial \Phi}{\partial r}, \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \right)$$

Potential Lines:

"Potential Lines have constant potential and are perpendicular to streamlines"

Laplace's Equation:

For an incompressible irrotational flow, the velocity potential  $\Phi$  satisfies Laplace's equation...

$$\nabla^2 \Phi = 0$$

Stream Function:

For a 2D incompressible flow, the stream function  $\psi(\underline{\mathbf{x}})$  is defined such that...

$$\underline{\mathbf{v}} = (v_x, v_y) = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \quad \underline{\mathbf{v}} = (v_r, v_\phi) = \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi}, -\frac{\partial \psi}{\partial r} \right)$$

$$\text{Irrotational: } \underline{\boldsymbol{\omega}} = 0 \implies \nabla^2 \psi = 0$$

"Lines of constant stream function are streamlines"

Superposition Theorem:

If  $\Phi_1$  and  $\Phi_2$  are solutions to Laplace's equation, then any linear superposition is also a solution...

$$\nabla^2 (a\Phi_1 + b\Phi_2) = a\nabla^2 \Phi_1 + b\nabla^2 \Phi_2 = 0$$

Elementary Flows:

Static:	$\underline{\mathbf{v}} = 0$	$\implies$	$\Phi = c : \text{const.}$
Uniform:	$\underline{\mathbf{v}} = (v_x, v_y)$	$\implies$	$\Phi = v_x x + v_y y$
Source & Sinks:	$\underline{\mathbf{v}} = (m/r, 0)$	$\implies$	$\Phi = m \ln(r)$
Free Vortex:	$\underline{\mathbf{v}} = (0, a/r)$	$\implies$	$\Phi = a\phi + b$

"In general, solve  $\nabla^2 \Phi = 0$ , then find  $\underline{\mathbf{v}} = \nabla \Phi$  and use Bernoulli's Theorem to find the pressure"