Introduction

Dimensions:

$$[Mass] = M \qquad [Length] = L \qquad [Time] = T$$

Coordinate Systems:

Cartesian:

$$\underline{\mathbf{x}} = x\widehat{\underline{\mathbf{e}}}_x + y\widehat{\underline{\mathbf{e}}}_y + z\widehat{\underline{\mathbf{e}}}_z = (x, y, z)$$

$$\mathrm{d}\underline{\mathbf{l}} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \qquad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y \longleftarrow \textit{Constant } z$$

Cylindrical Polar:

$$\underline{\mathbf{x}} = r\widehat{\underline{\mathbf{e}}}_r(\phi) + z\widehat{\underline{\mathbf{e}}}_z = (r, \phi, z)$$

$$\mathrm{d}\underline{\mathbf{l}} = (\mathrm{d}r, r\,\mathrm{d}\phi, \mathrm{d}z) \qquad \mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$$

$$\mathrm{d}S = \underbrace{r\,\mathrm{d}r\,\mathrm{d}\phi}_{\textit{Constant } z} = \underbrace{r\,\mathrm{d}\phi\,\mathrm{d}z}_{\textit{Constant } r}$$

$$\widehat{\underline{\mathbf{e}}}_r = \cos(\phi)\widehat{\underline{\mathbf{e}}}_x + \sin(\phi)\widehat{\underline{\mathbf{e}}}_y \qquad \widehat{\underline{\mathbf{e}}}_\phi = -\sin(\phi)\widehat{\underline{\mathbf{e}}}_x + \cos(\phi)\widehat{\underline{\mathbf{e}}}_y$$

Density:

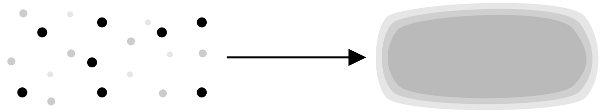
$$\mathrm{d}m = \rho\,\mathrm{d}V \implies m = \iiint_V \rho(\underline{\mathbf{x}})\,\mathrm{d}V$$

The Continuum Approximation:

We assume the fluid is a continuous distribution of matter whose properties are defined everywhere in space and time...

$$\rho = \rho(\underline{\mathbf{x}}, t) \qquad \underline{\mathbf{v}} = \underline{\mathbf{v}}(\underline{\mathbf{x}}, t)$$

Microscopic: Macroscopic:



*"The **Fluid Parcel** is an infinitesimal volume of fluid large enough to be valid under the continuum approximation"*

Kinematics

The Flow Velocity:

$$\underline{\mathbf{v}} = \underline{\mathbf{v}}(\underline{\mathbf{x}}, t)$$

"$\mathbb{R}^n$ flow is a function of n-spatial and 1-temporal dimensions, with no components in any remaining dimensions."

Uniform Flow:

$$\frac{\partial\underline{\mathbf{v}}}{\partial x} = \frac{\partial\underline{\mathbf{v}}}{\partial y} = \frac{\partial\underline{\mathbf{v}}}{\partial z} = 0$$

Steady Flow:

$$\frac{\partial\underline{\mathbf{v}}}{\partial t} = 0$$

Stagnation Points:

Stagnation points are where the flow stagnates...

$$\underline{\mathbf{v}}(\underline{\mathbf{x}}) = 0$$

Streamlines:

Streamlines are tangential to the instantaneous flow velocity...

$$\frac{\mathrm{d}\underline{\mathbf{x}}(s)}{\mathrm{d}s} = \underline{\mathbf{v}}(\underline{\mathbf{x}}(s), t)$$

Implicit 3D:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = v_x \quad \frac{\mathrm{d}y}{\mathrm{d}s} = v_y \quad \frac{\mathrm{d}z}{\mathrm{d}s} = v_z$$

Explicit 2D:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v_y}{v_x}$$

Pathlines:

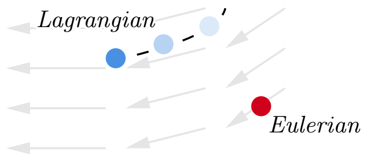
Pathlines are the trajectory of a fluid parcel moving with the flow...

$$\frac{\mathrm{d}\underline{\mathbf{x}}(t)}{\mathrm{d}t} = \underline{\mathbf{v}}(\underline{\mathbf{x}}(t), t)$$

"If the flow is steady, the streamlines are the pathlines"

Eulerian and Lagrangian Views:

"Eulerian observer's reference point is fixed (streamline), whereas Lagrangian observer's moves with the flow (pathline)"



Material Derivative:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \underbrace{\frac{\partial f}{\partial t}}_{\substack{\textit{Temporal Rate of Change}\\ \textit{Eulerian}}} + \overbrace{(\underline{\mathbf{v}}\cdot\nabla)f}^{\substack{\textit{Convective Rate of Change}\\ \textit{Lagrangian}}}$$

The Fluid Acceleration:

$$\underline{\mathbf{a}} = \frac{\mathrm{D}\underline{\mathbf{v}}}{\mathrm{D}t} = \frac{\partial\underline{\mathbf{v}}}{\partial t} + (\underline{\mathbf{v}}\cdot\nabla)\underline{\mathbf{v}}$$

"Always calculate fluid parcel acceleration by component"

Vortices and Vorticity:

A vortex is a rotating flow about some axis, characterised by its vorticity $\underline{\boldsymbol{\omega}}$...

$$\underline{\boldsymbol{\omega}} = \nabla\times\underline{\mathbf{v}}$$

*"**Vortex Lines** are tangential to the vorticity"*

Circulation:

$$\Gamma = \oint_C \underline{\mathbf{v}}\cdot\mathrm{d}\underline{\boldsymbol{\ell}} \equiv \iint_S \underline{\boldsymbol{\omega}}\cdot\widehat{\underline{\mathbf{n}}}\,\mathrm{d}S$$

Static Fluids

Pressure Forces:

$$\underline{\mathbf{F}}_P = -\iint_S P\,\widehat{\underline{\mathbf{n}}}\,\mathrm{d}S \equiv -\iiint_V \nabla P\,\mathrm{d}V$$

Body Forces:

$$\underline{\mathbf{F}}_B = \iiint_V \rho\,\underbrace{\underline{\mathbf{f}}}_{\textit{Body force per unit mass}}\,\mathrm{d}V$$

Conservative:

$$\underline{\mathbf{f}} = -\nabla\phi$$

Gravitational:

$$\underline{\mathbf{f}} = \underline{\mathbf{g}} = -g\,\widehat{\underline{\mathbf{e}}}_z$$

Fluid Equilibrium:

$$\underline{\mathbf{F}}_P + \underline{\mathbf{F}}_B = 0 \implies \nabla P = \rho\,\underline{\mathbf{f}}$$

Incompressible Fluids: $P_{\text{atm}} \approx 10^5$ Pa

$$\rho = \rho_0 \implies P(z) = P_0 - \rho_0 g z$$

Pascal's Theorem and Transfer of Pressure:

"A change of pressure ΔP occurring anywhere in a fluid is transmitted throughout the fluid instantaneously so that ΔP is felt everywhere"

Pressure as Weight of a Fluid:

$$\underline{\mathbf{F}}_W = \iiint_V \rho(\underline{\mathbf{x}})\underline{\mathbf{g}}\,\mathrm{d}V \qquad P = \frac{|\underline{\mathbf{F}}_W|}{A}$$

Buoyancy Forces:

A body immersed in a fluid under gravity experiences a buoyancy force $\underline{\mathbf{F}}_b$, which is equal and opposite to the weight of the displaced fluid...

$$\underline{\mathbf{F}}_b = -\underline{\mathbf{F}}_W^{\text{fluid}} = m^{\text{fluid}} g\, \widehat{\underline{\mathbf{e}}}_z$$

$\cdots \implies$ *Sink Vs Rise?*

Inviscid Fluid under Motion

Mass Flux:

$$\mu = \iint_S \rho\, \underline{\mathbf{v}} \cdot \widehat{\underline{\mathbf{n}}}\, \mathrm{d}S$$

Conversation of Mass:

Mass cannot be created or destroyed...

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \text{Total Mass Flux } \mu \textbf{ into } V$$

"For an incompressible fluid in a steady flow, $\dot{m} = -\mu = 0$"

Continuity Equation:

Integral Form:

$$\iiint_V \frac{\partial \rho}{\partial t}\, \mathrm{d}V = -\iint_S \rho\, \underline{\mathbf{v}} \cdot \widehat{\underline{\mathbf{n}}}\, \mathrm{d}S$$

Differential Form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\mathbf{v}}) = 0$$

Incompressible Flow:

"Incompressible fluids have constant density, whereas in incompressible flows fluid parcels remain at uniform density as they move along pathlines"

$$\text{Incomp. Fluids} : \rho = \rho_0 \implies \text{Incomp. Flow} : \nabla \cdot \underline{\mathbf{v}} = 0$$

The Euler Equation:

For a fluid parcel of density ρ with a body force per unit mass $\underline{\mathbf{f}}$ acting on it, the net force is...

$$\text{Net Force on the Fluid Parcel} = -\nabla P\, \mathrm{d}V + \rho \underline{\mathbf{f}}\, \mathrm{d}V$$

$$\text{"}\underline{\mathbf{F}} = m\underline{\mathbf{a}}\text{"} \implies \rho\, \mathrm{d}V \underline{\mathbf{a}} = \rho\, \mathrm{d}V \frac{\mathrm{D}v}{\mathrm{D}t} = -\nabla P\, \mathrm{d}V + \rho \underline{\mathbf{f}}\, \mathrm{d}V$$

$$\frac{\mathrm{D}\underline{\mathbf{v}}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \underline{\mathbf{f}}$$

The Incompressible Euler Equation:

For an incompressible fluid with a conservative body force...

$$\frac{\partial \underline{\mathbf{v}}}{\partial t} + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}} = -\nabla \left(\frac{P}{\rho_0} + \frac{1}{2}|\underline{\mathbf{v}}|^2 + \phi \right) = -\nabla B$$

$$\text{Bernoulli's Function: } B = \frac{P}{\rho_0} + \frac{1}{2}|\underline{\mathbf{v}}|^2 + \phi$$

Vorticity Transport Equation:

$$\frac{\mathrm{D}\underline{\boldsymbol{\omega}}}{\mathrm{D}t} = \frac{\partial \underline{\boldsymbol{\omega}}}{\partial t} + (\underline{\mathbf{v}} \cdot \nabla)\underline{\boldsymbol{\omega}} = (\underline{\boldsymbol{\omega}} \cdot \nabla)\underline{\mathbf{v}}$$

Bernoulli's Theorem:

*"For a steady flow of an incompressible fluid subject to a conservative body force, the Bernoulli function B is constant along a **streamline**"*

Internal

$$B = \frac{P}{\rho_0} + \frac{1}{2}|\underline{\mathbf{v}}|^2 + \phi \equiv \text{Energy of a Fluid Parcel per unit mass}$$

Kinetic *Potential*

Irrotational Bernoulli's Theorem:

*"If a steady flow is irrotational, incompressible and subject to a conservative body force, the Bernoulli function B is constant **everywhere** in the fluid"*

The Bernoulli Effect:

"For a constant potential ϕ, under Bernoulli's Theorem, regions of higher flow speed have reduced fluid pressure"

Potential Flow Theory

"Irrotational and Incompressible Flow"

Velocity Potential:

The velocity potential $\Phi(\underline{\mathbf{x}})$ is defined such that...

$$\underline{\mathbf{v}} = \nabla \Phi$$

$$\underline{\mathbf{v}} = (v_x, v_y) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right) \qquad \underline{\mathbf{v}} = (v_r, v_\phi) = \left(\frac{\partial \Phi}{\partial r}, \frac{1}{r}\frac{\partial \Phi}{\partial \phi} \right)$$

Potential Lines:

"Potential Lines have constant potential and are perpendicular to streamlines"

Laplace's Equation:

For an incompressible irrotational flow, the velocity potential Φ satisfies Laplace's equation...

$$\nabla^2 \Phi = 0$$

Stream Function:

For a 2D incompressible flow, the stream function $\psi(\underline{\mathbf{x}})$ is defined such that...

$$\underline{\mathbf{v}} = (v_x, v_y) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \qquad \underline{\mathbf{v}} = (v_r, v_\phi) = \left(\frac{1}{r}\frac{\partial \psi}{\partial \phi}, -\frac{\partial \psi}{\partial r} \right)$$

$$\text{Irrotational: } \underline{\boldsymbol{\omega}} = 0 \implies \nabla^2 \psi = 0$$

"Lines of constant stream function are streamlines"

Superposition Theorem:

If Φ_1 and Φ_2 are solutions to Laplace's equation, then any linear superposition is also a solution...

$$\nabla^2(a\Phi_1 + b\Phi_2) = a\nabla^2\Phi_1 + b\nabla^2\Phi_2 = 0$$

Elementary Flows:

$$\begin{aligned}
\text{Static: } & \underline{\mathbf{v}} = 0 && \implies && \Phi = c : \textit{const.} \\
\text{Uniform: } & \underline{\mathbf{v}} = (v_x, y_y) && \implies && \Phi = v_x x + v_y y \\
\text{Source \& Sinks: } & \underline{\mathbf{v}} = (m/r, 0) && \implies && \Phi = m \ln(r) \\
\text{Free Vortex: } & \underline{\mathbf{v}} = (0, a/r) && \implies && \Phi = a\phi + b
\end{aligned}$$

"In general, solve $\nabla^2\Phi = 0$, then find $\underline{\mathbf{v}} = \nabla\Phi$ and use Bernoulli's Theorem to find the pressure"