Distance and Metrics

Planar Geometry Metrics ($\mathrm{d}s$):

Cartesian: $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ Polar: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$

Relativity

Reference Frame:

"Set of spatial coordinates x, y, z and a time coordinate t."

Inertial Frame:

"A reference frame in which $\vec{\mathbf{F}} = m\underline{\mathbf{a}}$."

Non-Inertial Frame:

"A reference frame in which $\vec{\mathbf{F}} \neq m\underline{\mathbf{a}}$...

$\implies$ a fictitious force $\vec{\mathbf{F}}_i = m\underline{\mathbf{a}} - \underline{\mathbf{F}}$ exists."

Transformations:

Time Shift:	Displacement:	Rotation:
$t' = t - T$	$\underline{\mathbf{x}}' = \underline{\mathbf{x}} - \underline{\mathbf{d}}$	$\mathcal{R}(\underline{\mathbf{a}}, \theta)$

Uniform Motion (Galilean Transform):

$\underline{\mathbf{x}} = \underline{\mathbf{x}} - \underline{\mathbf{v}}t \qquad t' = t$

"All of these transformations don't change $\vec{\mathbf{F}} = m\underline{\mathbf{a}}$."

The Relativity Principle:

"Identical experiments carried out in different inertial frames give identical results."

Basic Postulates of Newtonian Relativity:

"1 - Identical experiments...

2 - Time always takes the same value up to a constant shift.

$\implies$ Universal Time "

Basic Postulates of Special Relativity:

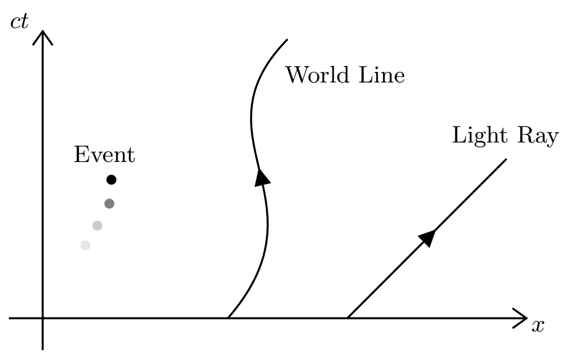
"1 - Identical experiments...

2 - The speed of light is the same in all inertial frames.

$\implies$ Moving Clocks Run Slowly "

Spacetime

Spacetime Diagrams:



Spacetime Interval:

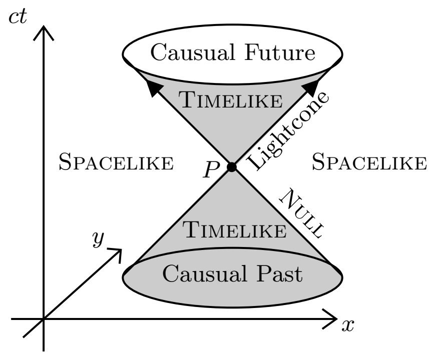
$$P : (\underline{\mathbf{x}}_p, t_p) \mapsto Q : (\underline{\mathbf{x}}_q, t_q) \implies (\Delta s)^2 = (\underline{\mathbf{x}}_q - \underline{\mathbf{x}}_p)^2 - c^2(t_q - t_p)^2$$

Separation:

$(\Delta s)^2 < 0$: Timelike $\qquad (\Delta s)^2 > 0$: Spacelike

$(\Delta s)^2 = 0$: $\underset{\text{Null}}{\text{Lightlike}}$

The Causal Structure of Spacetime:



Lorentz Transforms

Minkowsky Metric:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2 = |\,\mathrm{d}\underline{\mathbf{x}}|^2 - c^2\,\mathrm{d}t^2$$

Lorentz Transformation:

"Any transformation between the inertial frames S and S' which preserve the minkowsky metric are Lorentz transforms."

$$\implies |\,\mathrm{d}\underline{\mathbf{x}}|^2 - c^2\,\mathrm{d}t^2 = |\,\mathrm{d}\underline{\mathbf{x}}'|^2 - c^2\,\mathrm{d}t'^2$$

Lorentz Boost:

"The Lorentz transformation between two inertial frames S and S' in uniform relative motion is called a Lorentz Boost."

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$

$$\underset{\text{Gamma}}{\text{Lorentz Factor:}}\ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Time Dilation

Proper Time:

"The proper time τ of a body is the time measured by a clock moving with the body."

Rest Frame:

"The rest frame of a body is an inertial frame moving instantaneously with the same velocity as the body, such that $\mathrm{d}\underline{\mathbf{x}} = 0$ and $\mathrm{d}t = \mathrm{d}\tau$."

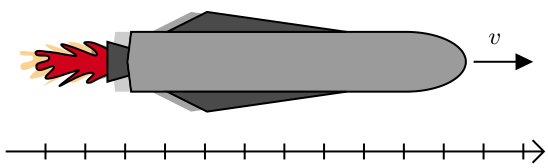
$$\implies \mathrm{d}s^2 = \cancelto{0}{\mathrm{d}\underline{\mathbf{x}}^2} - c^2\,\mathrm{d}t = -c^2\,\mathrm{d}\tau^2$$

Time Dilation:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

Constant Velocity $\implies \Delta t = \gamma \Delta\tau$

Length Contraction



Proper Length:

"The proper length $\ell_\star$ of a body is its length measured in its rest frame."

Lorentz Contraction:

$$\ell = \frac{\ell_\star}{\gamma}$$

$\implies$ Moving Bodies Appear Shorter

Four-Vectors

4-Vectors:

$$\underline{\mathbf{a}} = a^x \widehat{\mathbf{e}}_x + a^y \widehat{\mathbf{e}}_y + a^z \widehat{\mathbf{e}}_z + a^t \widehat{\mathbf{e}}_t$$

3-Position:

$$\underline{\mathbf{r}} = x\widehat{\mathbf{e}}_x + y\widehat{\mathbf{e}}_y + z\widehat{\mathbf{e}}_z$$

4-Position:

$$\underline{\mathbf{x}} = x\widehat{\mathbf{e}}_x + y\widehat{\mathbf{e}}_y + z\widehat{\mathbf{e}}_z + ct\widehat{\mathbf{e}}_t$$

Spatial Projection:

$$\underline{\mathbf{a}} = \vec{\mathbf{a}} + a^t\widehat{\mathbf{e}}_t \qquad \underline{\mathbf{a}} = \underline{\mathbf{r}} + ct\widehat{\mathbf{e}}_t$$

Scalar Products:

$$\widehat{\mathbf{e}}_x \cdot \widehat{\mathbf{e}}_x = \widehat{\mathbf{e}}_y \cdot \widehat{\mathbf{e}}_y = \widehat{\mathbf{e}}_z \cdot \widehat{\mathbf{e}}_z = 1 \qquad \widehat{\mathbf{e}}_t \cdot \widehat{\mathbf{e}}_t = -1$$

Spacetime classification:

$$\underline{\mathbf{a}} \cdot \underline{\mathbf{a}} < 0 : \text{TIMELIKE} \qquad \underline{\mathbf{a}} \cdot \underline{\mathbf{a}} > 0 : \text{SPACELIKE}$$

$$\underline{\mathbf{a}} \cdot \underline{\mathbf{a}} = 0 : \underbrace{\text{LIGHTLIKE}}_{\text{Null}}$$

Kinematics

4-Velocity:

$$\underline{\mathbf{u}} = \frac{\mathrm{d}\underline{\mathbf{x}}}{\mathrm{d}\tau}$$

$$(3+1): \ \underline{\mathbf{u}} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\underline{\mathbf{x}}}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mathbf{r}} + ct\widehat{\mathbf{e}}_t) = \gamma\underline{\mathbf{v}} + \gamma c\widehat{\mathbf{e}}_t$$

4-Acceleration:

$$\underline{\mathbf{a}} = \frac{\mathrm{d}\underline{\mathbf{u}}}{\mathrm{d}\tau}$$

Normalisation and Orthogonality:

$$\underline{\mathbf{u}} \cdot \underline{\mathbf{u}} = -c^2 \qquad \underline{\mathbf{u}} \cdot \underline{\mathbf{a}} = 0$$

Forces

4-Force: *"Bootstrap Formula"*

$$\underline{\mathbf{f}} = m\underline{\mathbf{a}}$$

Momentum and Energy

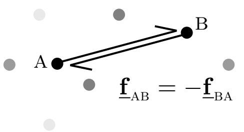
4-Momentum:

$$\underline{\mathbf{p}} = m\underline{\mathbf{u}} \qquad \underline{\mathbf{f}} = \frac{\mathrm{d}\underline{\mathbf{p}}}{\mathrm{d}\tau}$$

$$(3+1): \underline{\mathbf{p}} = \gamma m\underline{\mathbf{v}} + \gamma mc\widehat{\mathbf{e}}_t$$

$$\implies \text{MOVING } \underbrace{\text{MASSES APPEAR}}_{m \mapsto \gamma m} \text{ HEAVIER}$$

Particle Momenta Conversation: *"In an isolated system..."*



$$\text{All Particles} \implies \sum_i \underline{\mathbf{f}}_i = 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\sum_i \underline{\mathbf{p}}_i$$

Energy Conversation:

$$E = cp^t = \gamma mc^2 \longrightarrow \underbrace{\text{Rest Frame}}_{v=0\to\gamma=1} \implies E = mc^2$$

$$\underline{\mathbf{p}} = \vec{\mathbf{p}} + \gamma mc\widehat{\mathbf{e}}_t = \vec{\mathbf{p}} + \frac{E}{c}\widehat{\mathbf{e}}_t$$

Normalisation:

$$\underline{\mathbf{p}} \cdot \underline{\mathbf{p}} = \underline{\mathbf{p}}^2 = -m^2c^2 \qquad E^2 = \vec{\mathbf{p}}^2c^2 + m^2c^4$$

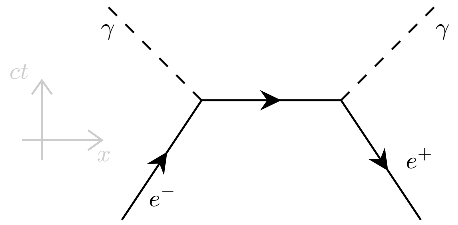
$\underbrace{\text{Massless Particles}}_{\textit{"photons"}}$: $\xcancel{\underline{\mathbf{p}} = m\underline{\mathbf{u}}}$

$$E^2 = \vec{\mathbf{p}}^2c^2 \implies |\vec{\mathbf{p}}| = \frac{E}{c}$$

Feynman Diagrams

The Feynman Rules:

- Particles are represented by straight lines with arrows, where electrons are solid and photons are dashed (wiggly)
- Interactions are represented by vertices where $2e^{\mp} \to \gamma$ meet
- External lines represent incoming or outgoing particles
- Internal lines joining vertices represent virtual particles and can be horizontal
- Momentum is conversed



Momentum Conversation:

"In general, try to use $\underline{\mathbf{p}}^2 = -mc^2$ with a reference frame of $\underbrace{\text{particle rest}}_{\vec{\mathbf{p}}=0}$ or $\underbrace{\text{centre of momenta}}_{\sum \vec{\mathbf{p}}_i = 0}$."

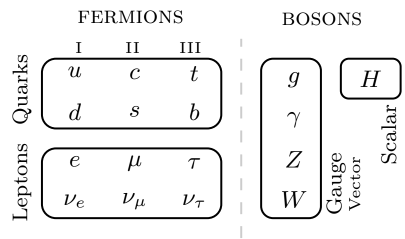
Particle Concepts

Fundamental Forces of Nature: $\underline{\mathbf{f}} = -e\nabla\phi$

$\underbrace{\text{Electromagnetism}}_{\text{Coulomb}}$ Strong $\underbrace{\text{Weak}}_{\text{Yukawa}}$ Gravity

$$\text{Coulomb:}\ \phi = \frac{e}{4\pi\varepsilon_0 r} \qquad \text{Yukawa:}\ \phi = \frac{q}{4\pi\varepsilon_0 r}e^{-\mu r}$$

The Standard Model:



"Each particle has an antiparticle of the same mass but opposite charge. Only the I*-generation fermions are stable."*

Conservation Rules:

- Electric Charge
- Energy & Momentum
- Angular Momentum: $\Delta s = \pm\{1, 2, 3\}$ $[\hbar]$
- Lepton Number: $+1$ for lepton particle; -1 for antiparticle
- Baryon Number: $+\frac{1}{3}$ per Quark; $-\frac{1}{3}$ per Antiquark

Klein-Gordon Theory

4-Gradient:

$$\underline{\boldsymbol{\nabla}} = \widehat{\mathbf{e}}_x\frac{\partial}{\partial x} + \widehat{\mathbf{e}}_y\frac{\partial}{\partial y} + \widehat{\mathbf{e}}_z\frac{\partial}{\partial z} - \widehat{\mathbf{e}}_t\frac{1}{c}\frac{\partial}{\partial t}$$

$$\nabla^2 = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Klein-Gordon Equation:

"For a field of mass m the Klein–Gordon equation is

$$\left(-\nabla^2 + \mu^2\right)\phi = 0,$$

where $\mu = mc/\hbar$."

Plane Wave Solutions:

$$\phi(\underline{\mathbf{x}}) = u_k e^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}} \qquad \underline{\mathbf{k}}^2 + \mu^2 = 0$$

Yukawa Potentials:

"Solve K-G in polar."

Vector Bosons

Massive Bosons:

"For a massive vector boson field $\underline{\mathbf{A}}(\underline{\mathbf{r}}, t)$ the field satisfies

$$\left(-\nabla^2+\mu^2\right)\underline{\mathbf{A}} = \left(-\nabla^2+\mu^2\right)\begin{bmatrix}A^t\\A^x\\A^y\\A^z\end{bmatrix} = 0, \qquad \underline{\boldsymbol{\nabla}}\cdot\underline{\mathbf{A}} = 0."$$

Plane Wave Solutions:

$$\underline{\mathbf{A}} = \underline{\mathbf{u}}_k e^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}} \qquad \underline{\mathbf{k}}\cdot\underline{\mathbf{u}}_k = 0 \qquad \underline{\mathbf{k}}^2+\mu^2 = 0$$

Single Particle: $\underline{\mathbf{p}} = \hbar\underline{\mathbf{k}}$

The Dirac Equation

Spinor Field:

$$\psi = \begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix}$$

"The spinor field ψ has four seperate complex field components. The field equation will now be a 4×4 matrix acting on the spinor."

Matrix of Vectors:

$$(\underline{\boldsymbol{\gamma}}\cdot\underline{\boldsymbol{\nabla}})^2 = -\mathrm{I}\nabla^2$$

The Dirac Equation: FERMIONS

"For a spinor field ψ of mass m the Dirac equation is

$$\big(-i\underbrace{\underline{\boldsymbol{\gamma}}\cdot\underline{\boldsymbol{\nabla}}}_{4\times4}+\mu\mathrm{I}\big)\psi = 0,$$

where $\mu = mc/\hbar$."

Plane Wave Solution:

$$\psi = ue^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}}$$

Constant Spinor

Pauli Matrices & Matrix Vectors:

$$\sigma^1 = \begin{bmatrix}0&1\\1&0\end{bmatrix} \quad \sigma^2 = \begin{bmatrix}0&-i\\i&0\end{bmatrix} \quad \sigma^3 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}} = \mathrm{I}\widehat{\mathbf{e}}_t + \sigma^1\widehat{\mathbf{e}}_x + \sigma^2\widehat{\mathbf{e}}_y + \sigma^3\widehat{\mathbf{e}}_z$$

$$\underline{\overline{\boldsymbol{\sigma}}} = \mathrm{I}\widehat{\mathbf{e}}_t - \sigma^1\widehat{\mathbf{e}}_x - \sigma^2\widehat{\mathbf{e}}_y - \sigma^3\widehat{\mathbf{e}}_z$$

Weyl Representation:

$$\underline{\boldsymbol{\gamma}} = \underbrace{\begin{bmatrix}[2\times2] & [2\times2]\\ [2\times2] & [2\times2]\end{bmatrix}}_{4\times4} = \begin{bmatrix}0 & \underline{\boldsymbol{\sigma}}\\ \underline{\overline{\boldsymbol{\sigma}}} & 0\end{bmatrix}$$

$$\gamma^x\gamma^y = -\gamma^y\gamma^x \qquad (\gamma^t)^2 = \mathrm{I} \qquad (\gamma^x)^2 = (\gamma^y)^2 = (\gamma^z)^2 = -\mathrm{I}$$

Charged Fermions:

"For a charged massive vector boson field $\underline{\mathbf{A}}(\underline{\mathbf{r}}, t)$ the field satisfies a modified Dirac equation...

$$\left[-i\underline{\boldsymbol{\gamma}}\cdot\left(\underline{\boldsymbol{\nabla}}+i\frac{q}{\hbar}\underline{\mathbf{A}}\right)+\mathrm{I}\mu\right]\psi = 0."$$

Neutrinos

Chiarlity:

"A spinor field can be decomposed into left & right chiral spinors." $\implies \psi = \begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix} = \begin{bmatrix}\psi_L\\\psi_R\end{bmatrix}$

$$(-i\underline{\boldsymbol{\gamma}}\cdot\underline{\boldsymbol{\nabla}}+\mu)\psi = 0 \longrightarrow \begin{cases}-i\underline{\boldsymbol{\sigma}}\cdot\underline{\boldsymbol{\nabla}}\psi_R = -\mu\psi_L\\ -i\underline{\overline{\boldsymbol{\sigma}}}\cdot\underline{\boldsymbol{\nabla}}\psi_L = -\mu\psi_R\end{cases}$$

"Massless spinors, $\mu = 0$, are realised in nature by neutrinos $\implies$ Neutrinos have two components."

Neutrino Masses:

" A different mass term is introduced, called the Majorana mass μ_m. This is observed in neutrino oscillations."

$$\nu_e \to \nu_\mu \to \nu_e \to \ldots$$

Higgs Boson

The Higgs Field:

"The Higgs field has two complex components,

$$H = \begin{bmatrix}h_1\\h_2\end{bmatrix},$$

and obeys a non-linear extension of the K-G equation...

$$-\nabla^2 h_1 - \mu_H^2 h_1 + \lambda(|h_1|^2+|h_2|^2)h_1 = 0,$$

$$-\nabla^2 h_2 - \mu_H^2 h_2 + \lambda(|h_1|^2+|h_2|^2)h_2 = 0."$$

$$\underset{\text{IMAGINARY}}{\textit{"Unphysical Mass"}} \xrightarrow{\text{NATURE}} \textit{"H has a non-zero vacuum constant value."}$$

$$\underset{\text{VEV}}{\text{Vacuum Expectation Value}}: v = \frac{\mu_H}{\sqrt{\lambda}}$$

The Higgs Particle:

"The Higgs particle is a fluctuation ϕ of the vacuum value, such that $h_1 = v+\phi$ and $h_2 = 0$."

$$-\nabla^2\phi + 2\lambda v^2\phi = 0 \qquad m_H = \frac{\hbar}{c}\sqrt{2\lambda v^2}$$

The Higgs Potential:

$$V(H) = -\frac{1}{2}\mu_H^2 H^\dagger H + \frac{1}{4}\lambda(H^\dagger H)^2$$

$$H^\dagger H = h_1h_1^* + h_2h_2^* = |h_1|^2+|h_2|^2$$

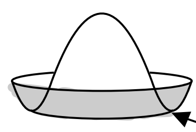
Electroweak Symmetry: $H \to UH$

2×2 Unitary Matrix $UU^\dagger = U^\dagger U = I$

The Higgs Doublet Field:

$$-\nabla^2 H + 2\frac{\partial V}{\partial H^\dagger} = 0$$

$$-\nabla^2 h_1 + 2\frac{\partial V}{\partial h_1^*} = 0 \qquad -\nabla^2 h_2 + 2\frac{\partial V}{\partial h_2^*} = 0$$



vev

"The Higg's field will sit at the bottom of its potentail"

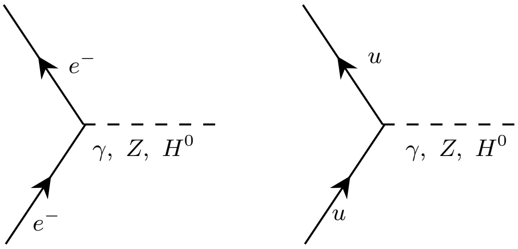
Electroweak Gauge Symmetry:

" The standard model is the most general theory that has $SU(3)\times SU(2)\times U(1)$ symmetry and allows for three generations of particles."

Electroweak Feynman Diagrams

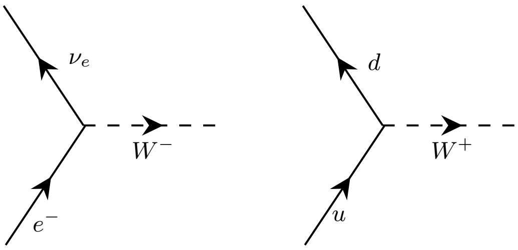
Electromagnetism:

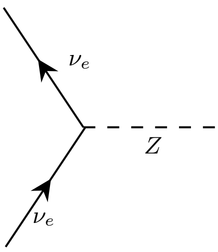
"The photon interactions with the charged particles: Leptons (e, μ, τ) and Quarks (u, d, s, c, b, t)."



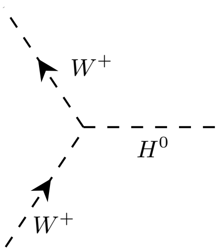
Weak Interactions:

" $W^\pm$ and Z bosons interact with matter particles. Z is the same as photon vertices. If W is involved, charge must be transferred."



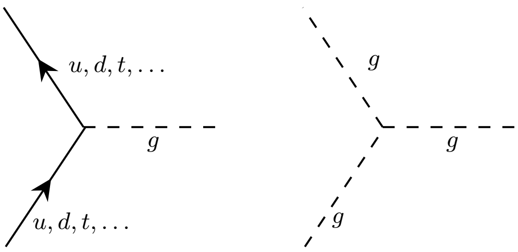


"The Higgs Boson H^0 is the same as photons, with an additional 3-boson interaction."



Strong Interactions:

"The gluon g interacts with quarks (or quark composites). There is an additional 3-boson interaction."



Decay Rates and Scattering Cross Sections:

"Feynman diagrams aren't just descriptive. We calculate the matrix element M by multiplying by $\sqrt{\alpha}$ for each vertex and $\min(1, E^2/m^2c^4)$ for each vertical particle."

Decay Rate: $\Gamma \approx \dfrac{mc^2}{\hbar}|M|^2$ Cross Section: $\sigma \approx \dfrac{(\hbar c)^2}{E^2}|M|^2$