

Minima and Maxima

Stationary Points:

Stationary points are points where the differential df vanishes...

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Fermat's Theorem:

"All local extrema are stationary points"

Difference Operator:

$$\Delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\Rightarrow \Delta f \approx \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \delta x^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \delta y^2 + \frac{\partial^2 f}{\partial x \partial y} \delta x \delta y$$

In matrix language...

$$\Delta f \approx \frac{1}{2} \begin{bmatrix} \delta x & \delta y \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

↖ Hessian: $H(f)$

$$\lambda_1, \lambda_2 > 0 : \text{Min}$$

$$\text{Mixed } \lambda : \text{Saddle}$$

$$\lambda_1, \lambda_2 < 0 : \text{Max}$$

"If an eigenvalue is zero, the hessian test fails"

The Fundamental Lemma of Functional Calculus:

Given,

1) A function $f(x)$ is continuous,

2) For every continuous piecewise differentiable function $\eta(x)$,

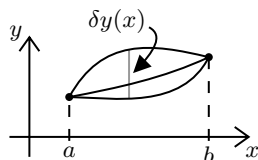
$$\int_a^b f(x) \eta(x) dx = 0,$$

then $f(x) = 0 \forall x \in [a, b]$.

Euler-Lagrange Equations

Functional:

We consider the extrema of the specific functional form $I[y]$...



$$I[y] = \int_a^b F(y, y', x) dx$$

...where $y(a)$ and $y(b)$ are fixed $\Rightarrow \delta y(a) = \delta y(b) = 0$

Functional Differential:

Stationary points are where the functional differential δI vanishes for all δy ...

$$\delta I = \lim_{\alpha \rightarrow 0} \frac{I[y + \alpha \delta y] - I[y]}{\alpha}$$

The Euler-Lagrange Equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

Geodesics

Geodesics in the Plane:

$$L[y] = \int_a^b \sqrt{1 + y'^2} dx \Rightarrow y = mx + c$$

"On a flat plane, the straight line is the shortest path"

Geodesics on the Sphere:

$$L[\phi] = \int_P^Q \sqrt{1 + \sin^2(\theta) \phi'^2} d\theta \Rightarrow \phi = 0$$

"On a sphere, the arc of a great circle is the shortest path"

Geodesics in General Relativity:

"Geodesics represent the world lines of particles in relativity"

World Line:

$$x^\mu(\tau); \mu = 0, 1, 2, 3$$

Metric:

$$c^2 d\tau = -g_{\mu\nu} dx^\mu dx^\nu$$

A **timelike geodesic** minimises the proper time τ between P and Q through 4D spacetime...

$$\int_P^Q c d\tau = \int_P^Q c \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau$$

First Integrals

First Integrals:

In the homogeneous case, $F = F(y, y')$, the euler-lagrange equations have a first integral...

$$y' \frac{\partial F}{\partial y'} - F = k : \text{const.}$$

"If $y' = 0$ then the first integral is invalid"

Multiple Functions:

$$y'_i \frac{\partial F}{\partial y'_i} - F = k \quad \sum_i y'_i \frac{\partial F}{\partial y'_i} - F = k$$

↖ Einstein Notation: Two i 's $\Rightarrow \sum$

Lagrange Multipliers

"Constrained Variation"

The Method of Lagrange Multipliers:

$$f(x, y) \text{ with } g(x, y) = k \Rightarrow \hat{f}(x, y) = f(x, y) - \lambda g(x, y)$$

Choose x , y , and λ such that...

$$\frac{\partial \hat{f}}{\partial x} = \frac{\partial \hat{f}}{\partial y} = 0 \quad g(x, y) = k$$

The Method of Lagrange Multipliers for Functionals:

$$I[y] = \int_a^b F(y, y', x) dx \text{ with } J[y] = \int_a^b G(y, y', x) dx = C$$

$$\Rightarrow \hat{I} = I - \lambda J$$

Multiple Functions

Multiple Functions:

$$y_1(x), y_2(x), \dots, y_n(x)$$

$$I[y_i] = \int_a^b F(y_i, y'_i, x) dx \Rightarrow \frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_i} \right) = 0$$

$$i = 1, 2, \dots, n$$

Multiple Dimensions

Multiple Dimensions: $u_i = \frac{\partial u}{\partial x^i}$ $u(x^1, x^2, \dots, x^n)$

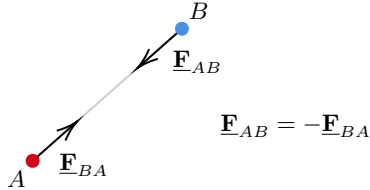
$$I[u] = \int_a^b F(u, u_i, x^i) dx \implies \frac{\partial F}{\partial u} - \frac{\partial}{\partial x^i} \left(\frac{\partial F}{\partial u_i} \right) = 0$$

Newton's Laws of Motion

Newton's 2nd Law:

$$\underline{\mathbf{F}} = \dot{\underline{\mathbf{p}}}$$

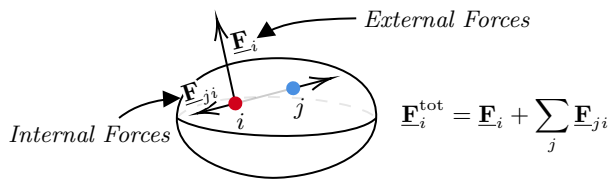
Newton's 3rd Law:



Centre of Mass:

$$\underline{\mathbf{R}} = \frac{1}{M} \sum_i m_i \underline{\mathbf{r}}_i \quad M = \sum_i m_i$$

Internal Forces and Solid Bodies:



"A solid Body has $|\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j| = k : \text{constant } \forall i, j$ "

Energy:

$$E = T + V = \frac{1}{2} m \dot{\underline{\mathbf{r}}}^2 + V(\underline{\mathbf{r}})$$

Conservative Force: $\underline{\mathbf{F}} = -\nabla V$

Hamilton's Principle

Action:

$$S[\underline{\mathbf{r}}] = \int_a^b L(\underline{\mathbf{r}}, \dot{\underline{\mathbf{r}}}, t) dt \quad \text{with } \underline{\mathbf{r}}(a) \text{ and } \underline{\mathbf{r}}(b) \text{ fixed.}$$

Lagrangian: $L = T - V$

Hamilton's Principle:

Solutions to the equations of motion are stationary points of the action functional $S[\underline{\mathbf{r}}]$...

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = 0$$

Coordinate Systems:

$$q^i = (x^1, x^2, \dots, x^N)$$

Cartesian:

Plane Polar:

$$\dot{\underline{\mathbf{r}}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\dot{\underline{\mathbf{r}}}^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

Spherical Polar:

$$\dot{\underline{\mathbf{r}}}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2(\theta) \dot{\phi}^2$$

Generalised Momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

"Momentum p_i is conserved if L is independent of q^i "

Generalised Energy:

$$E = \sum_{i=1}^N \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$$

Simplified Coordinates:

Forces of Constraint:

"Forces of constraint do no work"

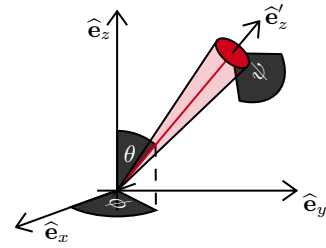
Constrained Systems:

"In simplified coordinates, the equations of motion are the E-L equations for the action with any forces of constraint ignored"

Euler Angles

Euler Angles:

3 angles can be used to describe the orientation of an object...



Precession of the Equinoxes:

Due to the equatorial bulge from rotation, the mean tidal potential from the Sun is...

$$V = \frac{3}{4} \omega_s^2 \Delta I \sin^2(\theta)$$

Including the moon's tidal forces, the Earth precesses with a period of $\sim 76,000$ years...

Hamiltonian Systems

Hamiltonian:

$$H(q^i, p_i, t) = \sum_i \dot{q}^i p_i - L(q^i, \dot{q}^i, t)$$

$$V = V(\underline{\mathbf{x}}) \implies H = T + V \quad p_i = \frac{\partial L}{\partial \dot{q}^i}$$

Hamilton's Equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$