PHY3029 - Variational Methods and Lagrangian Dynamics

Jake McMillan

Minima and Maxima

Stationary Points:

Stationary points are points where the differential $\mathrm{d}f$ *vanishes...*

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = 0$$

Fermat's Theorem:

"All local extrema are stationary points"

Difference Operator:

$$\Delta f = f(x+\delta x, y+\delta y) - f(x,y)$$

$$\implies \Delta f \approx \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\delta x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\delta y^2 + \frac{\partial^2 f}{\partial x \partial y}\delta x \delta y$$

In matrix language...

Hessian: $H(f)$

$$\Delta f \approx \frac{1}{2}\begin{bmatrix}\delta x & \delta y\end{bmatrix}\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta y\end{bmatrix}$$

$\lambda_1, \lambda_2 > 0$: *Min*
$\Delta f > 0$

Mixed λ : *Saddle*
$\Delta f = 0$

$\lambda_1, \lambda_2 < 0$: *Max*
$\Delta f < 0$

"If an eigenvalue is zero, the hessian test fails"

The Fundamental Lemma of Functional Calculus:

Given,

1) A function $f(x)$ *is continuous,*

2) For every continuous piecewise differentiable function $\eta(x)$*,*

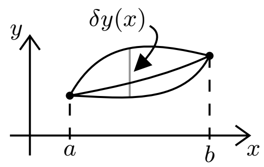
$$\int_a^b f(x)\eta(x)\,\mathrm{d}x = 0,$$

then $f(x) = 0 \ \forall \ x \in [a,b]$.

Euler-Lagrange Equations

Functional:

We consider the extrema of the specific functional form $I[y]$*...*



$$I[y] = \int_a^b F(y, y', x)\,\mathrm{d}x$$

...where $y(a)$ *and* $y(b)$ *are fixed* $\implies \delta y(a) = \delta y(b) = 0$

Functional Differential:

Stationary points are where the functional differential δI *vanishes for all* δy*...*

$$\delta I = \lim_{\alpha \to 0}\frac{I[y+\alpha\delta y] - I[y]}{\alpha}$$

The Euler-Lagrange Equation:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Geodesics

Geodesics in the Plane:

$$L[y] = \int_a^b \sqrt{1+y'^2}\,\mathrm{d}x \underset{\cdots}{\implies} y = mx + c$$

"On a flat plane, the straight line is the shortest path"

Geodesics on the Sphere:

$$L[\phi] = \int_P^Q \sqrt{1+\sin^2(\theta)\phi'^2}\,\mathrm{d}\theta \underset{\cdots}{\implies} \phi = 0$$

"On a sphere, the arc of a great circle is the shortest path"

Geodesics in General Relativity:

"Geodesics represent the world lines of particles in relativity"

World Line: $x^\mu(\tau);\ \mu = 0, 1, 2, 3$

Metric: $c^2\,\mathrm{d}\tau = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$

*A **timelike geodesic** minimises the proper time* τ *between* P *and* Q *through 4D spacetime...*

$$\int_P^Q c\,\mathrm{d}\tau = \int_P^Q c\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,\mathrm{d}\tau$$

First Integrals

First Integrals:

In the homogeneous case, $F = F(y, y')$*, the euler-lagrange equations have a first integral...*

$$y'\frac{\partial F}{\partial y'} - F = k : \text{const.}$$

"If $y' = 0$ *then the first integral is invalid"*

Multiple Functions:

$$y_i'\frac{\partial F}{\partial y_i'} - F = k \qquad \sum_i y_i'\frac{\partial F}{\partial y_i'} - F = k$$

Einstein Notation: *Two i's* $\implies \sum$

Lagrange Multipliers

"Constrained Variation"

The Method of Lagrange Multipliers:

$$f(x,y) \ \ \textit{with} \ \ g(x,y) = k \implies \widehat{f}(x,y) = f(x,y) - \lambda g(x,y)$$

Choose x, y, *and* λ *such that...*

$$\frac{\partial \widehat{f}}{\partial x} = \frac{\partial \widehat{f}}{\partial y} = 0 \qquad g(x,y) = k$$

The Method of Lagrange Multipliers for Functionals:

$$I[y] = \int_a^b F(y, y', x)\,\mathrm{d}x \ \ \textit{with} \ \ J[y] = \int_a^b G(y, y', x)\,\mathrm{d}x = C$$

$$\implies \widehat{I} = I - \lambda J$$

Multiple Functions

Multiple Functions: $y_1(x),\ y_2(x),\ \ldots,\ y_n(x)$

$$I[y_i] = \int_a^b F(y_i, y_i', x)\,\mathrm{d}x \implies \frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_i'}\right) = 0$$

$$i = 1,\ 2,\ \ldots,\ n$$

Multiple Dimensions

Multiple Dimensions: $u_i = \dfrac{\partial u}{\partial x^i}$ $\qquad u(x^1,\ x^2,\ \ldots,\ x^n)$

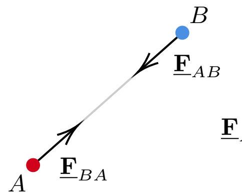
$$I[u] = \int_a^b F(u, u_i, x^i)\,\mathrm{d}^n x \implies \frac{\partial F}{\partial u} - \frac{\partial}{\partial x^i}\left(\frac{\partial F}{\partial u_i}\right) = 0$$

Newton's Laws of Motion

Newton's 2$^{\text{nd}}$ Law:

$$\underline{\mathbf{F}} = \dot{\underline{\mathbf{p}}}$$

Newton's 3$^{\text{rd}}$ Law:

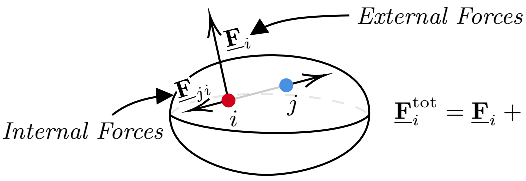


$$\underline{\mathbf{F}}_{AB} = -\underline{\mathbf{F}}_{BA}$$

Centre of Mass:

$$\underline{\mathbf{R}} = \frac{1}{M}\sum_i m_i \underline{\mathbf{r}}_i \qquad M = \sum_i m_i$$

Internal Forces and Solid Bodies:



$$\underline{\mathbf{F}}_i^{\text{tot}} = \underline{\mathbf{F}}_i + \sum_j \underline{\mathbf{F}}_{ji}$$

"A solid Body has $|\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j| = k$: constant $\forall\ i, j$"

Energy:

$$E = T + V = \frac{1}{2}m\dot{\underline{\mathbf{r}}}^2 + V(\underline{\mathbf{r}})$$

Conservative Force: $\underline{\mathbf{F}} = -\nabla V$

Hamilton's Principle

Action:

$$S[\underline{\mathbf{r}}] = \int_a^b L(\underline{\mathbf{r}}, \dot{\underline{\mathbf{r}}}, t)\,\mathrm{d}t \qquad \textit{with } \underline{\mathbf{r}}(a) \textit{ and } \underline{\mathbf{r}}(b) \textit{ fixed.}$$

Lagrangian: $L = T - V$

Hamilton's Principle:

Solutions to the equations of motion are stationary points of the action functional $S[\underline{\mathbf{r}}]$...

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0$$

Coordinate Systems: $\qquad q^i = (x^1, x^2, \ldots, x^N)$

Cartesian:

$$\dot{\underline{\mathbf{r}}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

Plane Polar:

$$\dot{\underline{\mathbf{r}}}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Spherical Polar:

$$\dot{\underline{\mathbf{r}}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + +r^2\sin^2(\theta)\dot{\phi}^2$$

Generalised Momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

"Momentum p_i is conserved if L is independent of q^i"

Generalised Energy:

$$E = \sum_{i=1}^N \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$$

Simplified Coordinates:

Forces of Constraint:

"Forces of constraint do no work"

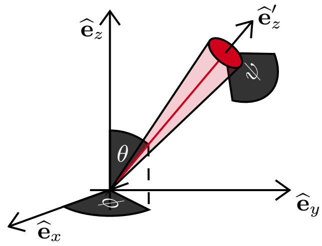
Constrained Systems:

"In simplified coordinates, the equations of motion are the E-L equations for the action with any forces of constraint ignored"

Euler Angles

Euler Angles:

3 angles can be used to describe the orientation of an object...



Precession of the Equinoxes:

Due to the equatorial bulge from rotation, the mean tidal potential from the Sun is...

$$V = \frac{3}{4}\omega_s^2 \Delta I \sin^2(\theta)$$

Including the moon's tidal forces, the Earth precesses with a period of $\sim 76,000$ years...

Hamiltonian Systems

Hamiltonian:

$$H(q^i, p_i, t) = \sum_i \dot{q}^i p_i - L(q^i, \dot{q}^i, t)$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

$$V = V(\underline{\mathbf{x}}) \implies H = T + V$$

Hamilton's Equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$