Observations of Stars

Stars:

*"A star is a body which is bound by its own gravity & radiations energy supplied internally by nuclear fusion and/or gravitational collapse $\implies$ **Stars must evolve**"*

Flux and Luminosity:

Flux $[\mathrm{erg \cdot s^{-1} \cdot cm^{-2}}]$ *is the measured radiation energy per time per area on a collector, whereas, Luminosity* $[\mathrm{erg \cdot s^{-1}}]$ *is the intrinsic radiation energy per time (total power)...*

$$F = \frac{L}{4\pi d^2}$$

Parallax and Distances:

$$d = \frac{1\ [\mathrm{AU}]}{\tan(\theta)} \approx \frac{1\ [\mathrm{AU}]}{\theta} \qquad d = \frac{1\ [\mathrm{pc}]}{\theta\ ['']}$$

Temperature:

Plank Function:

$$B(\lambda, T) = \frac{2hc^2}{\lambda^2}\frac{1}{e^{hc/\lambda k_B T} - 1} \simeq \frac{2hc^2}{\lambda^5 e^{hc/\lambda k_B T}}$$

Wien's Law:

$$\lambda_{\max} = \frac{0.29\ [\mathrm{cm \cdot K}]}{T\ [\mathrm{K}]}$$

Stefan–Boltzmann Law:

$$L = A\sigma T^4 = 4\pi R^2 \sigma T^4$$

$$F = \frac{L}{A} = \int_0^\infty B\, \mathrm{d}\lambda$$

Spectral Classification:

Oh **B**e **A** **F**ine **G**host **K**ill **M**e

Chemical Abundance

Stellar Spectra:

$\implies$ *"Composition of a Star's Atmosphere"*

Hydrogen Energy Levels:

$$E_n = -\frac{13.6}{n^2}\ [\mathrm{eV}]$$

The Boltzmann Distribution:

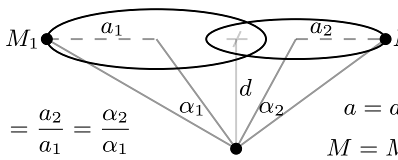
$$\frac{N_i}{N_j} = \frac{g_i}{g_j}\exp\left[-\frac{E_i - E_j}{k_b T}\right]$$

The Saha Equation:

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left[\frac{2\pi m_e k_B T}{h^2}\right]^{3/2} e^{-\chi/k_B T}$$

Measuring Mass From Binaries

Kepler's Laws:



$$\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1} \qquad a = a_1 + a_2 \qquad M = M_1 + M_2$$

I *Orbits are ellipses with the centre of mass at one focus.*

II *As the bodies orbit, the line connecting them sweeps out equal areas in equal times.*

III *The period P of the orbits is related to its semi-major axis a by...*

$$P^2 = \frac{4\pi^2 a^3}{GM}$$

Visual and Spectroscopic Binaries:

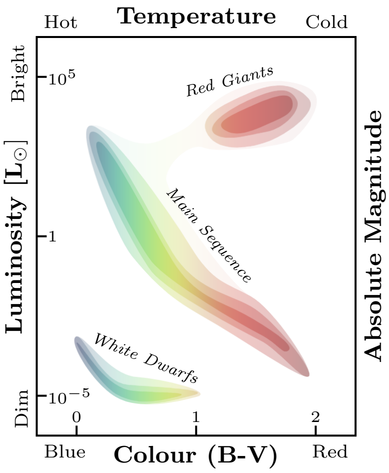
"Visual binaries can be seen as two distinct stars, whereas spectroscopic binaries can be inferred from observed periodic Doppler effects"

Measuring Mass:

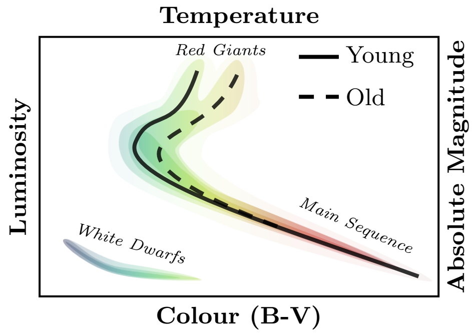
"By measuring α_1 and α_2 for the inferred ellipses and the full orbital period, the masses of both stars can be calculated. Any inclination means we only measure a lower limit of the masses"

The Hertzsprung-Russell diagram

The Observer's HR Diagram:



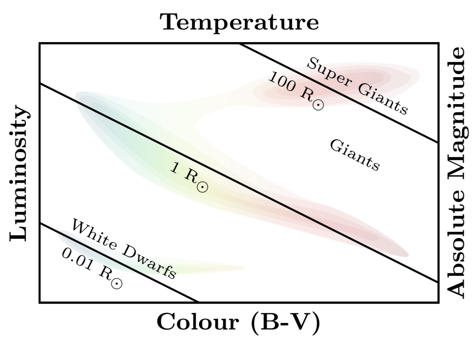
Clusters:



In younger clusters, the main sequence turns off towards the brighter blue side and vanishes for extremely old clusters...
$\implies$ *The brighter and bluer the star, the shorter it lives*

Stellar Radii:

$$L = 4\pi R^2\sigma T^4 \implies \log(L) = 4\log(T) + \log(4\pi\sigma) + 2\log(R)$$



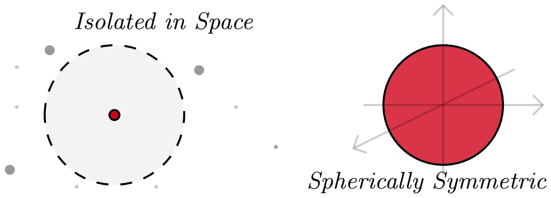
Stellar Masses:

For the main sequence, the stellar mass determines where it falls on the HR diagram (its luminosity) and basically all properties of the star...

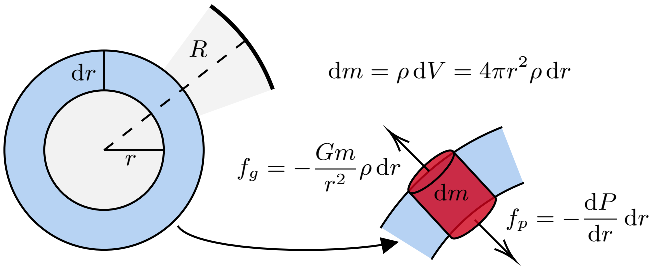
$$L \propto M^{3.5}$$

Hydrostatic Equilibrium and Lagrangian Coordinates

Basic Assumptions:



Shell's Equation of Motion:



$$\mathrm{d}m = \rho\,\mathrm{d}V = 4\pi r^2 \rho\,\mathrm{d}r$$

$$F = ma \implies \ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}$$

Dynamical Timescale:

If there was no pressure forces, the star would collapse under its own gravity...

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{Gm}{r^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}r}$$

$$\implies \int_0^v v\,\mathrm{d}v = -Gm\int_0^r \frac{\mathrm{d}r}{r^2} \implies v = \sqrt{\frac{2Gm}{r}}$$

For the whole star, a free fall velocity is found...

$$v_{\mathrm{ff}} = \sqrt{\frac{2GM}{R}}$$

The time taken for gravitational collapse is the dynamical timescale...

$$t_{\mathrm{dyn}} \approx \frac{R}{v_{\mathrm{ff}}} \approx \frac{1}{\sqrt{G\bar{\rho}}} \qquad t^{\odot}_{\mathrm{dyn}} = 10 \text{ hrs}$$

Hydrostatic Equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{Gm}{r^2} = -\rho g \qquad \frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}$$

Lagrangian Coordinates:

$$\textit{Euler} \quad \frac{\mathrm{d}}{\mathrm{d}r} \rightleftarrows \frac{\mathrm{d}}{\mathrm{d}m} \quad \textit{Lagrange}$$

"Lagrange has an advantage when thinking about advection, the transport of a substance by bulk motion of a fluid"

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}m}{\mathrm{d}r}\frac{\mathrm{d}f}{\mathrm{d}m} = 4\pi r^2\rho\frac{\mathrm{d}f}{\mathrm{d}m}$$

Virial Theorem

Gravitational Binding Energy:

$$\Omega(r) = -\frac{GMm}{R} = -\int_0^{m(r)} \frac{Gm}{r}\,\mathrm{d}m \qquad \Omega(R) = \Omega$$

Ideal Gas:

$$P = \frac{\mathcal{R}}{\mu}\rho T \qquad u = \frac{3}{2}k_B T = \frac{3}{2}\frac{\mathcal{R}}{\mu}T$$

The Virial Theorem:

By considering the lagrangian form of hydrostatic equilibrium...

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \implies \int_0^{P(r)} V\,\mathrm{d}P = -\frac{1}{3}\int_0^{m(r)} \frac{Gm}{r}\,\mathrm{d}r = \frac{1}{3}\Omega(r)$$

Using integration by parts, the pressure integral is evaluated...

$$\int_0^{P(r)} V\,\mathrm{d}P = [PV]_r - \int_0^{V(r)} P\,\mathrm{d}V = [PV]_r - \int_0^{m(r)} \frac{P}{\rho}\,\mathrm{d}m$$

Choosing $r = R$, an impressive result is found...

$$\frac{2}{3}\int_0^M u\,\mathrm{d}m = \frac{1}{3}\int_0^M \frac{Gm}{r}\,\mathrm{d}m$$

The total internal energy of the star is $-1/2 \times$ its total gravitational binding energy...

$$U = -\frac{1}{2}\Omega \qquad E = U + \Omega = \frac{1}{2}\Omega$$

Conservation of Energy:

Consider the energy change in a mass element $\mathrm{d}m$ over a small amount of time δt...

$$\delta E = \delta(u\,\mathrm{d}m) = \delta u\,\mathrm{d}m \overset{1^{st}\ \text{Thermo.}}{=} \delta Q + \delta W$$

For work on a gas...

$$\delta W = -P\delta V = -P\delta\left(\frac{\mathrm{d}V}{\mathrm{d}m}\,\mathrm{d}m\right) = -P\delta\left(\frac{1}{\rho}\right)\mathrm{d}m$$

For heat, there are both nuclear reactions at a rate per unit mass of energy released q and the transfer of heat flux $\Delta F = F(m) - F(m+\mathrm{d}m)$ across shells...

$$\delta Q = q\,\mathrm{d}m\delta t + \Delta F\delta t = \left(q - \frac{\partial F}{\partial m}\right)\mathrm{d}m\delta t$$

Thus, the first law of thermodynamics in a star is...

$$\frac{\mathrm{d}u}{\mathrm{d}t} + P\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) = q - \frac{\partial F}{\partial m}$$

Static Stars:

$$0 = q - \frac{\partial F}{\partial m}$$

$$\implies L_{nuc} = \int_0^M q\,\mathrm{d}m = \int_0^M \frac{\partial F}{\partial m}\,\mathrm{d}m = F(M) - \cancelto{0}{F(0)} = L$$

For static stars in equilibrium, the total energy leaving the star is from the nuclear reactions within it...

$$L_{\mathrm{nuc}} = L$$

Kelvin-Helmholtz Timescale:

Considering that in general, the gravitational potential energy is...

$$\Omega = -\alpha\frac{GM^2}{R}$$

If the nuclear reactions shut off instantly, how long would the star live if it radiated away at its current luminosity?

$$t \sim \frac{\Omega}{L} \sim \frac{GM^2}{RL}$$

The time for which a star could be powered by gravity alone without its radius changing is the Kelvin-Helmholtz timescale...

$$t_{\rm kh} = \frac{GM^2}{RL} \qquad t_{\rm kh}^{\odot} \approx 30 \text{ Myrs}$$

Nuclear Timescale:

How long can $L_{nuc} \approx L$? Assuming 4 Hydrogen $\mapsto$ Helium...

$$\Delta m = 4m_H - m_{He} \implies \varepsilon = \frac{\Delta m}{4m_H} \approx 0.0066$$

The time for which a star is powered by hydrogen fusion is the nuclear timescale...

$$t_{\rm nuc} = \frac{\varepsilon M c^2}{L} \qquad t_{\rm nuc}^{\odot} \approx 100 \text{ Gyrs}$$

Hierarchy of Timescales:

$$t_{\rm dyn} \leqslant t_{\rm kh} \leqslant t_{\rm nuc}$$

Stellar Pressure

Equation of State:

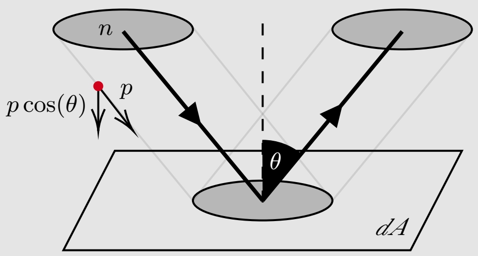
Chemical Composition

$$P = P(\rho, T, X)$$

Stellar Material:

$$E \ll k_B T \implies \textit{Gas}$$

The Kinetic Theory Model of Pressure:



Pressure is the rate of momentum change per unit area...

$$\text{1D}: \frac{\mathrm{d}^2 \rho_{surf}}{\mathrm{d}t\,\mathrm{d}A} = 2nvp\cos^2(\theta)$$

By using a solid angle integral and introducing the distribution of particles in momenta space $\mathrm{d}n(p)/\mathrm{d}p$ this can be generalised...

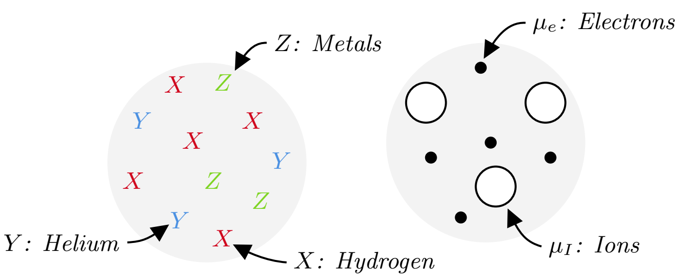
$$P = \frac{1}{3}\int_0^\infty \frac{\mathrm{d}n(p)}{\mathrm{d}p} pv\,\mathrm{d}p$$

Non-Relativistic Ideal Gas:

$$\textit{Boltzmann Dist.}: E = \frac{p^2}{2m} \implies \frac{\mathrm{d}n(p)}{\mathrm{d}p} \propto 4\pi p^2 \mathrm{e}^{-p^2/2mk_BT}$$

$$P = nk_BT$$

Gases with Multiple Species:



Stars are composed mainly of hydrogen, helium and other elements known as metals. Their mass fractions follow...

$$X + Y + Z = 1$$

Stellar material is also highly ionised, giving two components of ions and electrons. The mean mass per particle μ is...

Mass Fraction

$$\frac{1}{\mu} = \sum_{i=1}^{N} \frac{X_i}{\mathcal{A}_i} \qquad \frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_I}$$

Atomic Mass $[m_H]$

$$\frac{1}{\mu_e} = X + \frac{Y}{2} + Z\left\langle\frac{\mathcal{Z}}{\mathcal{A}}\right\rangle_{metals} \qquad \frac{1}{\mu_I} = X + \frac{Y}{4} + \frac{Z}{\langle\mathcal{A}\rangle_{metals}}$$

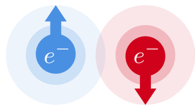
Ultra-Relativistic Gas:
Photons/Radiation

$$E = pc \implies \textit{Plank's Law}: \frac{\mathrm{d}n(p)}{\mathrm{d}p} = \frac{8\pi p^2}{h^3}\frac{1}{\mathrm{e}^{pc/k_BT} - 1}$$

$$P = \frac{1}{3}aT^4 \qquad a = \frac{4\sigma}{c}$$

Degenerate Gasses:

Fermions (electrons) cannot occupy the same quantum state in momentum space $\implies$ degeneracy pressure



Low T or High ρ

$$(\Delta x)^3(\Delta p)^3 \geqslant h^3$$

$$\implies \frac{\mathrm{d}n(p)}{\mathrm{d}p} = \frac{8\pi}{h^3}p^2$$

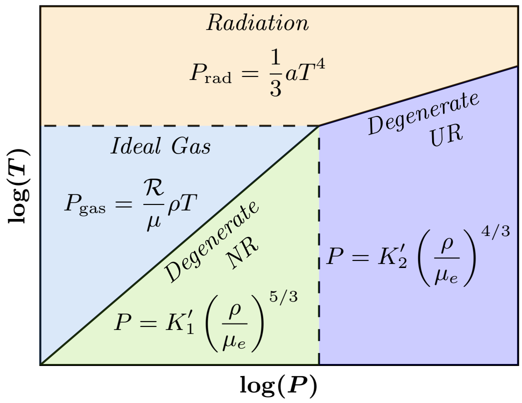
Non-Relativistic: $v = p/m$

$$P = K_1'\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Ultra-Relativistic: $v \approx c$

$$P = K_2'\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

Regimes of Pressure:



Internal Energy

Internal Energy per Unit Mass:

$$u = \frac{1}{\rho}\int_0^\infty \frac{\mathrm{d}n(p)}{\mathrm{d}p}\varepsilon(p)\,\mathrm{d}p$$

Kinetic Energy:

$$\varepsilon(p) = mc^2\left[\sqrt{1 + \frac{p^2}{m^2c^2}} - 1\right]$$

The non-relativistic, $p \ll mc$, energy expands to...

$$\varepsilon(p) \approx mc^2\left(1 + \frac{p^2}{2m^2c^2} - 1\right) = \frac{p^2}{2m}$$

The ultra-relativistic, $p \gg mc$, energy tends to...

$$\varepsilon(p) \approx mc^2\left(\sqrt{\frac{p^2}{m^2c^2}}\right) = pc$$

Regimes of Internal Energy:

Non-Relativistic:

$$u = \frac{3}{2}\frac{P}{\rho}$$

Ultra-Relativistic:

$$u = 3\frac{P}{\rho}$$

Adiabatic Processes and Adiabatic Index:

"Adiabatic processes are thermodynamic processes without heat transfer"

By using our first law of thermodynamics, an adiabatic process can be described...

$$\frac{\mathrm{d}u}{\mathrm{d}t} + P\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) = q - \frac{\partial F}{\partial m} = 0$$

By assuming a general form of the internal energy, the adiabatic index can be derived...

$$u = \phi\frac{P}{\rho}$$

The result is the adiabatic power law...

$$P = K_a\rho^{\gamma_a}$$

Adiabatic Index

$$\gamma_a = \frac{\phi + 1}{\phi}$$

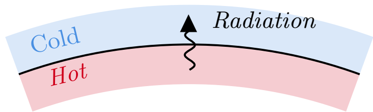
Adiabatic Constant

The adiabatic index γ_a describes how hard it is to compress the gas. The higher, the harder...

$$\text{Relativistic}: \frac{4}{3} \leqslant \gamma_a \leqslant \frac{5}{3}: \text{Ideal Gas}$$

Radiative Transfer

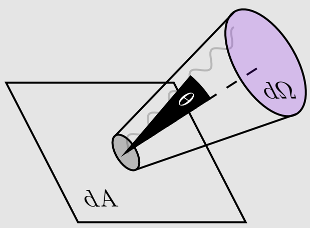
Energy Transfer:



"Energy is transported from the hot matter into the cold via the diffusion of radiation"

Beam of Radiation:

The radiation intensity I is the energy per unit area per unit time per unit frequency, including the direction and number of photons at each frequency...

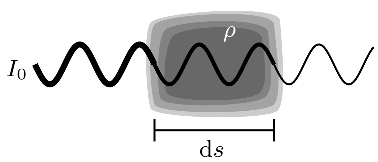


Flux H is just the average intensity over all directions...

$$H = \int I\cos(\theta)\,\mathrm{d}\Omega$$

Opacity:

The opacity κ describes how much photons are absorbed by the stellar material...



$$\kappa = -\frac{1}{\rho I}\frac{\mathrm{d}I}{\mathrm{d}s}$$

For a uniform slab, the intensity attenuation obeys...

$$I = I_0\mathrm{e}^{-\kappa\rho s} = I_0\mathrm{e}^{-\tau}$$

κ: Optical Depth

Emission and the Radiative Transfer Equation:

Sadly, radiation can also be emitted...

Emission Rate

$$\frac{\mathrm{d}I}{\mathrm{d}s} = -\kappa\rho I + j = -\kappa\rho I + \kappa\rho S \implies \frac{\mathrm{d}I}{\mathrm{d}\tau} = -I + S$$

Source Function

$$S = \frac{j}{\kappa\rho}$$

For a uniform opaque medium in thermal equilibrium, the distribution doesn't vary point to point...

$$\implies \frac{\mathrm{d}I}{\mathrm{d}\tau} = 0 \implies I = S = B(\nu, t)$$

Diffusion Approximation:

The interior of a star is pretty close to uniform, so its intensity is...

$$I(z,\theta) = S(T) - \frac{\cos(\theta)}{\kappa\rho}\frac{\mathrm{d}I(z,\theta)}{\mathrm{d}z}$$

Using the Rosseland approximation, in a nearly uniform medium like the centre of a star, I is nearly constant...

$$\implies \frac{\cos(\theta)}{\kappa\rho}\frac{\mathrm{d}I(z,\theta)}{\mathrm{d}z} \ll S(T) \equiv B(T)$$

The intensity is the plank function with a small perturbation...

$$I(z,\theta) = B(T) + \varepsilon I^{(1)}(z,\theta)$$

After integration, the radiation flux per unit area is...

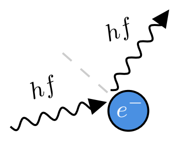
$$H = -\frac{16\sigma T^3}{3\kappa_R\rho}\frac{\partial T}{\partial z}$$

By considering a shell, the radiation temperature gradient is...

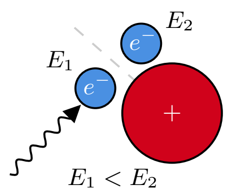
$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\sigma}\frac{\kappa_R\rho}{T^3}\frac{F}{4\pi r^2}$$

Opacity Sources:

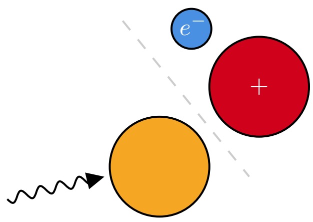
Electron Scattering:



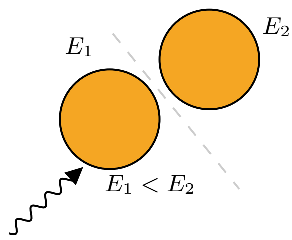
Free-Free Absorption:



Bound-Free Absorption:



Bound-Bound Absorption:



Nuclear Fusion Basics

Energy Release:

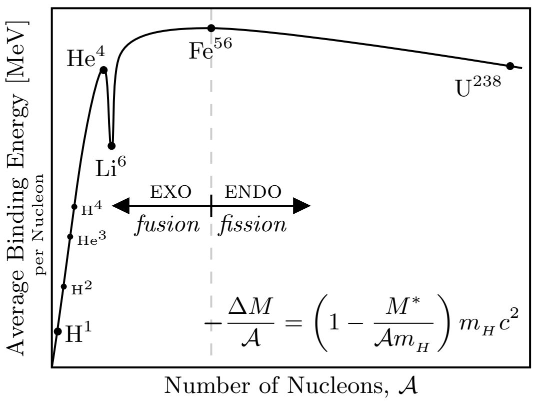
All nuclear reactions fundamentally convert mass into energy. The mass effect is...

$$E = \Delta mc^2$$

The rate of nuclear release per unit mass can be shown to be...

$$q_{nuc} = \frac{\rho^2}{m_H}\sum_{ijk}\left(\frac{1}{1+\delta_{ij}}\right)\frac{X_iX_j}{\mathcal{A}_I\mathcal{A}_j}R_{ijk}Q_{ijk}$$

Binding Energy Per Nucleon:

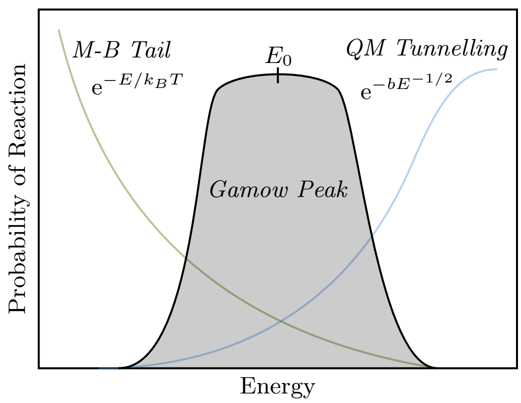


Quantum Tunnelling:

For nuclear fusion to occur, particles must get close enough and overcome the coulomb barrier. This can only happen due to Quantum Tunnelling...

$$T_{\mathrm{QT}}^{\odot} \gtrsim 10^7 \text{ K}$$

The Gamow Peak:



"The Gamow peak is a combination of decreasing reaction rate and increasing quantum tunnelling, telling us the energy when nuclear reactions are most probable. The reaction is very temperature-sensitive for increasing atomic number nuclei."

Nuclear Reactions in Stars

P-P Chain:

$${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{H} + e^+ + \nu_e$$
$${}^2_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$$

I: $${}^3_2\mathrm{He} + {}^1_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + \gamma$$

$${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$$

II:
$${}^7_4\mathrm{Be} + e^- \rightarrow {}^7_3\mathrm{Li} + \nu_e$$
$${}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \rightarrow 2\ {}^4_2\mathrm{He}$$

III:
$${}^3_4\mathrm{Be} + {}^1_1\mathrm{H} \rightarrow {}^8_5\mathrm{B} + \gamma$$
$${}^8_5\mathrm{B} \rightarrow {}^8_4\mathrm{Be} + e^+ + \nu_e$$
$${}^8_4\mathrm{Be} \rightarrow 2\ {}^4_2\mathrm{He}$$

In general, the proton-proton fusion chains result in...

$$4\ {}^1_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + \text{photons, neutrinos \& "light" particles}$$

Depends on pp branch

$$\Delta E_{\mathrm{pp}} \approx 27 \text{ Mev}$$

The first step is the throttle as it requires a spontaneous weak nuclear reaction, with an ignition temperature of...

$$T_{\mathrm{pp}} \sim 10^6 \text{ K}$$

CNO Cycle:

Cycle of reactions that require Carbon, Nitrogen and Oxygen to be present to begin burning...

$$\Delta E_{\mathrm{CNO}} \approx 25 \text{ MeV}$$

The limiting factor is the step...

$${}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{15}_8\mathrm{O} + \gamma$$

The ignition temperature is...

$$T_{\mathrm{CNO}} \sim 10^7 \text{ K}$$

Triple-α Process:

$${}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightleftharpoons {}^8_4\mathrm{Be}$$
$${}^{12}_6\mathrm{C} + {}^4_2\mathrm{He} \rightarrow {}^{16}_8\mathrm{O} + \gamma$$

This process is off the main sequence as it burns helium into oxygen...

$$\Delta E_{3\alpha} \approx 7 \text{ MeV}$$

The ignition temperature is...

$$T_{3\alpha} \sim 10^8 \text{ K}$$

Characterising Reactions:

Given the reaction rate coefficient is...

$$R \sim \frac{S(E_0)}{\mathcal{Z}_i \mathcal{Z}_j}\, \tau^2 e^{-\tau} \quad \textit{with} \quad \tau = \frac{3E_0}{k_B T}$$

A power law approximation can be made...

$$R \approx R_0 T^{\nu} \qquad \nu = \frac{\tau - 2}{3}$$

The rate of energy generation per unit mass is...

$$q \approx q_0 \rho T^{\nu}$$
$$\nu_{\mathrm{pp}} = 4 \qquad \nu_{\mathrm{CNO}} = 20 \qquad \nu_{3\alpha} = 41$$

Polytropes

The Equations of Stellar Structure:

Mass Conservation:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2$$

Momentum Conservation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{Gm}{r^2}$$

Temperature Gradient:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2}$$

Energy Conservation:

$$\frac{\mathrm{d}F}{\mathrm{d}r} = 4\pi r^2 \rho q$$

Pressure: $P = P(T)$

Opacity: $\kappa = \kappa_0 \rho^a T^b$

Energy: $q = q_0 \rho^m T^n$

Using boundary conditions, these equations can be solved either numerically using simulations or analytically using approximations. Simple models often include the stellar atmosphere $F = 4\pi R^2 \sigma T^4$ or other assumptions.

Polytropic Model:

An equation of state approximation can be made...

$$P = K_P \rho^{\gamma_P} \qquad \gamma_P = 1 + \frac{1}{n}$$

n: Polytrope Index

"Polytropes describes how pressure changes as you move through a star, whereas, the adiabatic equation of state describes how a given shell responds to being compressed. This is valid if a star is fully convective, enforcing constant entropy, or dominated by degenerate electron pressure." $\implies 1.5 \leqslant n \leqslant 3$

The Lane-Emden Equation:

Hydrostatic equilibrium and mass conservation can be combined...

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] = -4\pi G\rho$$

Using the polytope approximation, this can be non-dimensionalised through a change of variables...

$$\Theta^n = \frac{\rho}{\rho_c} \qquad \xi = \frac{r}{\alpha}$$

The underlying equation only depends on n (γ_P)...

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi^2\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right] = -\Theta^n$$

From this, mass and scale with n...

$$M \sim R^{n-3/n-1}$$

The Chandrasekhar Mass and Relativistic Gases:

For white dwarfs, the polytropic index is $n = 1.5$ as they are dominated by non-relativistic degenerate gases...

$$R \propto M^{-1/3}$$

More massive white dwarfs have smaller radii... If we keep adding mass can the radius shrink indefinitely? After the mass is greater than the Chandrasekhar Mass limit, the white dwarf is out of equilibrium and collapses on a dynamical timescale, resulting in a Type Ia supernova...

$$M_{\text{CH}} = 1.46\ M_\odot$$

Very Massive Stars:

Very massive stars can be close to $n = 3$ polytopes as they are dominated by relativistic photon gasses (radiation pressure). They are unstable and eject material to try reestablish stability.

Applications of Polytropes

The Eddington Limit:

By reconsidering temperature dependence at sufficient temperatures, where radiation pressure dominates...

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2}$$

$$\implies -\frac{\kappa\rho}{c}\frac{F}{4\pi r^2} = \frac{4}{3}aT^3\frac{\mathrm{d}T}{\mathrm{d}r} \equiv \frac{\mathrm{d}P_{gas}}{\mathrm{d}r}$$

In hydrostatic balance, $P = P_{rad} + P_{gas}$...

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P_{rad}}{\mathrm{d}r} + \frac{\mathrm{d}P_{gas}}{\mathrm{d}r} = -\rho\frac{Gm}{r^2} \implies \frac{\mathrm{d}P_{gas}}{\mathrm{d}r} > -\rho\frac{Gm}{r^2}$$

Thus, a maximum radiation flux is found...

$$F < \frac{4\pi cGm}{\kappa}$$

The result is the Eddington Luminosity...

$$L_{\text{Edd}} = \frac{4\pi cGm}{\kappa}$$

"Fundamental limit on luminosity for any object in hydrostatic equilibrium, such as stars, galaxies and black holes"

An upper limit on the nuclear energy generation at the centre follows...

$$F = m\frac{\mathrm{d}F}{\mathrm{d}m} + \cdots \approx m\frac{\mathrm{d}F}{\mathrm{d}m} \implies \frac{\mathrm{d}F}{\mathrm{d}m} = q_c < \frac{4\pi cG}{\kappa}$$

The Eddington Model:

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P} = \frac{\kappa F}{4\pi cGm} = \frac{F}{L_{\text{Edd}}} = \text{Const.}$$

The Eddington model assumes that the flux through a star ÷ Eddington luminosity is constant...

$$\implies P_{\text{rad}} = \frac{F}{L_{\text{Edd}}}P$$

This model implies that the pressure coefficient β is constant throughout the star...

$$\beta = \frac{P_{gas}}{P} = 1 - \frac{F}{L_{Edd}} = Const.$$

From this, our first theoretical mass-luminosity relation can be found...

$$0 = -1 + \beta + 0.0004\left(\frac{M}{M_\odot}\right)^2\left(\frac{\mu}{0.61}\right)^4\beta^4$$

At the surface of the star...

$$L = (1-\beta)L_{edd} = 5.5\beta^4\left(\frac{\mu}{0.61}\right)^4\left(\frac{1\ \text{cm}^2\text{g}^{-1}}{\kappa_s}\right)\left(\frac{M}{M_\odot}\right)^3 L_\odot$$

The result is a mass-luminosity scaling relation...

Low Mass $\gtrsim M_\odot$: $L \propto M^3$

High Mass $\gtrsim 100\ M_\odot$: $L \propto M$

Convection in Stars

The Adiabatic Temperature Gradient:

"Convection is a way to transport energy in a star. It involves the mass motion of fluid and is very efficient"

Assuming the gas is ideal...

$$P = \frac{\mathcal{R}}{\mu}\rho T \implies \mathrm{d}P = \left(\frac{P}{P}\frac{\mathrm{d}T}{\mathrm{d}r} + \frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)\mathrm{d}r$$

Since $t_{kh} \gg t_{dyn}$, the gas behaves adiabatically on short timescales...

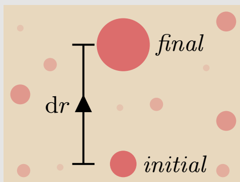
$$P = K_a\rho^{\gamma_a} \implies \mathrm{d}P = \gamma_a\frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\mathrm{d}r$$

By combining the ideal and adiabatic equations of state, the adiabatic temperature gradient is...

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\text{ad}} = \frac{(\gamma_a - 1)}{\gamma_a}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(\gamma_a - 1)}{\gamma_1}\frac{\mu}{\mathcal{R}}\frac{Gm}{r^2}$$

"If the temperature gradient is equal to the adiabatic temperature gradient, convection doesn't do anything"

The Brunt-Väisälä Frequency:



The equation of motion is ...

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathrm{d}r) = Ag\mathrm{d}r$$

$$\implies \textit{Harmonic Oscillator...} \quad \mathrm{d}r = Ce^{\mathrm{i}Nt}$$

The Brunt-Väisälä Frequency N is the oscillating frequency of a vertically displaced fluid element in a statistically stable environment...

$$N = \pm\sqrt{-Ag} = \pm\sqrt{\left(\frac{1}{\gamma_a P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)g}$$

If N is...

- *$\mathbb{R}$eal: Stable oscillator with internal gravity waves*
- *$\mathbb{I}$mag: Unstable system resulting in convection ($A > 0$)*

Convective Stability:

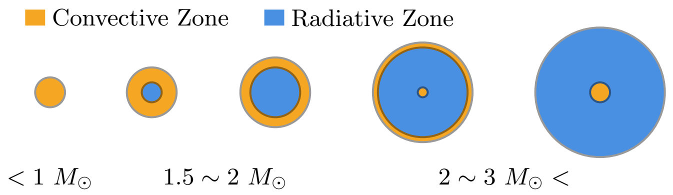
Convection occurs when the temperature gradient is super-adiabatic, meaning it is steeper than the adiabatic temperature gradient...

$$\left|\frac{\mathrm{d}P}{\mathrm{d}r}\right| > \left|\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)_{\mathrm{ad}}\right|$$

When convection is happening, the energy transport equation is the adiabatic convection gradient rather than the radiative diffusion gradient.

"In general, convection happens when the opacity is high or energy generation is very large"

Location of Convection:



"Low mass stars are fully convective. Medium mass stars are convective in the envelope and radiative in the core. High mass stars are convective in the core due to the CNO cycle and radiative in the envelope."

Convective Energy Transport:

The convective heat flow in a star is...

$$F_c = 4\pi r^2 \rho c_p \delta \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right) \ell \overline{v}_c$$

Unfortunately, we lack a "spherically symmetric theory of convection", so an empirical approach called mixing length theory is used to evaluate the mixing length ℓ and bubble velocity $\overline{v}_c$...

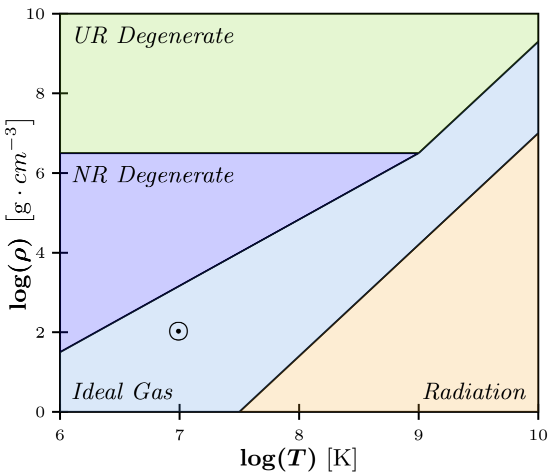
$$\ell = \alpha \frac{\mathcal{R}}{\mu}\frac{T}{g} \qquad \overline{v}_c = \alpha \frac{\mathcal{R}}{\mu}\left[\beta \frac{T}{g}\delta\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)\right]^{1/2}$$

Schematics of The Evolution of Stellar Cores

The $\log(\mathrm{T}), \log(\rho)$ Plane:

$$P_{\mathrm{gas}} \sim T \quad P_{\mathrm{rad}} \sim T^4 \quad P_{\mathrm{nrd}} \sim \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad P_{\mathrm{urd}} \sim \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

By logging these equations, boundary lines between the different pressure types can be found and plotted...



"In reality, the regimes are smooth and continuous"

Mass Lines: