Propagation of Light Through Free Space

Luminosity and Specific Luminosity:

Luminosity is the energy radiated from a source per time, with a specific luminosity at a given frequency...

$$L = \frac{\mathrm{d}E}{\mathrm{d}t} \qquad L_\nu = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}\nu}$$

Flux and Specific Flux:

Flux is the energy crossing a surface per time per area, with a specific flux at a given frequency...

$$F = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} \qquad F_\nu = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}\nu}$$

Solid Angle: *"3D Opening Angle"*

$$\mathrm{d}\Omega = \frac{\mathrm{d}A}{r^2}$$

$$[\underset{\mathrm{sr}}{\mathrm{steradian}} \equiv \mathrm{radian}^2]$$

"We model light as bundles of rays with some solid angle"

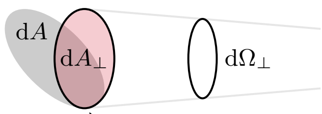
Simple 1D Ray:

$$\mathrm{d}\Omega = \pi\,\mathrm{d}\theta^2$$

General 3D Ray:

$$\mathrm{d}\Omega = \mathrm{d}\cos(\theta)\,\mathrm{d}\phi = \mathrm{d}\mu\,\mathrm{d}\phi$$

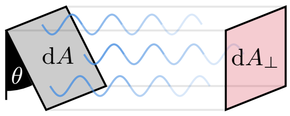
Solid Angle Subtended by a Source:



Projected Area

$$\mathrm{d}\Omega_\perp = \frac{\mathrm{d}A_\perp}{r^2}$$

Projected Area:



$$\mathrm{d}A_\perp = \cos(\theta)\,\mathrm{d}A = \mu\,\mathrm{d}A$$

Intensity and Specific Intensity:

The specific intensity is the energy radiated from a source per time per projected area per solid angle per frequency. The total intensity is integrated across all frequencies...

$$I_\nu = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A_\perp\,\mathrm{d}\Omega\,\mathrm{d}\nu} \qquad I = \int_0^\infty I_\nu\,\mathrm{d}\nu = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A_\perp\,\mathrm{d}\Omega}$$

"Without relativity, the specific intensity I_ν is constant as light propagates through empty space"

Luminosity from Intensity:

$$\mathrm{d}L = \iint_\Omega \mathrm{d}A_\perp I\,\mathrm{d}\Omega = \iint_{\phi\mu} \mathrm{d}A_\perp I\,\mathrm{d}\mu\,\mathrm{d}\phi$$

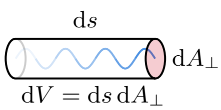
Flux from Intensity:

$$F = I\,\mathrm{d}\Omega_\perp \qquad \mathrm{d}F = \mathrm{d}I\,\mathrm{d}\Omega_\perp$$

Propagation of Light Through a Medium I

Emission Coefficient:

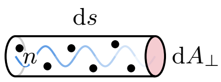
$$j_\nu = \frac{\mathrm{d}E}{\mathrm{d}V\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\nu}$$



$$\mathrm{d}I_\nu = j_\nu\,\mathrm{d}s$$

Increase in beam intensity due to emission over a path length ds

Absorption Coefficient: *"Increase" in beam intensity due to absorption over a path length ds*



$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu\,\mathrm{d}s$$

Particle Cross Section

$$\alpha_\nu = n\sigma_\nu = \frac{1}{\ell_\nu}$$

Mean Path Length

The Radiative Transfer Equation:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu$$

Optical Depth:

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s \qquad \tau_\nu = \int_0^s \alpha_\nu(s')\,\mathrm{d}s'$$

"In an opaque optically thick medium, $\tau_\nu \gg 1$, whereas in a transparent optically thin medium, $\tau_\nu \ll 1$"

Source Function:

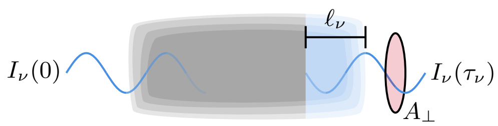
$$S_\nu = \frac{j_\nu}{\alpha_\nu} \implies \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu$$

The general solution is...

$$I_\nu(\tau_\nu) = I_\nu(0)\mathrm{e}^{-\tau_\nu} + \mathrm{e}^{-\tau_\nu}\int_0^{\tau_\nu} S_\nu(\tau'_\nu)\mathrm{e}^{\tau'_\nu}\,\mathrm{d}\tau'_\nu$$

Transmitted intensity — *Absorption diminishing* — *Integrated emission throughout the medium diminished by absorption*

Optically Thick Homogeneous Medium: $\tau_\nu \gg 1$



$$I_\nu(\tau_\nu) = S_\nu$$

"We only see photons emitted from the volume of a single ℓ_ν"

Propagation of Light Through a Medium II

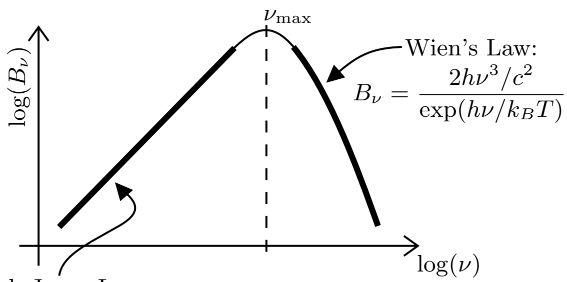
Blackbody Radiation:

Blackbody radiation is emitted by optically thick mediums in thermal equilibrium, caused by many photon interactions. The source function is the Planck function...

$$I_\nu = S_\nu = B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_BT) - 1}$$

Photon Temperature

The Plank Function:



Rayleigh-Jeans Law:

$$B_\nu = \frac{2\nu^2}{c^2}k_BT$$

Brightness Temperature

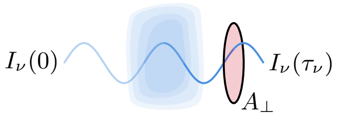
Stefan-Boltzmann Law:

$$L = \sigma A T^4$$

Wien's Displacement Law:

$$h\nu_{\max} = 2.82\,k_BT$$

Optically Thin Homogeneous Medium: $\tau_\nu \ll 1$



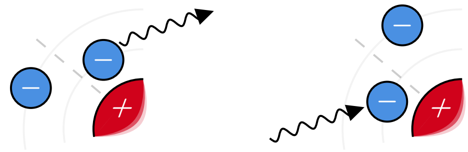
$$I_\nu(\tau_\nu) = I_\nu(0) + \big[S_\nu - I_\nu(0)\big]\tau_\nu$$

"We see photons from everywhere"

Emission: $S_\nu > I_\nu(0)$ Absorption: $I_\nu(0) > S_\nu$

Emission and Absorption Mechanics:

Emission: Absorption:

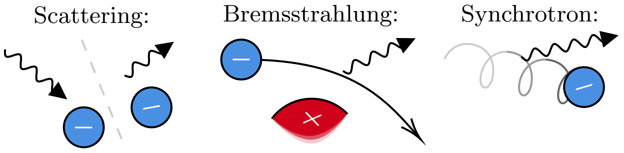


"Bound-Bound"

Recombination: Ionisation:

"Bound-Free"

Scattering: Bremsstrahlung: Synchrotron:

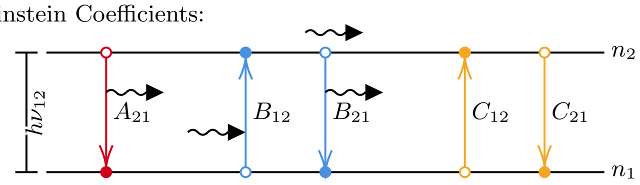


"Free-Free"

"Emission and absorption lines originate from bound electrons moving to lower and higher bound levels respectively."

The Two-Level Atom

Einstein Coefficients:



"Transitions are quantified by the Einstein coefficients"

A - Spontaneous Radiative Decay:

The energy of emitted photons are drawn from a line profile...

$$\phi(\nu) \approx \delta(\nu - \nu_{12}) \implies j_\nu = \frac{\phi(\nu)}{4\pi} h\nu n_2 A_{21}$$

B - Absorption and Stimulated Emission:

$$\mathbb{P}_{12} = B_{12}\int_0^\infty u_\nu \phi(\nu)\,\mathrm{d}\nu \approx B_{12}u_\nu$$

Radiative Energy Density:

$$u_\nu = \frac{1}{c}\int_0^\infty I_\nu\,\mathrm{d}\Omega = \frac{4\pi}{c}I_\nu$$

C - Collisional Excitation and Deexcitation:

$$\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1}\exp\left[-\frac{h\nu_{12}}{k_B T_e}\right]$$

Electron Temperature

Statistical Equilibrium:

In Statistical Equilibrium, the number of electrons leaving level one is the same as the number entering...

$$n_2\big[A_{21} + u_{\nu_{12}}B_{21} + n_e C_{21}\big] = n_1\big[u_{\nu_{12}}B_{12} + n_e C_{12}\big]$$

Thermodynamic Equilibrium:

Thermodynamic Equilibrium (TE) is the balance of processes in blackbodies, such that there are no lines. This is rare, but Local Thermodynamic Equilibrium (LTE) can occur in nature.

Optically Thin Emission Line-Dominated Medium:

> ***Planetary Nebula*** *are emission line strong diffuse clouds of gas lost from a proto-white dwarf, that can be modelled as an optically thin, emission line-dominated medium.*

Emission Line Flux:

$$L_{21} = n_2 A_{21}\ h\nu_{21}\ V \implies F_{21} = \frac{n_2 A_{21}\ h\nu_{21}\ V}{4\pi D^2}$$

Critical Decay Density:

$$n_e^* = \frac{A_{ij}}{C_{ij}} \qquad i > j$$

Low Density: $n_e < n_e^*$ High Density: $n_e > n_e^*$

"In low-density regimes, spontaneous decay dominates collisional. In high-density, the opposite applies, but, there are still enough spontaneous decays to produce emission lines."

Atomic Physics

Occupation Rules:

"No two electrons can have the same quantum numbers"

$n = 1,\ 2,\ 3,\ \ldots \qquad \ell = 0,\ 1,\ \ldots,\ n-1 \qquad -\ell \leqslant m_\ell \leqslant +\ell$

Term Symbol:

$$n^{(2s+1)}\mathrm{L}_j$$

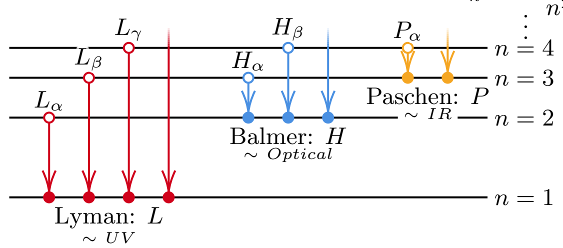
L : S P D F G $\cdots$
ℓ : 0 1 2 3 4

Degeneracy:

$$g = (2\ell+1)(2s+1) \qquad g = (2j+1)$$

Hydrogen Transitions:

$$E_n \approx -\frac{13.6}{n^2}\ \mathrm{eV}$$



Selection Rules for Transitions:

High order electric dipole mechanisms enable likely transitions, whereas low order fine structure magnetic dipole mechanisms unlock unlikely transitions...

High A Permitted > *Medium A* Semi-Forbidden > *Low A* Forbidden

Nomenclature:

Singly Ionised Semi-Forbidden

X I X II] [X III]

Neutral Permitted *Doubly Ionised Forbidden*

Measuring Elemental Abundances:

> ***Supernovas*** *are emission line strong explosions from a dying star. Using observed flux ratios of forbidden emission lines, elemental abundances can be measured.*

The Three-Level Atom

Flux Ratios:

$$\frac{F_{ik}}{F_{km}} = \frac{n_i}{n_k}\frac{A_{ik}}{A_{km}}\frac{\nu_{ik}}{\nu_{km}} \qquad i > k > m$$

Diagnostics:

Flux Ratios allow temperature T_e and density n_e diagnostics...

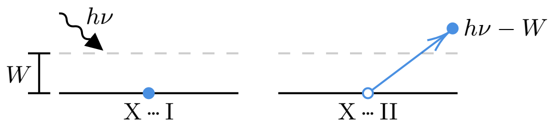
$$T_e : \frac{F_{32}}{F_{21}} \qquad n_e : \frac{F_{31}}{F_{21}}$$

*Low-density regimes can model **Planetary Nebula**, which allows for testing of the expected temperature and density...*

$$T_e^{PN} \approx 10^4 \text{ K} \qquad n_e^{PN} \approx 10^{10} \text{ m}^{-3}$$

Ionisation and Recombination

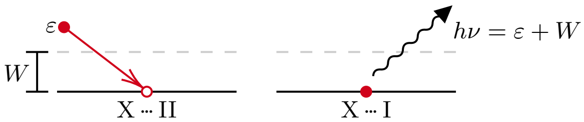
Photoionisation:



$$\begin{array}{c}\text{Ionisations from X - I} \to \text{X - II} \\ \text{per second}\end{array} = \int_0^\infty \frac{4\pi I_\nu}{h\nu}\, \sigma_I^{\rm bf}(\nu)\, \mathrm{d}\nu$$

Bound-Free cross-section of X - I

Photorecombination:



$$\begin{array}{c}\text{Recombinations from X - II} \to \text{X - I} \\ \text{per second per volume}\end{array} = n_e n_{I+1}\, \alpha_r(T_e)$$

Recombination Coefficient $\alpha_r \propto T_e^{-1/2}$

Collisional:

"There are also collisional mechanisms for ionisation and recombination, however, they activate at high n_e."

Ionisation Balance:

$$\text{Ionisation Rate} = \text{Recombination Rate}$$

*In low n_e diffuse **Planetary Nebula** with a central star emitting photons with intensity $I_\nu(0)$ at a rate $\dot{N}_\gamma$...*

$$\dot{N}_\gamma \geqslant \int_0^\infty \frac{4\pi I_\nu(0)}{h\nu}\, \sigma_I^{bf}(\nu)\, \mathrm{d}\nu = n_e n_{I+1}\, \alpha_r(T_e) V$$

Hydrogen Recombination:

"Hydrogen recombination lines occur when electrons cascade to the groundstate, due to dominant A coefficients, after recombining with ionised Hydrogen H II*"*

Recombination into k — *Spont. Decay out of k*

$$n_e n_{\rm HII}\, \alpha_{r,k} + \sum_{i>k} n_i A_{ik} = n_k \sum_{m<k} A_{km}$$

Spont. Decay into k $\quad \ldots \implies \dfrac{F_{ik}}{F_{km}} = \text{const.}$

Balmer Decrement:

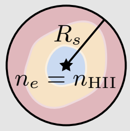
$$\frac{F(H_\alpha)}{F(H_\beta)} = 2.86 \left(\frac{T_e}{10^4 \text{ K}}\right)^{-0.07}$$

Hydrogen Alpha Photons:

$$\frac{\dot{N}_\gamma}{\dot{N}_{H_\alpha}} \approx 2.2$$

Strömgren Sphere:

*In **Planetary Nebula**, the* H II *regions have strong hydrogen recombination lines. The Strömgren sphere model assumes a sphere of purely ionised hydrogen gas...*



$$\dot{N}_\gamma \approx n_{\rm HII}^2 \alpha_r \frac{4}{3}\pi R_s^3 \approx 2.2\, \frac{L_{\rm H\alpha}}{h\nu_{\rm H\alpha}} = 2.2\, \frac{4\pi D^2 F_{\rm H\alpha}}{h\nu_{\rm H\alpha}}$$

$$\implies R_s = \left(\frac{6.6 D^2 F_{\rm H\alpha}}{\alpha_r n_{\rm HII}^2 h\nu_{\rm H\alpha}}\right)^{1/3}$$

Absorption Lines

Transmitted Specific Intensity:

$$\text{Absorption: } S_\nu \ll I_\nu(0) \implies I_\nu(\tau_\nu) = I_\nu(0)\mathrm{e}^{-\tau_\nu}$$

Column Density:

Column density N_1 describes everything material intrinsic...

$$\tau_\nu = \int_0^s \alpha_\nu(s')\, \mathrm{d}s' = \sigma_\nu \int_0^s n_1(s')\, \mathrm{d}s' = \sigma_\nu N_1$$

Relation to the Einstein Coefficients:

Stimulated Emission

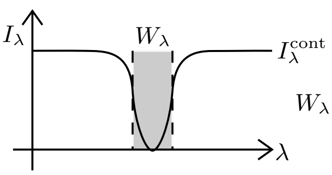
$$\alpha_\nu = [n_1 B_{12} - n_2 B_{21}]\, \frac{h\nu}{c}\phi(\nu)$$

$$\implies \sigma_\nu = \frac{\alpha_\nu}{n_1} = B_{12}\frac{h\nu}{c}\phi(\nu)\left[1 - \frac{n_2}{n_1}\frac{B_{21}}{B_{12}}\right]$$

Pure Absorption

Radio:	Optical:
Small $\nu_{12} \implies n_2 \sim n_1$	Large $\nu_{12} \implies n_2 \ll n_1$
"Anti-Absorption Line"	*"Absorption Line"*
	$\implies \sigma_\nu = B_{12}\dfrac{h\nu}{c}\phi(\nu)$

Equivalent Width:



$$W_\lambda = \int_0^\infty \frac{I_\lambda^{\rm cont} - I_\lambda}{I_\lambda^{\rm cont}}\, \mathrm{d}\lambda = \int_0^\infty 1 - \mathrm{e}^{-\tau_\lambda}\, \mathrm{d}\lambda$$

$$\underset{\tau_\lambda = \tau_\nu \ll 1}{\textit{Optically Thin}} \implies W_\lambda = \int_0^\infty \tau_\nu\, \mathrm{d}\lambda = N_1 B_{12} h \int_0^\infty \frac{\delta(\nu - \nu_{12})}{\nu}\, \mathrm{d}\nu$$

$$W_\lambda = N_1 B_{12}\frac{h}{\nu_{12}} = N_1 B_{12}\frac{h\lambda_{12}}{c}$$

Excitation Temperature:

We define the excitation temperature T_X as temperature if a medium were in Local Thermal Equilibrium...

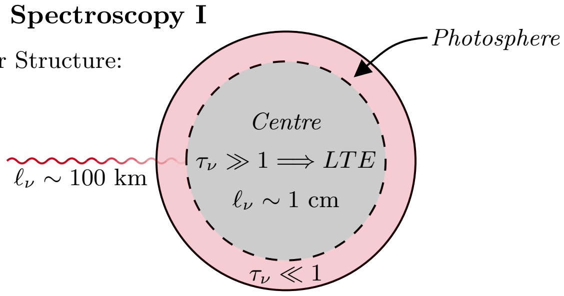
$$T_X : \frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\frac{h\nu_{12}}{k_B T_X}\right]$$

Lyman Alpha Forest:

"When observing a distant object, clouds of gas at different redshifts each contribute their own Lyman alpha absorption lines, create a forest of them."

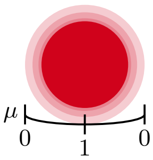
Stellar Spectroscopy I

Stellar Structure:



Limb Darkening:

"Limb darkening is where a star appears brighter and hotter at its centre than at its edges because sight lines to the centre see deeper into the star than the edges, with the temperature of the blackbody reducing with radius"



Radial Optical Depth:

$$\mathrm{d}\mathcal{T}_\nu = -\mu\,\mathrm{d}\tau_\nu \implies \mu\frac{\mathrm{d}I_\nu}{\mathrm{d}\mathcal{T}_\nu} = -(S_\nu - I_\nu)$$

$$\textit{Optically Thick Star} \implies I_\nu^{\mathrm{out}}(\mu) = \frac{1}{\mu}\int_0^\infty S_\nu(\mathcal{T}_\nu)\,\mathrm{e}^{-\mathcal{T}_\nu/\mu}\,\mathrm{d}\mathcal{T}_\nu$$

"Weighted to small radial optical depths"

The Eddington-Barber Relation:

We assume a source function of the form $S_\nu(\mathcal{T}_\nu) = a_\nu + b_\nu \mathcal{T}_\nu \ldots$

$$I_\nu^{\mathrm{out}}(\mu) = a_\nu + b_\nu\mu \equiv S_\nu(\mu)$$

"We see down to a radial optical depth of $\mathcal{T}_\nu = \mu$"

The Grey Atmosphere Limb Darkening Law:

We approximate to a grey atmosphere (drop the ν) and Eddington star (isothermal ideal gas), which gives the grey source function...

$$S(\mathcal{T}) = \frac{3F_s}{4\pi}\left[\frac{2}{3} + \mathcal{T}\right] \implies \frac{I^{out}(\mu)}{I^{out}(1)} \equiv \frac{S(\mu)}{S(1)} = \frac{3}{5}\left[\frac{2}{3} + \mu\right]$$

$$I^{\mathrm{out}} \propto 2/3 + \mu \propto T^4$$

Effective Temperature:

$$T_{\mathrm{eff}} = T(\mu = 2/3)$$

Stellar Spectroscopy II

Fraunhofer Lines:

"Fraunhofer lines are optical absorption lines in stellar spectra, composed of different absorption lines"

Spectral Classification:

O$_{0-9}$ **B**$_{0-9}$ **A**$_{0-9}$ **F**$_{0-9}$ **G**$_{0-9}$ **K**$_{0-9}$ **M**$_{0-9}$

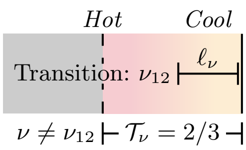
$40,000$ K $\longleftarrow T_{\mathrm{eff}}$ — $1,500$ K

Compactness: $g = \dfrac{GM_*}{R_*^2}$

Least Dense I II III IV V VI VII *Most Dense*

"Our Sun is **G**2V*"*

The Origin of Stellar Absorption Lines:



"We typically observe a blackbody at a depth of $\mathcal{T}_\nu = 2/3$ into the star. But, for transition frequencies with larger absorption coefficients α_ν, we see less deeply into the star, corresponding to a cooler blackbody and a drop in observed intensity I_ν."

Balmer Absorption Lines:

The equivalent width of a Balmer ($k > 2$) line is...

$$W_\lambda \propto \lambda_{2k} B_{2k} n_2 = \lambda_{2k} B_{2k}\,\underbrace{\frac{n_2}{n_{\mathrm{HI}}}}_{\textit{Excitation Fraction}}\,\underbrace{\frac{n_{\mathrm{HI}}}{n_{\mathrm{H}}}}_{\textit{Bound Fraction: } 1-\zeta}\,n_{\mathrm{H}}$$

Excitation Fraction:

Using the Boltzmann distribution, n_2/n_{HI} can be calculated...

$$n_j \propto g_j \exp\left(\frac{|E_j|}{k_B T}\right) \implies \frac{n_2}{n_{\mathrm{HI}}} = \frac{4\exp\left[3.4\text{ eV}/k_B T_{\mathrm{eff}}\right]}{\sum_i i^2 \exp\left[13.6\text{ eV}/i^2 k_B T_{\mathrm{eff}}\right]}$$

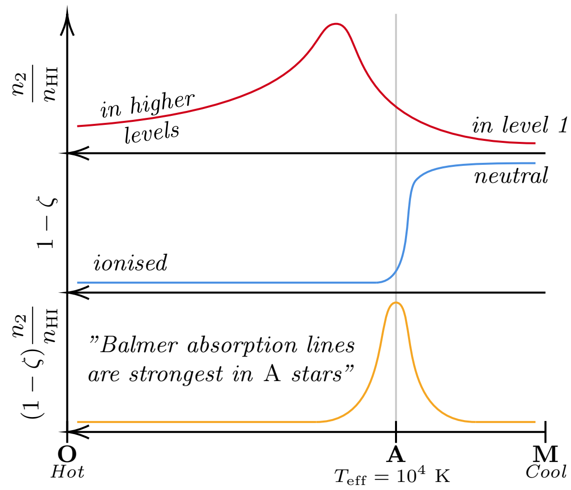
Ionisation Fraction:

Using the Saha equation, ξ can be solved...

$$\frac{n_{\mathrm{HII}}}{n_{\mathrm{HI}}} = \frac{1}{n_e}\left(\frac{2\pi m_e k_B T_{\mathrm{eff}}}{h^2}\right)^{3/2}\exp\left[-\frac{13.6\text{ eV}}{k_B T_{\mathrm{eff}}}\right]$$

$$\cdots \implies \frac{\xi^2}{1-\xi} = \frac{1}{n_{\mathrm{H}}}\left(\frac{2\pi m_e k_B T_{\mathrm{eff}}}{h^2}\right)^{3/2}\exp\left[-\frac{13.6\text{ eV}}{k_B T_{\mathrm{eff}}}\right]$$

Resulting Equivalent Width:



Relativistic Effects

Doppler Beaming:

Doppler beaming is the increase (decrease) of observed flux if the emitter is moving towards (away from) the observer...

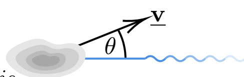
$$F^o_{\nu_o} = \delta^3 I^e_{\nu_e}\,\mathrm{d}\Omega_e$$

"Length Contraction: δ^2; Time Dilation: δ"

$$\delta = \frac{\nu_o}{\nu_e} = \frac{1}{1+z}$$

Approaching:	Receding:
$\delta > 1 \implies$ *Blue Shift*	$\delta < 1 \implies$ *Red Shift*

The Doppler Factor:



In special relativity, the Doppler factor is...

$$\delta = \frac{1}{\gamma\left[1 - \beta\cos(\theta)\right]}$$

$$\text{Lorentz Factor: } \gamma = \frac{1}{1 - v^2/c^2} = \frac{1}{1-\beta^2}$$

Specific Intensity in Special Relativity:

The total observed specific intensity is...

$$I^o = \delta^4 I^e$$

"Photon Energy: δ"

Relativistic Aberration:

$$\theta_{\max} = \frac{1}{\gamma}$$

"Rapidly moving emitters collapse their photons into a tiny beam of half-opening angle $1/\gamma$"

Shocks

Taylor-Sedov Blast Wave:

Explosion → Expanding Fireball → Shock

Consider a blast wave after an explosion of energy E into a medium of ambient density ρ after a time t...

$$[r] = \mathrm{L} \qquad [E] = \mathrm{ML^2T^{-2}} \qquad [\rho] = \mathrm{ML^{-3}} \qquad [t] = \mathrm{T}$$

Using dimensional analysis, the radius is...

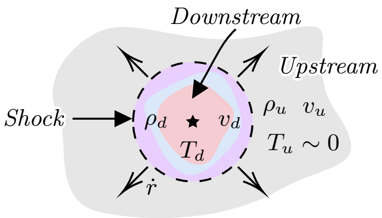
$$r = C\left(\frac{Et^2}{\rho}\right)^{1/5} \qquad C \approx 1$$

*With this, energy of the **Trinity Atomic Test** and velocity of the **Grab Supernova Remnant** can be estimated*

Shocks:

"Shocks form when the blast wave moves faster than the speed of sound in the medium. Shocks are physically a discontinuity in the density, pressure, etc..."

Strong Shock Jump Conditions:



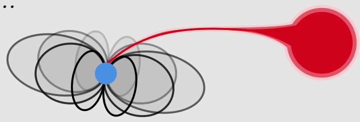
Consider conserving mass, momentum and energy per unit area flowing across the shock front...

$$\rho_d = 4\rho_u \qquad v_d = \frac{1}{4}v_u$$

"After the blast wave, the shocked material is 4 times as dense but slows to be 1/4 times as fast"

Polar White Dwarf:

*Material accredited from a companion star can be streamed onto a **Polar White Dwarf**, causing a strong shock on the surface...*



Shock Acceleration

Cosmic Rays:

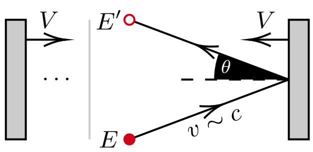
Highly relativistic elementary particles from space. We observe a power law spectrum and peak energy...

$$\frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{-k} \qquad E_{\max} \sim 10^{20}\ \mathrm{eV}$$

$k \approx 2.5$

Fermi Acceleration:

Fermi's mechanism considers particles reflecting off constantly incoming walls, causing a fractional energy change...



$$\implies \frac{\Delta E}{E} = \frac{V}{c}\cos(\theta)$$

After n collisions, there are $N_n = N_0 P^n$ remaining particles with energy $E_n = E_0\Theta^n$. The resulting spectrum is...

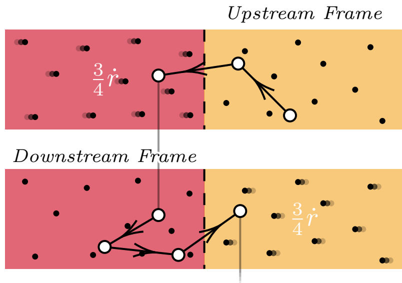
$$\frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{\ \ln(P)/\ln(\Theta) - 1} \qquad k = 1 - \frac{\ln(P)}{\ln(\Theta)}$$

Second Order Fermi Acceleration:

"Fermi's original astrophysical model of Fermi acceleration employs randomly moving magnetic mirrors. However, as $\langle \Delta E/E\rangle \propto (V/c)^2$, this cannot explain the universally observed cosmic ray spectrum."

Diffuse Shock Acceleration: *"First Order"*

"Diffusive shock acceleration involves particles repeatedly crossing a strong shock front, gaining energy from reflections off the material approaching in each rest frame."



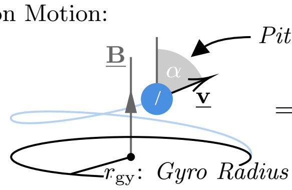
$$\langle \Delta E/E\rangle \propto \frac{V}{c} \implies \frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{-2}$$

Synchrotron Radiation I

Synchrotron Radiation:

*When ultra-relativistic electrons spiral around statics magnetic field lines, they emit synchrotron Radiation. These high-energy electrons can be created by diffusive shock acceleration from **supernovae** or **black hole jets**.*

Electron Motion:



$$\implies \underline{\mathbf{a}} = -\frac{evB\sin(\alpha)}{\gamma m_e}\,\hat{\mathbf{r}}$$

Gyro Radius:

$$r_{\mathrm{gy}} = \frac{\gamma m_e v\sin(\alpha)}{eB}$$

Gyro Period:

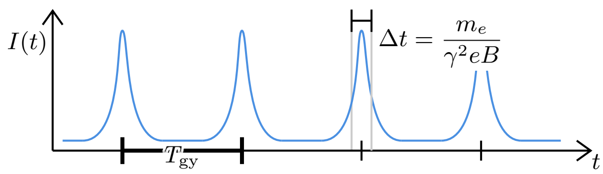
$$T_{\mathrm{gy}} = \frac{2\pi\gamma m_e}{eB}$$

Synchrotron Luminosity:

By applying a Lorentz transformation to the classical Larmor formula, the average luminosity radiated by an electron over all pitch angles is...

$$L = \frac{e^4}{9\pi m_e^2 \varepsilon_0 c}\gamma^2\beta^2B^2 \propto \gamma^2\beta^2B^2$$

Single Electron Spectrum:

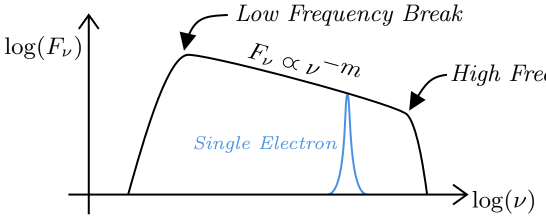


The specific intensity $I_\nu(\nu)$ is found using a Fourier transform of $I(t)$. The result is a fairly narrow peak centred about the observed synchrotron frequency...

$$\nu_c = \frac{1}{\Delta t} \approx \frac{\gamma^2 eB}{m_e}$$

Synchrotron Radiation II

Observed Spectrum:



Power Law:

Consider specific flux from specific luminosity...

$$L_\nu = L(\gamma)\frac{\mathrm{d}n_e}{\mathrm{d}\nu}V \implies F_\nu \propto L_\nu \propto L(\gamma)\frac{\mathrm{d}n_e}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\nu}$$

From diffusive shock acceleration with $E = \gamma m_e c^2$...

$$\frac{\mathrm{d}n_e}{\mathrm{d}E} \propto E^{-k} \implies \frac{\mathrm{d}n_e}{\mathrm{d}\gamma} \propto \gamma^{-k}$$

$$F_\nu \propto \nu^{-(k-1)/2} = \nu^{-m} \qquad m = \frac{k-1}{2}$$

"From DSA, where $k = 2$, we find $m = 1/2$"

Low Frequency Break: *"Self Absorption Break"*

From self-absorption, the plasma is optically thick for low-frequency photons and optically thin for those above the break.

High Frequency Break:

"$\mathrm{d}n_e/\mathrm{d}\gamma \propto \gamma^{-k}$ breaks down near the highest γ electrons"

Limits of DSA:

Immediately after DSA, the electron gyro radius sets to the shock front acceleration radius R...

$$r_{\text{gy}} = \frac{\gamma_{\max} m_e v \sin(\alpha)}{eB} \equiv R \xRightarrow[v\approx c]{\alpha\approx\pi/2} \gamma_{\max} = \frac{ReB}{m_e c}$$

$$\nu_{\max} = \left(\frac{eB}{m_e}\right)^3 \left(\frac{R}{c}\right)^2 \propto B^3 R^2$$

Cooling Break:

Emitting synchrotron radiation drains electron energy...

$$t_{\text{cool}} = \frac{E}{\mathrm{d}E/\mathrm{d}T} = \frac{\gamma m_e c^2}{L(\gamma)} = \frac{9\pi m_e^3 c^3 \varepsilon_0}{e^4 \gamma B^2}$$

"Highest energy electrons radiate away their energy fastest"

$$\nu_{\max} \propto t^{-2} B^{-3}$$

Spectral Ageing:

*Spectral ageing exploits the time evolution of the high frequency break to measure the time since plasma was last accelerated. By applying this to galaxies with supermassive black holes, such as **Cygnus A**, we see jet ejection is a short-lived phase of $\sim 10^8$ years...*

Black Holes

Event Horizon:

A Black Hole has all its mass in a singularity, and within its event horizon not even light can escape...

$$r_h = \frac{2GM}{c^2} = 2r_g$$

$$\text{Gravitational Radius: } r_g = \frac{GM}{c^2}$$

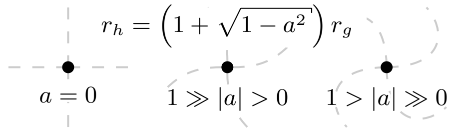
Spin Parameter:

The spin parameter a describes the BH's angular momentum...

$$a = \frac{J}{Mcr_g} \qquad -1 \leqslant a \leqslant +1$$

For a spinning black hole, the event horizon gets closer to the singularity due to frame dragging...

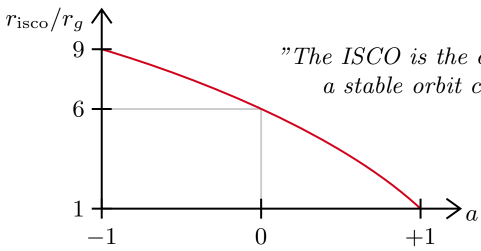
$$r_h = \left(1 + \sqrt{1-a^2}\right) r_g$$



Keplerian Velocity:

$$v_\phi = \sqrt{\frac{GM}{r}}$$

Innermost Stable Circular Orbit:



"The ISCO is the closest radius a stable orbit can exist"

Accreting Black Holes:

***Active Galactic Nuclei** (AGNs) are supermassive BHs at the centre of a galaxy accreting mass from the host galaxy. **X-ray Binaries** (XBs) are stellar BHs accreting mass from a binary companion star.*

The accretion luminosity from losing gravitational energy is...

$$L = \eta \dot{M} c^2$$

$$\text{Accretion Efficiency: } \eta = \frac{1}{2}\frac{1}{r_{\text{isco}}/r_g}$$

"Accretion on a BH is by far the most efficient rest mass to energy conversion mechanism"

Eddington Luminosity:

Assuming the accretion disk is a proton-electron plasma, the gravitational force acts on the heavier proton...

$$F_{grav} = \frac{GMm_p}{r^2}$$

Whereas the radiation force on an electron is...

$$F_{rad} = \frac{\delta_T}{4\pi r^2}\frac{L}{c}$$

As they're both strongly electrically bound and the disk is neutral, they can be considered together...

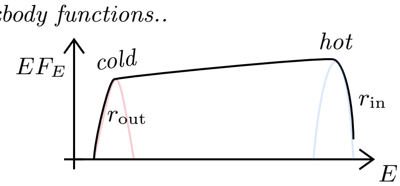
The maximum theoretical luminosity of an accreting BH is...

$$L_{\text{Edd}} = \frac{4\pi GMm_pc}{\delta_T} \implies \dot{M}_{\text{Edd}} = \frac{L_{\text{Edd}}}{\eta c^2}$$

Accretion Disks

Disc Spectrum:

The spectrum of an accretion disk is a sum of different temperature blackbody functions..



Peak Disk Temperature:

$$\sigma T_{\text{in}}^4 = \frac{L}{4\pi r_{\text{in}}^2}$$

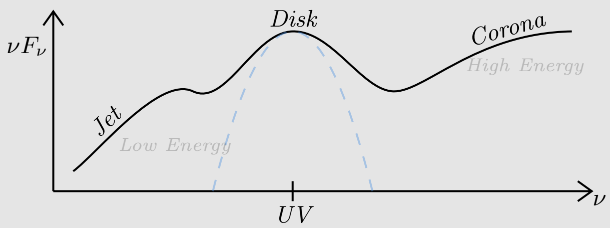
$$\text{Eddington Ratio: } \ell \equiv \frac{L}{L_{\text{Edd}}}$$

For any BHs with fixed spin (r_{isco}/r_g) and Eddington ratio ℓ...

$$T_{\text{in}} \sim M^{-1/4}$$

"AGNs have a much lower temperature than XBs"

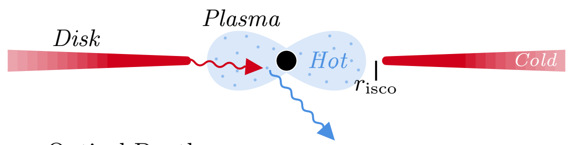
Observations of Disks:

> *AGNs never display a pure disk spectrum...*
>
> 
>
>
> *However, XBs do when in a soft state, allowing for the measurement of its spin using observed disk temperature and luminosity...*

Compton Scattering

X-Ray Binary Hard State:

The hard state spectrum is dominated by Compton scattering of photons by hot electrons in the BH's corona...



Thomson Optical Depth:

$$\tau_T = \int_0^s n_e \sigma_T \, \mathrm{d}s' \qquad \textit{Isotropic: } \tau_T = n_e \sigma_T s$$

"The fraction of photons that pass through a medium without experiencing a scattering is $\mathrm{e}^{-\tau_T}$, whereas those that experience at least k scatterings is $(1-\mathrm{e}^{-\tau_T})^k$"

Fractional Energy Transfer:

$$\left\langle \frac{\Delta\varepsilon}{\varepsilon} \right\rangle = \left\langle \frac{\Delta\varepsilon}{\varepsilon} \right\rangle_T - \left\langle \frac{\Delta\varepsilon}{\varepsilon} \right\rangle_{\text{rec}} = \frac{4}{3}\gamma^2\beta^2 - \frac{h\nu}{m_e c^2}$$

Thomson *Compton*

Thermal Comptonisation:

In the Corona, the electrons are in local thermal equilibrium...

$$\frac{1}{2} m_e \langle v \rangle^2 = \frac{3}{2} k_B T_e \implies \beta^2 = \frac{\langle v \rangle^2}{c^2} = 3\frac{k_B T_e}{m_e c^2} = 3\Theta_e$$

$$\left\langle \frac{\Delta\varepsilon}{\varepsilon} \right\rangle = \frac{4}{3}\beta^2 - \frac{h\nu}{m_e c^2} = 4\Theta_e - \frac{h\nu}{m_e c^2} \qquad \Theta_e = \frac{k_B T_e}{m_e c^2}$$

Corona Spectrum:

The emergent Corona spectrum is a power law with a high-frequency cut off...

$$F \propto h\nu \frac{\mathrm{d}N}{\mathrm{d}\nu} \propto \nu^{-m} \qquad m = -\frac{\log(1-e^{-\tau_T})}{\log(1+4\Theta_e)}$$

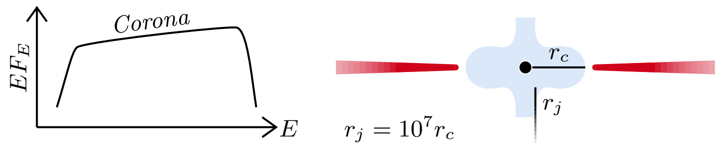
"Cut off at $h\nu \sim 4k_B T_e$"

Jets

X-Ray Binary Jets:

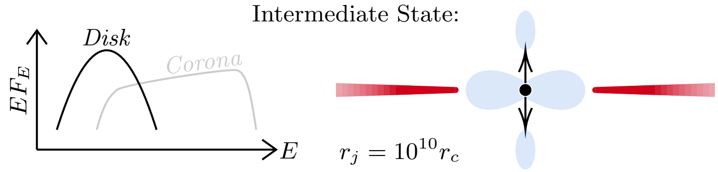
X-Ray Binary's emit Jets during outburst. The system goes from quiescence to a hard state, intermediate state then soft state; looping back again...

Hard State:

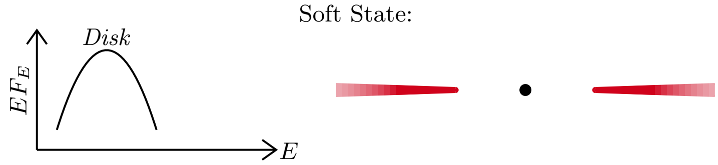


"Jet is ~ 10 million times the size of the corona"

Intermediate State:



Soft State:



Superluminal Jet Motion:

"Transient jet ejections can appear to move across the sky faster than the speed of light due to it approaching us"

Apparent Speeds:

$$\beta_{\text{app}} = \frac{\beta \sin(\theta)}{1-\beta\cos(\theta)} \qquad \beta_{\text{red}} = \frac{\beta \sin(\theta)}{1+\beta\cos(\theta)}$$

"The approaching blob moves faster than the receding blob"

> *By comparing the relative speeds of the approaching and receding blobs, we can infer upper limits ($\beta = 1$) for the inclination angle and the distance to the X-ray binary.*

Blob Brightness:

$$\frac{I_\nu^{\text{app}}}{I_\nu^{\text{rec}}} = \left(\frac{\delta_{\text{app}}}{\delta_{\text{rec}}}\right)^{3+m} = \left(\frac{1+\beta\cos(\theta)}{1-\beta\cos(\theta)}\right)^{3+m}$$

"The approaching blob appears brighter than the receding due to Doppler boosting"

AGN Unification:

"Different classes of AGN are the same system viewed from different angles and/or at different times"

Jet Production Mechanisms:

"Jets are thought to be launched by magnetic fields being twisted into a corkscrew shape"