Distance and Metrics

Planar Geometry Metrics ($\mathrm{d}s$):

Cartesian: $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ Polar: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$

Relativity

Reference Frame:

"Set of spatial coordinates x, y, z and a time coordinate t"

Inertial Frame:

"A reference frame in which $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$"

Non-Inertial Frame:

"A reference frame in which $\underline{\mathbf{F}} \neq m\underline{\mathbf{a}}$...

$\implies$ *a fictitious force $\underline{\mathbf{F}}_i = m\underline{\mathbf{a}} - \underline{\mathbf{F}}$ exists"*

Transformations:

Time Shift: $t' = t - T$ Displacement: $\underline{\mathbf{x}}' = \underline{\mathbf{x}} - \underline{\mathbf{d}}$ Rotation: $\mathcal{R}(\underline{\mathbf{a}}, \theta)$

Uniform Motion (Galilean Transform):
$\underline{\mathbf{x}} = \underline{\mathbf{x}} - \underline{\mathbf{v}}t$ $t' = t$

"All of these transformations don't change $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$"

The Relativity Principle:

"Identical experiments carried out in different inertial frames give identical results"

Basic Postulates of Newtonian Relativity:

Identical experiments and time always takes the same value up to a constant shift...

"Universal Time"

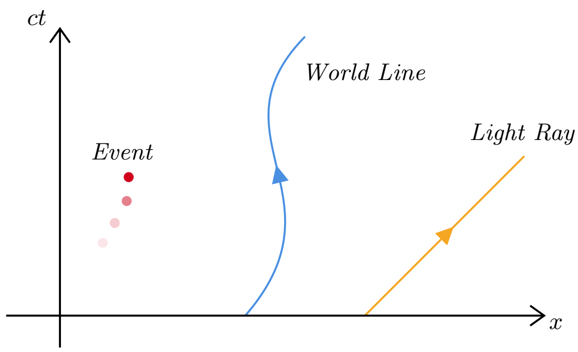
Basic Postulates of Special Relativity:

Identical experiments and the speed of light is the same in all inertial frames...

"Moving Clocks Run Slowly"

Spacetime

Spacetime Diagrams:



Spacetime Interval:

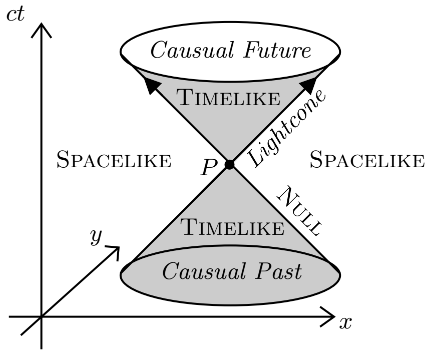
$$P : (\underline{\mathbf{x}}_p, t_p) \mapsto Q : (\underline{\mathbf{x}}_q, t_q) \implies (\Delta s)^2 = (\underline{\mathbf{x}}_q - \underline{\mathbf{x}}_p)^2 - c^2(t_q - t_p)^2$$

Separation:

$(\Delta s)^2 < 0$: *Timelike* $(\Delta s)^2 > 0$: *Spacelike*

$(\Delta s)^2 = 0$: *Null*

The Causal Structure of Spacetime:



Lorentz Transforms

Minkowsky Metric:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2 = |\,\mathrm{d}\underline{\mathbf{x}}|^2 - c^2\,\mathrm{d}t^2$$

Lorentz Transformation:

Any transformation between the inertial frames S and S' which preserve the Minkowski metric are Lorentz transforms...

$$|\,\mathrm{d}\underline{\mathbf{x}}|^2 - c^2\,\mathrm{d}t^2 = |\,\mathrm{d}\underline{\mathbf{x}}'|^2 - c^2\,\mathrm{d}t'^2$$

Lorentz Boost:

The Lorentz transformation between two inertial frames S and S' in uniform relative motion is called a Lorentz Boost...

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$

$$\text{Lorentz Factor: } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Time Dilation

Proper Time:

"The proper time τ of a body is the time measured by a clock moving with the body"

Rest Frame:

The rest frame of a body is an inertial frame moving instantaneously with the same velocity as the body, such that $\mathrm{d}\underline{\mathbf{x}} = 0$ and $\mathrm{d}t = \mathrm{d}\tau$...

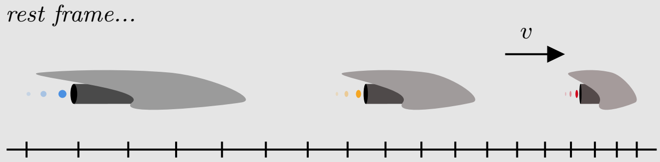
$$\mathrm{d}s^2 = \cancelto{0}{\mathrm{d}\underline{\mathbf{x}}^2} - c^2\,\mathrm{d}t = -c^2\,\mathrm{d}\tau^2$$

Time Dilation: $$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

Constant Velocity $\implies \Delta t = \gamma \Delta\tau$

Length Contraction

*The **proper length** $\ell_\star$ of a body is its length measured in its rest frame...*



Lorentz Contraction: $\ell = \dfrac{\ell_\star}{\gamma}$

"Moving Bodies Appear Shorter"

Four-Vectors

4-Vectors:

$$\underline{\mathbf{b}} = b^x\, \widehat{\mathbf{e}}_x + b^y\, \widehat{\mathbf{e}}_y + b^z\, \widehat{\mathbf{e}}_z + b^t\, \widehat{\mathbf{e}}_t$$

3-Position:

$$\underline{\mathbf{r}} = x\, \widehat{\mathbf{e}}_x + y\, \widehat{\mathbf{e}}_y + z\, \widehat{\mathbf{e}}_z$$

4-Position:

$$\underline{\mathbf{x}} = x\, \widehat{\mathbf{e}}_x + y\, \widehat{\mathbf{e}}_y + z\, \widehat{\mathbf{e}}_z + ct\, \widehat{\mathbf{e}}_t$$

Spatial Projection:

$$\underline{\mathbf{b}} = \vec{\mathbf{b}} + b^t\, \widehat{\mathbf{e}}_t = \vec{\mathbf{b}} + ct\, \widehat{\mathbf{e}}_t$$

Scalar Products:

$$\widehat{\mathbf{e}}_x \cdot \widehat{\mathbf{e}}_x = \widehat{\mathbf{e}}_y \cdot \widehat{\mathbf{e}}_y = \widehat{\mathbf{e}}_z \cdot \widehat{\mathbf{e}}_z = 1 \qquad \widehat{\mathbf{e}}_t \cdot \widehat{\mathbf{e}}_t = -1$$

Spacetime classification:

$\underline{\mathbf{b}} \cdot \underline{\mathbf{b}} < 0$: *Timelike* $\qquad$ $\underline{\mathbf{b}} \cdot \underline{\mathbf{b}} > 0$: *Spacelike*

$\underline{\mathbf{b}} \cdot \underline{\mathbf{b}} = 0$: *Null*

Kinematics

4-Velocity:

$$\underline{\mathbf{u}} = \frac{\mathrm{d}\underline{\mathbf{x}}}{\mathrm{d}\tau}$$

$$(3+1): \ \underline{\mathbf{u}} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\underline{\mathbf{x}}}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mathbf{r}} + ct\, \widehat{\mathbf{e}}_t) = \gamma\underline{\mathbf{v}} + \gamma c\, \widehat{\mathbf{e}}_t$$

4-Acceleration:

$$\underline{\mathbf{a}} = \frac{\mathrm{d}\underline{\mathbf{u}}}{\mathrm{d}\tau}$$

Normalisation and Orthogonality:

$$\underline{\mathbf{u}} \cdot \underline{\mathbf{u}} = -c^2 \qquad \underline{\mathbf{u}} \cdot \underline{\mathbf{a}} = 0$$

Forces

4-Force: *"Bootstrap Formula"*

$$\underline{\mathbf{f}} = m\underline{\mathbf{a}}$$

Momentum and Energy

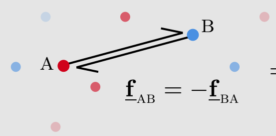
4-Momentum:

$$\underline{\mathbf{p}} = m\underline{\mathbf{u}} \qquad \underline{\mathbf{f}} = \frac{\mathrm{d}\underline{\mathbf{p}}}{\mathrm{d}\tau}$$

$$(3+1): \underline{\mathbf{p}} = \gamma m\underline{\mathbf{v}} + \gamma mc\, \widehat{\mathbf{e}}_t$$

"Moving Masses Appear Heavier"

Particle Momenta Conversation: *In an isolated system...*



$$\underline{\mathbf{f}}_{\mathrm{AB}} = -\underline{\mathbf{f}}_{\mathrm{BA}} \implies \sum_i \underline{\mathbf{f}}_i = 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\sum_i \underline{\mathbf{p}}_i$$

Energy Conversation:

$$E = cp^t = \gamma mc^2 \longrightarrow \textit{Rest Frame} \implies E = mc^2$$

$$\underline{\mathbf{p}} = \vec{\mathbf{p}} + \gamma mc\, \widehat{\mathbf{e}}_t = \vec{\mathbf{p}} + \frac{E}{c}\, \widehat{\mathbf{e}}_t$$

Normalisation:

$$\underline{\mathbf{p}}^2 = \underline{\mathbf{p}} \cdot \underline{\mathbf{p}} = -m^2c^2 \qquad E^2 = \vec{\mathbf{p}}^2c^2 + m^2c^4$$

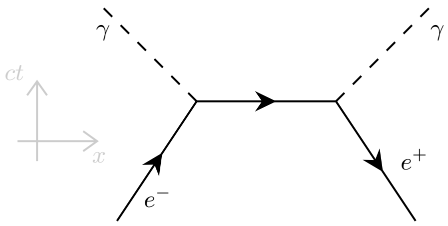
Massless Particles: *"Photons"*

$$\xcancel{\underline{\mathbf{p}} = m\underline{\mathbf{u}}} \qquad E^2 = \vec{\mathbf{p}}^2c^2 \implies |\vec{\mathbf{p}}| = \frac{E}{c}$$

Feynman Diagrams

The Feynman Rules:

- *Particles are represented by straight lines with arrows, where electrons are solid and photons are dashed (wiggly).*
- *Interactions are represented by vertices where $2e^{\mp} \to \gamma$ meet.*
- *External lines represent incoming or outgoing particles.*
- *Internal lines joining vertices represent virtual particles and can be horizontal.*
- *Momentum is conserved.*



Momentum Conversation:

"In general, try to use $\underline{\mathbf{p}}^2 = -mc^2$ with a reference frame of $\underbrace{\textit{particle rest}}_{\vec{\mathbf{p}}=0}$ or $\underbrace{\textit{centre of momenta}}_{\sum \vec{\mathbf{p}}_i = 0}$"

Particle Concepts

Fundamental Forces of Nature: $\underline{\mathbf{f}} = -e\nabla\phi$

Electromagnetism (*Coulomb*) $\longrightarrow \phi = \dfrac{e}{4\pi\varepsilon_0 r}$ $\qquad$ *Strong* $\qquad$ *Weak* (*Yukawa*) $\longrightarrow \phi = \dfrac{q}{4\pi\varepsilon_0 r}e^{-\mu r}$ $\qquad$ *Gravity*

The Standard Model:

	Fermions I	II	III	**Bosons**	
Quarks	u	c	t	g	H (Scalar)
	d	s	b	γ	
Leptons	e	μ	τ	Z	
	ν_e	ν_μ	ν_τ	W (Gauge Vector)	

*Each particle has an **antiparticle** of the same mass but opposite charge. Only the first generation fermions are stable.*

Conservation Rules:

- *Electric Charge*
- *Energy & Momentum*
- *Angular Momentum: $\Delta s = \pm\{1,2,3\}\hbar$*
- *Lepton Number: $+1$ for lepton particle; -1 for antiparticle*
- *Baryon Number: $+\frac{1}{3}$ per quark; $-\frac{1}{3}$ per antiquark*

Klein-Gordon Theory

4-Gradient:

$$\underline{\boldsymbol{\nabla}} = \widehat{\mathbf{e}}_x\frac{\partial}{\partial x} + \widehat{\mathbf{e}}_y\frac{\partial}{\partial y} + \widehat{\mathbf{e}}_z\frac{\partial}{\partial z} - \widehat{\mathbf{e}}_t\frac{1}{c}\frac{\partial}{\partial t}$$

$$\nabla^2 = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Klein-Gordon Equation:

For a field of mass m the Klein–Gordon equation is...

$$\left(-\nabla^2 + \mu^2\right)\phi = 0$$

where $\mu = mc/\hbar$

*"Solving this in polar yields the **Yukawa Potentials**"*

Plane Wave Solutions:

$$\phi(\underline{\mathbf{x}}) = u_k e^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}} \qquad \underline{\mathbf{k}}^2 + \mu^2 = 0$$

Vector Bosons

Massive Bosons:

For a massive vector boson field $\underline{\mathbf{A}}(\underline{\mathbf{r}}, t)$ the field satisfies...

$$\left(-\nabla^2 + \mu^2\right)\underline{\mathbf{A}} = \left(-\nabla^2 + \mu^2\right)\begin{bmatrix} A^t \\ A^x \\ A^y \\ A^z \end{bmatrix} = 0 \qquad \underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{A}} = 0$$

Plane Wave Solutions:

$$\underline{\mathbf{A}} = \underline{\mathbf{u}}_k e^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}} \qquad \underline{\mathbf{k}} \cdot \underline{\mathbf{u}}_k = 0 \qquad \underline{\mathbf{k}}^2 + \mu^2 = 0$$

Single Particle: $\underline{\mathbf{p}} = \hbar\underline{\mathbf{k}}$

The Dirac Equation

Spinor Field:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

"The spinor field ψ has four seperate complex field components. The field equation will now be a 4×4 matrix acting on the spinor."

Matrix of Vectors:

$$(\underline{\boldsymbol{\gamma}} \cdot \underline{\boldsymbol{\nabla}})^2 = -\mathrm{I}\nabla^2$$

The Dirac Equation: *"Fermions"*

For a spinor field ψ of mass m the Dirac equation is...

$$\big(-i\underbrace{\underline{\boldsymbol{\gamma}} \cdot \underline{\boldsymbol{\nabla}}}_{4\times4} + \mu\mathrm{I}\big)\psi = 0$$

where $\mu = mc/\hbar$

Plane Wave Solution:

$$\psi = ue^{i\underline{\mathbf{k}}\cdot\underline{\mathbf{x}}}$$

Constant Spinor

Pauli Matrices & Matrix Vectors:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}} = \mathrm{I}\ \widehat{\mathbf{e}}_t + \sigma^1\ \widehat{\mathbf{e}}_x + \sigma^2\ \widehat{\mathbf{e}}_y + \sigma^3\ \widehat{\mathbf{e}}_z$$

$$\underline{\overline{\boldsymbol{\sigma}}} = \mathrm{I}\ \widehat{\mathbf{e}}_t - \sigma^1\ \widehat{\mathbf{e}}_x - \sigma^2\ \widehat{\mathbf{e}}_y - \sigma^3\ \widehat{\mathbf{e}}_z$$

Weyl Representation:

$$\underline{\boldsymbol{\gamma}} = \left[\begin{array}{c|c} 0 & \underline{\boldsymbol{\sigma}} \\ \hline \underline{\overline{\boldsymbol{\sigma}}} & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & \sigma_{11} & \sigma_{12} \\ 0 & 0 & \sigma_{21} & \sigma_{22} \\ \overline{\sigma}_{11} & \overline{\sigma}_{12} & 0 & 0 \\ \overline{\sigma}_{21} & \overline{\sigma}_{22} & 0 & 0 \end{bmatrix}$$

$$\gamma^x\gamma^y = -\gamma^y\gamma^x \qquad (\gamma^t)^2 = \mathrm{I} \qquad (\gamma^x)^2 = (\gamma^y)^2 = (\gamma^z)^2 = -\mathrm{I}$$

Charged Fermions:

For a charged massive vector boson field $\underline{\mathbf{A}}(\underline{\mathbf{r}}, t)$ the field satisfies a modified Dirac equation...

$$\left[-i\underline{\boldsymbol{\gamma}} \cdot \left(\underline{\boldsymbol{\nabla}} + i\frac{q}{\hbar}\underline{\mathbf{A}}\right) + \mathrm{I}\mu\right]\psi = 0.\text{"}$$

Neutrinos

Chiarlity:

"A spinor field can be decomposed into left & right chiral spinors."

$$\implies \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

$$(-i\underline{\boldsymbol{\gamma}} \cdot \underline{\boldsymbol{\nabla}} + \mu)\psi = 0 \longrightarrow \begin{cases} -i\underline{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{\nabla}}\psi_R = -\mu\psi_L \\ -i\underline{\overline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{\nabla}}\psi_L = -\mu\psi_R \end{cases}$$

"Massless spinors, $\mu = 0$, are realised in nature by neutrinos"

Neutrino Masses:

A new mass term is introduced, called the Majorana mass μ_m. This is observed in neutrino oscillations...

$$\nu_e \to \nu_\mu \to \nu_e \to \ldots$$

Higgs Boson

The Higgs Field:

The Higgs field has two complex components and obeys a non-linear extension of the Klein-Gordon equation...

$$H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$-\nabla^2 h_1 - \mu_H^2 h_1 + \lambda(|h_1|^2 + |h_2|^2)h_1 = 0$$
$$-\nabla^2 h_2 - \mu_H^2 h_2 + \lambda(|h_1|^2 + |h_2|^2)h_2 = 0$$

$$\underset{\text{imaginary}}{\text{"Unphysical Mass"}} \xrightarrow{\text{Nature}} \text{"H has a non-zero vacuum constant value"}$$

$$\underset{\text{VEV}}{\text{Vacuum Expectation Value}}: v = \frac{\mu_H}{\sqrt{\lambda}}$$

The Higgs Particle:

The Higgs particle is a fluctuation ϕ of the vacuum value, such that $h_1 = v + \phi$ and $h_2 = 0$...

$$-\nabla^2\phi + 2\lambda v^2\phi = 0 \qquad m_H = \frac{\hbar}{c}\sqrt{2\lambda v^2}$$

The Higgs Potential:

$$V(H) = -\frac{1}{2}\mu_H^2 H^\dagger H + \frac{1}{4}\lambda(H^\dagger H)^2$$

$$H^\dagger H = h_1 h_1^* + h_2 h_2^* = |h_1|^2 + |h_2|^2$$

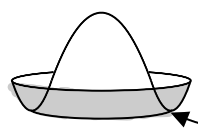
Electroweak Symmetry: $H \to UH$

2×2 Unitary Matrix $UU^\dagger = U^\dagger U = I$

The Higgs Doublet Field:

$$-\nabla^2 H + 2\frac{\partial V}{\partial H^\dagger} = 0$$

$$-\nabla^2 h_1 + 2\frac{\partial V}{\partial h_1^*} = 0 \qquad -\nabla^2 h_2 + 2\frac{\partial V}{\partial h_2^*} = 0$$



"The Higg's field will sit at the bottom of its potentail"

vev

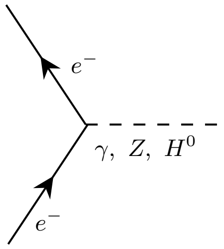
Electroweak Gauge Symmetry:

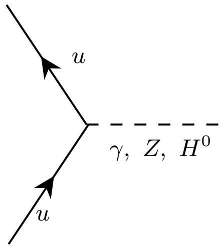
*"The **standard model** is the most general theory that has $SU(3) \times SU(2) \times U(1)$ symmetry and allows for three generations of particles"*

Electroweak Feynman Diagrams

Electromagnetism:

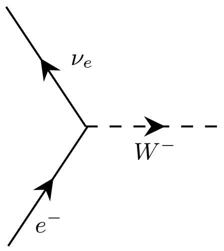
The photon interactions with the charged particles: Leptons (e, μ, τ) and Quarks (u, d, s, c, b, t)...

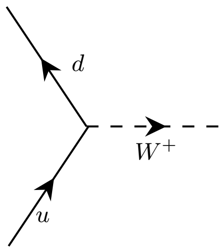


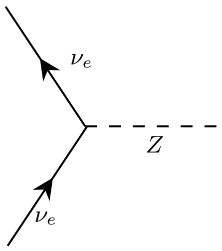


Weak Interactions:

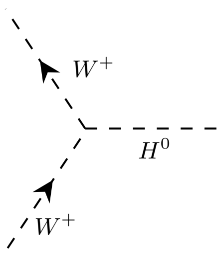
$W^\pm$ and Z bosons interact with matter particles. Z is the same as photons. If W is involved, charge must be transferred...





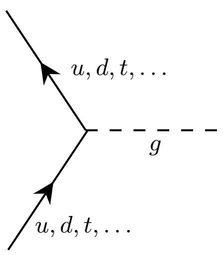


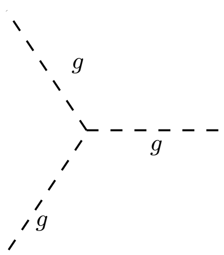
The Higgs Boson H^0 is the same as photons, with an additional 3-boson interaction...



Strong Interactions:

The gluon g interacts with quarks (or quark composites). There is an additional 3-boson interaction...





Decay Rates and Scattering Cross Sections:

Feynman diagrams aren't just descriptive. We calculate the matrix element M by multiplying by $\sqrt{\alpha}$ for each vertex and $\min(1, E^2/m^2c^4)$ for each vertical particle...

Decay Rate: $\Gamma \approx \dfrac{mc^2}{\hbar}|M|^2$ *Cross Section:* $\sigma \approx \dfrac{(\hbar c)^2}{E^2}|M|^2$