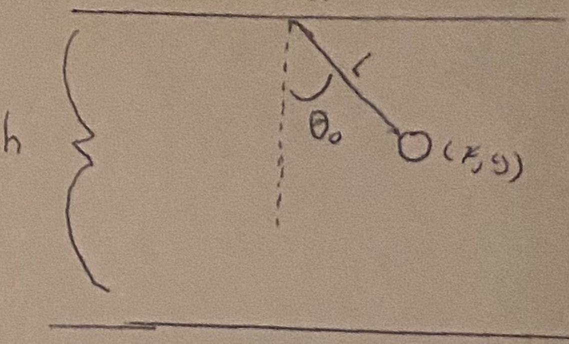
Simple Pendulum Problem

What is the height of the bob after 19.5 seconds?

(not to Scale)



First find equation of motion using the Lagrangian:

$h = 5m$, $L = 1m$, $\theta_0 = 40°$

$$\mathcal{L} = \underset{K.E.}{T} - \underset{P.E.}{V} \quad ; \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad V = mgy$$

we can simplify the Lagrangian by using θ:

$$x = L\sin\theta, \quad y = -L\cos\theta \quad \rightarrow \quad \dot{x} = L(\cos\theta)\dot{\theta}, \quad \dot{y} = L(\sin\theta)\dot{\theta}$$

$$T = \frac{1}{2}m(L^2\sin^2\theta\,\dot{\theta}^2 + L^2\cos^2\theta\,\dot{\theta}^2) = \frac{1}{2}mL^2\dot{\theta}^2, \quad V = -mgL\cos\theta$$

so $$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

using Euler-Lagrange $$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\frac{d}{dt}(mL^2\dot{\theta}) + mgL\sin\theta = 0 \quad \rightarrow \quad mL^2\ddot{\theta} + mgL\sin\theta = 0$$

$$L\ddot{\theta} + g\sin\theta = 0$$

$$\boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$$

$\theta_0 = 40°$ is too large for the small angle approximation

But, we can accurately solve the coupled ODE system using a ~~Rung~~ Runge-Kutte 4th order scheme

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Using derived equation of motion I will set up the RK4 scheme:

- Equation of motion:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

- Define State vectors from Coupled ODE

$$q = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}, \quad q' = f(q) = \begin{bmatrix}\dot{\theta} \\ -\frac{g}{L}\sin\theta\end{bmatrix}$$

- RK4 step (from $t_n \rightarrow t_{n+1} = t_n + h$)
  - $k_1 = f(q_n)$
  - $k_2 = f(q_n + \frac{h}{2}k_1)$
  - $k_3 = f(q_n + \frac{h}{2}k_2)$
  - $k_4 = f(q_n + h k_3)$

- Written out with $q = [\theta, \dot{\theta}]^T$
  - $k_1^{\theta} = \dot{\theta}_n$;
  - $k_2^{\theta} = \dot{\theta}_n + \frac{h}{2}k_1^{\dot{\theta}}$;
  - $k_3^{\theta} = \dot{\theta}_n + \frac{h}{2}k_2^{\dot{\theta}}$;
  - $k_3^{\theta} = \dot{\theta}_n + h k_3^{\dot{\theta}}$;
  - $k_1^{\dot{\theta}} = -\frac{g}{L}\sin\theta_2$
  - $k_2^{\dot{\theta}} = -\frac{g}{L}\sin(\theta_n + \frac{h}{2}k_1^{\theta})$
  - $k_3^{\dot{\theta}} = -\frac{g}{L}\sin(\theta_n + \frac{h}{2}k_2^{\theta})$
  - $k_4^{\dot{\theta}} = -\frac{g}{L}\sin(\theta_n + h k_3^{\theta})$

And update θ, $\dot{\theta}$ with:

$$\theta_{n+1} = \theta_n + \frac{h}{6}(k_1^{\theta} + k_2^{\theta} + k_3^{\theta} + k_4^{\theta})$$

$$\dot{\theta}_{n+1} = \dot{\theta}_n + \frac{h}{6}(k_1^{\dot{\theta}} + k_2^{\dot{\theta}} + k_3^{\dot{\theta}} + k_4^{\dot{\theta}})$$

This has been implemented in the attached python code with

~~h=0.0005~~ $h = 0.005$ seconds, $t_{final} = 18.5$ seconds, $nt = t_{final}/h$,

$\theta_0 = 40°$, $g = 9.81\,m/s^2$

yielding a final $y_{18.5} = -0.7944$. Adding 5m to account for the cieling height gives final answer $\boxed{4.21\,m}$