1

NUMBERS, ARITHMETIC, AND MATHEMATICS

OUTLINE

OBJECTIVES

1. Discuss the origin of complex numbers.
2. Represent complex numbers in polar form and rectangular form.
3. Define and interpret Euler's identity.
4. Conduct numerical calculations with complex numbers.
5. Describe properties of a function of a complex variable.
6. Interpret and resolve indeterminate values.
7. Use MATLAB for arithmetic operations.

INTRODUCTION

Electrical, mechanical, hydraulic, and other systems contain components that obey calculus-based laws. Determining the behavior of such a system in response to an input signal involves solving differential equations. There is, however, one particular type of signal, e^{st}, for which the linear differential equations reduce to simple algebraic equations. Fortunately, this exponential function can represent the kinds of signals of interest to us. While this signal is a simple function of time, it unfortunately returns values that are complex numbers.

Since complex numbers are central to this text, a brief review of how they arise and the arithmetic they obey is likely to be helpful. A few other mathematical topics related to functions, and especially complex functions, will also be discussed. Otherwise, we will generally introduce mathematical concepts as they are needed.

1.1 THE NUMBER SYSTEM

The names that have been given to different types of numbers show that it is a common reaction to view new concepts with astonishment and possibly skepticism. After reading this section you should be able to:

- Discuss the origin of complex numbers.
- Explain why "imaginary" numbers are necessary.
- Recognize that you are not alone in finding new concepts difficult.

Much speculation has occurred on the origins of our numbering system, but it most certainly started with counting. A prehistoric hunter might have recorded the success of his efforts by the number of skins obtained, or the length of his efforts by the number of sunrises that occurred while he was away. Some of his skins might have been bartered away for roots and berries or for tools and weaponry, "I'll give you two of these for one of those" types of transactions. All of these processes required positive integers, a concept that is both natural and easy for us to appreciate.

Manuscripts dating from about 2000 B.C. show that simple fractions were also understood early on. Apparently, questions of how a lord could divide his territory uniformly among his subjects so that they might be equally and fairly taxed were among the most pressing early mathematical issues. Some priorities never change.

If a man had 5 skins and traded 3 of them for a new spear, he could determine how many skins he had left by counting out those needed for the trade and then counting those that remained. No new skills or concepts needed to be developed. It was unthinkable that 5 skins would be traded if only 3 were available. Such a transaction would be impossible! Several millennia would pass before that changed.

It is difficult, today, to fully appreciate some of the discoveries of the early mathematicians. The concepts with which they struggled are now so ingrained in our experience that they are second nature to us. The notations and algebraic laws that allow complicated questions to be addressed systematically were not available to them.

Yet, once the philosophers of the day postulated that no transaction should be impossible, the existence of an entirely new set of numbers was required. Thus negative numbers were discovered, the number system doubled in size, and debt became an element in some transactions.

The concept of the number *zero* was especially difficult to establish. After all, why create a number to count nothing? It is only when man began to philosophize about numbers that meaning could be attached to such an abstraction. The discovery that there were numbers that could not be represented as a ratio of integers, numbers like those special ones we denote today as π or e, did not come from everyday experience. Indeed, the claim that there were such numbers was greeted with much skepticism and disbelief. Such numbers were inconceivable, irrational! We know now that the set of *irrational* numbers is infinitely larger than all those that were known previously.

Finally, but not until around the sixteenth century, it was recognized that there had to be yet again an entirely new set of numbers, in fact a new kind of number, if equations like $x^2 = -1$ were to have solutions. These new numbers were called *imaginary*—a regrettable though understandable label—to distinguish them from the *real* numbers known previously. A j (mathematicians prefer i) is used to indicate the imaginary part of a number, where $j = +\sqrt{-1}$.

Today we conveniently think of the real number system as being represented by a continuous line of numbers running from $-\infty$ to $+\infty$, with every point along the line being a unique number. We think of the imaginary numbers in much the same way, but with two exceptions: (1) Each imaginary number is tagged with a j to distinguish it from the real numbers, and (2) the line of imaginary numbers is independent of, or perpendicular to, the line of real numbers. Together the real and imaginary numbers form a *complex-number plane*, each point of which represents a generalized number that could be used in, or result from, a mathematical calculation. Every equation conceived of by man has yielded a result somewhere in this number plane . . . so far.

It is not necessary to introduce complex numbers into the study of electrical technology. All of our variables—voltage, current, resistance, inductance, etc.—are physical quantities that are described by ordinary real numbers. Ultimately, complex numbers gain their meaning through their application and interpretation, not through the arbitrary tags of "real" or "imaginary" that mathematicians use to distinguish between their two component parts. The fact that these components are independent allows a single complex number to hold information about two separate physical properties. We introduce complex numbers because they can make many of our calculations easier.

1.2 RECTANGULAR/POLAR CONVERSIONS

Complex numbers may be represented in rectangular or polar form. After reading this section you will be able to:

- Represent complex numbers in polar form and rectangular form.
- Convert from one form to the other.

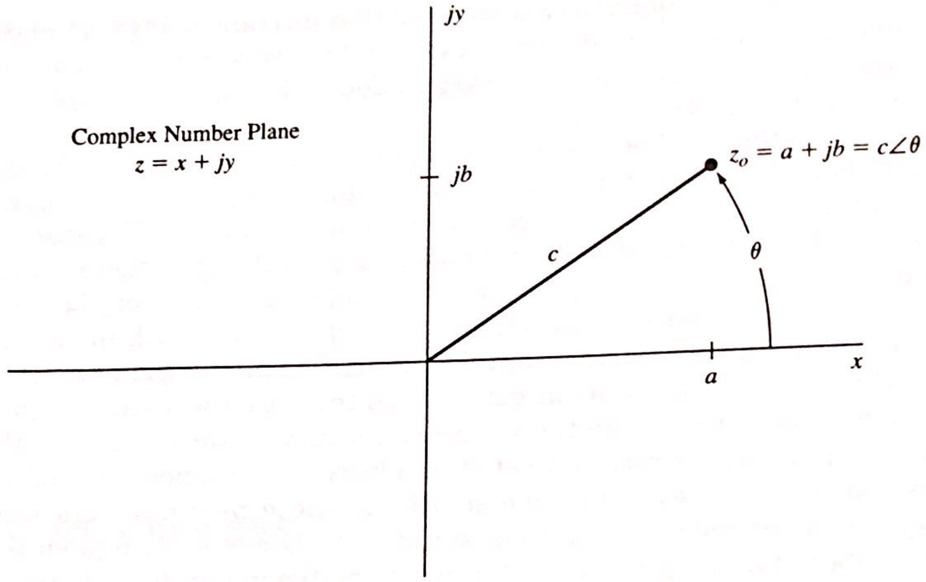


Figure 1.1 A specific complex number may be described either by its rectangular or its polar coordinates in a complex number plane.

A complex number can be identified by giving it real and imaginary parts to identify its location in the complex-number plane of Figure 1.1. This is called the *rectangular* form of the number. If $z = a + jb$, where a and b are real numbers, a is called *the real part* of z, and b is called *the imaginary part* of z. To emphasize this terminology, whenever we speak of the imaginary part of a complex number, we are talking about a real number that is being multiplied by j.

Some people like to think of j as a *rotational operator*, since it converts a real number into an imaginary number by rotating it 90° to the j axis in the complex-number plane. We prefer simply to define j to be a number, namely, the positive square root of -1, and to treat it like any other number. To us, there is no difference between the numbers $j2$ and $2j$. Some of the products of j are:

$$j = +\sqrt{-1} \qquad j^2 = -1 \qquad j^3 = -j \qquad j^4 = 1 \qquad j^5 = j$$

A complex number may alternatively be specified by giving its distance from the origin along a radius line having a specified angle relative to the positive real axis. This is called the *polar* form of the number, and is often stated using the notation $c\angle\theta$, read as "a magnitude of c at an angle of θ." By convention, θ is taken as positive in the counterclockwise direction. Both c and θ are real numbers.

It is essential to be able to convert between polar and rectangular forms. The required relationships are easily deduced from Figure 1.1. The conversion to polar form is complicated slightly by the fact that the inverse tangent function must be evaluated in the proper quadrant. A sketch is often useful in establishing the proper angle.

Polar/Rectangular Conversions

$$a + jb = c\angle\theta$$

$P \rightarrow R$

$a = c\cos\theta$ **(1.1a)**

$b = c\sin\theta$ **(1.1b)**

$P \leftarrow R$

$c = \sqrt{a^2 + b^2}$ **(1.1c)**

$\phi = \arctan(|b/a|)$ **(1.1d)**

Quadrant	a	b	θ
1st	+	+	ϕ
2nd	−	+	$\pi - \phi$
3rd	−	−	$\pi + \phi$
4th	+	−	$-\phi$

EXAMPLE 1.1A

Find the rectangular form of the complex number $z_o = 2\angle 140° = 2\angle 2.444$.

Solution

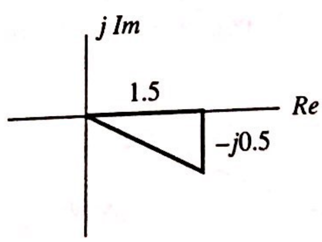
$$a = 2\cos 140° = 2(-0.7664) = -1.533$$
$$b = 2\sin 140° = 2(0.6428) = 1.286$$
$$\therefore z_o = -1.533 + j1.286$$

EXAMPLE 1.1B

Find the polar form of the complex number $z_o = 1.5 - j0.5$.

Solution

A sketch is usually helpful:



$$c = \sqrt{1.5^2 + (-0.5)^2} = \sqrt{2.25 + .25} = 1.581$$
$$\phi = \arctan\left(\left|\frac{-0.5}{1.5}\right|\right) = \arctan\left(\frac{1}{3}\right) = 0.322 = 18.43°$$

A z_o with a positive real part and a negative imaginary part is in the 4th quadrant, so $\theta = -18.43°$.

$$\therefore z_o = 1.581\angle -18.43°$$

Although angles in radians are, strictly speaking, correct, either degrees or radians are generally accepted. It is important, however, to make sure you enter numerical values corresponding to the units your calculator is expecting!

The conversion procedures of Equations 1.1a–d and Examples 1.1a–b are rarely used in practice. Modern technical calculators have built-in rectangular/polar functions that take care of making these conversions for you, including finding the proper quadrant angle information. It is essential that you consult your calculator manual and become proficient in making these conversions.

1.3 EULER'S IDENTITY

In preparing to investigate arithmetic operations involving complex numbers, we need to establish the proper mathematical equivalence of our polar notation $c\angle\theta$. After completing this section you will be able to:

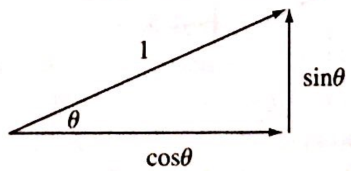
- State and interpret Euler's identity.
- Describe a polar-form complex number using standard math functions.
- Express a cosine as a sum of complex exponentials.
- Express a sine as a sum of complex exponentials.

A very important relationship, known as *Euler's identity*, is

$$e^{\pm j\theta} \equiv \cos\theta \pm j\sin\theta \tag{1.2}$$

If Euler's identity is converted from its rectangular form to polar form, it becomes

$$e^{j\theta} = \cos\theta + j\sin\theta = \sqrt{\cos^2\theta + \sin^2\theta}\angle\arctan(\tan\theta) = 1\angle\theta$$



$$e^{j\theta} \equiv 1\angle\theta \tag{1.3}$$

Thus, the true mathematical interpretation of $A\angle\theta$ is $Ae^{j\theta}$, and the algebraic rules for exponents apply to the angle information of polar-form complex numbers (Table 1.1).

Table 1.1 A Few Laws of Exponents

$e^{jx} e^{jy} = e^{j(x+y)}$	$(e^{jx})^y = e^{jxy} = (e^{jy})^x$	$1/e^{jx} = e^{-jx}$

A few additional special cases follow by direct substitution into Equation 1.2:

$$e^{\pm j0} = 1 \qquad e^{\pm j\pi} = -1 \qquad e^{\pm j2\pi} = 1 \qquad e^{\pm j\pi/2} = \pm j$$

The two signs in Euler's identity give us two equations, which may be solved for the cosine or sine function. For instance,

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \\ \hline e^{j\theta} + e^{-j\theta} &= 2\cos\theta \qquad \text{(adding the 2 equations)} \\ e^{j\theta} - e^{-j\theta} &= 2j\sin\theta \qquad \text{(subtracting the 2 equations)} \end{aligned}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{1.4}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{1.5}$$

Using these relationships, any sinusoidal signal may be expressed in terms of the even more elementary functions of complex exponentials. This is, in fact, the origin of the *phasor* concept, which is fundamental to our study of alternating current (a-c) circuits. It is difficult to overstate the importance of Euler's identity; we will spend the rest of this text making use of it.

EXAMPLE 1.2

Express $f(x)$ in terms of cosine functions:

$$f(x) = 2e^{j2x} + 4e^{-jx} + 4e^{jx} + 2e^{-j2x}$$

Solution

Regrouping the terms to compare with Equation 1.4, we get

$$f(x) = 2[e^{j2x} + e^{-j2x}] + 4[e^{jx} + e^{-jx}]$$

and

$$f(x) = 4\cos 2x + 8\cos x$$

1.4 COMPLEX-NUMBER ARITHMETIC

Complex numbers are "complex" only in the sense that their arithmetic is more involved than that for real numbers. Adding two complex numbers involves the same amount of effort as adding two sets of two real numbers. This doubles the chances for error as well as the time required to complete the operation. After completing this section you will be able to:

- Conduct numerical calculations with complex numbers.
- Calculate the sum, difference, product, or ratio of complex numbers.
- Raise a complex number to a power.
- Find the conjugate of a complex number.
- Combine a number with its conjugate to find its real or imaginary part, its magnitude, or its angle.

1.4.1 Addition/Subtraction

If $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Similarly,

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

The j term serves to remind us that two completely independent types of numbers are present and that they must be handled separately. We also like the final results to be recombined into a single real part and a single imaginary part.

If complex numbers are given in polar form, the usual way to add them is to immediately convert both of them to rectangular form. Most calculators will accept polar-form numbers, convert and add them, and put the result back in polar form for you, all in a single operation. Of course, if two complex numbers have exactly the same angle, their magnitudes may be added directly as if they were vectors.

EXAMPLE 1.3A

Find $z_1 - z_2$ if $z_1 = 2 + j5$ and $z_2 = 2\angle 140°$.

Solution

Since a subtraction is called for, the numbers must be in rectangular form. From Example 1.1a, $z_2 = (2\underline{/140°}) = -1.533 + j1.286$. Then, $z_1 - z_2 = 2 + j5 - (-1.533 + j1.286) = 3.533 + j3.714$.

1.4.2 Multiplication

Complex numbers may be multiplied in either rectangular or polar form. The procedure used depends primarily on what form the numbers are in initially and what form we want for the result. Starting with rectangular-form numbers, we may simply multiply them out, remembering that $j^2 = -1$:

$$\text{If } z_1 = x_1 + jy_1 \text{ and } z_2 = x_2 + jy_2, \text{ then}$$
$$z_1z_2 = (x_1 + jy_1)(x_2 + jy_2) = x_1(x_2 + jy_2) + jy_1(x_2 + jy_2)$$
$$z_1z_2 = x_1x_2 + jx_1y_2 + jy_1x_2 + j^2y_1y_2$$
$$z_1z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2)$$

You are not expected to memorize these results, but rather, when faced with making a hand calculation, to carry out this multiplication process in an expeditious fashion. It is really no different from carrying out the multiplication of real factors like $(x + a)(y + b)$ except that with complex numbers we like to finish things off by regrouping the results into their real and imaginary parts.

The multiplication of two complex numbers is particularly easy if they are in polar form. Their magnitudes are multiplied and their angles are added:

$$\text{If } z_1 = c_1\angle\theta_1 \text{ and } z_2 = c_2\angle\theta_2, \text{ then } z_1z_2 = c_1e^{j\theta_1}\,c_2e^{j\theta_2} = c_1c_2e^{j(\theta_1+\theta_2)}$$

Raising a complex number to a power may be regarded as a special case of repeated multiplication. Powers are most easily found in polar form, since

$$z^n = (ce^{j\theta})^n = c^ne^{jn\theta} = c^n\angle n\theta$$

1.4.3 Conjugation

The conjugate of a complex number z, denoted with the symbol z^*, is defined as follows:

$$\text{If } z = a + jb, \text{ then } z^* = a - jb; \text{ or if } z = ce^{j\theta}, \text{ then } z^* = ce^{-j\theta}$$

Notice that the conjugate is formed by simply replacing j by $-j$ in these basic complex-number forms. This continues to be true when the numbers are not fully simplified, and may be generalized in the following theorem.

The Conjugation Theorem

If z results from evaluating an arithmetic expression involving the addition (subtraction), multiplication, and/or division of complex numbers, z^* may be formed simply by replacing every j with a $-j$ in that expression.

Notice that if $z = a + jb$, then adding or subtracting its conjugate will separate out the real and imaginary parts:

$$z + z^* = 2a = 2\mathrm{Re}(z) \qquad \text{(Read as "twice the real part of } z\text{")}$$

and

$$z - z^* = 2jb = 2j\mathrm{Im}(z) \qquad \text{(Read as "}2j\text{ times the imaginary part of } z\text{")}$$

Multiplying a number by its conjugate separates out the magnitude information:

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 \qquad \text{or} \qquad zz^* = (ce^{j\theta})(ce^{-j\theta}) = c^2$$

1.4.4 Division

Division of rectangular-form numbers is converted to a multiplication problem by *rationalizing*. This involves multiplying the numerator and denominator by the conjugate of the denominator, which converts the denominator to a real number and moves all of the angle information to the numerator. Here again, it is the process that should be remembered, not the results.

If $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2 + j(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

Division is easiest if the numbers are initially in polar form. The procedure is evident immediately from the rules for exponentials (see Table 1.1):

$$\frac{z_1}{z_2} = \frac{c_1\angle\theta_1}{c_2\angle\theta_2} = \frac{c_1e^{j\theta_1}}{c_2e^{j\theta_2}} = \frac{c_1}{c_2}e^{j(\theta_1-\theta_2)}$$

Dividing a complex number by its conjugate allows us to sort out the number's angle information, since

$$\text{If } z_2 = z_1^*, \text{ then } \frac{z_1}{z_1^*} = e^{j2\theta_1} = 1\angle 2\theta_1$$

EXAMPLE 1.3B

Find $\dfrac{z_1z_2}{z_2^*} + z_1^*$ if $z_1 = 1 - j3$ and $z_2 = 2 - j1$.

Solution

Since the final operation will be the addition of z_1^*, we will retain rectangular form throughout the calculations.

$$\frac{(1-j3)(2-j1)}{2+j1}+(1+j3)=\underbrace{\frac{(1-j3)(2-j1)^2}{5}}_{\text{rationalizing}}+(1+j3)$$

$$=\frac{(1-j3)(3-j4)}{5}+(1+j3)=-\frac{9}{5}-j\frac{13}{5}+\left(\frac{5}{5}+j\frac{15}{5}\right)$$

$$=-\frac{4}{5}+j\frac{2}{5}$$

EXAMPLE 1.3C

Find the polar form of $\dfrac{(z_1+z_2^2)}{z_2{}^*}$ if $z_1 = 1 - j3$ and $z_2 = 2 - j1$.

Solution

In this case there are several ways to proceed. Since z_2 is only squared, we will retain rectangular form in the numerator until it has been fully simplified. (If z_2 had been raised to a higher power, we would be tempted first to convert it to polar form, then to raise it to the required power, and finally to convert it back to rectangular form for the addition to z_1.)

$$\frac{z_1+z_2^2}{z_2{}^*}=\frac{(1-j3)+(2-j1)^2}{\sqrt{5}\angle 0.4636}=\frac{(1-j3)+(3-j4)}{\sqrt{5}\angle 0.4636}=\frac{4-j7}{2.236\angle 0.4636}$$

$$=\frac{8.062\angle -1.0517}{2.236\angle 0.4636}=3.606\angle\underbrace{-1.5153}_{-86.82^\circ}$$

Each of the numerical examples of this section might have been accomplished on your calculator without worrying about the procedures to be used. However, these procedures are also important in manipulating complex variables that have not been assigned specific values.

1.5 FUNCTIONS OF A COMPLEX VARIABLE

Throughout this text we will encounter functions consisting of polynomials of a complex variable. The theory of functions of complex variables can be a rather challenging mathematical investigation, but our concerns are fairly basic. After completing this section you will be able to:

- Describe properties of a function of a complex variable.
- Properly interpret questions related to complex functions.

- Identify a function's poles and zeros.
- Represent the function with a pole-zero diagram.

Real functions have, at most, a sign and an amplitude. Questions related to the properties of such functions are usually unambiguous as a result. We can seek to find where the real function has a particular value, or look for its maximum or minimum values. A function of a complex variable, however, is usually itself complex, so it is important to exercise due care in asking questions about it. A complex function, just like a complex number, has a real part, an imaginary part, a magnitude, and an angle. Our questions must clearly reflect these properties. Asking, for instance, if a function F meets the condition that $F = 1$ is really asking: Is there a value of the independent variable for which the real part of F equals $+1$ while the imaginary part is simultaneously zero? It is definitely not the same as asking if $|F| = 1$ or even if $\mathrm{Re}(F) = 1$. We must ask the question correctly to be able to answer it correctly.

EXAMPLE 1.4

For the function $F(z) = 6/z$, where $z = x + jy$, compare the conditions under which $F = 3$, $|F| = 3$, and $\mathrm{Re}(F) = 3$.

Solution

Expressing F in rectangular form identifies all of these conditions.

$$F(z) = \frac{6}{z} = \frac{6}{x + jy} = \frac{6(x - jy)}{x^2 + y^2} = \frac{6x}{x^2 + y^2} - j\frac{6y}{x^2 + y^2}$$

For $F = 3$, the imaginary part must vanish, which requires that $y = 0$. Under that condition, $F(z) = \left.\dfrac{6x}{x^2 + y^2}\right|_{y=0} = \dfrac{6}{x}$. This equals 3 only at the single point $x = 2, y = 0$, or $z = 2 + j0$.

By inspection, $\mathrm{Re}(F) = \dfrac{6x}{x^2 + y^2} = 3$ when $x^2 + y^2 = 2x$. Rearranging and completing the square gives $(x - 1)^2 + y^2 = 1$, which is the equation of a circle centered at $z = 1 + j0$ with unit radius. $\mathrm{Re}(F) = 3$ everywhere on this circle except at the single point $z = 0$, where F is undefined.

Finally, we can obtain an expression for $|F|$ and set it equal to 3.

$$|F(z)| = \left|\frac{6(x - jy)}{x^2 + y^2}\right| = \frac{6|x - jy|}{x^2 + y^2} = \frac{6\sqrt{x^2 + y^2}}{x^2 + y^2} = \frac{6}{\sqrt{x^2 + y^2}} = 3$$

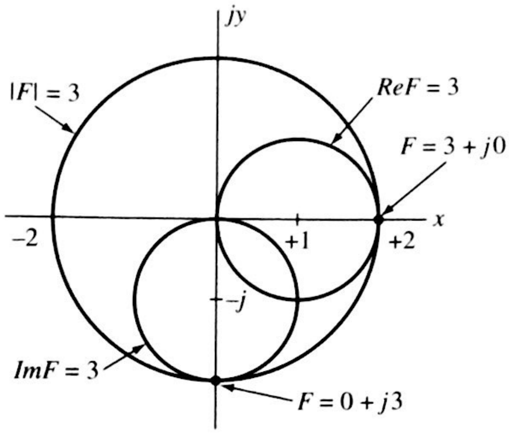


Figure 1.2 Locations where various properties of the function $F = 6/z = +3$.

Squaring both sides and simplifying gives $x^2 + y^2 = 4$, which is also a circle centered at $z = 0$ and with a radius of 2. (Can you find an easier way to get this result?)

These and other conditions for this function are summarized in Figure 1.2.

We have used Example 1.4 to emphasize the importance of asking the right questions when dealing with functions of a complex variable. As our functions become more complicated, the task of trying to show all locations in the z plane where the function takes on a specific value becomes formidable.

The general problem is that a complex variable, z, can take on any value within the two-dimensional complex-number plane. If $F(z)$ is also complex, two additional dimensions are required to show its values. Since four-dimensional graphs are impossible to draw, we would probably take the two components of $F(z)$, either its real and imaginary parts or its magnitude and angle, and draw them separately over the complex-number plane. Figure 1.3 shows one such graph for the real part of the function $F(z) = \sin(z)$. While such graphs provide a visual overview of the function's behavior, the process of creating them is calculation intensive, and it is difficult to recover the numerical values from the graph itself. On those occasions when we do investigate a function over the entire complex-number plane, a *pole-zero plot* provides the kind of information we are looking for.

The pole-zero plot of a function identifies the extreme points in its magnitude. Poles are locations (values of z) where the function is infinite, while zeros are locations where the function is zero. Zeros are indicated with an "o" and poles with an "x." If a function contains a repeated root, such as would result from a term like $(z - a)^n$, the *multiplicity* of the root is indicated by placing the value of n next to it on the pole-zero plot. The pole-zero diagram (Figure 1.4) is like a topographical map showing points of elevation extremes, but without the added detail provided by intermediate contour lines of constant elevation.

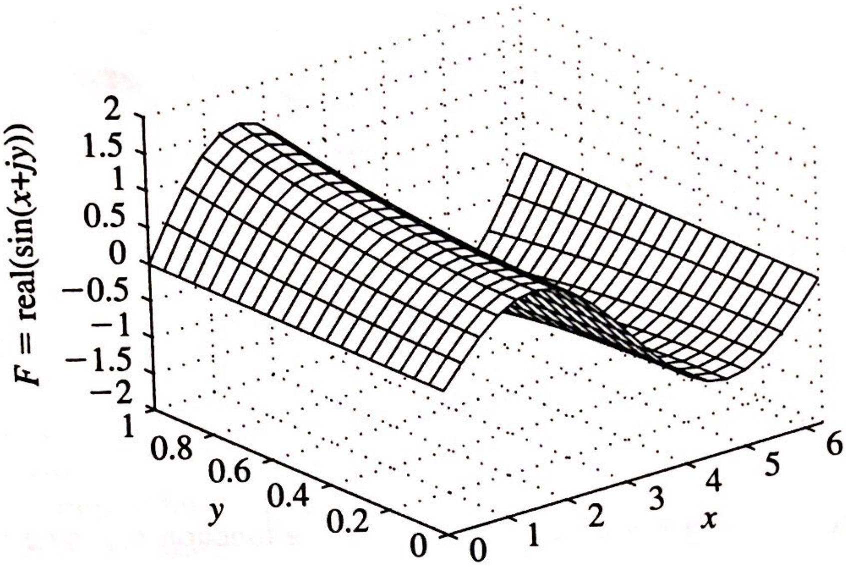


Figure 1.3 A three-dimensional plot of the real part of the function $F = \sin(z)$, where $z = x + jy$ using the MATLAB **mesh** command. The complex sine function is defined by Equation 1.5, where θ is replaced by z.

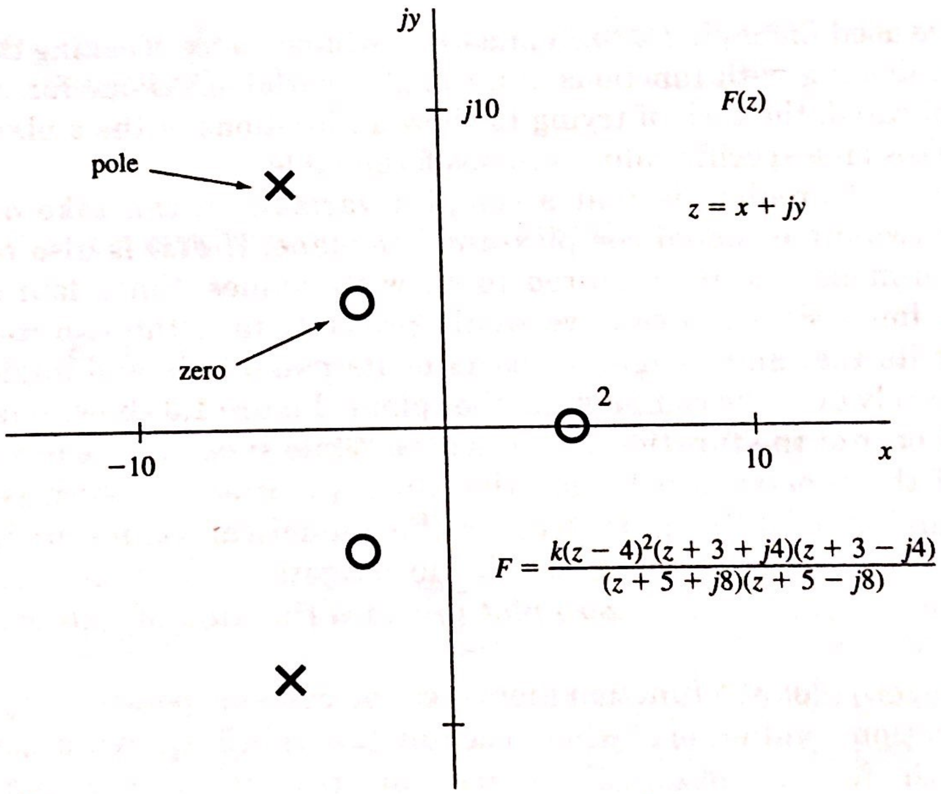


Figure 1.4 A *pole-zero diagram* indicates locations in the z plane where the function has a magnitude of infinity or zero, respectively. For the functions encountered in this text, complex poles or zeros will always occur as conjugate pairs. In this figure, a double zero is located at $z = 4 + j0$. The original function can be recreated from its pole-zero diagram to within a multiplicative constant.

Most of the complex functions we will encounter will consist of a ratio of polynomials. For such functions, the finite zeros are simply the zeros of the numerator polynomial, and the finite poles are the zeros of the denominator polynomial. Functions having this form may be represented by Equation 1.6, where N is the degree of the denominator polynomial whose roots represent poles, p_n, of the function. M is the degree of the numerator polynomial whose roots, z_m, are the zeros of the function. If the multiplier, k, is added to the pole-zero plot, the plot describes F as completely as does the equation.

$$F(z) = \frac{k\prod_{m=1}^{M}(z - z_m)}{\prod_{n=1}^{N}(z - p_n)} = \frac{k(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \tag{1.6}$$

If the number of finite poles and zeros are not equal ($M \neq N$), then the function will have poles or zeros at infinity. At very large values of z, only the highest powers of z need be considered. Then

$$\lim_{|z|\to\infty} F(z) = \lim_{|z|\to\infty}\frac{k\prod_{m=1}^{M}(z - z_m)}{\prod_{n=1}^{N}(z - p_n)} = \lim_{|z|\to\infty} kz^{M-N} = \begin{cases}\infty & M > N\\ 0 & N > M\end{cases}$$

Every function has an equal number of poles and zeros if those at infinity are included. Poles and zeros at infinity are not shown on pole-zero diagrams.

EXAMPLE 1.5

Provide a pole-zero plot for the following function:

$$F(z) = \frac{2(z^2 + 4)}{(z^2 - z)(z + 2)}$$

Solution

Placing F in factored form gives

$$F(z) = \frac{2(z + j2)(z - j2)}{z(z - 1)(z + 2)}$$

which shows that the numerator of F goes to zero if $z = \pm j2$. Similarly, the denominator of F goes to zero if $z = 0, +1,$ or -2. A zero also exists at $|z| = \infty$, since the total number of zeros must equal the number of poles. See Figure 1.5.

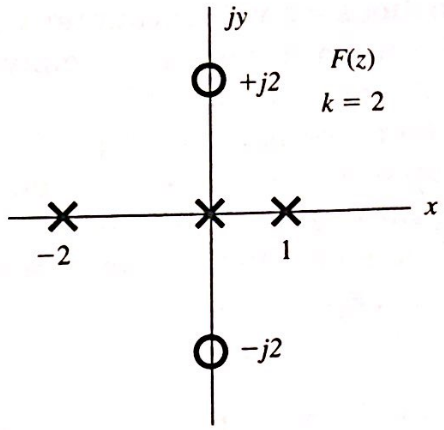


Figure 1.5 The pole-zero diagram for the function of Example 1.5.

Frequently we simply want to know the value of a function along a specific path in the complex-number plane. Most often the path of interest will be either the real axis, the imaginary axis, or a circle centered at the origin of the z plane. In these cases, the path can be specified in the independent real variable x, y, or θ, and the results shown in a pair of two-dimensional plots with which we are more familiar. Example 1.6 demonstrates this type of problem.

EXAMPLE 1.6

Find the magnitude of $F(z) = \dfrac{2(z+4)}{(z-1)}$ along the path $z = x + j2$.

Solution

The pole-zero diagram for this function (Figure 1.6) shows that the indicated path will take us close to a zero and a pole as we move from $z = -\infty + j2$ to $+\infty + j2$. It seems reasonable that the function will dip as we pass the zero, and peak as we pass the pole. The zero affects the landscape in the vicinity of the pole, and vice versa, so minimum and maximum points do not necessarily occur at points on the evaluation path closest to the poles or zeros.

To check this, we evaluate F along the specified path:

$$F(z)\big|_{z=x+j2} = \left.\frac{2(z+4)}{(z-1)}\right|_{z=x+j2} = 2\frac{(x+j2+4)}{(x+j2-1)} = 2\frac{[(x+4)+j2]}{[(x-1)+j2]}$$

$$F(x+j2) = 2\frac{[(x+4)+j2][(x-1)-j2]}{(x-1)^2+4} = \frac{2x^2+6x-j20}{x^2-2x+5}$$

The result is clearly a complex function of the real variable x. Evaluating the magnitude of the function over a range of x where the pole and zero exercise their

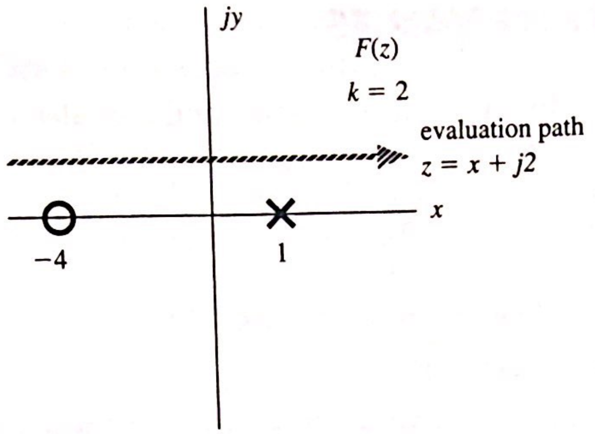


Figure 1.6 Pole-zero diagram and evaluation path for Example 1.6.

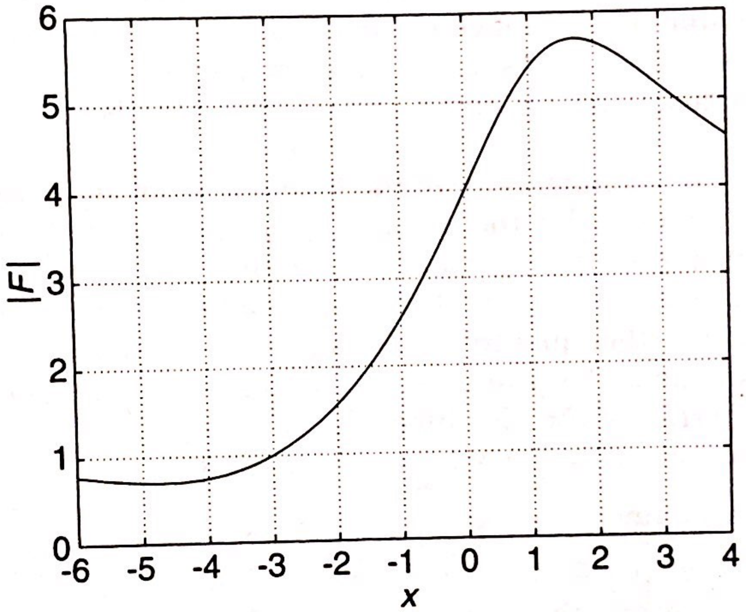


Figure 1.7 The magnitude of the function in Example 1.6 is shown evaluated along the path $z = x + j2$. Note that the maximum and minimum points do not correspond to the points of closest approach to the pole and the zero, respectively. The equation of this curve is $|F(x + j2)| = \dfrac{\sqrt{4x^2(x+3)^2 + 400}}{x^2 - 2x + 5}$.

greatest influence confirms our predictions and augments them with specific numerical information (Figure 1.7). Note that a relatively simple function of the complex variable z has become a quite messy function of the real variable x. Having a calculator or software package that routinely handles complex variables would allow the simpler complex function to be evaluated along a specified path without complicating the problem with a lot of unnecessary algebra.

1.6 INDETERMINATE VALUES

Division by zero is undefined in mathematics and can also create havoc with some computer operating systems. However, important physical events can be associated with this mathematical condition, and measurable properties can be calculated by resolving indeterminate values. After completing this section you will be able to:

- Interpret and resolve indeterminate values.
- Recognize common indeterminate forms.
- Use L'Hopital's rule to define a value for an indeterminate form.
- Predict conditions under which special events may occur.

If a class of students is asked the value of the function $F(m) = (\sin am)/m$ at $m = 0$, about half of them would first notice the numerator and declare the result to be zero. The rest would first notice the denominator, and declare the result to be *infinite* or *undefined*. This very substantial discrepancy occurs because the value of the function is, in fact, *indeterminate* at $m = 0$. Three kinds of situations arise.

Situation 1. If the numerator of the function is not zero, we say the function becomes *infinite*, denoted by the symbol ∞, as the divide-by-zero condition is approached. We interpret this symbol as indicating that the value of the function increases without bound as the divide-by-zero condition is approached, but we are forbidden to try and further quantify or evaluate it. We have called these points *poles* in our complex functions. They are also called *singular points*, or *singularities*. In investigating properties of complex functions, we are permitted to get as close to a pole as we wish, as long as we do not actually reach it.

Situation 2. If a function takes the form 0/0 at a point (∞/∞, $0 \cdot \infty$, and $\infty - \infty$ are a few related forms), we say that the value of the function is *indeterminate* at that point. We will interpret this as meaning that the function is hiding its true value from us at this point and that further investigation is required to assign it the appropriate value.

It is not uncommon to obtain different functions as a result of solving simultaneous equations, depending on the procedure used. One method might lead to a result $f(x)$, for instance, whereas using a rote procedure involving determinants might lead to the result $g(x) = (x - 1)f(x)/(x - 1)$. In this case f and g are identical everywhere, except that g is indeterminate at $x = 1$. This situation arises if opportunities to cancel common factors are not noticed during a derivation. Clearly, the physical property being represented by these functions is correctly given by f, and the indeterminacy of g at a particular point should not be allowed to cloud the issue.

A function like

$$F(m) = \int_0^a (\cos mx)\, dx = \frac{\sin ma}{m}$$

becomes indeterminate at $m = 0$ after the integration, but does not if the special case of $m = 0$ is entered before the integral is evaluated:

$$F(0) = \int_0^a (\cos 0x)\,dx = \int_0^a 1\,dx = a$$

For the most part, indeterminate forms result from a failure to simplify expressions when the opportunity presents itself or from trying to generalize a complicated result in a single formula. In these circumstances we assume that the correct value of the function at the indeterminacy can be found by taking the limit of the function as we approach arbitrarily close to the indeterminacy. A procedure routinely used for resolving this value is *L' Hopital's rule.*

L' Hopital's Rule

If

$$f(x) = \frac{n(x)}{d(x)}$$

becomes indeterminate at $x = a$, then

$$f(a) = \lim_{x \to a} f(x) = \frac{n'(a)}{d'(a)} \qquad \textbf{(1.7)}$$

where $n'(a)$ and $d'(a)$ are the respective derivatives of $n(x)$ and $d(x)$ evaluated at $x = a$.

If the indeterminacy is still not resolved, the second derivatives of numerator and denominator are taken, and so on until it is resolved. Remember, L' Hopital's rule requires you to take the derivative of the numerator and denominator *separately*, which is much easier than taking the derivative of the whole function!

EXAMPLE 1.7

Evaluate $F(z)$ at $z = 1, 2$, and 3, where

$$F(z) = 3\frac{z^2 - 3z + 2}{z^2 - 5z + 6}$$

Solution

Substituting directly we find

$$F(1) = \frac{0}{2} = 0, \qquad F(2) = \frac{0}{0} = ?, \qquad \text{and} \qquad F(3) = \frac{6}{0} = \infty$$

Neither $F(1)$ nor $F(3)$ is indeterminate, but note that both the numerator and the denominator values must be found before that fact can be ensured.

Factoring the polynomials resolves the indeterminacy at $z = 2$:

$$F(z) = 3\frac{z^2 - 3z + 2}{z^2 - 5z + 6} = 3\frac{(z-1)(z-2)}{(z-2)(z-3)} = 3\frac{z-1}{z-3} \qquad \therefore F(2) = -3$$

The resulting simplified equation for F applies at all values of z and is used in place of the original expression.

Alternatively, we could have applied L' Hopital's rule to resolve the issue at the single indeterminate point $z = 2$. In doing so, we assume that differentiation with respect to a complex variable is the same as for real variables.

$$n' = \frac{d}{dz}3(z^2 - 3z + 2) = 3(2z - 3) \qquad d' = \frac{d}{dz}(z^2 - 5z + 6) = (2z - 5)$$

$$\lim_{z\to 2} F(z) = 3\frac{2z-3}{2z-5}\bigg|_{z=2} = -3$$

Situation 3. Sometimes our only interest in a function is to determine the condition under which it becomes indeterminate. Suppose, for example, a function is given by $F(z) = 0/D(z)$. Since its numerator is identically zero, the function is zero . . . unless the denominator also happens to be zero. Then there is at least a possibility that F will not be zero at locations where it normally has poles. In these cases it is sufficient for us to know that something may be going on, even if we cannot immediately have more detailed information about it.

This final interpretation of an indeterminate form probably seems bizarre at this point, but it eventually will prove to have great physical significance.

1.7 INTRODUCTION TO MATLAB

MATLAB is a mathematical software package that has gained considerable favor in the engineering community. By using its function calls (subroutines) related to signals and systems, we can easily and quickly investigate many of the concepts introduced in this text. In this section you will:

- Use MATLAB for arithmetic operations.
- Observe MATLAB's response to typical numerical calculations.
- Observe MATLAB's handling of complex numbers.
- Create arrays and make calculations with arrays.
- Use some plotting functions to create and label a graph.
- Identify your workplace variables and array dimensions.
- Use the MATLAB help facility.

To complete this section, you need to be seated at a computer with the MATLAB Command Window active. Type the commands indicated between the prompt (>) and comment field indicator (%), and follow with a return. (Actual MATLAB prompts may include >>, EDU>, and other variations.) If you are using a networked computer that restricts writing to its hard drive, you will need to have a floppy disk handy should you want to save any of your results.

You cannot modify lines in the command window after the return has been executed. If you make an error in entering a command, just retype the command correctly. Usually Ctrl-C will abort an apparent infinite loop.

MATLAB EXAMPLES

We will start with some ordinary arithmetic operations. The symbols +, −, *, /, and ^ indicate the matrix operations of addition, subtraction, multiplication, division, and exponentiation, respectively. Operations are performed left to right. The highest-priority operation is exponentiation, then multiplication/division, and finally addition/subtraction. Use parentheses to group operations to produce the result intended.

```
> 2*3/4               % calculations do not need an "=" if you do not want
                        to save the result
> y=2+4/3             % saves the result as y
> Y=(2+4)/3           % names are case sensitive, both y and Y now have
                        values
> y=2^3/4             % 2 cubed and divided by 4 (original y was
                        overwritten)
> y=2^(3/4)           % 2 raised to the 3/4 power, use parentheses to make
                        the interpretation clear
> y1=2*(1+j)/(1-j)    % complex numbers are OK too
> b=2j                % MATLAB recognizes this as an imaginary number
> c=j2                % but not this
> c=j*2               % but you can always think of j (or i) as a number
                        by itself
> y2=2*exp(j*pi/4)    % the usual standard math functions are available, a
                        few are:
```

sin cos tan asin acos atan exp log log10 sqrt

```
> y3=tan(pi*(1+j)/2)  % angles are in radians, complex angles allowed
                        but have no meaning to us
> y4=(1+j)^2          % complex numbers are presented in rectangular form
```

real imag conj abs angle

These and all other MATLAB functions consist of a name, and in parentheses one or more arguments that are separated by commas.

```
> y5=real(y4)         % select out the real part of a previous calculation
> mag=abs(y4)         % find the magnitude of the complex number y4
```

```
> phase=angle(y4)    % get the angle of y4
> y_star=conj(y4)    % take the conjugate of a complex number
> y=1/0              % infinity is recognized
> y=0/0              % indeterminate results are marked with NaN: Not a
                       (known) Number
```

Variable names must start with a letter. They can be longer than you would want to type. Use good computer programming practice for variable names. MATLAB will let you know if it does not like your choice.

In MATLAB, variables are held in arrays. Our applications will involve mostly single-row or single-column arrays, which are called *vectors*. Arithmetic operations can be performed on two or more vectors, provided the vectors have the same number of rows or columns. Vector arithmetic is done on an element-by-element basis and is identified by a dot (period) preceding the operation symbol. Exception: the dot is omitted in an addition or subtraction of arrays.

```
> x=[0 1 2 ;3 4 5]    % creates a 2-row, 3-column array; extra spacing is
                        optional
> y=[1 2 3; 1 2 3]    % creates another one
> z1=x.*y             % each element of x is multiplied by the
                        corresponding element of y
> z2=x./y             % each element of x is divided by the corresponding
                        element of y
> z3=y.\x             % an alternate way to express z2
> z4=x.^y             % each element of x is raised to the power of the
                        corresponding element in y
> z5=x.+y             % the dot is inappropriate in addition or
                        subtraction
> 2.^y                % operations by constants apply to all elements of y
> y.^2
> y+2
```

[ ]   linspace   logspace   colon notation   help FunctionName

One of the most common applications we will have for a row vector is to represent an independent variable in an equation. The equation is used to calculate a second array containing the values of the dependent variable for each point in the first array. Then we will plot the resulting curves. Suppose we want an independent variable running from −20 to +30. Any of the following commands can be used to create it:

```
> x1=[-20 -10 0 10 20 30]    % manually creates a six-element row vector
> x2=linspace(-20,30,6)      % uses the linspace function: x2 is identical
                               to x1
> x3=linspace(-20,30)        % gives 100 points (default value) over the
                               specified range
> x4=-20:0.2:30              % colon notation:
                               starting value:increment:ending value
> x5=(-100:150)/5            % x5=x4 The default increment is unity in
                               colon notation
```

The **logspace** function would be inappropriate for this independent variable specification, but demonstrates the help facility. If you know a MATLAB function name but are not sure what it does or what its argument options are, you can be reminded using the help facility.

```
> help logspace          % help function_name
```

Up to now we have wanted to see the calculation results at each step. As we start dealing with large arrays and more involved command sequences, we will usually want to suppress having the results printed to the screen until we specifically ask to see them. *Concluding a command with a semicolon causes the calculation to be made but not printed to the screen.*

Suppose we want to calculate the formula $y = x^2/4$ for the set of points in x3. The dependent variable statement is

```
> y1=(x3.*x3)/4;     % the parentheses are only for clarity in expressions
                       y1, y2, and y3
> y2=(x3.^2)/4;      % y1, y2, and y3 are all the same result
> y3=(x3.^2)./4;     % no dot is needed for division by the scalar, but it
                       would be accepted
```

Let's reward ourselves by plotting the result.

```
> plot(x3,y1)             % plots y1 versus x3
```

plot grid title xlabel ylabel gtext axis hold

```
> grid                    % add an optional grid and proper documentation for
                            your graph
> title ('anything you want to say at the top of the graph')
> xlable ('x axis variable and units')
> ylable ('y axis variable and units')
> gtext ('puts this text wherever you click on the graph')
> axis ([0 20 0 400])     % sets graph axes: axis ([xstart xend ystart yend])
> hold                    % holds the current graph while more curves are
                            added to it
> plot (x3, y2, 'r+')     % same plot using just data points marked by a red +
> hold                    % the hold command cycles the hold action on and off
```

The **plot** command interpolates between actual calculated points in order to produce a smooth curve. Sometimes that interpolation is not wanted. To plot data points only, specify a data point marker without any linetype. Try some of the line/data point options suggested by the help facility.

```
> help plot
```

Commands are available to remind you what variable names you have used, the dimensions of your arrays, or both, and to allow you to clear all or some of your variables from the command window.

who size whos length clear

```
> who                  % identifies existing variable names
> size (variable_name) % identifies the row and column dimensions of the
                         variable_name array
> whos                 % does both of the above, including bytes allocated
> length(x)            % identifies the largest dimension of array x. If
                         x is a vector, this gives the number of elements
                         in x
> clear x y Y          % clears variables, x, y, and Y
                       % clear without a qualifier clears all variables;
                         watch out for this!
```

diary load workplace save workplace

In future MATLAB sessions, or when doing MATLAB problem assignments, you may wish to keep a record of your Command Window activities. If you do not understand MATLAB's response to your commands, obtaining a printout of the Command Window will allow you to seek an explanation from your instructor or from other experienced MATLAB users. Your instructor may also require such a record for graded homework or test problems. The **diary** command will duplicate the Command Window activities in a text file. You could also just cut and paste the Command Window to a text editor.

```
> diary filename       % on a networked Microsoft Windows system the
                         filename usually must include an A: prefix.
```

If you have filled your Command Window with unnecessary array elements because you forgot the semicolon at the end of an instruction, your instructor will probably appreciate your editing those sections from your text file before obtaining a hard copy.

If you must interrupt a MATLAB session, and want to be able to pick up where you left off, you can select **save workplace** under the **file** menu and be prompted for a filename and destination. The workplace variable names and values are stored in a digital format MAT-file. You cannot **open** this file, but you can **load** it while in MATLAB. If you want to try this out, do the following:

```
> whos                        % shows the current workplace variables
save workplace A: saver.mat   % (assumes a Microsoft Windows operating
                                system)
> clear
> whos                        % verify that the workplace is now empty
> load A: saver.mat           % or select load workplace from the file
                                menu
> whos                        % verify that the workplace variables have
                                been restored
```

EXAMPLE 1.8

Use MATLAB to plot the phase of $F(z) = \dfrac{2\,(z+4)}{(z-1)}$ along the path $z = x + j2$.

Solution

This is the same function evaluated in Example 1.6. If we select the same range for x as in that example, the programming is identical no matter what property of F is desired. See Figure 1.8.

```
> x=linspace(−6, 4);            % create a 100-point independent variable
> z=x+2j;                       % now the range of z has been defined
> F=2*(z+4)./(z−1);             % calculate the function for each value of z
> plot(x,180*angle(F)/pi)       % plot phase in degrees
> grid
> xlable('x')
> ylable('phase in degrees')
```

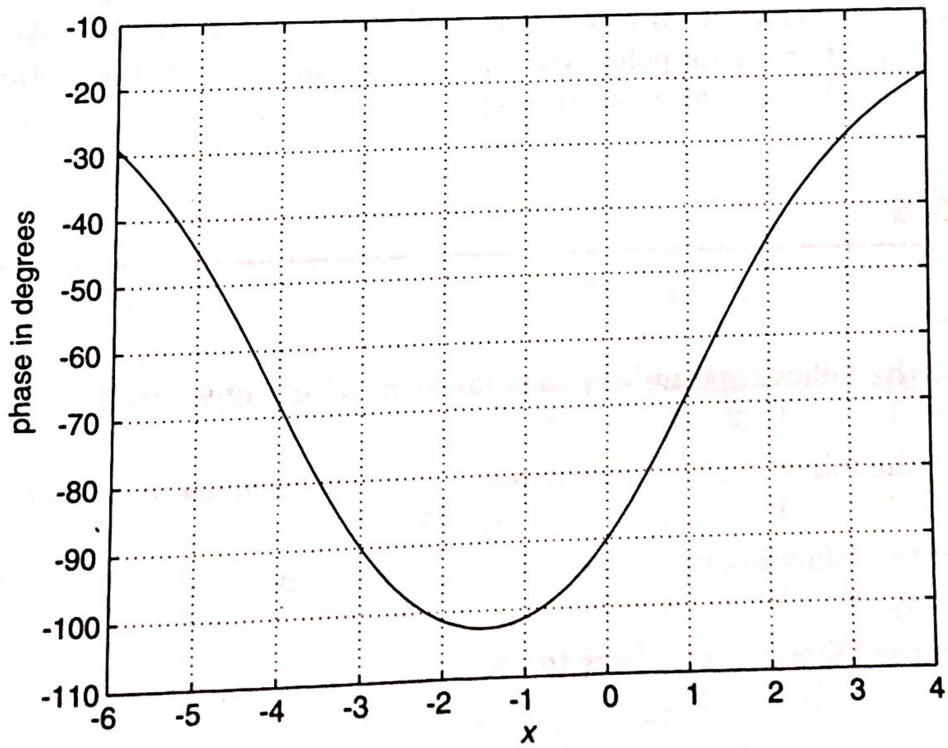


Figure 1.8 The phase of the function of Examples 1.6 and 1.8 evaluated along the path $z = x + j2$.

In this example, F was just specified in a formula, as it might have been in any general-purpose programming language. A much simpler way of describing this type of function is available with MATLAB; it will be introduced in the next chapter.

SUMMARY

Complex numbers are like aliens from the planet of Mathematics. They are foreign to our everyday experience, but they have many beneficial characteristics once we become friendly with them. This chapter has attempted to review how to make arithmetic calculations with complex numbers and how to convert between their polar and rectangular forms, and it has introduced a software package that routinely works with them. We have also encountered Euler's identity, which establishes the mathematically correct form for our shorthand polar-number notation. Euler's identity also shows us how a sinusoid can be expressed in terms of a more elementary function, namely, the exponential.

Complex variables lead to complex functions. Unlike real functions, which have only a magnitude and a sign, complex functions have a magnitude and an angle, or a real part and an imaginary part. It becomes important to be clear which of these parts of the function are the desired features. We have considered ways to present information about a complex function graphically, and have settled on a simple method called a pole-zero diagram.

Finally, we have reviewed the significance of a division by zero in functions, and distinguished between infinite values and indeterminate values. We will avoid evaluating a function at its poles. We will use the methods of L'Hopital to assign functions values at points of indeterminacy.

PROBLEMS

Section 1.2

1. Convert the following numbers to polar form with angles in degrees.
   a. $-2 + j1$ b. $2 - j4$ c. $1 - j3$
2. Convert the following numbers to polar form with angles in radians.
   a. $-1 + j3$ b. $3 + j2$ c. $3 - j4$
3. Convert the following numbers to rectangular form.
   a. $\pi\underline{/2 \text{ rad}}$ b. $2\underline{/-4 \text{ rad}}$ c. $5\underline{/18 \text{ rad}}$
4. Convert the following numbers to rectangular form.
   a. $4\underline{/27^\circ}$ b. $2\underline{/-120^\circ}$ c. $5\underline{/180^\circ}$

Section 1.3

5. Determine the magnitude and angle of the following complex numbers.
   a. $3e^{j\pi}$ b. $-2e^{j2}$ c. $0.25e^{-j5}$

6. Express these numbers in rectangular form.
   a. $e^{-j\pi/4}$ b. $3e^{j4}$ c. $2\pi e^{j1}$
7. Express the given sum as a sum of sine or cosine functions.
   a. $-3e^{-j2} + 2e^{-j} + 2e^{j} - 3e^{j2}$ b. $4e^{-j3} - e^{-j\pi/2} + e^{j\pi/2} - 4e^{j3}$

Section 1.4

8. Evaluate the expressions and leave the result in rectangular form for $z_1 = 1 + j2$, $z_2 = 2 - j1$, and $z_3 = -1 + j$.
   a. $z_1 + \dfrac{z_2}{z_3}$ b. $\dfrac{z_1 z_2{}^*}{z_3}$ c. $\dfrac{z_1 - z_3{}^*}{z_2}$
9. Evaluate the expressions and leave the result in rectangular form for $z_1 = 2 + j3$, $z_2 = 1 - j3$, and $z_3 = 0 + j2$.
   a. $z_1 z_2 z_3$ b. $\dfrac{z_1 + z_2}{z_3{}^*}$ c. $(z_1/z_2)^3$
10. Evaluate the expressions of Problem 9 and put the result in polar form for $z_1 = \pi\underline{/35°}$, $z_2 = 2\underline{/-140°}$, and $z_3 = 1\underline{/-50°}$.
11. Evaluate the indicated expressions and give the answer in rectangular form for $z_1 = 0.5\angle 0.24$, $z_2 = 4\angle 1.20$, and $z_3 = 3\angle -4.0$.
   a. $\dfrac{z_1 z_2}{z_2 + z_3}$ b. $\left(\dfrac{z_1}{z_2 + 1}\right)^2$ c. $z_1 + z_2{}^* + z_3$

Section 1.5

12. For the function $F(z) = z^2$, where $z = x + jy$:
   a. Determine the locations where $\mathrm{Re}F = 1$.
   b. Determine the locations where $F = 1$.
13. For the function $F(z) = 1/(z - 1)$, where $z = x + jy$:
   a. Determine the conditions under which $|F| = 2$.
   b. Determine the conditions under which $F = j2$.
14. a. Sketch a fully labeled pole-zero diagram for the following complex function:

$$F(z) = \frac{2}{z + 2}$$

   b. Obtain expressions for the real and imaginary parts of $F(z)$ along the path $z = -1 + jy$, and sketch each for $0 < y < \infty$. Identify actual values at $y = 1$.
15. a. Sketch a fully labeled pole-zero diagram for the following complex function:

$$F(z) = \frac{3z}{z + 2}$$

   b. Obtain expressions for the magnitude and angle of $F(z)$ along the path $z = 0 + jy$, and sketch each for $0 < y < \infty$. Identify actual values at $y = 2$.

Section 1.6

16. a. Evaluate $F(x)$ at $x = -1$ and $+1$:

$$F(x) = \frac{x^2 - 1}{x - 1}$$

b. Evaluate $G(x)$ at $x = 0$:

$$G(x) = \frac{1 - \cos 2x}{2x}$$

17. a. Evaluate $F(z)$ at $z = 0, 2,$ and 10:

$$F(z) = \frac{z^3 - 6z^2 - 40z}{z^2 - 12z + 20}$$

b. Evaluate $G(z)$ at $z = 2$:

$$G(z) = \frac{\sin[2(z - 2)/5]}{(z - 2)}$$

18. In numerical problems a calculator can often resolve indeterminate values. Simply calculate the function at a point extremely close to the value causing the indeterminate result. For example, if a function is indeterminate at $z = 1$, calculate the function at $z = 1 \pm 10^{-4}$. Use this technique to resolve any indeterminate values in Problem 17.

Section 1.7

19. Show two ways of producing a MATLAB row vector that starts at 0 and ends at π, in 13 equally spaced points.
20. Given the row vectors $x = [3\ 6\ 9]$ and $y = [5\ 3\ 0]$, what are the results of the following MATLAB operations?
    a. x + y   b. x.*y   c. x.^y   d. x./y
21. Describe the result of the following MATLAB command sequence:

```
>t=linspace(0,pi/2,123);
>y=sin(2*t);
>plot(t,y,'r')
```

Additional Problems

22. a. Sketch a fully labeled pole-zero diagram for the following complex function:

$$F(z) = \frac{z^2 + 1}{z^2 - 1}$$

b. Based on your pole-zero diagram, sketch roughly how you would expect the magnitude of F to vary if the function were evaluated along the path

$z = 1.1\angle\theta$. Don't try to specify any actual magnitudes, just where the peaks and valleys are expected for $-\pi \geq \theta \geq \pi$.

c. Use MATLAB to plot $|F|$ along the path indicated, and compare to your expectations.

d. Does the real part of F pass through zero over this path?

23. Use MATLAB to demonstrate that if $F = 6/z$, the real part of $F = 3$ almost everywhere on a circle of unit radius centered at $z = 1$ (refer to Example 1.4).

24. Given the function $F(z) = \dfrac{z-1}{z}$,

a. Determine the z plane locations where $\text{Re}(F) = 0$.

b. Determine the z plane locations where $\text{Re}(F) = 1$.

c. Determine the z plane locations where $\text{Re}(F) = 2$.

d. Use MATLAB to verify your results.