5

STANDARD FILTERS

OUTLINE

OBJECTIVES

1. Define characteristics of ideal low-pass and bandpass filters.
2. Discuss the characteristics of standard filter prototypes.
3. Denormalize prototype filter polynomials.
4. Transform a prototype to any desired filter type.
5. Develop simple prototype circuits.
6. Use MATLAB to find, denormalize, and transform filter polynomials.

INTRODUCTION

A common problem in signal processing is to isolate a desired signal from an interfering signal. When the signals occupy different frequency ranges, a filter is used to accomplish this task. Before we can implement a filter, we must first identify the properties that an ideal filter should have. Then we can search for a transfer function that will best approximate as many of those ideal characteristics as possible. This process has led to the Butterworth, Chebyshev, and other standard filter polynomials detailed in design handbooks.

In the past, the next stage in the design process was to match the filter transfer function to a circuit that could implement it. In power applications, this is still a necessary step. Today, however, the filtering of low-power signals can be accomplished with software rather than hardware, and with superior results. Some filter circuits are still needed to get the signal into and out of the digital processor, but these can be relatively simple and inexpensive.

The filter design handbook has also been made obsolete by software. The standard filter polynomials can be defined easily in the *s* domain, and denormalized or transformed to the particular filter type desired. Although these transformations involve only algebra, they are very difficult to accomplish in high-order filters without the assistance of appropriate computer software.

5.1 IDEAL FILTERS

An ideal filter cannot be built, but it establishes the properties to be strived for in real filters. After completing this section you will be able to:

- Define characteristics of ideal low-pass and bandpass filters.
- Define *cutoff frequency, passband,* and *stopband.*
- Explain the importance of constant gain in the passband.
- Explain the importance of linear phase in the passband.

In the frequency domain, the properties of an ideal filter seem obvious. Every sinusoid in the desired signal should emerge from the filter unchanged, while any other sinusoids should not emerge from the filter at all. The ideal filter frequency response is divided into passbands and stopbands, with the boundaries between them identified as cutoff frequencies.

Actually, there is no objection to signals in the passband emerging amplified, so long as all emerging sinusoids receive exactly the same gain. If this is the *only* change experienced by the signal, the output is an amplified duplicate of the input. If sinewaves in the passband are not amplified equally, the output suffers from *amplitude distortion.*

It is impossible to build a frequency-selective circuit that does not shift the phase of sinewaves passing through it. Phase shift is also not a problem, provided the passband sinusoids are all delayed the same amount *in time*. This requires a phase shift that decreases linearly with frequency. The output is still an exact replica of the input, but it experiences a *phase delay*, τ_p, given by Equation 5.1. Any other phase shift relationship introduces *phase distortion* in the filter output.

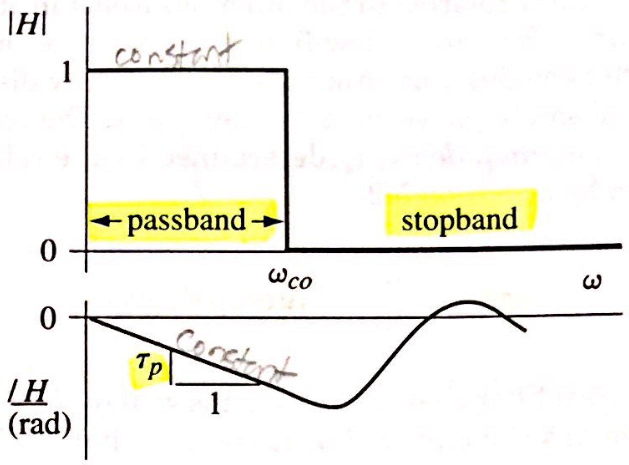


Figure 5.1 Ideal low-pass filter characteristics: constant amplitude gain and constant phase delay over the passband, no signal passage in the stopband. The edge of the passband is defined by the cutoff frequency ω_{co}.

$$\cos \omega(t - \tau_p) = \cos(\omega t - \omega\tau_p) = \cos(\omega t + \theta)$$

$$\tau_p = -\frac{\theta}{\omega} \qquad \text{(phase delay)} \tag{5.1}$$

An ideal low-pass filter introduces neither amplitude nor phase distortion and has the characteristic shown in Figure 5.1.

The characteristics of an ideal bandpass filter (Figure 5.2) are essentially the same as for the low-pass case, except it is only necessary for the time delays experienced in

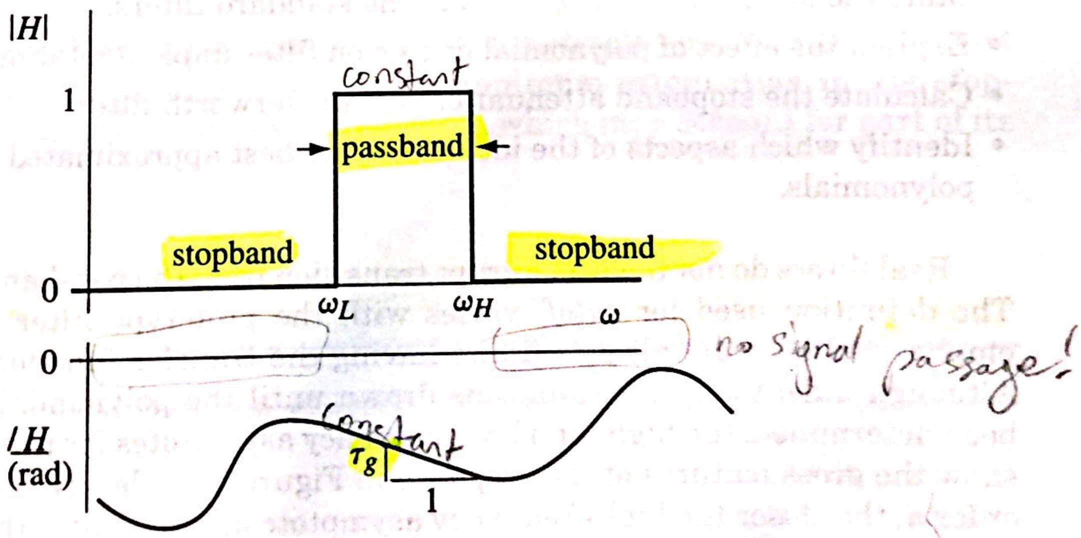


Figure 5.2 Ideal bandpass filter characteristics: constant amplitude gain and constant group delay over the passband, no signal passage in the stopband. The edges of the passband are defined by the cutoff frequencies ω_L and ω_H.

the passband to be constant relative to the other sinusoids in that region. Bandpass-type signals usually arise from using low-frequency information to modulate a high-frequency carrier. The demodulation process will eventually discard the carrier, so it is unimportant what absolute phase shift it experiences. The modulation envelope at the output experiences a *group delay*, τ_g, determined by the relative phase shift over the passband as given by Equation 5.2.

$$\tau_g = -\frac{d\theta}{d\omega} \qquad \text{(group delay)} \tag{5.2}$$

In defining these ideal filter characteristics, it is well to remember that the natural response of the system has been ignored. It is possible that the ideal filter for steady-state sinusoids may not be the best indicator of performance in applications where abrupt changes in the input signal occur regularly and excite the circuit's natural response. Standard filters arise from compromising one or another aspect of the ideal filter to achieve a practical result. The "best" filter is the one that achieves the most acceptable compromises in performance, as well as in cost, for the application intended.

5.2 FILTER PROTOTYPES

Mathematicians have identified polynomials that provide an optimal match to some features of the ideal filter. To provide a unified design procedure, their results are summarized in a prototype filter, which is a low-pass filter with a cutoff frequency of 1 rad/s. After completing this section, you will be able to:

- Discuss the characteristics of standard filter prototypes.
- State the definition of *cutoff* used in the standard filters.
- Explain the effect of polynomial degree on filter implementation.
- Calculate the stopband attenuation of a Butterworth filter.
- Identify which aspects of the ideal filter are best approximated by which polynomials.

Real filters do not have an abrupt transition from the passband to the stopband. The definition used for *cutoff* varies with the prototype filter. Two that will be emphasized here are all-pole filters having the transfer function of Equation 5.3. Although a detailed SLA cannot be drawn until the polynomial coefficients have been determined, the high- and low-frequency asymptotes for this transfer function show the gross features of the response in Figure 5.3. Clearly, the higher the filter order n, the closer the high-frequency asymptote approximates the infinite drop-off rate of the ideal low-pass filter. One may also assume that higher-order polynomials would, because of their greater number of terms, permit a tighter fit to the ideal characteristics. Actual filters are named for the investigators who specified the polynomial coefficients that best approximate some feature of the ideal filter.

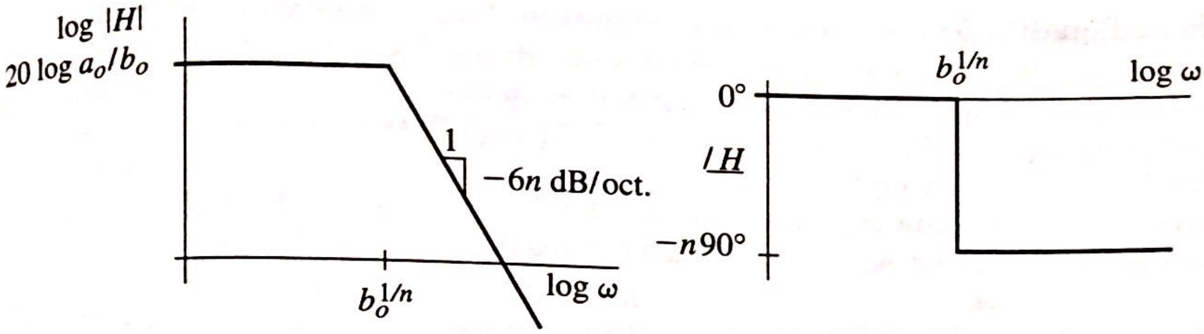


Figure 5.3 High- and low-frequency asymptotes for the all-pole filter characteristic of Equation 5.3. The frequency at which these asymptotes intersect does not necessarily have any physical significance.

$$H(s) = \frac{a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0} \tag{5.3}$$

The *Butterworth* filter is one of the most popular of the all-pole filters. It does the best job of approximating the flat gain desired in the passband. An nth-order Butterworth prototype has a "maximally flat" passband and a transfer function that reduces to the simple form of Equation 5.4. This equation shows that cutoff for the Butterworth filter is defined by the −3-dB (0.7071) point, which is the standard definition of cutoff used in most circuits. The only parameter to specify for a Butterworth prototype is the filter order, n.

$$|H(\omega)|^2 = \frac{1}{1 + (\omega/\omega_{co})^{2n}} \tag{5.4}$$

Since the Butterworth transfer function takes this simple form, it may be solved to find the required n to meet a particular minimum attenuation in the stopband. It is the only one that takes a simple form, which may account for part of its popularity.

EXAMPLE 5.1

What is the required order for a Butterworth low-pass filter if the gain must be down 20 dB at $\omega = 2\omega_{co}$?

Solution

A gain of −20 dB is equivalent to an $|H|$ of

$$-20 = 20 \log |H| \qquad \text{or} \qquad |H| = 0.1$$

Using Equation 5.4,

$$1 + \left(\frac{\omega}{\omega_{co}}\right)^{2n} = \frac{1}{(0.10)^2} = 100$$
$$(2)^{2n} = 99$$
$$2n \log 2 = \log 99$$
$$n = 3.31$$

A fourth-order filter is required.

The frequency response for a third-order Butterworth is shown in Figure 5.4. The poles of a Butterworth prototype can be shown to be equally spaced in the LHP along a unit circle centered on the origin of the s plane.

Another popular all-pole prototype is the *Chebyshev* type 1 filter. By adjusting the polynomial coefficients, Chebyshev was able to trade off flatness in the passband for a higher *initial rate* of attenuation in the stopband. The Chebyshev polynomials create an *equiripple passband* in which the gain is allowed to vary a specified amount. The more variation allowed in the passband, the higher the initial attenuation in the stopband. The concept of an "equiripple passband" suggests that the appropriate definition for cutoff would be the frequency at which the gain finally exceeds the variation specified. In the prototype, this always occurs at 1 rad/s.

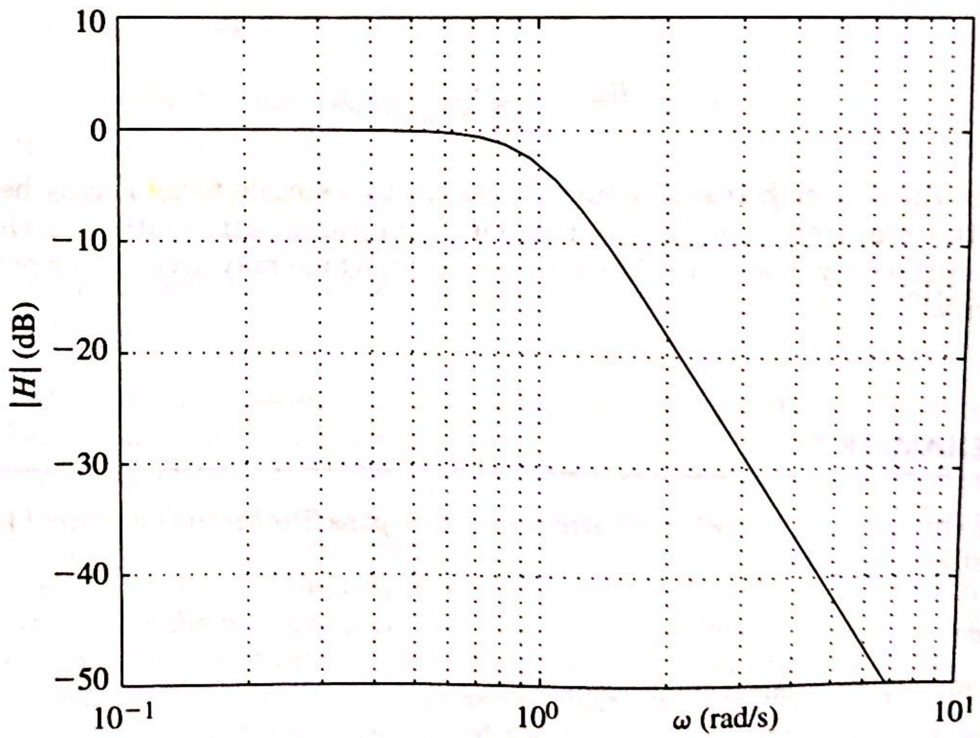


Figure 5.4 Response of a third-order Butterworth prototype.

An nth-order Chebyshev (type 1) prototype response has a total of n maximums and minimums in the passband. All the maximums are the same height, and all the minimums are the same depth, which is the reason for the *equi*ripple designation. The poles of a Chebyshev (type 1) prototype are in the LHP and are equally spaced along an ellipse whose eccentricity is related to the allowed passband ripple. Specifications for a Chebyshev prototype include the filter order, n, and the allowed passband ripple, in decibels. A third-order Chebyshev prototype with 1-dB passband ripple is shown in Figure 5.5.

The Bessel prototype is another all-pole filter, but its polynomial coefficients are chosen to best approximate the linear phase characteristic of the ideal filter. It, like the Butterworth, is fully specified by the filter order, n, and the design process is identical for both. Although the phase response of a filter may be more important than the amplitude response in some situations, we will find that a linear phase characteristic is easily achieved in digital filtering. As a result, the Bessel polynomials will be of little future interest to us, and we will not discuss them further.

Placing zeros in the stopband region of a low-pass transfer function leads to another interesting class of filters. One scheme that results in such a transfer

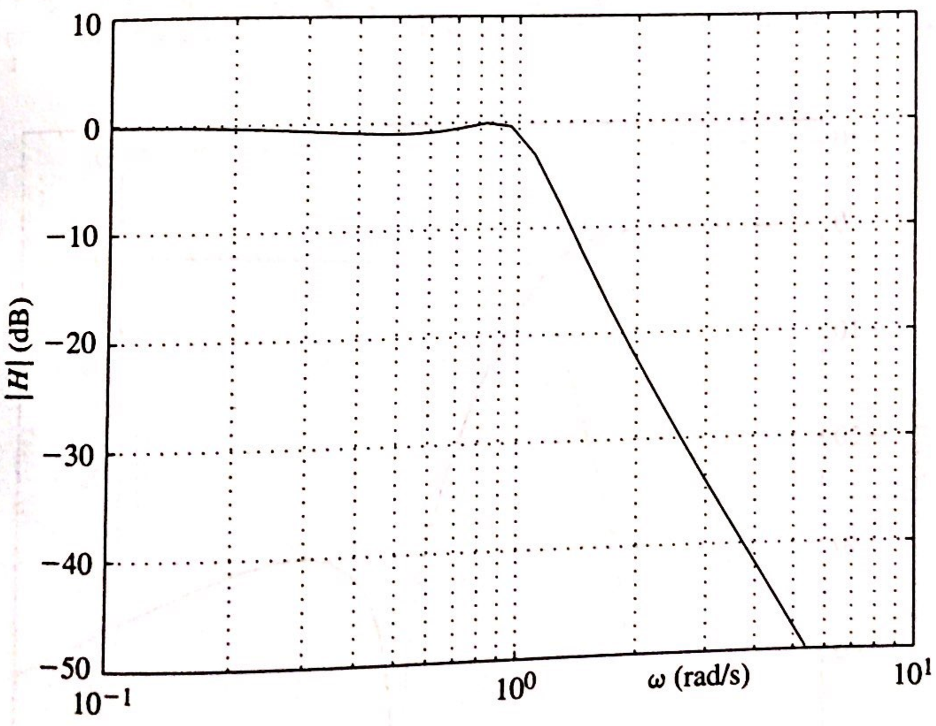


Figure 5.5 A third-order Chebyshev (type 1) prototype with 1-dB passband ripple. Cutoff is defined as the frequency where the gain drops below −1 dB. Although equal-order all-pole filters have the same ultimate *rate* of attenuation, the high-frequency asymptote of the Chebyshev filter has a lower break frequency than the equivalent Butterworth filter. In addition, the Chebyshev response approaches its high-frequency asymptote from above, while the Butterworth approaches its asymptote from below. This gives the Chebyshev filter a higher initial rate of stopband attenuation, and a greater amount of stopband attenuation.

don't worry about type #2 Chebyshev ONLY #1

function is to start with a Chebyshev all-pole prototype, transform it into a high-pass filter, and subtract it from unity. The result is a low-pass prototype with the ripple moved from the passband into the stopband. This prototype retains the abrupt transition from the passband to the stopband that the Chebyshev polynomials provide, and it is called a Chebyshev type 2 filter. Its polynomial coefficients are directly related to the type 1 filter. The specifications for this filter type include the order, n, and the minimum attenuation allowed in the stop band. The cutoff frequency of the type 1 Chebyshev prototype translates into the starting frequency of the type 2 stopband. Although it is a viable filter option, it is not particularly popular, primarily because the design does not define the end of the pass region. When we refer to a Chebyshev filter, it will mean the type 1 filter unless otherwise stated. A third-order type 2 Chebyshev filter with a minimum attenuation of 30 dB at $\omega = 1$ rad/s is shown in Figure 5.6.

Zeros may be placed right on the $j\omega$ axis by using, for example, series resonant traps to shunt the transmission path. Equation 5.5 shows the transfer function of such a low-pass filter.

$$H(s) = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0} \tag{5.5}$$

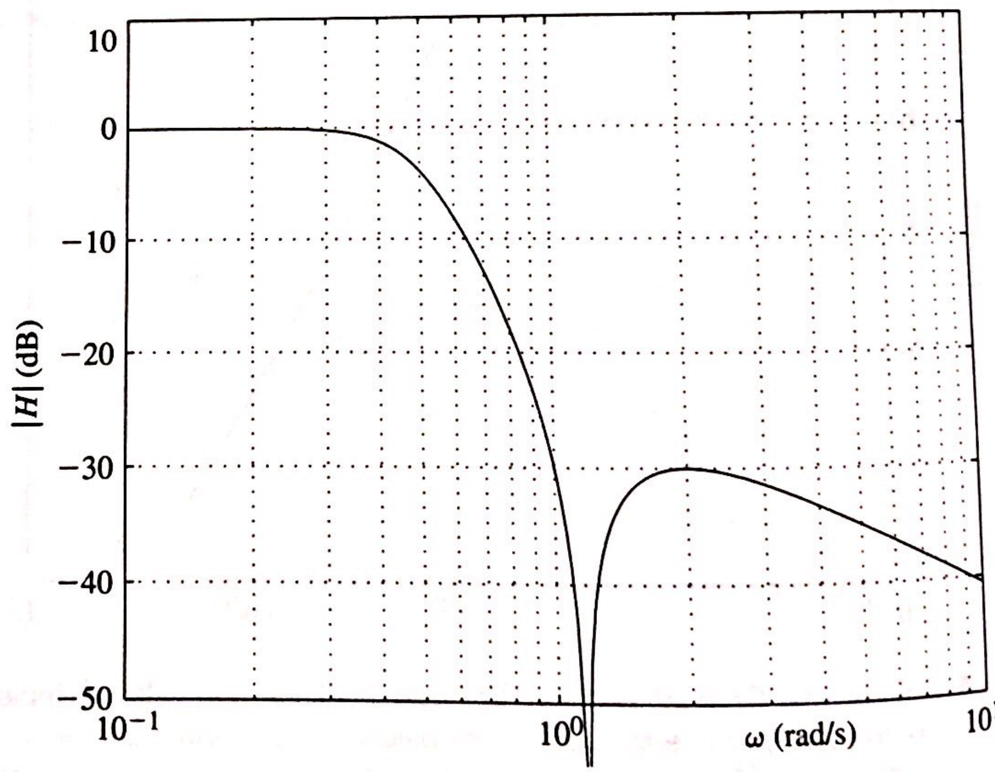


Figure 5.6 A third-order Chebyshev (type 2) prototype with a 30-dB minimum stopband attenuation. The prototype cutoff specification is replaced by the frequency where the minimum stopband attenuation is first achieved.

The high-frequency asymptote is $a_m s^{m-n}$, where m is an even integer if resonant traps provide the zeros. The high-frequency asymptote will provide increasing attenuation with increasing frequency as long as $n > m$. For the limiting case of $n = m$, the high-frequency asymptote is horizontal, but the successive zeros keep pulling the gain back to zero to create the stopband. By adjusting a_m and properly spacing the zeros, it should be possible to achieve an equiripple stopband. Allowing ripple in both the passband and the stopband provides a filter with the most abrupt transition at cutoff.

The polynomials that provide equiripple pass and stopbands are elliptical functions, of which the Chebyshev polynomials are a special case. The filters are often referred to as *Cauer filters* in honor of another prominent investigator in this area. The specifications for a Cauer filter includes n, the allowed passband ripple, in decibels, and the minimum stopband attenuation, in decibels. A third-order Cauer filter with 1-dB passband ripple and 30-dB minimum stopband attenuation is shown in Figure 5.7.

Selecting among Butterworth, Chebyshev, Bessel, and Cauer filters is also influenced by the sensitivity each polynomial has to errors in circuit component values. Some designers favor the Butterworth because its flaws tend to be less obvious than an "unequiripple" Chebyshev response. In digital filters, however, the polynomials can be made very precise, and this criterion does not enter the filter selection process.

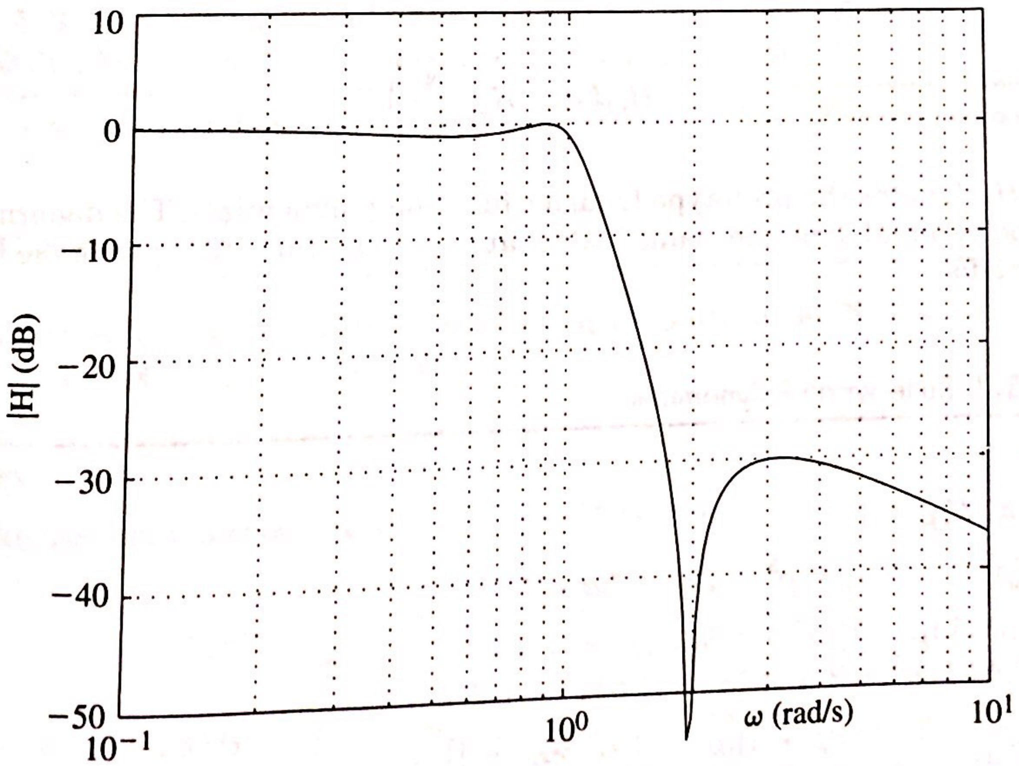


Figure 5.7 A third-order Cauer prototype with a 1-dB passband ripple and a minimum of 30 dB stopband attenuation. Cutoff is defined as the frequency where the gain leaves the equiripple passband, as in the case of a Chebyshev filter.

5.3 DENORMALIZATION

Denormalization is the process of converting a prototype filter to a low-pass filter with the desired cutoff frequency. After completing this section you will be able to:

- Denormalize prototype filter polynomials.
- Obtain the transfer function of a low-pass filter with a desired ω_{co}.
- Modify a circuit to have the cutoff frequency desired.

A filter handbook gives the prototype polynomials for a reasonable selection of filters. The Butterworth and Bessel prototype polynomials, for example, depend only on the polynomial degree, n, so they would require, at most, a page for each. Chebyshev prototypes, however, require a table of polynomials for every possible passband ripple. Even worse, Cauer prototypes would require a full set of tables for each Chebyshev table. A limited selection of prototype polynomials is given in Tables 5.1–5.3. The frequency variable is represented by p instead of s in these tables to reduce the chance for error during denormalization. If the prototype polynomial is to be denormalized for a cutoff frequency of ω_{co}, the p variable must be replaced everywhere by the expression (s/ω_{co}). The denormalization process can also be described in terms of functional notation as

$$H_{LP}(s) = H_p\left(\frac{s}{\omega_{co}}\right) \tag{5.6}$$

where H_p denotes the prototype transfer function polynomials. The denormalized filter polynomial has the same value at $s = j\omega_{co}$ that the prototype has at $p = j1$ rad/s.

Table 5.1 Butterworth Polynomials

$(p + 1)$
$p^2 + 1.4142p + 1$
$(p + 1)(p^2 + p + 1) = p^3 + 2p^2 + 2p + 1$
$(p^2 + 0.7654p + 1)(p^2 + 1.8478p + 1) =$ $p^4 + 2.6131p^3 + 3.4142p^2 + 2.6131p + 1$
$(p + 1)(p^2 + 0.6180p + 1)(p^2 + 1.6180p + 1) =$ $p^5 + 3.2361p^4 + 5.2361p^3 + 5.2361p^2 + 3.2361p + 1$

For cutoff frequency of 1 rad/s.
$a_o = 1$ for a maximum gain of 0 dB (see Equation 5.3).

Table 5.2 Chebyshev Polynomials: 0.5-dB Ripple; $\omega_{co} = 1.0$ rad/s

$p + 2.8628$
$p^2 + 1.4256p + 1.5162$
$(p + 0.6265)(p^2 + 0.6265p + 1.1424) = p^3 + 1.2529p^2 + 1.5349p + 0.7157$
$(p^2 + 0.3507p + 1.0635)(p^2 + 0.8467p + 0.3564) = p^4 + 1.1974p^3 + 1.7169p^2 + 1.0255p + 0.3791$
$(p + 0.3623)(p^2 + 0.5862p + 0.4768)(p^2 + 0.2239p + 1.0358) = p^5 + 1.1725p^4 + 1.9374p^3 + 1.3096p^2 + 0.7525p + 0.1789$

To set the maximum gain to 0 dB, make $a_o/b_o = 0.9441$ if n is even and $a_o/b_o = 1$ if n is odd (see Equation 5.3).

Table 5.3 Chebyshev Polynomials: 2.0-dB Ripple; $\omega_{co} = 1.0$ rad/s

$p + 1.3076$
$p^2 + 0.8038p + 0.8231$
$(p + 0.3689)(p^2 + 0.3689p + 0.8861) = p^3 + 0.7378p^2 + 1.0222p + 0.3269$
$(p^2 + 0.2098p + 0.9287)(p^2 + 0.5064p + 0.2216) = p^4 + 0.7162p^3 + 1.2565p^2 + 0.5168p + 0.2058$
$(p + 0.2183)(p^2 + 0.3532p + 0.3932)(p^2 + 0.1349p + 0.9523) = p^5 + 0.7065p^4 + 1.4995p^3 + 0.6935p^2 + 0.4593p + 0.0817$

To set the maximum gain to 0 dB, make $a_o/b_o = 0.7943$ if n is even and $a_o/b_o = 1$ if n is odd (see Equation 5.3).

EXAMPLE 5.2

Determine the transfer function for a third-order 0.5-dB ripple Chebyshev low-pass filter with a cutoff frequency of 1 kHz.

Solution

The prototype filter transfer function is, from Table 5.2,

$$H_p(p) = \frac{0.7157}{p^3 + 1.2529p^2 + 1.5349p + 0.7157}$$

To move cutoff to 1 kHz, each p in the prototype is replaced by the expression s/ω_{co} to obtain the denormalized low-pass filter polynomial. The desired filter is

$$H_{LP}(s) = \frac{0.7157}{(s/2\pi\cdot 10^3)^3 + 1.2529(s/2\pi\cdot 10^3)^2 + 1.5349(s/2\pi\cdot 10^3) + 0.7157}$$

Multiplying through by $(2\pi \cdot 10^3)^3$ gives a final form of

$$H_{LP}(s) = \frac{178 \cdot 10^9}{s^3 + 7.872 \cdot 10^3 \cdot s^2 + 60.60 \cdot 10^6 \cdot s + 178 \cdot 10^9}$$

There is, of course, a great deal of difference between knowing what the filter transfer function should be and having a circuit with that transfer function. Often, filter handbooks will give circuits that implement a particular prototype, and the designer will have to denormalize the circuit. Circuits also have an impedance level, which is independent of the transfer function polynomial coefficients. The load will usually be normalized to 1 Ω. In past chapters we have also normalized our circuits to impedance levels and frequencies that keep the numbers simple so that we could concentrate on the ideas presented without becoming bogged down with unwieldy powers of 10. Denormalizing our own, or handbook, circuits for a practical design is straightforward. The first step is to select new capacitors and inductors that will have the same impedance at the desired cutoff frequency as they presently have at the normalized cutoff frequency. Using primes to indicate the new values:

$$\omega'_{co}\, C' = \omega_{co}\, C \qquad \text{and} \qquad \omega'_{co}\, L' = \omega_{co}\, L$$

$$C' = \omega_{co} \frac{C}{\omega'_{co}} \qquad L' = \omega_{co} \frac{L}{\omega'_{co}} \tag{5.7}$$

where ω_{co} is usually 1. Next, the impedance level is increased by some arbitrary factor k. The value of k is selected to make one component, usually the load resistor, a specified value:

$$R' = kR, \qquad L'' = kL', \qquad \text{and} \qquad C'' = \frac{C'}{k} \tag{5.8}$$

Naturally, Equations 5.7 and 5.8 could be combined into a single operation.

EXAMPLE 5.3

The circuit of Figure 5.8 is a third-order 2-dB Chebyshev prototype. Modify it to provide a 1-kHz cutoff frequency while driving a 600-Ω load.

Solution

The new cutoff frequency is $\omega' = 2\pi \cdot 10^3$, so applying Equation 5.7 gives

$$L'_1 = \frac{1.772}{2\pi \cdot 10^3} = 0.282 \text{ mH}$$

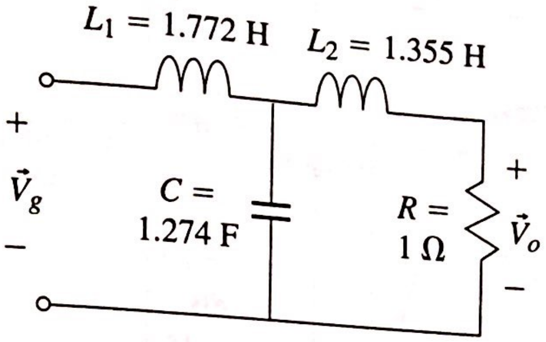


Figure 5.8

$$L_2' = \frac{1.355}{2\pi \cdot 10^3} = 0.215 \text{ mH}$$

$$C' = \frac{1.274}{2\pi \cdot 10^3} = 202.8\ \mu\text{F}$$

Now the impedance level must be raised by a factor of 600 to give the load resistance specified. Applying Equation 5.8 gives

$$R' = 600\ \Omega$$

$$L_1'' = 600(0.282 \text{ mH}) = 169 \text{ mH}$$

$$L_2'' = 600(0.215 \text{ mH}) = 129 \text{ mH}$$

$$C'' = \frac{202.8\ \mu\text{F}}{600} = 338 \text{ nF}$$

5.4 FILTER TRANSFORMATIONS

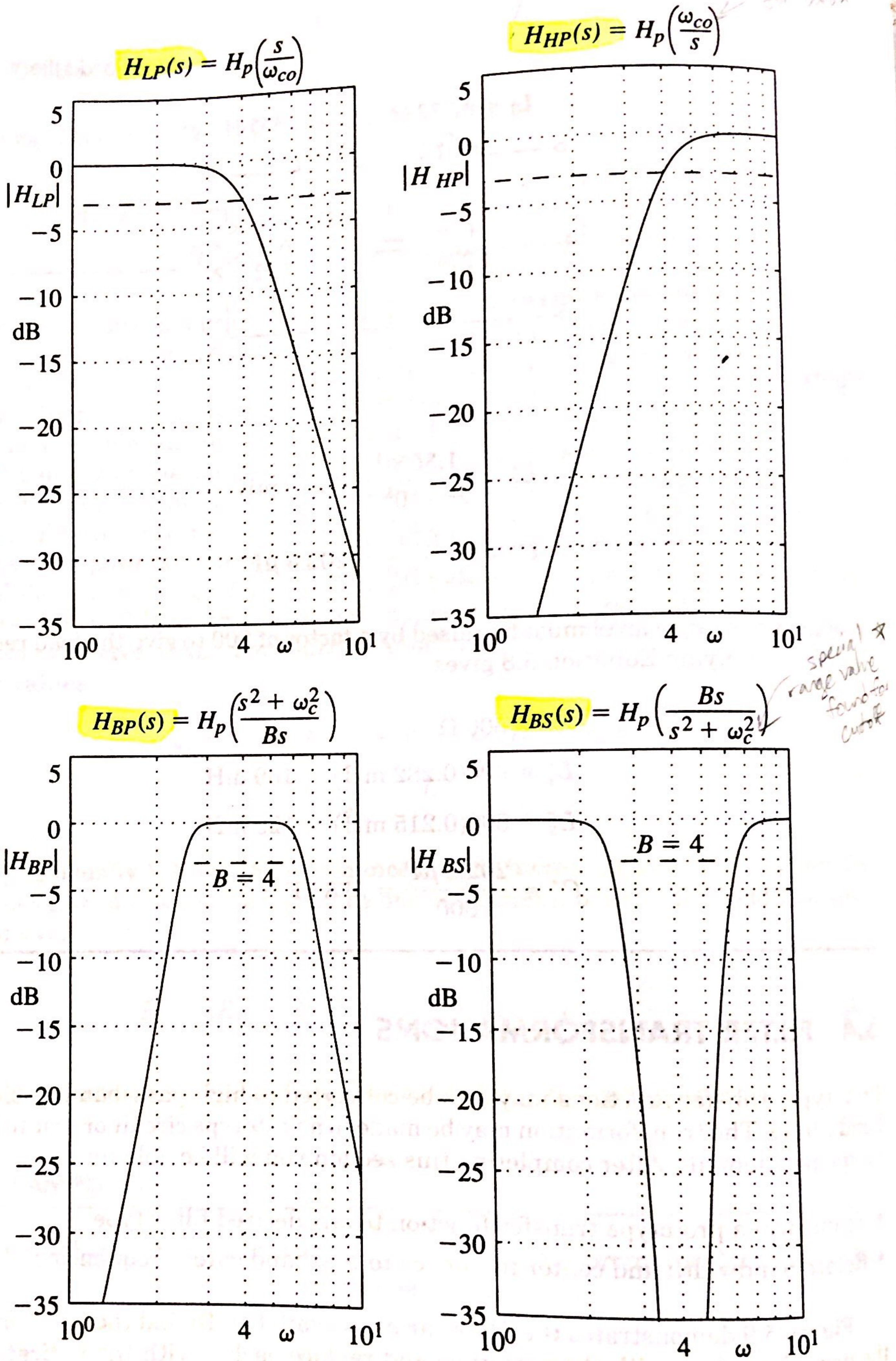
Prototype transfer functions may also be converted to high-pass, bandpass, or stopband filters. The transformation may be made to a prototype circuit or just to the prototype polynomials. After completing this section you will be able to:

- Transform a prototype transfer function to any desired filter type.
- Relate bandwidth and center frequency to passband cutoff frequencies.

Figure 5.9 demonstrates the filter varieties available. To find the polynomials for the new filter, start with the prototype and replace each p with the indicated argument. In the case of bandpass or stopband filters, the bandwidth and center frequency need to be determined. These are related to the cutoff frequencies, ω_H and ω_L, by

$$B = \omega_H - \omega_L \qquad \text{and} \qquad \omega_c = \sqrt{\omega_H \omega_L} \tag{5.9}$$

shows what "p" becomes!



special range value found for cutoff

Figure 5.9 Characteristics of low-pass, high-pass, bandpass, and bandstop filters derived from a fourth-order Butterworth prototype. Cutoff for the low- and high-pass units is 4 rad/s. Bandpass and bandstop units were designed for a center frequency of 4 rad/s and a bandwidth of 4 rad/s.

EXAMPLE 5.4

Find $H(s)$ for a second-order Chebyshev bandpass filter with 2 dB of passband ripple and cutoff frequencies of 5 and 8 rad/s.

Solution

We start with the appropriate prototype. From Table 5.3:

$$H(p) = \frac{0.6538}{p^2 + 0.8038p + 0.8231}$$

We calculate $B = 8 - 5 = 3$ and $\omega_c = \sqrt{40} = 6.325$, giving a transformation of

$$p \rightarrow \frac{s^2 + 40}{3s}$$

Then

$$H_{BP}(s) = \frac{0.6538}{\left(\frac{s^2+40}{3s}\right)^2 + 0.8038\left(\frac{s^2+40}{3s}\right) + 0.8231} * \frac{(9s^2)}{(9s^2)}$$

$3s^2 = 9s$

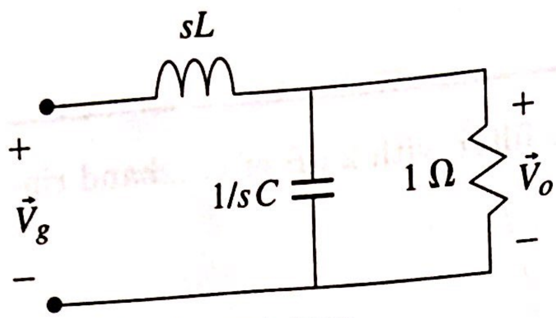
$$= \frac{0.6538(9s^2)}{(s^2+40)^2 + 0.8038(3s)(s^2+40) + 0.8231(9s^2)}$$

After a little bookkeeping,

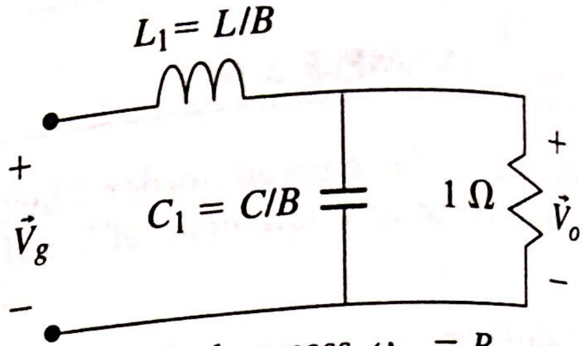
$$H_{BP}(s) = \frac{5.884s^2}{s^4 + 2.411s^3 + 87.41s^2 + 96.46s + 1600}$$

Although the process is routine, it is clear that with realistic frequencies and higher-order prototypes, the process becomes very tedious and prone to error!

Prototype circuits may also be transformed directly to, for instance, a bandpass circuit. Figure 5.10 demonstrates this process with a second-order prototype. It is of interest primarily because it shows the origin of the bandpass transformation. To achieve the correct bandwidth, the prototype must first be denormalized to a cutoff frequency equal to the bandwidth eventually desired. The inductor that passes low



2nd-order prototype.

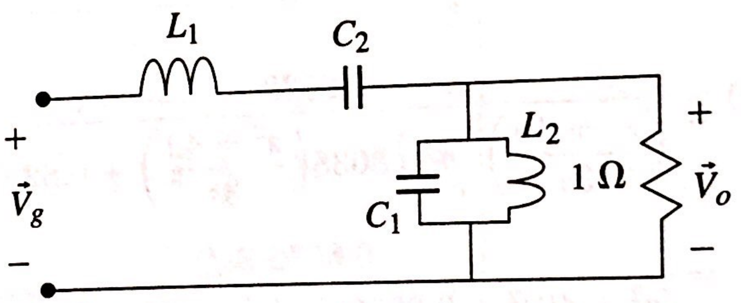


2nd-order low-pass, $\omega_{co} = B$.

$$sL_1 \rightarrow \left(sL_1 + \frac{1}{C_2 s}\right) = L_1\left(\frac{s^2 + 1/C_2L_1}{s}\right) = L\left(\frac{s^2 + \omega_c^2}{Bs}\right)$$

$$\frac{1}{sC_1} \rightarrow \frac{(1/sC_1)(sL_2)}{sL_2 + 1/sC_1} = \frac{sL_2/C_1}{L_2(s^2 + 1/L_2C_1)} = \frac{1}{C\left(\frac{s^2 + \omega_c^2}{Bs}\right)}$$

$$\omega_c^2 = \frac{1}{L_1C_2} = \frac{1}{L_2C_1}$$



2nd-order bandpass, center frequency ω_c,
and bandwidth B.

Figure 5.10 Prototype-to-bandpass transformation. First the prototype is denormalized to the proper bandwidth. Then the series path is made a series resonant circuit ($sL_1 + 1/sC_2$) and the shunt path is made a parallel resonant circuit (sL_2 in parallel with $1/sC_1$). Mathematically, the final result involves only replacing the s variables in the prototype with the bandpass transformation.

frequencies must then be replaced with a series resonant circuit that only passes signals around ω_c. Similarly, the capacitor that shunts the signal to ground at high frequencies must be replaced by a parallel resonant circuit that shunts the signal at all frequencies except those around ω_c.

The same process is used to obtain a stopband filter except that the series path must become a parallel resonant circuit while the shunt path becomes a series resonant circuit. The prototype should again be denormalized to the proper bandwidth first.

5.5 PROTOTYPE CIRCUIT DEVELOPMENT

We have been able to modify prototype polynomials or circuits to meet a design requirement. While prototype polynomials can be computer generated, prototype circuits have to be described in a handbook. We will develop our own mini-handbook to suggest the process and to provide a few prototype circuits. After reading this section you will be able to:

- Develop simple prototype circuits.
- State the advantages of active filters.

The general process of finding a circuit that implements a particular $G(s)$ is known as *circuit synthesis,* and it is usually a graduate-level course for circuit theorists. Fortunately, we do not need to be general and can just propose a low-pass circuit topology. As long as we can achieve the desired polynomial coefficients without having to provide negative resistors, inductors, or capacitors, the design can be considered successful.

The all-pole prototype filter is shown in Figure 5.11. The order of the filter equals the number of independent inductors and capacitors present. The design procedure is to obtain the transfer function for the filter order desired and to determine the relationship between the circuit components and the polynomial coefficients. This task becomes more difficult as higher-order filters are attempted.

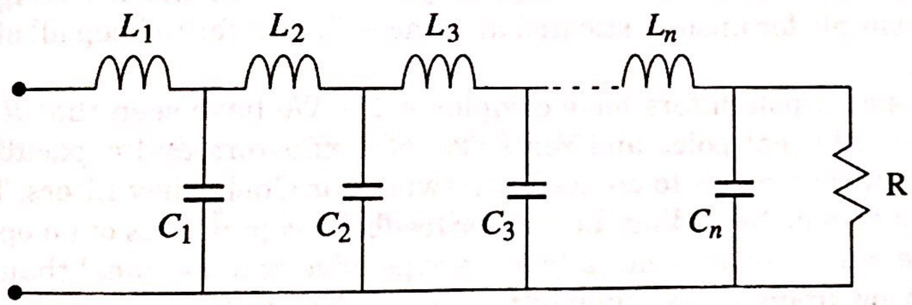


Figure 5.11 Ladder implementation of an all-pole prototype of order $2n$.

EXAMPLE 5.5

Determine the circuit components needed for a second-order Butterworth prototype filter.

Solution

The circuit of Figure 5.12 gives a second-order all-pole filter. The transfer function can be found with the voltage divider:

$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{\dfrac{R/sC}{R + 1/sC}}{sL + \dfrac{R/sC}{R + 1/sC}} = \frac{\dfrac{R}{sCR + 1}}{sL + \dfrac{R}{sCR + 1}} = \frac{1/LC}{s^2 + s/CR + 1/LC}$$

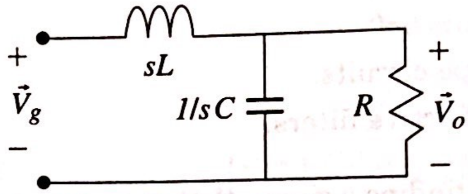


Figure 5.12

For a Butterworth prototype, $1/CR = \sqrt{2}$ and $1/LC = 1$. One of the many ways this could be accomplished is $R = 1$, $C = 0.7071$, and $L = 1.4142$. (See Problem 11, Chapter 3, page 90, for a potential third-order prototype.)

The results of Example 5.5 might be incorporated in a handbook either as a prototype circuit or as a sequence of design steps as shown in Figures 5.13a and b. Neither approach reflects the full range of options available for the design, but they keep things simple for unsophisticated audiences. We prefer to keep all of the design options open.

The classic all-pole filters have complex poles. We have seen that R-C and R-L circuits have only real poles and zeros, so both inductors and capacitors must be present in passive circuits to create Butterworth or Chebyshev filters. This limitation can be overcome by adding the gain or feedback capabilities of an op amp to the circuit. Since inductors are generally more expensive and less ideal than capacitors, especially at low frequencies, schemes that allow the synthesis of complex roots with R-C circuits are very attractive. An op amp circuit that provides this capability is shown in Figure 5.14.

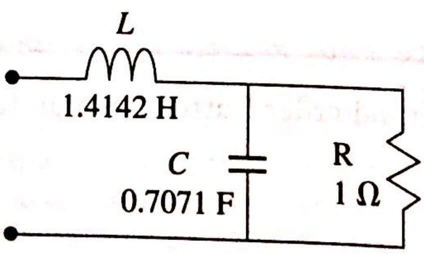


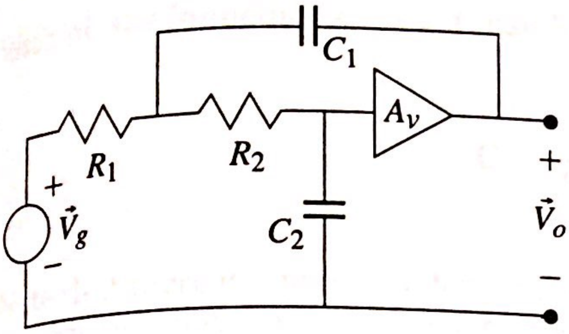
Figure 5.13a Second-order Butterworth prototype circuit.

1. Select R.

2. $C = \dfrac{1}{\sqrt{2}R}$

3. $L = \dfrac{1}{C}$

Figure 5.13b Second-order Butterworth prototype design equations.



$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{\dfrac{A_V}{R_1R_2C_1C_2}}{s_2 + \left[\dfrac{1}{C_1R_1} + \dfrac{1}{C_1R_2} + \dfrac{1-A_V}{C_2R_2}\right]s + \dfrac{1}{R_1R_2C_1C_2}}$$

Figure 5.14 An active filter quadratic section.

There are five components to adjust and only two polynomial coefficients to set. This means that several arbitrary decisions can be made in the design. The op amp also prevents loading on the circuit, so filter coefficients are independent of loading effects, and higher-order filters may be created by simply cascading second-order sections. Circuits using op amps to achieve these advantages are called *active filters*.

EXAMPLE 5.6

Design an active fourth-order Butterworth prototype filter.

Solution

Table 5.1 shows that the required polynomials are

$$(p^2 + 0.7654p + 1)(p^2 + 1.8478p + 1)$$

One popular option is to let $A_v = 1$ and $R_1 = R_2 = 1$. This reduces the transfer function of a quadratic section to

$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{1/C_1C_2}{s^2 + (2/C_1)s + 1/C_1C_2}$$

which requires $C_1 = 2/0.7654 = 2.613$ F, and $C_2 = 0.3827$ F for the first section. The values for the second section are $C_1 = 2/1.8478 = 1.0824$ F, and $C_2 = 0.9239$ F.

Active circuits are limited by the frequency capability of the op amps used and are suitable only for low-power applications. They also require power supplies for biasing the op amp, which may make them uneconomical unless other electronic circuits are already present. Note from Figure 5.14 that a large value of A_v could result in poles in the RHP! (Test #1 of the Routh-Hurwitz criterion.)

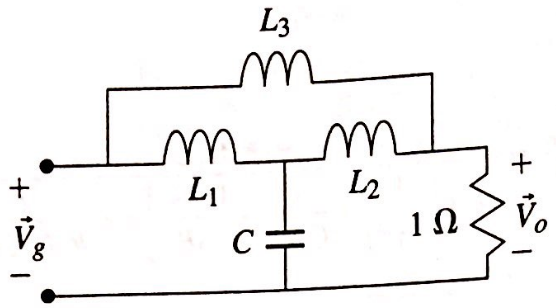


Figure 5.15 A circuit topology suitable for a Cauer third-order prototype.

Cauer filters require a different topology, since they have both poles and zeros. A third-order Cauer filter requires a minimum of five components, because it has to be able to set five polynomial coefficients independently. Its transfer function is given as Equation 5.10.

$$H(p) = \frac{a_2 s^2 + a_0}{s^3 + b_2 s^2 + b_1 s + b_0} \tag{5.10}$$

A circuit with this transfer function is shown in Figure 5.15. It does require that a_o equal b_o, but this condition only sets the circuit's d-c gain to unity, which is exactly what is desired in a third-order Cauer prototype. When the load resistance is normalized to unity, the circuit's transfer function becomes as shown in Equation 5.11. Comparing the two transfer functions provides the design equations.

$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{\frac{1}{L_3}s^2 + \frac{1}{C}\left(\frac{1}{L_1L_2} + \frac{1}{L_1L_3} + \frac{1}{L_2L_3}\right)}{s^3 + \left(\frac{1}{L_2} + \frac{1}{L_3}\right)s^2 + \frac{1}{C}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)s + \frac{1}{C}\left(\frac{1}{L_1L_2} + \frac{1}{L_1L_3} + \frac{1}{L_2L_3}\right)} \tag{5.11}$$

$$R = 1 \quad L_3 = \frac{1}{a_2}$$

and

$$\frac{1}{L_2} + a_2 = b_2 \quad \text{or} \quad L_2 = \frac{1}{b_2 - a_2}$$

Dividing b_0 by b_1 eliminates C, leaving L_1 as the only unknown. After some algebra, the result is

$$L_1 = \frac{b_1 b_2 - b_0}{(b_2 - a_2)(b_0 - a_2 b_1)}$$

With all the inductors known, C can be found from

$$C = \frac{1}{b_1}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

Note that there is a potential for some of the inductors to come out with negative values. Should that happen, another circuit topology would have to be tried or the filter specifications relaxed.

Cauer filters are not generally used in simple filtering applications and have been discussed only for the sake of completeness. They are often implemented as digital filters, however.

5.6 MATLAB LESSON 5

MATLAB will serve as our filter polynomial handbook and save us from transformation drudgery. It also allows us to reduce any self-imposed drudgery by creating scripts to execute a sequence of commands repeatedly. After completing this section you will be able to:

- Use MATLAB to find, denormalize, and transform filter polynomials.
- Write and execute MATLAB scripts.

MATLAB EXAMPLES

Matlab commands exist for finding prototype filter polynomials and for transforming them into the filter types desired. The *ap* ending to the following commands indicates *analog prototype*.

buttap **cheb1ap** **cheb2ap** **ellipap**

```
> z,p,k]=buttap(6);     % returns the zeros, poles, and multiplier for a
                          6th-order Butterworth
> num=k*poly(z);        % creates the Butterworth numerator, z is an empty
                          set
> den=poly(p);          % creates the Butterworth denominator
> w=logspace(-1,1);     % go a decade down and up from cutoff
> freqs(num,den,w);     % displays the frequency response of the Butterworth
                          prototype
> subplot(1,1,1)        % restores single graph format
```

The above sequence may be used to plot the response of any of the prototypes. The **ellipap** command is for the Cauer prototype. Command formats are

```
[z,p,k]=cheb1ap(order, ripple in dB)
[z,p,k]=cheb2ap(order, minimum stopband attenuation in dB)
[z,p,k]=ellipap(order, ripple in dB, minimum stopband attenuation in dB)
```

Try one of your choice.

The task of denormalizing or transforming to a different filter type, which gets very messy for realistic filter specs, can also be turned over to MATLAB.

lp2lp lp2hp lp2bp lp2bs

```
> tnum,tden]=lp2bp(num,den,2*pi*1000, 2*pi*500)
> % returns the polynomials for a bandpass filter centered at 1 kHz and
  500 Hz wide.
> % based on the num and den polynomials of your prototype
> format short e                    % will need to change format to
                                    see all the polynomial terms
> tnum
> tden
> w=logspace(3,5);                  % make a frequency response run
> h=freqs(tnum,tden,w);             % use this format to retain
                                      control of what is graphed
> semilogx(w,20*log10(abs(h)))
> axis([2000 20000 -80 10])         % curve shows straight-line
                                      segments, need more points or
                                      shorter range
> w=logspace(3.5,4.5, 100);
> h=freqs(tnum,tden,w);
> semilogx(w,20*log10(abs(h)))
> axis([2000 20000 -80 10])
```

In many situations a particular command sequence is used repeatedly. It can quickly become tedious retyping the same commands or even recovering them with the up-arrow key. An alternative is to create a text file consisting of the command sequence, just as you would normally enter them in the Command Window. The file is named, and given an **.m** extension. Once it has been saved, it may be called by typing its name in the Command Window.

You may create an M-file with any text editor, or by selecting **New** under the MATLAB **File** menu. Once the desired command sequence has been entered, you must **Save** the file either to the hard drive or to a floppy, and give it a name. When that name is used in the Command Window, MATLAB goes searching for an M-file of that name, and if it finds it, executes the instructions it contains. If the file is saved in the MATLAB directory, it will be found. If it is saved in another directory or on a floppy, MATLAB must be directed to include that directory or floppy in its search path. This is accomplished by following the

Set Path procedure under the **File** menu. On a Microsoft Windows system, entering **cd a:** while in the MATLAB Command Window will direct MATLAB to search the floppy before going through its usual search path.

Now that MATLAB can find your script, you can call it and edit it if necessary. *Remember to resave it after any editing.*

Variables in a script become part of the workspace and must not conflict with variable names already in the Command Window. Scripts basically are used for a special purpose and then discarded. MATLAB also allows users to create their own functions, which are similar to scripts but have many advantages. We will discuss function M-files later, but if a script has fairly widespread application, it should probably be rewritten as a function. If a script is not discarded after use, make sure it contains enough commentary to remind you what the file does and what information has to be provided before it can be called.

EXAMPLE 5.7

Your company sells 1-dB Chebyshev low-pass filters. Prepare a family of curves showing the stopband attenuation from cutoff out to four times cutoff, for even-ordered filters out to the tenth order.

Solution

Start by creating an M-file **File** **New** **M-File**

```
% This script plots the roll-off of Chebyshev 1 dB filters
% call after defining filter order and frequency range w
[z,p,k]=cheblap(n,1); num=k*poly(z); den=poly(p);
h=freqs(num,den,w);
semilogx(w,20*log10(abs(h)))
```

File **Save** (to floppy) **As** **ez1.m**

Back in the Command Window (Microsoft system)

```
> cd a:                            % put the floppy in the search path
> w=logspace (0,.7); n=2; ez1      % define n and w and execute the script
> hold
> n=4; ez1
> n=6; ez1
> n=8; ez1
> n=10; ez1
> axis([1 4, -100 10])
> grid                             % complete the labeling, and use gtext to
                                     label individual curves
```

See Figure 5.16 for the result.

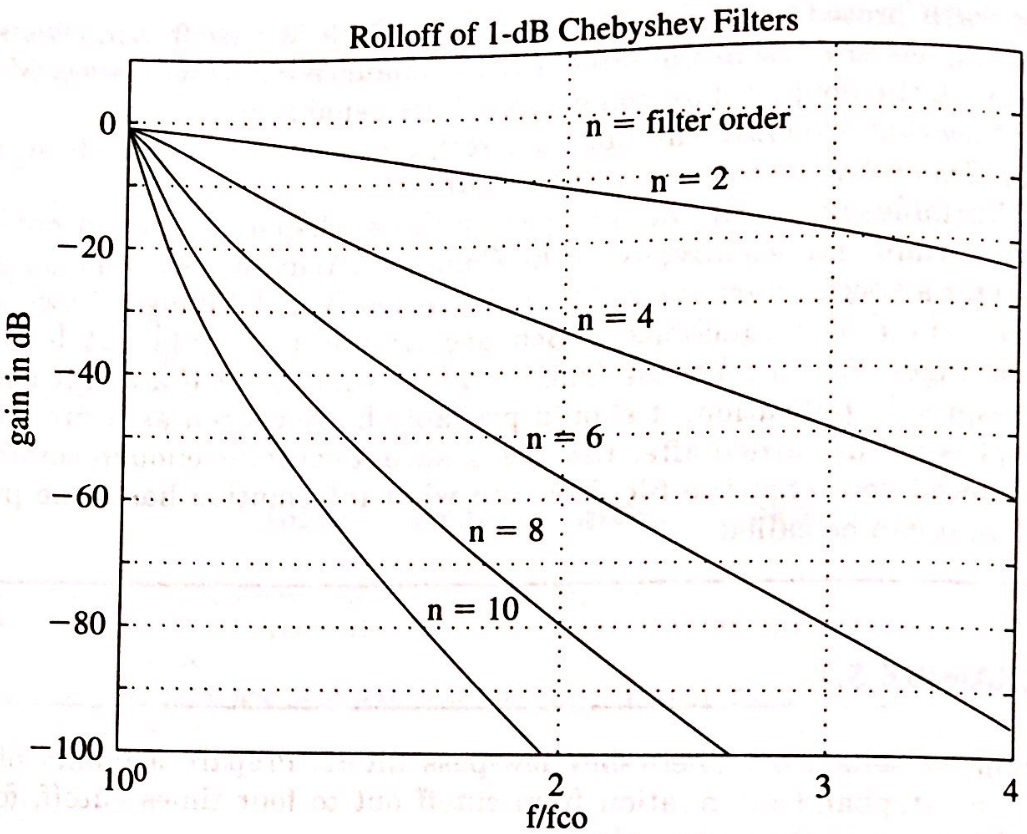


Figure 5.16

CHAPTER SUMMARY

For a signal to pass through a filter without distortion, the filter must have a constant gain and linearly decreasing phase over its passband. For interfering signals to be completely eliminated by the ideal filter, an abrupt transition to the stopband and zero gain in the stopband is needed. Butterworth, Bessel, Chebyshev, and Cauer filters all seek to approximate one aspect of these ideal characteristics by compromising the others.

Prototype filters are low-pass filters with a cutoff frequency of 1 rad/s. Actual filter polynomials are obtained from the prototype by denormalization and transformation. For implementation, circuit transfer functions are compared to the filter polynomials to arrive at design equations.

In recent years, handbooks of filter polynomials have been replaced with computer software that can generate the prototypes directly. In addition, low-power signals are being filtered by a digital program rather than a circuit. This technique allows much higher-order filters than are practical with circuits and thus much closer approximations to the ideal.

PROBLEMS

Section 5.1

1. A signal having frequency components below 2 rad/s is passed through a filter with the characteristics shown in Figure P5.1. Compare the input and output signals.

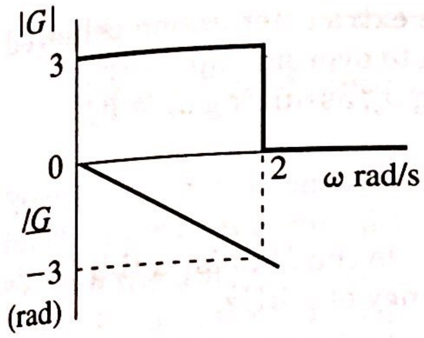


Figure P5.1

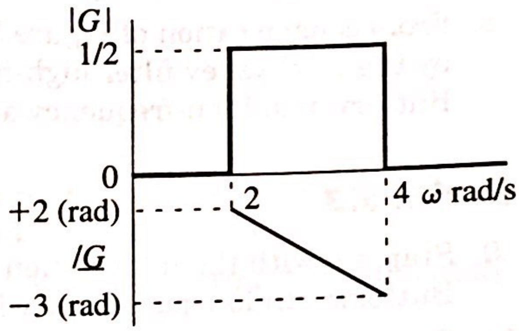


Figure P5.2

2. A signal having frequency components between 2 and 4 rad/s passes through the filter of Figure P5.2. Compare the output and input signals.

3. To demonstrate the effects of phase and group delay, create the function $f = \cos 9t + \cos 10t - \cos 10.5t$ for $0 \le t \le 12$, using about 500 points. This signal is input to a bandpass filter with a constant phase delay of 2 seconds. The output is $g = \cos(9t - 18) + \cos(10t - 20) - \cos(10.5t - 21)$.
   a. Verify that the expression for g corresponds to f with a phase delay of 2 seconds. Plot f and g to show they are *identical* except for the 2-second delay.
   b. Create a new signal, x, by adding any arbitrary number (say, 5) to each phase term in the expression for g. This signal no longer has a constant phase delay, but it does still have a constant group delay of 2 seconds. Create two more new signals, y and z, by adding different arbitrary constants. Plot g, x, y, and z on the same graph. Observe that all the functions are different but that they all fit in the same envelope and that the envelope is delayed by 2 seconds.

4. Using the identity $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$, show analytically that if the amplitude-modulated signal $f(t) = (1 + m \cos \omega_m t) \cos \omega_c t$ passes through a unity-gain bandpass filter whose phase plot is as shown in Figure P5.4, then the output is $g(t) = [1 + m \cos \omega_m(t - \tau_g)] \cos \omega_c(t - \tau_p)$.

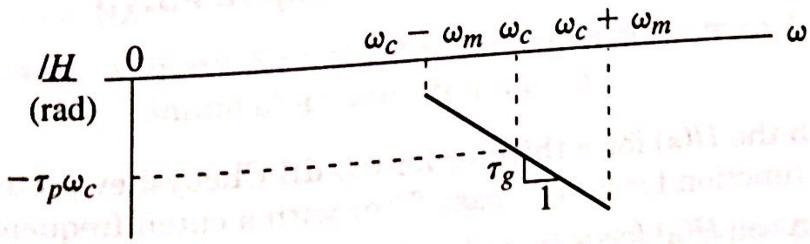


Figure P5.4

Section 5.2

5. Insert the third-order Butterworth prototype coefficients into Equation 5.3 and show that it reduces to the form of Equation 5.4.
6. Determine the order of a Butterworth low-pass filter required if the gain is to be down 80 dB at $f = 2f_{co}$.
7. What order Butterworth low-pass filter can be used if the gain needs to be down 20 dB at $f = 1.5f_{co}$?
8. From consideration of Figure 5.3, show that the extra attenuation achieved by the Chebyshev filter high-frequency asymptote over an equal-order Butterworth high-frequency asymptote is 20 log b_o, assuming $a_o = b_o$.

Section 5.3

9. Starting with the information in Table 5.1, provide the $H(s)$ for a third-order Butterworth low-pass filter with a cutoff frequency of 4 kHz.
10. Starting with the information in Table 5.2, provide the $H(s)$ for a third-order Chebyshev 0.5-dB low-pass filter with a cutoff frequency of 1 kHz.
11. Starting with the third-order 2-dB Chebyshev prototype filter of Figure 5.8, provide a schematic of a third-order 2-dB Chebyshev low-pass filter with a cut-off frequency of 20 kHz and a 4.7-kΩ-load resistor.
12. The circuit of Figure P5.12 is a notch filter whose notch is at 2 rad/s. Show how you would modify the circuit so that it notches at 120 Hz using a 10-H inductor.
13. The circuit of Figure P5.13 is a Butterworth low-pass filter with a cutoff frequency of 4 rad/s. Modify the circuit so that it cuts off at 4.0 kHz and supplies a 1.2-kΩ load.

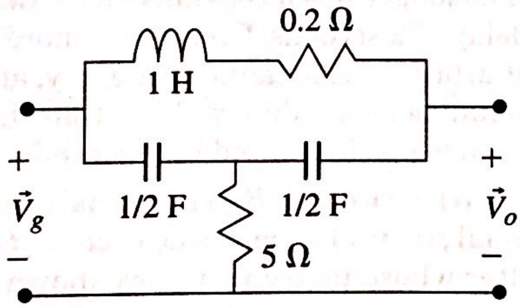


Figure P5.12

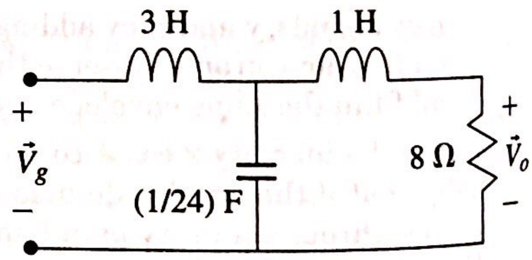
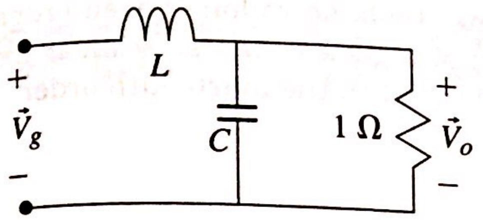


Figure P5.13

Section 5.4

14. Starting with the $H(s)$ for a third-order 0.5-dB Chebyshev prototype, obtain the transfer function for a high-pass filter with a cutoff frequency of 1 kHz.
15. Starting with the $H(s)$ for a second-order Butterworth prototype, obtain the transfer function for a bandpass filter with cutoff frequencies of 3 and 5 rad/s.

16. Provide a schematic for a second-order 2-dB Chebyshev bandpass filter having cutoff frequencies of 1 kHz and 4 kHz and a 1-kΩ-load resistor (see Figure P5.16).



$$H(p) = \frac{1LC}{p^2 + p/C + 1/LC}$$

Figure P5.16 Second-order prototype.

17. Starting with a second-order 2-dB Chebyshev prototype, obtain the transfer function for a stopband filter with cutoff frequencies of 1 and 4 rad/s.
18. Provide the schematic of a third-order high-pass Butterworth filter with cutoff at 400 Hz and a 1-kΩ load.
19. Provide the schematic of a second-order Butterworth bandpass filter with cutoff frequencies of 0.5 and 2 rad/s.

Section 5.5

20. Provide component values for the circuit of Figure P5.20 to serve as a third-order prototype 2-dB Chebyshev low-pass filter.

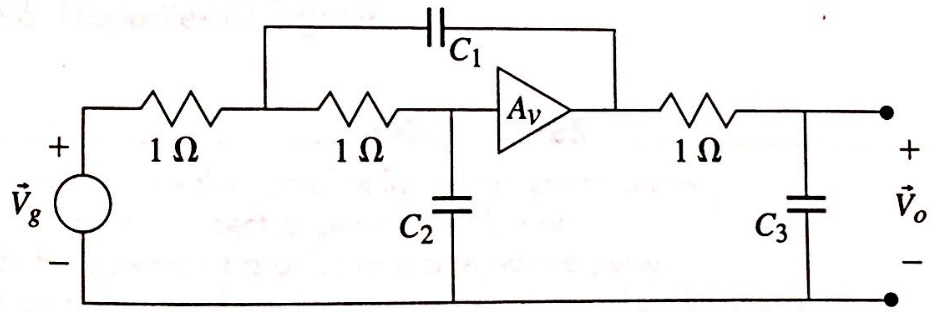


Figure P5.20

21. Provide a prototype circuit for a third-order Butterworth filter using only passive circuit elements.
22. Provide a prototype circuit for a third-order Cauer filter with 1-dB passband ripple and 30-dB minimum stopband attenuation.

Section 5.6

23. Obtain the polynomials of $H(s)$ for a Butterworth sixth-order low-pass filter with cutoff at 10 kHz. Plot $|H(s)|$ vs. f.

24. Plot the transfer function, magnitude, and phase, for a Butterworth bandpass filter derived from an eighth-order prototype and with cutoff frequencies of 8 kHz and 12 kHz. Estimate the group delay for the filter.
25. Prepare a family of curves showing the stopband rolloff of even-ordered Chebyshev 2-dB filters similar to that of Figure 5.16. Use a linear frequency range of $1 \leq f/f_{co} \leq 2$, and include filters out to the fourteenth order.

Chapter 5

5.1 The output will be amplified by a factor of 3 and delayed 1.5 seconds but otherwise will be identical to the input.

5.3 a) If you want to see it, do as instructed.

5.3 b) Use plot (t,g,t,x,t,y,t,z) in MATLAB for different-colored curves.

5.5 $$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

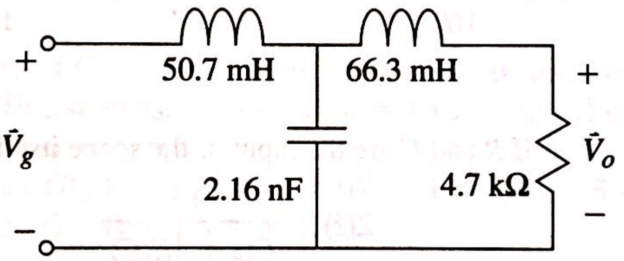
$$|H(j\omega)| = \left|\frac{1}{1 - 2\omega^2 + j\omega(2s - \omega^2)}\right| = \frac{1}{\sqrt{(1-2\omega^2)^2 + \omega^2(2-\omega^2)^2}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{(1 - 4\omega^2 + 4\omega^4) + \omega^2(4 - 4\omega^2 + \omega^4)}} = \frac{1}{\sqrt{1+\omega^6}}$$

5.7 $n = 5.65 \rightarrow 6$

5.9 $p \rightarrow \frac{s}{2\pi 4000}$ $\quad H(s) = \frac{15.88 \times 10^{12}}{s^3 + 50.27 \times 10^3\, s^2 + 1.263 \times 10^9\, s + 15.88 \times 10^{12}}$

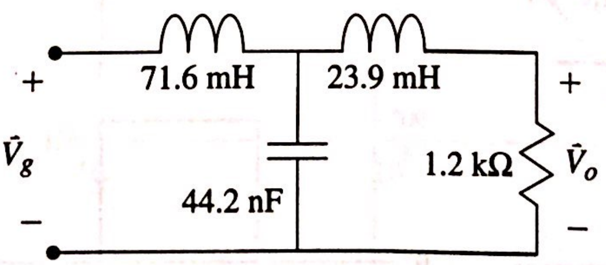
5.11



$$L_1 \rightarrow 1.355\frac{4700}{40{,}000\pi} = 50.68 \text{ mH}$$

$$C \rightarrow \frac{1.274}{4700(40{,}000\pi)} = 2.157 \text{ nF}$$

5.13



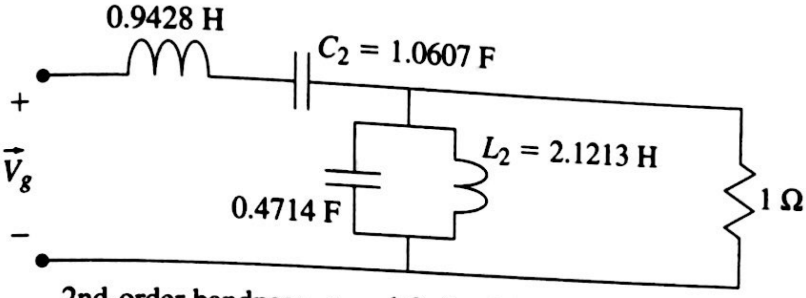
Scanned by CamScanner

$$R' \rightarrow 150R = 1200\,\Omega$$
$$L'' \rightarrow \frac{150}{2000\pi} L$$
$$C'' \rightarrow \frac{C}{2000\pi(150)}$$

5.15 $p \rightarrow \dfrac{s^2 + 15}{2s}$ $\qquad H(s) = \dfrac{4s^2}{s^4 + 2.828s^3 + 34s^2 + 42.43s + 225}$

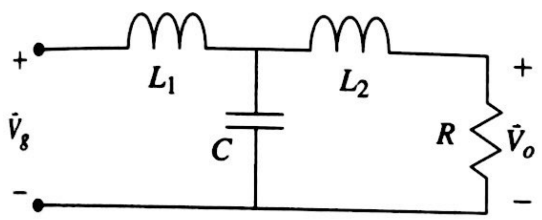
5.17 $p \rightarrow \dfrac{3s}{s^2 + 4}$ $\qquad H_{BS}(s) = \dfrac{0.7943s^4 + 6.3546s^2 + 12.709}{s^4 + 2.9299s^3 + 18.935s^2 + 11.719s + 16}$

5.19



2nd-order bandpass, $\omega_c = 1.0$, $B = 1.5$

5.21 We need a low-pass filter with three reactive elements. One such circuit is



See Problem 3.11

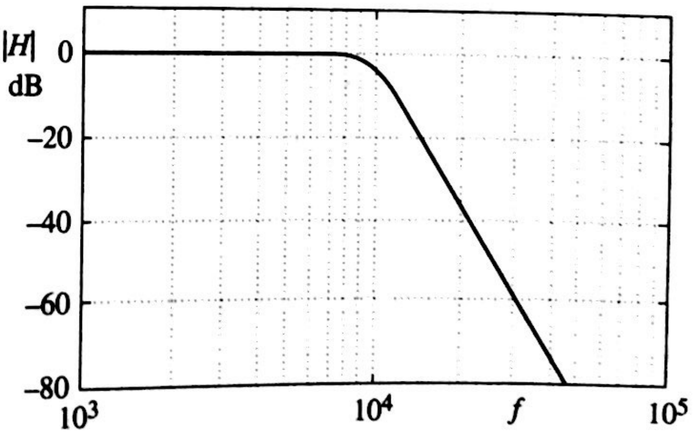
The required tranfer function is $\dfrac{\vec{V}_o}{\vec{V}_g} = \dfrac{1}{s^3 + 2s^2 + 2s + 1}$

a_2: $\dfrac{R}{L_2} = 2$ $\quad a_0$: $\dfrac{2}{L_1 C} = 1$ $\quad a_1$: $\dfrac{L_1 + L_2}{2L_2} = 2$

If $R = 1\,\Omega$, then $L_2 = 1/2$ H, $L_1 = 3/2$ H, and $C = 4/3$ F.

5.23

```
num = 6.1529e + 28
den = 1.0
2.4276e + 05
2.9467e + 10
2.2676e + 15
1.1633e + 20
3.7836e + 24
6.1529e + 28
```



5.25 Use the same *ez1.m* file used for Example 5.7, but change the first instruction to [z,p,k]=cheb1ap(n,2); and use **plot** instead of **semilogx** in the last instruction.

Chapter 6

6.1 $$c_m = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jm\omega_0 t}\,dt = \frac{A\tau}{T}\,\frac{\sin m\pi\tau/T}{m\pi\tau/T}$$

6.3 a) $f(t) = \cdots -0.2387je^{-j4\pi t/3} - 0.4775je^{-j2\pi t/3} + 0 + 0.4775je^{+j2\pi t/3} + 0.2387je^{+j4\pi t/3} + \cdots$

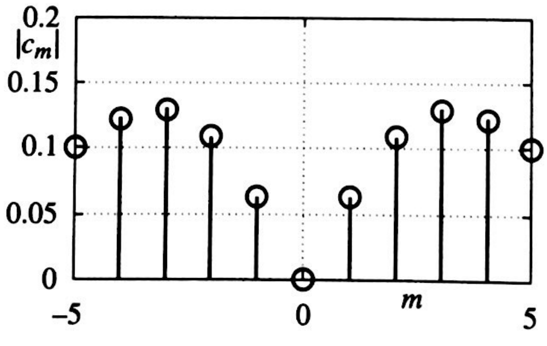
b) $f(t) = -0.9550 \sin 2\pi t/3 - 0.4774 \sin 4\pi t/3 - 0.2388 \sin 8\pi t/3 - \cdots$

6.5 a) $$c_m = \frac{1}{8}\int_0^2 \sin\left(\frac{\pi}{4}t\right)e^{-jm\frac{\pi}{4}t}\,dt$$

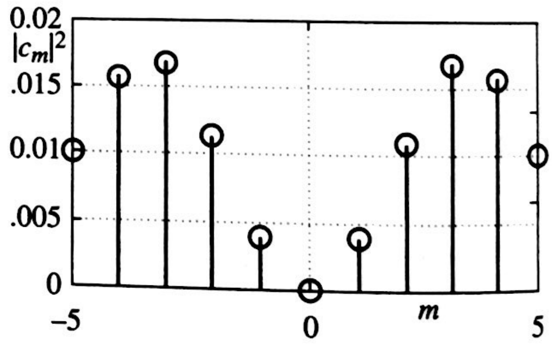
b) $$c_m = \frac{1}{9}\left[\int_{-2}^{-1} 3\,e^{-jm\frac{2\pi}{9}t}\,dt + \int_1^2 3\,e^{-jm\frac{2\pi}{9}t}\,dt\right] = \frac{2}{3}\int_1^2 \cos m\frac{2\pi}{9}t\,dt$$

6.7 $$c_m = \frac{\cos 0.25\pi m - \cos 0.75\pi m}{j\pi m}$$

6.9 a)



b)



6.11 a) $$F(\omega) = \int_0^2 e^{-j\omega t}\,dt - \int_2^3 2e^{-j\omega t}\,dt = \frac{1 - 3e^{-j\omega 2} + 2e^{-j\omega 3}}{j\omega}$$

b) $f(t) = \cdots 0.4135e^{-j(4\pi t/3 - \pi/3)} + 0.8270e^{-j(2\pi t/3 - 2\pi/3)} + 0 + 0.8270e^{+j(2\pi t/3 - 2\pi/3)} + 0.4135e^{+j(4\pi t/3 - \pi/3)} + \cdots$

6.13 $X(\omega) = \int_{-3ms}^{0} -2e^{-j\omega t}\,dt + \int_{0}^{3ms} 2e^{-j\omega t}\,dt = \frac{4j}{\omega}(\cos(3 \times 10^{-3}\omega) - 1)$

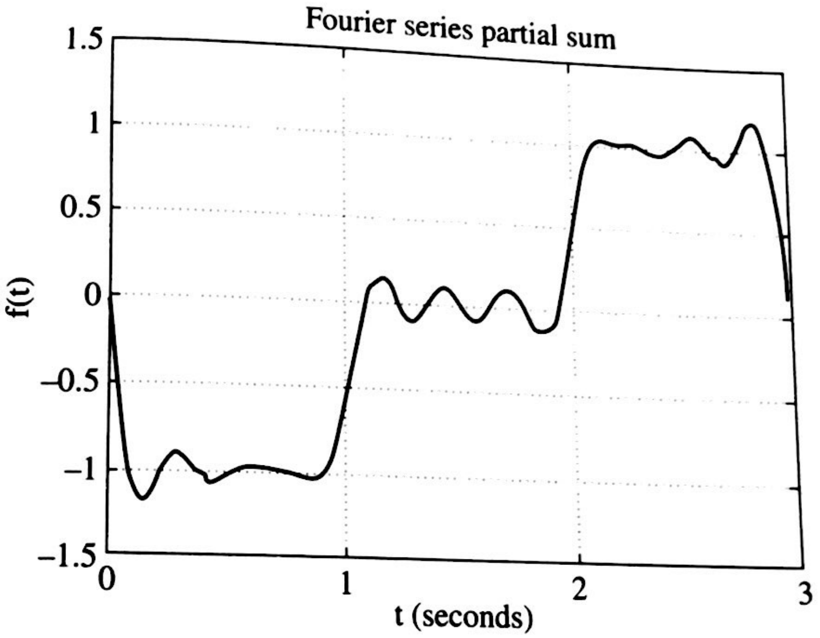
6.15

c_{-5}	c_{-4}	c_{-3}	c_{-2}	c_{-1}	c_0
−0.0320	−0.0839	−0.2652	−0.6821	−1.8243	1.0000

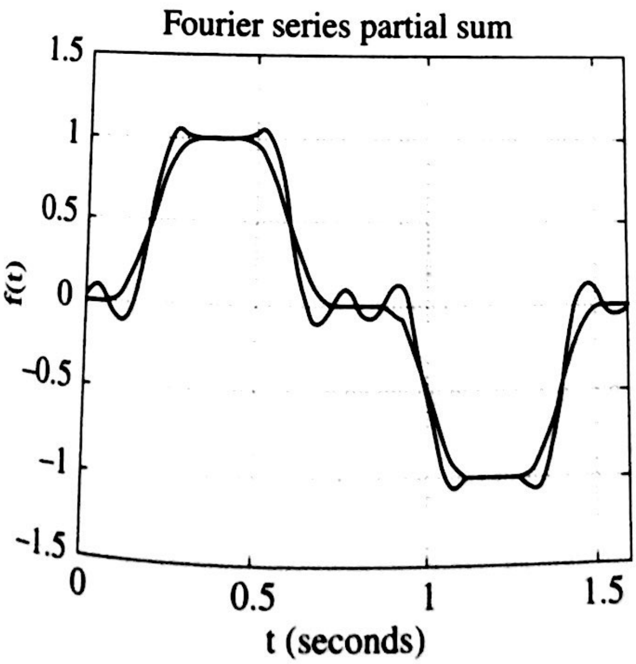
c_1	c_2	c_3	c_4	c_5
1.8243	0.6821	0.2652	0.0839	0.0320

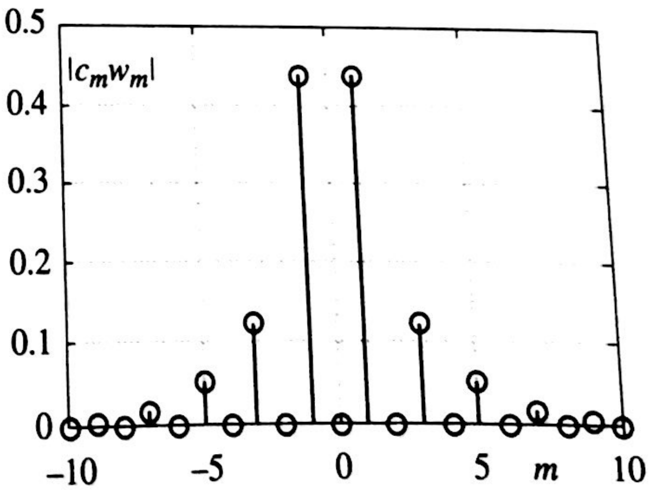
6.17 Waveforms c and d are discontinuous and need a window.

6.19



6.21 a) b)





c) $P = 0.4259$