2

CONTINUOUS-TIME SYSTEMS

OUTLINE

OBJECTIVES

1. Identify a linear differential equation.
2. Interpret variations of the exponential signal.
3. Establish an appropriate phasor transformation for each problem.
4. Transform between differential equations and transfer functions.
5. Determine aspects of the natural response of a linear system.
6. Use MATLAB for efficient polynomial operations.

differential equations. Given the initial status of the system, these equations prescribe the subsequent behavior of the system at any instant in time.

The general solution of differential equations involves processes significantly different from those of algebraic equations. However, linear differential equations with constant coefficients reduce to algebraic equations for an exponential time signal of the form e^{st}, where s is a constant. Such a function repeats itself under differentiation $d(e^{st})/dt = se^{st}$. The interpretation of this signal for complex s leads to the definition of the phasor.

The solutions of linear differential equations consist of the superposition of a forced response and a natural response. The natural response represents the system's reaction to any disturbance; the forced response is the system's reaction to a specific applied input signal. The forced response is generally the desired feature, but an unacceptable natural response may make the system unusable.

2.1 SYSTEM EQUATIONS

Physical systems obey the laws of physics. The exact formulation of these laws involves calculus-based relationships. This section will demonstrate how these relationships result in differential equations and will discuss the problem of solving such equations. After reading this section you will be able to:

- Identify a linear differential equation with constant coefficients.
- Write differential equations for simple mechanical and electrical systems.
- State the characteristics that define a linear system.
- Appreciate the problem of solving a differential equation.

Most systems obey a central law or laws and have system components whose variables are related to the central law. In electrical systems, the central laws are those of Kirchhoff:

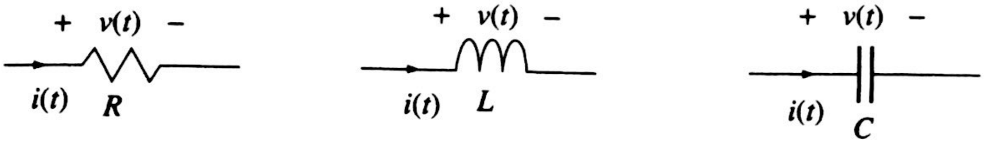
The algebraic sum of the voltages around any closed path equals zero:

$$\sum v(t) = 0 \tag{2.1a}$$

The algebraic sum of the currents entering a junction equals zero:

$$\sum i(t) = 0 \tag{2.1b}$$

The term *algebraic sum* means that voltage rises must be distinguished from voltage drops by giving one a plus sign and the other a minus sign. Similarly, currents



$v(t) = R\,i(t)$ $\qquad$ $v(t) = L\dfrac{di(t)}{dt}$ $\qquad$ $v(t) = \dfrac{1}{C}\int_{-\infty}^{t} i(t)dt$

$i(t) = \dfrac{v(t)}{R}$ $\qquad$ $i(t) = \dfrac{1}{L}\int_{-\infty}^{t} v(t)dt$ $\qquad$ $i(t) = C\dfrac{dv(t)}{dt}$

Figure 2.1 The integro-differential relationships between currents and voltages in linear passive circuit elements. The equations are positive for currents entering the positive voltage end of the circuit element.

entering a junction must be distinguished by their sign from those leaving. The criterion governing which type of term is given the plus sign is immaterial.

The passive devices used to construct a circuit are shown in Figure 2.1. A circuit results when some of these devices are connected together and attached to a voltage or current source.

EXAMPLE 2.1

Write the differential equation governing the behavior of the circuit shown in Figure 2.2.

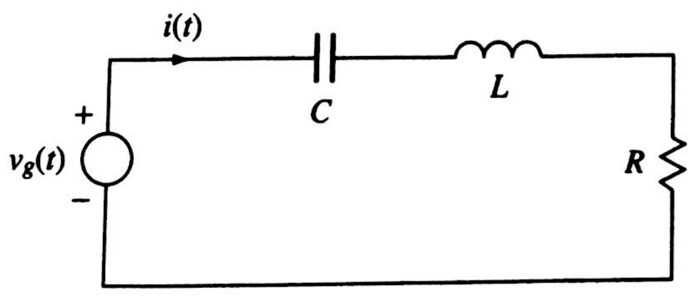


Figure 2.2

Solution

In this case, Kirchhoff's voltage law is needed (Eq. 2.1a):

$$v_g(t) = \frac{1}{C}\int i(t)\,dt + L\frac{di(t)}{dt} + i(t)R$$

A mechanical system is governed by Newton's laws. In a purely rotational system, with J representing the system's moment of inertia, θ the angular displacement, and T the torques present, Newton's second law states that

$$J\frac{d^2\theta}{dt^2} = \sum T \tag{2.3}$$

where those torques that contribute to the motion must be algebraically distinguished from those opposing the motion. Figure 2.3 shows the most common elements. The differential equation of this system is summarized as

$$J\frac{d^2\theta(t)}{dt^2} + k_f\frac{d\theta(t)}{dt} + k_s\theta(t) = T_i(t) \tag{2.4}$$

While electrical and mechanical systems are important of themselves, systems often involve an interaction of the two. Suppose, for instance, that we wish to predict the response of the permanent magnet d-c motor shown in Figure 2.4 to an input voltage. Both Newton's and Kirchhoff's laws provide equations.

A current will flow in the armature (rotor) circuit of the d-c motor if a voltage is applied to it. The resulting current in the armature wires, which are in the field produced by the permanent magnets, produces a torque, $T_m = k_m i_a$, that causes the motor to begin turning. As these same wires move through the magnetic field, they also experience a counterelectromotive force opposing the buildup of current, $e_b = k_m(d\theta/dt)$.

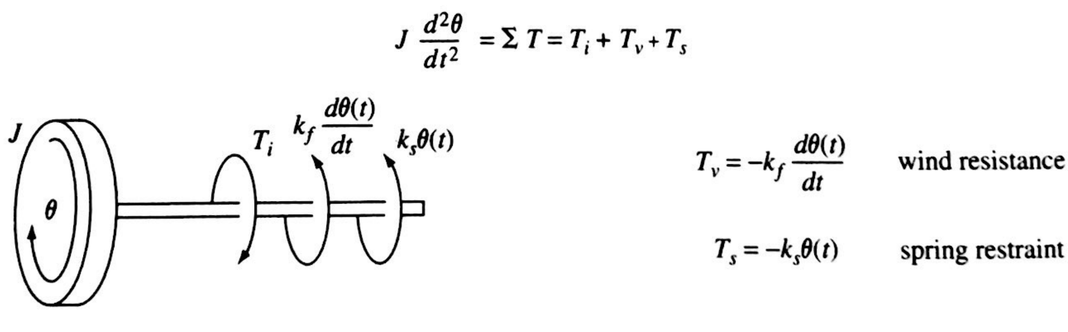


Figure 2.3 In a rotational system, the torque, T_i, causing the motion defines the positive direction of θ. Opposing the motion is a frictional force proportional to the velocity of rotation. In some systems, such as in D'Arsonval meter movements, the system may also be restrained by a spring, which produces a countertorque proportional to the angular displacement.

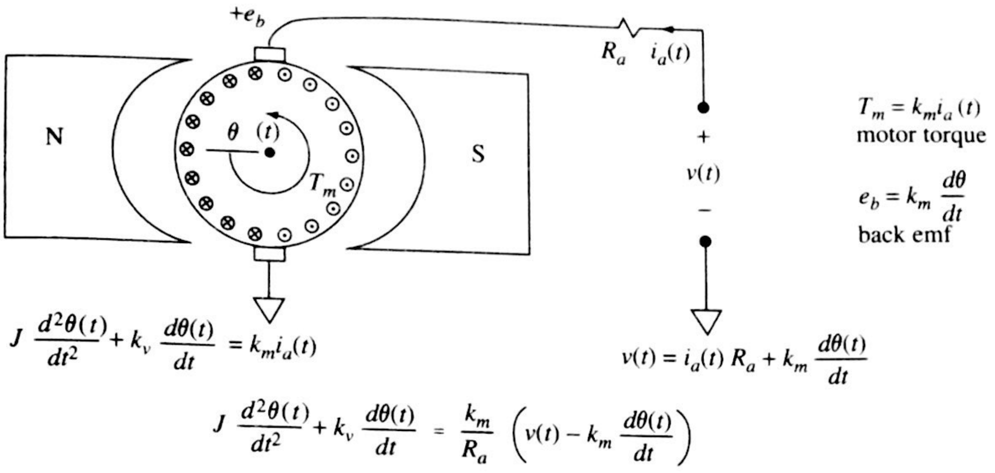


Figure 2.4 In a permanent magnet d-c motor, the action of the commutator and brushes combine to keep the armature-current spatial distribution constant (⊗ represents a current entering the paper, ⊙ represents a current coming up out of the paper). For the currents shown, the armature looks like a coil producing a north magnetic pole downward. Since like poles repel and unlike poles attract, the motor will rotate counterclockwise. The equations for the two systems are coupled by the *Blv-Bli* laws of physics, where *B* is the magnetic flux density, and *l* is the effective length of the conductor in the magnetic field. These values, as well as geometrical factors, have been lumped into the motor constant, k_m.

These two effects couple the mechanical and electrical equations. Rearranging them so that input and output variables are on opposite sides of the equation gives the following result:

$$J\frac{d^2\theta(t)}{dt^2} + \left[k_v + \frac{k_m^2}{R_a}\right]\frac{d\theta(t)}{dt} = \frac{k_m}{R_a}v(t) \tag{2.5a}$$

Using ω to represent the motor's angular velocity, where $\omega = d\theta/dt$, we could also write this differential equation as

$$J\frac{d\,\omega(t)}{dt} + \left[k_v + \frac{k_m^2}{R_a}\right]\omega(t) = \frac{k_m}{R_a}v(t) \tag{2.5b}$$

All of the systems examined so far have been described in terms of ordinary, linear differential equations with constant coefficients. This is the only type of differential equation that will be considered in this text. Systems described by such an equation are said to be continuous-time, linear, and time invariant, or CTLTI for short. The continuous-time designation simply means that the independent variable is time, and that it may take on all possible real values. The general form for an Mth order equation of this type is corresponding increase in the output. With nonlinear systems, the shape of the output waveform depends not only on the shape of the input waveform but also on the size of the input waveform. In a d-c motor like that of Figure 2.4, for example, static friction between the brushes and the commutator prevents the motor from beginning to rotate if the input voltage is too small. At the other extreme, if input voltages get too large in electric circuits, capacitors arc over, inductors with iron cores saturate, and resistors get too hot and burn up. We do not include these nonlinear effects in our differential equations because we would not be able to solve the resulting equations. By approximating the real systems with linear models, we obtain results that give us insight as to how various parameters affect system behavior.

Solving the differential equations of nonlinear systems requires a numerical integration approach, such as is produced by a SPICE transient analysis; the nature of the results will vary as the input amplitude changes. It is, with few exceptions, impossible to describe the behavior of such systems with an algebraic equation.

The most important result of linearity is that it allows us to break complicated input waveforms into their components and to apply the simpler components individually to the system. The individual results may then be superimposed to obtain the full response. Breaking a complex system into smaller, simpler pieces is always an attractive analysis option. For the purposes of this text, and all introductory a-c circuit texts, linearity/superposition is essential.

Even with a linear system, the task of solving a differential equation is not easy.

EXAMPLE 2.2

Solution

Since the current is known everywhere in the series circuit for $t > 0$, the voltages across each circuit element may be found immediately:

$$v_R = iR = 12te^{-t} \qquad v_L = L\frac{di}{dt} = [-6te^{-t} + 6e^{-t}]$$

$$v_C = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \int_0^t 6te^{-t}\,dt + v_C(0) = 6e^{-t}(-t-1)\Big|_0^t = 6 - 6e^{-t} - 6te^{-t}$$

$$\therefore\ v_g(t) = v_R + v_L + v_C = \underbrace{12te^{-t}}_{v_R} + \underbrace{-6te^{-t} + 6e^{-t}}_{v_L} + \underbrace{6 - 6e^{-t} - 6te^{-t}}_{v_C} = 6$$

The result, perhaps surprising, is that a 6-V battery is the input source that causes the current specified.

The typical differential equation problem is just the opposite of Example 2.2. Given that the input source is a 6-V d-c battery, how can we determine the resulting current? Obviously this is not a simple algebraic problem, but rather a search for a special function whose integrals and derivatives meet just the right conditions to satisfy Kirchhoff's laws for the circuit.

A good way for a newcomer to approach this type of problem is to assume a form for the solution that encompasses most of the functions commonly encountered in technical studies. A *power series* expansion exists for many trigonometric, exponential, or logarithmic functions, and takes the form

$$i(t) = \sum_{n=0}^{\infty} a_n t^n \tag{2.7}$$

This constitutes a very general guess for the solution of the differential equation. The task remaining is to determine the conditions on the a_n so that it does satisfy the differential equation. The following example demonstrates this procedure.

EXAMPLE 2.3

Find the solution of the differential equation $\dfrac{dy(t)}{dt} + y(t) = 2$.

Solution

We hope to find a solution of the form

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

This makes $\frac{dy}{dt} = y' = 0 + a_1 + 2a_2t + 3\,a_3t^2 + \cdots = \sum_{n=1}^{\infty} na_nt^{n-1}$ and the differential equation requires that $\sum_{n=0}^{\infty} a_nt^n + \sum_{n=1}^{\infty} na_nt^{n-1} = 2.$

We need to make the summations run over the same range so that we can combine them. If the n index of the summation is reduced by k, while each n in the summation is increased by k, the equation remains unchanged. Then y' can be expressed as

$$y' = \sum_{n=1}^{\infty} na_nt^{(n-1)} = \sum_{n=0}^{\infty-1} (n+1)\,a_{(n+1)}t^n$$

Using this second form for y' makes the differential equation

$$\sum_{n=0}^{\infty} a_nt^n + \sum_{n=0}^{\infty} (n+1)a_{(n+1)}t^n = 2 \qquad \text{or} \qquad \sum_{n=0}^{\infty} [(n+1)a_{n+1} + a_n]t^n = 2$$

For $n = 0$, which is the only term *not* containing a t, we have $(n+1)a_1 + a_0 = 2$. All of the other terms are multiplied by some power of t, and since t does not appear on the right, it must be that $(n+1)a_{n+1} + a_n = 0$ for all $n > 0$. This leads to a *recursion formula*: $a_{n+1} = -\frac{a_n}{n+1} \qquad (n > 0)$

Calculating the first few values gives

$$a_2 = -\frac{a_1}{2} \qquad a_3 = -\frac{a_2}{3} = \frac{a_1}{2(3)} \qquad a_4 = -\frac{a_3}{4} = -\frac{a_1}{2(3)4} \qquad a_5 = -\frac{a_4}{5} = \frac{a_1}{5!}$$

and the sum becomes

$$y(t) = (2 - a_1) + a_1t - a_1\frac{t^2}{2!} + a_1\frac{t^3}{3!} - a_1\frac{t^4}{4!} + a_1\frac{t^5}{5!} \cdots$$

The solution for a first-order differential equation has one arbitrary constant, a_1, that can be used to set the initial value of the function. For example, to make $y(0) = 0$, let $a_1 = 2$. In general, an nth order differential equation has n arbitrary constants, which are normally used to set the initial value of y and its first $n - 1$ derivatives.

With a little practice, and knowing the forms of a few standard series expansions, we could recognize that the result for this example is very close to the series expansion for an exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \textbf{(2.8)}$$

The thing that complicates our search for the solution of a differential equation is that most functions do not reproduce themselves under differentiation or integration. It is consequently difficult to imagine what function could be combined with its derivatives such that they would all add up to a constant, as they did in the last two examples. The power series method of solving differential equations overcomes this difficulty for us, but in more complicated problems it may not be possible to identify the individual functions making up the series. Because of this drawback, we will not pursue this approach further. Instead, we will specialize in the one function that does repeat itself under differentiation and integration: the exponential. It is a decision that subsequent events will fully justify.

2.2 THE EXPONENTIAL SIGNAL

A time function of the form Ae^{st}, where A and s are constants, repeats itself under differentiation and integration with respect to time. Linear differential equations are easily solved for such a function. We will specialize in this type of signal. After completing this section, you will be able to:

- Interpret variations of the exponential signal.
- Define the Euler phasor.
- Use a superposition of exponentials to represent a sinusoid.
- Solve linear differential equations for their forced response to an exponential signal.

If s is real, the exponential signal takes one of the forms shown in Figure 2.6. The degenerate case of $s = 0$ results in a signal of constant amplitude, as encountered in d-c circuits. Finding the forced response of a differential equation with a real exponential forcing function is very straightforward.

EXAMPLE 2.4

0 1 2 3 4 5 0 1 2 3 4 5

t/τ t/τ

Figure 2.6 The real decreasing exponential is often expressed in terms of its time constant, τ. It decreases to 36.8% of its initial value after 1 time constant. If it were to continue decreasing at its initial rate, it would be down to zero after 1 time constant. It is essentially negligible after 5 time constants. The increasing exponential also starts at unity, but grows to about 150 times its initial value in 5 time constants.

Solution

Since exponential functions repeat themselves under differentiation, the forced response must have the same s as the forcing function. The solution must therefore be $y(t) = Ke^{-4t}$. Substituting this into the differential equation requires that

$$-4Ke^{-4t} + 3Ke^{-4t} = 3e^{-4t} \qquad \text{or} \qquad K = -3$$

The solution for the forced response is consequently

$$y(t) = -3e^{-4t}$$

Find the forced response of the linear differential equation

$$\frac{dy(t)}{dt} + 3y(t) = 6 \cos 2t$$

Solution

Replacing the cosine function using Euler's identity, we will actually solve the differential equation for the exponential forcing function

$$\frac{dy_1(t)}{dt} + 3y_1(t) = 3e^{j2t} \quad \text{where} \quad y_1(t) = \vec{Y}e^{j2t}$$

A subscript has been added to the y variable to remind us that we are using superposition to find one component of the full solution.

Substituting the proposed solution into the differential equation gives

$$\begin{aligned} j2[\vec{Y}e^{j2t}] + 3[\vec{Y}e^{j2t}] &= 3e^{j2t} \\ [3 + j2]\vec{Y} &= 3 \\ \therefore \vec{Y} &= \frac{3}{3 + j2} = 0.832\angle - 33.7° \end{aligned}$$

This makes $y_1(t) = 0.832\, e^{j(2t - 33.7°)}$, and the solution to the original problem is

$$y(t) = 2\text{Re}\, y_1(t) = 1.664 \cos(2t - 33.7°)$$

Both nepers and radians are actually dimensionless. Lab instruments are calibrated to indicate frequency, f, in hertz, while ω is the frequency variable of choice in theoretical work. These frequency variables are related by

$$\omega = 2\pi f \tag{2.14}$$

Note that the definition of the Euler phasor is not affected by this more generalized signal. Only the value of s has changed. The procedure for handling this type of signal is, therefore, exactly the same as that for the pure sinusoid.

EXAMPLE 2.6

Solve for the forced response of the differential equation

$$\frac{dy(t)}{dt} + 3y(t) = 2e^{-3t}\cos(2t + 50°)$$

Solution

The input forcing function for this problem can be represented by a Euler phasor of $1\underline{/50°}$, and its complex frequency is $s = -3 + j2$. We therefore need to solve the equation

$$\frac{dy_1(t)}{dt} + 3y_1(t) = e^{j\,50°}\,e^{(-3+j2)t}$$

2.3 PHASOR TRANSFORMATIONS

The Euler phasor has been defined from the cosine function. It is not necessary to retain a rigorous mathematical link between the differential equation forcing function and the phasor. After reading this section you will:

- Establish an appropriate phasor transformation for each problem.

So far we have retained the mathematical equality between a sinusoidal time function and its complex components defined from

$$X_m e^{\sigma t}\cos(\omega t + \theta) = \underbrace{\left[\frac{X_m}{2}e^{j\theta}\right]}_{\bar{X}} e^{st} + \underbrace{\left[\frac{X_m}{2}e^{-j\theta}\right]}_{\bar{X}^*} e^{s^*t}$$

We have, however, never actually solved for the conjugate component, but simply recognized what its contribution would be. By replacing the equality with a bidirectional arrow, $\Leftrightarrow$, we can establish a one-to-one correspondence between a sinusoid and an e^{st} signal that will represent it. This mathematically less formal procedure releases us from always using the Euler phasor. Instead of halving X_m to get the magnitude of the Euler phasor, and then doubling the result to get back to the sinusoid, we can just define the phasor to have an amplitude X_m, and skip the doubling step. This phasor transformation would be stated as

$$X_m e^{\sigma t}\cos(\omega t + \theta) \Leftrightarrow \underbrace{[X_m e^{j\theta}]}_{\bar{X}}\, e^{st}$$

Suppose further that the forcing function for a differential equation is a sine wave rather than a cosine. Since $\sin\beta = \cos(\beta - 90°)$, we could use the following phasor transform:

$$X_m e^{\sigma t}\sin(\omega t + \theta) \Leftrightarrow \underbrace{[X_m e^{j(\theta - 90°)}]}_{\vec{X}} e^{st}$$

Here again, there is no advantage to carrying the extra $-90°$ along in the phasor angle just because the Euler phasor is referenced to a cosine function. Instead, we will choose one of the following transformations, depending on whether the forcing function is given as a sine or a cosine function:

$$X_m e^{\sigma t}\sin(\omega t + \theta) \Leftrightarrow \underbrace{[X_m e^{j(\theta)}]}_{\vec{X}} e^{st} \qquad s = \sigma + j\omega \tag{2.15}$$

$$X_m e^{\sigma t}\cos(\omega t + \theta) \Leftrightarrow \overbrace{[X_m e^{j(\theta)}]} e^{st}$$

As long as the transformation establishes the relationship between a signal and its phasor, any linear relationship is acceptable. The only restriction is that whatever definition we use for one phasor must be applied to all phasors in a given problem. If a problem involves both sine and cosine forcing functions, the cosine is the preferred reference.

EXAMPLE 2.7

Find the forced response of the differential equation

$$\frac{d^2y(t)}{dt^2} + 3y(t) = 4\sin(2t + 20°)$$

Solution

The equation we will solve is

$$\frac{d^2y(t)}{dt^2} + 3y(t) = \vec{X}e^{st}$$

where we choose the phasor transformation to be

$$X_m e^{\sigma t}\sin(\omega t + \theta) \Leftrightarrow [X_m e^{j(\theta)}] e^{st}$$

so

$$\vec{X} = 4e^{j20°} \qquad \text{and} \qquad s = 0 + j2$$

The expected solution is $y(t) = \vec{Y}e^{j2t}$; substituting it in the differential equation gives

$$s^2Ye^{j2t} + 3Ye^{j2t} = 4e^{j20^\circ}e^{j2t}$$

$$\vec{Y} = \frac{4\angle 20^\circ}{s^2+3} = \frac{4\angle 20^\circ}{(j2)^2+3} = -4\angle 20^\circ$$

Reversing the phasor transformation gives the forced response as

$$y(t) = -4\sin(2t + 20^\circ) = 4\sin(2t - 160^\circ) = 4\sin(2t + 200^\circ)$$

Although we have found that it is not necessary to use the Euler phasor when solving differential equations, it is still the mathematically correct link between sinusoidal and exponential signals. It will surface again in Fourier analysis.

2.4 TRANSFER FUNCTIONS

If the s value of an exponential forcing function is left general, it is still possible to find the relationship for the ratio of a system's output and input phasor from the system differential equation. The result is a transfer function describing the system as a function of s. After completing this section you will be able to:

- Transform between differential equations and transfer functions.
- Distinguish between the s domain and the time domain.
- Provide a pole-zero diagram of a transfer function.

Starting with the general linear differential equation with constant coefficients, we may propose an arbitrary input signal $x(t) = \vec{X}e^{st}$ and a corresponding forced output $y(t) = \vec{Y}e^{st}$:

$$\frac{d^My(t)}{dt^M} + \cdots + b_1\frac{dy(t)}{dt} + b_0y(t) = a_N\frac{d^Nx(t)}{dt^N} + \cdots + a_1\frac{dx(t)}{dt} + a_0x(t)$$

$$s^MYe^{st} + \cdots + b_1s\vec{Y}e^{st} + b_0\vec{Y}e^{st} = a_Ns^N\vec{X}e^{st} + \cdots + a_1\,s\,\vec{X}e^{st} + a_0\vec{X}e^{st}$$

Canceling the common e^{st} terms and arranging as a phasor ratio gives

$$\frac{\vec{Y}}{\vec{X}} = \frac{a_Ns^N + \cdots + a_1s + a_0}{s^M + b_{M-1}s^{M-1} + \cdots + b_1s + b_0} \tag{2.16}$$

This transformation between the differential equation and the transfer function is readily made by inspection.

domain. When the system description is in terms of phasors, or transfer functions of the complex frequency s, the description is said to be in the *s domain*, or the *frequency domain*, or the *phasor domain*. These three terms are used interchangeably. The only instances when the t and s variables should appear in the same equation is in the definition of a phasor, $x(t) = \bar{X}e^{st}$ or temporarily while making the transition between domains.

The transfer function depends both on the differential equation coefficients and on the value of s for the forcing function. We may consequently explore the response of any differential equation for a whole range of signals by plotting its transfer function's pole-zero diagram over the complex s plane. At zeros of the transfer function, an input produces no output. At s values close to poles of the transfer function, a small input may produce very large outputs. If we are particularly interested in the response to sinusoids, we need to evaluate the transfer function along the path $s = j\omega$, which gives what is generally meant by the term *frequency response*.

EXAMPLE 2.8

Determine the transfer function of the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$$

and briefly discuss the system's forced response.

Solution

By inspection, the transfer function is

$$\frac{\bar{Y}}{\bar{X}} = \frac{3s + 1}{s^2 + 3s + 2} = \frac{3(s + 1/3)}{(s + 1)(s + 2)}$$

A forcing function like $e^{-t/3}$ produces no output, while forcing functions close to e^{-t} or e^{-2t} should produce a very large output. We can only speculate about what happens

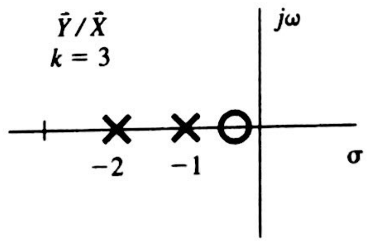


Figure 2.7 The pole-zero diagram for the system of Example 2.8.

In the future, we will spend more time working with transfer functions than with the differential equations from which they are derived. While working with the algebraic transfer function has the advantage of simplicity, questions that are unanswerable from the vantage point of the transfer function can always be answered in the more fundamental differential equation. It can be shown, for instance, that if a forcing function has exactly the same frequency as a transfer function pole, the assumption of a solution in the form e^{st} is too restrictive. An additional term of the form te^{st} also satisfies the differential equation under this condition. This and other special cases that arise in the solution of linear differential equations will be temporarily ignored. The Laplace transform techniques of Chapter 12 includes these special cases routinely.

2.5 THE NATURAL RESPONSE

When a system is disturbed (something changes), it reacts in a way that is characteristic of the system. Any forcing function is irrelevant. After completing this section you will be able to:

- Determine aspects of the natural response of a linear system.
- Distinguish between the forced and natural response of the system.
- Characterize the stability of the system.

In addition to its forced response, every linear differential equation has a natural response. This is found by solving the equation with the forcing functions set to zero. Mathematicians call this the *homogeneous equation:*

$$\frac{d^M y(t)}{dt^M} + \cdots + b_1 \frac{dy(t)}{dt} + b_0 y(t) = 0 \qquad \textbf{(2.17)}$$

We can accomplish the same thing from the point of view of the transfer function. If $\tilde{X} = 0$, $\tilde{Y}$ must normally also be zero. However, at its poles the expression for $\tilde{Y}$ becomes indeterminate, and a nonzero output is possible:

$$\frac{\tilde{Y}}{\tilde{X}} = \frac{a_N s^N + \cdots + a_1 s + a_0}{s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}$$

The denominator of the transfer function is called the *characteristic equation* of the system, and the poles occur at the system's *natural frequencies,* s_n. Signals of the form $e^{s_n t}$ can and will be generated any time the forcing function is turned on or off or the system is otherwise disturbed.

Determine the full response of the system whose differential equation is

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 2\frac{dx(t)}{dt} + x(t)$$

when $x(t) = 2 \cos t$.

Solution

The transcriber function for this system is

$$\frac{\vec{Y}}{\vec{X}} = \frac{2s + 1}{s^2 + 2s + 4}$$

For the forced response, the phasor transform gives

$$\vec{X} = 2\angle 0° \qquad \text{and} \qquad s = j1$$

$$\therefore\ \vec{Y} = \frac{1 + 2j}{3 + 2j} 2\angle 0° = 1.2403\angle 29.74° \Leftrightarrow y_f(t) = 1.2403 \cos(t + 29.74°)$$

For the natural response, the transfer function poles are at

$$s^2 + 2s + 4 = 0 \qquad s = -1 \pm j\sqrt{3}$$

The natural response terms are consequently

$$y_n(t) = K_1 e^{-t} e^{j\sqrt{3}t} + K_2 e^{-t} e^{-j\sqrt{3}t}$$

where K_1 and K_2 are, in general, complex conjugates. If K_1 is thought of as $(K/2)e^{j\theta}$, the natural response can be written as

$$y_n(t) = Ke^{-t}\cos(\sqrt{3}t + \theta)$$

The full response of this system is

$$y(t) = 1.2403 \cos(t + 29.74°) + Ke^{-t} \cos(\sqrt{3}t + \theta)$$

where K and θ are arbitrary real constants.

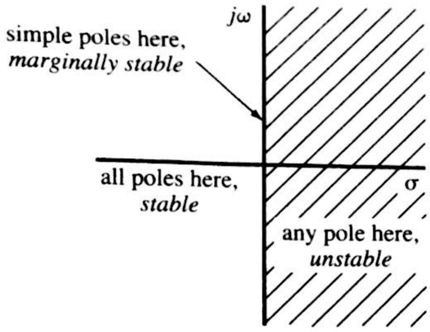


Figure 2.8 Transfer functions having poles in the right-hand side of the s plane (RHP) have growing natural responses and are *unstable*. Simple poles on the $j\omega$ axis produce a natural response that is present but neither growing nor decaying. Such a system, called *marginally stable,* represents an oscillator. Higher-order poles on the $j\omega$ axis also produce an unstable natural response.

Designating the system poles by $s_n = \sigma_n + j\omega_n$, there are three possible system categories based on the pole locations in the complex s plane (see Figure 2.8):

1. If all poles have $\sigma_n < 0$, the natural response dies out with time and is said to be *transient*. The forced response then eventually dominates the system performance. Such systems are *stable*.
2. If any one pole has a $\sigma_n = 0$, the natural response of that pole does not die out with time. Most often this happens with a conjugate pair of poles, and the result is a sinusoidal oscillator. The system is described as *marginally stable*.
3. If any one pole has a $\sigma_n > 0$, its natural response grows exponentially with time and completely overwhelms any forced response. Such a system is *unstable* and unusable. This also occurs if a multiple (repeated) pole has $\sigma_n = 0$. In short, for a system to be stable, all of its poles must be in the left half of the s plane.

2.6 MATLAB LESSON 2

This chapter has shown that linear systems may be described by a ratio of polynomials in s called a transfer function. After completing this section you will be able to:

- Use MATLAB for efficient polynomial operations.
- Correct command typo errors easily.
- Modify MATLAB's numerical display format.

We start with a reminder that you will get nothing out of this section unless you are seated at a computer and entering the MATLAB commands indicated. Feel free to experiment with them beyond the instructions.

MATLAB allows a polynomial to be described simply by entering its coefficients into an array. The final coefficient must be the coefficient of the s^0 term. It also provides a variety of commands that recognize this simplified notation:

roots **poly** **polyval** **polyder** **conv** **deconv**

```
> p1=[1 0 0 2]      % defines the polynomial s^3 + 2. The last entry must be the
                      s^0 term
> r1=roots(p1)      % creates a vector containing the roots of polynomial p1
> p2=poly(r1)       % given the roots, find the polynomial form, p2 = p1
> r2=[-1 -1 -1]     % defines three identical roots at s = -1
> p2=poly(r2)       % p2 = (s + 1)^3 = s^3 + 3s^2 + 3s + 1
```

In the last command we overwrote the original value of p2. Assume this was an error, and that we had intended to name this last polynomial p3. Instead of retyping these commands, hit the up-arrow key three times. As you do, you will see the previous commands appear sequentially on the active command line where they can be edited. If you go too far up the list of commands, use the down-arrow key to move back down. When the active command line shows p2 = poly(r1), hit return to re-execute it. This restores the original p2 polynomial. Now hit the up-arrow key again two times and p2 = poly(r2) will be on the active command line. Use the side-arrow keys to move the insertion point to p2, change the 2 to a 3, and execute the command. You now have polynomials p1, p2, and p3, as intended.

If you execute a command and get an error statement, using the up-arrow key allows you to immediately get that command back and fix it with a minimum of typing. You may also copy and paste a previous command to the active command line for editing or re-executing, but be careful not to copy the prompt along with the command.

```
> z=[-2 -1 0 1 j];          % defines a set of 5 points
> polyval(p2,z),            % calculates p2 at the set of points in z. We
                              did the same thing in Chapter 1, but had to
                              describe the polynomial in a formula. With
                              polyval we can do it with this simpler
                              notation.
> p3=[1 -1]                 % defines the polynomial p3 = s - 1
> p4=conv(p1,p3)            % takes the product of polynomials p1 and p3
> [p5,rem]=deconv(p1,p3)    % divides p1 by p3, result is p5 with remainder
                              rem
> [p6,re]=deconv(p4,p3)     % divides p4 by p3, should get back p1 with no
                              remainder
> r1                        % let's see the r1 roots (roots of p1) again
```

The elements of any array may be addressed individually. In an array called X, individual elements can be addressed by giving their row/column position as X(row,column). For row or column vectors, only the column or row number is needed. We will use this technique to select the pair of conjugate roots of p1 and find the quadratic polynomial they come from:

```
> q1=[r1(2) r1(3)]   % form an array consisting of the 2nd and 3rd elements
                       of r1 (these should be the conjugate roots)
> p7=poly(q1)        % forms the quadratic in p1=(s + 1.2599) (s^2 - 1.2599s
                       + 1.5874)
```

We can combine this procedure with colon notation to select out a subset of a vector:

```
> p8=linspace(-5,5,11);   % create the vector -5 -4 -3 -2 -1 0 1 2 3 4 5
> p9=p8(6:11)             % selects out elements 6 thru 11 of p8
> p10=p8(1:3)             % selects out the first 3 elements of p8
```

What do you think the command **polyder** does? How would you find out? Do it!

zplane

```
> p2=[1 3 2 0]     % create a new polynomial s(s + 1) (s + 2)
> zplane(p1,p2)    % plots the pole-zero diagram of F(s)=p1/p2
```

MATLAB wakes up in a *short, fixed-number display* format. The systems we have been working with and will continue to work with are normalized and have nice, simple coefficients. We will learn later how to denormalize them. Once we do, the fixed format display will not be acceptable. Keep in mind that we are discussing how the results are *displayed* by MATLAB, not the number of digits used in calculations.

format

```
> p8=[1 1000]           % creates the polynomial s + 1000
> p8=conv(p8,p8);       % creates the polynomial (s + 1000)^2
> (up arrow, return)    % hitting the up-arrow key brings the last command
                          back up, hitting return executes it again. Now p8
                          is (s + 1000)^4
> p8                    % see the result in fixed format
> format short e        % change to short floating point
> p8                    % see the actual value of p8
> format short          % change back
> help format           % will show all the options
```

EXAMPLE 2.10

if the forcing function is $x = -6 + 2e^{-t} + 2\cos(4t + 20°)$.

Solution

The transfer function polynomials are

```
> num=[3 -12 0];    den=[1 2 8 18 5];
```

To find the natural response terms we use the *roots* function

```
> r=roots (den)

r = 0.1364 + 2.8242i
    0.1364 - 2.8242i
    -1.9525
    -0.3203
```

In subsequent presentations of MATLAB results, column vectors will be displayed as if they were row vectors, to conserve space.

These roots show the natural response is

$$y_n(t) = K_1 e^{-0.3203t} + K_2 e^{-1.9525t} + K_3 e^{0.1364t}\cos(2.8242t + \theta)$$

Since the conjugate roots are in the RHP, the system is unstable and the forced response is meaningless. We will find it, for the practice.

```
> s=[0 -1 4j];                          % List the phasor frequencies of x
> T=polyval(num,s)./polyval(den,s)      % Evaluate the transfer function at
                                          these frequencies

T = 0    -2.5000     -0.1775 -0.4356i

> X=[-6 2 2*exp(j*20*pi/180)];          % Create the phasors of x

> Yforced=X.*T
Yforced = 0    -5.0000    -0.0356- 0.9401i

> mag=abs(Yforced(3))                   % find Yforced(3) in polar form
mag = 0.9408
> theta=180*angle(Yforced(3))/pi
theta = -92.1663
```

The forced response is consequently

$$y_f(t) = 0 - 5e^{-t} + 0.9408\cos(4t - 92.2°)$$

and the complete response is $y(t) = y_n(t) + y_f(t)$.

CHAPTER SUMMARY

We have seen how mechanical, electrical, and electromechanical systems obey physical laws that relate system variables through differential equations. If the equations are linear, they are easily solvable for an exponential forcing function, e^{st}. We

function provided a fixed ratio
We were able to relate the output phasor back to a sinusoid formally using Euler phasors or informally using a more convenient phasor transformation.

The poles of the transfer function were shown to be the natural frequencies of the system, and indicate the general form of the system's natural response to any disturbance. If any of the system poles are in the RHP, the natural response grows exponentially with time and the system is unstable and unusable.

The realm of the differential equation is called the time domain. The realm of the transfer function is called the frequency domain or the s domain or the phasor domain.

In future continuous-time systems, we will work almost exclusively with transfer functions and, consequently, with polynomials in s. We have seen that MATLAB has a simplified method for describing polynomials in an array and a variety of functions that interpret an array as a polynomial. Many more such functions will be introduced later.

PROBLEMS

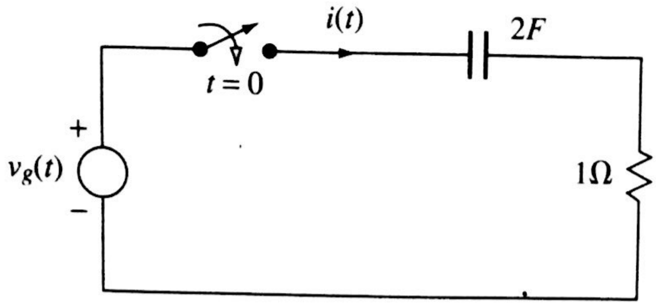


Figure P2.3

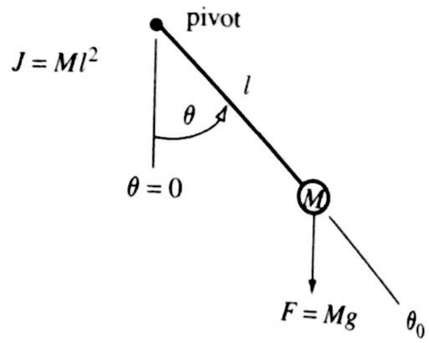


Figure P2.4

4. The pendulum shown in Figure P2.4 is displaced to an initial angle θ_0 and released at $t = 0$.
   a. Ignoring friction and wind resistance, determine the differential equation describing $\theta(t)$. Is it linear?
   b. State an approximation that will make the pendulum equation linear for small θ.

Section 2.2

5. For the following conditions, find the forced response of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = x(t)$$

   a. $x(t) = e^{-t}$ b. $x(t) = 3e^{2t}$

6. Find the forced response of the given differential equation to the $x(t)$ specified,

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = x(t)$$

   a. $x(t) = 2\cos t$ b. $x(t) = e^{-t}\cos t$

7. Identify the Euler phasor and s value of each forcing function.
   a. $x(t) = 3\cos(2t + 12°)$ b. $x(t) = e^{-3t}\cos t$
   c. $x(t) = 12\sin(2t + 20°)$ d. $x(t) = 2e^{t}\cos 3t$

Section 2.3

8. Select an appropriate phasor transformation and use it to find the forced response given that $x(t) = 10\cos(2t + 35°)$ and the system equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2\frac{dx}{dt} + x(t)$$

9. Find the output phasor given that $x(t) = 3e^{-2t}\sin t$ and the system is described by

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \frac{dx}{dt} + x$$

State the phasor transformation used, and identify $\vec{X}$ and s.

10. Determine the transfer functions of the following systems.

a. $\dfrac{d^3y}{dt^3} + 4\dfrac{d^2y}{dt^2} + 16\dfrac{dy}{dt} + 8y = 2\dfrac{d^2x}{dt^2} + 3x$

b. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} = \dfrac{d^2x}{dt^2} - 9x$

11. Determine the transfer function of the systems indicated.

a. $\dfrac{d^3y}{dt^3} + 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + y = 4\dfrac{dx}{dt} + 4x$ b. $\dfrac{d}{dt}\left[\dfrac{dy}{dt} + 4y\right] = \dfrac{d^2x}{dt^2}$

12. Sketch pole-zero diagrams for the systems of Problem 2.10.

13. Sketch pole-zero diagrams for the following systems.

a. $\dfrac{\vec{Y}}{\vec{X}} = 2\dfrac{s+1}{s+2} - \dfrac{s+2}{s+4}$ b. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{s^2 - 1}{s^3 + 2s^2 + 2s + 1}$

Section 2.5

14. Determine the complete response of the systems for $x(t) = 2\cos 2t$.

a. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{1}{s^3 + 2s^2 + 2s + 1}$ b. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{s-2}{s^2 + 6s + 8}$

15. Determine the natural response of the systems. Classify each as stable, unstable, or marginally stable.

a. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{s^3 - 2s}{s^2 + 4s + 8}$ b. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{s^2 + 16}{s^3 + 4s^2 + 8s + 32}$

c. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{s + 24}{(s+8)(s^2 - s + 4)}$ d. $\dfrac{\vec{Y}}{\vec{X}} = \dfrac{2s(s+2)^2}{s^3 + 2s^2 + 3s + 2}$

Section 2.6

16. Show the MATLAB programming needed to multiply together the polynomials $x^2 - 3x + 4$ and $x^5 + 2x^3 + 3x^2 + 4x + 2$.

17. Show the MATLAB programming that would be used to evaluate the polynomial $y = x^5 + 2x^2 + x - 4$ at the points $x = -2, 3j$, and 4.

18. Show the commands that are needed to correct the z vector after the following command sequence:

```
> x = -2:8; y = x.^2 - 4; z = sin(y)./y
```

19. Show the command needed to correct the y vector after the following command sequence:

```
> x = 0:20; num = [1 -2 1]; den = [1 -1];
y = polyval(num, x)./polyval(den, x)
```

Additional Problems

20. Verify that $y = K_1e^{-t} + K_2te^{-t}$ is a solution of the differential equation $dy/dt + y = e^{-t}$, where $K_2 = 1$ and K_1 is arbitrary.

21. Given $x(t) = 2 + 4e^{-3t} + 6\cos 3t$, use MATLAB to assist you in finding the complete solution of the differential equation

$$\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 16\frac{d^2y}{dt^2} + 18\frac{dy}{dt} + 2y = \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x$$

22. Using a power series expansion for $y(t)$, solve the differential equation $\dfrac{dy(t)}{dt} + y(t) = 2t$ if $y(0) = 0$.

23. Using a power series expansion for $y(t)$, solve the differential equation $\dfrac{dy(t)}{dt} + y(t) = e^{-t}$ if $y(0) = 0$.