3

ANALYSIS TECHNIQUES

OUTLINE

OBJECTIVES

1. Define impedance.
2. Use impedances and phasors to analyze circuits.
3. Introduce feedback systems using a block diagram.
4. Simplify block diagrams to obtain the system transfer function.
5. Explain ways to test a system for stability.
6. Use MATLAB for block diagram reduction and system simulation.

INTRODUCTION

Although physical systems are fundamentally described with differential equations, they can also be described by algebraic equations for exponential forcing functions. In circuit theory, this process is expedited by converting the calculus-based laws of the circuit components immediately into phasor ratios. This leads to the definition of *impedance,* and allows circuit properties to be derived directly in the *s* domain.

Most systems are not purely electrical or mechanical, but rather a mix of amplifier circuits, actuators such as motors and hydraulic pistons, and sensors such as tachometers and anemometers. Each of these elements individually may be described with a transfer function and represented with an input/output block. The blocks are then interconnected to form a system, usually in a feedback arrangement where the desired output is compared to the actual output. Feedback has the potential to make a system's forced response very precise, but it simultaneously may make the system's natural response unacceptable or even unstable. The design of elementary feedback systems must consider both responses.

3.1 DRIVING POINT IMPEDANCE

For specialists in electrical systems, it is not necessary to start each problem with a differential equation. By investigating the properties of phasors you will be able to:

- Define impedance.
- Convert circuits directly to the phasor or s domain.
- Combine impedances to simplify circuits.

Suppose we have the three currents i_1, i_2, and i_3 meeting at a junction in a circuit, where

$$i_1(t) = I_1 e^{\sigma t}\cos(\omega t + \theta_1)$$
$$i_2(t) = I_2 e^{\sigma t}\cos(\omega t + \theta_2)$$
$$i_3(t) = I_3 e^{\sigma t}\cos(\omega t + \theta_3)$$

As shown in Equations 2.10–2.12, each of these currents may be written as

$$i(t) = \vec{I}e^{st} + \vec{I}^{\,*}\, e^{s^*t}$$

Kirchhoff's law requires that at every instant $i_1(t) + i_2(t) + i_3(t) = 0$, which means

$$\underbrace{\vec{I}_1 e^{st} + \vec{I}_1^{\,*} e^{s^*t}}_{i_1(t)} + \underbrace{\vec{I}_2 e^{st} + \vec{I}_2^{\,*} e^{s^*t}}_{i_2(t)} + \underbrace{\vec{I}_3 e^{st} + \vec{I}_3^{\,*} e^{s^*t}}_{i_3(t)} = 0$$

$$\underbrace{(\vec{I}_1 + \vec{I}_2 + \vec{I}_3)\, e^{st}}_{0} + \underbrace{(\vec{I}_1^{\,*} + \vec{I}_2^{\,*} + \vec{I}_3^{\,*})\, e^{s^*t}}_{0} = 0$$

Because neither e^{st} nor e^{s^*t} is zero for all time, it follows that the current phasors (and their conjugates) must sum to zero. This establishes a very important fact: *If a set of signals obey Kirchhoff's laws, then the phasors representing those signals also obey Kirchhoff's laws.*

$\vec{I}_1 + \vec{I}_2 + \vec{I}_3 = 0 \qquad \vec{I}_1^* + \vec{I}_2^* + \vec{I}_3^*$

The voltage across an inductor or capacitor is related to the current through that device by a calculus expression. What is the corresponding relationship between the device's phasors? Taking the inductor as an example, if $i(t) = \vec{I}e^{st}$, then

$$v(t) = L\frac{d}{dt}\vec{I}e^{st} = L\vec{I}\,se^{st} = \vec{V}e^{st}$$

and

$$\frac{v(t)}{i(t)} = \frac{\vec{V}e^{st}}{\vec{I}e^{st}} = sL \qquad \textbf{(3.1)}$$

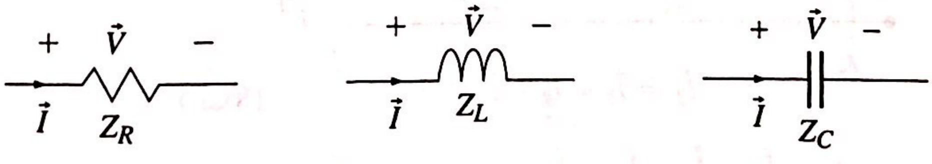
The notation in Equation 3.1 is deceptive and is considered poor practice, because it is only true for the exponential signal assumed initially. Compare it to Ohm's law for a resistor, for instance, where $v(t) = i(t)R$ is true for *any* waveform. The presence of either a phasor or the s variable automatically means that the signal is e^{st}. The $v(t)$ and $i(t)$ notation implies a generality that does not exist, and it is best to avoid using it.

To write Equation 3.1 in a proper form, we retain just the second half of the equation, cancel the e^{st} terms, and define the ratio of the inductor's voltage and current phasors to be the inductive impedance, Z_L:

ohms law for phasors

$$Z_L = \frac{\vec{V}}{\vec{I}} = sL \qquad \textbf{(3.2)}$$

This result is often thought of as an Ohm's law for phasors. The impedances for all of the passive circuit elements are shown in Figure 3.1. Impedance is defined only for e^{st} signals, not for sinusoids, although sinusoids may be represented by an e^{st} signal through a phasor transformation.



$\frac{\vec{V}}{\vec{I}} = Z_R = R \qquad \frac{\vec{V}}{\vec{I}} = Z_L = sL \qquad \frac{\vec{V}}{\vec{I}} = Z_C = \frac{1}{sC}$

$\frac{v(t)}{i(t)} = R \qquad v(t) = L\frac{di(t)}{dt} \qquad i(t) = C\frac{dv(t)}{dt}$

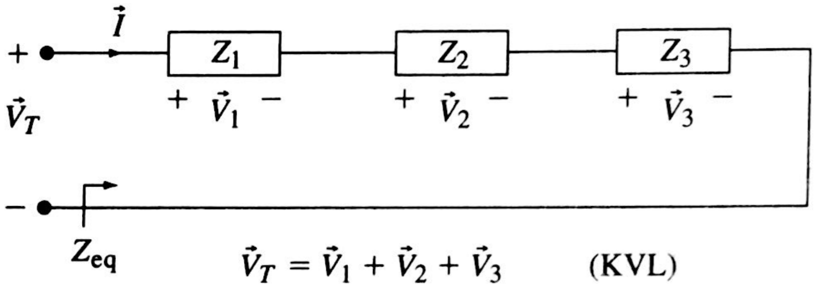
Figure 3.1 An "Ohm's law" exists for voltage and current phasors. The ratio of the voltage and current phasor for each circuit element is defined as the element's impedance and is designated by the symbol Z.

(prof. additional example...)

All information may be given in the phasor or s domain, and phasor transformations may remain unspecified until a link to the time domain needs to be established. Frequently this link is never stated, because all the important information is contained in the phasors. Since ratios of variables are involved, it makes absolutely no difference if the amplitudes are interpreted as peak, peak to peak, or rms.

It is easy to show (Figs. 3.2a–c) from Kirchhoff's laws and the relationships for phasors that impedances in series add. Admittances, which are the inverse of impedances, add for parallel combinations. The voltage and current divider relationships follow immediately, and are often sufficient for solving simple circuit problems.

The types of circuits of primary interest in this text have a single input signal. We can characterize the effects of the input signal in terms of a transfer function,

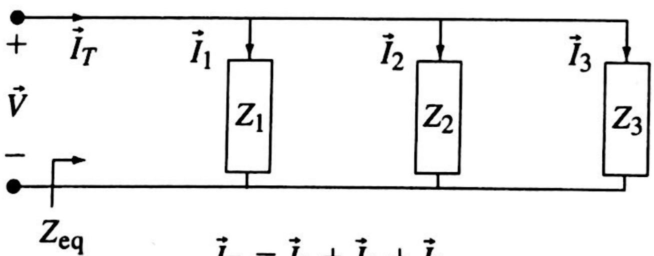


$$\vec{V}_T = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 \qquad \text{(KVL)}$$

$$Z_{eq} = \frac{\vec{V}_T}{\vec{I}} = \frac{\vec{V}_1}{\vec{I}} + \frac{\vec{V}_2}{\vec{I}} + \frac{\vec{V}_3}{\vec{I}} = Z_1 + Z_2 + Z_3$$

$$\frac{\vec{V}_3}{\vec{V}_T} = \frac{\vec{I}Z_3}{\vec{I}Z_{eq}} = \frac{Z_3}{Z_1 + Z_2 + Z_3} \qquad \text{(Voltage Divider)}$$

Figure 3.2a Phasor relationships in a series circuit.

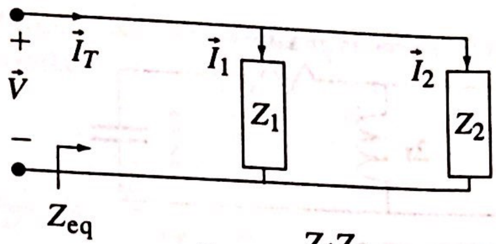


$$\vec{I}_T = \vec{I}_1 + \vec{I}_2 + \vec{I}_3 \qquad \text{(KCL)}$$

$$\frac{1}{Z_{eq}} = \frac{\vec{I}_T}{\vec{V}} = \frac{\vec{I}_1}{\vec{V}} + \frac{\vec{I}_2}{\vec{V}} + \frac{\vec{I}_3}{\vec{V}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

$$\frac{\vec{I}_3}{\vec{I}_T} = \frac{\vec{V}/Z_3}{\vec{V}/Z_1 + \vec{V}/Z_2 + \vec{V}/Z_3} = \frac{Y_3}{Y_1 + Y_2 + Y_3} \qquad \text{(Current Divider)}$$

Figure 3.2b Phasor relationships in a parallel circuit: $Y = 1/Z$ = admittance.



$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \text{product over sum}$$

$$\frac{\vec{I}_2}{\vec{I}_T} = \frac{Z_1}{Z_1 + Z_2} = \text{opposite over sum}$$

Figure 3.2c Special case of two impedances in parallel.

which relates the output signal at one set of terminals to the input signal at another set of terminals. We might, for instance, determine the output voltage caused by an input current. The result would be a ratio of an output voltage phasor to an input current phasor. Such a transfer function would have the units of impedance and would be called a *transimpedance*, to emphasize that the voltage and current involved are at different points in the circuit. Generally the term *impedance* is understood to apply to a ratio of phasors at the same set of terminals. If there is danger of confusion, the term *driving point impedance* can be used to emphasize that only one set of terminals is involved.

EXAMPLE 3.1

Find the driving point impedance of the circuit in Figure 3.3a.

Solution

The circuit components must be converted to impedances or admittances, as in Figure 3.3b. Next the series combination can be combined, as in Figure 3.3c.

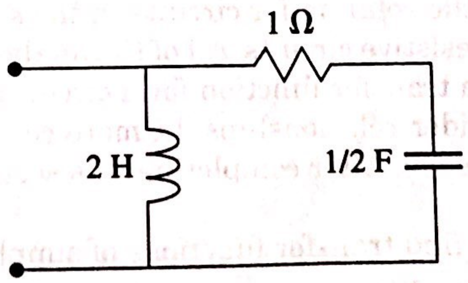


Figure 3.3a

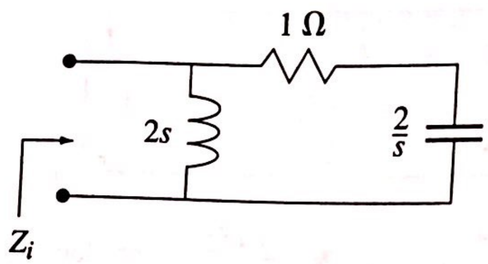


Figure 3.3b

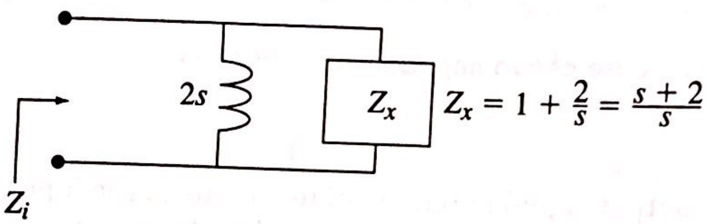


Figure 3.3c

The equivalent admittance is then found by adding the admittances of the parallel paths:

$$Y_i = \frac{1}{2s} + \frac{s}{s+2} = \frac{2s^2 + s + 2}{2s(s+2)} \quad \text{or} \quad Z_i = \frac{2s(s+2)}{2s^2 + s + 2} = \frac{s(s+2)}{s^2 + s/2 + 1}$$

Some algebra is involved in this process, so it is always a good idea to see if the result makes sense. At d-c ($s = 0$) the inductor makes the circuit a short. At high frequencies ($s = \infty$) the inductor is open, the capacitor is a short, and the circuit looks like a 1-Ω resistor. These properties are confirmed by the expression for Z_i.

3.2 CIRCUIT ANALYSIS IN THE *s* DOMAIN

With phasors representing voltages and currents, and impedance representing the passive circuit elements, the solution for circuits in the *s* domain is superficially no different than for purely resistive circuits. All of the analysis techniques for resistive circuits still apply. Often a transfer function for a circuit can be deduced using simple voltage or current divider relationships. In more complicated circuits, mesh or node equations may be needed. After completing this section you will be able to:

- Combine impedances to find transfer functions of simple circuits.
- Apply voltage and current dividers.
- Select among various analysis procedures.

For a simple series circuit, the voltage divider may be applied directly to obtain a voltage transfer function.

EXAMPLE 3.2 EXAM

Find $\vec{V}_o / \vec{V}_g$ for the circuit of Figure 3.4.

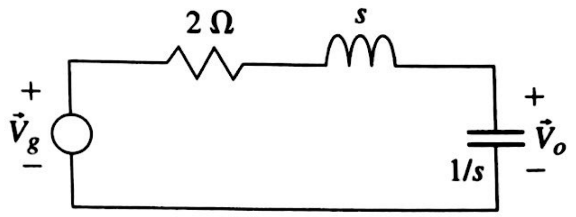


Figure 3.4

Solution

In this case the circuit has already been converted to the phasor domain, and the circuit elements are given as impedances. The result is obtained by inspection from the voltage divider (Fig. 3.2a).

$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{1/s}{2 + s + 1/s} = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}$$

In some problems it may be necessary to combine some impedances before trying to apply the voltage divider.

EXAMPLE 3.3

Find $\vec{V}_o/\vec{V}_g$ for the circuit of Figure 3.5a.

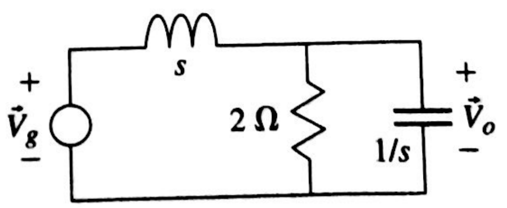
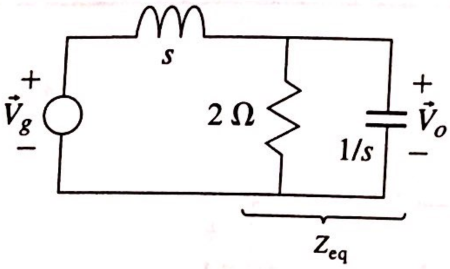


Figure 3.5a

Solution

Combining the parallel combination of 2 Ω and $1/s$ Ω into an equivalent impedance reduces the circuit to a series equivalent, and the voltage divider may be applied, as in Figure 3.5b:



$$Z_{eq} = \frac{2/s}{2 + 1/s} = \frac{2}{2s + 1} = \frac{1}{s + 1/2}$$

Figure 3.5b

$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{Z_{eq}}{s + Z_{eq}} = \frac{1}{s(s + 1/2) + 1} = \frac{1}{s^2 + s/2 + 1}$$

The current divider is also often useful. A typical procedure is to find the driving point impedance of the circuit to find the total current delivered to the circuit. Then the current divider is used to determine how the total current divides up among several paths.

EXAMPLE 3.4

Find $\vec{I}_o/\vec{V}_g$ in the circuit of Figure 3.6a.

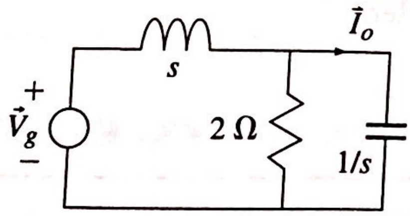


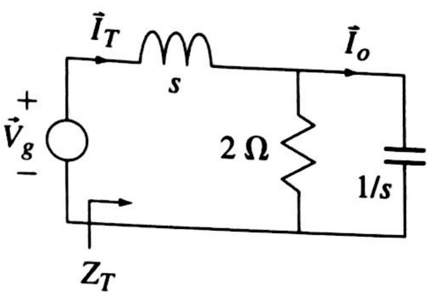
Figure 3.6a

Solution

The driving point impedance seen by $\vec{V}_g$ is

$$Z_T = s + \frac{2/s}{2 + 1/s} = s + \frac{2}{2s + 1} = \frac{2s^2 + s + 2}{2s + 1}$$

The total current into the circuit consequently is as shown in Figure 3.6b.



$$\vec{I}_T = \frac{\vec{V}_g}{Z_T} = \frac{\vec{V}_g(2s+1)}{2s^2+s+2}$$

Figure 3.6b

Using the special case of the 2-branch current divider (Fig. 3.2c),

$$\vec{I}_o = \frac{2}{2+1/s}\vec{I}_T = \frac{2s}{2s+1}\left(\frac{\vec{V}_g(2s+1)}{2s^2+s+2}\right) = \frac{s\vec{V}_g}{s^2+s/2+1}$$

This result could also have been obtained from taking Example 3.3 one step further, since the voltage across the capacitor was found in that example.

All of the reduction techniques normally taught in circuit theory can be used to find transfer functions in the s domain. For complicated electrical systems, the more general procedures of writing mesh or node equations may provide easier and more direct routes to the desired result than simple tools like voltage and current dividers. The inclusion of dependent sources requires such an approach. There are also circuit topologies for which the divider equations cannot be applied (see Fig. 3.7).

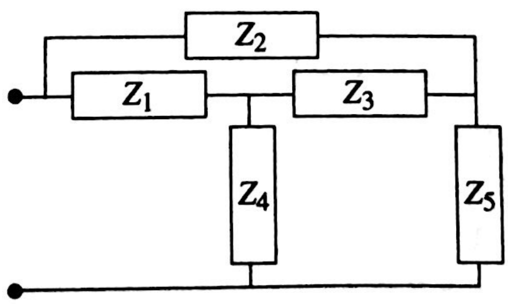


Figure 3.7 A circuit topology that does not lend itself to voltage or current division techniques because there are no series or parallel connections.

Find $\vec{V}_o/\vec{V}_g$ for the circuit of Figure 3.8a.

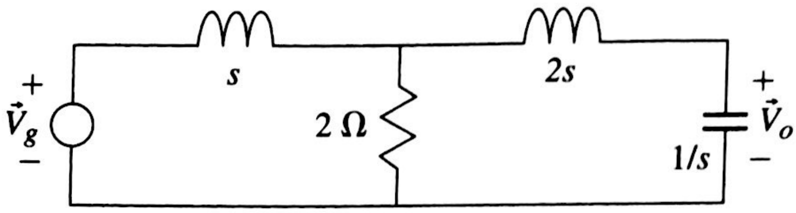


Figure 3.8a

Solution

This circuit could be solved by combining impedances and using divider equations, but it may be easier to assume mesh currents and write simultaneous equations. In this case, the mesh equations are combined to eliminate $\vec{I}_1$ (Fig. 3.8b):

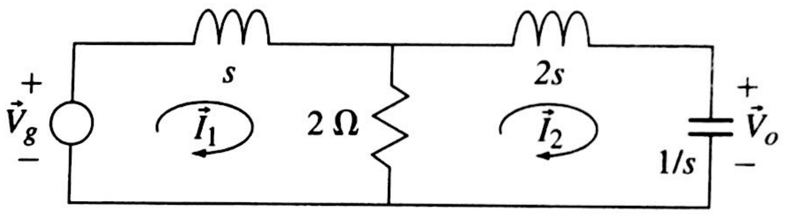


Figure 3.8b

$$\left.\begin{aligned}\vec{V}_g &= (s+2)\vec{I}_1 - 2\vec{I}_2 \\ 0 &= -2\vec{I}_1 + (2+2s+1/s)\vec{I}_2\end{aligned}\right\} \vec{V}_g = (s+2)(1+s+1/2s)\vec{I}_2 - 2\vec{I}_2$$

$$\vec{I}_2 = \frac{\vec{V}_g}{(s+2)(1+s+1/2s)-2} = \frac{\vec{V}_g}{s+s^2+1/2+2s+1/s}$$

$$\vec{I}_2 = \frac{s\vec{V}_g}{s^3+3s^2+s/2+1}$$

$$\therefore \frac{\vec{V}_o}{\vec{V}_g} = \frac{1}{s^3+3s^2+s/2+1}$$

It is always wise to check the result at $s = 0$ and $s = \infty$ to identify potential algebraic errors. The circuit under these conditions is as sketched in Figures 3.8c and 3.8d; the sketches support the transfer function equation at these s values.

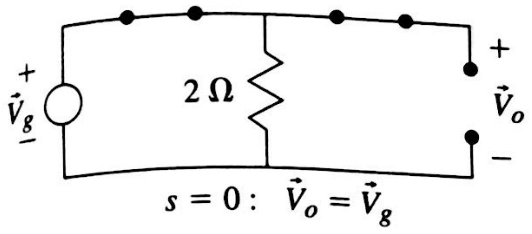


Figure 3.8c

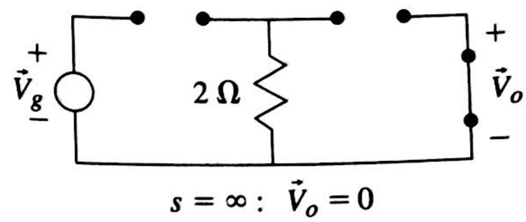


Figure 3.8d

Whatever analysis technique is used to achieve it, the characteristics of a particular circuit can be reduced to those of a transfer function. It may then be simply represented as a mathematical input/output block. In this way a complex system may be created and analyzed without cluttering up the system diagram with circuit details. The same process may be used to provide blocks representing mechanical, electromechanical, hydraulic, and other devices.

3.3 FEEDBACK DIAGRAM ELEMENTS

As different types of devices are combined to form a major system, each device is represented by its own s domain transfer function and shown as a block. The flow of signals from device to device is shown in the form of a block diagram. Some devices monitor the system's behavior and feed this information back to the devices that control the behavior. By comparing the actual behavior to the desired behavior, it is possible to achieve automatic control. After completing this section you will be able to:

- Recognize a basic feedback loop and describe the role of its elements.
- Apply the feedback viewpoint to two basic op-amp circuits.
- State the main advantage provided by feedback.
- State the main danger introduced by feedback.

The elements of a basic feedback system are shown in Figure 3.9. The notation used for signals is generic. A reference input $\vec{R}$ is compared to $\vec{F}$, a fed-back sample

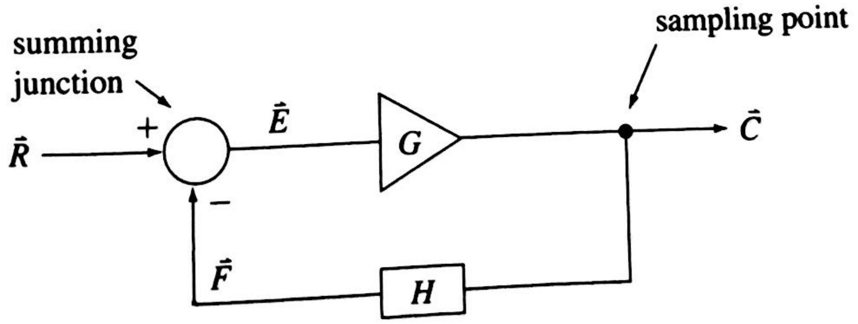


Figure 3.9 The block diagram of an elemental negative feedback loop. Signals are represented by phasors, while G and H are transfer functions.

of the controlled output $\vec{C}$. The difference between $\vec{R}$ and $\vec{F}$ is an error signal $\vec{E}$ used to adjust the output. Forward transfer functions are denoted by G, while H is used for reverse transfer functions. The signals are related as follows:

$$G = \frac{\vec{C}}{\vec{E}} \qquad H = \frac{\vec{F}}{\vec{C}} \qquad \vec{E} = \vec{R} - \vec{F} = \vec{R} - \vec{C}H$$

controlled output

Note that if $\vec{E} = 0$, then $\vec{R} = \vec{C}H$ or

$\vec{C}/\vec{R} = \frac{V_o}{V_{in}}$

$$\frac{\vec{C}}{\vec{R}} = \frac{1}{H} \quad \text{(ideal)} \tag{3.3a}$$

reference input

Since zero error is always the desired result, this may be regarded as the goal of feedback. With error present, the output is found as $\vec{C} = G\vec{E} = G(\vec{R} - \vec{C}H)$ or

$$\frac{\vec{C}}{\vec{R}} = \frac{G}{1 + GH} \quad \text{(actual)} \tag{3.3b}$$

We can see that the ideal case will be approached if $GH >> 1$. The implication of this is that we have the potential for trading a high but possibly widely varying gain, G, for a lower overall system gain set accurately by H.

Frequently the signals from the input and output sensors are electrical. In such cases the feedback summing junction can be implemented with the circuit variations of Figure 3.10. The presence of a summing junction does not necessarily mean feedback is present. It is also used to create specific transfer functions by combining signals from smaller transfer functions.

The sampling point is just a connection point for the input of H. Either G needs to be derived with the loading of H in place, or H needs to present a very high driving

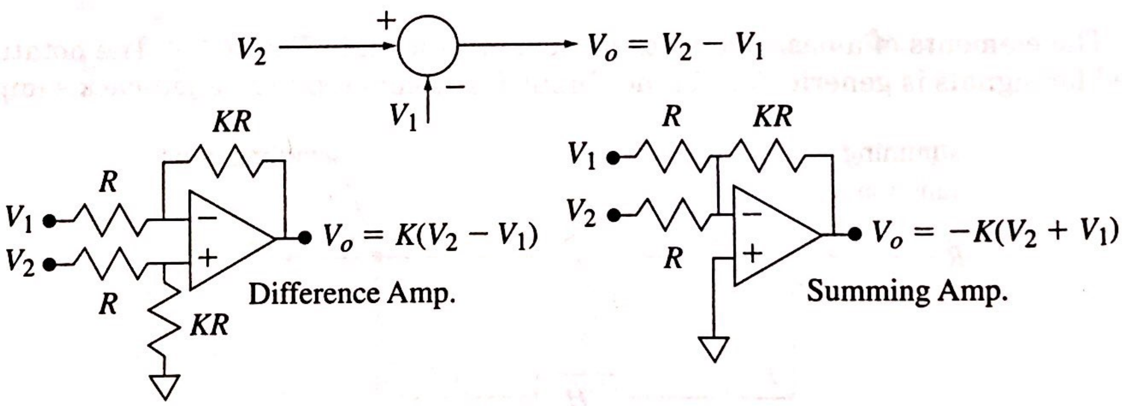


Figure 3.10 Summing junctions are implemented electronically with op-amp summing or differencing amplifiers. The input voltages must be of opposite polarity to achieve a difference voltage with the summing amplifier.

point impedance to G. Equation 3.3b shows that G does not matter as long as it makes the *loop gain*, GH, very large. On the other hand, H must be a precise and reliable device since it will set the overall system behavior.

In order to provide a few specific examples of feedback systems and the feedback point of view, consider the linear circuit model for the op amp shown in Figure 3.11a. Keep in mind that the op amp is actually an electronic circuit containing transistors that must be properly biased to function. Our model assumes the appropriate d-c supplies have been connected to bias the internal amplifier stages in their linear range. Any time a sample of the output is brought back and used to modify the input, it must be done with care. The feedback diagram (Fig. 3.9) requires that the sample from the output be *subtracted* from the input to achieve the results predicted in Equation 3.3. This is called *negative feedback*, and it provides a corrective action. Consider the connection shown in Figure 3.11b. If the output tended to go positive, for whatever reason, a positive voltage would be brought back to the inverting input, driving the output negative, that is, countering the original tendency. As a result, the output remains at zero volts, and internal amplifier stages remain properly biased. On the other hand, the connection shown in Figure 3.11c would bring a slight positive voltage at the output back to the noninverting

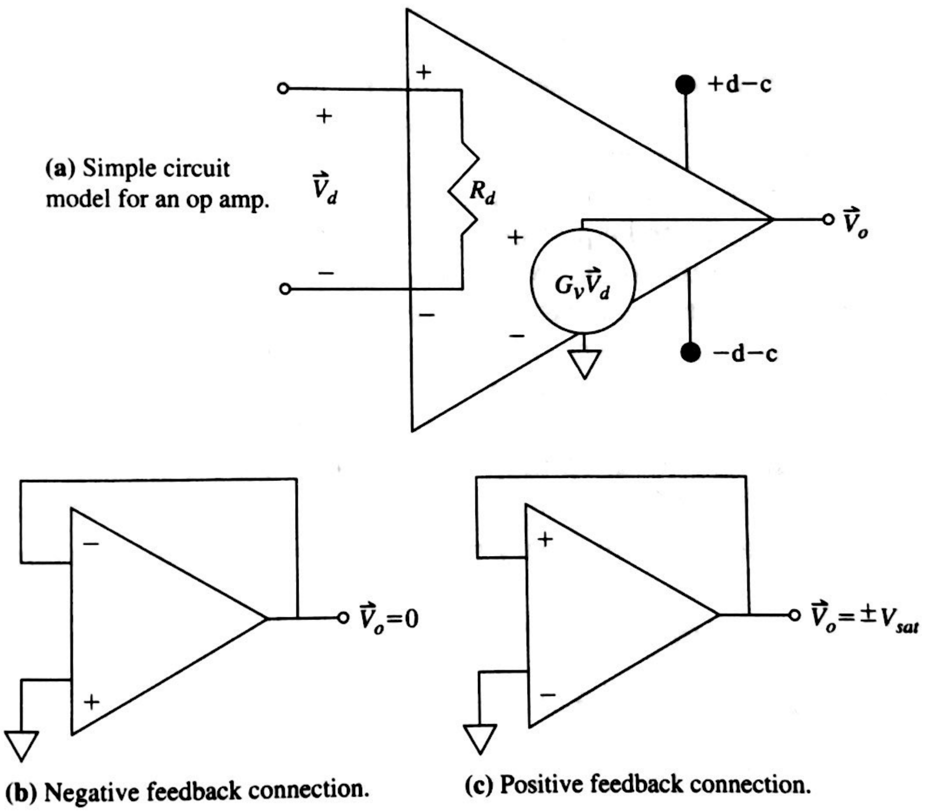


Figure 3.11 **(a)** A simple circuit model for an operational amplifier consists of a differential input resistance R_d, and an inherent voltage gain, G_v. The negative labeled input terminal is called the inverting input, while the positive labeled input terminal is called the noninverting input. A positive voltage on the inverting input drives the output negative, while a positive voltage on the noninverting input drives the output positive. **(b)** A negative feedback path is provided. **(c)** A positive feedback path is provided.

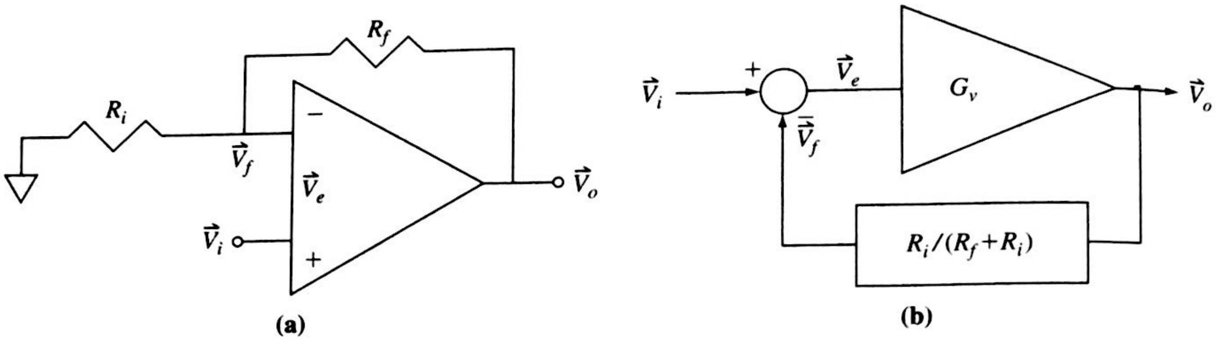


Figure 3.12 (a) The noninverting amplifier and (b) its feedback diagram.

input, driving the output even more positive. This is *positive feedback*, and the output will end up saturated at its maximum output in whatever direction it first departs from zero. Such a device has two stable states, and possible binary applications, but is unusable as an amplifier. For linear applications, the negative feedback connection must be present at d-c to keep the op-amp stages properly biased.

just see notes...

Figure 3.12a shows an op amp connected as a noninverting amplifier. In this circuit Kirchhoff's voltage law provides the summing junction since $\vec{V}_e = \vec{V}_i - \vec{V}_f$. We have three options for analyzing this circuit: ideal op-amp theory, the feedback viewpoint, or circuit theory. Ideal op-amp theory assumes that R_d and G_v are infinite, so no current enters the op-amp inputs, and V_d is zero. These assumptions make the analysis very easy, but do not show the limitations imposed by realistic op-amp properties. Circuit theory gives exact results, consistent with our model for the op-amp's internal circuitry, but provides a very complicated equation for the voltage transfer function. It does not readily show us which components are controlling the circuit behavior. The feedback viewpoint provides a compromise between these extremes, and focuses our attention on those elements that dominate the circuit's performance. Figure 3.12b shows the basic feedback loop for a noninverting amplifier.

Voltage summing is used in the noninverting amplifier, so we will not let $\vec{V}_e$ go to zero, but we will retain the ideal op-amp assumption that no current will enter the input terminals. Then $\vec{V}_o = G_v\vec{V}_e$ and $\vec{V}_f$ is given by the voltage divider equation $\vec{V}_f = \vec{V}_o R_i/(R_f + R_i)$. Ideal op-amp theory says that $\vec{V}_i = \vec{V}_f$, giving the familiar result

$$A_v = \vec{V}_o/\vec{V}_i = 1 + R_f/R_i = 1/H$$

The feedback model suggests that a more accurate result is

$$A_v = \vec{V}_o/\vec{V}_i = G_v/(1 + G_v R_i/(R_f + R_i)) \tag{3.4}$$

EXAMPLE 3.6

Data sheet information on a μA741 op amp indicates that its minimum d-c voltage gain is 20 V/mV, while its typical voltage gain is 200 V/mV. If it is used to create a noninverting amplifier using precision resistors with $R_f/R_i = 49$, what range of d-c voltage gain might it have? Repeat if $R_f/R_i = 4999$.

Solution

If an op amp is used that has the minimum G_v value, the overall gain will be $A_v = G_v/(1 + G_v/50) = 20{,}000/(1 + 400) = 49.88$. In this case $GH = 400$, and we will be within 0.24% of our target gain of 50. Op amps having the typical G_v value or higher will have overall gains even closer to the goal of 50.

If we try for an overall gain of 5000, an op amp with the minimum G_v will only have a GH product of 4 at d-c. The calculation becomes

$$A_v = G_v/(1 + G_v/5000) = 20{,}000/(1 + 4) = 4000$$

This is significantly different than the expectations of ideal op amp theory.

The basic inverting amplifier is shown in Figure 3.13a. This can be investigated from two viewpoints. An obvious summing junction is provided by Kirchhoff's current law, where $\vec{I}_e = \vec{I}_i - \vec{I}_f$. Here G is a transimpedance, while H is a transadmittance, emphasizing that the nature of G and H is defined by the variables sampled and summed. The feedback loop forms a current-to-voltage converter and R_i is not directly involved in the feedback. Since our goal is to determine the voltage transfer function $\vec{V}_o/\vec{V}_i$, we will need to find how much $\vec{I}_i$ is produced by $\vec{V}_i$, which means we must find the input resistance of the converter. This is easily accomplished and is approximately the same as the *Miller equivalent resistance* $R_f/(G_v + 1)$.

An alternate approach that is more direct is to assume that the currents into the op amp are negligible, and to find $\vec{V}_d$ by superposition. While it is not quite so easy to identify an error signal in this case, the circuit equations show a feedback look-alike result. Solving

$$\vec{V}_d = -\vec{V}_i \frac{R_f}{R_i + R_f} - \vec{V}_o \frac{R_i}{R_i + R_f} \quad \text{and} \quad \vec{V}_o = G_v \vec{V}_d$$

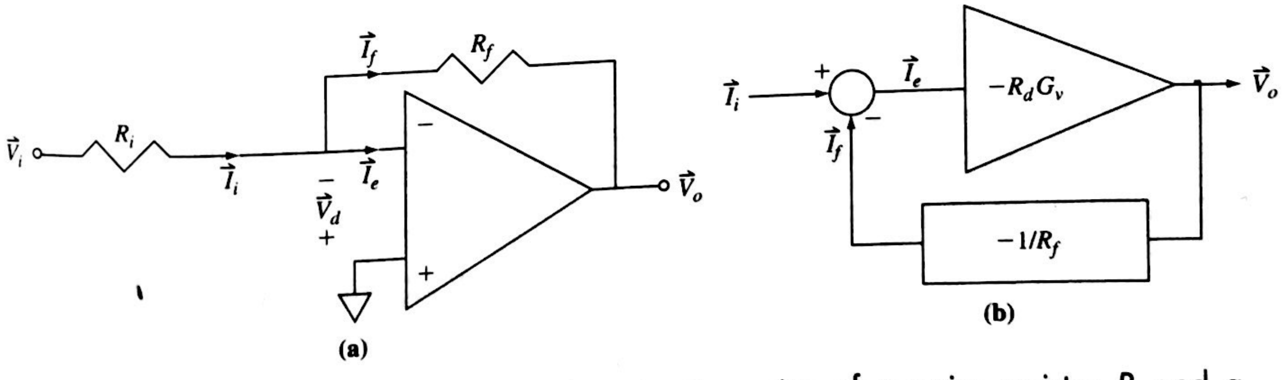


Figure 3.13 (a) The inverting amplifier circuit consists of a series resistor R_i and a current-to-voltage converter. **(b)** The current-to-voltage converter feedback diagram.

results in

$$\frac{\vec{V}_o}{\vec{V}_i} = A_v = \frac{-G_v'}{1 + G_v'R_i/R_f} \qquad \text{where} \qquad G_v' = \frac{R_f}{R_i + R_f} G_v \tag{3.5}$$

Slightly different assumptions are used in these two viewpoints, but essentially the same results are obtained from them.

EXAMPLE 3.7

A poor μA741 ($R_d = 300$ kΩ, $G_v = 20$ V/mV @ d-c) is connected as an inverting amplifier with $R_f = 2.2$ MΩ and $R_i = 1$ kΩ. Determine its d-c voltage gain.

Solution

Using Equation 3.5, we have $G_v' = \dfrac{2200}{2201}(-20000) = -19991$ and $A_v = \dfrac{-19991}{1 + 19.991/2.2} = -1982.$

The current-to-voltage converter would have

$$\frac{\vec{V}_o}{\vec{I}_i} = \frac{-R_dG_v}{1 + R_dG_v/R_f} = \frac{-6 \times 10^9}{1 + 6 \times 10^9/2.2 \times 10^6} = \frac{-6 \times 10^9}{2728} = -2.1992 \times 10^6$$

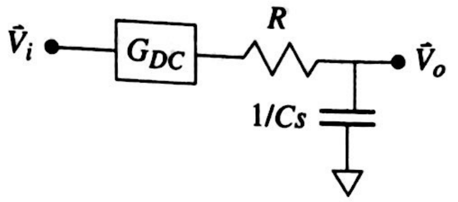
This result depends on the transimpedance of the op amp, which is not listed in the data sheet information. We have estimated a minimum value for it and it is still 6000 MΩ. For values of R_f that we might typically use, the feedback has so much excess gain that its transfer function may simply be taken as $-R_f$. This makes both approaches independent of R_d.

$$R_{in} = \frac{2.2 \times 10^6}{2 \times 10^4 + 1} = 110 \qquad \text{so} \qquad \vec{I}_i = \frac{\vec{V}_i}{1110}$$

$$\text{and} \qquad A_v = \frac{-2.1992 \times 10^6}{1110} = -1981$$

Including the effect of R_d will lower R_{in} by about 0.05 Ω. The two viewpoints lead to essentially the same numerical results, despite differences in the assumptions used.

So far we have considered both G and H to be simple constants. If that is true and we connect the feedback up correctly to start with, it will always remain negative feedback and provide us with a precise transfer function. Unfortunately, real devices have a frequency response that drops off at high frequencies. If G is an amplifier, capacitive effects start shorting the signal to ground, or bypassing it around the transistors as



$$\frac{\vec{V}_o}{\vec{V}_i} = G_{DC}\frac{1/Cs}{\mathrm{R} + 1/Cs} = \frac{G_{DC}}{RCs + 1} = \frac{G_{DC}}{s/\omega_1 + 1}$$

Figure 3.14 A model for an amplifier stage used to establish the stage's frequency response in terms of its cutoff frequency ω_1

frequency increases. Mechanical and thermal devices exhibit inertial effects that prevent them from following rapid input changes.

We can model the frequency response of one stage of an amplifier by the circuit of Figure 3.14. For sinusoidal signals ($s = j\omega$), each stage has a phase shift that approaches $-90°$ as ω approaches ∞. An op amp containing 3 or more such stages will consequently reach a phase shift of $-180°$ at some finite frequency; call it f_x. At this frequency, the op amp's inverting input becomes a noninverting input, and vice versa. The connections that provided negative feedback at d-c now provide positive feedback at f_x, and the entire circuit will tend to oscillate. No input is required for this to happen, the circuit will supply its own input. The $(1 + GH)$ factor reduces to zero and the onset of oscillation occurs when $GH = -1$. If $|GH| > 1$ when the phase is $-180°$ so that $(1 + GH)$ is negative, there is more gain than necessary to start the oscillations. The system is still unstable, but the linear model for the amplifier becomes invalid. In a sense we could say that the amplifier will overdrive itself, which automatically drops its effective gain to maintain the $|GH| = 1$ condition.

EXAMPLE 3.8

A basic negative feedback loop has a gain $G = \dfrac{10^7}{(s + 10)^3}$. If H is a constant, for what value of H will the system become unstable?

Solution

The system will become unstable if $GH = -1$. For this particular G we can see that the $-180°$ phase condition occurs when $10 + j\omega$ has an angle of 60° by expressing G in polar form. In a more general situation, however, we do the following:

$$GH = \frac{10^7}{(s + 10)^3}H = -1$$

$$-10^7 H = s^3 + 30s^2 + 300s + 1000 = 1000 - 30\omega^2 + j\omega(300 - \omega^2)$$

The imaginary term must vanish, so the frequency where oscillation occurs is $\omega = \sqrt{300}$ rad/s. Equating the real parts at this frequency gives

$$-10^7 H = 1000 - 30\omega^2 = 1000 - 30(300) = -8000$$
$$\therefore H = 8 \times 10^{-4} \qquad 1/H = 1250$$

A larger H just makes the $(1 + GH)$ term a negative number, which still represents an unstable condition. We must therefore limit ourselves to overall gains larger than 1250.

To check the above conclusions, the poles of the transfer function (zeros of $(1 + GH)$) are found. We expect poles will move into the RHP if $H > 8 \times 10^{-4}$.
Poles occur when $s^3 + 30s^2 + 300s + 1000 - 10^7 H = 0$.
If $H = 7 \times 10^{-4}$:

$$0 = s^3 + 30s^2 + 300s + 8000 = (s + 28.2)(s + 0.91 \pm j15.7) \text{ stable}$$

If $H = 9 \times 10^{-4}$:

$$0 = s^3 + 30s^2 + 300s + 10000 = (s + 30.8)(s - 0.40 \pm j18.0) \text{ unstable}$$

To avoid the instability problem, feedback systems need to be *compensated* to prevent the phase from reaching $-180°$ before $|G|$ is down to unity. A brute force method of accomplishing this is used with internally compensated op amps. One amplifier stage is forced to cut off at a very low frequency so that the overall $|G|$ is close to or below unity before the other stages start adding their extra lagging phase to the system. This makes the phase stay at about $-90°$ over the useful range of the op amp. Stability is guaranteed, although much of the inherent high frequency gain of the op amp is sacrificed. This will means that the goal of having $GH >> 1$ will be compromised, but without stability, other goals are meaningless. Externally compensated op amps allow the user to customize the compensation to their application so that less high frequency gain is wasted.

EXAMPLE 3.9

The G of Example 3.8 is cascaded with a compensator whose transfer function is

$$G_c = \frac{10^{-4}(s + 10)}{(s + 0.001)}$$

Determine the overall system gain at $\omega = 0.1$ rad/s if $H = 0.05$.

Solution

The transfer function of cascaded blocks is the product of individual block transfer functions. Note that the compensator's gain is unity at d-c ($s = 0$). The effective gain is

$$G_{eff} = G_cG = \frac{10^3}{(s + 0.001)(s + 10)^2}$$

G_{eff} has the same d-c gain as in Example 3.8, but it starts decreasing at $\omega = 0.001$ rad/s. Using the same procedure as in that example, we find that the $GH = -1$ condition occurs when

$$s^3 + 20s^2 + 100.2s + 0.1 = -10^3H = 0 = 0.1 - 20\omega^2 + j\omega(100.2 - \omega^2)$$

For the imaginary part to vanish, $\omega \approx 10$, so the instability will occur if

$$-1000H \approx -2000 \qquad \text{or} \qquad 1/H = 0.5$$

This system will be stable at any gain, since G is less than one when $GH = -1$.
At $\omega = 0.1$,

$$G_{eff} = \frac{10^3}{(j0.1 + 0.001)(j0.1 + 10)^2} \approx -100j \quad \text{and}$$

$$\frac{\vec{C}}{\vec{R}} = \frac{G_{eff}}{1 + G_{eff}H} = \frac{-100j}{1 - 100j\,(0.05)} = \frac{100\angle -90°}{5.099\angle -78.7°} = 19.6\angle -11.3°$$

This is within 2% of the desired gain of 20 at 0.1 rad/s. At d-c, where $G_{eff} = 10{,}000$, $\vec{C}/\vec{R} = 19.96$, for an error of only 0.2%.

Feedback offers us an opportunity to set the system transfer function with great precision by providing a large, but not necessarily precise, G. On the other hand, any connection from the output back to the input must be made with care. If G can have a phase shift of more than 180°, the feedback that is negative at low frequency can become positive at higher frequencies and lead to instability. If a positive feedback path exists, the system will find that path all by itself, and supply its own signal.

3.4 BLOCK DIAGRAM REDUCTION

Most systems do not fit the form of the elemental feedback block diagram. Multiple summing junctions and sampling points may be present, and the system transfer function is not immediately evident. After completing this section you will be able to:

- Simplify block diagrams to obtain the system transfer function.
- Include systems with multiple inputs.

Generally the more variables sampled in a control system, the more control that can be exercised. Additional summing junctions are then also needed, and the block diagram complexity increases. If the block diagram can be reduced to the

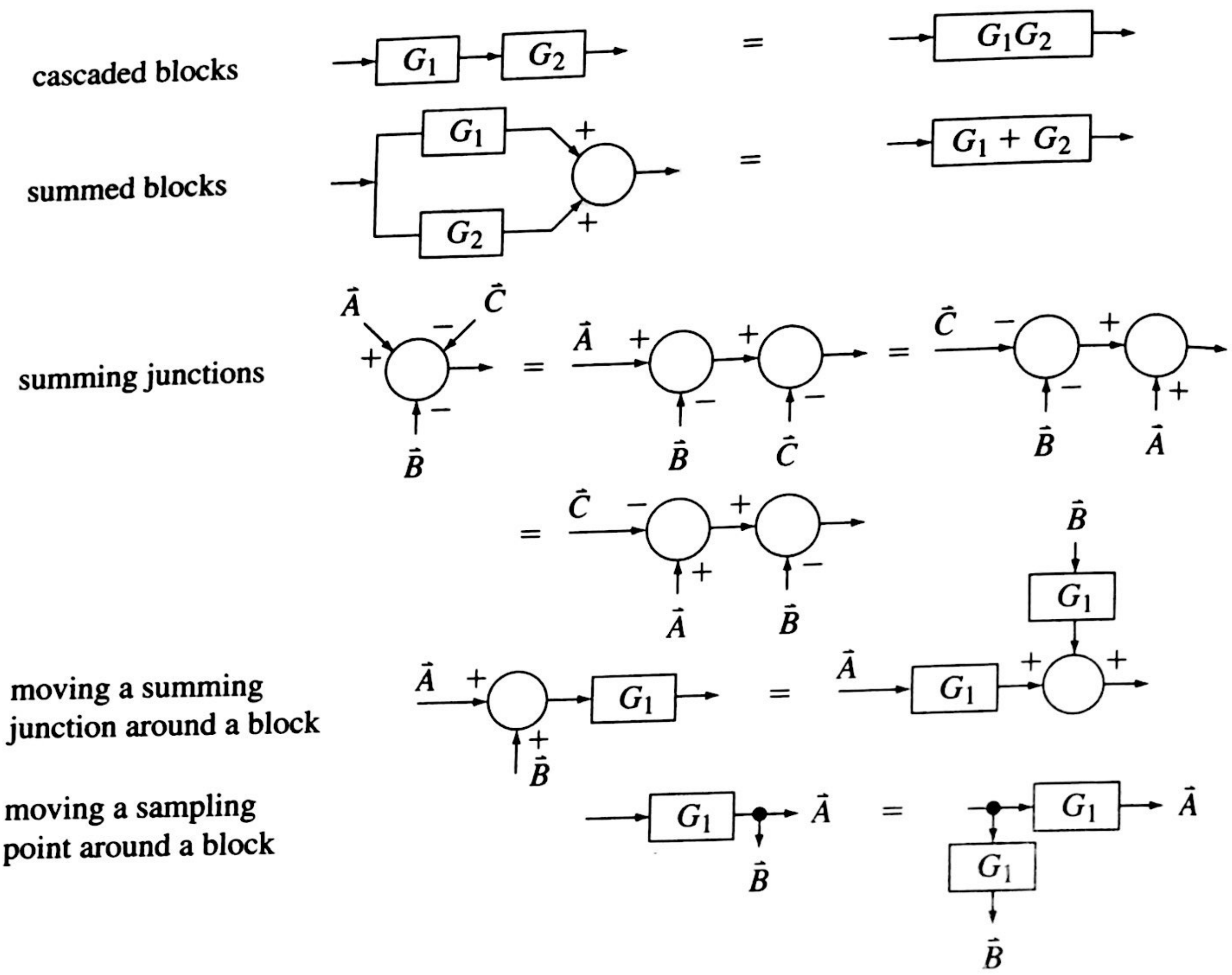


Figure 3.15 Block diagram relationships used to reduce and simplify complicated systems.

basic elements of Figure 3.9, the system transfer function can be found by inspection. Figure 3.15 shows legitimate consolidation operations.

Although we will limit our concern to simplifying given block diagrams, it is instructive to see how the permanent magnet d-c motor whose differential equation was derived in Chapter 2 could be represented by block diagrams. Figures 3.16a and 3.16b show the two main equations of the system and their corresponding block diagram representation. We have added a load torque so that we might investigate how the motor speed reacts to loading.

Combining the electrical and mechanical diagrams gives the overall block diagram (Fig. 3.16c). The torque summing junction has been broken down into two summing junctions. This makes the inner feedback loop more obvious so that it can be immediately replaced by a single block ($G = 1/Js$ and $H = k_v$). The sign of the load torque is also changed, with a corresponding change in the sign at its summing junction (Fig. 3.16d).

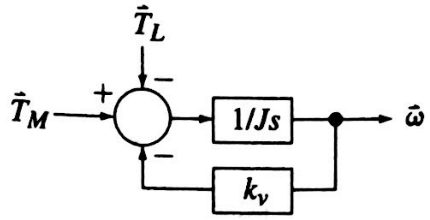
Since there are two input signals, we will use superposition to determine how each affects the motor speed. We first let $\vec{T}_L$ be zero, which allows the torque

$$J\frac{d\omega}{dt} = \Sigma T = T_m - k_v\omega - T_l$$

T_m = motor torque

T_l = a variable loading torque

$$\omega = \tilde{\omega}e^{st},\ T_m = \tilde{T}_M e^{st},\ \text{and}\ T_l = \tilde{T}_L e^{st}$$



$$Js\tilde{\omega} = \tilde{T}_M - k_v\tilde{\omega} - \tilde{T}_L$$

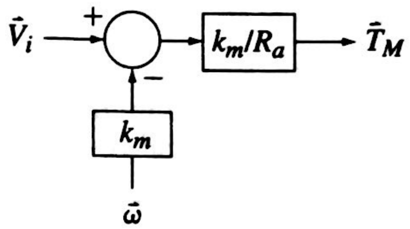
Figure 3.16a Mechanical portion of the block diagram for a permanent magnet d-c motor.

$$T_m = k_m i_a = \frac{k_m}{R_a}(v_i - e_b)$$

v_i = motor input voltage

$e_b = k_m\omega$ induced back *emf*

$$v_i = \tilde{V}_i e^{st}$$



$$\tilde{T}_M = \frac{k_m}{R_a}(\tilde{V}_i - k_m\tilde{\omega})$$

Figure 3.16b Electrical portion of the block diagram for a permanent magnet d-c motor.

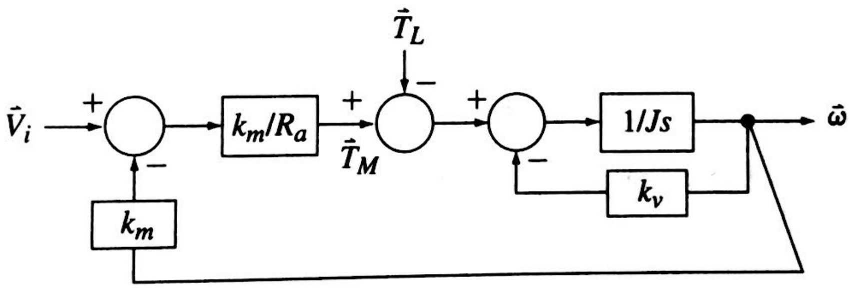


Figure 3.16c Block diagram of a permanent magnet d-c motor.

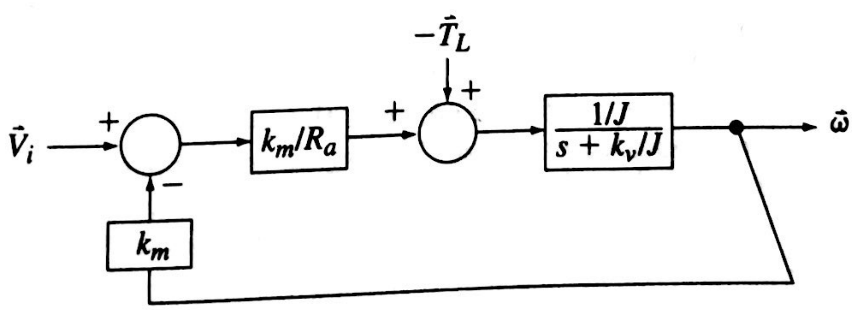


Figure 3.16d Reduced block diagram of a permanent magnet d-c motor.

summing junction to be removed and the effective forward and reverse transfer functions to be identified as

$$G = \frac{k_m/JR_a}{s + k_v/J} \qquad \text{and} \qquad H = k_m$$

The resulting transfer function is

$$\frac{\tilde{\omega}}{\tilde{V}_i} = \frac{\left(\dfrac{k_m/JR_a}{s + k_v/J}\right)}{1 + k_m\left(\dfrac{k_m/JR_a}{s + k_v/J}\right)} = \frac{k_m/JR_a}{s + k_v/J + k_m^2/JR_a}$$

Next $\tilde{V}_i$ is reduced to zero. The minus sign on the voltage summing junction can be transferred to the torque summing junction. Now with the voltage summing junction removed, the forward and reverse transfer functions can be identified as

$$G = \frac{1/J}{s + k_v/J} \qquad \text{and} \qquad H = k_m^2/R_a$$

giving

$$\frac{\tilde{\omega}}{-\tilde{T}_L} = \frac{\dfrac{1/J}{s + k_v/J}}{1 + (k_m^2/R_a)\dfrac{1/J}{s + k_v/J}} = \frac{1/J}{s + k_v/J + k_m^2/JR_a}$$

The superposition of these results gives the motor's speed in reaction to applied voltage and load torque as

$$\tilde{\omega} = \frac{(k_m/JR_a)\tilde{V}_i - (1/J)\tilde{T}_L}{s + k_v/J + k_m^2/JR_a} \tag{3.6}$$

Although the derivation of Equation 3.6 has provided an opportunity to demonstrate some block diagram reduction techniques, the result has further significance. First, notice that the poles of the system are the same for either input. That means that an abrupt change in the input voltage, or in the mechanical loading will induce exactly the same type of natural response. This again confirms that a system's natural response does not depend on the forcing function. Secondly, we note that feedback is inherent in nature and is just a way of looking at the equations of a system. The feedback provided by the motor's back emf term can be duplicated and enhanced by having the motor drive an unloaded generator (tachometer) and adding its signal to the voltage summing junction. This is the basis of many automated speed control systems.

Finally, if the input voltage and load torque do not depend on time, then $s = 0$, and phasors become identical to the variables they represent ($v_i = \bar{V}_i e^{st} = \bar{V}_i$, etc.). Under these conditions Equation 3.6 for a given motor reduces to

$$\omega = k_1 v_i - k_2 T_l$$

which simply says a motor's steady state speed is proportional to the applied d-c voltage, and decreases with loading. This is consistent with our everyday experiences.

Suggestions for reducing block diagrams are about the same as for reducing circuits. Make the major simplifications one at a time to avoid confusion, and fully simplify intermediate transfer functions. Identify each simplification as an operation permitted by Figure 3.15. Note that there is no simplification that moves a summing junction around a sampling point. Different but legitimate simplification routes always lead to the same final result.

EXAMPLE 3.10

Find the transfer function for the system in Figure 3.17a.

Solution

The sampling point for H_1 and the summing junction next to it prevent us from seeing any elemental feedback loops. Either the summing junction must be moved around G_1 or the sampling point moved around G_2. Taking the latter approach, the signal sampled will be G_2 larger than before, so it must be reduced by the same factor in the feedback path (Fig. 3.17b).

Now the positive feedback loop formed by G_2 and H_2 is obvious, and reduced to a single block (Fig. 3.17c).

At this point the final transfer function can be obtained as

$$\frac{\bar{C}}{\bar{R}} = \frac{\dfrac{G_1G_2}{1 - G_2H_2}}{1 + \dfrac{G_1G_2}{1 - G_2H_2}\dfrac{H_1}{G_2}} = \frac{G_1G_2}{1 + G_1H_1 - G_2H_2}$$

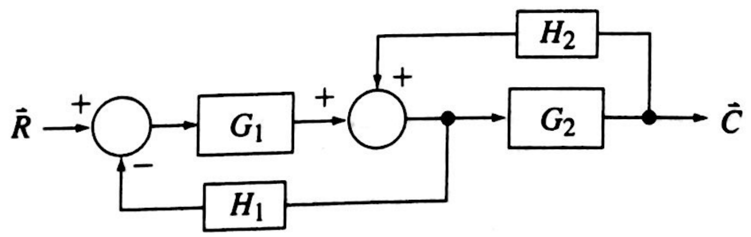


Figure 3.17a

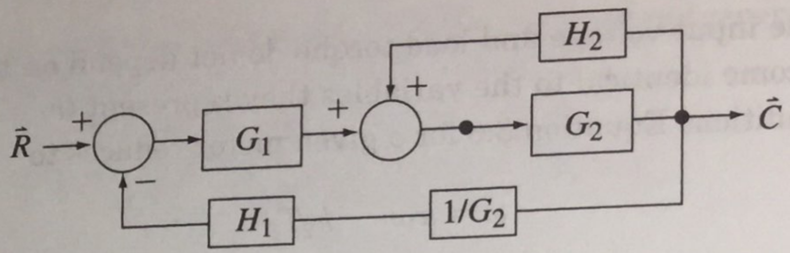


Figure 3.17b

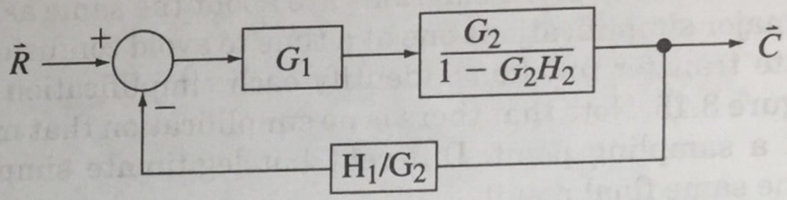


Figure 3.17c

3.5 STABILITY TESTS

right half-plane

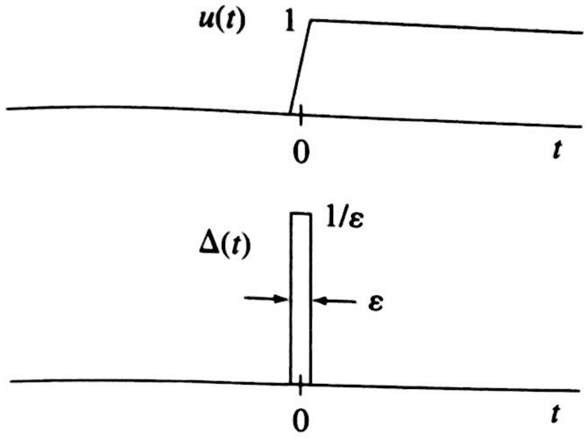
Any system with poles in the RHP is unusable. It would be unfortunate to spend a considerable amount of time investigating the forced response of a system only to find that it has an unstable natural response. The importance of the stability question has led to the development of a number of tests for stability. After completing this section you will be able to:

- Explain ways to test a system for stability.
- Define the unit impulse function.
- Use the Routh–Hurwitz stability test for polynomials.

One of the things that usually cannot be done to test a system's stability is to turn it on and see what happens. Watching a motor snap its shaft in an unstable system is both expensive and potentially dangerous to test personnel. Instead, a great deal of effort has been expended in developing computer simulation tools to evaluate stability.

Given a mathematical model of a system, we can test it for stability simply by finding where its poles are located. If the system gain is a parameter, the poles will move as the gain varies. This movement is smooth and gradual, with each pole tracing a path on the s plane. A plot of these paths as a function of the gain is called a *root locus diagram*. This diagram is particularly useful in control theory, because it not only identifies when poles move in or out of the RHP, but also shows how close poles come to the $j\omega$ axis, which measures the system's relative stability. We will explore this technique briefly in the next section.

If a rectangular pulse is applied to a system, its application stimulates the system's natural response along with the forced response. When it turns off, it again stimulates the natural response, but is no longer forcing a response. By reducing the



unit step $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

$u(t) = \int \delta(t)dt$

unit impulse $\delta(t) = \lim_{\varepsilon \to 0} \Delta(t)$

Figure 3.18 Step and impulse functions are frequently used test signals. When they turn on, they activate a circuit's natural response. The step also produces a d-c ($s = 0$) forced response, while the impulse dumps energy into the circuit at $t = 0$ and then lets the circuit's natural response take over. Although the ideal impulse occurs in zero time, any pulse whose duration is short compared to the time constants of the circuit under test will produce the same effect.

pulse's width to zero while simultaneously increasing its amplitude to retain a finite energy, we obtain an impulse function. The impulse is over so quickly that we only see the results of its turning off, which is the natural response. Using simulation, therefore, we can determine the stability of a system by finding its impulse response and seeing if it dies out or just keeps growing.

Mathematically, the unit impulse has properties that make it very useful in other areas of system or signal analysis. Its definition is given in Figure 3.18.

The *Routh–Hurwitz Criterion* is a stability test that does not require use of a computer. It can, however, become cumbersome for polynomials higher than the fourth degree, so we will limit our use to polynomials of up to the fourth degree. For such a polynomial (i.e., $s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$) to have no roots in the RHP, it must meet two tests:

Test 1: All the lower a_n must be present and have the same sign. This test is *sufficient* for polynomials of the first or second degree. This test simply recognizes that a root in the RHP would have the form $(s - r)$. Its presence will produce a negative term in first- or second-degree polynomials, but its sign may be swamped out by the contributions of LHP roots in larger polynomials.

Test 2: For polynomials of the third or fourth degree, form the array

$$\begin{matrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{matrix} \qquad (a_5 = 0, \text{ but helps show the array pattern})$$

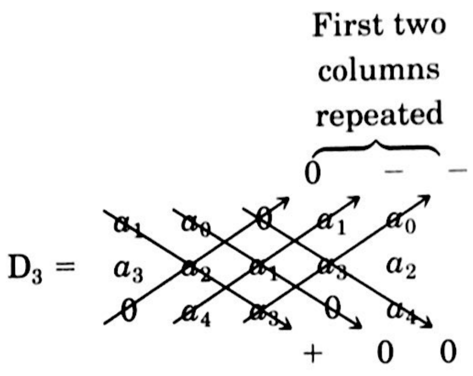
For a third-degree polynomial, evaluate the determinant for the 2×2 square array whose upper left-hand corner is a_1. For a fourth-degree polynomial, evaluate the determinant of the 3×3 array whose upper left-hand corner is a_1. *The value of these determinants must be greater than zero for stability*. The determinants needed for these two cases are shown next. (Determinants for arrays of the fifth or higher degree polynomials require use of a technique called the *method of minors*. See an appropriate mathematics text.)

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_0a_3$$
$$\qquad\qquad\qquad\qquad - \quad +$$

The arrows show the terms multiplied and the sign of the product.

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} = a_1a_2a_3 - a_1^2a_4 - a_0a_3^2$$

The pattern for finding D_3 is most easily remembered by repeating the first two columns and considering only three-term diagonals. Remember, the procedure given here has been specialized to polynomials of up to the fourth degree.



EXAMPLE 3.11

Determine the conditions on K for stability if a system has the characteristic equation $s^3 + 2s^2 + Ks + 8$.

Solution

Test 1: The cubic polynomial has all of its lower powers of s terms present, and they are all positive, provided $K > 0$.

Test 2: The array is

$$\begin{matrix} K & 8 & 0 \\ 1 & 2 & K \\ 0 & 0 & 1 \end{matrix}$$

and since the polynomial is of the third degree, we only need to evaluate the 2×2 determinate

$$D_2 = \begin{vmatrix} K & 8 \\ 1 & 2 \end{vmatrix} = 2K - 8$$

$$2K - 8 > 0 \text{ for stability}$$

$$\therefore K > 4 \text{ for stability}$$

The most restrictive requirement must be met.

Do not gloss over Test 1 just because it is simple. Its requirements are essential if incorrect conclusions are to be avoided. The conditions set by Test 2 can become a little trickier to deduce in a fourth-degree polynomial.

EXAMPLE 3.12

A particular system is tested for stability using Routh–Hurwitz. Test 1 requires $K > 0$ for stability. Test 2 results in the requirement that $-K^2 + 10K + 11 > 0$. Provide an unambiguous statement on the K values for which the system is stable.

Solution

The Test 2 requirement is converted to

$$K^2 - 10K - 11 < 0$$

and factored, temporarily assuming an equality. The result is

$$(K + 1)(K - 11) < 0$$

The inequality says that the product of the two factors must be negative for stability. That, in turn, requires the factors to be opposite in sign, which occurs if $-1 < K < +11$. However, Test 1 requires $K > 0$, so the system is stable only as long as $0 < K < 11$. If $K = 0$ or 11, at least one pole is on the $j\omega$ axis.

3.6 MATLAB LESSON 3

This lesson will demonstrate some tools for simplifying block diagrams, investigating stability, and expanding our graphing options. In addition, conditional branching options will be demonstrated. After completing this section you will be able to:

- Use MATLAB for block diagram reduction and system simulation.
- Select a multiple-graph option.
- Create a root locus plot.
- Discuss conditional branching options.

As with previous MATLAB lessons, you need to perform the command statements indicated to benefit from the lesson.

MATLAB EXAMPLES

The core of MATLAB consists of mathematical operations on matrices. These inherent operations can be expanded and applied to many specialty fields through Toolboxes that are sold separately. The topics of this text require commands from the Controls and Signal Processing Toolboxes. If you are in an unfamiliar computer lab where MATLAB is available, you may still find that the commands you are accustomed to using are not available because the corresponding Toolbox has not been installed.

When two blocks are cascaded, we know that the overall transfer function is the product of their numerator polynomials over the product of their denominator polynomials. We already know how to find these products separately using the **conv** command. The **series** command from the Controls Toolbox is a short program that uses **conv** twice, making the operation appear like a single step for us. Even more complicated transfer function reductions can be made using other commands from this Toolbox.

series **feedback** **cloop**

```
> gnum=10; gden=[1 2];                        % defines the transfer
                                                function G = 10/(s + 2)
> hnum=1; hden=[1 0];                         % defines the transfer
                                                function H = 1/s
> [nu,de]=series(gnum,gden,hnum,hden)         % finds the transfer
                                                function GH
> [num,den]=feedback(gnum,gden,hnum,hden)     % finds the transfer
                                                function G/(1 + GH)
> [num,den]=feedback(gnum,gden,hnum,hden,1)   % finds the transfer
                                                function G/(1 − GH)
```

not installed

The fifth argument of the **feedback** function describes the sign used for the feedback connection at the summing junction. A −1 defines negative feedback and is the default value. In control theory, the H transfer function often is unity. The **cloop** command is this special case of **feedback**.

```
> [num,den]=feedback(gnum,gden,1,1,−1)    % finds the function G/(1 + G)
> [num,den]=feedback(gnum,gden,1,1)       % does the same thing
> [num,den]=cloop(gnum,gden)              % so does this
```

impulse **step**

The response of a system to a step or impulse input may be obtained by specifying the transfer function numerator and denominator and the time range of interest. The impulse response is a particularly good way to check a system for stability.

```
> num=[1 0]; den=[1 1 4];  % a slightly resonant transfer function
> impulse(num,den)         % or step(num,den)-lets program determine range
                             of t.
> t=linspace(0,4);         % if you want to see details of first two peaks
> impulse(num,den,t)
> den=[1 -1 4];            % set up an unstable system
> impulse(num,den)         % instability is obvious
```

Although it is rarely of interest, we can fool the program into giving the forced response to a ramp or parabolic input by multiplying the denominator by s or s^2, respectively, and using the step command.

```
> num=4              % get a low-pass filter
> den=[1 1 4];       % regain a stable system
> step(num,den)      % gives the response to a unit step input
> den=[den 0];       % multiply denominator by s
> step(num,den)      % this is the response to a unit ramp forcing function
> den=[den 0];       % multiply denominator by s again
> step(num,den)      % this is the response to a 0.5* unit parabola forcing
                       function
```

subplot

The Figure Window may be subdivided into multiple plots. Each plot may then be addressed separately, and all of the graph-related commands are available for modifying and labeling each plot. **subplot(a,b,c)** specifies that the Figure Window be divided into **a** rows and **b** columns of plots, and selects plot **c** as the active plot. The plots are numbered sequentially from left to right across the first row, followed by the second row, and so on. **subplot(3,3,5)** would specify nine graphs, arranged in 3 rows and 3 columns, and selects the plot in the center of the array as the active plot.

```
> x=linspace(-2,2);          % define an independent variable
> y=2*x;                     % define a dependent variable
> z=y.^2;                    % define another dependent variable
> subplot(1,2,1);            % sets up two side-by-side plots, and makes the
                               left plot active
> plot(x,y); title('y=2x')   % all of the graphical commands apply only to
                               the active plot
                             % check the next command carefully before exe-
                               cuting!
> subplot(1,2,2);            % the right-hand graph is now the active one
> plot(x,z); ylabel('z')     % all of the graphical commands apply only to
                               the active plot
```

If we change the first two arguments of the subplot command, even unintentionally, it forces a change in the graph format and may destroy any existing graphs.

```
> subplot(2,1,1) % wanted to make labeling changes on the left graph,
                   but goofed
```

When dealing with complex numbers, we frequently want to graph both magnitude and phase (or real and imaginary parts) on the same sheet of paper. The subplot feature allows us to do this. Many MATLAB functions also use the subplot feature, and we need to know the subplot command if we want to modify the axis scale or labeling on a graph it has created. After printing or saving the page, you need to restore the original single-graph format.

```
> subplot(1,1,1) % returns the Figure Window to single-graph format
```

for-end while-end if-elseif-else-end

MATLAB permits the same types of conditional programming loops found in general computer languages. When they are used within the command window, the loop execution is suspended until the **end** command is entered. Errors will consequently not be detected until then. The MATLAB cursor is also turned off within the loop.

For loops evaluate a set of commands a fixed number of times.

```
> for i=0:306
     x(i+1)=sin(i*pi/307);      % lowest allowed element number is 1 (the
                                  indenting is optional)
end
> plot(x)
> clear
```

For loops can be nested.

```
> for i=1:3
     for j=1:3
          x(i,j)=i*j;
     end
end
> x
> clear
```

While loops evaluate a group of expressions until a test condition is met.

```
> t=0; x=0; y=10; % make sure loop starts
while x<y
     t=t+.0001;    % Solve to find t, where 1.5t=sin πt, t ≠ 0
     x=1.5*t;
     y=sin(pi*t);
end
> t
> clear
```

If-else constructs provide alternative paths based on the test condition. The "=" sign is used as an assignment operator in MATLAB. It assigns the value of the expression on its right to the variable named on its left. To test if an integer variable k "equals" zero, we need to use "==".

```
for t=1:300
    if t==100
        y(t)=pi/10;          % correct the indeterminate value of y(100)
    elseif t==200
        y(t)=pi/10;          % correct the indeterminate value of y(200)
    else y(t)=1000*sin(pi*t/100)/((t-100)*(t-200));   % no array here
    end
end
> plot(y,'o')
```

The following commands will plot the location of the roots of a polynomial. We want them all on one line, so we can repeat the entire line of commands.

```
> k=0; p=[1 3 10 k+1 k]; r=roots(p); plot(r,'*')  % the polynomial is of
                                                    fourth degree
> hold
```

Now use the up arrow to bring the commands back, modify the value of *k*, and execute the line.

```
> k=1; p=[1 3 10 k+1 k]; r=roots(p); plot(r,'*')
> k=2; p=[1 3 10 k+1 k]; r=roots(p); plot(r,'*')  % etc. continue as long
                                                    as you wish
> hold
```

The result is an example of a root locus plot. This is also an obvious application for a **for** loop. We may as well add a few embellishments.

```
> k = 0; p=[1 3 10 k+1 k]; r=roots(p); plot(r,'o') % mark the k = 0 points
                                                     with a 'o'
> hold
> for k = 1:100
p=[1 3 10 k+1 k]; r=roots(p); plot(r,'.')          % use '.' to draw the
                                                     locus
end
p=[1 3 10 k+1 k]; r=roots(p); plot(r,'*')          % mark the k = max end-
                                                     ing points with '*'
> hold
```

CHAPTER SUMMARY

Specializing the voltage–current relationships of the three passive circuit elements for the e^{st} signal allowed their characteristics to be expressed in the form of a ratio of voltage and current phasors called an impedance. It was shown that if a set of signals obey Kirchhoff's laws, then the phasors representing those signals also obey them. With signals represented by phasors and circuit elements represented by impedance, all of the circuit simplification techniques typically introduced in a d-c circuits course can be used. Transfer functions can be derived entirely in the s domain without reverting to differential equations.

Systems are formed by interconnecting various kinds of devices. The systems may be described using block diagrams, where each block states the transfer function of the device it represents. These interconnections usually take the form of a negative feedback system in which the actual and desired outputs are compared so that the system can automatically correct itself. Adding gain to such a system can set the system's transfer function at a reliable value, but may also make its natural response unstable. Block diagrams may be systematically simplified and reduced to a final overall transfer function for the system.

If a system is unstable, it is unusable. The Routh–Hurwitz test provides one method of predicting values of a parameter, usually gain, that can cause the system to become unstable. A unit impulse is a good simulation input to use when testing for stability.

MATLAB provides tools that can assist us in simplifying feedback systems. It also provides a means of testing for stability through simulation or through pole locations in the s plane.

PROBLEMS

Section 3.1

1. Determine the driving point impedance of the circuit in Figure P3.1.

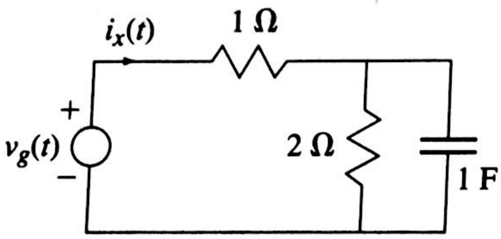


Figure P3.1

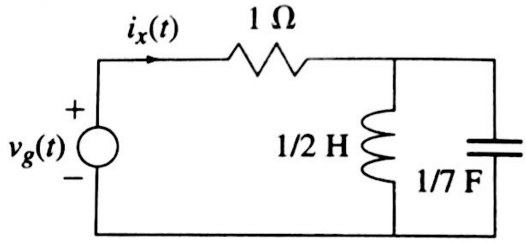


Figure P3.2

2. Determine the driving point impedance of the circuit in Figure P3.2.
3. Determine the driving point admittance of the circuit of Figure P3.3.

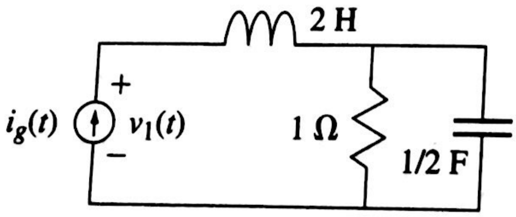


Figure P3.3

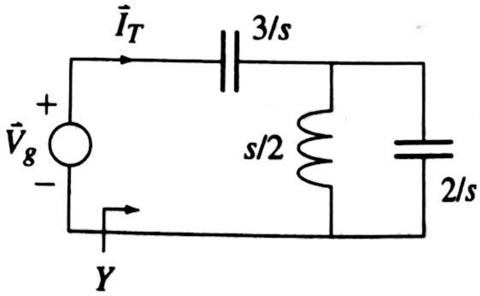


Figure P3.4

4. Determine the driving point admittance of the circuit of Figure P3.4.

Section 3.2

5. For the circuit of Figure P3.5, find and sketch the pole-zero diagram for:
   a. the transimpedance $\vec{V}_o/\vec{I}_g$.
   b. the driving point impedance Z.

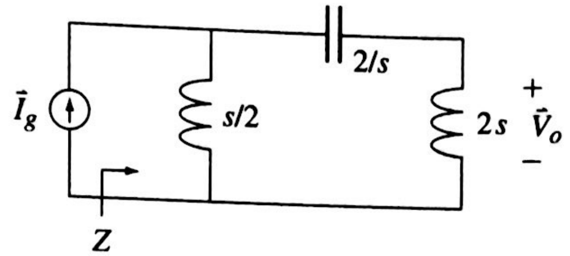


Figure P3.5

6. Find the transfer function $\vec{I}_x/\vec{V}_g$ in the circuit of Figure P3.6.

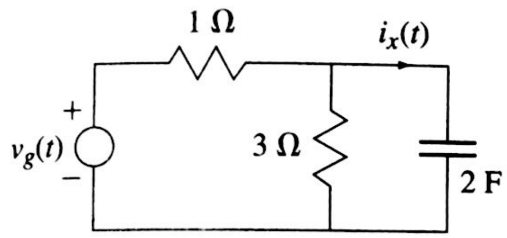


Figure P3.6

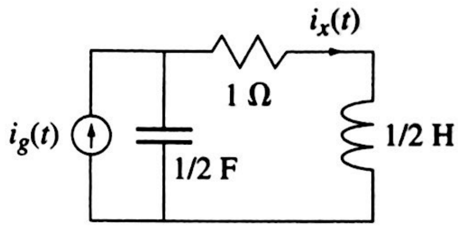


Figure P3.7

7. Find the transfer function $\vec{I}_x/\vec{I}_g$ in the circuit of Figure P3.7.
8. Find the transimpedance $\vec{V}_o/\vec{I}_g$ in the circuit of Figure P3.8.

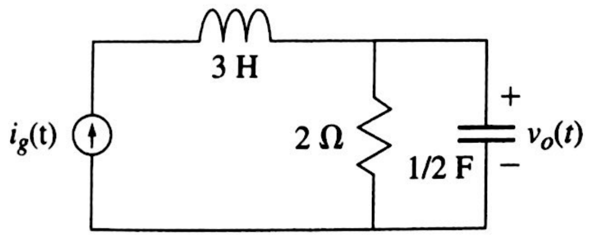


Figure P3.8

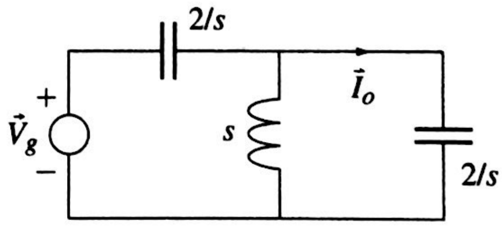


Figure P3.9

9. Find the transadmittance $\vec{I}_o/\vec{V}_g$ in the circuit of Figure P3.9.
10. Find the transfer function for the Wien bridge notch filter of Figure P3.10.

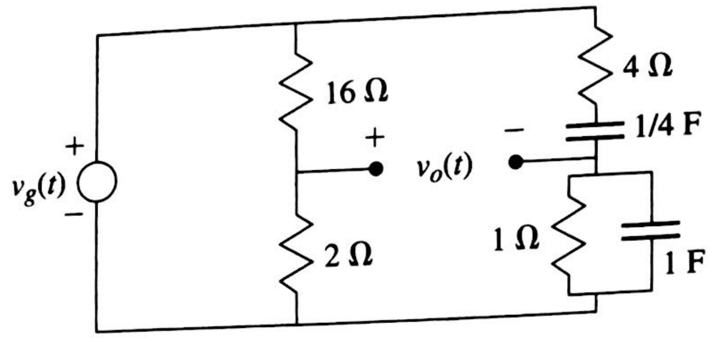
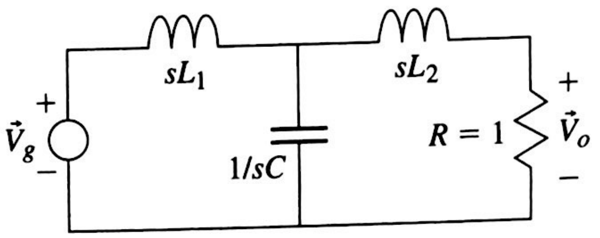


Figure P3.10

11. The circuit of Figure P3.11 is to be used to implement the transfer function indicated. Find expressions for the values of L_1, L_2, and C in terms of the a coefficients.



$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{a_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Figure P3.11

12. Find $\vec{V}_o(s) / \vec{V}_g(s)$ for the low-pass filter of Figure P3.12.

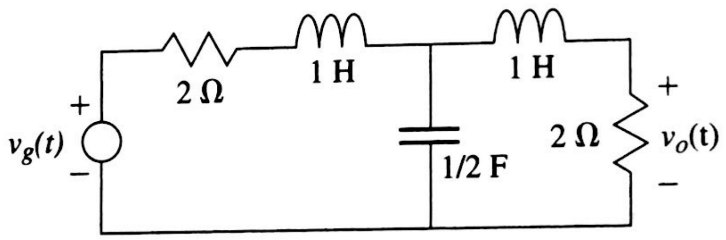


Figure P3.12

13. The circuit of Figure P3.13 is a Colpitts oscillator and contains a dependent current source. Determine the conditions necessary for oscillation. (*Hint:* $s = j\omega$ for an oscillator.)

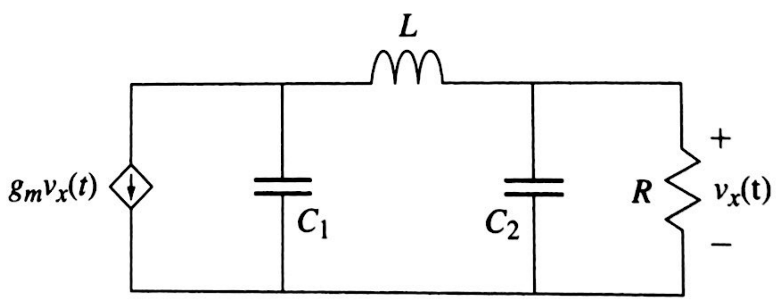


Figure P3.13

Section 3.3

14. The system of Figure P3.14 has $100 < G < 10{,}000$. Determine the range in $\vec{C}/\vec{R}$ if $H = 0.10$.

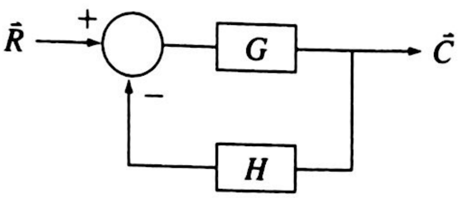


Figure P3.14

15. In the system of Figure P3.14, what value of G is needed if $\vec{C}/\vec{R}$ is to be within 1% of $1/H$?

16. In the system of Figure P3.14,

$$G = \frac{10^3}{s^3 + 3s^2 + 8s + 5}$$

Determine the d-c gain and system stability if:
a. $H = 0.10$ b. $H = 0.002$

17. The speed control system of Figure P3.17 is tested to obtain the calibration chart on the left at unit loading. The motor's back-emf may be assumed to equal the tach voltage magnitude. The motor armature resistance is 0.01 Ω, and the power amp has unity gain. The tach is connected such that $\vec{V}_{tach}$ and $\vec{V}_i$ are always of opposite polarity.
    a. With the tach disconnected and $\vec{V}_x$ set to 12.0 V, the motor runs at 1100 RPM. If the loading now doubles so that the armature current doubles, determine the new motor speed.

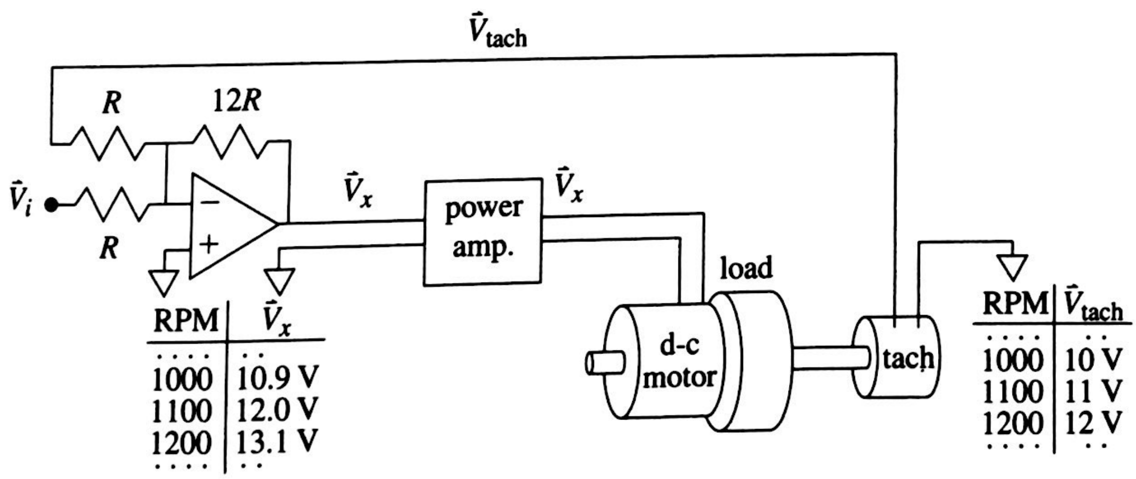


Figure P3.17

b. With the tach connected, the system is readjusted for 1100 RPM at unit load. Determine the $\vec{V}_i$ required.
c. Determine the new motor speed if the loading doubles in the closed loop system.

Section 3.4

18. Find an expression for $\vec{C}/\vec{R}$ in the system of Figure P3.18.

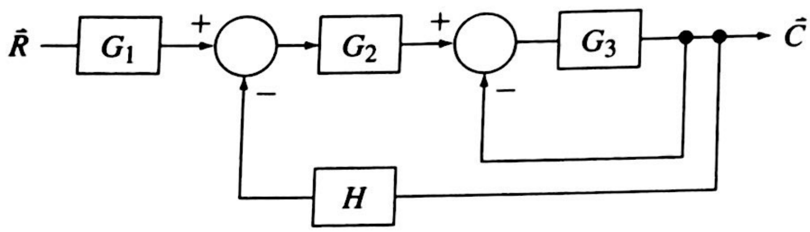


Figure P3.18

19. Find an expression for $\vec{C}/\vec{R}$ in the system of Figure P3.19.

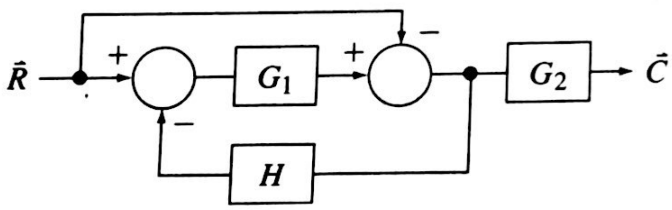


Figure P3.19

20. Find an expression for $\vec{C}$ in the system of Figure P3.20.

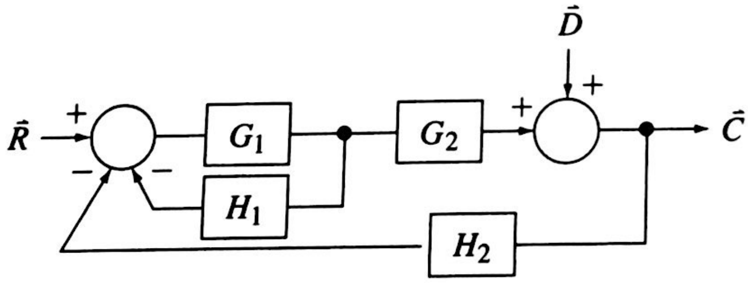


Figure P3.20

21. The transfer functions in the motor speed control system of Figure P3.21 are based on speed in radians/second, torque in newton-meters, and voltage in volts. K is provided by a variable-gain amplifier.
a. Given $K = 5$, determine the motor's no-load speed for a d-c input of 6 V.

b. Given the full-load torque is 20 N-m, determine the full-load speed.
c. If K is increased to 50, what new input voltage would be required to achieve the same no-load speed, and what would be the new full-load speed?

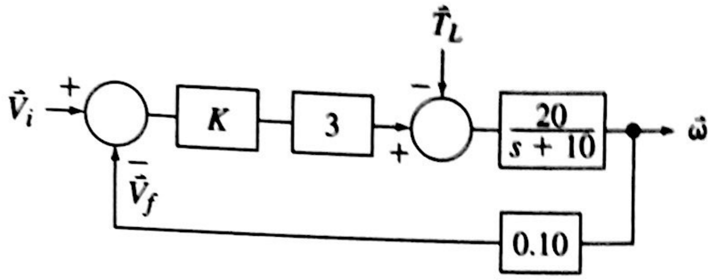


Figure P3.21

Section 3.5

22. Use the Routh–Hurwitz test to determine the range of K for which the following negative feedback systems are stable.

a. $G = \dfrac{K}{s(s^2 + 2s + 6)} \qquad H = 1$

b. $G = \dfrac{K}{s(s^2 + 2s + 6)} \qquad H = \dfrac{1}{s+1}$

23. Use the Routh-Hurwitz test to find the range of K for which the following polynomials will have no roots in the RHP.

a. $s^3 + 2Ks^2 + Ks + 24$ b. $s^3 + 10s^2 + Ks + 24$
c. $s^4 + Ks^3 + 20s^2 + 8s + 16$ d. $s^4 + 8s^3 + 20s^2 + 8s + K$
e. $s^4 + 8s^3 + 20s^2 + Ks + 24$

Section 3.6

24. Provide a set of MATLAB instructions that will produce the graph shown in Figure P3.24 using a 900-element vector for $0 \le x \le 3.9$.

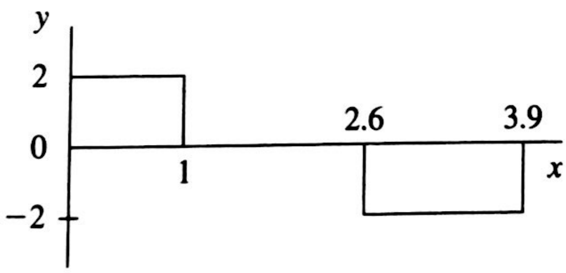


Figure P3.24

25. A transfer function $H(s)$ has been evaluated along the path $s = j\omega$ where ω runs from 0 to 20. Give a MATLAB command sequence that will plot $|H|$ and $\angle H$ vs. ω in two properly titled graphs arranged in a column.

26. Given numG, denG, numH, and denH for a negative feedback system, give a MATLAB two-command sequence that will determine if the system is stable.

27. Explain what is produced by the indicated command sequence.

a.
```
>x=[ ];
>for i=1:10
      x=[x i 0];
end
>x
```

b.
```
>x=[]; i=3;
>while length(x) < 5
i=i-1;
x=[x i];
end
>x
```

Additional Problems

28. An internally compensated op amp has a $G_v = 100{,}000\pi/(s + 20\pi)$. If used in a noninverting amplifier configuration with $H = 0.01$, determine the resulting voltage gain at

a. d-c b. 1 kHz

29. An op amp has an internal voltage gain given by $G_v = \dfrac{10^{10}}{(s+10)(s+100)(s+1000)}$. If it is used in an noninverting amplifier, for what range of H is it stable?

c) $y(t) = K_1e^{-8t} + K_2e^{t/2}\cos(1.937t + \phi)$; unstable
d) $y(t) = K_1e^{-t} + K_2e^{-t/2}\cos(1.323t + \phi)$; stable

2.17
```
>y = [1 0 0 2 1 -4]; x=[-2 3j 4]; polyval(y,x)
```
2.19
```
>y(2)=0                          % identifies the indeterminate value at x = 1
```
2.21 $y_f(t) = 1.0 + 0.5378e^{-3t} + 0.9836\cos(3t + 36.87°)$
$y_n(t) = K_1e^{-0.3998t}\cos(3.840t + \phi) + K_2e^{-1.076t} + K_3e^{-0.1247t}$

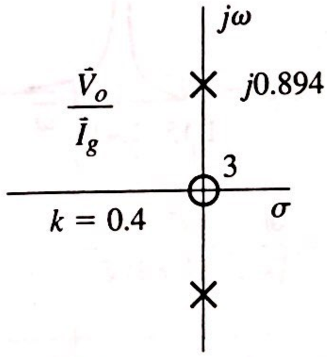
2.23 $y(t) = t - 1 + e^{-t}$

Chapter 3

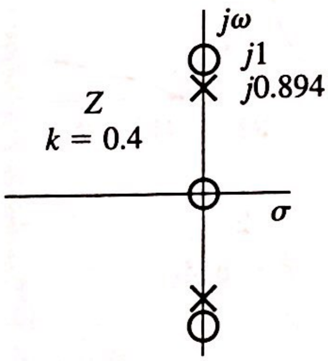
3.1 $Z(s) = 1 + \dfrac{2/s}{2 + 1/s} = \dfrac{2 + 3/s}{2 + 1/s} = \dfrac{s + 3/2}{s + 1/2}$

3.3 $Y_T = \dfrac{0.5(s + 2)}{(s + 1)^2}$

3.5 a)



b)



3.7 $\dfrac{\bar{I}_x}{\bar{I}_g} = \dfrac{4}{s^2 + 2s + 4}$

3.9 $\dfrac{\bar{I}_o}{\bar{V}_g} = \dfrac{s^3}{4(s^2 + 1)}$

3.11 $L_2 = \dfrac{1}{a_2}$, $L_1 = \dfrac{a_1}{a_0} - \dfrac{1}{a_2}$, $C = \dfrac{a_2^2}{a_1a_2 - a_0}$

3.13 $g_m = \dfrac{1}{R}\left(\dfrac{C_1}{C_2}\right)$, $\omega_o^2 = \dfrac{C_1 + C_2}{C_1C_2L}$

3.15 $G > 99/H$

3.17 a) 1000 rpm b) 12 V c) 1092 rpm

3.19 $\dfrac{\bar{C}}{\bar{R}} = \dfrac{G_2(G_1 - 1)}{1 + G_1H}$

3.21 $\bar{\omega} = \dfrac{60K\bar{V}_i - 20\bar{T}_L}{s + 10 + 6K}$

a) $V_i = 6$ V, $s = 0$, $\omega_{NL} = 45$ rad/s
b) $T_L = 20$, $\omega_{FL} = 45 - 10 = 35$ rad/s
c) $V_i = 4.650$ V for 45 rad/s, $\omega_{FL} = 43.71$ rad/s

3.23 a) $K > \sqrt{12}$ b) $K > 2.4$ c) $0.4174 < K < 9.583$
d) $0 < K < 19$ e) $10.26 < K < 149.7$

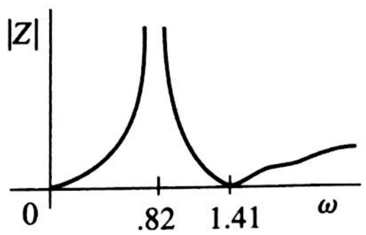
3.25
```
>subplot(2,1,1); plot(w,abs(H))
>title('|H| vs. w')
>subplot(2,1,2); plot(w,angle(H))
>title('angle of H vs w')
```

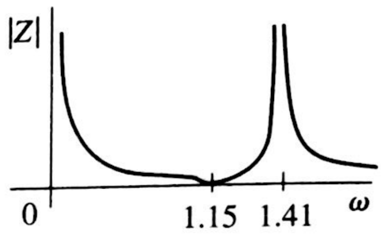
3.27
```
a) x = 0 1 0 2 0 3 0 4 0 5 0 6 0 7 0 8 0 9 0 10
b) x = 2 1 0 -1 -2
```

3.29 $H \leq 0.01222$ $\quad 1/H \geq 81.8$

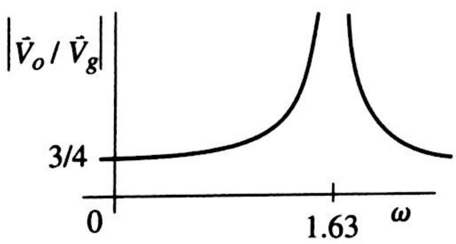
Chapter 4

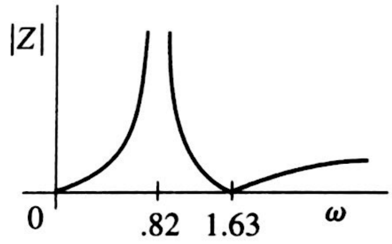
4.1 a) $Z(s) = \dfrac{2s(s \pm j1.4142)}{3(s \pm j0.8165)}$ b) $Z(s) = \dfrac{3(s \pm j1.1547)}{s(s \pm j1.4142)}$





4.3 a) $\dfrac{\tilde{V}_o}{\tilde{V}_g} = \dfrac{2}{s^2 + 8/3}$ b) $Z(s) = \dfrac{s(s \pm j1.633)}{(s \pm j0.8165)}$





4.5 $Z(s) = \dfrac{2(s + 0.0167 \pm j0.577)}{s(s + 0.0167 \pm j)}$

a) $|Z(j1)| = \dfrac{2(0.423)(1.577)}{1(2)(0.0167)} = 39.9\ \Omega$;

$B \approx 2(0.0167) = 0.0334$ rad/s

b) $|Z(j0.577)| = \dfrac{2(0.0167)(1.154)}{0.423(1.577)(0.577)} = 0.10\ \Omega$

$B \approx 0.0334$ rad/s