6

SPECTRAL ANALYSIS

OUTLINE

OBJECTIVES

1. Mathematically state the complex form of Fourier's series.
2. Determine a signal's spectral power distribution.
3. Compare the spectra of a pulse and a repetitive pulse.
4. Improve partial sums of the Fourier series with window functions.
5. Create MATLAB functions.
6. Explore relationships between signals and signal spectra.

INTRODUCTION

By specializing in e^{st} forcing functions, system differential equations become algebraic equations and are easily solved. Work originating with Joseph Fourier (1768–1830) and subsequently expanded upon by many others has shown that any signal waveform may be created by superimposing appropriate amounts of e^{st} signals (Laplace transform) and/or $e^{j\omega t}$ signals (Fourier transform). There are two major implications of these facts:

1. If we can decompose an arbitrary forcing function into its $e^{j\omega t}$ components, we can apply each one individually to a linear system, use the system frequency response to determine its magnitude and phase at the output, and superimpose the output signals to find the resulting output signal waveform. This removes almost all restrictions on the types of input signals for which we can find the output.
2. The decomposition of a signal into $e^{j\omega t}$ components describes the signal's spectrum. The spectrum tells us the magnitude and phase of the $e^{j\omega t}$ signal present at each frequency. That, in turn, tells us how the signal's power is distributed versus frequency, and will influence the design of circuits intended to detect or pass such signals.

Today the work of Fourier and Laplace have implications of profound importance to many areas of science, as well as to engineering and technology. The Laplace transform will be considered in Chapter 12.

6.1 FOURIER SERIES

Fourier's original work claimed that an infinite series of sinusoids could be used to represent any periodic function, even one with square corners. The idea that smoothly varying functions like sinewaves could be used to create a square corner was greeted with much skepticism by his colleagues, but Fourier was right. After reading this section you will be able to:

- Mathematically state the complex form of the Fourier series.
- Identify the period and fundamental frequency of a periodic waveform.
- Find the Euler phasors in a periodic waveform.

A periodic function is shown in Figure 6.1. One cycle of the waveform occurs in the *period T*. As a result of Fourier's work, his successors showed that this type of periodic waveform could be expressed in an infinite sum of the form

$$f(t) = \sum_{m=-\infty}^{m=+\infty} c_m e^{jm\omega_o t} \tag{6.1}$$

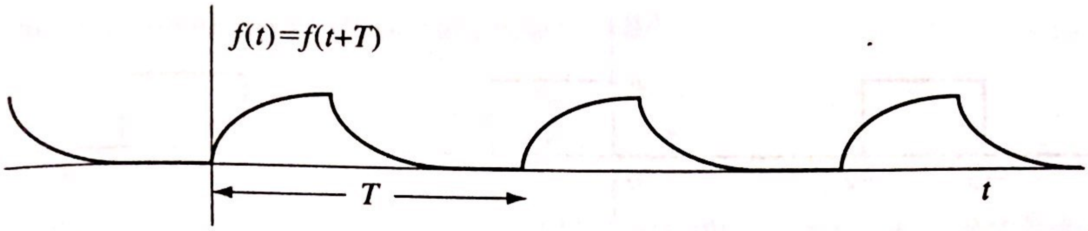


Figure 6.1 A periodic signal with period T has the property that $f(t) = f(t + T)$ for all t. It has a fundamental frequency or repetition rate of $\omega_o = 2\pi/T$.

where $\omega_o = 2\pi/T$, m is an integer, and

$$c_m = \frac{1}{T}\int_T f(t)e^{-jm\omega_o t}\,dt \qquad \textbf{(6.2)}$$

This is known as the complex form of the Fourier series.

The frequencies present, $\pm\omega_o$, are called the *fundamental frequencies*, and all integer multiples of the fundamental are called *harmonics*. The integer m gives the harmonic number; the c_m values are, in general, complex numbers that describe how much of each exponential time function is present and what phase it has at $t = 0$. In fact, the last time we encountered these numbers, we called the c_{+m} values *Euler phasors*. (Since $f(t)$ is generic and the study of the Fourier series spans many disciplines where the term *phasor* is meaningless, the phasor arrow notation is not used here.)

Equation 6.2 shows that

$$c_{-m} = c_m^* \qquad \textbf{(6.3)}$$

which is exactly the condition required for the positive and negative frequency exponentials to combine to give a real, sinusoidal $f(t)$. We can determine the magnitude and phase of that sinusoid either by taking equal $\pm m$ terms together or by just using the $+m$ value and the Euler phasor transformation.

There are two reasons for evaluating the Fourier series. One is to obtain an expression for $f(t)$ that applies everywhere, rather than only over a single period. The other is to determine the phasors, which indirectly tell how much power is available at each harmonic of the waveform.

EXAMPLE 6.1

Find c_m for the periodic waveform of Figure 6.2.

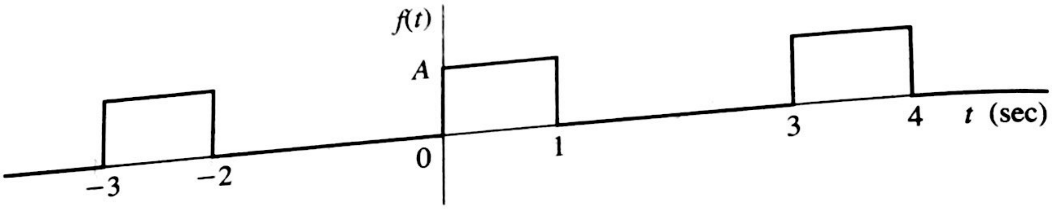


Figure 6.2

Solution

There are basically four steps involved in evaluating Equation 6.2.

1. *Identify the period.* In this case $T = 3$. (Take *any* 3-second interval, duplicate it, concatenate the duplicates, and the waveform of Figure 6.2 will result.)
2. *Find an expression for f(t) that is good over any one period.* In this example we choose

$$f(t) = \begin{cases} A & 0 < t < 1 \\ 0 & 1 < t < 3 \end{cases}$$

(The function of this example is extremely easy to describe in a piecewise fashion. The Fourier series will provide us with a single equation that applies for any value of t.)

3. *Evaluate the integral.* (Use a good set of integral tables, and any "tricks" you know to simplify the process.)

$$c_m = \frac{1}{3}\int_0^3 f(t)e^{-jm\frac{2\pi}{3}t}\,dt = \frac{1}{3}\left[\int_0^1 Ae^{-jm\frac{2\pi}{3}t}\,dt + \int_1^3 0\,dt\right]$$

$$c_m = \frac{A}{3}\left.\frac{e^{-jm\frac{2\pi}{3}t}}{-jm\frac{2\pi}{3}}\right|_0^1 = \frac{A}{3}\left[\frac{e^{-jm2\pi/3} - 1}{-jm2\pi/3}\right]$$

$$c_m = \frac{A}{m\pi}e^{-jm\pi/3}\sin\frac{m\pi}{3}$$

4. *Resolve any indeterminate cases.* (It is not unusual for the resulting expression for c_m to be indeterminate at one or two m values. Check to see if there is any m value for which the denominator goes to zero. A finite-sized wave cannot possibly have an infinite component, so it must be indeterminate at that m value.)

$$c_0 = \frac{0}{0}$$

This can be resolved using L'Hopital's rule or from

$$\lim_{m\to 0} \sin\left(\frac{m\pi}{3}\right) = \frac{m\pi}{3}$$

so that $\lim_{m\to 0} c_m = \frac{A}{m\pi} e^{-jm\pi/3}\left(\frac{m\pi}{3}\right) = \frac{A}{3}$, or by returning to the integral specifically for c_0, which is

$$c_0 = \frac{1}{3}\int_0^3 f(t)\,dt = \frac{1}{3}\left[\int_0^1 A\,dt + \int_1^3 0\,dt\right] = \frac{A}{3}$$

Note that c_0 is the d-c or average value of the function, and may often be found by inspection.

The resulting Fourier series is

$$f(t) = \sum_{m=-\infty}^{\infty} \left(\frac{A}{m\pi} \sin\frac{m\pi}{3}\right) e^{jm(\omega_o t - \pi/3)}$$

or, if you prefer,

$$f(t) = \frac{A}{3} + \sum_{m=1}^{\infty} \left(\frac{2A}{m\pi} \sin\frac{m\pi}{3}\right) \cos\left(m\omega_o t - \frac{m\pi}{3}\right)$$

Generalizing waveforms, when possible, gives the best return on the time invested. The results can be collected in tables and applied to specific numerical problems without rederiving them from scratch. The period sets the fundamental frequency and consequently tells what frequencies are potentially present. The position of the waveform relative to the time origin affects only the phase of the Euler phasors. It can be chosen to simplify the integration and then adjusted to match the waveform position desired. For many applications, the phase is unimportant.

EXAMPLE 6.2

Determine the Euler phasors for the waveform of Figure 6.3.

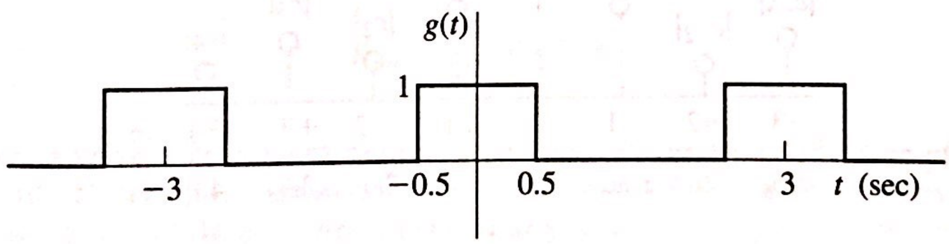


Figure 6.3

Solution

If $g(t)$ is delayed by 1/2 second, it is identical to the $f(t)$ of Example 6.1, with $A = 1$. From the properties of ideal filters we know that a pure delay is represented by a phase shift of

$$e^{-j\tau_p\omega} = e^{-j\tau_p m\omega_o} = e^{-j2\pi m\tau_p/T}$$

With $\tau_p = 1/2$ and $T = 3$, multiplying each Euler phasor of $g(t)$ by the factor $e^{-j\pi m/3}$ delays $g(t)$ by 1/2 second; or, equivalently, multiplying the Euler phasors of $f(t)$ by $e^{+j\pi m/3}$ moves it ahead in time by 1/2 second. Since the Fourier series for $f(t)$ is

$$f(t) = \sum_{-\infty}^{\infty}\left(\frac{A\sin m\pi/3}{m\pi}\right)e^{jm(\omega_o t - \pi/3)}$$

the Fourier series for $g(t)$ must be

$$g(t) = \sum_{-\infty}^{\infty}\left(\frac{\sin m\pi/3}{m\pi}\right)e^{jm\omega_o t}$$

The spectrum of an $f(t)$ is a plot of its c_m vs. m or $m\omega_o$. For theoretical work we prefer to allow m to range over the positive and negative integers, which gives the amplitudes and phases of exponential signals. If we are only given the c_m for positive m values, we know the rest are just the conjugate of their positive frequency mate.

Nontechnical personnel are more comfortable with real signals than with complex exponentials, so commercial spectrum analyzers modify the c_m values to give the amplitudes and phases of the corresponding sinusoids. The different representations are shown in Figures 6.4a–c. The phase of each amplitude spectrum component may also be provided in a second graph. The power spectrum is entirely real and has no phase.

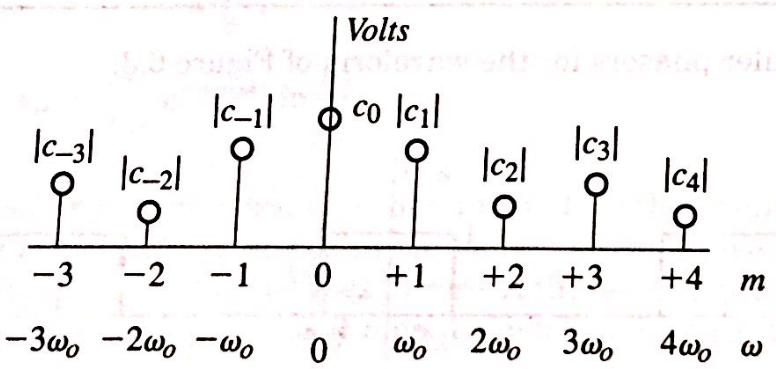


Figure 6.4a The amplitude spectrum of the exponential components of a periodic waveform.

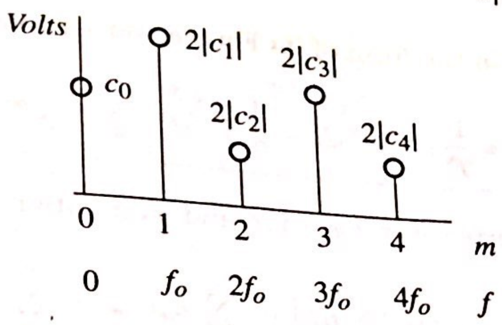


Figure 6.4b The amplitude spectrum of sinusoids as it would be displayed on a commercial spectrum analyzer.

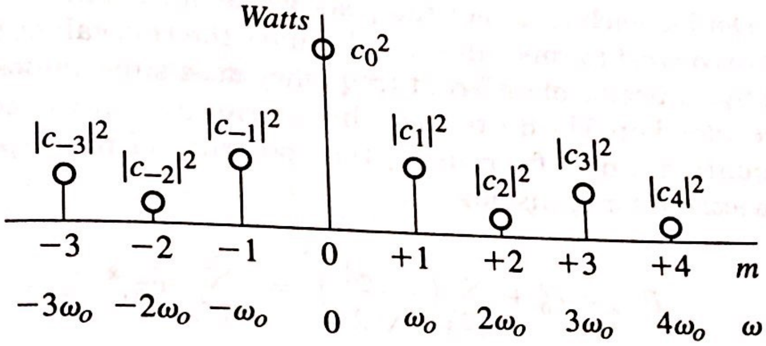


Figure 6.4c The power spectrum of a periodic waveform as it would be displayed for exponential components.

6.2 PARSEVAL'S THEOREM

The average power of a periodic waveform may be found in either the time domain or the frequency domain. After completing this section you will be able to:

- Determine a signal's spectral power distribution.
- Find the percentage of total power in each frequency component.

In the time domain; we define the average power of the function $f(t)$ as

$$P_{ave} = \frac{1}{T}\int_T f(t)^2\, dt$$

If $f(t)$ were a voltage or current applied to a 1-Ω resistor, this definition of average power would be familiar. However, we will use this definition for the power of the signal whether or not its units are known or any resistor is implied. This way we will be able to discuss the "power" of a signal that consists of just a sequence of numbers in a computer.

Expressing one $f(t)$ in the form of its Fourier series gives

$$P_{ave} = \frac{1}{T}\int_T f(t)^2\,dt = \frac{1}{T}\int_T f(t)\left[\sum_{-\infty}^{\infty} c_m e^{jm\omega_o t}\right]dt$$

and interchanging the order of integration and summation results in

$$P_{ave} = \sum_{-\infty}^{\infty} c_m \underbrace{\left[\frac{1}{T}\int_T f(t)e^{jm\omega_o t}\,dt\right]}_{c_{-m}} = \sum_{-\infty}^{\infty} c_m c_{-m} = \sum_{-\infty}^{\infty} c_m c_m^* = \sum_{-\infty}^{\infty} |c_m|^2$$

A less elegant but more familiar approach is to add the powers of the sinusoids in Figure 6.4b. Double each c_m to get the peak value of the sinusoids, divide by the square root of 2 to convert to rms values, and square the rms values to get power. The power provided by sinusoids of different frequency does superimpose, so the components can just be added up. The d-c term is already rms and only needs to be squared. The simplest equation comes from using the spectrum of the exponential signals instead of the spectrum for sinusoids.

$$P_{ave} = c_0^2 + \sum_{m=1}^{\infty}\left(\frac{2|c_m|}{\sqrt{2}}\right)^2 = \sum_{m=-\infty}^{\infty} |c_m|^2$$

That the signal power can be determined in either the time or the frequency domain is called Parseval's theorem:

$$P_{ave} = \frac{1}{T}\int_T f(t)^2\,dt = \sum_{m=-\infty}^{\infty} |c_m|^2 \qquad \text{Parseval's Theorem} \qquad \textbf{(6.4)}$$

EXAMPLE 6.3

Determine the percentage of the total power represented by the fourth harmonic in the waveform of Figure 6.5 given that

$$c_m = \frac{1}{T}\int_0^{T/4} \sin\left(\frac{8\pi t}{T}\right)e^{-jm2\pi t/T}\,dt = \frac{2}{\pi}\frac{1 - e^{-jm\pi/2}}{16 - m^2}$$

Solution

The expression for c_m becomes indeterminate at $m = 4$. Using L'Hopital's rule, we get

$$c_4 = \frac{2}{\pi}\frac{(j\pi/2)e^{-jm\pi/2}}{-2m} = -j\frac{1}{\pi}\frac{\pi}{8} = \frac{-j}{8}$$

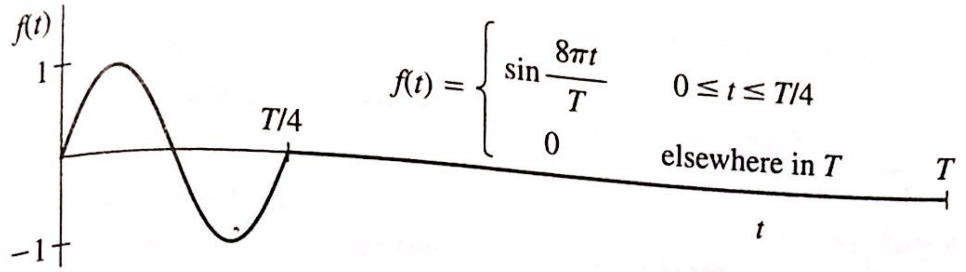


Figure 6.5

$$P_4 = \left(2\frac{1/8}{\sqrt{2}}\right)^2 = \frac{1}{32}$$

The total power in the waveform is found from the time domain as

$$P_{ave} = \frac{1}{T}\int_T f(t)^2\,dt = \frac{1}{T}\int_0^{T/4} \sin^2\left(\frac{8\pi}{T}t\right)dt = \frac{1}{2T}\int_0^{T/4}\left[1 - \cos\left(\frac{16\pi t}{T}\right)\right]dt = \frac{1}{8}$$

$$\therefore 100\frac{P_4}{P_{ave}} = 100\frac{1/32}{1/8} = 25\%$$

The total power could also be found in the frequency domain by adding up the contributions from all of the significant harmonics.

6.3 THE FOURIER TRANSFORM

Signals that consist of a single pulse have a finite energy but zero average power. The Fourier transform is a spectral distribution function from which the signal's energy can be determined in the frequency domain. After reading this section you will be able to:

- State the Fourier transform and the inverse Fourier transform.
- Compare the spectra of a pulse and a repetitive pulse.
- Determine the Fourier series from the Fourier transform.

The Fourier transform gives the magnitudes and phases of a *continuous sum* (integral) of $e^{j\omega t}$ signals that combine to form an *aperiodic* (not periodic) pulse:

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \qquad \text{Inverse Fourier Transform} \qquad \textbf{(6.5)}$$

where

$$F(\omega) = \int_{t=-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \qquad \text{Fourier Transform} \tag{6.6}$$

If $f(t)$ is a voltage, Equation 6.6 shows $F(\omega)$ has the units of volt second, or volts per unit bandwidth, which describes a signal's spectral density. The energy provided by $f(t)$ can be found from

$$W = \int_{t=-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |F(\omega)|^2\,d\omega \tag{6.7}$$

which is derived in the same fashion as for the periodic case, and represents Parseval's theorem for an energy signal.

EXAMPLE 6.4

Find the Fourier transform of the unit impulse $\delta(t)$.

Solution

The unit impulse is zero everywhere, except when its argument is zero. It is infinite at $t = 0$ but has an area of unity:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{0-}^{0+} \delta(t)\,dt = 1$$

Inserting the impulse in Equation 6.6 gives

$$F(\omega) = \int_{t=-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{0-}^{0+} \delta(t)e^{-j\omega t}\,dt = \int_{0-}^{0+} \delta(t)e^{-j\omega 0}\,dt = 1$$

The unit impulse makes integrations easy because it exists only when its argument is zero. Over that infinitesimal range, the other integrand terms may be considered constant. We introduced the unit impulse function earlier as a good input test signal to use in checking the stability of a system, since it activates the system's natural response without imposing any forced response. The output signal that results when using the unit impulse as the input is designated $h(t)$.

If we multiply the spectral density $X(\omega)$ of an input signal by $H(\omega)$, the transfer function of a system, we obtain the spectral density of the output signal, $Y(\omega) = X(\omega)H(\omega)$. If a unit impulse is the input signal, its spectral density is $1\angle 0°$ everywhere, and the output spectral density equals the frequency response of the system, $Y(\omega) = H(\omega)$. Clearly, $H(\omega)$ and $h(t)$ are related by the Fourier transforms, and either one provides a complete mathematical description of the system. The term *impulse response of a system* is used to refer to either $h(t)$ or $H(\omega)$. Once you know one, you know how to find the other.

The Fourier transform is actually the most general way of describing the frequency domain characteristics of signals. Even the Fourier transform of a periodic waveform can be defined:

$$F(\omega) = 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega + m\omega_o) \qquad \text{Fourier Transform of a Periodic Function}$$

It shows the expected infinite energy at multiples of the fundamental frequency; if inserted into the inverse Fourier transform, it reproduces the Fourier series.

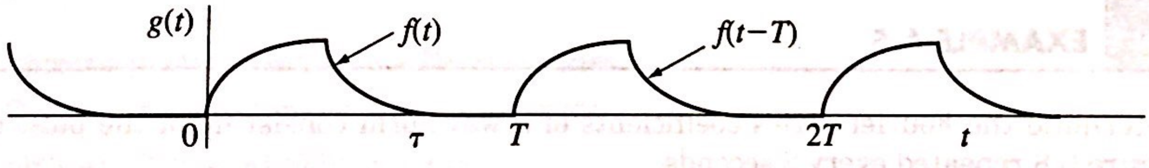
Most of the theorems developed for signals come from the Fourier transform. Our interest is primarily in the Fourier series, but we want to explore further what Fourier's series and transform have in common.

Consider the waveforms and relationships summarized in Figure 6.6. The integrals for c_mT and $F(\omega)$ are identical except for the way their parameter is stated: $m\omega_o$ for one and ω for the other. Neither of these parameters affects the integration with respect to time, so the way we label them can be altered before or after the integration is carried out. The conclusion is that

$$c_mT = F(\omega)\big|_{\omega=m\omega_o} = F(m\omega_o) \tag{6.8}$$

The significance of this result is that $F(\omega)$, which is based only on the pulse shape, establishes all possible amplitudes of the Fourier series components. The

$$g(t) = \sum_{-\infty}^{\infty} c_m e^{jm\omega_o t} \qquad \text{where} \qquad c_mT = \int_0^{\tau} f(t)e^{-jm\omega_o t}dt$$



$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad \text{where} \qquad F(\omega) = \int_0^{\tau} f(t)e^{-j\omega t}dt$$

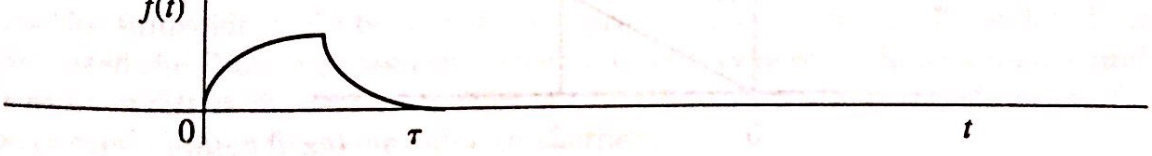


Figure 6.6 A periodic signal $g(t)$ is generated when the single pulse $f(t)$ is repeated every T seconds. The integrals for $F(\omega)$ and c_mT are identical.

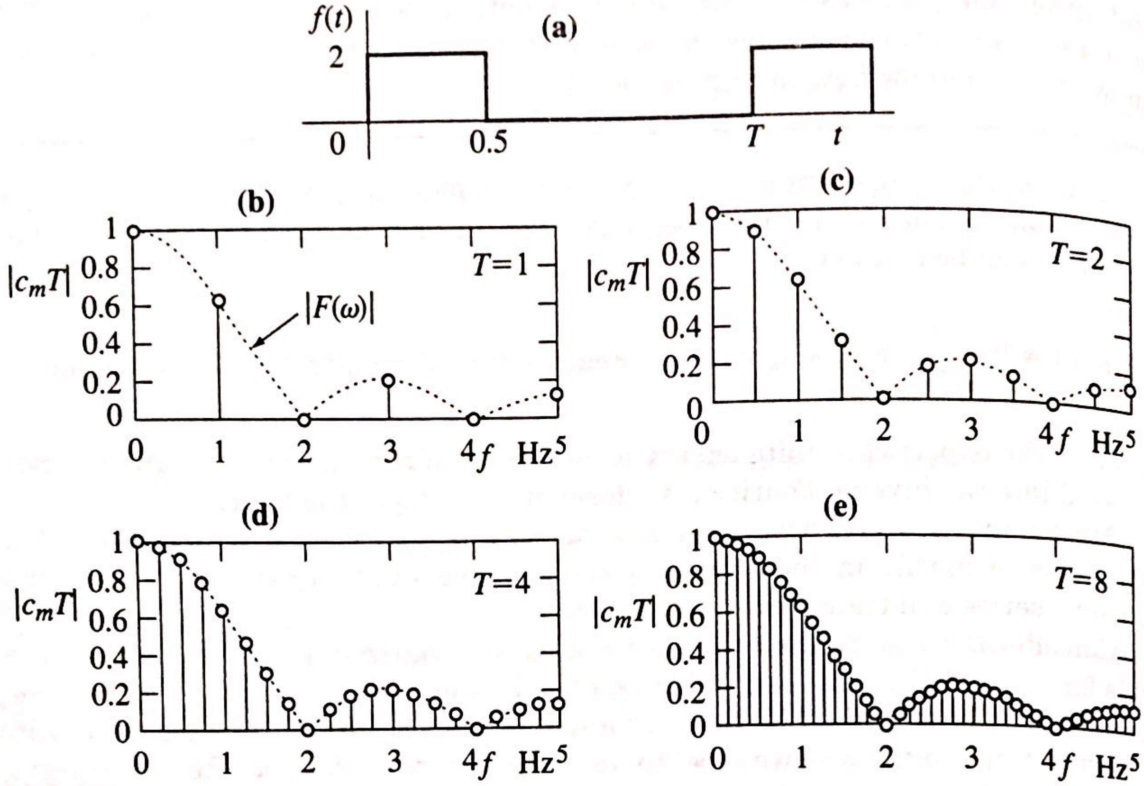


Figure 6.7 The spectra resulting from a fixed 0.5-second rectangular pulse is observed as the period T increases. The $c_m T$ values of the Fourier series are seen to be samples of the Fourier transform $F(\omega)$ taken at frequencies m/T Hz.

period determines only where the $F(\omega)$ is sampled to obtain a specific series component. Moreover, if we want to see $F(\omega)$ for a particular pulse, we can get it from the Fourier series by making the pulse periodic, with a very large period. Both of these concepts are demonstrated for the rectangular pulse of Figure 6.7a.

EXAMPLE 6.5

Determine the Fourier series coefficients of a waveform consisting of the pulse of Figure 6.8 repeated every 2 seconds.

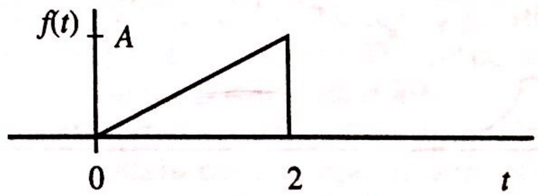


Figure 6.8

Solution

$$f(t) = \begin{cases} \dfrac{At}{2} & 0 < t < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \frac{A}{2}\int_0^2 te^{-j\omega t}\,dt = \frac{A}{2}\left[\frac{e^{-j\omega t}}{-\omega^2}(-j\omega t - 1)\right]_0^2 \quad \omega \neq 0$$

$$F(\omega) = \frac{A}{2}\left[\frac{e^{-j\omega t}}{\omega^2}(j\omega t + 1)\right]_0^2 = \frac{A}{2\omega^2}[e^{-j2\omega}(j2\omega + 1) - 1]$$

Now from Equation 6.8, the effect of the period is entered: ($T = 2$, $\omega_o = \pi$)

$$2c_m = F(\pi m) = \frac{A}{2(\pi m)^2}[e^{-j2(\pi m)}(j2\pi m + 1) - 1]$$

$$c_m = \frac{1}{2}F(\pi m) = \frac{A}{4\pi^2 m^2}[j2\pi m] = \boxed{\frac{jA}{2\pi m}} \quad m \neq 0$$

The integral formula used to evaluate $F(\omega)$ breaks down at ω=0, and thus for m=0. The easiest way to obtain c_0 is by inspection of the waveform's average value, which is $A/2$ in this example.

6.4 WINDOWS

One application of the Fourier series is to provide an equation to represent a particular shape. To make it practical to use, the series must be limited to a finite number of terms. The Fourier series coefficients may need to be tweaked with a window function to give best results in these partial sums. After reading this section you will be able to:

- Improve partial sums of the Fourier series.
- Discuss how the Fourier series converges.
- Identify several window functions.
- Recommend when a window function should be used.

While many of Fourier's contemporaries scoffed at the notion that smooth functions like sinusoids could be added up to provide a sharp corner, Dirichlet showed that indeed the Fourier series converges to $f(t)$ everywhere $f(t)$ is continuous, and to the mean of $f(t)$ at the discontinuities. A *discontinuity* is any point where the function abruptly jumps from one value to another.

For functions without discontinuities, the Fourier series coefficients need no modification. The only decision is how many terms to use in the partial sum.

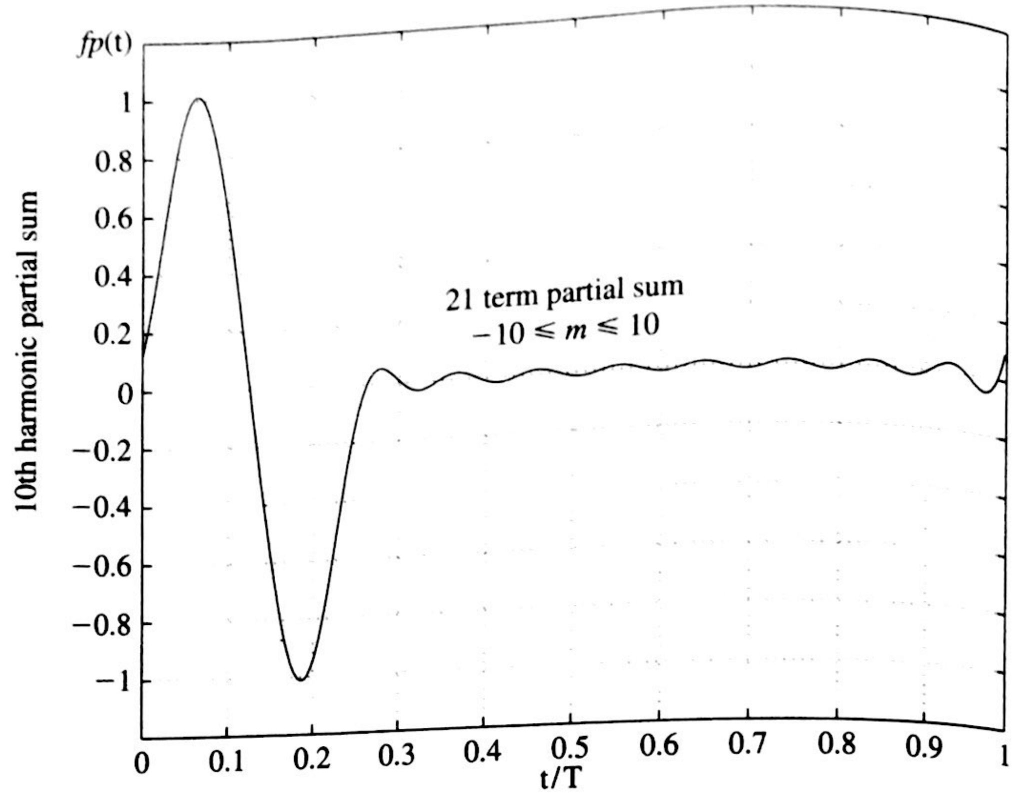


Figure 6.9 Partial sum representation of the waveform of Figure 6.5, including terms out to the tenth harmonic.

The waveform of Figure 6.5 is continuous. By writing a program to evaluate and plot its Fourier series, the results of Figure 6.9 are obtained for a 21-term sum. The highest error in the partial sum occurs where the curve suddenly changes direction. High-frequency terms are needed for corners, and they are what partial sums limit.

Discontinuous functions behave less well when approximated by a Fourier partial sum. In order to produce the jumps required at discontinuities, the Fourier series retains excessive amplitudes in its high-frequency terms. When a partial sum is used, the discarded high-frequency terms are no longer present to cancel out the excess amplitudes of terms retained in the partial sum. As a result, the partial sum waveform shows rippling, as if it has too much high-frequency content. As more terms are included in the partial sum, the rippling does not disappear or even decrease substantially, but instead it is pushed closer and closer to the point of discontinuity, as shown in Figure 6.10. In the limit, as the partial sum becomes the infinite sum, the rippling is all pushed to the discontinuity, leaving the waveform with "ears" known as *Gibb's phenomenon* (Figure 6.11).

The process of obtaining a partial sum approximation for a function is equivalent to putting the original function through an ideal low-pass filter. This *rectangular window* keeps all of the Fourier coefficients out to the Mth harmonic, but multiplies all the harmonics higher than the Mth harmonic by zero. We have

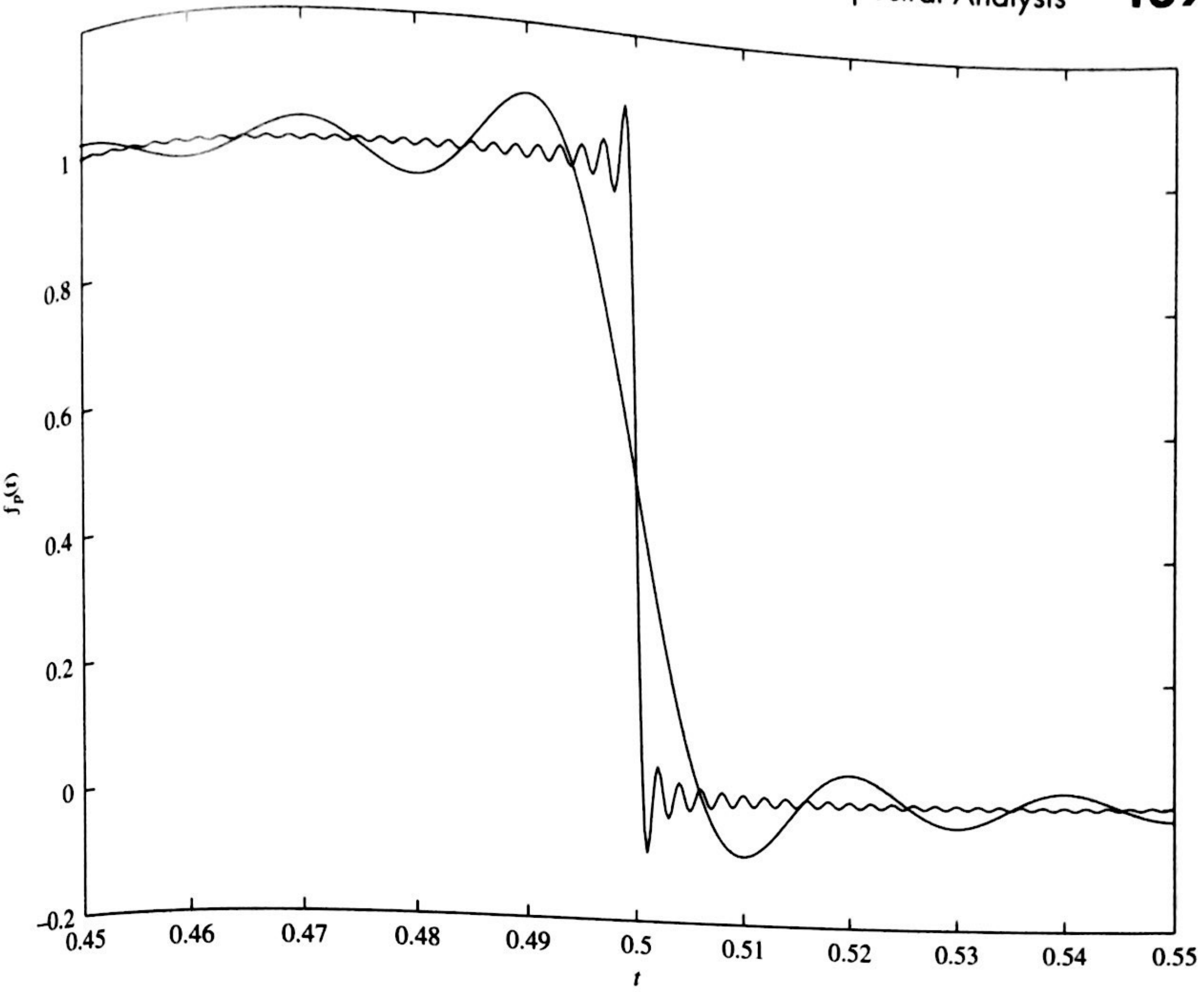


Figure 6.10 Close-up of the transition in a square wave for a 50 and 500 harmonic Fourier series showing that the ripple does not decrease, but just gets pushed to the transition edge.

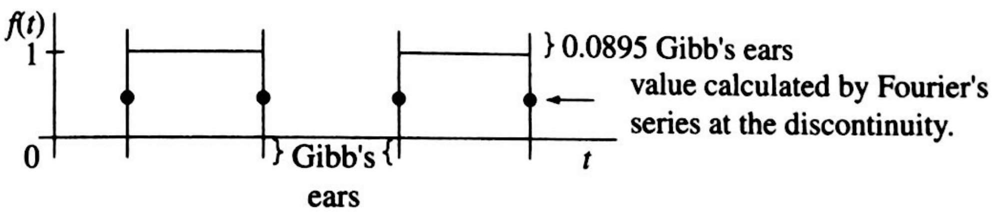
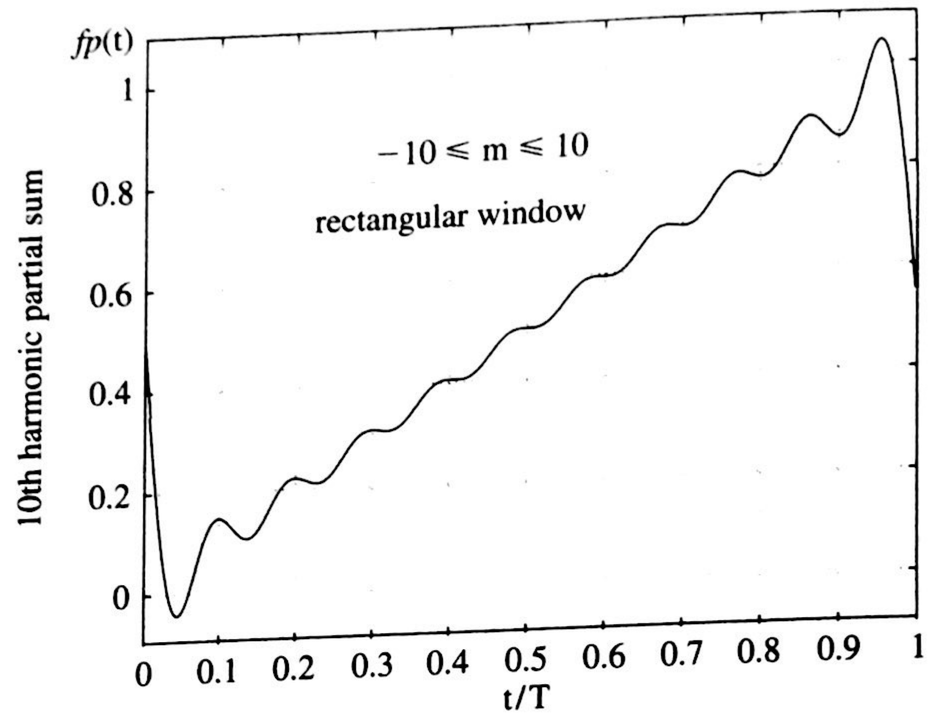


Figure 6.11 Gibb's ears at points of discontinuity in a square wave. The waveform still averages to the mean at the jumps, satisfying Dirichlet's conditions.

seen that this works fine for continuous functions, but results in rippling in discontinuous functions. Since this rippling is the result of having excess amplitude in the high-frequency terms that are retained, it is reasonable to seek a way to systematically taper off the high frequencies. In other words, for discontinuous functions, the Fourier series coefficients will be taken as a starting point; but instead of passing them through a rectangular window, we will pass them through a tapered window that gradually reduces each successively higher harmonic out to the last one retained.

The result of doing this for the sawtooth waveform of Example 6.5 is demonstrated in Figure 6.12. The windowed partial sum appears to be identical to the



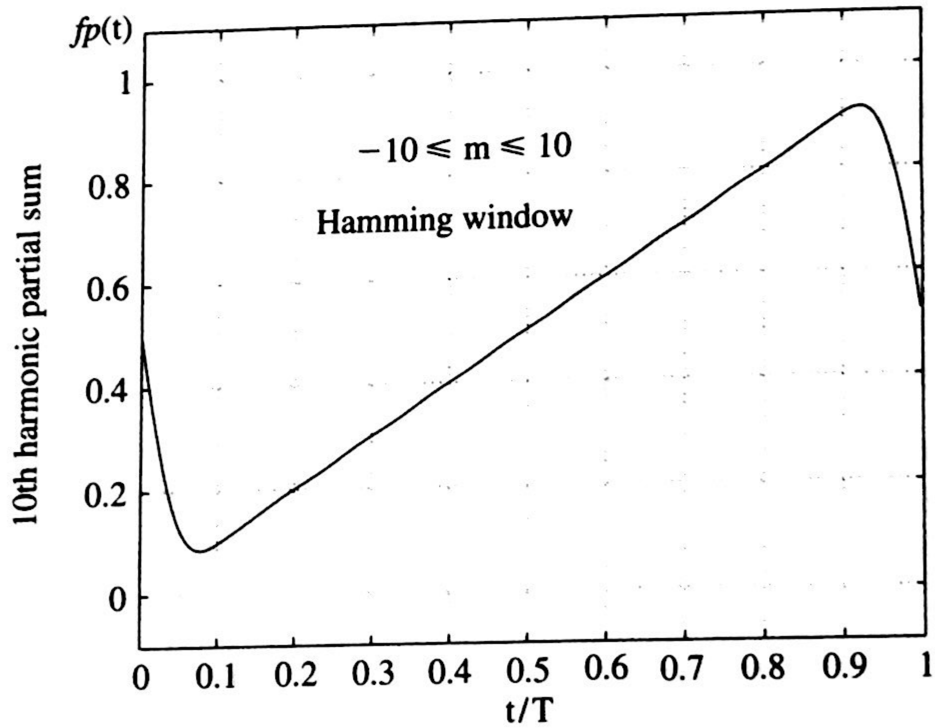


Figure 6.12 The partial Fourier series of a sawtooth waveform is demonstrated with and without windowing.

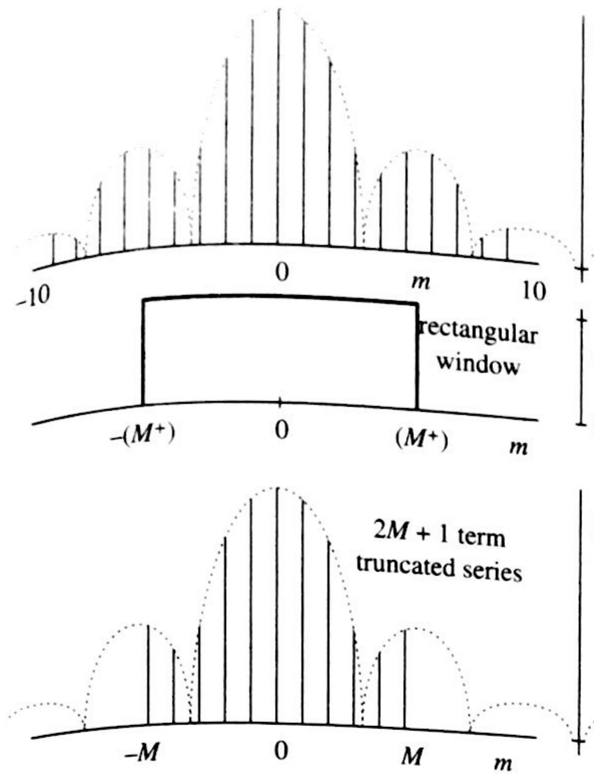


Figure 6.13a Truncating the Fourier series after $2M + 1$ terms is equivalent to multiplying the spectrum by a rectangular window.

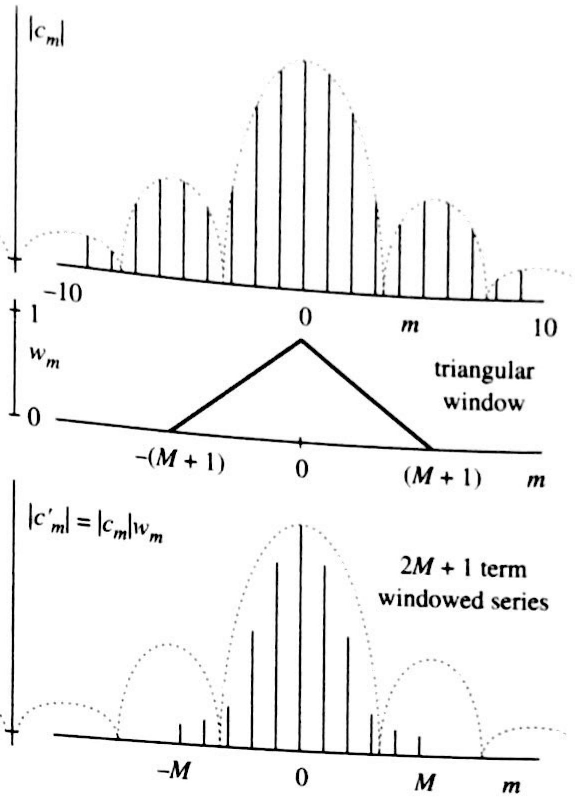


Figure 6.13b Reducing the series coefficients to zero by a linearly increasing percentage is equivalent to multiplying the spectrum by a triangular window.

desired straight line over 80% of the period, but does have a wider transition region than the rectangular windowed series.

The windowing process is pictured in Figures 6.13a and b for rectangular and triangular windows. There is a wide variety of window functions. They all multiply the d-c term by unity. The d-c term only sets the level of the waveform but does not affect its shape. At this point the only criteria we have for a window function is that it should systematically taper off the Fourier coefficients and be easily calculated for whatever number of harmonics are to be retained in the partial sum. A few popular window functions are given in Table 6.1.

Table 6.1 Common Window Functions

Name	***Window Function***
Triangular	$w(m) = 1 - \lvert m \rvert/(M + 1)$
Hamming	$w(m) = 0.54 + 0.46 \cos(m\pi/M)$
Hanning (Von Hann)	$w(m) = 0.5[1 + \cos(m\pi/(M + 1))]$

M = highest harmonic number; $-M \le m \le M$ ($2M + 1$ terms).

EXAMPLE 6.6

The Fourier series of a discontinuous function is

$$f(t) = \cdots 0.2e^{-j4t} + 0.6e^{-j3t} + e^{-j2t} + 2e^{-jt} - 2 + 2e^{+jt} + e^{+j2t} + 0.6e^{+j3t} + 0.2e^{+j4t} + \ldots$$

The series is to be approximated by a five-term partial sum and modified with a triangular window to reduce rippling. Determine the required partial sum.

Solution

The unmodified (rectangular window) five-term partial sum is

$$f_p(t) = \underbrace{1}_{c_{-2}}e^{-j2t} + \underbrace{2}_{c_{-1}}e^{-jt}\underbrace{-2}_{c_0} + \underbrace{2}_{c_1}e^{+jt} + \underbrace{1}_{c_2}e^{+j2t}$$

The triangular window of Table 6.1 gives, for $M = 2$ (five terms),

$$w(m) = 1 - \frac{|m|}{3} = \underbrace{\frac{1}{3}}_{w_{-2}} \quad \underbrace{\frac{2}{3}}_{w_{-1}} \quad \underbrace{\frac{1}{1}}_{w_0} \quad \underbrace{\frac{2}{3}}_{w_1} \quad \underbrace{\frac{1}{3}}_{w_2}$$

The triangular windowed five-term partial sum consequently is

$$f_p(t) = \underbrace{c_{-2}w_{-2}}_{c'_{-2}}e^{-j2t} + \underbrace{c_{-1}w_{-1}}_{c'_{-1}}e^{-jt} + \underbrace{c_0w_0}_{c'_0} + \underbrace{c_1w_1}_{c'_1}e^{+jt} + \underbrace{c_2w_2}_{c'_2}e^{+j2t}$$

or

$$f_p(t) = \frac{1}{3}e^{-j2t} + \frac{4}{3}e^{-jt} - 2 + \frac{4}{3}e^{+jt} + \frac{1}{3}e^{+j2t}$$

There are some annoying inconsistencies in the window functions provided by different references. If we are going to compare a rectangular window with some other window, we ought to at least be comparing equal-length series. The more terms we include in a partial sum, the better we expect the function to be approximated by the series. If a series has a nonzero c_M term, the window function should not force that Mth harmonic term to zero, or we would be comparing an Mth harmonic series to a windowed $(M-1)$th harmonic series. The equations in Table 6.1 have been written so that this does not happen.

6.5 MATLAB LESSON 6

It would be useful to have a program that would plot a Fourier partial sum to demonstrate issues that have arisen in this chapter. Creating a MATLAB function is one way to accomplish this. After completing this section you will be able to:

- Create MATLAB functions.
- Graph a Fourier partial sum with or without a window function.
- Perform a matrix multiplication.
- Find the output of a circuit with a periodic input signal.

MATLAB EXAMPLES

We want to develop a MATLAB function that we can use to explore Fourier partial sums. In preparation for that, we need to learn a little matrix algebra.

A **matrix multiply** is defined for a row vector and a column vector of the same length:

$$a = (a_1 \quad a_2 \quad a_3) \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad a{*}b \equiv a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_ib_i$$

The order is important since $a{*}b \neq b{*}a$. Both are legitimate operations but the one we want requires that the second vector be the column vector, and the result is a single constant. Note the summation. Think $\boldsymbol{a} = \boldsymbol{c_m}$ and $\boldsymbol{b} = \boldsymbol{e^{jm\omega_0 t}}$, and it might start to look like a Fourier series.

Since the matrix multiplication involves both row and column vectors, we may need to convert between them. This is called taking the *transpose* of the vector, and was used in Chapter 4 to obtain a listing of paired data points (page 117). We need to distinguish between two types of transposes. An *array transpose* simply converts a row vector to a column vector, or vice versa, with no other changes.

$$a = (a_1 \quad a_2 \quad a_3) \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad a.' = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad b.' = (b_1 \quad b_2 \quad b_3)$$

A *matrix transpose* switches between row and column vectors, but also takes the complex conjugate of each element. When all the elements are real numbers, the array transpose and the matrix transpose are identical.

$$a = (a_1 \quad a_2 \quad a_3) \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad a' = \begin{pmatrix} a_1{*} \\ a_2{*} \\ a_3{*} \end{pmatrix} \qquad b' = (b_1{*} \quad b_2{*} \quad b_3{*})$$

We will use the array transpose to switch between rows and columns when we want to use a matrix multiply to obtain the sum of products. The matrix transpose is useful if we want a sum of squared magnitudes, as in

$$P_{ave} = \sum_m c_m c_m{}^*$$

stem

Stem plots are ideal for functions of a discrete variable such as the spectrum of a periodic signal. Its format is stem(discrete_variable_x, any_variable_y)

hamming hanning triang

MATLAB provides a variety of window functions for partial sums of the complex Fourier series. We will restrict ourselves to the Hamming, Hanning (Von Hann), or triangular windows. For these windows we need only specify the size of the window. For unknown reasons, MATLAB delivers its window functions as column vectors instead of row vectors, so we will always end up transposing them. They are real functions and may be transposed with the simpler notation of the matrix transpose. The following programming demonstrates most of the above comments.

```
>m=-3:3                          % show it on screen
>cm=sin(m*pi/3)./(m*pi)          % waveform of Fig.6.3
>cm(4)=1/3                       % fix the NaN
>w=hamming(7)                    % select a Hamming window
>cwin=cm.*w'                     % use the matrix transpose (w is real)

>stem(m,abs(cm))                 % plot the spectral components before
>hold
>plot(m,abs(cwin),'x')           % and after windowing
>hold

>Pave=cm*cm'                     % note the matrix multiply and transpose

>t=0                             % calculate the value of g at t = 0
>e=exp(j*m*2*pi*t/3)
>g=cwin*e.'                      % note the matrix multiply and array transpose

>for t=0:6                       % calculate 7 values of g over a period
tt=t/2;
e=exp(j*m*2*pi*tt/3);
g(t+1)=cwin*e.';
x(t+1)=tt;                       % prepare x axis variable
end
>plot(x,real(g),'o')             % may want a few more points
```

We have created M-file scripts so that a sequence of commands could be executed repeatedly. MATLAB *function* calls are special versions of M-files that allow us to create a toolbox for our own area of specialization. A function file is created with any text editor and has the following characteristics:

1. The file and function names must be identical, but the file name has the *.m* extension.
2. The first nonblank line of the function file *must be*

```
function [out1, out2,..] = function_name(in1, in2, . . . )
```

where out1, out2 . . . are output variables to be calculated by the function using the input variables in1, in2 . . . . If there is only one output variable, the square bracket may be omitted.

3. Successive comment lines following the function line constitute the information brought to the screen if *help function_name* is entered. The first line of this sequence is the only line searched by the *Find* command.
4. The first noncomment line begins the programming, which describes how the output variables are calculated from the input variables. Blank lines are ignored, and may help isolate programming sections.

There are hundreds of examples of function files within the MATLAB directory of any computer on which it is installed, and they are just as accessible as any other text file. Viewing a few will give you an idea of how the professionals do it. The function file reacts with the workplace only through the variables passed to it or returned back from it. This prevents workplace variables from being duplicated and overwritten by the function, which can be a problem with ordinary script files.

We have also used enough function calls by now to appreciate that the names given to input and output variables used by the function are immaterial. It will assign the numerical values of the symbols we use to call the function to the symbols defined in its function file according to their position in the function argument. We have also seen functions that accept a variable number of input variables or that either deliver an output listing or a graph, depending on how the function call is made. These options can be made available by using MATLAB's inherent *nargin* and *nargout* variables. They count the number of input or output arguments specified in the function call. The function file can include branching to accommodate the instructions implied by the number of input or output arguments that have been specified.

Function files may call other functions. When the function file is first run, it is compiled along with any other function calls included within the function file. Functions consequently execute very efficiently. As with other M-files, functions must be located within the MATLAB search path.

If your M-files are on a floppy, remember to add the floppy to the MATLAB search path.

The following function file demonstrates the format. It should be duplicated and copied to a floppy. It will graph a Fourier partial sum with or without a window function. The user is required to define the series coefficients before calling the function, and provide the window coefficients.

① File and function names must agree

② This is the *function* declarative; it must be on the first file line used.

③ This is the *help FS* output. Its first line is the only one searched by the *Find* command. Remember your own experiences and make this information *helpful!*

```
                         file: FS.m
function [yy,t]=FS(c,w,T)
% FS Fourier Partial Sum Plotter
%     FS calculates the Fourier series partial
%     sum given the complex Fourier series
%     coefficients c, the window w, and the
%     period T. Default values are a rectangular
%     window and T = 1. The command
%     format is [part_sum,time]=FS(c,w,T).
%     FS(c,w,T) used alone plots the partial sum.
```

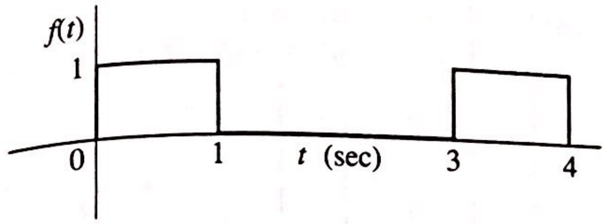
④ The program starts here. If two input arguments are provided, the period is set to 1. If only one argument is provided, a rectangular window is used.

```
if nargin==2
    T=1;                   % if omitted, set T = 1
    elseif nargin==1       % if omitted, set w = 1
    w=1; T=1;
end

cm=c.*w';                        % calculate windowed coefficients
n=length(c);                     % determine the number of harmonics
M=(n-1)/2; m=-M:M;
for t=1:400                      % use 400 points for the plot
    tt=(t-1)/400;
    arg=j*2*pi*m*tt;             % calculate the exponents
    y(t)=exp(arg)*cm.';          % note the matrix multiply
    x(t)=tt;
end
if nargout==0                    % decide what output format is desired
    plot(x*T,real(y))
    title('Fourier series partial sum')
    ylabel('f(t)')
    xlabel('t(seconds)')
elseif nargout==1
    yy=real(y);
elseif nargout==2
    yy=real(y); t=x*T;
end
```

EXAMPLE 6.7

Retaining terms out to the fifth harmonic, plot the partial sum of the waveform in Figure 6.14 using a rectangular window. Also, plot the exponential spectrum and determine the percentage of the total power in these harmonics.



$$c_m = \frac{e^{-j\pi m/3}}{m\pi}\sin m\pi/3$$

Figure 6.14

Solution

(Assumes the FS.m file is on a floppy and has been put in the search path.)

```
>m=-5:5; cm=exp(-j*m*pi/3).*sin(m*pi/3)./(m*pi);
>cm(6)=1/3;        % fix indeterminate values
>FS(cm,1, 3)       % call the function
```

See the resultant graph in Figure 6.15.

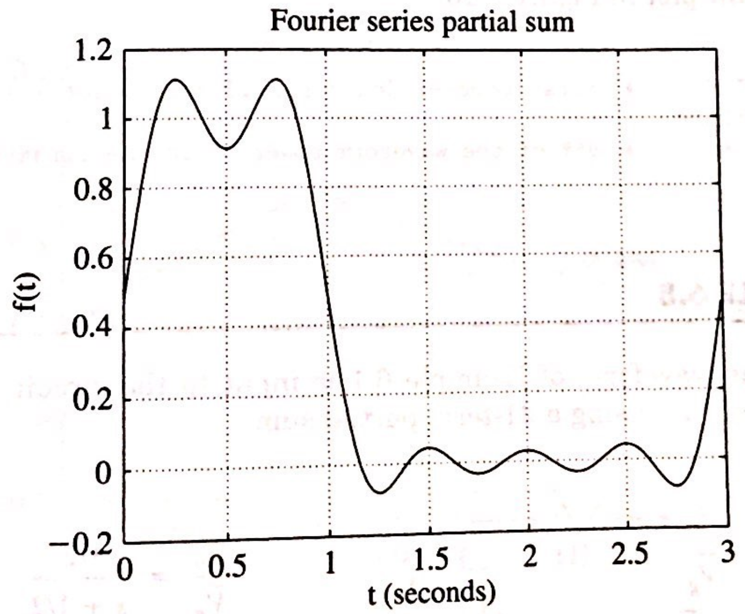


Figure 6.15 Eleven term partial sum Fourier series representation of the waveform of Figure 6.14 using a rectangular window.

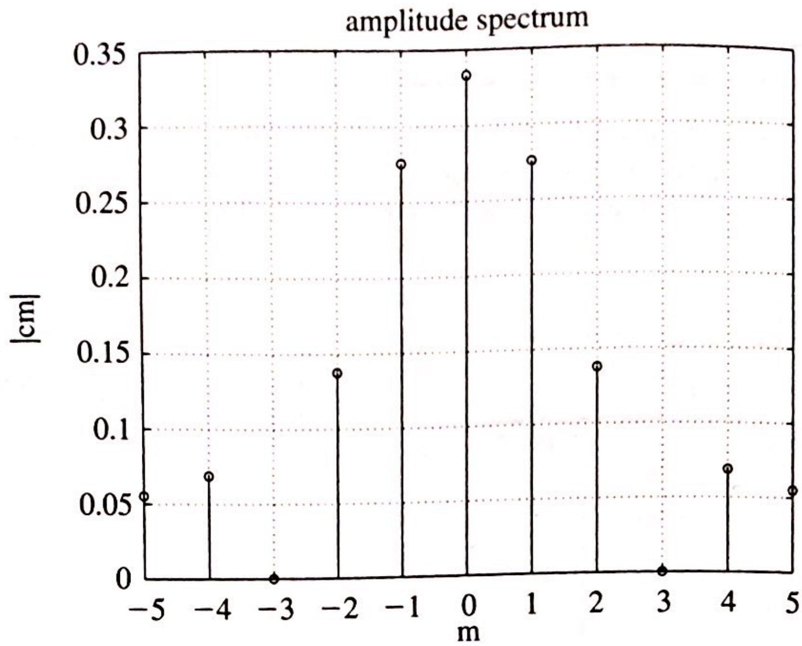


Figure 6.16 The exponential signal amplitude spectrum containing 95% of the power in the waveform of Figure 6.14.

To plot the spectrum:

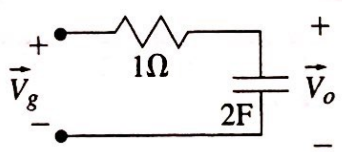
```
>stem(m,abs(cm))
>grid; title('amplitude spectrum');ylabel('|cm|');xlabel('m')
```

See the resultant plot in Figure 6.16.

```
>Pave=cm*cm'
Pave = 0.3167     % total power=1 joule spread over 3 sec = 1/3 watt
>PartP =3*Pave
PartP = 0.95      % 95% of the waveform power is in the first five harmonics.
```

EXAMPLE 6.8

The rectangular waveform of Example 6.1 is input to the circuit of Figure 6.17. Determine the output using a 21-term partial sum.



$$\frac{\vec{V}_o}{\vec{V}_g} = \frac{1/2}{s + 1/2}$$

Figure 6.17

Solution

The circuit transfer function must be expressed in terms of the harmonic frequencies:

$$\frac{\tilde{V}_o}{\tilde{V}_g} = \frac{1/2}{s + 1/2} = \frac{1/2}{jm\omega_o + 1/2} = \frac{1/2}{1/2 + jm2\pi/3}$$

Since the circuit is a low-pass filter, it will tend to remove the rippling, so no window will be used on the Fourier coefficients of $v_g(t)$.

```
>m=-10:10;in=exp(-j*m*pi/3).*sin(m*pi/3)./(m*pi);
>in(11) = 1/3;
>FS(in,1,3)                              % plot v_g(t)
>hold
>h=0.5./(.5 + 2j*pi*m/3);               % calculate transfer
                                          function
>out=in.*h;
>FS(out,1,3)                             % plot v_o(t)
>gtext('input')
>gtext('output')
```

The resultant output is displayed in Figure 6.18.

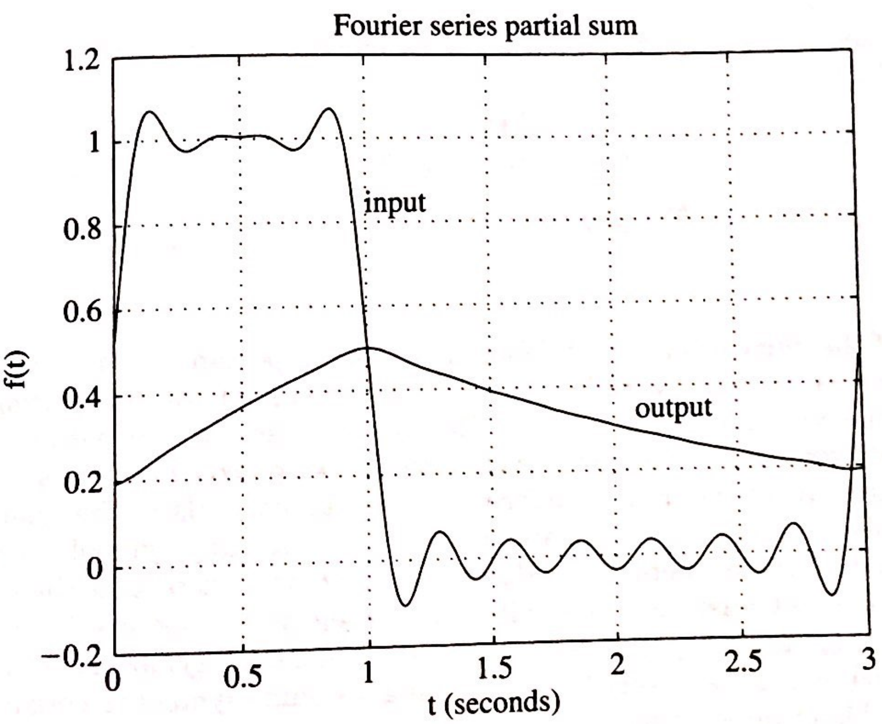


Figure 6.18 Tenth harmonic Fourier partial sum representations of the input and output waveforms in the circuit of Figure 6.17.

6.6 PROPERTIES OF SIGNALS

Much of the content of this chapter has involved evaluating Fourier's integral to find the Euler phasors in a particular waveform, resolving indeterminate values, and modifying the results to give better partial sum approximations for that waveform. These are mechanical issues that must be addressed if we are to make use of the tools Fourier has provided. The real beauty of Fourier's work lies in what it tells us about general relationships between signals and their spectra. These are not obtained by cranking through some horrendously complicated integrals, but rather by observing subtle effects eloquently expressed through functional notation. After completing this section you will be able to

- Cite the modulation and time delay theorems.
- Use convolution to find the output of a circuit, given its input and impulse response.
- Measure the cross-correlation between two signals.
- Better appreciate the power of functional notation.

We start by repeating the Fourier transform pair:

parameter ω

$$F(\omega) = \int_{t=-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad (6.6)$$

variable of integration t

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (6.5)$$

In most of the derivations which follow, it will be important to pay attention to the distinction between the variable that is a parameter of the integration and that which is the variable of integration. The symbol used for the parameter may be changed at any time, as long as it is changed in two places. Both the Fourier transform and the inverse Fourier transform are definite integrals, so the symbol used for the variable of integration may also be replaced with any symbol except the one being used for the parameter. Note that the ω and t symbols reverse their roles going between the two transforms. Basically, these integrals tell us how to find the function F given the function f or vice versa. We will change the symbols used for the variables only when it is necessary to avoid having the same symbol be both a parameter and a variable of integration.

Scaling: Given that $f(t)$ has the Fourier transform $F(\omega)$, what can be said about the Fourier transform of $g(t) = f(at)$? By definition,

$$G(\omega) = \int_{t=-\infty}^{\infty} f(at)e^{-j\omega t}\,dt$$

We can make a change in variable by letting $x = at$, where a is a positive constant. Then $dx = a\,dt$, and the range of integration remains $\pm\infty$ whether x or t is the variable of integration. We also note that

$$\int_{x=-\infty}^{\infty} f(x)e^{-j(parameter)x}\,dx$$

defines the function F, so

$$G(\omega) = \int_{x=-\infty}^{\infty} f(x)e^{-j\left(\frac{\omega}{a}\right)x}\,\frac{dx}{a} = \frac{1}{a}F\left(\frac{\omega}{a}\right) \qquad \textbf{(6.9)}$$

Equation 6.9 says that if f has the spectrum F, shown in Figure 6.19, then when the argument of f is 6, the pulse is over. The pulse of $f(2t)$ is also over when its argument is 6, which occurs when $t = 3$. So f and g are identical except that g occurs twice as fast as f. The spectra of f and g differ in that G is twice as wide and half as large as F. In all other aspects they are identical. How the widths of the spectra are measured is arbitrary. We could measure spectral bandwidths as the frequency range where the spectra are 3dB down from their maximums. The bandwidth of G would be twice that of F on that basis, or any other basis used.

These conclusions also make sense from an energy viewpoint. g has half the energy of f because it is otherwise identical to f but occurs in half the time. $|F|^2$ and $|G|^2$ are proportional to energy density (Parseval's theorem), so the average

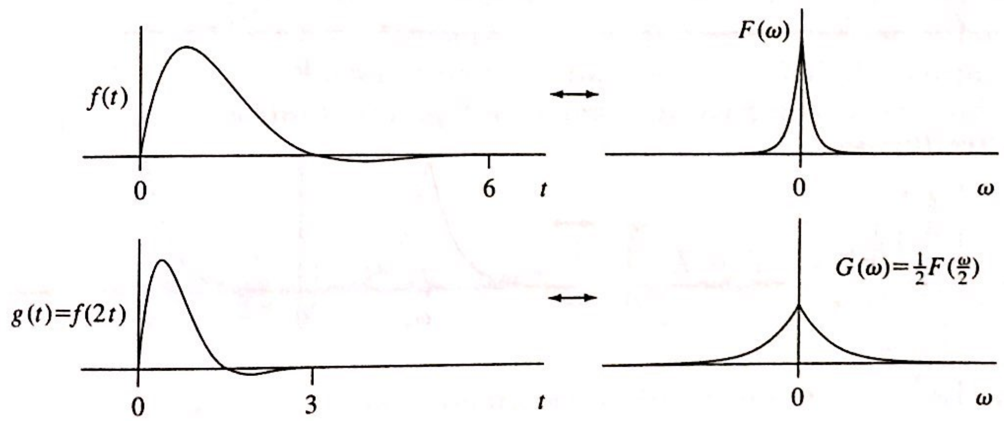


Figure 6.19 An $f(t)$ has a spectrum $F(\omega)$. If $g(t)$ has exactly the same shape as $f(t)$ but occurs twice as fast, its spectrum is the same shape as $f(t)$ but has half the amplitude and twice the bandwidth.

energy density in G is one-quarter that of F, but G is twice as wide, for a net energy reduction of 2.

The scaling theorem demonstrates that we do not need to specify a particular pulse shape or carry out specific integrals in order to learn important relationships between signals and their spectra. The symmetry between the Fourier transform and its inverse is so intense that we usually get two theorems for the price of one. In the case of scaling, we would have reached the same final conclusions if we had asked how g was related to f given that $G(\omega) = F(a\omega)$.

Frequency (time) displacement: Given that $f(t)$ has the Fourier transform $F(\omega)$, what can be said about the Fourier transform of $g(t) = e^{j\omega_o t} f(t)$? Again, we start with the definition of G, and conclude that only the equation parameter is affected.

$$G(\omega) = \int_{t=-\infty}^{\infty} \left(e^{j\omega_o t} f(t)\right) e^{-j\omega t} dt = \int_{t=-\infty}^{\infty} f(t) e^{-j(\omega-\omega_o)t} dt = F(\omega - \omega_o) \qquad \textbf{(6.10)}$$

In words, this theorem says that multiplying a signal by $e^{j\omega_o t}$ just shifts the spectrum of f so that it is centered on the frequency ω_o. The spectrum of f is not changed in amplitude, shape, or spread, but is simply translated in frequency. This is often called the *modulation theorem*. Figure 6.20 shows an example of this theorem using a real (cosine) function as the carrier.

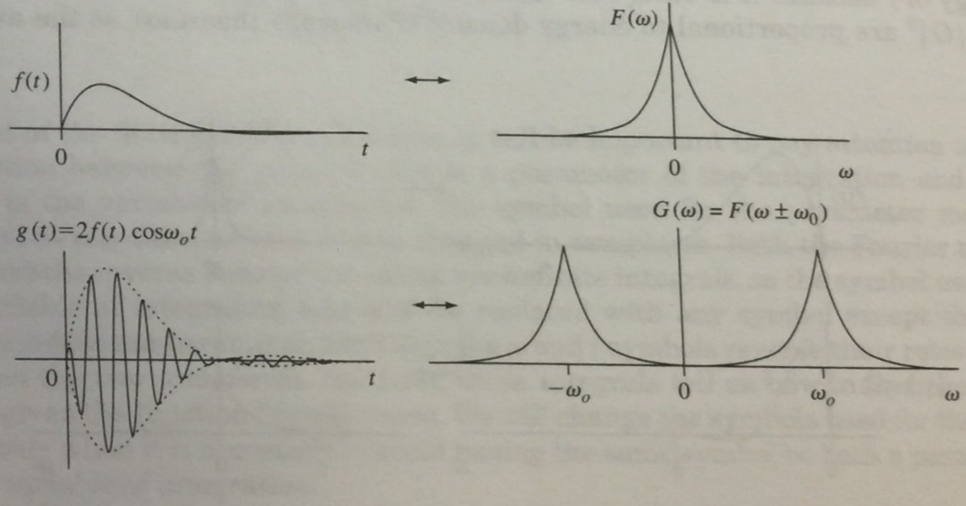


Figure 6.20 An $f(t)$ has a spectrum $F(\omega)$. If $g(t)$ has exactly the same shape as $f(t)$ but is multiplied by $\cos\omega_o t$, the spectrum of $g(t)$ consists of replications of the spectrum of $f(t)$ centered on the frequencies of $\pm\omega_o$. This is the modulation theorem.

Table 6.2 Selected Signal Properties Derived from the Fourier Transform

	$g(t) \Leftrightarrow G(\omega)$
Scaling theorem	$f(at) \leftarrow \mathrm{F1} \rightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right)$
Modulation theorem	$e^{j\omega_o t} f(t) \leftarrow \mathrm{F2} \rightarrow F(\omega - \omega_o)$
Time-delay theorem	$f(t-\tau) \leftarrow \mathrm{F3} \rightarrow e^{-j\tau\omega} F(\omega)$
Time-convolution theorem	$\int_{-\infty}^{\infty} x(\sigma) h(t-\sigma) d\sigma \leftarrow \mathrm{F4} \rightarrow X(\omega) H(\omega)$
Frequency-convolution theorem	$x(t)\, h(t) \leftarrow \mathrm{F5} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma) H(\omega - \sigma) d\sigma$
Correlation theorem	$\int_{-\infty}^{\infty} x(\sigma) y(\sigma + t) d\sigma \leftarrow \mathrm{F6} \rightarrow X^*(\omega) Y(\omega)$

We might have started this problem with the question: given that $F(\omega) \leftrightarrow f(t)$, then what is the signal corresponding to $F(\omega)e^{-j\tau\omega}$? Pursuing this question would have led us to the *time delay theorem,* which tells us that multiplying a signal's spectrum by $e^{-j\tau\omega}$ just delays the signal in time by τ seconds. We arrived at the same conclusion back in Chapter 5 considering just sinusoids.

A few other relationships that can be established between signals and their spectra are indicated in Table 6.2. We will consider two of them in more detail.

Convolution: We have on numerous occasions taken an input signal's phasor, and put it through a transfer function to find the output phasor. Using a phasor transformation, we then concluded what the output sinusoid's amplitude and phase would be. Fourier allows us to generalize that process. Given an input signal's spectrum, $X(\omega)$, and a transfer function, $H(\omega)$, we can find the output signal's spectrum as $Y(\omega) = X(\omega)H(\omega)$. Taking the inverse Fourier transform would then give us the output signal.

$$y(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega$$

Since we are looking for general relationships between signals, we will resist the temptation to pick a specific $x(t)$ and transfer function. Instead, let us replace either X or H by its Fourier transform representation. We must be careful not to get the t parameter in our equation mixed up with the t variable of integration in the Fourier transform. Replacing $X(\omega)$ gives

$$X(\omega) = \int_{\sigma=-\infty}^{\infty} x(\sigma)e^{-j\omega\sigma}d\sigma \qquad y(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} H(\omega)\left[\int_{\sigma=-\infty}^{\infty} x(\sigma)e^{-j\omega\sigma}d\sigma\right]e^{j\omega t}d\omega$$

Normally we would be expected to carry out the integration over σ first, but that would just take us backwards. Instead, we will interchange the order of integration by collecting all the terms that depend on ω and plan on doing that integral first. We cannot actually do the integral over ω because no specific $H(\omega)$ has been given. Fortunately, however, with a little regrouping of terms, we will recognize the inverse Fourier transform for $h(t-\sigma)$.

$$y(t) = \frac{1}{2\pi}\int_{\sigma=-\infty}^{\infty} x(\sigma)\left[\int_{\omega=-\infty}^{\infty} H(\omega)e^{j\omega(t-\sigma)}\right]d\omega d\sigma = \int_{\sigma=-\infty}^{\infty} x(\sigma)\left[\underbrace{\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} H(\omega)e^{j(t-\sigma)\omega}\,d\omega}_{h(t-\sigma)}\right]d\sigma$$

The final result shows that the output $y(t)$ can be found by staying entirely within the time domain. This is called the *convolution* of x and h. If we had substituted for H instead of for X, the functions would have just switched arguments.

$$y(t) = \int_{\sigma=-\infty}^{\infty} x(\sigma)h(t-\sigma)d\sigma = \int_{\sigma=-\infty}^{\infty} x(t-\sigma)h(\sigma)d\sigma \tag{6.11}$$

The notation $h(t-\sigma)$ indicates that $h(\sigma)$ is to be folded about the vertical axis, and then displaced by an amount t. It is often necessary to sketch the situation in order to have the correct limits of integration for the different amounts of displacement. Since this folding and shifting can be done to either function, it is best to do it to the simpler function.

EXAMPLE 6.9

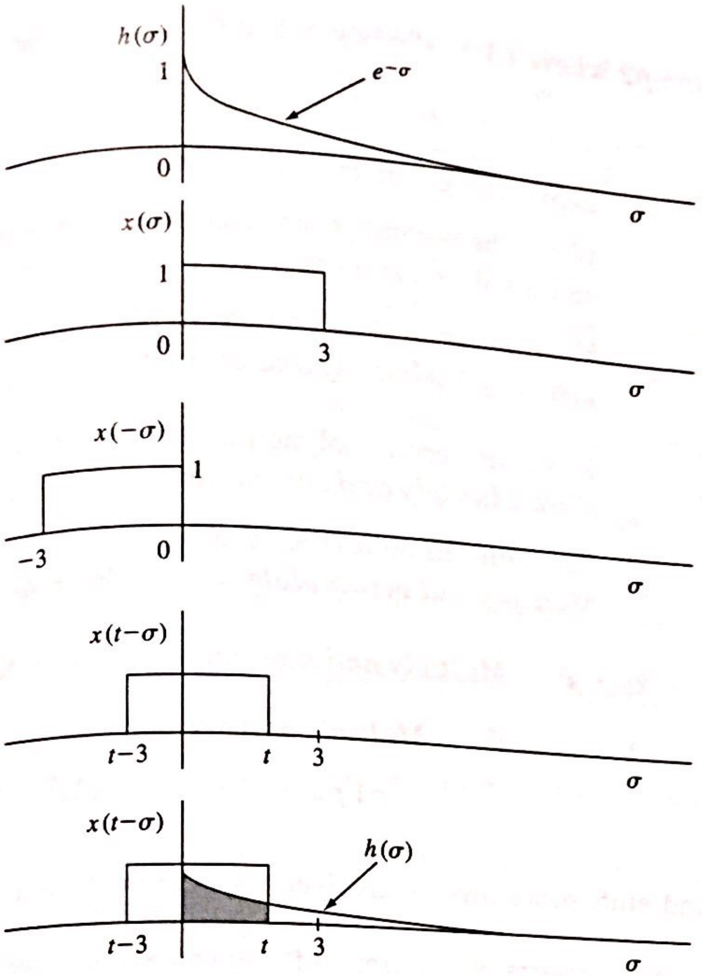
A 1 V pulse lasting 3 seconds is input to a circuit whose voltage transfer function is $H(\omega) = 1/(1+j\omega)$. Use convolution to find the output voltage.

Solution

The student is asked to verify that the Fourier transform of

$$h(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

is the $H(\omega)$ given, and that $h(t)$ is therefore the impulse response of the circuit. Since the input waveform is a simple rectangular pulse, it will be the function folded and shifted. The process of convolving the waveforms is visualized in Figure 6.21. From



1. h and x are sketched with the independent variable changed to σ.

2. The simpler function is folded about the vertical axis.

3. After folding, the function is shifted an amount t.

4. Carry out the integral appropriate to the t value being considered. The area captured by the integral is shown shaded.

Figure 6.21 Evaluating the convolution integral of Example 6.9 is visualized in order to establish the correct range of integration as x is shifted.

this we can see that there are 3 possible ranges of overlap of the functions. The integrand does not change in the convolution integral, but the limits over which the integration is taken do. Obviously, the integrand is zero whenever the product of the two functions is zero.

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \displaystyle\int_0^t e^{-\sigma} d\sigma = 1 - e^{-t} & \text{for } 0 \le t \le 3 \left\{ \begin{array}{c} \text{condition shown} \\ \text{in Fig. 6.21} \end{array} \right. \\ \displaystyle\int_{t-3}^t e^{-\sigma} d\sigma = (e^3 - 1)e^{-t} & \text{if } t \ge 3 \end{cases}$$

We have encountered the MATLAB command **conv** used for taking the product of two polynomials. You may be wondering why its name suggests convolution. The reason is that both operations involve the same principle. Suppose we want to multiply the polynomials $p1$ and $p2$ where $p1 = s^2 + 2s + 1$ and $p2 = s^2 + 3s + 4$. We could do it as follows:

$p2 = s^2 + 3s + 4$ Leave one polynomial unchanged.

$p1 = 1 + 2s + s^2$ Write the second polynomial in reverse order and displace for no overlap.

$1 + 2s + s^2$ Move the second polynomial to overlap the first term of $p2$. *Multiply and accumulate* $1s^4$

$1 + 2s + s^2$ Move the second polynomial to overlap two terms of $p2$. *Multiply and accumulate* $2s^3 + 3s^3 = 5s^3$

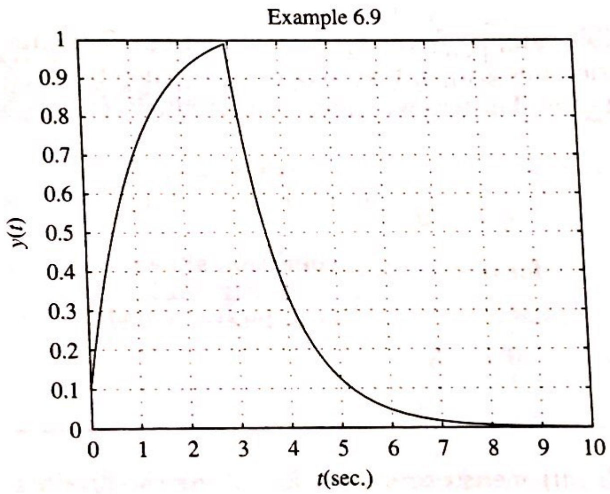
$1 + 2s + s^2$ Continue till no further overlap *Multiply and accumulate* $s^2 + 6s^2 + 4s^2 = 11s^2$

$1 + 2s + s^2$ *Multiply and accumulate* $3s + 8s = 11s$

$1 + 2s + s^2$ *Multiply and accumulate* 4

$$p1\,p2 = s^4 + 5s^3 + 11s^2 + 11s + 4$$

Since the same fold and shift procedure is involved as in convolution, we can have MATLAB do Example 6.9 for us numerically. Our only job is to take care of scale factors. The dt of the convolution integral is the time difference between two successive points. If we use 10 points per second, then dt is 1/10 sec. The programming and result is shown in Figure 6.22.




```
>t=linspace(0,10);
>h=exp(-t);
>x=ones(1,30);
%slightly better to let x=[0 ones(1,29)]
%      so it turns on.
>y=conv(x,h)/10;     % dt=1/10
>plot(t,y(1:100));
>ylabel('y(t)'); xlabel('t(sec.)')
>title('Example 6.9'); grid
```

Figure 6.22 The MATLAB solution for Example 6.9 using the **conv** command.

Convolution can also occur in the frequency domain. If two time functions are multiplied together, the resulting signal's spectrum is the convolution of the individual spectra. The derivation follows the same pattern as for the time convolution theorem, emphasizing again the symmetry of the Fourier transforms.

Correlation: Correlation involves a statistical measure that is related to convolution, but is looked at from a different perspective. It provides a measure of the similarity between two waveforms as a function of a search parameter. The derivation for the correlation theorem follows the same approach used to derive the convolution theorem. The cross-correlation between signals x and y over the interval T as a function of the search parameter t is calculated from

$$\phi_{xy}(t) = \int_T x(\tau)y(\tau + t)d\tau$$

If we let $\sigma = \tau + t$, then $d\sigma = d\tau$, the equation may also be written as

$$\phi_{xy}(t) = \int_T x(\sigma - t)y(\sigma)d\sigma$$

Since σ is a dummy variable, we can revert to the original notation to show that

$$\phi_{xy}(t) = \int_T x(\tau)y(\tau + t)d\tau = \int_T x(\tau - t)y(\tau)d\tau \qquad \textbf{(6.12)}$$

Figure 6.23 shows a common application. The envelope for a radar pulse is shown as $x(\sigma)$, while the demodulated envelope of the return pulse is $y(\sigma)$. A correlation processor takes the integral of the return pulse and a shifted version of the original transmitted pulse. If the two pulses do not overlap, zero correlation is measured. As the pulses begin to overlap, the area picked up by the integral (shown shaded) steadily increases until the pulses are fully aligned, and then begins to decrease again as x is shifted beyond y. The peak in the correlation occurs when t equals the time delay experienced by the echo pulse, which in turn tells the distance

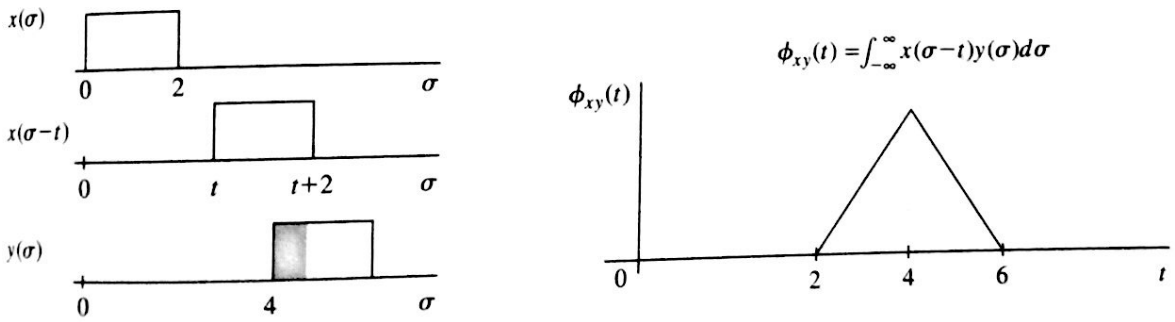


Figure 6.23 The correlation function $\phi_{xy}(t)$ measures the similarity between the two waveforms x and y as one of them is shifted an amount t. The correlation peak provides the best indicator of pulse alignment, particularly when random noise is present.

to the target. (We would have obtained the same information by shifting y in the opposite direction, which is what the first integral for $\phi_{xy}(t)$ says to do.)

Why not just measure the time delay between the leading edges of the pulses? Think random noise. The received pulse is not the nice clean version depicted in Figure 6.23, but is likely to be buried in noise to the point that it is almost impossible to tell where the pulse is, much less where its leading edge is. The integration involved in correlation tends to wipe out the effects of the random noise, making the correlation peak the best indicator of the target's range. A full discussion of correlation is intimately connected to the noise issue, and beyond the scope of this text.

Other properties of signals not included in Table 6.2 can be easily deduced from the continuous-time Fourier transform. We will investigate some of these properties further once we have found some additional computer tools like **conv** that can help us avoid evaluating integrals.

CHAPTER SUMMARY

Fourier showed how any repetitive waveform can be created by superimposing sinusoids with the proper amplitude and phase. The sinusoids have frequencies that are integer multiples of the repetition rate, called the fundamental frequency. The Fourier series converges to the function where it is continuous and to the average value where it is discontinuous.

The spectrum of a repetitive signal is a display of the amplitude, and possibly phase, present at each frequency. The symmetrical display of exponential frequencies is preferable mathematically, but the positive frequency display of sinusoidal components is provided in commercial equipment. A power spectrum is a similar display but gives the average power of the signal at each frequency.

In many applications, the Fourier series is terminated to provide a practical approximation to a desired function. If that function contains a discontinuity, the partial sum will tend to oscillate around the ideal function because the Fourier series of a discontinuous function retains excessive amounts of high-frequency components. By tapering off the amplitudes of the Fourier coefficients with a window function, a smoother result is obtained for the partial sum. Windows are not normally used with continuous functions.

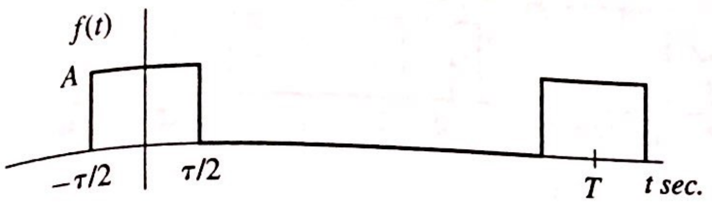
The Fourier transform is the general tool for determining the frequency spectrum of a pulse. Such signals have a finite energy and therefore zero average power. They also have a continuous spectrum, with infinitesimal components available at all frequencies but with a relative distribution given by $F(\omega)$. If the pulse is repeated so that a repetitive signal is generated, the c_m will show the identical relative frequency distribution as for the single pulse.

Computer-based tools for plotting a Fourier series partial sum make it possible to observe firsthand many of the issues discussed in this chapter. MATLAB functions can be created to perform this task, and they allow users to personalize their toolboxes. The FS MATLAB function created in this chapter is essential to verifying the concepts presented.

PROBLEMS

Section 6.1

1. Show that the waveform of Figure P6.1 has the Euler phasors indicated.



$$c_m = A\frac{\tau}{T}\frac{\sin \pi m\tau/T}{\pi m\tau/T}$$

Figure P6.1

2. The Euler phasors of a waveform periodic in 2π seconds are

$$c_m = \frac{2\cos(m\pi/2)}{1-m^2}$$

   a. Write the five-term partial sum of the complex Fourier series for this waveform.
   b. Write out the first three nonzero terms of the cosine Fourier series for this waveform.

3. The Euler phasors of a waveform periodic in 3 seconds are

$$c_m = \frac{-j}{m\pi}\left[\cos\left(\frac{2\pi m}{3}\right) - 1\right]$$

   a. Write out the five-term partial sum ($-2 \le m \le 2$) of the complex Fourier series for this waveform.
   b. Write out the first three nonzero terms of a trigonometric Fourier series for this waveform.

4. For the waveforms of Figure P6.4, set up the integral expression for c_m, including all specific numerical information, but do not integrate.

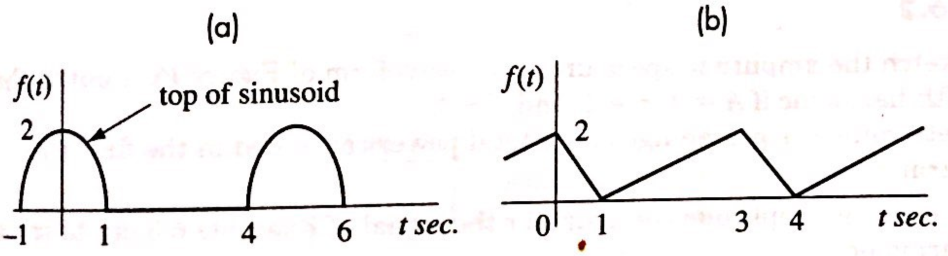


Figure P6.4

5. For the waveforms of Figure P6.5, set up the integral expression for c_m, including all specific numerical information, but do not integrate.

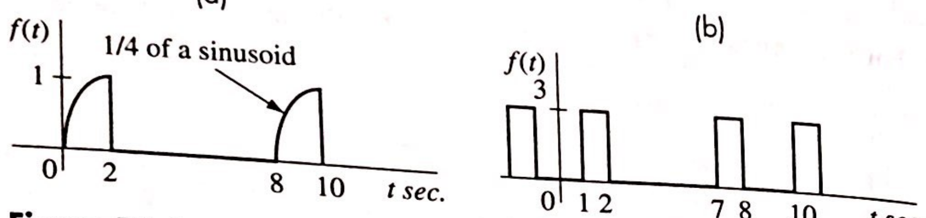


Figure P6.5

6. From the results of Problem 1, determine the Euler phasors for the waveform of Figure P6.6.

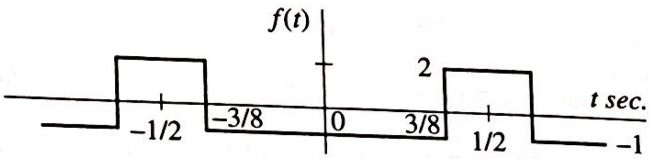
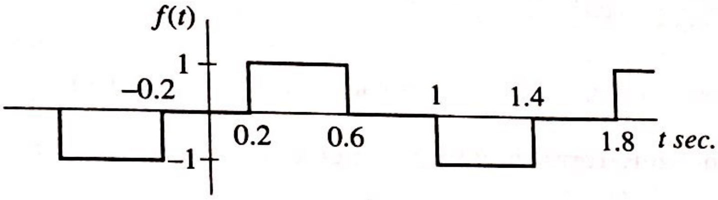


Figure P6.6

7. Show that the waveform of Figure P6.7 has the Euler phasors indicated:



$$c_m = \frac{\cos m\pi/4 - \cos 3\,m\pi/4}{j\pi m}$$

Figure P6.7

Section 6.2

8. a. Sketch the amplitude spectrum of the waveform of Figure P6.1 out to the fifth harmonic if $A = 1$, $\tau = 1$, and $T = 4$.
   b. Determine the percentage of the total power contained in the first five harmonics.
9. a. Sketch the amplitude spectrum for the signal of Example 6.3 out to the fifth harmonic.
   b. Sketch the power spectrum for the signal out to the fifth harmonic.

10. Using the results of Example 6.5, determine the percentage of the total waveform power that exists in the first three harmonics for the waveform of Figure P6.10.

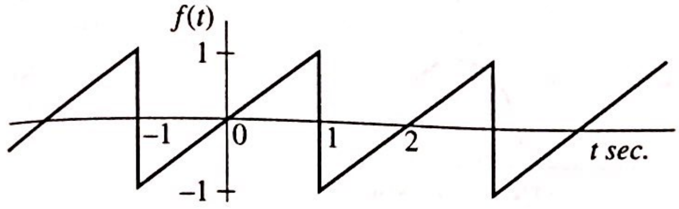
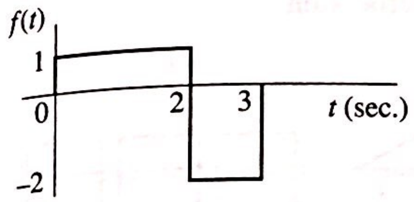


Figure P6.10

Section 6.3

11. The pulse shown in Figure P6.11 has the $F(\omega)$ indicated.
    a. Verify the expression for $F(\omega)$.
    b. Determine the complex Fourier series ($-2 \leq m \leq 2$) for the waveform that results if the pulse repeats every 3 seconds.



$$\longleftrightarrow \qquad F(\omega) = \frac{1 - 3e^{-j2\omega} + 2e^{-j3\omega}}{j\omega}$$

Figure P6.11

12. The pulse described here has the Fourier transform indicated:

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq 2\pi \\ 0 & \text{elsewhere} \end{cases} \quad \leftrightarrow \quad F(\omega) = 2je^{-j\pi\omega}\frac{\sin \pi\omega}{1 - \omega^2}$$

Determine a few terms ($-2 \leq m \leq 2$) of the Fourier series of the periodic waveform that results if the pulse repeats in a period of
a. 2π seconds. b. 4π seconds.

13. Find the Fourier transform $X(\omega)$ for the pulse of Figure P6.13.

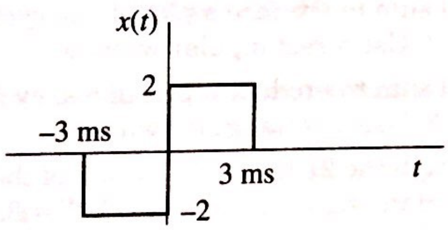


Figure P6.13

14. Find the Fourier transform $X(\omega)$ for the sinusoidal pulse of Figure P6.14.

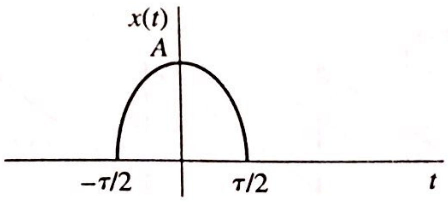


Figure P6.14

Section 6.4

15. The coefficients of a Fourier series are $c_m = 2/m$ $(m \neq 0)$ and $c_0 = 1$. List the coefficients of the 11-term Fourier partial sum using a Hamming window.
16. The coefficients of a Fourier series are $c_m = \sin(m)/2m$. List the values of coefficients c_0 through c_4 if a 21-term partial sum is used with a Hanning window.
17. State which of the five waveforms in Figure P6.17 would best be approximated by a nonrectangular windowed Fourier partial sum.

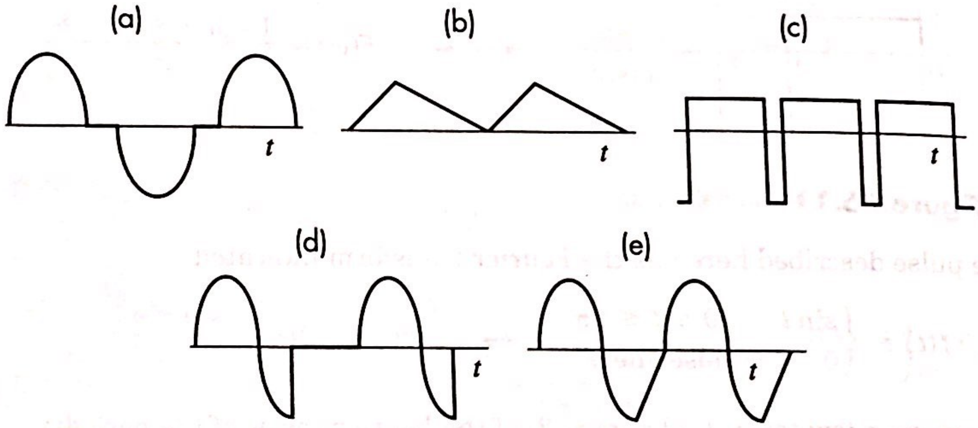


Figure P6.17

Section 6.5

18. Use a 31-term partial sum to produce a plot of one cycle of the $f(t)$ whose c_m are given in Problem 2. Use a rectangular window.
19. Use a 21-term partial sum to produce a plot of one cycle of the $f(t)$ whose c_m are given in Problem 3. Use a rectangular window.
20. a. Plot on the same graph the 21-term partial sum of the waveform in Figure P6.1 with and without a Hanning window for $\tau = 1, T = 3$, and $A = 2$.
    b. Plot the exponential spectrum of the windowed signal.
    c. Determine the average power of the windowed signal.

21. a. Plot on the same graph the 21-term partial sum of the waveform in Figure P6.7 with and without a Hanning window.
b. Plot the exponential spectrum of the windowed signal.
c. Determine the average power of the windowed signal.

22. Plot on the same graph the window functions of Table 6.1. Use 41 terms for each window, and identify each curve.

23. For the MATLAB vector $x = [1{-}j \quad 2 \quad j]$, determine the result of the operations indicated. (One will be an error statement.)
a. `a=x*x'` b. `b=x.*x` c. `c=x.*x.'` d. `d=x*x.'`

Section 6.6

24. Carry out the inverse Fourier transform to determine $f(t)$ given

$$F(\omega) = \delta(\omega - \omega_o) + \delta(\omega + \omega_o)$$

25. For the waveforms in Figure P6.25, given that $f(t) \leftrightarrow F(\omega)$, express $G(\omega)$ in terms of F.

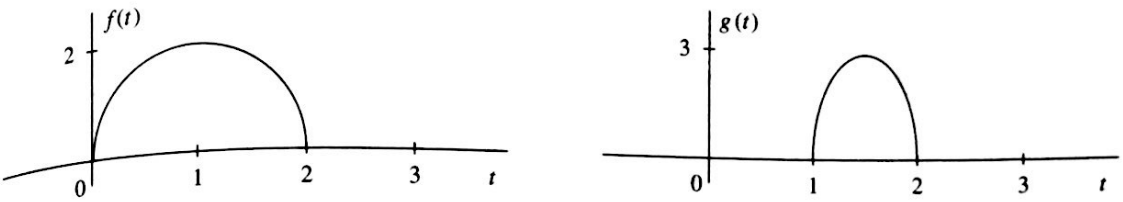


Figure P6.25

26. For the spectra in Figure P6.26, given that $f(t) \leftrightarrow F(\omega)$, express $g(t)$ in terms of f.

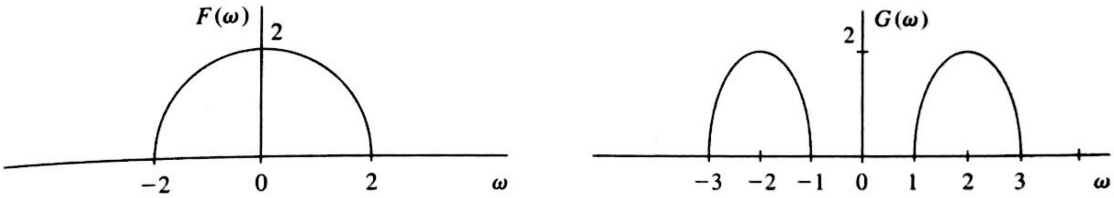


Figure P6.26

27. The ramping pulse $v_i(t)$ is applied to a circuit whose unit impulse response is e^{-t}.

$$v_i(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 4 \\ 0 & t > 4 \end{cases}$$

a. Provide integral expressions for determining $v_o(t)$ over all t.
b. Obtain a scaled plot of the circuit output over 8 seconds.

28. A 2-second-long rectangular pulse of 1 V amplitude is input to a circuit whose unit impulse response is $h(t)$.
   a. Determine the transfer function $H(\omega)$ for the circuit.
   b. Find integral expressions for the circuit's output.
   c. Use **conv** to obtain a properly scaled plot of the circuit output.

$$h(t) = \begin{cases} 0 & t < 0 \\ 2e^{-t}\cos 4t & t \geq 0 \end{cases}$$

29. a. Obtain analytical expressions for the convolution of the functions indicated in Figure P6.29.
   b. Use **conv** to obtain a properly scaled graph of the convolution of x and h.

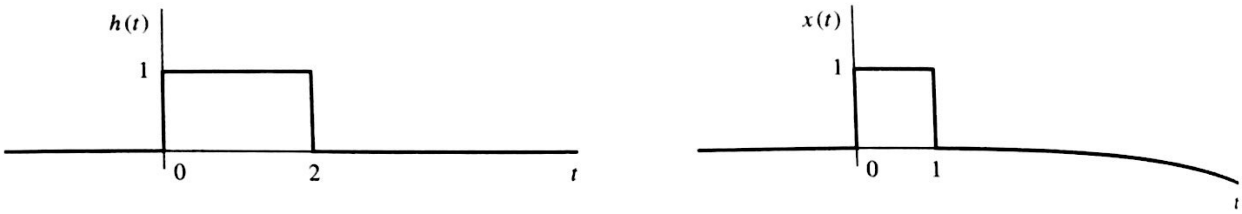


Figure P6.29

30. Show that if $\Phi(\omega) = X(\omega)Y^*(\omega)$, then the inverse Fourier transform of $\Phi(\omega)$ is

$$\phi_{xy}(t) = \int_{\tau=-\infty}^{\infty} x(t+\tau)y(\tau)d\tau$$

31. A circuit with a unit impulse response of $h(t) = e^{-t} - e^{-2t}$ has an input signal $v_i(t) = \sin 2\pi t$ turned on at $t = 0$.
   a. Determine the transfer function of the circuit.
   b. Use convolution to provide integral expressions for the output voltage.
   c. Use **conv** to obtain a properly scaled graph of $v_o(t)$ from $0 \leq t \leq 8$ seconds.

32. A correlation phase meter is used to determine the phase of a signal $x(t)$ relative to an internally generated reference signal $y(t)$. Since periodic signals are involved, the range of integration is the period T. Assume $x(t) = \cos(t + 1.2)$ and $y = \cos t$, and evaluate $\phi_{xy}(t) = \int_T x(\sigma)y(t+\sigma)d\sigma$ to show that the cross-correlation peaks when $t = 1.2$. (The significance of this result will be more meaningful when we can bury x in random noise.)

Additional Problems

33. The signal of Problem 6.1 has $A = 2$, $\tau = 1$, and $T = 6$. It is input to the circuit shown in Figure P6.33. Use a 31-term Fourier partial sum to determine the output waveform.

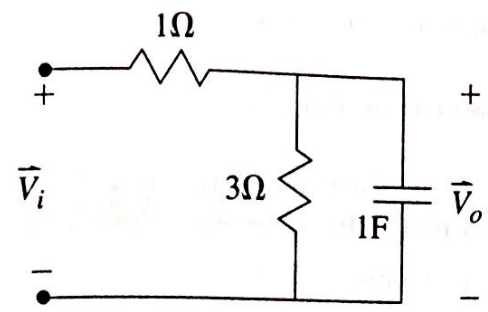


Figure P6.33

34. The signal of Example 6.5 repeats every 4 seconds and has an amplitude of 12 V. It is input to the circuit of Figure P6.34. Determine the output if a 31-term Fourier partial sum with a Hamming window is used to represent the input.

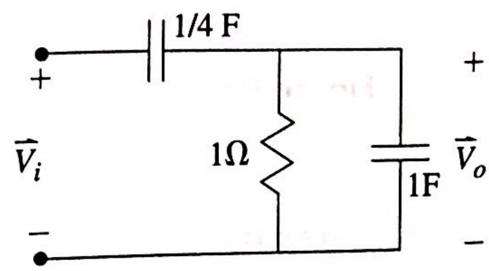


Figure P6.34

5.25 Use the same *ez1.m* file used for Example 5.7, but change the first instruction to [z,p,k]=cheb1ap(n,2); and use **plot** instead of **semilogx** in the last instruction.

Chapter 6

6.1 $c_m = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jm\omega_0 t}\,dt = \frac{A\tau}{T}\frac{\sin m\pi\tau/T}{m\pi\tau/T}$

6.3 a) $f(t) = \cdots -0.2387je^{-j4\pi t/3} - 0.4775je^{-j2\pi t/3} + 0 + 0.4775je^{+j2\pi t/3} + 0.2387je^{+j4\pi t/3} + \cdots$

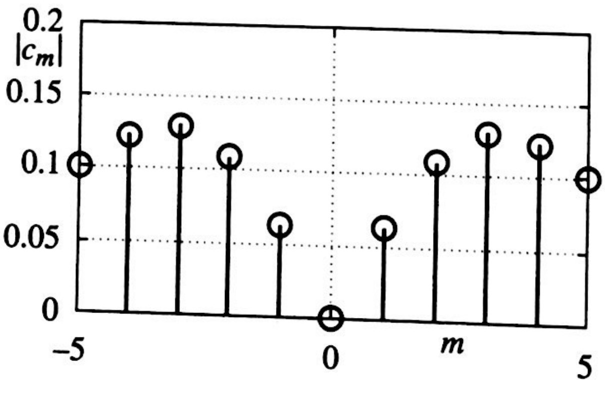
b) $f(t) = -0.9550\sin 2\pi t/3 - 0.4774\sin 4\pi t/3 - 0.2388\sin 8\pi t/3 - \cdots$

6.5 a) $c_m = \frac{1}{8}\int_0^2 \sin\left(\frac{\pi}{4}t\right)e^{-jm\frac{\pi}{4}t}\,dt$

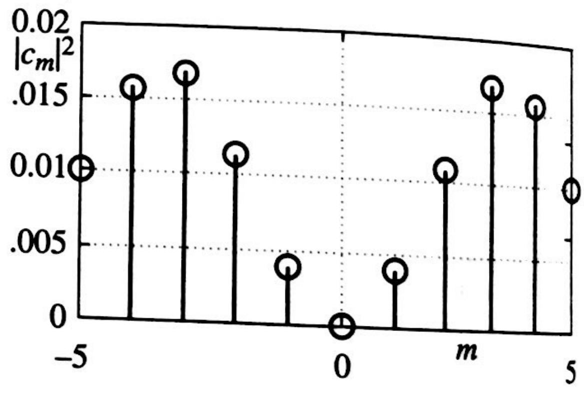
b) $c_m = \frac{1}{9}\left[\int_{-2}^{-1} 3\,e^{-jm\frac{2\pi}{9}t}\,dt + \int_1^2 3\,e^{-jm\frac{2\pi}{9}t}\,dt\right] = \frac{2}{3}\int_1^2 \cos m\frac{2\pi}{9}t\,dt$

6.7 $c_m = \dfrac{\cos 0.25\pi m - \cos 0.75\pi m}{j\pi m}$

6.9 a)



b)



6.11 a) $F(\omega) = \int_0^2 e^{-j\omega t}\,dt - \int_2^3 2e^{-j\omega t}\,dt = \dfrac{1 - 3e^{-j\omega 2} + 2e^{-j\omega 3}}{j\omega}$

b) $f(t) = \cdots 0.4135e^{-j(4\pi t/3-\pi/3)} + 0.8270e^{-j(2\pi t/3-2\pi/3)} + 0 + 0.8270e^{+j(2\pi t/3-2\pi/3)} + 0.4135e^{+j(4\pi t/3-\pi/3)} + \cdots$

6.13 $X(\omega) = \int_{-3ms}^{0} -2e^{-j\omega t}\,dt + \int_{0}^{3ms} 2e^{-j\omega t}\,dt = \frac{4j}{\omega}(\cos(3 \times 10^{-3}\omega) - 1)$

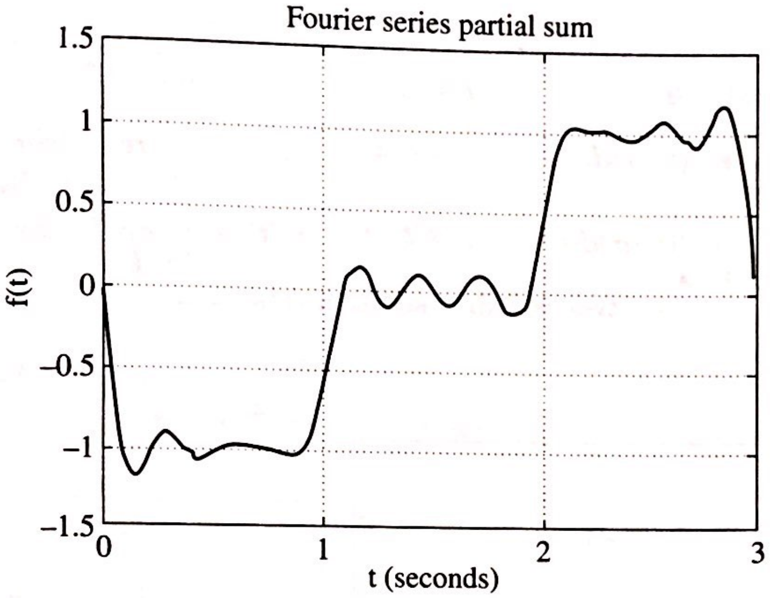
6.15

c_{-5}	c_{-4}	c_{-3}	c_{-2}	c_{-1}	c_0
−0.0320	−0.0839	−0.2652	−0.6821	−1.8243	1.0000

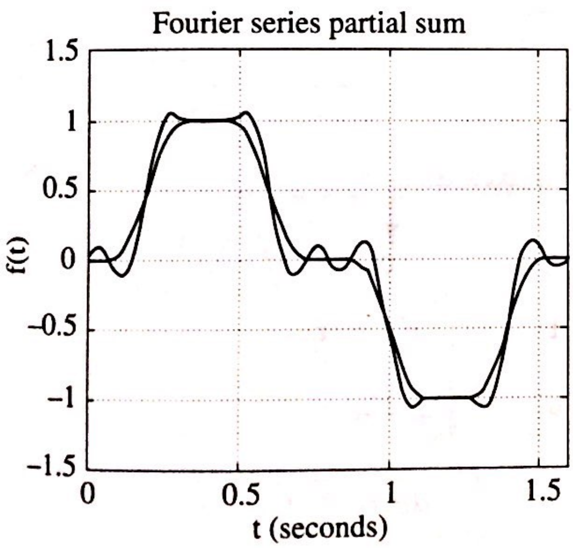
c_1	c_2	c_3	c_4	c_5
1.8243	0.6821	0.2652	0.0839	0.0320

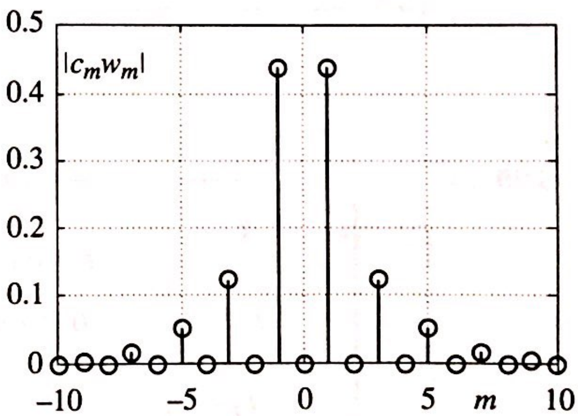
6.17 Waveforms c and d are discontinuous and need a window.

6.19



6.21 a) b)



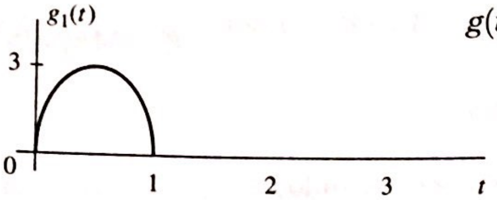


c) $P = 0.4259$

6.23 $a = 7 \quad b = 0 - 2.0000i \quad 4.000 \quad -1.0000 \quad c = \text{Error} \quad d = 3.0000 - 2.0000i$

6.25 $g_1(t) = \frac{3}{2}f(2t) \leftrightarrow \frac{3}{2}[\frac{1}{2}F(\omega/2)] = \frac{3}{4}F(\omega/2)$

$$g(t) = g_1(t-1) \leftrightarrow e^{-j\omega}G_1(\omega) = \tfrac{3}{4}e^{-j\omega}F(\omega/2)$$



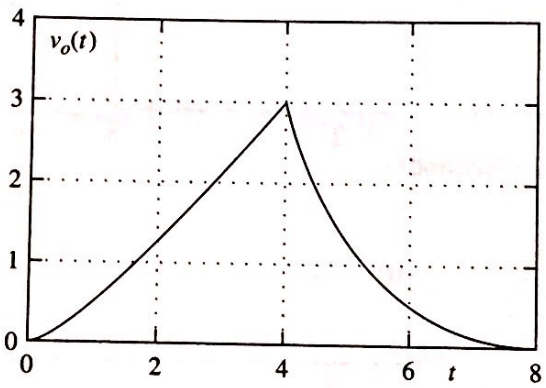
Suggestion: Perform scaling before time shifts to avoid making multiple time shifts.

6.27 a)

$$\begin{array}{lll}
v_o(t) = 0 & t < 0 & v_o(t) = 0 \\
v_o(t) = \int_0^t e^{-\sigma}(t-\sigma)d\sigma & 0 \le t \le 4 & v_o(t) = \int_0^t \sigma e^{-(t-\sigma)}d\sigma \\
v_o(t) = \int_{t-4}^t e^{-\sigma}(t-\sigma)d\sigma & t > 4 & v_o(t) = \int_0^4 \sigma e^{-(t-\sigma)}d\sigma
\end{array}$$

two possible solution sets

b)

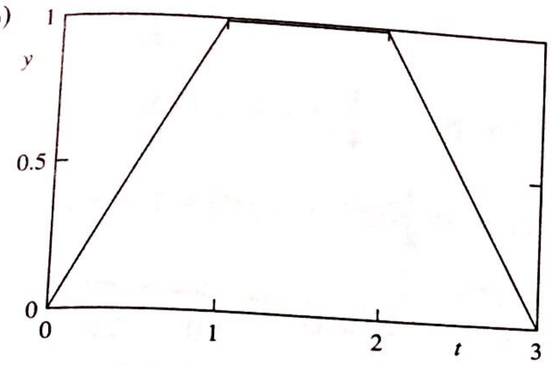


6.29 a)

or

$$y = \begin{cases}
0 & t < 0, t > 3 & 0 \\
\int_0^t d\sigma = t & 0 \le t \le 1 & \int_0^t d\sigma = t \\
\int_{t-1}^t d\sigma = 1 & 1 \le t \le 2 & \int_0^1 d\sigma = 1 \\
\int_{t-1}^2 d\sigma = 3-t & 2 \le t \le 3 & \int_{t-2}^t d\sigma = 3-t
\end{cases}$$

b)

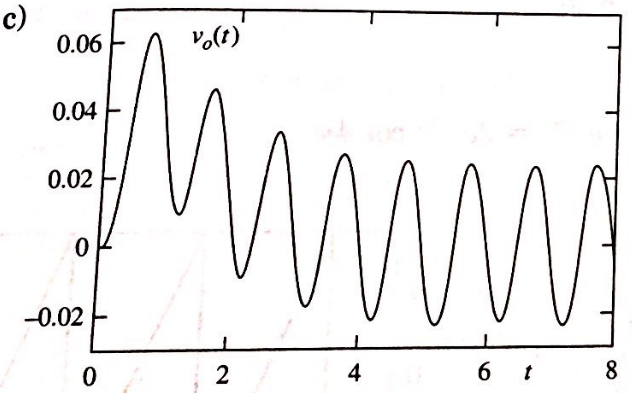


6.31 a) $H(\omega) = \dfrac{1}{2 + 3(j\omega) + (j\omega)^2}$

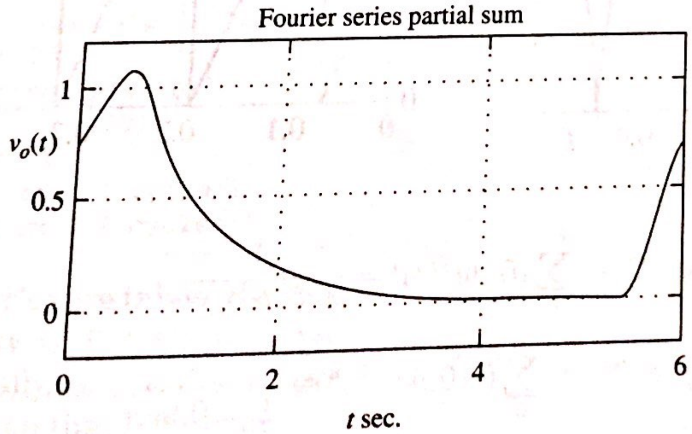
b) $$v_o(t) = \begin{cases} 0 & t < 0 \\ \int_0^t (e^{-(t-\sigma)} - e^{-2(t-\sigma)}) \sin 2\pi\sigma & t \geq 0 \end{cases}$$

or $$v_o(t) = \begin{cases} 0 & t < 0 \\ \int_0^t (e^{-\sigma} - e^{-2\sigma}) \sin 2\pi(t-\sigma) d\sigma & t \geq 0 \end{cases}$$

c)



6.33



$$c_m = \frac{1}{3}\left(\frac{\sin(m\pi/6)}{m\pi/6}\right)$$

$$\omega_o = 2\pi/6 = \pi/3$$

$$\frac{\bar{V}_o}{\bar{V}_i} = \frac{3}{4 + jm\pi}$$