Quantitative analysis of the complexity of dynamical systems*

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(Dated: August 10, 2025)

Abstract

Usage: Secondary publications and information retrieval purposes.

Structure: The paper presents a comprehensive framework for quantifying complexity in chaotic systems.

I. INTRODUCTION

The Lorenz attractor arises from a simplified model of atmospheric convection developed by Edward N. Lorenz in 1963. It is also studied in information theory, complexity science, and geometric analysis.
It is given by three coupled differential equations. The solution to these equations traces a beautiful butterfly-shaped trajectory in xyz-space. By simply observing this trajectory, we recognize that it is not a simple system. Yet quantifying this complexity remains difficult. There are some defined ways to measure complexity such as Kolmogorov complexity, López-Ruiz-Mancini-Calbet (LMC) complexity, etc. We can compute existing complexity measures, but this immediately raises a vital question: "Are these measures even sufficient for capturing the true complexity of this system? If not, what alternative approach can we discover?" If we succeed in establishing a meaningful new quantity for measuring complexity, it must carry genuine physical significance.
Simultaneously, we can explore other measurements that can describe the system like entropy, as entropy in general quantifies the unpredictability and the lack of information. One finds several entropies such as Boltzmann entropy, Shannon entropy, Kolmogorov–Sinai entropy (KS entropy), approximate entropy, sample entropy, permutation entropy, and many more. With so many entropy definitions available, we must ask: Do all entropies tell the same story, or do they reveal fundamentally different aspects? Moreover, we could seek novel kinds of entropy that might describe this system more effectively.
Furthermore, the Lorenz attractor is an example of deterministic chaos. The term deterministic chaos means that the system originates from a set of deterministic equations, yet produces behavior that is unpredictable in the long term although it is predictable in the short term. Given its chaotic nature, we might also calculate quantities like Lyapunov exponents, which explain how sensitive a dynamical system is to initial conditions, and multifractal spectrum (need to write why we use it), etc. Beyond dynamics, we may also look for geometric properties – extracted not only from the trajectory itself, but also from phase space, velocity space, and beyond. (need to write what kind of geometry we may find)
In order to calculate all these, we have three time series (x, y, z). One may study them statistically or using information theory. In either case, each time series can be studied individually, but we must understand their interdependence. Therefore, these series demand mutual analysis.
To uncover relationships between key quantities, we can employ tools like mutual information, transfer mutual information, etc. Even though the system is chaotic, we can search for periodicity using autocorrelation which measures how a signal relates to itself over time and Higuchi Fractal Dimension for estimating fractal dimension. Moreover, we need to understand how we can find geometric properties from these time series. In order to understand it we could try to find if there is any conserved quantity as energy, work etc. However as it is a bound orbit we might think if there exist any homoclinic or heteroclinic orbit. If they exist how many of them are there.

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II. LITERATURE SURVEY

A. Shannon Entropy

Shannon entropy (SE), introduced by Claude Shannon (site) that quantifies the average uncertainty of a random variable. For a discrete variable X with outcome x_i, is defined as:

$$H(X) = -\sum_i p(x_i)\log_2 p(x_i) \tag{1}$$

where $p(x_i)$ is the probability of outcome x_i.where,

$$H(X) \geq 0$$

SE calculates the entropy of entropy of symbolic sequences. High SE indicates high chaoticity. However, SE ignores geometric structure. As a result same SE values can be arrived from geometrically different structure. Besides it can not capture directional information flow.

B. Kolmogorov-Sinai Entropy

The Kolmogorov-Sinai (KS) entropy, introduced independently by Kolmogorov (1958) and Sinai (1959). It measure the rate of information production in deterministic dynamical systems which refers to the rate at which a system generates uncertainty about its future state. KS entropy is defined as:

$$h_u = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} h(P_n) \tag{2}$$

C. Rényi entropy

Rényi entropy [] is generalization of Shannon entropy which quantify uncertainty or randomness of a system using a parameter α. For a discrete probability distribution $P = (p_1, p_2, ..., p_k)$ this is defined as:

$$H_\alpha(P) = \frac{1}{1-\alpha}\log(\sum_{i=1}^{k} p_i^\alpha) \tag{3}$$

Rényi entropy is particularly useful in analyzing where a single scaling entropy measure may not suffice []. here:
$\alpha \geq 0$
$\alpha \neq 1$ (for $\alpha = 1$, it converge to Shannon entropy)

D. Transfer Entropy

Transfer entropy (TE) [] quantifies the directional flow of information between two time series. It is widely used in chaos theory, neuroscience and complex system making it highly relevant for studying Lorenz attractor. For two time series X and Y the transfer entropy from Y to X is:

$$T_{Y\to X} = \sum p(x_{t+1}, x_t^{(k)}, y_t^{(k)}) \log\left(\frac{p(x_{t+1} \mid x_t^{(k)}, y_t^{(k)})}{p(x_{t+1} \mid x_t^{(k)})}\right) \tag{4}$$

TE can identify dominant driving variable. It also detects dynamic causation. Besides Lyapunov exponents [] measure local instability and KS entropy calculates global unpredictability but not coupling structure but TE complements these by quantifying information transfer pathway.

E. Permutation Entropy

Permutation entropy (PE), introduced by Bandt and Pompe in 2002 []. It is a simple method for quantifying complexity. Unlike traditional entropy PE calculates the unpredictability or randomness of a signal by analyzing the ordinal patterns of its values. PE is broadly use is the study of chaotic system.

$$H_{PE} = -\sum_i P(\pi_i)\log P(\pi_i) \tag{5}$$

Here $P(\pi_i)$ is the probability of pattern π_i.
High PE refers irregular behavior and low PE indicates deterministic nature. PE ignores amplitude as a result weighted permutation entropy is used. Moreover, it can not handle equal values is neighborhood. For this reason we can use weighted permutation entropy.

F. Weighted Permutation Entropy

Weighted Permutation Entropy (WPE) is an advanced variant of PE. Unlike PE it includes amplitude information into complexity analysis. WPE is a powerful tool for chaotic system like Lorenz attractor as both ordinal patterns and amplitude variations play critical role here.

$$H_{WPE} = -\sum_\pi p_w(\pi)\ln p_w(\pi) \tag{6}$$

Where $p_w(\pi)$ is weighted probability of observing an ordinal pattern π in a time series.
(need to make sure the eq for wpe)
where w_i is variance $w_i = \frac{1}{m}\sum_{k=1}^{m}(x_{i+(k-1)\tau} - \overline{x}_i)$

G. Mutual Information

Mutual Information (MI) measures the nonlinear dependence between two random variables. It estimates how much knowing one variable reduces uncertainty about another. In the context of dynamical systems , it's used to study interdependence between different time series or system components. Let X and Y be two random variable and $p(x, y)$ ber joint probability and $p(x)$ and $p(y)$ be marginal probability. In that case MI is defined as:

$$I(X, Y) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} \tag{7}$$

It can also be interpreted as:

$$I(X, Y) = H(X) + H(Y) - H(X, Y) \tag{8}$$

where $H(X)$, $H(Y)$ and $H(X, Y)$ and Shannon entropy of X, Y and joint SE of X and Y respectively.

H. Higuchi Fractal Dimension

Higuchi Fractal Dimension (HFD) [] is an efficient way to estimate fractal dimention (FD) [] of a time series. It was introduced by T. Higuchi (1988). It gives a quantitative measure of complexity. Besides chaos analysis it is widely used in neuroscience and signal processing.
need to finish it

I. Autocorrelation

Autocorrelation [] computes how much a signal correlates with a time-shifted or delayed version of itself. It can be used for detecting periodicity and predictability by evaluating how its past values influence the future values of a time series.
For a time series $x_t{}_{t=1}^{N}$ and lag τ, autocorrelation $R(\tau)$ is:

$$R(\tau) = \frac{\sum_{t=1}^{N-\tau} (x_t - \overline{x})(x_{t+\tau} - \overline{x})}{\sum_{t=1}^{N} (x_t - \overline{x})^2} \tag{9}$$

Here range of $R(\tau) \in [-1, 1]$
We can use it to find periodicity in Lorenz system. Furthermore, rapid decay of $R(\tau)$ indicates high chaos and slow decay of it refers predictability.

J. Homoclinic Orbit

Its a trajectory that starts and ends at the same saddle point. Which means it is said to be Homoclinic orbit if

$$\lim_{t \to -\infty} x(t) = x_0 \tag{10}$$

and

$$\lim_{t \to \infty} x(t) = x_0 \tag{11}$$

where x_0 is a saddle equilibrium.
This types of orbits create a closed loop in phase space. Besides it is sensitive to perturbation as a result a tiny change destroy the orbit.

K. Heteroclinic Orbit

This types of trajectory connects two different saddle equilibria. It starts at a saddle point x_a as $t \to -\infty$ and ends at another saddle x_b as $t \to \infty$:

$$\lim_{t \to -\infty} x(t) = x_a and \lim_{t \to \infty} x(t) = x_b \tag{12}$$

It forms a bridge between two saddle points. However it is often seen in symmetric system.
To find these types of orbits first we need to calculate the saddle equilibrias. we need to solve the values of x, y and z for $\dot{x} = 0$, $\dot{y} = 0$ and $\dot{z} = 0$.
after solving it we get,

$$(x, y, z) = (0, 0, 0) \tag{13}$$

and

$$(x, y, z) = (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1) \tag{14}$$

L. Citations and References

1. Citations

a. Syntax
b. The options of the cite command itself

2. Example citations

3. References

4. Example references

M. Footnotes

III. METHODOLOGY

To find the geometric properties of the Lorenz system, firstly, we take cross sections of the phase space as p_x vs x, p_y vs p_z etc and observed them. We avoid terms like p_x vs y. However, we also checked if the kinetic energy of the system is conserved or not. Since, for a system usually the total energy is conserved sometimes we therefore try to calculate the work on a unit mass particle. After this we summed and subtracted this from the kinetic energy to find if these results are constant. we also try to find some average values like average of x, v_x etc. Besides, for a range of parameter's value homoclinic and heteroclinic

orbits are found. We took the values of ρ, σ and β to be [0,30] as the perameters are positive of lorenz system. With step 1, 29791 trajectories was examined for being homoclinic or heteroclinic or none of them. As these types of trajectories are vary rare we expected to find a few trajectories of these types.
As its known that for a straight line the $\ddot{\vec{r}}$ is zero and for a circle $\dot{\vec{r}}$ is zero for this reason dot product and cross product of these terms was calculated over time. If this terms converge it might be a property of the system.

IV. RESULTS AND DISCUSSION

If we analysis the slices of phase space we find that each of the 2D slices have some kind symmetry or anti-symmetry of the trajectory on that about a line. For different slices the lines are different. We believe that if it was possible to let the time goes for infinity one might find that each point on a bound space were covered.

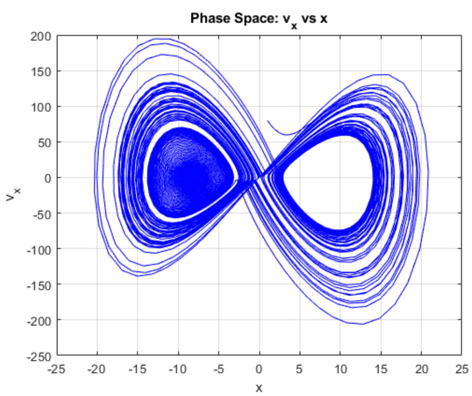

FIG. 1: Phase Space: v_x vs x showing symmetric patterns

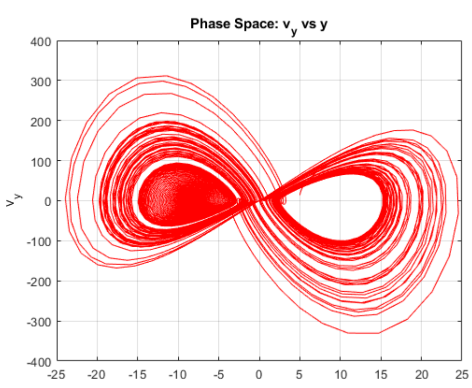

FIG. 2: Phase Space: v_y vs y demonstrating reflection symmetry

None of kinetic energy, work and their summation or subtraction are conserved. However $\langle x \rangle$, $\langle y \rangle$ and $\langle z \rangle$

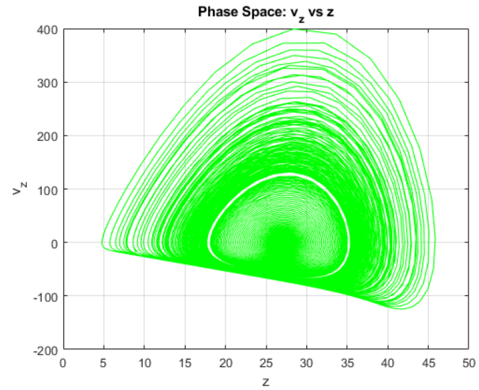

FIG. 3: Phase Space: v_z vs z with rotational symmetry

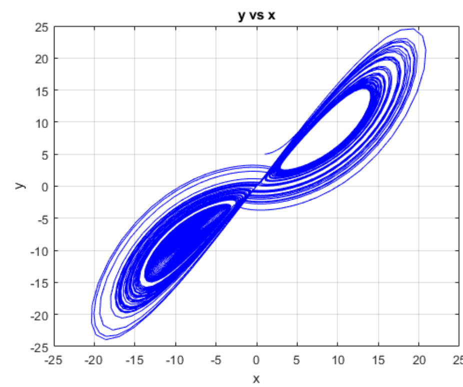

FIG. 4: Position Space: y vs x showing spatial correlation

converge to some values. which is due to their nature of orbiting around some attractor.
With step=1 for the values of ρ, σ and β in the range [0,30] we have found some values of (σ, β, ρ) for which we get the homoclinic orbits and hetero clinic orbits. Neither the dot product nor the cross product of $\ddot{\vec{r}}$ and $\dot{\vec{r}}$ converged.

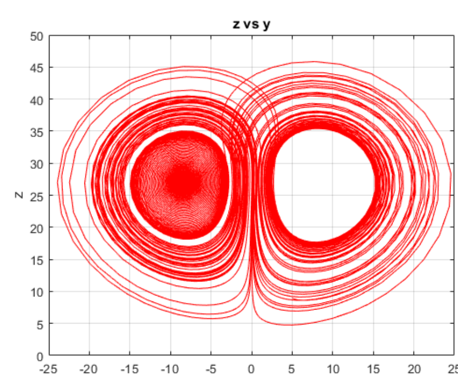

FIG. 5: Position Space: z vs y with complex interdependence

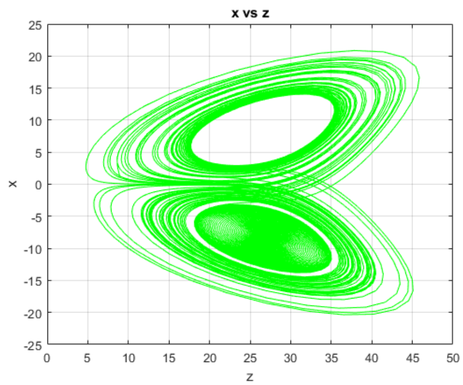

FIG. 6: Position Space: x vs z revealing trajectory patterns

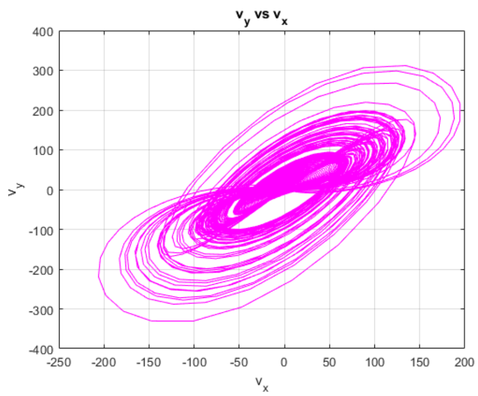

FIG. 7: Velocity Space: v_y vs v_x showing velocity correlations

Here is the plot of permutaion entropy and Shanon entropy over time

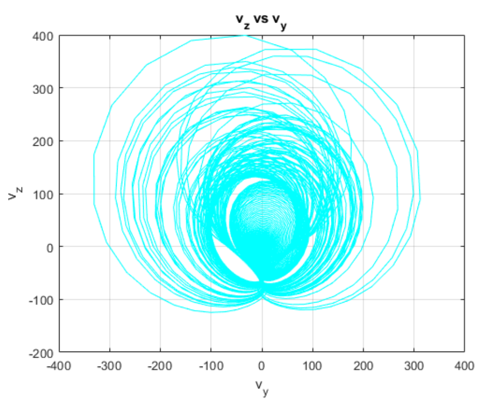

FIG. 8: Velocity Space: v_z vs v_y with rotational patterns

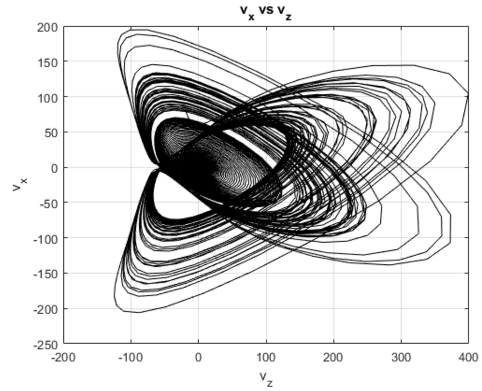

FIG. 9: Velocity Space: v_x vs v_z demonstrating symmetric structure

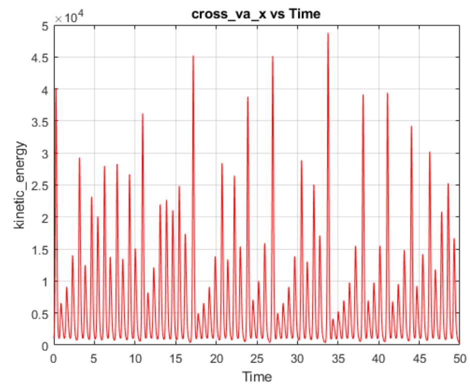

FIG. 10: Velocity Space: v_x vs v_z demonstrating symmetric structure

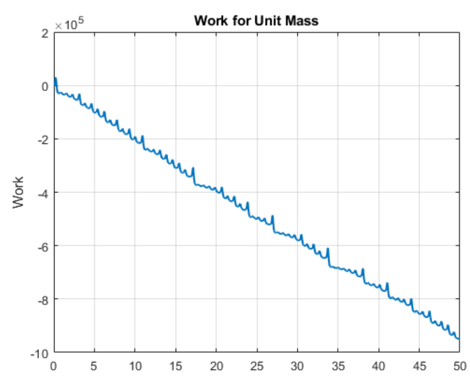

FIG. 11: Velocity Space: v_x vs v_z demonstrating symmetric structure

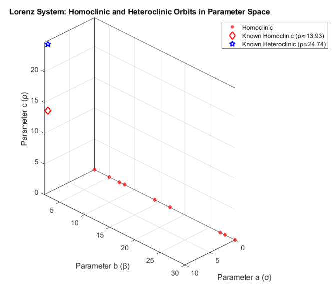

FIG. 12: Velocity Space: v_x vs v_z demonstrating symmetric structure

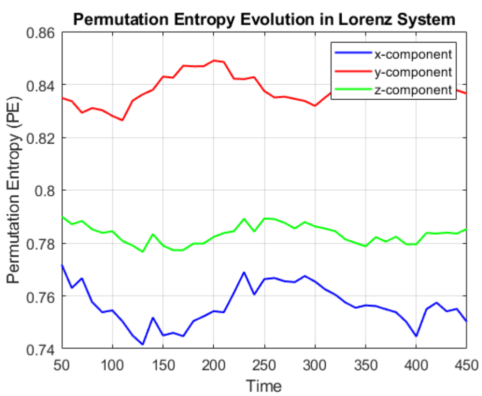

FIG. 13: Velocity Space: v_x vs v_z demonstrating symmetric structure

V. CONCLUSION

ACKNOWLEDGMENTS

We acknowledge helpful discussions with colleagues and support from our institutions. This work was supported by the XYZ Foundation (Grant No. 12345).

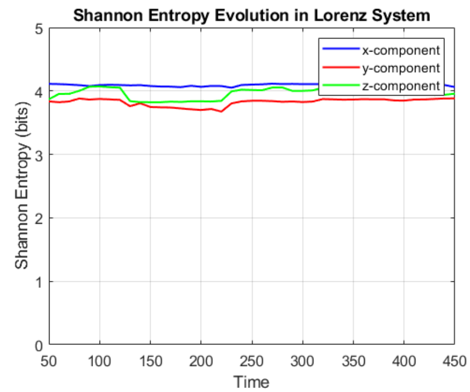

FIG. 14: Velocity Space: v_x vs v_z demonstrating symmetric structure

Appendix A: Technical Details

The appendix provides additional mathematical details omitted from the main text for readability.

$$\mathcal{R} = \sum_{i=1}^{N} \frac{x_i^2}{2\sigma^2}. \tag{A1}$$