,

Fast Fourier Transform

Revolutionizing algorithm

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Introduction to FFT

- The Fast Fourier Transform (FFT) is an algorithm to compute the Discrete Fourier Transform (DFT) and its inverse.
- It drastically reduces the computational complexity of computing DFT, making it feasible for real-time processing.
- Developed by Cooley and Tukey in 1965, FFT has become a fundamental tool in various fields such as signal processing, image processing, and more.
- This presentation aims to provide an overview of FFT, its significance, and applications.

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1. Background
2. Motivation
3. The Fast Fourier Transform (FFT)
4. Polynomial Representation
5. Evaluation
6. Interpolation
7. Conclusion

Fourier Transform

- Fourier Transform decomposes a signal into its frequency components.
- It represents a signal in terms of sinusoidal basis functions.
- The continuous Fourier Transform is given by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

where $f(t)$ is the signal, and $F(\omega)$ is its frequency domain representation.

Discrete Fourier Transform (DFT)

- DFT is the discrete counterpart of the continuous Fourier Transform.
- It is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-i2\pi kn/N}$$

  where $x[n]$ is the discrete signal, and $X[k]$ is its frequency domain representation.
- Direct computation of DFT is of $O(N^2)$ complexity.

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Motivation

- Analyze signals in the frequency domain to understand their composition.
- Limitations of the Discrete Fourier Transform (DFT):
  - Quadratic time complexity
    $$O(N^2)$$
    computationally expensive for large signals.
  - Need for a faster and more efficient algorithm.
- Applications:
  - Signal processing (noise removal, filtering,...)
  - Image processing (compression, feature extraction,...)
  - Speech and audio processing (compression, synthesis,...)
  - Scientific computing (solving differential equations, analyzing time-series data)

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The Fast Fourier Transform (FFT)

- FFT is an efficient algorithm for computing DFT.
- It exploits the periodicity and symmetry properties of sinusoidal functions to reduce the number of computations.
- The Cooley-Tukey algorithm is the most popular FFT algorithm.
- It divides the DFT computation into smaller DFTs, recursively applies FFT, and combines the results to obtain the final DFT.
- FFT reduces the computational complexity to $O(N \log N)$, making it feasible for real-time applications.

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An Optimization Problem

An Optimization Problem

Let's multiply two quadratic polynomials:

An Optimization Problem

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$$
\begin{aligned}
P(x) &= (a_0 + a_1 x + a_2 x^2) \times (b_0 + b_1 x + b_2 x^2) \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4
\end{aligned}
$$

An Optimization Problem

Let's multiply two quadratic polynomials:

$$\begin{aligned} P(x) &= (a_0 + a_1x + a_2x^2) \times (b_0 + b_1x + b_2x^2) \\ &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \end{aligned}$$

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ c_3 &= a_1 b_2 + a_2 b_1 \\ c_4 &= a_2 b_2 \end{aligned}$$

$$O(n^2)$$

An Optimization Problem

Let's multiply two quadratic polynomials:

$$P(x) = (a_0 + a_1x + a_2x^2) \times (b_0 + b_1x + b_2x^2)$$
$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

$$\begin{aligned}
c_0 &= a_0 b_0 \\
c_1 &= a_0 b_1 + a_1 b_0 \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
c_3 &= a_1 b_2 + a_2 b_1 \\
c_4 &= a_2 b_2
\end{aligned}$$

$$O(n^2)$$

Can we do better?

Polynomial Representation

Two Unique Representation for Polynomials:

$$P(x) = p_0 + p_1 x + p_1 x^2 + ... + p_d x^d$$

- Coefficient Representation:

$$[p_0, p_1, p_2, ...p_d]$$

- Value Representation:

$$[(x_0, P(x_0)), (x_1, P(x_1)), ...(x_d, P(x_d))]$$

Why Value Representation

$$A(x) = x^2 + 2x + 1$$

[(-2, 1), (-1, 0), (0,1), (1,4), (2, 9)]

$$B(x) = x^2 - 2x + 1$$

[(-2, 9), (-1,4), (0,1), (1,0), (2, 1)]

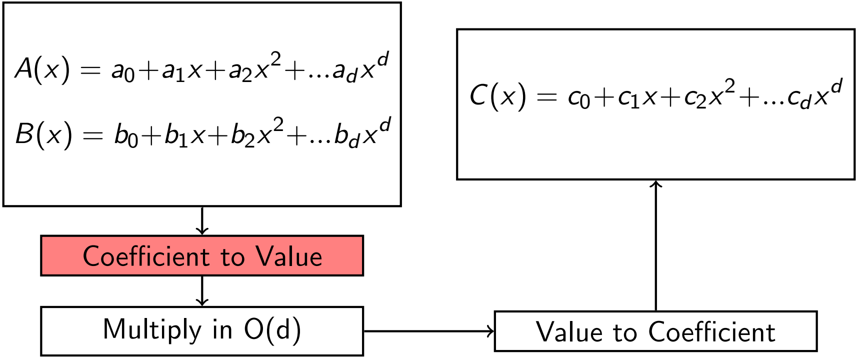
$$C(x) = A(x).B(x)$$

(-2, 1), (-1, 0), (0,1), (1,4), (2, 9)
x (-2, 9), (-1,4), (0,1), (1,0), (2, 1)

C(x) =(-2, 9), (-1,0), (0,1), (1,0), (2, 9)

Multiplication only needs O(d) time

Work Flow


$A(x) = a_0 + a_1 x + a_2 x^2 + \dots a_d x^d$
$B(x) = b_0 + b_1 x + b_2 x^2 + \dots b_d x^d$
Coefficient to Value
Multiply in O(d)
Value to Coefficient
$C(x) = c_0 + c_1 x + c_2 x^2 + \dots c_d x^d$

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Evaluation

$$P(x) = x^2$$

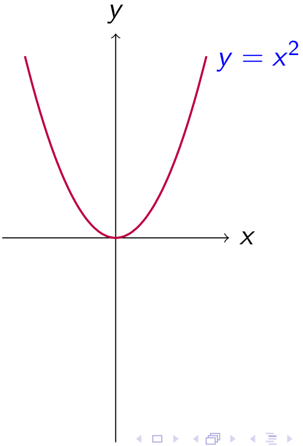
Evaluate at n = 8 points

Which Point Should We Pick?

(1,1)	(-1,1)
(2,4)	(-2,4)
(3,9)	(-3,9)
(4,16)	(-4,16)

$$P(x) = P(-x)$$

We need only 4 points!



Evaluation

$$P(x) = x^3$$

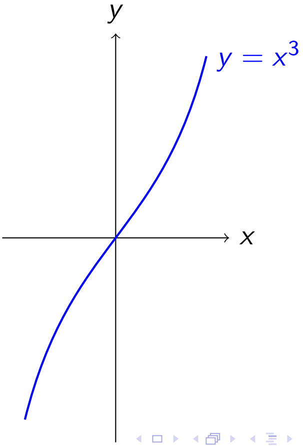
Evaluate at n = 8 points

Which Point Should We Pick?

(1,1)	(-1,-1)
(2,4)	(-2,-4)
(3,9)	(-3,-9)
(4,16)	(-4,-16)

$$P(x) = -P(-x)$$

We need only 4 points!



Evaluation

$$P(x) = p_0 + p_1 x + p_2 x^2 + ... + p_{n-1} x^{n-1}$$

Evaluate at n points $\pm x_1$,$\pm x_1$, $\pm x_2 ... \pm x_{n/2}$

$$P(x) = P_e(x^2) + xP_o(x^2)$$

$$P(x_i) = P_e(x_i^2) + xP_o(x_i^2)$$

$$P(-x_i) = P_e(x_i^2) - xP_o(x_i^2)$$

$P_e(x_i^2)$ and $P_o(x_i^2)$ have degree $n/2 - 1$

Evaluate $P_e(x_i^2)$ and $P_o(x_i^2)$ each at x_1^2, x_2^2, x_3^2, ..., $x_{n/2}^2$

Evaluation

$$P(x) = P_e(x^2) + xP_o(x^2)$$

$$P(x_i) = P_e(x_i^2) + xP_o(x_i^2)$$

$$P(-x_i) = P_e(x_i^2) - xP_o(x_i^2)$$

Points[$\pm\ x_1$, $\pm\ x_2$, $\pm\ x_3$, ... ,$\pm\ x_{n/2}$,] are $\pm$ paired.

Points[x_1^2, x_2^2, x_3^2, ... ,$x_{n/2}^2$,] are not $\pm$ paired.

Evaluation

$$P(x) = P_e(x^2) + xP_o(x^2)$$

$$P(x_i) = P_e(x_i^2) + xP_o(x_i^2)$$

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Points[$\pm\ x_1$, $\pm\ x_2$, $\pm\ x_3$, ... ,$\pm\ x_{n/2}$,] are $\pm$ paired.

Points[x_1^2, x_2^2, x_3^2, ... ,$x_{n/2}^2$,] are not $\pm$ paired.

Recursion breaks!!!

Is it possible to make [x_1^2, x_2^2, x_3^2, ... ,$x_{n/2}^2$,]$\pm$ paired?

Evaluation

$$P(x) = P_e(x^2) + xP_o(x^2)$$

$$P(x_i) = P_e(x_i^2) + xP_o(x_i^2)$$

$$P(-x_i) = P_e(x_i^2) - xP_o(x_i^2)$$

Points[$\pm\ x_1$, $\pm\ x_2$, $\pm\ x_3$, ... ,$\pm\ x_{n/2}$,] are $\pm$ paired.

Points[x_1^2, x_2^2, x_3^2, ... ,$x_{n/2}^2$,] are not $\pm$ paired.

Recursion breaks!!!

Is it possible to make [x_1^2, x_2^2, x_3^2, ... ,$x_{n/2}^2$,]$\pm$ paired?
Some of original [$\pm\ x_1$, $\pm\ x_2$, $\pm\ x_3$, ... ,$\pm\ x_{n/2}$,] need to be complex numbers!

Evaluation

$$P(x) = x^3 + x^2 - x - 1$$

We take 4 points : $\pm x_1, \pm x_2$

Evaluation

$$P(x) = x^3 + x^2 - x - 1$$

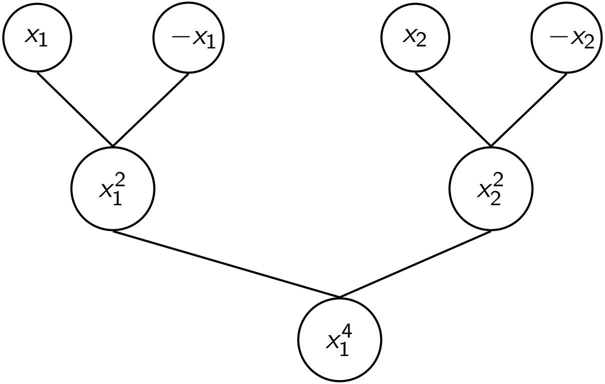
We take 4 points : $\pm x_1, \pm x_2$

x_1 $-x_1$ x_2 $-x_2$

Evaluation

$$P(x) = x^3 + x^2 - x - 1$$

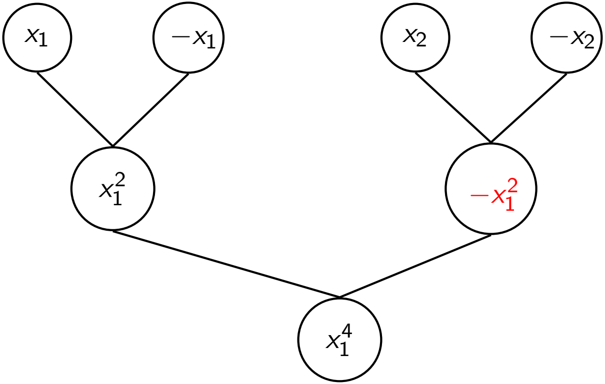
We take 4 points : $\pm x_1, \pm x_2$



Evaluation

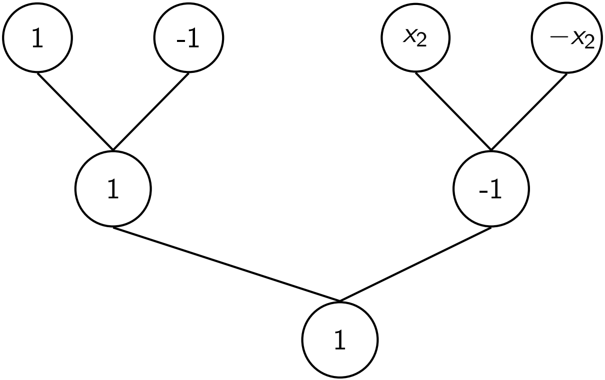
$$P(x) = x^3 + x^2 - x - 1$$

We take 4 points : $\pm x_1, \pm x_2$



Evaluation

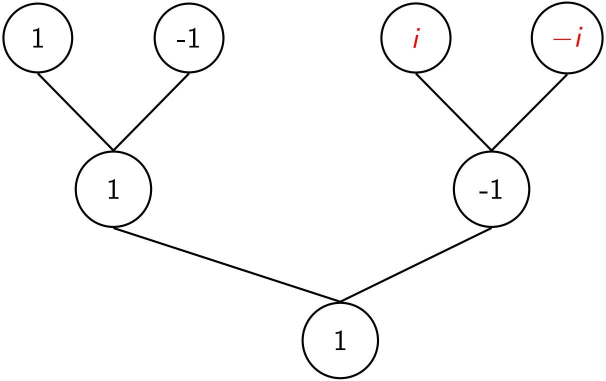
$$P(x) = x^3 + x^2 - x - 1$$



let $x_1 = 1 \implies x_2 = i$

Evaluation

$$P(x) = x^3 + x^2 - x - 1$$

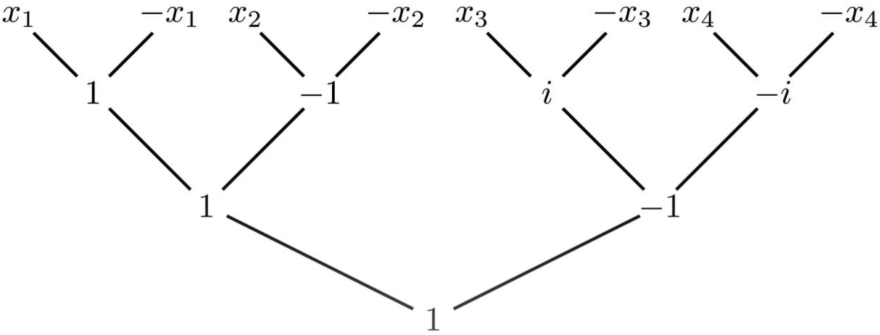


We take roots of $x^4 = 1$

Evaluation

$$P(x) = x^5 + 2x^4 - x^3 + x^2 + x + 1$$

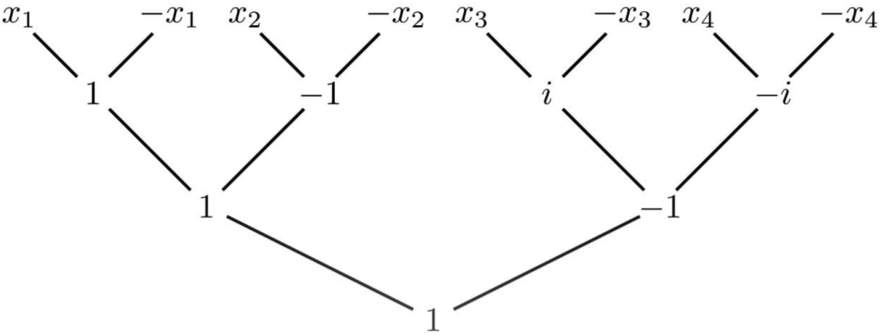
Need $n >= 6$ points, We take 8 points (powers of 2 are convenient)
Points are 8th roots of unity.



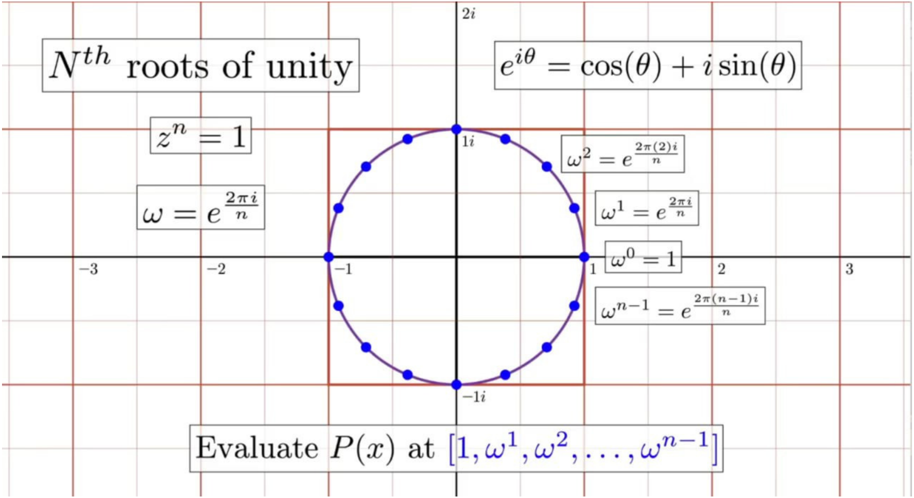
Evaluation

$$P(x) = p_0 + p_1 x + p_2 x^2 + ... + p_d x^d$$

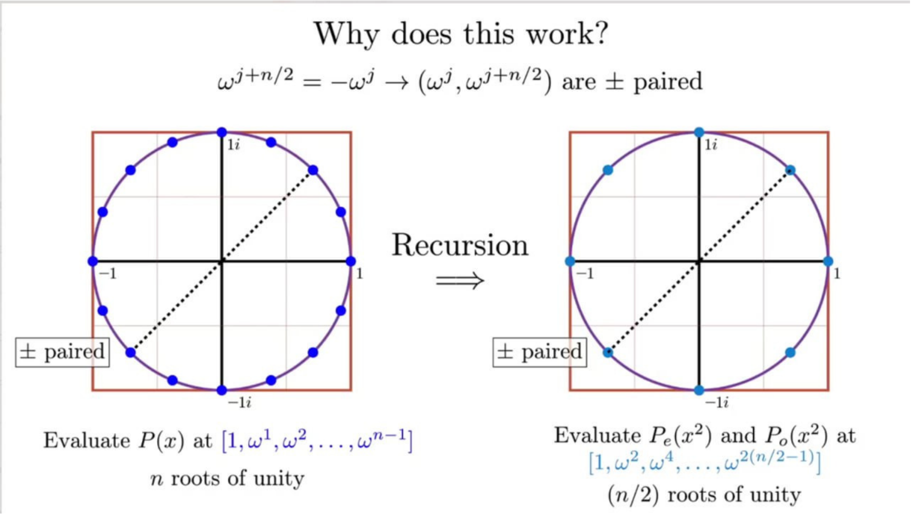
Need $n >= d + 1$ points to evaluate
where $n = 2^k, k \epsilon \mathbb{Z}$
The Points are nth roots of unity.



nth Roots of Unity



nth Roots of Unity



Implementation

$$\boxed{\text{FFT} \quad \begin{matrix} P(x) : [p_0, p_1, \ldots, p_{n-1}] \\ \omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \ldots, \omega^{n-1}] \end{matrix}}$$

$$\boxed{n = 1 \Rightarrow P(1)}$$

$$\boxed{\text{FFT} \quad \begin{matrix} P_e(x^2) : [p_0, p_2, \ldots, p_{n-2}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{matrix}} \qquad \boxed{\text{FFT} \quad \begin{matrix} P_o(x^2) : [p_1, p_3, \ldots, p_{n-1}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{matrix}}$$

$$y_e = [P_e(\omega^0), P_e(\omega^2), \ldots, P_e(\omega^{n-2})] \qquad y_o = [P_o(\omega^0), P_o(\omega^2), \ldots, P_o(\omega^{n-2})]$$

$$\boxed{\begin{matrix} P(x_j) = P_e(x_j^2) + x_j P_o(x_j^2) \\ P(-x_j) = P_e(x_j^2) - x_j P_o(x_j^2) \\ j \in \{0, 1, \ldots (n/2 - 1)\} \end{matrix}}$$

Implementation

$$\boxed{\text{FFT} \quad \begin{array}{c} P(x) : [p_0, p_1, \ldots, p_{n-1}] \\ \omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \ldots, \omega^{n-1}] \end{array}}$$

$$\boxed{n = 1 \Rightarrow P(1)}$$

$$\boxed{\text{FFT} \quad \begin{array}{c} P_e(x^2) : [p_0, p_2, \ldots, p_{n-2}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{array}} \qquad \boxed{\text{FFT} \quad \begin{array}{c} P_o(x^2) : [p_1, p_3, \ldots, p_{n-1}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{array}}$$

$$y_e = [P_e(\omega^0), P_e(\omega^2), \ldots, P_e(\omega^{n-2})] \qquad y_o = [P_o(\omega^0), P_o(\omega^2), \ldots, P_o(\omega^{n-2})]$$

$$x_j = \omega^j$$

$$\boxed{\begin{array}{c} P(\omega^j) = P_e(\omega^{2j}) + \omega^j P_o(\omega^{2j}) \\ P(-\omega^j) = P_e(\omega^{2j}) - \omega^j P_o(\omega^{2j}) \\ j \in \{0, 1, \ldots (n/2 - 1)\} \end{array}}$$

Implementation

FFT $\quad P(x) : [p_0, p_1, \ldots, p_{n-1}]$

$\omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \ldots, \omega^{n-1}]$

$$\boxed{n = 1 \Rightarrow P(1)}$$

FFT $\quad P_e(x^2) : [p_0, p_2, \ldots, p_{n-2}]$

$[\omega^0, \omega^2, \ldots, \omega^{n-2}]$

$$y_e = [P_e(\omega^0), P_e(\omega^2), \ldots, P_e(\omega^{n-2})]$$

FFT $\quad P_o(x^2) : [p_1, p_3, \ldots, p_{n-1}]$

$[\omega^0, \omega^2, \ldots, \omega^{n-2}]$

$$y_o = [P_o(\omega^0), P_o(\omega^2), \ldots, P_o(\omega^{n-2})]$$

$$x_j = \omega^j$$

$$-\omega^j = \omega^{j+n/2}$$

$$P(\omega^j) = P_e(\omega^{2j}) + \omega^j P_o(\omega^{2j})$$

$$P(\omega^{j+n/2}) = P_e(\omega^{2j}) - \omega^j P_o(\omega^{2j})$$

$$j \in \{0, 1, \ldots (n/2 - 1)\}$$

$$y_e[j] = P_e(\omega^{2j})$$

$$y_o[j] = P_o(\omega^{2j})$$

Implementation

$$\boxed{\text{FFT} \quad \begin{matrix} P(x) : [p_0, p_1, \ldots, p_{n-1}] \\ \omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \ldots, \omega^{n-1}] \end{matrix}}$$

$$\boxed{n = 1 \Rightarrow P(1)}$$

$$\boxed{\text{FFT} \quad \begin{matrix} P_e(x^2) : [p_0, p_2, \ldots, p_{n-2}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{matrix}} \qquad \boxed{\text{FFT} \quad \begin{matrix} P_o(x^2) : [p_1, p_3, \ldots, p_{n-1}] \\ [\omega^0, \omega^2, \ldots, \omega^{n-2}] \end{matrix}}$$

$$y_e = [P_e(\omega^0), P_e(\omega^2), \ldots, P_e(\omega^{n-2})] \qquad y_o = [P_o(\omega^0), P_o(\omega^2), \ldots, P_o(\omega^{n-2})]$$

$$x_j = \omega^j$$

$$-\omega^j = \omega^{j+n/2}$$

$$\boxed{\begin{matrix} P(\omega^j) = y_e[j] + \omega^j y_o[j]) \\ P(\omega^{j+n/2}) = y_e[j] - \omega^j y_o[j]) \\ j \in \{0, 1, \ldots (n/2 - 1)\} \end{matrix}}$$

$$y_e[j] = P_e(\omega^{2j})$$

$$y_o[j] = P_o(\omega^{2j})$$

$$\boxed{y = [P(\omega^0), P(\omega^1), \ldots, P(\omega^{n-1})]}$$

Implementation

```
def FFT(P):
    # P - [p_0, p_1, ..., p_{n-1}] coeff representation
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = e^(2πi/n)
    P_e, P_o = [p_0, p_2, ..., p_{n-2}], [p_1, p_3, ..., p_{n-1}]
    y_e, y_o = FFT(P_e), FFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] − ω^j y_o[j]
    return y
```

FFT $P(x) : [p_0, p_1, \ldots, p_{n-1}]$, $\omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \ldots, \omega^{n-1}]$

$n = 1 \Rightarrow P(1)$

FFT $P_e(x^2) : [p_0, p_2, \ldots, p_{n-2}]$, $[\omega^0, \omega^2, \ldots, \omega^{n-2}]$

$y_e = [P_e(\omega^0), P_e(\omega^2), \ldots, P_e(\omega^{n-2})]$

FFT $P_o(x^2) : [p_1, p_3, \ldots, p_{n-1}]$, $[\omega^0, \omega^2, \ldots, \omega^{n-2}]$

$y_o = [P_o(\omega^0), P_o(\omega^2), \ldots, P_o(\omega^{n-2})]$

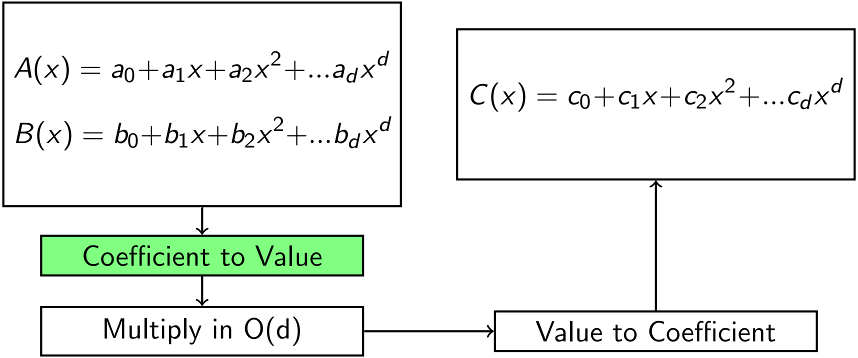
$P(\omega^j) = y_e[j] + \omega^j y_o[j])$

$P(\omega^{j+n/2}) = y_e[j] - \omega^j y_o[j])$

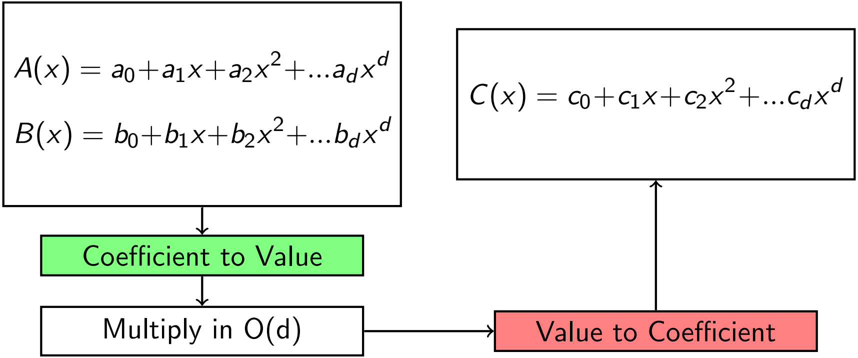
$j \in \{0, 1, \ldots (n/2 - 1)\}$

$y = [P(\omega^0), P(\omega^1), \ldots, P(\omega^{n-1})]$

Work Flow


$A(x) = a_0 + a_1x + a_2x^2 + \ldots a_dx^d$
$B(x) = b_0 + b_1x + b_2x^2 + \ldots b_dx^d$
Coefficient to Value
Multiply in O(d)
Value to Coefficient
$C(x) = c_0 + c_1x + c_2x^2 + \ldots c_dx^d$

Work Flow



$A(x) = a_0 + a_1x + a_2x^2 + \dots a_dx^d$
$B(x) = b_0 + b_1x + b_2x^2 + \dots b_dx^d$
Coefficient to Value
Multiply in O(d)
Value to Coefficient
$C(x) = c_0 + c_1x + c_2x^2 + \dots c_dx^d$

Work Flow

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_d x^d$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots b_d x^d$$

Coefficient to Value

Multiply in O(d)

Value to Coefficient

$$C(x) = c_0 + c_1 x + c_2 x^2 + \ldots c_d x^d$$

HOW!

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Interpolation

Alternative Perspective on Evaluation/FFT

$$P(x) = p_0 + p_1 x + p_2 x^2 + ... + p_{n-1} x^{n-1}$$

$$P(x_0) = p_0 + p_1 x_0 + p_2 x_0^2 + ... + p_{n-1} x_0^{n-1}$$

$$P(x_1) = p_0 + p_1 x_1 + p_2 x_1^2 + ... + p_{n-1} x_1^{n-1}$$

$$P(x_2) = p_0 + p_1 x_2 + p_2 x_2^2 + .. + p_{n-1} x_2^{n-1}$$

$$P(x_{n-1}) = p_0 + p_1 x_{n-1} + p_2 x_{n-1}^2 + ... + p_{n-1} x_{n-1}^{n-1}$$

Alternative Perspective on Evaluation

$$P(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_{n-1} x^{n-1}$$

$$\begin{bmatrix} P(x_0) \\ P(x_1) \\ P(x_2) \\ \vdots \\ P(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix}$$

$$x_k = \omega^k, \text{ where } \omega = e^{\frac{2\pi i}{n}}$$

Interpolation

Alternative Perspective on Evaluation/FFT

$$P(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_{n-1} x^{n-1}$$

$$\begin{bmatrix} P(\omega^0) \\ P(\omega^1) \\ P(\omega^2) \\ \vdots \\ P(\omega^{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}}_{\text{Discrete Fourier Transform (DFT) matrix}} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix}$$

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Interpolation

$$x_k = \omega^k \text{ where } \omega = e^{\frac{2\pi i}{n}}$$

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$$\Downarrow$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} P(\omega^0) \\ P(\omega^1) \\ P(\omega^2) \\ \vdots \\ P(\omega^{n-1}) \end{bmatrix}$$

The inverse matrix and original matrix look quiet similar!

Every ωin original matrix is now $\frac{1}{n}\omega^{-}1$

Interpolation

$$x_k = \omega^k \text{ where } \omega = e^{\frac{2\pi i}{n}}$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} P(\omega^0) \\ P(\omega^1) \\ P(\omega^2) \\ \vdots \\ P(\omega^{n-1}) \end{bmatrix}$$

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The inverse matrix and original matrix look quiet similar!

Every ωin original matrix is now $\frac{1}{n}\omega^-1$

$$
\begin{aligned}
&\text{def FFT}(P):\\
&\quad \#\ P\ \text{-}\ [p_0, p_1, \ldots, p_{n-1}]\ \text{coeff rep}\\
&\quad n = \text{len}(P)\ \#\ n \text{ is a power of } 2\\
&\quad \text{if } n == 1:\\
&\quad\quad \text{return } P\\
&\quad \omega = e^{\frac{2\pi i}{n}}\\
&\quad P_e, P_o = P[::2], P[1::2]\\
&\quad y_e, y_o = \text{FFT}(P_e), \text{FFT}(P_o)\\
&\quad y = [0] * n\\
&\quad \text{for } j \text{ in range}(n/2):\\
&\quad\quad y[j] = y_e[j] + \omega^j y_o[j]\\
&\quad\quad y[j + n/2] = y_e[j] - \omega^j y_o[j]\\
&\quad \text{return } y
\end{aligned}
$$

Interpolation

$$\text{IFFT}(\textless\text{values}\textgreater) \Leftrightarrow \text{FFT}(\textless\text{values}\textgreater) \text{ with } \omega = \frac{1}{n} e^{\frac{-2\pi i}{n}}$$

```
def FFT(P):
    # P - [p_0, p_1, ..., p_{n-1}] coeff rep
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = e^(2πi/n)
    P_e, P_o = P[::2], P[1::2]
    y_e, y_o = FFT(P_e), FFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

```
def IFFT(P):
    # P - [P(ω^0), P(ω^1), ..., P(ω^(n-1))] value rep
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = (1/n) * e^(-2πi/n)
    P_e, P_o = P[::2], P[1::2]
    y_e, y_o = IFFT(P_e), IFFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

Example Problems

All possible sums

We are given two arrays $a[]$ and $b[]$. We have to find all possible sums $a[i] + b[j]$, and for each sum count how often it appears.
For example for $a = [1,\ 2,\ 3]$ and $b = [2,\ 4]$ we get: then sum 3 can be obtained in 1 way, the sum 4 also in 1 way, 5 in 2, 6 in 1, 7 in 1.

Hint :

Construct for the arrays a and b two polynomials A and B. The numbers of the array will act as the exponents in the polynomial ($a[i] \Rightarrow x^{a[i]}$); and the coefficients of this term will be how often the number appears in the array.

Example Problems

String matching

We are given two strings, a text T and a pattern P, consisting of lowercase letters. We have to compute all the occurrences of the pattern in the text.

Hint :

Create a polynomial for each string ($T[i]$ and $P[I]$ are numbers between 0 and 25 corresponding to the 26 letters of the alphabet):

$$A(x) = a_0x^0 + a_1x^1 + \cdots + a_{n-1}x^{n-1}, \quad n = |T|$$

with $a_i = \cos(\alpha_i) + i\sin(\alpha_i), \quad \alpha_i = \frac{2\pi T[i]}{26}$.
And

$$B(x) = b_0x^0 + b_1x^1 + \cdots + b_{m-1}x^{m-1}, \quad m = |P|$$

with $b_i = \cos(\beta_i) - i\sin(\beta_i), \quad \beta_i = \frac{2\pi P[m-i-1]}{26}$.

More Problems to try:

POLYMUL - Polynomial Multiplication

MAXMATCH - Maximum Self-Matching

ADAMATCH - Ada and Nucleobase

Yet Another String Matching Problem

Lightsabers (hard)

Running Competition

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1. Background
2. Motivation
3. The Fast Fourier Transform (FFT)
4. Polynomial Representation
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7. Conclusion

Conclusion

- FFT is a powerful algorithm for efficiently computing the Discrete Fourier Transform.
- It has revolutionized various fields by enabling fast and accurate frequency domain analysis.
- Understanding FFT and its applications is essential for anyone working in signal processing, communications, image processing, and related domains.

Thank You for your patience!