# Lecture 2: The Bellman Equation

## Dynamic Programming

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# The Simplest Consumption Model

- **Model:**

$$\begin{aligned} V_t^\star(M) &= \max_{C_t, C_{t+1}, \dots, C_T} \left\{ C_t + \beta C_{t+1} + \dots + \beta^{T-t} C_T \right\} \\ &\text{s.t.} \\ M &= C_t + C_{t+1} + \dots + C_T \\ C_t &\in \mathbb{N} \end{aligned}$$

- **Solution by backwards induction:**

$$\begin{aligned} V_T^\star(M) &= \sqrt{M} \\ V_{T-1}^\star(M) &= \max_{C_{T-1}} \left\{ \sqrt{C_{T-1}} + \beta V_T^\star(M - C_{T-1}) \right\} \\ V_{T-2}^\star(M) &= \max_{C_{T-2}} \left\{ \sqrt{C_{T-2}} + \beta V_{T-1}^\star(M - C_{T-2}) \right\} \\ V_{T-3}^\star(M) &= \max_{C_{T-3}} \left\{ \sqrt{C_{T-3}} + \beta V_{T-2}^\star(M - C_{T-3}) \right\} \\ &\dots \\ V_t^\star(M) &= \max_{C_t} \left\{ \sqrt{C_t} + \beta V_{t+1}^\star(M - C_t) \right\} \end{aligned}$$

**Algorithm 4:** Find $V_t^\star$ given $V_{t+1}^\star$ [find_V]

**input :** $M$
$V_{t+1}^\star[\bullet]$
**output:** $V_t^\star[\bullet]$ (value of optimal choice)
$C_t^\star[\bullet]$ (optimal choice)

**for** $m = 0$ **to** $M$ **do**
- $i_M = m + 1$
- $V_t^\star[i_M] = -\infty$
- **for** $C_t = 0$ **to** $m$ **do**
  - $V = \sqrt{C_t} + \beta V_{t+1}^\star[(m - C_t) + 1]$
  - **if** $V > V_t^\star[i_M]$ **then**
    - $V_t^\star[i_M] = V$
    - $C_t^\star[i_M] = C_t$

**Algorithm 5:** Find all $V_t^\star$ (algorithm 4 inserted)

**input** : $M$
**output:** $V_t^\star[\bullet]$ for all $t$
$C_t^\star[\bullet]$ for all $t$

$V_T^\star[m+1] = \sqrt{m}$ for all $m = 0$ **to** $M$
**for** $t = T-1$ **to** $1$ **do**
    **for** $m = 0$ **to** $M$ **do**
        $i_M = m + 1$
        $V_t^\star[i_M] = -\infty$
        **for** $C_t = 0$ **to** $m$ **do**
            $V = \sqrt{C_t} + \beta V_{t+1}^\star[(m - C_t) + 1]$
            **if** $V > V_t^\star[i_M]$ **then**
                $V_t^\star[i_M] = V$
                $C_t^\star[i_M] = C_t$

[From last lecture: **b.socrative.com**, room: **DynamicProgramming** (all but the last question)]

# Slight reformulation I

- **Instead of:**

$$
\begin{aligned}
V_t^\star(M) &= \max_{C_t, C_{t+1}, \ldots, C_T} \left\{ \sqrt{C_t} + \beta\sqrt{C_{t+1}} + \beta^2\sqrt{C_{t+2}} + \cdots + \beta^{T-t}\sqrt{C_T} \right\} \\
&\text{s.t.} \\
M &= C_t + C_{t+1} + \cdots + C_T \\
C_t &\in \mathbb{N}
\end{aligned}
$$

- **We will write:**

$$
\begin{aligned}
V_t^\star(M_t) &= \max_{C_t, C_{t+1}, \ldots, C_T} \left\{ \sqrt{C_t} + \beta\sqrt{C_{t+1}} + \beta^2\sqrt{C_{t+2}} + \cdots + \beta^{T-t}\sqrt{C_T} \right\} \\
&\text{s.t.} \\
M_{t+1} &= M_t - C_t \\
C_t &\leq M_t \\
C_t &\in \mathbb{N} \\
M_t &= M
\end{aligned}
$$

# Slight reformulation II

- Start from

$$V_t^{\star}(M_t) = \max_{C_t, C_{t+1}, \dots, C_T} \left\{ \sqrt{C_t} + \beta\sqrt{C_{t+1}} + \beta^2\sqrt{C_{t+2}} + \cdots + \beta^{T-t}\sqrt{C_T} \right\}$$

- As $C_{t+1}, \dots, C_T$ does not affect $C_t$ we can write

$$\begin{aligned} V_t^{\star}(M_t) &= \max_{C_t} \left\{ \sqrt{C_t} + \beta \left\{ \max_{C_{t+1}, \dots, C_T} \sqrt{C_{t+1}} + \beta\sqrt{C_{t+2}} + \cdots + \beta^{T-(t+1)}\sqrt{C_T} \right\} \right\} \\ &= \max_{C_t} \left\{ \sqrt{C_t} + \beta V_{t+1}^{\star}(M_{t+1}) \right\} \end{aligned}$$

- **What do I use in the second equality?**
- **Mathematical details:** See Lucas and Stokey theorem 4.2.

# Bellman's Principle of Optimality

***Principle of Optimality:*** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)*

# Model with stochastic shocks

- Consider the following **Bellman equation**

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t[V_{t+1}(M_{t+1})] \\
&\text{s.t.} \\
C_t &\leq M_t \\
M_{t+1} &= \begin{cases} M_t - C_t + 1 & \text{with probability } \pi \in (0,1) \\ M_t - C_t & \text{else} \end{cases}
\end{aligned}
$$

where $\mathbb{E}_t[\bullet] \equiv \mathbb{E}[\bullet | M_t, C_t]$

- **Question 1**: Can we try out all combinations of $C_t, C_{t+1}, \ldots, C_T$?
- **Question 2**: Can we write down a formula for $\mathbb{E}_t[V_{t+1}(M_{t+1})]$?

Go to **b.socrative.com**, room name **DynamicProgramming [Q1 only]**

# Formula for $\mathbb{E}_t[V_{t+1}(M_{t+1})]$

- **We have:**

$$M_{t+1} = \begin{cases} M_t - C_t + 1 & \text{with probability } \pi \in (0,1) \\ M_t - C_t & \text{else} \end{cases}$$

- **Possibility 1:**

$$\mathbb{E}_t[V_{t+1}(M_{t+1})] = \pi V_{t+1}(M_t - C_t + 1)$$

- **Possibility 2:**

$$\mathbb{E}_t[V_{t+1}(M_{t+1})] = V_{t+1}(\pi(M_t - C_t + 1) + (1-\pi)(M_t - C_t))$$

- **Possibility 3:**

$$\mathbb{E}_t[V_{t+1}(M_{t+1})] = \pi V_{t+1}(M_t - C_t + 1) + (1-\pi) V_{t+1}(M_t - C_t)$$

**Algorithm 6:** Find $V_t^\star$ given $V_{t+1}^\star$ [find_V]

**input :** $M$
$V_{t+1}^\star[\bullet]$
**output:** $V_t^\star[\bullet]$ (value of optimal choice)
$C_t^\star[\bullet]$ (optimal choice)

**for** $m = 0$ **to** $M + t$ **do**
  $i_M = m + 1$
  $V_t^\star[i_M] = -\infty$
  **for** $C_t = 0$ **to** $m$ **do**
    $i\_next = (m - C_t) + 1$
    $V = \sqrt{C_t} + \beta V_{t+1}^\star[i\_next]$ (if deterministic)
    $i\_next\_plus = (m - C_t) + 1 + 1$
    $EV_{t+1} = (1 - \pi) V_{t+1}^\star[i\_next] + \pi V_{t+1}^\star[i\_next\_plus]$
    $V = \sqrt{C_t} + \beta EV_{t+1}$ (if stochastic)
    **if** $V > V_t^\star[i_M]$ **then**
      $V_t^\star[i_M] = V$
      $C_t^\star[i_M] = C_t$

# Model with continuous consumption

- Consider the following **Bellman equation**

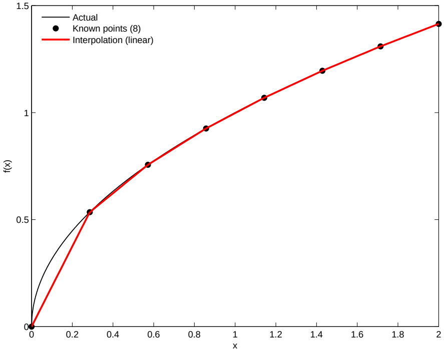
$$\begin{aligned} V_t(M_t) &= \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t[V_{t+1}(M_{t+1})] \\ &\text{s.t.} \\ C_t &\leq M_t \\ M_{t+1} &= M_t - C_t \end{aligned}$$

  where we now allow $C_t$ to be continuous, i.e. $C_t \in \mathbb{R}$ and not just $C_t \in \mathbb{N}$.
- **Remember:** We only know $V_{t+1}(M_{t+1})$ at some *grid*, $M_{t+1} \in \{0, 1, 2, 3, \ldots, M\} \equiv \mathcal{G}$
- **Question:** What is $V_{t+1}(0.5)$?

# Linear Interpolation

- **Information:**
  1. $f(x)$ is known for $x$ in some set $\mathcal{G}$
  2. $\mathcal{G}$ is ordered *low to high*
- **Task:** linear interpolation at $x \notin \mathcal{G}$:

Figure: Linear interpolation



# Linear Interpolation

- **Information:**
  1. $f(x)$ is known for $x$ in some set $\mathcal{G}$
  2. $\mathcal{G}$ is ordered *low to high*
- **Task:** linear interpolation at $x \notin \mathcal{G}$:
- **Step 1: Neighbors.** Find $x_n \in \mathcal{G}$ such that

$$x_n \leq x < x_{n+1}.$$

- **Step 2: Relative distance**. Compute

$$\omega = \frac{x - x_n}{x_{n+1} - x_n}$$

- **Step 3: Interpolation value**. Compute

$$\begin{aligned} f(x) &\approx (1-\omega)f(x_n) + \omega f(x_{n+1}) \\ &= f(x_n) + \omega(f(x_{n+1}) - f(x_n)) \end{aligned}$$

**Algorithm 7:** Linear interpolation [called "interp" later]

**input :** $x$ (point to be interpolated)
$\mathcal{G}$ (grid of length # where the function is known)
$f[\bullet]$ (function values at grid points)
**output:** $y$ (interpolated value)
1 **if** $x < \mathcal{G}[1]$ **then**
2 $\quad n = 1$ (extrapolate below)
3 **else if** $x > \mathcal{G}[\#]$ **then**
4 $\quad n = \# - 1$ (extrapolate above)
5 **else**
6 $\quad n = 1$
7 $\quad$ **while** $x$ ? $\quad \mathcal{G}[n+1]$ **do**
8 $\quad\quad n = n + 1$
9 $\omega = \frac{x - x[n]}{x[n+1] - x[n]}$
10 $y = f[n] + \omega(f[n+1] - f[n])$

- What should be instead of "?" on line 7: $\leq, \geq$?
  - $\geq$ because when this requirement is violated for the first time, $x$ is just below $\mathcal{G}[n+1]$ and above $\mathcal{G}[n]$

**Algorithm 8:** Find $V_t^\star$ given $V_{t+1}^\star$ [find_V]

**input** : $\mathcal{G}_M$ (grid for $M$ with $\#_M$ elements)
$V_{t+1}^\star[\bullet]$
**output:** $V_t^\star[\bullet]$ (value of optimal choice)
$C_t^\star[\bullet]$ (optimal choice)

```
for i_M = 1 to #_M do
    V*_t[i_M] = -∞
    M_t = G_M[i_M]
    for C_t = 0 to M_t do
        V = sqrt(C_t) + β · interp(M_t − C_t, G_M, V*_{t+1})
        if V > V*_t[i_M] then
            V*_t[i_M] = V
            C*_t[i_M] = C_t
```

$$
\begin{aligned}
&\textbf{for } i_M = 1 \textbf{ to } \#_M \textbf{ do}\\
&\quad V_t^\star[i_M] = -\infty\\
&\quad M_t = \mathcal{G}_M[i_M]\\
&\quad \textbf{for } C_t = 0 \textbf{ to } M_t \textbf{ do}\\
&\quad\quad V = \sqrt{C_t} + \beta \cdot \text{interp}(M_t - C_t, \mathcal{G}_M, V_{t+1}^\star)\\
&\quad\quad \textbf{if } V > V_t^\star[i_M] \textbf{ then}\\
&\quad\quad\quad V_t^\star[i_M] = V\\
&\quad\quad\quad C_t^\star[i_M] = C_t
\end{aligned}
$$

- **Or is something not quite right?**

# Summary

- **The Bellman equation** was introduced. In general notation

$$V_t(M_t) = \max_{C_t \in \mathcal{C}(M_t)} u(C_t) + \beta \mathbb{E}_t[V_{t+1}(\Gamma(M_t, C_t))]$$

- You can now solve models with:
  1. Stochastic shocks
  2. Continuous choices and/or states
- You can also do both at once...

**Algorithm 9:** Find $V_t^\star$ given $V_{t+1}^\star$ [find_V]

**input** : $V_{t+1}^\star[\bullet]$
$\mathcal{G}_M$ (grid for $M$ with $\#_M$ elements)
$\mathcal{G}_C$ (grid for C (as a share of M) with $\#_C$ elements in $(0,1)$)

**output:** $V_t^\star[\bullet]$ (value of optimal choice
$C_t^\star[\bullet]$ (optimal choice)

**for** $i_M = 1$ **to** $\#_M$ **do**
  $V_t^\star[i_M] = -\infty$
  $M_t = \mathcal{G}_M[i_M]$
  **for** $i_c = 1$ **to** $\#_C$ **do**
    $C_t = \mathcal{G}_C[i_C]M_t$
    $EV_{t+1} = \pi \text{interp}(M_t - C_t + 1, \mathcal{G}_M, V_{t+1}^\star)$
    $\quad + (1-\pi)\text{interp}(M_t - C_t, \mathcal{G}_M, V_{t+1}^\star)$
    $V = \sqrt{C_t} + \beta EV_{t+1}$
    **if** $V > V_t^\star[i_M]$ **then**
      $V_t^\star[i_M] = V$
      $C_t^\star[i_M] = C_t$

- **Could you do something different?**

# Until next

- **Ensure that you understand:**
  1. Algorithm 7-9
  2. And why we had

$$\mathbb{E}_t[V_{t+1}(M_{t+1})] = \pi V_{t+1}(M_t - C_t + 1) + (1-\pi)V_{t+1}(M_t - C_t)$$

- Go to **PadLet** and ask or answer a question (https://padlet.com/thomas_jorgensen1/DP)
- **Think about:**
  1. How can we compute $\mathbb{E}_t[V_{t+1}(M_{t+1})]$ if:

$$\begin{aligned} M_{t+1} &= R(M_t - C_t) + Y_{t+1} \\ Y_{t+1} &= \exp(\xi_{t+1}) \\ \xi_{t+1} &\sim \mathcal{N}(0, \sigma_\xi^2) \end{aligned}$$

- **Now:** go to **b.socrative.com** room name "**DynamicProgramming**" and answer a few questions