Lecture 7-8: Consumption-Saving

Dynamic Programming

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Introduction

- **My two guest lectures:** Consumption-saving
  1. Important topic in itself (70 percent of GDP)
  2. Central aspect of many other decisions
     a) Labor supply
     b) Portfolio choice
     c) Housing
- **Dynamic programming** essential for recent advances
  1. Idiosyncratic and aggregate uncertainty
  2. Ex ante and ex post heterogeneity
  3. Internal and external optimization frictions (bounded rationality, adjustment costs etc.)
- **My own research** in a nutshell
  1. Macro questions (with a focus on consumption-saving)
  2. Micro data
  3. Computational methods
- **My own teaching:** Introduction to programming and numerical analysis (in Python)
- **Plan**
  1. **First part:** The buffer-stock consumption model
  2. **Second part:** General equilibrium with heterogeneous agents

Permanent Income Hypothesis (PIH)

- **Household problem**

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho}, \;\beta < 1, \rho \geq 1 \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
B_{t+1} &= R \cdot A_t \\
M_{t+1} &= B_{t+1} + P_{t+1} \\
P_{t+1} &= G \cdot P_t \\
A_T &\geq 0
\end{aligned}
$$

- **Assumptions**
  1. Return impatience (RI): $(\beta R)^{1/\rho}/R < 1$
  2. Finite human wealth (FHW): $G/R < 1$
- **What do you think is missing?**

The Intertemporal Budget Constraint (IBC)

- **Substitution** implies

$$
\begin{aligned}
A_T &= M_T - C_T = (RA_{T-1} + P_T) - C_T \\
&= R(M_{T-1} - C_{T-1}) + P_T - C_T \\
&= R^2 A_{T-2} + RP_{T-1} - RC_{T-1} + P_T - C_T \\
&= R^{T+1} A_{-1} + \sum_{t=0}^{T} R^{T-t}(P_t - C_t)
\end{aligned}
$$

- Use **terminal condition** (why equality?)

$$
\begin{aligned}
A_T &= 0 \\
RA_{-1} + \sum_{t=0}^{T} R^{-t}(P_t - C_t) &= 0 \\
B_0 + H_0 &= \sum_{t=0}^{T} R^{-t} C_t
\end{aligned}
$$

where $H_0 \equiv \sum_{t=0}^{T} (G/R)^t P_0 = \frac{1-(G/R)^{T+1}}{1-G/R} P_0$

Static problem → Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T} R^{-t} C_t - (B_0 + H_0) \right]$$

- **First order conditions**

$$\forall t:\ 0 = \beta^t C_t^{-\rho} - \lambda R^{-t}$$

- ***Short-run* Euler** equation: $\frac{C_{t+1}}{C_t} = (\beta R)^{1/\rho}$
- ***Long-run* Euler** equation: $\frac{C_t}{C_0} = (\beta R)^{t/\rho}$

Consumption function

- Insert **Euler** into **IBC**

$$\begin{aligned}\sum_{t=0}^{T} R^{-t}(\beta R)^{t/\rho} C_0 &= B_0 + H_0 \Leftrightarrow \\ C_0 \sum_{t=0}^{T}((\beta R)^{1/\rho}/R)^t &= B_0 + H_0\end{aligned}$$

- **Solve** for C_0

$$C_0 = \frac{1-(\beta R)^{1/\rho}/R}{1-((\beta R)^{1/\rho}/R)^{T+1}}(B_0 + H_0)$$

  - **MPC:** $\frac{\partial C_0}{\partial B_0} \approx 1 - [(\beta R)^{1/\rho}/R] \approx 1 - R^{-1}$
  - **MPCP:** $\frac{\partial C_0}{\partial P_0} \approx 1 - [(\beta R)^{1/\rho}/R] \frac{\partial H_0}{\partial P_0} \approx \frac{1-1/R}{1-G/R} \approx 1$

Side note: Value function

- **Analytical expression** for the value function

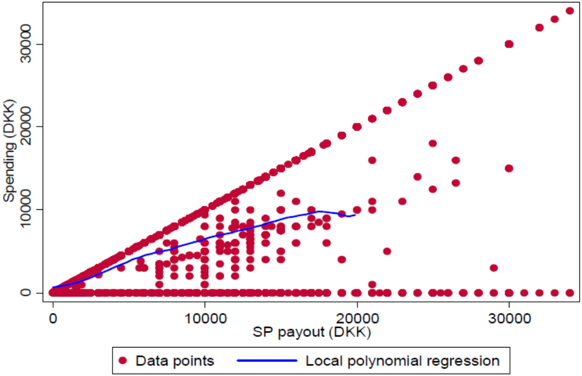
$$
\begin{aligned}
V_0(M_0, P_0) &= \sum_{t=0}^{T} \beta^t u((\beta R)^{t/\rho} C_0) \\
&= \sum_{t=0}^{T} \beta^t (\beta R)^{(1-\rho)t/\rho} \frac{C_0^{1-\rho}}{1-\rho} \\
&= \sum_{t=0}^{T} ((\beta R)^{1/\rho}/R)^t \frac{C_0^{1-\rho}}{1-\rho} \\
&= \frac{1-((\beta R)^{1/\rho}/R)^{T+1}}{1-(\beta R)^{1/\rho}/R} \frac{C_0^{1-\rho}}{1-\rho}
\end{aligned}
$$

Empirical evidence

- **Pro**
  1. Micro-founded consumption-saving
     - Theoretically appealing (humans are intentional)
     - Empirically appealing (testable implications on micro-data)
  2. Larger responses to permanent than to transitory shocks
  3. Consumption smoothing - save for retirement (low income)
- **Con**
  1. *Households seems to have a high MPC in the range 0.20-0.40*
     - Survey studies
     - Tax rebates studies
     - Lottery studies
     - ARM payments studies
  2. *Consumption responds to anticipated income changes*
  3. *Households with more volatile income have larger savings*
  4. *Consumption tracks income over the life-cycle*

High MPC: Danish SP payout

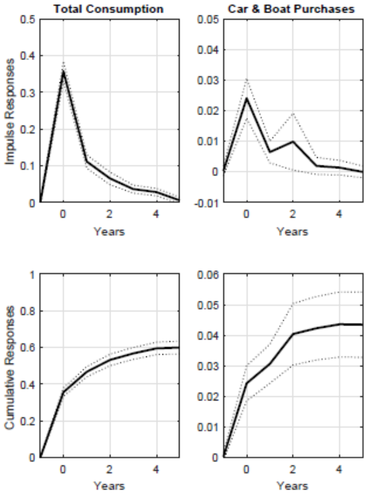
Figure 4: Spending and the size of the SP payout



NOTE: 5055 observations.

Source: Kreiner, Lassen og Leth-Petersen (AEJ:Pol, 2019)

High MPC: Norwegian lottery winners



Source: Fagereng, Holm, Natvik (WP, 2018)

Buffer-stock model (Deaton-Carroll)

$$
\begin{aligned}
V_0(M_0, P_0) &= \max_{\{C_t\}_{t=0}^T} \mathbb{E}_t \sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
& \epsilon_t \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
P_{t+1} &= GP_t\psi_{t+1}, \quad \psi_t \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

1. Borrowing constraints
2. Income uncertainty (analytical results: $\mu = 0$ and $\pi > 0$)

How to solve the model?

- **Borrowing constraints** $\rightarrow$ inequalities $\rightarrow$ high-dimensional **Kuhn-Tucker problem**
- **Uncertainty** $\rightarrow$ fully dynamic problem $\rightarrow$ no Lagrangian
- **No analytical solution with CRRA preferences**
  - Quadratic or CARA utility, which give some analytical results, have implausible properties

$$\text{CRRA: } u(c) = \frac{c^{1-\rho}}{1-\rho} \rightarrow \text{RRA} = \rho$$

$$\text{Qudratic: } u(c) = ac - \frac{b}{2}c^2 \rightarrow \text{ RRA} = \frac{b}{a-bc}c$$

$$\text{CARA: } u(c) = \frac{1}{\alpha}e^{-\alpha c} \rightarrow \text{ RRA} = \alpha c$$

where RRA = relative risk aversion $= \frac{-u''(c)}{u'(c)}c$

- **Solution:** Set up Bellman equation $\rightarrow$ apply numerical dynamic programming

Bellman equation

$$
\begin{aligned}
V_t(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}, P_{t+1}) \right] \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
P_{t+1} &= GP_t \psi_{t+1} \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Normalization I

- **Defining** $c_t \equiv C_t/P_t$, $m_t \equiv M_t/P_t$ etc. implies

$$
\begin{aligned}
A_t = M_t - C_t \quad &\Leftrightarrow \quad A_t/P_t = M_t/P_t - C_t/P_t \\
&\Leftrightarrow \quad a_t = m_t - c_t
\end{aligned}
$$

$$
\begin{aligned}
M_{t+1} = RA_t + Y_{t+1} \quad &\Leftrightarrow \quad M_{t+1}/P_{t+1} = RA_t/P_{t+1} + Y_{t+1}/P_{t+1} \\
&\Leftrightarrow \quad m_{t+1} = Ra_tP_t/P_{t+1} + \xi_{t+1} \\
&\Leftrightarrow \quad m_{t+1} = \frac{R}{G\psi_{t+1}}a_t + \xi_{t+1}
\end{aligned}
$$

The **adjustment factor** $\frac{1}{G\psi_{t+1}}$ is due to changes in permanent income

Normalization II

- **Defining** $v_t(m_t) = V_t(M_t, P_t)/P_t^{1-\rho}$ implies

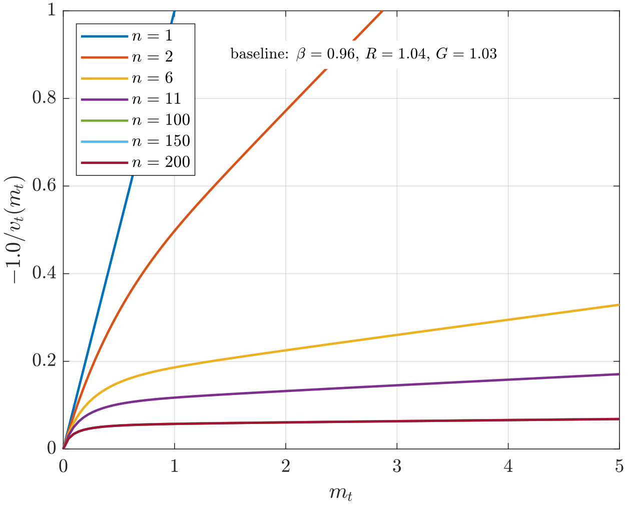
$$
\begin{aligned}
V_t(M_t,P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[V_{t+1}(M_{t+1},P_{t+1})\right] \\
&= \max_{c_t} \frac{(c_t P_t)^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[V_{t+1}(M_{t+1},P_{t+1})\right] \Leftrightarrow \\
V_t(M_t,P_t)/P_t^{1-\rho} &= \max_{c_t} \frac{(c_t P_t)^{1-\rho}/P_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[V_{t+1}(M_{t+1},P_{t+1})/P_t^{1-\rho}\right] \Leftrightarrow \\
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[V_{t+1}(M_{t+1},P_{t+1})/P_{t+1}^{1-\rho} \cdot P_{t+1}^{1-\rho}/P_t^{1-\rho}\right] \\
&= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})\right]
\end{aligned}
$$

Bellman equation in ratio form

$$
\begin{aligned}
v_t(m_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G\psi_{t+1}} R a_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq -\lambda \\
a_T &\geq 0
\end{aligned}
$$

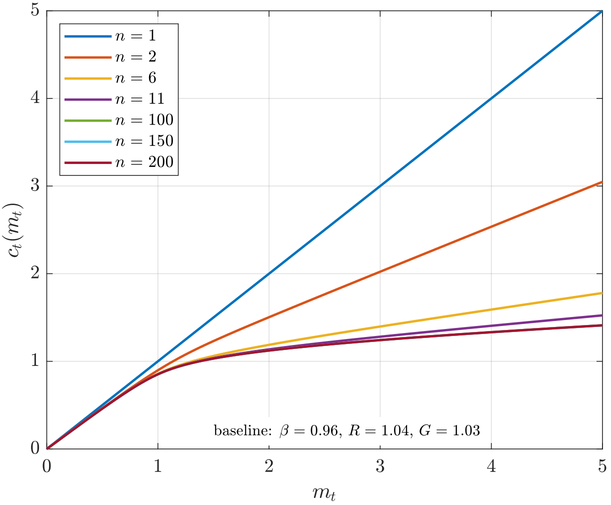
- **Benefit:** Dimensionality of state space reduced
  Can this always be done?
- Easy to solve by **VFI**

Convergence of $-1.0/v_t(m_t)$



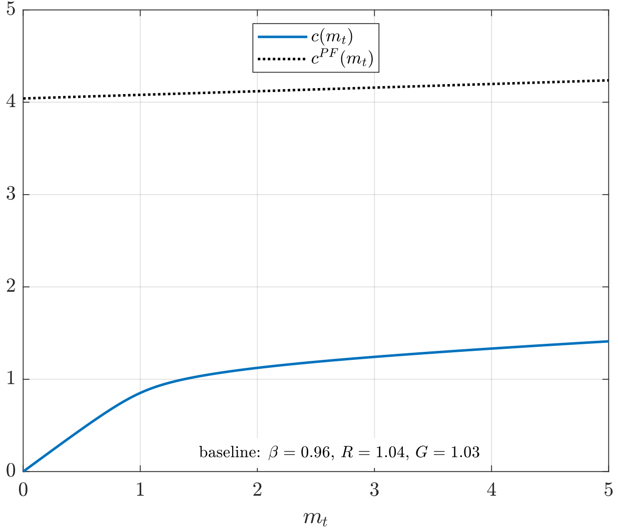
Other parameters: $\rho = 2$, $\pi = 0.005$, $\sigma_\psi = \sigma_\xi = 0.10$

Convergence of $c_t(m_t)$

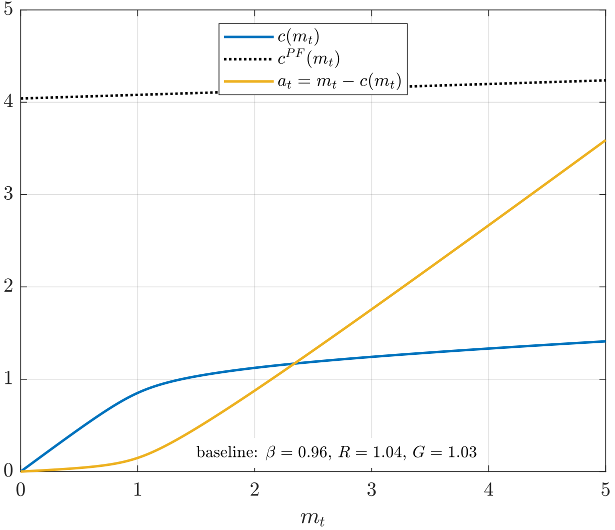


- What is the MPC?

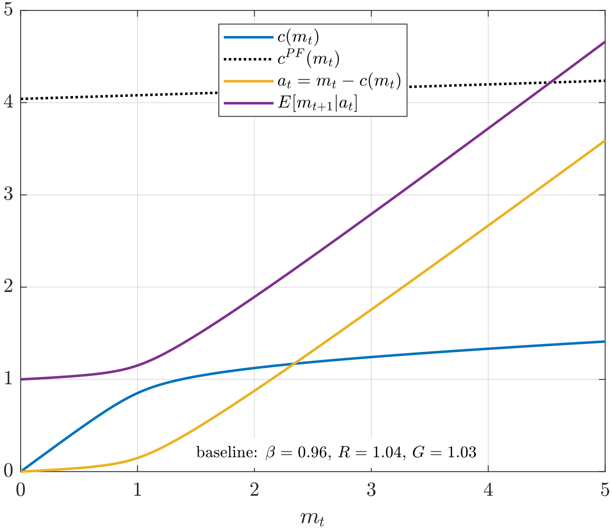
$T \to \infty$: The buffer-stock target



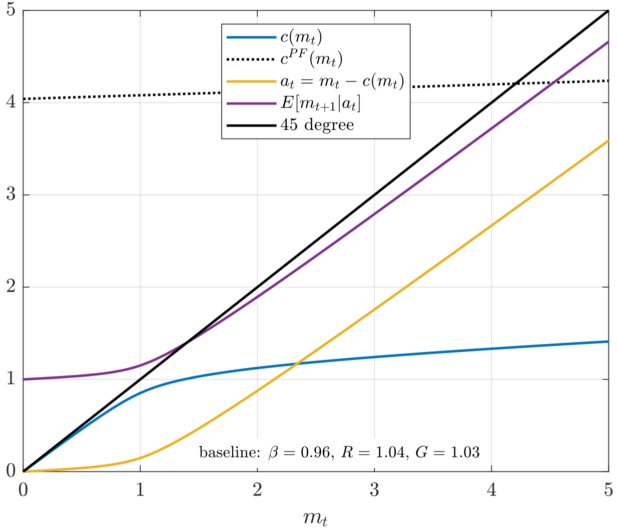
$T \to \infty$: The buffer-stock target



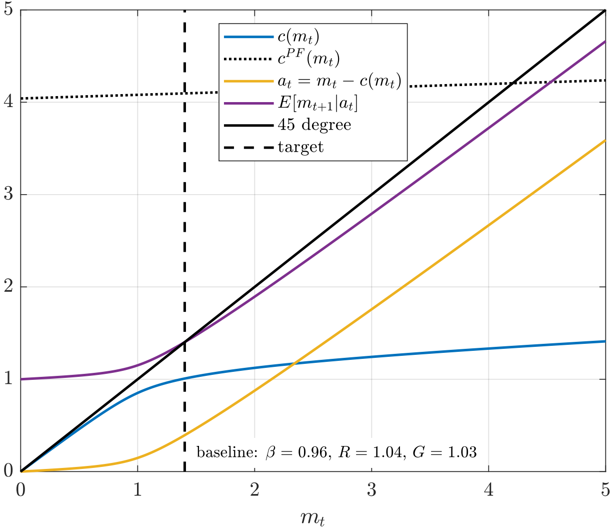
$T \to \infty$: The buffer-stock target



$T \to \infty$: The buffer-stock target



$T \to \infty$: The buffer-stock target



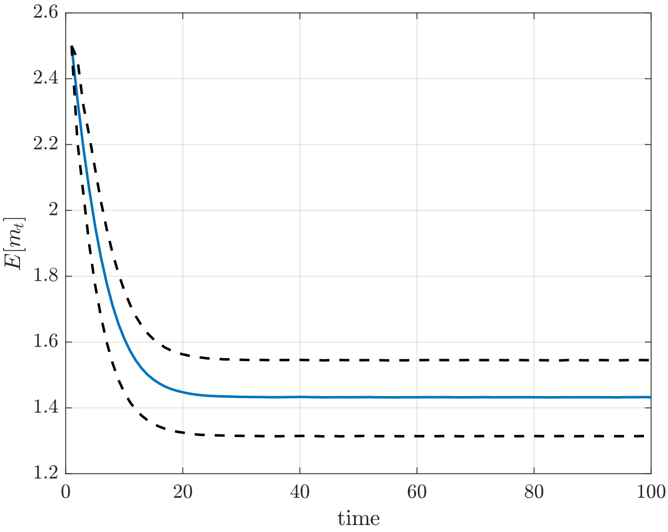
Simulation

- **Solution:** The consumption function $c^\star(m_t)$
- **Simulation** for $t \in \{1, 2, \dots, T\}$:
  1. Choose m_1 and set $t = 1$
  2. Calculate $c_t = c^\star(m_t)$
  3. Calculate $a_t = m_t - c_t$
  4. Draw (pseudo-)random numbers

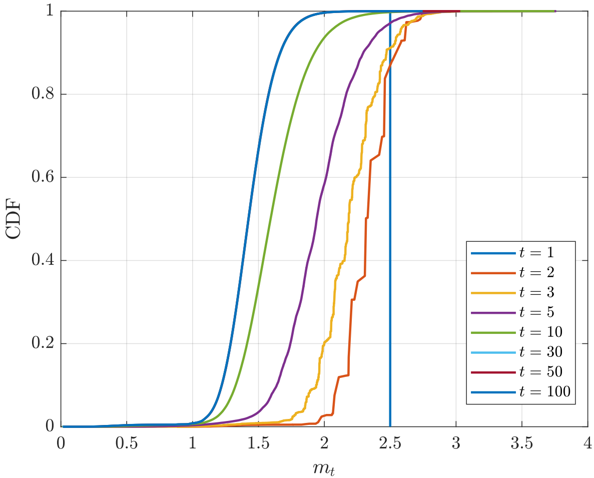
$$
\begin{aligned}
\epsilon_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
\psi_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\
\eta_{t+1} &\sim \mathcal{U}(0,1)
\end{aligned}
$$

  5. Calculate $\xi_{t+1} = \begin{cases} \mu & \text{if } \eta_{t+1} < \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases}$
  6. Calculate $m_{t+1} = \frac{R}{G\psi_{t+1}} a_t + \xi_{t+1}$
  7. Set $t = t + 1$
  8. Stop if $t > T$ else go to step 2

Simulation: Avg. cash-on-hand

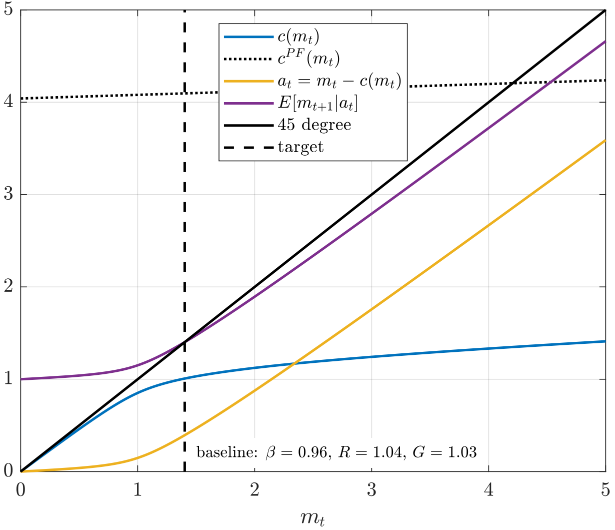


Simulation: Distribution of cash-on-hand

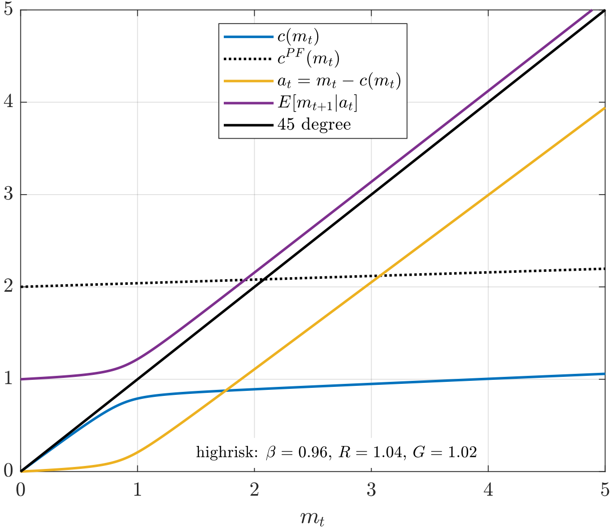


- **Proof of convergence:** Szeidl (2006)

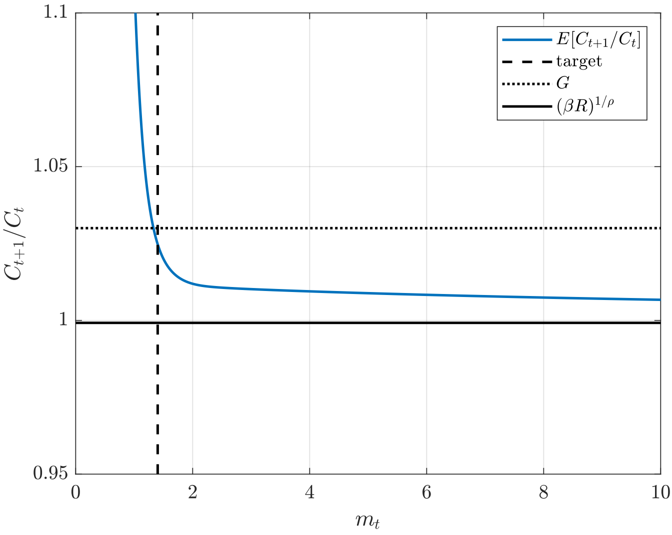
$\sigma_\psi = 0.10$



$\sigma_\psi = 0.15$



Consumption growth I



Consumption growth II

- Remember **Euler-equation**

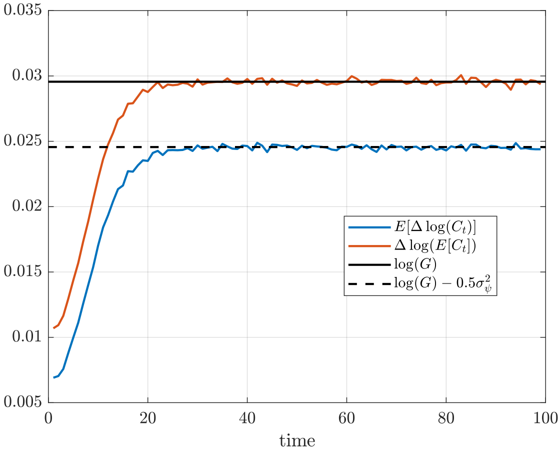
$$C_t^{-\rho} = \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right]$$

  no uncertainty $\Rightarrow C_{t+1}/C_t = (\beta R)^{1/\rho}$
- **Results**
  1. C_{t+1}/C_t is declining in m_t
  2. $\lim_{m_t \to \infty} C_{t+1}/C_t = (\beta R)^{1/\rho} = \text{RI}$
  3. $\lim_{m_t \to 0} C_{t+1}/C_t = \infty$
  4. $C_{t+1}/C_t < G$ at buffer-stock target
- **Intuition** for $C_{t+1}/C_t > (\beta R)^{1/\rho}$
  1. Uncertainty $\Rightarrow$ expected marginal utility $\uparrow$ (because $C_{t+1}^{-\rho}$ is convex function)
  2. Consumer must be lowered today, $C_t \downarrow$
  3. Consumption growth will increase, $C_{t+1}/C_t \uparrow$

Further: *The above arguments are stronger for lower cash-on-hand relative to permanent income*

Consumption growth III

1. Growth of average consumption $= G$
2. Average consumption growth $= G - 0.5\sigma_\psi^2$



Always a buffer-stock target? I

1. **Utility impatience (UI):**

$$\beta < 1$$

2. **Return impatience (RI):**

$$(\beta R)^{1/\rho}/R < 1$$

3. **Weak return impatience (WRI):**

$$\pi^{1/\rho}(\beta R)^{1/\rho}/R < 1$$

4. **Growth impatience (GI)** ($\mathbb{E}_t \psi_{t+1}^{-1} > 1$)**:**

$$(\beta R)^{1/\rho}\mathbb{E}_t \psi_{t+1}^{-1}/G < 1$$

5. **Absolute impatience (AI):**

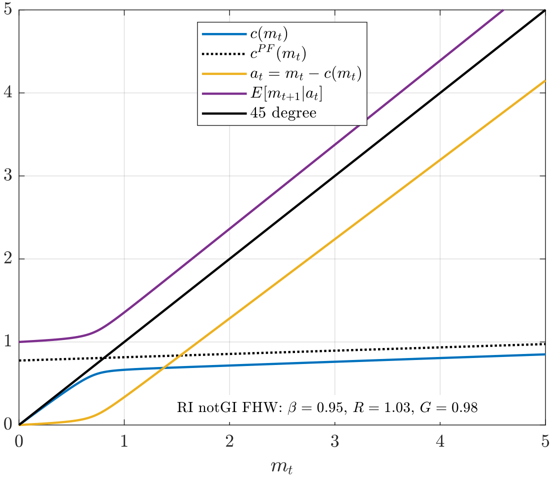
$$(\beta R)^{1/\rho} < 1$$

6. **Finite value of autarky (FVA)** ($\mathbb{E}_t \psi_{t+1}^{1-\rho} < 1$)**:**

$$\beta \mathbb{E}_t (G\psi_{t+1})^{1-\rho} < 1$$

Always a buffer-stock target? II

- **GI ensures buffer-stock target**
- If not GI then something like



Addendum: More analytical results

- **Natural borrowing constraint**

$$\lim_{c_t \to 0} \frac{c_t^{1-\rho}}{1-\rho} = -\infty \Rightarrow c(m_t) < m_t \Rightarrow \lambda \text{ does not matter}$$

- **Liquidity constrained model reached in the limit:**

$$\lim_{\pi \to 0} c(m_t; \pi) = c(m_t; \pi = 0, \lambda = 0)$$

- **Existence of solution:** WRIC + FVA
  - **Proof:** Use *Boyds weighted contraction mapping theorem*
  - **Standard assumptions:** FHW, RI, GI
- The **consumption function** is twice continuously differentiable**, increasing** and **concave**

The borrowing constraint

- Assume **perfect foresight** ($\sigma_\psi = \sigma_\epsilon = \pi = 0$), but **no borrowing**, $\lambda = 0$.
- **Solution:** RI + FHW is still *sufficient* (with $\lambda = \infty$ it is necessary)
- **Standard solutions:** RI + FHW
  1. **GI** $\Rightarrow$ *constraint will eventually be binding*

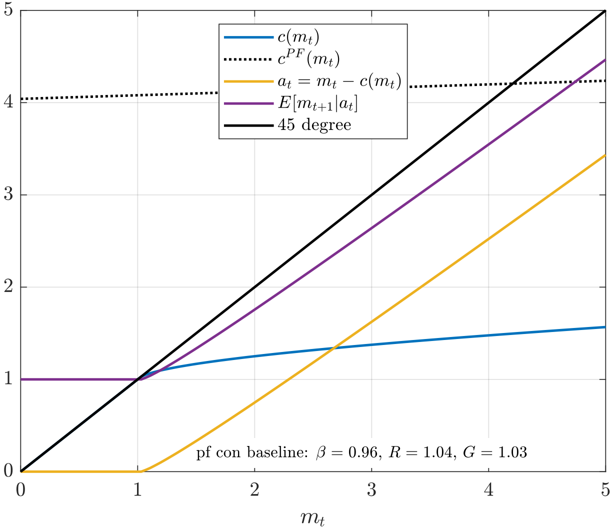
     $$c(m_t) \text{ converge to } c^{PF}(m_t) \text{ from below as } m_t \to \infty$$

  2. **Not GI** $\Rightarrow$ *constraint is never reached*

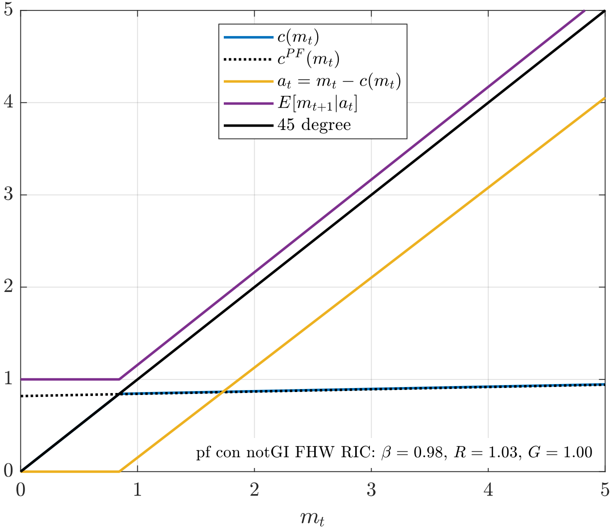
     $$c(m_t) = c^{PF}(m_t) \text{ for } m_t \geq 1$$

- **Exotic solutions without FHW** exists (GI necessary)

Perfect foresight with $\lambda = 0$ and GI



Perfect foresight with $\lambda = 0$, but not GI



Adding a life-cycle (normalized)

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{v(z_t) c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G L_{t+1} \psi_{t+1})^{1-\rho} v_{t+1}(\bullet) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G L_t \psi_{t+1}} R a_t + \xi_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob. } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq \lambda_t = \begin{cases} -\lambda & \text{if } t < T_R \\ 0 & \text{if } t \geq T_R \end{cases}
\end{aligned}
$$

- **Demographics**: z_t (exogenous)
- **Income profile:** $P_{t+1} = G L_t P_t \psi_{t+1}$
- **No shocks in retirement:** $\psi_t = \xi_t = 1$ if $t > T_R$
- **Euler equation:** $C_t^{-\rho} = \beta R \mathbb{E}_t [\frac{v(z_{t+1})}{v(z_t)} C_{t+1}^{-\rho}]$

Consumption functions ($v(z_t) = 1$)

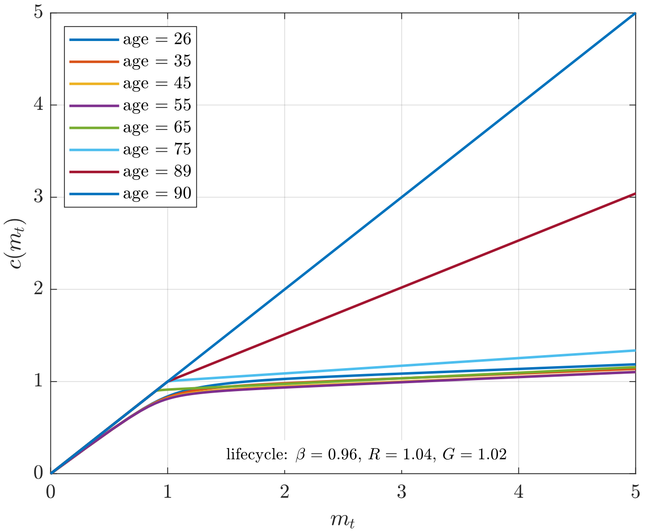
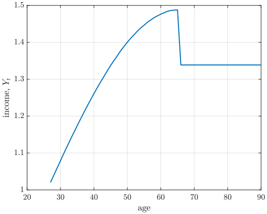
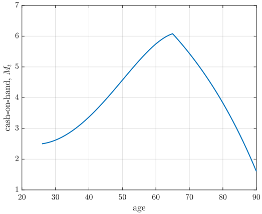


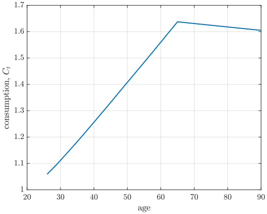
Figure: Life-cycle profiles ($v(z_t) = 1$)



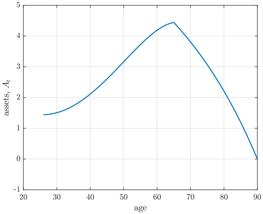
(a) Income, Y_t



(b) Cash-on-hand, M_t



(c) Consumption, C_t



(d) End-of-period assets, A_t

Euler-equation

- All optimal **interior choices** must satisfy

$$\begin{aligned} C_t^{-\rho} &= \beta R \mathbb{E}_t\left[C_{t+1}^{-\rho}\right] \Leftrightarrow \\ c_t^{-\rho} &= \beta R \mathbb{E}_t\left[(G\psi_{t+1}c_{t+1})^{-\rho}\right] \end{aligned}$$

- Else optimal choice is **constrained**

$$\begin{aligned} C_t^{-\rho} &\geq \beta R \mathbb{E}_t\left[C_{t+1}^{-\rho}\right] \Leftrightarrow \\ C_t &= M_t + \lambda P_t \Leftrightarrow \\ c_t &= m_t + \lambda \end{aligned}$$

- **For simplicity:** Assume $\lambda = 0$ then we must have $a_t > 0$
  - Note that $\lim_{c_t \to 0} u'(c_t) = \infty$
  - But $a_t \leq 0 \Rightarrow \Pr[m_{t+1} \leq 0] > 0$ where $c_{t+1} = 0$

Endogenous grid method: Intuition

- **Obs.:** Given $C^{\star}_{t+1}(M_{t+1},P_{t+1})$ and A_t and P_t we have

$$
\begin{aligned}
C_t^{-\rho} &= \beta R \mathbb{E}_t\left[\left(C^{\star}_{t+1}(M_{t+1},P_{t+1})\right)^{-\rho}\right] \Leftrightarrow \\
C_t &= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(M_{t+1},P_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(RA_t+Y_{t+1},P_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&= \mathbb{E}_t\left[\beta R\left(C^{\star}_{t+1}(RA_t+P_t\psi_{t+1}\xi_{t+1},P_t\psi_{t+1})\right)^{-\rho}\right]^{-\frac{1}{\rho}} \\
&\equiv F(A_t,P_t)
\end{aligned}
$$

- **Endogenous grid:** $A_t = M_t - C_t \Leftrightarrow M_t = C_t + A_t$
- **Conclusion:** (M_t, P_t, C_t) is a solution to the Bellman equation because it satisfies the Euler equation
- **Perspectives:** Varying A_t (and P_t) we can map out the consumption function without using any numerical solver!
- **The borrowing constraint** is binding below the lowest M_t points we find from $A_t = 0$

... in ratio form with $\lambda = 0$

- **Prerequisites** ($\lambda = 0 \Rightarrow \underline{a}_t = 0$)
  1. Next-period **consumption function**: $c^{\star}_{t+1}(m_{t+1})$
  2. **Asset grid:** $\mathcal{G}_a = \{a_1, a_2, \dots, a_{\#}\}$ with $a_1 = \underline{a}_t + 10^{-6}$
- **Algorithm:** For each $a_i \in \mathcal{G}_a$
  1. Find consumption using Euler equation

$$c_i = \mathbb{E}_t \left[\beta R \left(G\psi_{t+1} c^{\star}_{t+1} \left(\frac{R}{G\psi_{t+1}} a_i + \xi_{t+1} \right) \right)^{-\rho} \right]^{-\frac{1}{\rho}}$$

  2. Find endogenous state

$$a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$$

- The **consumption function**, $c_t(m_t)$, is given by

$$\{0, c_1, c_2, \dots, c_{\#}\} \text{ for } \{\underline{a}_t, m_1, m_2, \dots, m_{\#}\}$$

- *We can find all consumption functions in this way!*

Addendum: The natural borrowing constraint ($\lambda > 0$)

- The **optimal end-of-period asset choice satisfies**

$$A_t \geq \underline{A}_t = \begin{cases} 0 & \text{if } t \geq T_R \\ -\min\{\Lambda_t, \lambda_t\} G L_t \underline{\psi} & \text{if } t < T_R \end{cases}$$

where

$$\Lambda_t \equiv \begin{cases} R^{-1} G L_t \underline{\psi}\underline{\xi} & \text{if } t = T_R - 1 \\ R^{-1}\left[\min\{\Lambda_{T-1}, \lambda_t\} + \underline{\xi}\right] G L_t \underline{\psi} & \text{if } t < T - 1 \end{cases}$$

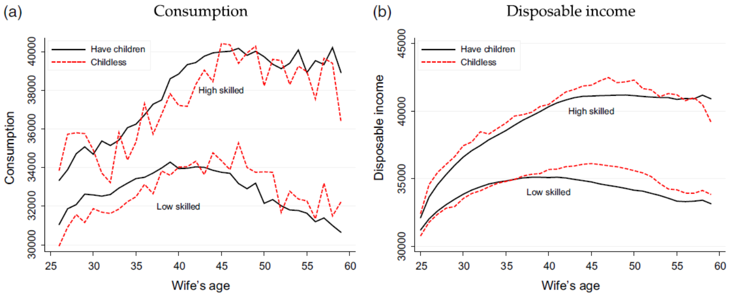
and $\underline{\psi}$ and $\underline{\xi}$ are the minimum realizations of ψ_{t+1} and ξ_{t+1}

- **Proof:** Can be shown as a consequence of the household wanting to avoid $C_t = 0$ at *any cost*

Three generations of models

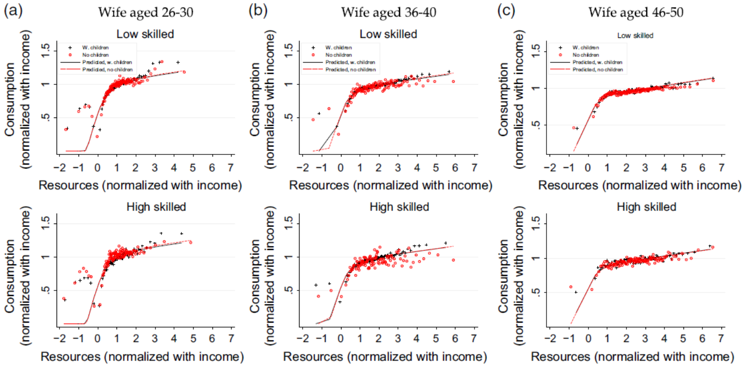
- **1st:** *Permanent income hypothesis* (Friedman, 1957) or *life-cycle model* (Modigliani and Brumberg, 1954)
- **2nd:** *Buffer-stock consumption model* (Deaton, 1991, 1992; Carroll 1992, 1997, 2012)
- **3nd:** *Multiple-asset buffer-stock consumption models* (e.g. Kaplan and Violante, 2014)

Denmark: Life-cycle profiles fit



Source: Jørgensen (2017)

Denmark: Consumption function fit



Source: Jørgensen (2017)

Level of wealth and MPC

- Consumption-saving models a few years ago **could not endogenously fit** both
  1. The level of wealth observed
  2. The high MPCs found in quasi experiments
- **Three solutions:**
  1. Exogenous **hands-too-mouth households** (Campbell and Mankiw, 1990)
  2. **Preference heterogeneity** (Carroll et al. 2017)
  3. **Wealthy hands-to-mouth** (Kaplan and Violante, 2014)
     Many households hold mostly illiquid assets with a high return $\rightarrow$ *consumption adjust in reponse to small income shock*

Kaplan-Violante model

$$V_t(M_t, N_t, P_t) = \max\left\{v_t^{keep}(M_t, N_t, P_t), v_t^{adj.}(M_t + N_t - \lambda, P_t)\right\}$$

$$\begin{aligned}
v_t^{keep}(M_t, N_t, P_t) &= \max_{C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
A_t &= M_t - C_t \\
B_t &= N_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

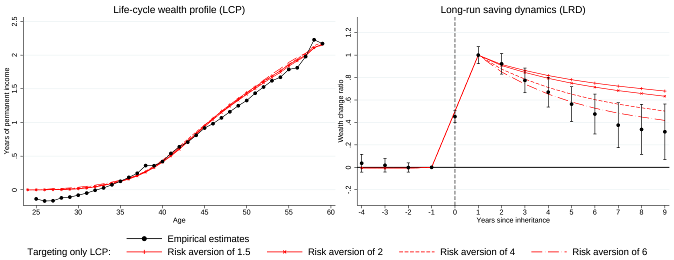
$$\begin{aligned}
\tilde{v}_t^{adj.}(X_t, P_t) &= \max_{B_t, C_t} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \text{ s.t.} \\
M_t &= X_t - B_t \\
N_t &= B_t \\
A_t &= M_t - C_t \\
A_t &\geq -\omega P_t.
\end{aligned}$$

$$W_t(A_t, B_t, P_t) = \mathbb{E}_t[V_t(RA_t + P_t\psi_{t+1}\xi_{t+1}, (1-\delta)B_t, P_t\psi_{t+1})]$$

Level of wealth and long-run dynamics I

- **Best test of a life-cycle consumption-saving model:**
  A *sudden*, *sizable* and *salient* shock to wealth
  + long panel to observe how the extra wealth is spend
- **My own research (with Alessandro Martinello):**
  Compare individuals in the Danish register data who
  1. Receive a similar inheritance, but at different points in time
  2. From parents dying due to heart attacks or car crashes

Level of wealth and long-run dynamics II



- **Net worth:** Good fit for different parametrizations
- **Also dynamics:** Good fit only if
  1. Substantial impatience
  2. Very strong precautionary saving motive (ρ)
  3. Higher impatient (β)

Frontier topics

- The dynamics of **durable consumption** (very volatile over the business cycle, involve non-convexities due to adjustment costs)
- The effects of **non-Gaussian** and **high frequency income uncertainty** (monthly Danish income since 2008 and machine learning estimation very interesting)
- **Housing** and a more detailed specification of the households' **balance sheets** (did expectations or credit availability drive the boom and bust in house prices?)
- Relevant **deviations from rationality** (learning, myopia, hypoerbolic discounting, reference dependence, mental accoutning)
- Fitting the **level and dynamics of inequality** – circumstances or behavior?
- **General equilibrium** with heterogeneous households (next up)

Introduction to GE

- **Topic:** Solving models in general equilibrium
- **Two cases:**
  1. Finding the *stationary equilibrium*, when there are *no* aggregate shocks
  2. Finding the *dynamic equilibrium path*, when there are aggregate shocks
- **Literature:**
  - **Stationary equilibrium:** E.g. Aiyagari (1994)
  - **Dynamic equilibrium:**

    Original: Krusell and Smith (1998)
    Comparison of methods: JEDC 34 (2010)
    Handbook: Algan, Allais, Den Haan and Rendahl (2014)
    Frontier: Winberry (2018) + Ahn et. al. (2018)
- **Today:** Simple illustrative model

Model

- **Components**
  1. Households facing idiosyncratic income risk
  2. Representative firm
  3. Government (balanced budget, tax τ)
  4. (Aggregate technology shocks)
- **Idiosyncratic variables**
  - m_{it}: cash-on-hand
  - c_{it}: consumption
  - $a_{it} = m_{it} - c_{it}$: end-of-period assets
  - u_{it}: employed/unemployed (hours worked, $\bar{l}$)
- **Aggregate variables**
  - Q: technology
  - K: capital
  - L: labor supply
  - W: wage rate
  - R: return factor
- κ: ***cdf*** of households over a_{it-1} and u_{it}

Household problem

$$
\begin{aligned}
v(m_t, u_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[v(m_{t+1}, u_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= Ra_t + W \cdot \begin{cases} \mu & \text{if } u_{t+1} = 1 \\ (1-\tau)\bar{l} & \text{if } u_{t+1} = 2 \end{cases} \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi(u_t) \\ 2 & \text{with prob. } 1 - \pi(u_t) \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

Firm problem (simple)

- **Production function**

$$Y_t = F(K, L) = QK^{\alpha}(\bar{l}L)^{1-\alpha}$$

- **Profit function**

$$\Pi(K, L) = F(K, L) - \delta K - (R-1)K - WL$$

- **FOC for K_t**

$$\frac{\partial \Pi(K, L)}{\partial K} = 0 \quad \leftrightarrow \quad R(Q, K, L) = 1 + \alpha Q(K/\bar{l}L)^{\alpha-1} - \delta$$

$$K(R, Q, L) = \left(\frac{R - 1 + \delta}{\alpha Q(\bar{l}L)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}$$

- **FOC for L_t**

$$\frac{\partial \Pi(K, L)}{\partial L} = 0 \quad \leftrightarrow \quad W(Q, K, L) = (1-\alpha)Q(K/\bar{l}L)^{\alpha}$$

Recursive *stationary* equilibrium

A *stationary equilibrium* is a set of quantities K and L, a cdf κ, a consumption function $c(m_{it}, u_t)$, and prices R and W such that

1. The prices are determined by optimal firm behavior, i.e. $R = R(Q, K, L)$ and $W = W(Q, K, L)$
2. $c(\bullet)$ solve the household problem given prices R and W
3. κ is the invariant cdf over a_{it-1} and u_{it} implied by the solution to the household problem
4. The labor market clears, i.e. $L = \int \mathbf{1}_{u_{it}=2} d\kappa = 1 - u^*$, where u^* is steady state unemployment
5. The capital market clears, i.e. $K = \int a_{it-1} d\kappa$

Solve for the *stationary* equilibrium

1. Guess on R below $1/\beta$
2. Calculate $K^d = K(R, Q, 1)$ and $W = W(Q, K^d, 1)$
3. Solve the household problem
4. Simulate a panel of N households for T periods
5. Compute $K^s = \frac{1}{N}\sum_{i=1}^{N} a_{iT}$ (from final period)
6. If for some tolerance ι

$$\left|K^s - K^d\right| < \iota$$

then stop, otherwise return to step 1 and update guess appropriately

Calibration

- **Preferences:** $\beta = 0.99$, $\rho = 1$ (i.e. log utility)
- **Income:** $u^* = 0.10$, $\pi(1) = 0.60$, $\mu = 0.15$, $\tau = \frac{\mu u^*}{\bar{l}(1-u^*)}$
- **Production function:** $Q = 1$, $\alpha = 0.36$, $\delta = 0.025$, $\bar{l} = 0.9$
- **Simulation:** $N = 10000$, $T = 2000$

Illustration of equilibrium

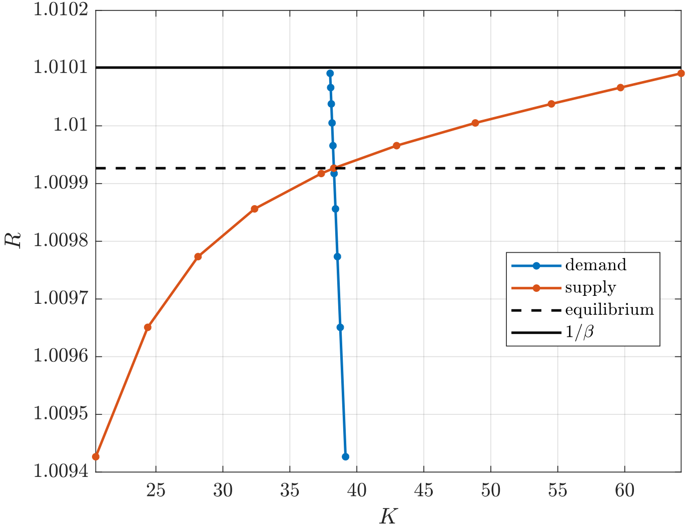
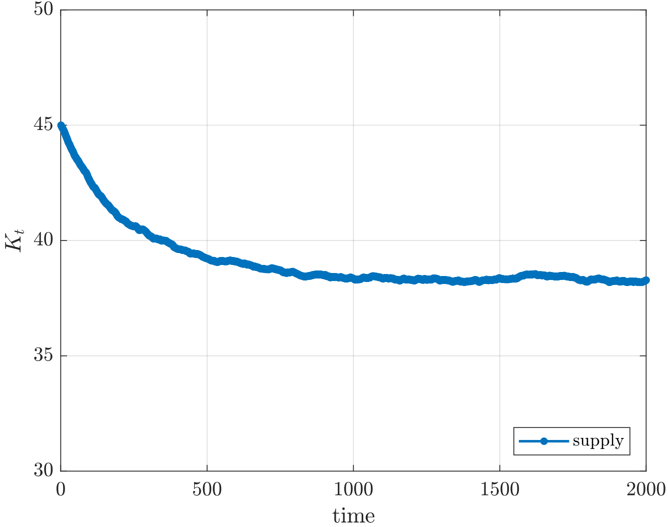


Illustration of convergence



Further perspectives

- **Note:** Like a Ramsey model, but with heterogeneity on the household side
- Easy to look at **steady state welfare effects** of various policies (taxes, social security etc.)
  ... including distributional effects
- **Extensions:**
  1. Durables and housing
  2. Heterogeneous firms
  3. Open economy with goods trade and capital flows
  4. Over-lapping generations
  5. Environmental trade-offs
  6. Transitional dynamics

Adding aggregate dynamics

- $z_t \in \{1,2\}$ denotes respectively *recession* and *boom*
- z_t follows a simple *symmetric Markov process*

$$z_t = \begin{cases} z_{t-1} & \text{with prob. } \pi_z \in (0,1) \\ \sim z_{t-1} & \text{else} \end{cases}$$

- **Technology** is time-varying

$$Q_t = Q(z_t)$$

- **Unemployment** is time-varying

$$u_{t+1} = \begin{cases} 1 & \text{with prob. } \pi(u_t, z_{t+1}) \\ 2 & \text{with prob. } 1 - \pi(u_t, z_{t+1}) \end{cases}$$

Time-varying distribution and aggregates

- The **distribution**, κ_t, of households over a_{it-1} and u_t, will be time-varying, i.e.

$$\kappa_{t+1} = \Gamma(\kappa_t, z_{t+1})$$

  where Γ is a *deterministic* transition function due to the *law of large numbers*

- Likewise with **aggregates** and **factor prices**

$$\begin{aligned} R(Q_t, K_t, L_t) &= 1 + \alpha Q_t (K_t / \bar{l} L_t)^{\alpha - 1} - \delta \\ W(Q_t, K_t, L_t) &= (1-\alpha) Q_t (K_t / \bar{l} L_t)^{\alpha} \end{aligned}$$

  where

$$K_t = \int a_{it-1} d\kappa_t$$

$$L_t = \int 1_{u_{it}=2} d\kappa_t$$

- **Trick:** Choose $\pi(\bullet)$ such that $L_t = L(z_t)$

True household problem

$$
\begin{aligned}
v(m_t, z_t, u_t, \kappa_t) &= \max_{C_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[v(m_{t+1}, z_{t+1}, u_{t+1}, \kappa_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= R_{t+1} a_t + W_{t+1} \cdot \begin{cases} \mu & \text{if } u_{t+1} = 1 \\ (1-\tau_t)\bar{l} & \text{if } u_{t+1} = 2 \end{cases} \\
R_{t+1} &= R(Q(z_{t+1}), K_{t+1}, L(z_{t+1})) \\
W_{t+1} &= W(Q(z_{t+1}), K_{t+1}, L(z_{t+1})) \\
K_{t+1} &= \int a_{it} d\kappa_{t+1} \\
\kappa_{t+1} &= \Gamma(\kappa_t, z_{t+1}) \\
a_t &\geq 0
\end{aligned}
$$

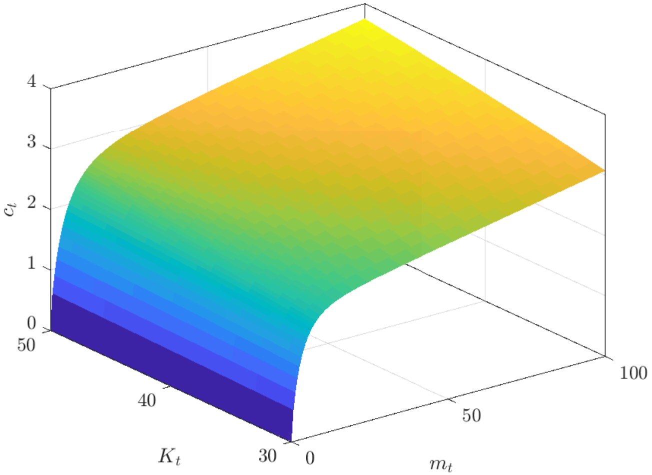
What makes this problem very hard to solve?

Approximate household problem

$$
\begin{aligned}
v(m_t, z_t, u_t, K_t) &= \max_{C_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[v(m_{t+1}, z_{t+1}, u_{t+1}, K_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= R_{t+1} a_t + W_{t+1} \cdot \begin{cases} \mu & \text{if } u_{t+1} = 1 \\ (1-\tau_t)\bar{l} & \text{if } u_{t+1} = 2 \end{cases} \\
R_{t+1} &= R(Q(z_{t+1}), K_{t+1}, L(z_{t+1})) \\
W_{t+1} &= W(Q(z_{t+1}), K_{t+1}, L(z_{t+1})) \\
K_{t+1} &= \begin{cases} \exp(a_1 + b_1 \log K_t) & \text{if } z_t = 1 \\ \exp(a_2 + b_2 \log K_t) & \text{if } z_t = 2 \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

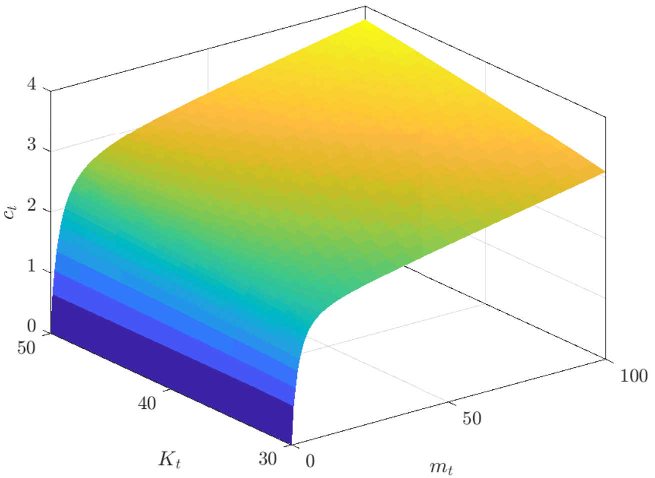
where a_1, b_1, a_2 and b_2 are parameters in the *perceived law of motion* (PLM) for capital

Consumption function ($z_t = 1, u_t = 1$)



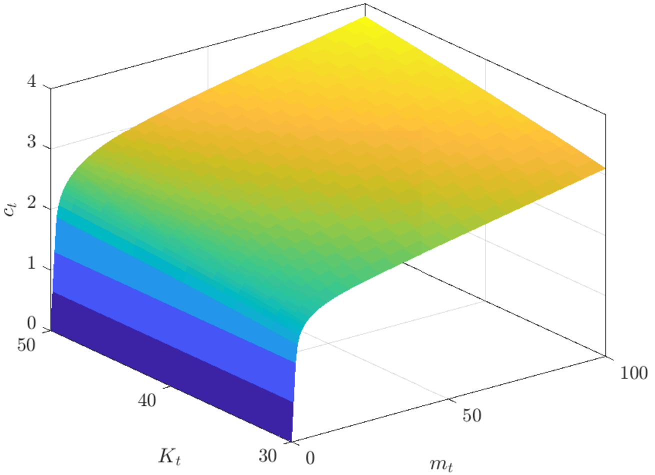
Using $a_1 = 0.124$, $a_2 = 0.137$, $b_1 = 0.966$, $b_2 = 0.964$

Consumption function ($z_t = 2, u_t = 1$)



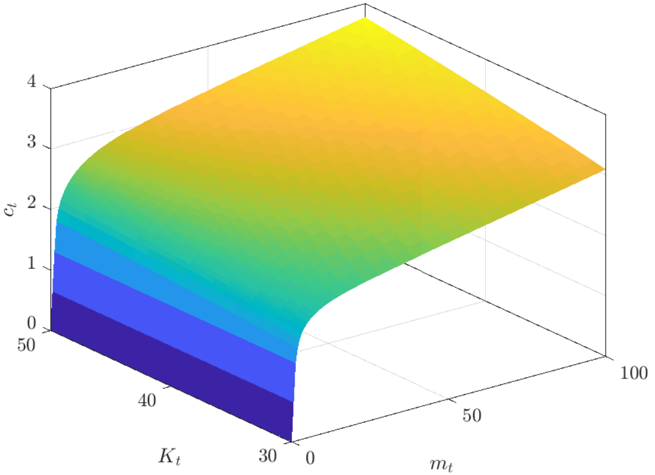
Using $a_1 = 0.124$, $a_2 = 0.137$, $b_1 = 0.966$, $b_2 = 0.964$

Consumption function ($z_t = 2, u_t = 2$)



Using $a_1 = 0.124$, $a_2 = 0.137$, $b_1 = 0.966$, $b_2 = 0.964$

Consumption function ($z_t = 1, u_t = 2$)



Using $a_1 = 0.124$, $a_2 = 0.137$, $b_1 = 0.966$, $b_2 = 0.964$

Recursive *dynamic* equilibrium path

For an exogenous path of z_t (and thus Q_t), and an initial cdf κ_0, a (approximate) *dynamic equilibrium* is a path of quantities K_t and L_t, a path of the cdf κ_t, a consumption function $c(m_{it}, z_t, u_t, K_t)$, and a path of prices R_t and W_t such that

1. The prices are determined by optimal firm behavior, i.e. $R_t = R(Q_t, K_t, L_t)$ and $W_t = W(Q_t, K_t, L_t)$
2. $c(m_{it}, z_t, u_t, K_t)$ solve the household problem
3. κ_t develops according to $\kappa_{t+1} = \Gamma(\kappa_t, z_{t+1})$
4. The labor market clears each period, i.e. $L_t = \int 1_{u_{it}=2} d\kappa_t$
5. The capital market clears each period, i.e. $K_t = \int a_{it-1} d\kappa_t$

Solve for the *dynamic* equilibrium path

1. Draw a T period sequence of z_t (and thus Q_t)
2. Guess on a_1, b_1, a_2 and b_2
3. Solve the (approximate) household problem
4. Simulate a panel of N households for T periods, where

$$K_t = \frac{1}{N}\sum_{i=1}^{N} a_{it-1}$$

   and $R_t = R(Q(z_t), K_t, L(z_t))$ and $W_t = W(Q_t, K_t, L(z_t))$
5. Using data from $\underline{T}$ to T (i.e. after a burn-in period) *estimate* the following equation by OLS

$$\log K_{t+1} = (\tilde{a}_1 + \tilde{b}_1 \log K_t)\mathbf{1}_{z_t=1} + (\tilde{a}_2 + \tilde{b}_2 \log K_t)\mathbf{1}_{z_t=2}$$

6. Calculate $\eta = (a_1 - \tilde{a}_1)^2 + (b_1 - \tilde{b}_1)^2 + (a_2 - \tilde{a}_2)^2 + (b_2 - \tilde{b}_2)^2$
   if $\eta < \iota$ then stop
   else set $a_1 = \tilde{a}_1$, $b_1 = \tilde{b}_1$ etc. and return to step 3

Convergence

- **Theoretically:** Hard to ensure that there is convergence to an equilibrium path
  1. The equilibrium might *not exist*
  2. The equilibrium might *not be (globally) stable* (will not be reached by the recursive algorithm proposed)
  3. There might be *multiple equilibria*
- **Practice**: Use various tips and tricks to help the recursion, including
  1. **Relaxtion:** $a_1 = \omega\tilde{a}_1 + (1-\omega)a_1, b_1 = \omega\tilde{b}_1 + (1-\omega)b_1$ etc.
  2. **Bounding:** Force K_t to be inside pre-specified range

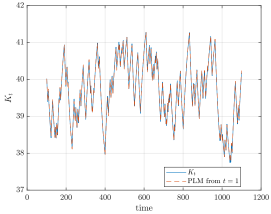
Dynamic calibration

- **Preferences:** $\beta = 0.99$, $\rho = 1$ (i.e. log utility)
- **Income:** $\pi_z = 0.875$, $\mathbb{E}[u_{it}|z_t = 1] = 0.1$, $\mathbb{E}[u_{it}|z_t = 2] = 0.04$
- **Production function:** $Q = 1$, $\alpha = 0.36$, $\delta = 0.025$, $\bar{l} = 0.9$
- **Simulation:** $N = 10000$, $T = 1100$
- **Initial forecasting rule:** $a_1, a_2 = 0$ and $b_1, b_2 = 1$

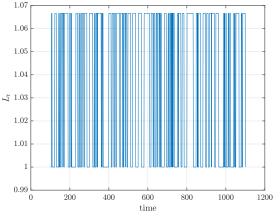
Updates of a_{z_t} and b_{z_t}

0: a = [0.000 0.000], b = [1.000 1.000]
1: a = [0.151 0.151], b = [0.959 0.959], eta = 0.7362345300
2: a = [0.138 0.143], b = [0.962 0.962], eta = 0.0519581489
3: a = [0.131 0.139], b = [0.964 0.963], eta = 0.0259560452
4: a = [0.128 0.137], b = [0.965 0.963], eta = 0.0129470685
5: a = [0.126 0.137], b = [0.965 0.963], eta = 0.0064824221
6: a = [0.125 0.137], b = [0.965 0.963], eta = 0.0032898298
7: a = [0.125 0.137], b = [0.965 0.964], eta = 0.0017045487
8: a = [0.124 0.137], b = [0.965 0.964], eta = 0.0009211280
9: a = [0.124 0.137], b = [0.965 0.964], eta = 0.0005238535
10: a = [0.124 0.137], b = [0.965 0.964], eta = 0.0003127121
(...)
26: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000001763
27: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000001077
28: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000000651
29: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000000394
30: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000000234
31: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000000138
32: a = [0.124 0.137], b = [0.966 0.964], eta = 0.0000000081

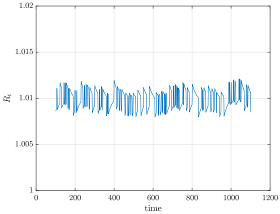
Figure: Paths of aggregate quantities and prices



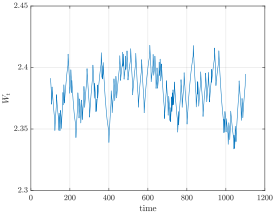
(a) K_t



(b) L_t

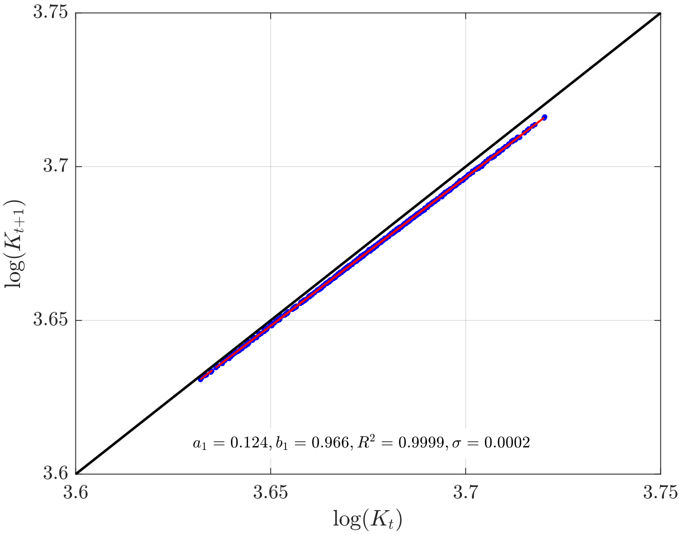


(c) R_t

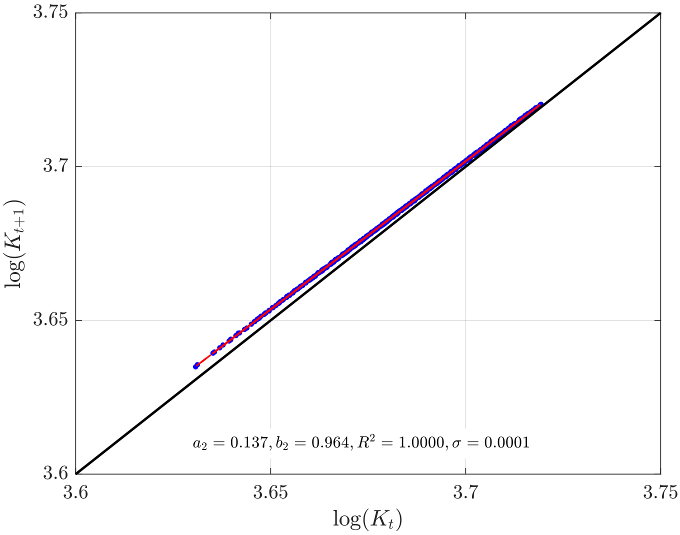


(d) W_t

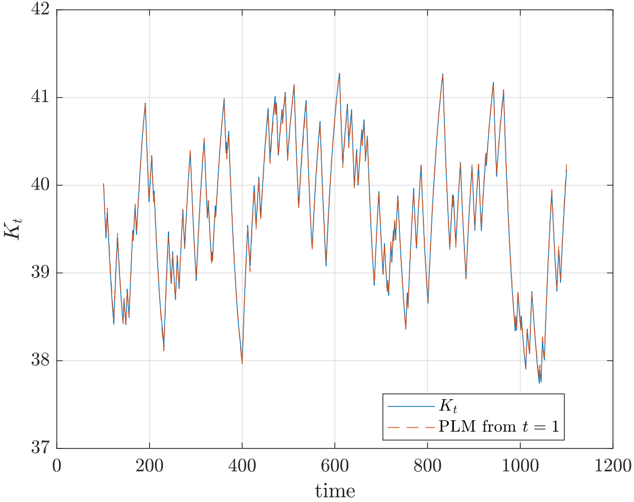
PLM - $z_t = 1$ (recession)



PLM - $z_t = 2$ (boom)

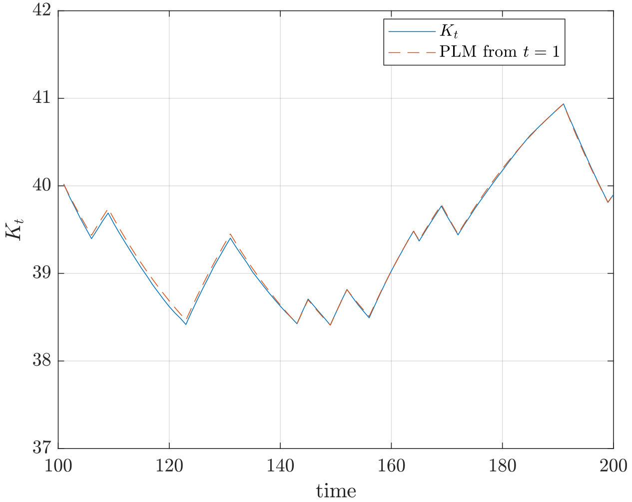

3.75
3.7
3.65
3.6
$\log(K_{t+1})$
$a_2 = 0.137, b_2 = 0.964, R^2 = 1.0000, \sigma = 0.0001$
3.6
3.65
3.7
3.75
$\log(K_t)$

Accuracy of forecast - long-term



$$K_t^{PLM} = \exp(a_{z_t} + b_{z_t} \log K_{t-1}^{PLM}), \text{ with } K_1^{PLM} = K_1$$

Accuracy of forecast - short-term



$$K_t^{PLM} = \exp(a_{z_t} + b_{z_t} \log K_{t-1}^{PLM}), \text{ with } K_1^{PLM} = K_1$$

Perspectives

- **Today:** Only real economy and shocks to technology
- **Many interesting developments:**
  1. Price setting frictions and non-neutrality of money
  2. Shocks to expectations, credit, sentiments etc.
  3. Financial markets and endogenous money
  4. Propgation of shocks and semi-endogenous fluctuations
  5. Further deviations from rational expectation and non-Walrassian equilibrium concepts

Summary

- **Dynamic programming** is needed to solve **empirically realistic consumption-saving models**
- The **buffer-stock consumption model**, and it's two asset cousin, can fit central stylized facts
  1. High MPC
  2. Responses to expected windfalls
  3. Households with more volatile income save more
  4. Consumption tracks income over the life-cycle
- Advances in micro-data, numerical methods and computational power are leading to **new discoveries**
- **EGM is a powerful solution method** (can be generalized, DCEGM, G2EGM, NEGM)
- Realistic consumption-saving behavior can be included in **general equilibrium models** $\rightarrow$ welfare analysis with full distributional effects