Term Paper: A replication study of Optimal Financial Knowledge and Wealth Inequality

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1 Introduction

This is where the introduction goes

$$\alpha + \beta \cdot \sum a_i \tag{1}$$

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2 Model

2.0.1 State & Choice space

Our state space contains the following variables:

$$s = (m_t, f_t, \mu_t, e) \tag{2}$$

Where e is the education level which stays constant throughout the life cycle of an agent, m is the assets accumulated at time t, and f is the accumulated financial knowledge accumulated at period t. μ_t is the persistent shock to income growth (or permanent income).

We make a couple of simplifications compared to the original paper, mainly due to reducing the computational load when solving the model. An elaboration of our choice variables are presented below: We have 4 choice variables:

c, i, κ The consumption is denoted by c, and is made with a grid of 40 points. These 40 points are fractions of the assets accumulated at time t. This follows the paper exactly. The variable $i \in \{0, 1\}$ is a binary choice to whether or not to invest in capital. This contrasts the original paper, where the investment is considered to be an continous variable, which are discretized into 25 points when implemented. The variable $\kappa_t \in \{0, 0.55\}$ is the share of assets to be invested in the risky asset conditional on having investing in the risky asset. This implies that $(1 - \kappa)$ will be invested in the risky asset. We assume in this implementation that κ is fixed. This is also a simplification compared to the original paper. The saving is denoted with ψ. The saving can be modelled as the residual i.e. we only need to consider the three choice variables:

$$\text{choice variables} = c, i, \kappa \tag{3}$$

2.0.2 Equations

$$
\begin{aligned}
V_t(s_t) &= \max_{c_t, i_t, \kappa_t} u(c_t) + \beta \mathbb{E}[V_{t+1}(s_{t+1})] && (4) \\
a_{t+1} &= \tilde{R}_t[m_t - c_t - \pi(i_t) - c_d I(k_t \neq 0)] && (5) \\
m_{t+1} &= \tilde{R}(f_t + 1)a_t + y_t + tr_{t+1} && (6) \\
f_{t+1} &= (1 - \delta)f_t + i_t && (7) \\
\tilde{R}(f_{t+1}) &= (1 - \kappa_t)R^{\text{riskfree}} + \kappa_t R^{\text{riskful}}(f_{t+1}) && (8) \\
\log R^{\text{riskful}}(f_{t+1}) &= \bar{r} + r(f_{t+1}) + \sigma_\epsilon \epsilon_t && \sigma_\epsilon = 0.16 \quad (9) \\
r(f_t) &= f_t^\alpha, && \alpha = 1, r(f_t) \in [0, 0.04] \quad (10) \\
\pi(i_t) &= \omega \cdot i_t && (11) \\
\log y_t &= g_{y,t}(t) + \mu_t + \nu_t && nu_t \sim\sim N(0, \sigma_\nu) \quad (12) \\
\mu_t &= \rho \mu_{t-1} + \omega_t && \omega_t \sim N(0, \epsilon_\omega) \quad (13) \\
g_{y,t}(t) &= \text{TO COME (Non parametric estimation)} && (14) \\
u(c_t) &= n_t \cdot \frac{(c_t/n_t)^{1-\rho} - 1}{1 - \rho} && (15) \\
& && (16)
\end{aligned}
$$

2.0.3 Variables & Parameters

All extra variables (neither choice nor state) is:

$$a_t, \cdots, \mu_t \tag{17}$$

And parameters are:

$$\sigma_y = 1, \tag{18}$$

2.0.4 Modelling decisions & parameter choices

This model is to a large extend an extension of the paper **Optimal Financial Knowledge and Wealth Inequality**(REF), but we have made a number of modifications on the model. These modifcations are primarily made, to make the model fit a danish context, and to reduce the size of the state space, that othervise would slow down the model-solving computations significantly. The state space reductions and is discussed in the section 2.0.1.

Since we want to emulate the danish economy, we have made a set of choices. 1) We have removed the out-of-pocket expendature, which is present in the original model. This is due to the fact, that the danish welfare system in covers expenditures from illness, which is what the *oop* should model. 2) We have chosen to model government transfers as a constant, since that is how "folkepension" (which is the retirement benefit in Denmark) works.

Other simplifications and additions to the model is: We have chosen to model the function $r(f_t)$ as a potentially concave function. In the original paper, they model acquiring more financial knowledge increasingly more expensive, and have a linear $r(\cdot)$ function. We have chosen to model investing in financial knowledge, i_t, as a binary variable, where agents incur a fixed cost when investing, and it can therefore be argued, that $r(\cdot)$ in our setup should be concave.

3 Solving the model

To solve this model we need to solve the following problem: find the function φ that satisfies:

$$\varphi : (\mu_t, f_t, a_t, e) \mapsto (c_t^*, t_t^*, \kappa_t^*), V^*(\cdot) \tag{19}$$

In other words, for a given state space which action set/choice set should the agent take, if the agent wishes to maximize its cummulative utility during its lifecycle.

The bellman equation, is what allows to only consider two consequtive periods: $(t, t+1)$. We can solve the model, by backwards induction. Solve the model in period T, where the model terminates.

3.0.1 Case $\kappa_t = 0.55$ and $i_t = 1$

In this case, we have

$$\tilde{R}_\kappa(f_{t+1}) = (1 - 0.55)\bar{R} + 0.55 \times \exp(\bar{r} + r(f_t) + \sigma_\epsilon \epsilon_t)$$

and thus

$$m_{t+1} = \tilde{R}_\kappa(f_{t+1}) \left[m_t - c_t - \pi(1) - c_d\right] + Y_{t+1} + tr_{t+1}$$

where all stochastic elements can be integreated out via Gauss-Hermite integration. Thus, given an education level e, for each point in the state space (m_t, p_t, f_t) we evaluate 4 discrete choices and optimize consumption given these choices. Then, we save this solution $(c_t^*, i_t^*, \kappa_t^*)$.

3.1 Evaluating $E_t[V_{t+1}(m_{t+1}, p_{t+1}, f_{t+1})]$

We are interested in evaluating

$$E_t[V_{t+1}(m_{t+1}, p_{t+1}, f_{t+1})] = \int_{\psi_t} \int_{\xi_t} \int_{\epsilon_t} V_{t+1}(m_{t+1}, p_{t+1}, f_{t+1}) d\epsilon_t d\xi_t d\psi_t$$

which can be done via Gauss-Hermite integration as all shocks are log-normally distributed. We draw 8 nodes for each of the three stochastic elements $\{\psi_j, \xi_j, \epsilon_j\}_{j=1}^{8}$, and we then calculate the expectation of the value function given a realization of the state space $s_{t+1} = (m_{t+1}, p_{t+1}, f_{t+1})$ and a choice (c_t, κ_t, i_t):

$$
\begin{gathered}
E_t[V_{t+1}(s_{t+1})] = E_t[\tilde{R}_\kappa(f_{t+1})a_t + Y_{t+1} + tr_{t+1}] = \\
E_t\left[\tilde{R}_\kappa(f_{t+1})a_t + (\xi_{t+1}\left[Gp_{y,t}\psi_{t+1}\right] + g_{y,e}(t+1)) + tr_{t+1}\right] = \\
E_t\left[\left\{(1-\kappa_t)\bar{R} + \kappa_t\tilde{R}((1-\delta)f_t + i_t)\right\}a_t + (\xi_{t+1}\left[Gp_{y,t}\psi_{t+1}\right] + g_{y,e}(t+1)) + tr_{t+1}\right] = \\
E_t\left[\left\{(1-\kappa_t)\bar{R} + \kappa_t\left[\bar{r} + r\left((1-\delta)f_t + i_t\right) + \epsilon_t\right]\right\}a_t + (\xi_{t+1}\left[Gp_{y,t}\psi_{t+1}\right] + g_{y,e}(t+1)) + tr_{t+1}\right]
\end{gathered}
$$

which we can approximate by

$$
\begin{gathered}
\sum_{j=1}^{8} w_j^\psi \sum_{i=1}^{8} w_i^\xi \sum_{k=1}^{8} w_k^\epsilon \\
\left\{(1-\kappa_t)\bar{R} + \kappa_t\left[\bar{r} + r\left((1-\delta)f_t + i_t\right) + \epsilon_{t+1}^{(k)}\right]\right\} \times \\
\left[m_t + tr_t - c_t - \pi(i_t) - c_d I(\kappa_t > 0)\right] + \left(\xi_{t+1}^{(i)}\left[Gp_{y,t}\psi_{t+1}^{(j)}\right] + g_{y,e}(t+1)\right)
\end{gathered}
$$

3.2 Notes

- One interesting reformulation to the Danish context would be to make pension tranfers fixed after age 65 as a function of the wealth at retirement.
- In their formulation, $\tilde{R}$ is only log-normally distributed meaning that we cannot obtain negative returns on our risky investments. Perhaps that's why the model predicts high participation, whereas this is not the case in real life?

4 Consumption preference

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As in the original text we formulate the consumption preference $n_{e,t}$ the following way:

$$
z(j,k) = (j + 0.7k)^{0.75} \tag{20}
$$

$$
n_{e,t} = z(j_{e,t}, k_{e,t})/z(2,1) \tag{21}
$$

where j denotes the number of parents and k denotes the number of kids. The notation $j_{e,t}$ denotes the average number of parents in a household conditioning on the time and education level. The same goes for $k_{e,t}$ which denotes the average number of kids in a household for a given age of agent and given education level.

In the original paper they use the PSID calculate $j_{e,t}, k_{e,t}$, and the results are not explicitely mentioned in the paper. We therefore simulate results that seem reasonable, where higher education level implies lower fertility. We find that $n_{e,t}$ is a humpshaped function with respect to t.