Problem 27. Let G, H be groups. Prove that the functors $BG \to BH$ are in bijection with group homomorphisms $G \to H$. Build on this to understand the natural transformations between them, and therefore the category $[BG, BH]$.

Solution.

Let ϕ be a homomorphism from $G \to H$. Then, ϕ induces a functor $f_\phi : \mathbf{Grp} \to \mathbf{Grp}$ taking the one element $\ast_G$ of BG to the one element $\ast_H$ of BH and operating on morphisms $g : \ast_G \to \ast_G$ via $f_\phi(g) = \phi(g)$ in BH. Throughout, I will not explicate any more sources and targets of morphisms, as there is only one object and all morphisms are from that object to itself. That is, there is one hom-set and so I can refer to things like "the" identity morphism, etc.

I will show that f_ϕ is a functor. Firstly, it maps identities to identities, as homomorphisms map identity elements to identity elements. To write it in symbols, $f_\phi(e_G) = \phi(e_G) = e_H$ and e_H is the identity morphism in BH. It also respects morphism composition. Suppose g, g' are morphisms of BG with $\phi(g) = h, \phi(g') = h'$. Then $f_\phi(gg') = \phi(gg') = \phi(g)\phi(g') = f_\phi(g)f_\phi(g')$. Thus, f_ϕ is a functor.

Now, take a functor $F : BG \to BH$. I will show that this induces a group homomorphism from $G \to H$. Given g, g' morphisms in BH, $F(gg') = F(g)F(g')$. Interpreting the morphism g in BG as an element of G and the morphism $F(g)$ in BH as an element of H, this is precisely what it means to be a group homomorphism.

So, we have that functors between the $B-$ categories are in bijection with group homomorphisms.

Suppose F, F' are functors $BG \to BH$. A natural transformation $\alpha : F \Rightarrow F'$ would be a collection of morphisms in BH, which is to say, elements of H with a commutative diagram as follows. Notice that because BG only has one element, there is only one component of α and so we need to only consider one commutative diagram and we know its objects.

$$\begin{tikzcd}
\boxed{F\ast_G = \ast_H} \arrow[r, "Fg"] \arrow[d, "\alpha_\ast"'] & \boxed{F\ast_G = \ast_H} \arrow[d, "\alpha_\ast"] \\
\boxed{F'\ast_G = \ast_H} \arrow[r, "F'g"'] & \boxed{F'\ast_G = \ast_H}
\end{tikzcd}$$

For this to commute, we need that $Fg \circ \alpha_\ast = \alpha_\ast \circ F'g$ for all g. We also know that $\alpha_\ast$ is a morphism in BH, which we will denote h. That is, we need $F(g)h = hF'(g)$ for all g. H is a group, though, so we can take the inverse on the left on both sides and conclude that $F(g) = hF'(g)h^{-1}$ for all g. We conclude that the natural transformations between homomorphisms are only those given by conjugation by some element of B.

Problem 30. Let $\alpha : F \Rightarrow G$ be a natural transformation, where $F, G : \mathcal{A} \to \mathcal{B}$. Prove that α is an isomorphism in the functor category $[\mathcal{A}, \mathcal{B}]$ if and only if every component $\alpha_A : FA \to GA$ is an isomorphism

Solution. and another solution. Hallelujah!

Problem 33. Let V be a vector space over a field $\mathbb{F}$ and define $V^* := [V, \mathbb{F}]$ (the space of linear transformation $V \to \mathbb{F}$).

1. Prove that $V \mapsto V^{**}$ is a functor $(-)^{**} : \mathbf{Vect} \to \mathbf{Vect}$
2. Show that the function $e : V \to V^{**}$ given by $e(v)(f) := f(v)$ defines a natural transformation.
3. Show that the natural transformation is an isomorphism if and only if V is finite dimensional

. Show that

Solution. and another solution. Hallelujah!