

**Problem 35.** Let  $\mathcal{C}$  be a category, and  $X \in \mathcal{C}$ . The slice category  $\mathcal{C}/X$  has objects  $(Y, f)$  where  $Y \in \mathcal{C}$  and  $f \in \mathcal{C}(Y, X)$ .

1. Construct the rest of the slice category
2. For a set  $S$  with corresponding discrete category  $dS$ ,  $[dS, \mathbf{Set}] \simeq \mathbf{Set}/S$

### Solution.

The first part of this problem is relatively straightforward. The second is also straightforward, but takes a lot of checking. I use **Bold** text to break up the work into steps.

The equivalence given in part 2 gives us motivation for the morphisms in the slice category: A Set mod S should look like the functor category between  $dS$  and  $\mathbf{Set}$ , which has morphisms natural transformations between functors from  $dS \rightarrow \mathbf{Set}$ .

#### What are the morphisms of $\mathcal{C}/X$ ?

So, given  $(Y, f), (Z, g) \in \mathcal{C}/X$ , we can say that a morphism  $F$  must take  $Y$  to  $Z$  (a morphism in  $\mathcal{C}$ ) and  $f : Y \rightarrow X$  to  $g : Z \rightarrow X$ . We draw:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow ? & & \downarrow 1_X \\ Z & \xrightarrow{g} & X \end{array}$$

So, our  $?$  ought to be the morphism  $h : Y \rightarrow Z$  such that  $g \circ h = f$ .

That is,  $F$  is a morphism  $F : Y \rightarrow Z$  with  $g \circ F = f$ . Morphism composition is given by morphism composition in  $\mathcal{C}$ , and the identity is the identity in the underlying category as well.

#### Showing that $\mathcal{C}/X$ is actually a category

We must show that  $\mathcal{C}/X$  is a category with this construction. We will first check that morphism composition is sensible, then that identities function how we want. We inherit associativity from the underlying category.

Given  $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2) \xrightarrow{G} (Y_3, f_3)$ ,  $G \circ F$  has the right sources and targets (it maps from  $Y_1 \rightarrow Y_3$ ). Then, using associativity of morphisms in the underlying category and the condition on morphisms in the slice category,  $f_3 \circ (G \circ F) = f_2 \circ F = f_1$ . Hence, the composition of two morphisms is a morphism!

Lastly, we check identities. Let  $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2)$ . We said that the identity on  $(Y_1, f_1)$  ought to be  $1_{Y_1}$ .  $f_2 \circ (F \circ 1_{Y_1}) = f_2 \circ F = f_1$ . This proves right identities. Left identities follow identically.

We conclude that  $\mathcal{C}/X$  is a category. □

#### Roadmap

To show equivalence, we can construct two functors, one from  $[dS, \mathbf{Set}]$  to  $\mathbf{Set}/S$  and one the other way. To define such a functor, we must first show what it does on objects, then morphisms, check that it respects composition and identities, and lastly we will show that the functors compose in a way that is naturally isomorphic to the identity (both ways).

### Building the functor $J : [dS, \mathbf{Set}] \rightarrow \mathbf{Set}/S$

Given a functor  $F : dS \rightarrow \mathbf{Set}$ , we must send it to a pair  $(X, f : X \rightarrow S)$  in  $\mathbf{Set}/S$ .  $F$  has two parts: on objects, it maps  $s \mapsto Y_s$ . On morphisms (of which there are only identities), it maps  $1_s \mapsto 1_Y$ . We can map  $F$  to the pair  $(\coprod_S F(s), \pi : \coprod_S F(s) \rightarrow S)$  where  $\pi(k) = s$  if  $k \in F(s)$ . This is well defined, because even if  $F$  maps two elements  $s, s'$  of  $dS$  to the same set, we use the disjoint product to identify where each  $s, s'$  came from. Let's call the functor performing this operation  $J$  for "Jake has no better ideas for notation."

We must see what  $J$  does on morphisms. Morphisms in the functor category are natural transformations. Given  $\alpha : F \Rightarrow G$ , we define  $J(\alpha) = \coprod_S \alpha_s$  where  $\alpha_s$  are morphisms  $p_s : F(s) \rightarrow G(s)$  in  $\mathbf{Set}$ . This has the proper source and target.  $dS$  is discrete, so the only  $f : s \rightarrow s'$  is the identity on  $s$ . The naturality square of  $\alpha$  just says that

$$\begin{array}{ccc} F_s & \xrightarrow{1_{F_s}} & F_s \\ \alpha_s \downarrow & \searrow \alpha_s & \downarrow \alpha_s \\ G_s & \xrightarrow{1_{G_s}} & G_s \end{array}$$

This obviously commutes.

Let  $JF_\pi$  be the projection part of  $JF$ , and similarly for  $JG$ . We need to first show that  $JG_\pi \circ J\alpha = JF_\pi$ . Suppose  $k \in F(s_0)$ . Then,  $p_s(k) \in G(s_0)$

Then  $JF_\pi(k) = s_0$ , and  $JG_\pi \circ \coprod_S p_s(k) = s_0$ .

#### Showing $J$ is a functor

Now, we show  $J$  is a functor: The identity natural transformation  $1_F$  has components identities  $1_{F_s} = 1_{F_s}$ . We need that  $J(1_F) = 1_{JF}$ . So, we write  $J(1_F) = \coprod_S 1_{F(s)} = 1_{\coprod_S F(s)} = 1_{JF}$

Now, consider  $\alpha, \beta : F \Rightarrow G$  where  $F, G \in [dS, \mathbf{Set}]$  and their vertical composition  $\alpha \circ \beta$ .  $J(\alpha \circ \beta) = \coprod_S \alpha_s \circ \beta_s = \coprod_S \alpha_s \circ \coprod_S \beta_s = J(\alpha) \circ J(\beta)$ .

Thus,  $J$  is a functor.

### Building the functor $K : \mathbf{Set}/S \rightarrow [dS, \mathbf{Set}]$

Given  $(Y, f : Y \rightarrow S)$  in the slice category, we must map it to a functor  $F$  from  $dS \rightarrow \mathbf{Set}$ . First, we build the functor  $F$ , then show that it is a functor. We define  $F$  on objects to take  $s \mapsto f^{-1}(s)$ . On morphisms, the only morphisms of  $dS$  are the identities, and so they go to the corresponding identities. The only morphism composition that  $F$  needs to respect is repeated identity composition, which it clearly does. Hence,  $F$  is a functor. That is,  $K(Y, f) = F$ .

Now,  $K$  must take morphisms of  $\mathbf{Set}/S$  to natural transformations between functors between  $dS \rightarrow \mathbf{Set}$ . Luckily, we had defined our morphisms of the slice category so that this would make sense! Given a morphism  $H : (Y, f) \rightarrow (Z, g)$ , where  $K(Y, f) = F$  and  $K(Z, g) = G$ , we think of  $H$  as a morphism  $h : Y \rightarrow Z$  with  $g \circ h = f$ .

So,  $K(H)$  must be a natural transformation given by components  $KH_s : F(s) \rightarrow G(s)$ . By the same argument with  $J$ , the naturality square automatically commutes as there is only the identity morphism from each  $s \rightarrow s$  in  $dS$ . We can let  $KH_s = h|_{f^{-1}(s)}$ . We know

that  $g \circ h = f$ , so if  $x \in f^{-1}(s)$ , then  $g(h(x)) = s$ , so  $h(x) \in g^{-1}$ . Hence  $KH_s$  has the proper source ( $f^{-1}(s)$ ) and target ( $g^{-1}(s)$ ).

**Showing  $K$  is a functor** Identities:  $K(1_X)_s = 1_X|_{f^{-1}(s)} = 1_{KX}$ .

Composition:  $K(X \xrightarrow{F} Y \xrightarrow{G} Z)_S = (g \circ f)_{(g \circ f)^{-1}(s)} : x \in (f \circ g)^{-1}(s) \iff fx \in g^{-1}(s)$ . So, we can split the restriction to  $g|_{g^{-1}(s)} \circ f|_{f^{-1}(s)}$ . This is  $KG \circ KF$ .

**Proving equivalence** We need to show that  $KJ$  and  $JK$  are both naturally isomorphic to the identity, beginning with  $JK$ .

$$\begin{aligned} JK((Y, f : Y \rightarrow S)) &\xrightarrow{F} (Z, g : Z \rightarrow S)) \\ &= J(f^{-1}(S)) \xrightarrow{F|_{f^{-1}(S)}} g^{-1}(S)) \\ &= (\coprod_S f^{-1}(s), \pi_f) \xrightarrow{\coprod_S F|_{f^{-1}(s)}} (\coprod_S g^{-1}(s), \pi_g) \end{aligned}$$

We must find an isomorphism  $\Phi$  from  $\mathbf{Set}/S \Rightarrow \mathbf{Set}/S$  taking  $(Y, f)$  to  $\coprod_S f^{-1}(s)$  such that the following diagram commutes.

$$\begin{array}{ccc} (\coprod_S f^{-1}(s), \pi_f) & \xrightarrow{\coprod_S h|_{f^{-1}(s)}} & (\coprod_S g^{-1}(s), \pi_g) \\ \Phi_{Y,f} \downarrow & & \downarrow \Phi_{Z,g} \\ (Y, f) & \xrightarrow{h} & (Z, g) \end{array}$$

Given some  $(s, y) \in \coprod_S f^{-1}(s)$  (thinking about the disjoint union as a tagged union), we take  $\Phi_{Y,f}((s, y), \pi_f) = (y, f)$ .

Then,  $(h \circ \Phi_{Y,f})((s, y), \pi_f) = (h(y), g)$  and  $(\Phi_{Z,g} \circ \coprod_S h|_{f^{-1}(s)})((s, y), \pi_f) = \Phi_{Z,g}((h(s), h(y)), \pi_g) = (h(y), g)$ , so the square commutes.

Moreover,  $\Phi$  is invertible: we can define  $\Phi_{Y,f}^{-1}(y, f) = ((s, y), \pi_f)$  where  $s$  is an element (choose one!) such that  $y \in f^{-1}(s)$ . Then,  $\Phi \circ \Phi^{-1}$  is clearly the identity, and  $\Phi^{-1} \circ \Phi$  is isomorphic to the identity (by choosing the "right"  $s$ ). We can just redefine  $\Phi$  to choose such  $s$ , and this gives us that the natural transformation  $\Phi$  is in fact a natural isomorphism from  $JK$  to the identity. Hence,  $JK \cong 1$ .

Lastly, we show that  $KJ \cong 1$ .

$$\begin{aligned} KJ(F \Rightarrow_\alpha G) &= \\ K\left(\left(\coprod_S F(s), \pi_F\right) \xrightarrow{\coprod_S \alpha_s} \left(\coprod_S G(s), \pi_G\right)\right) &= \\ \pi_F^{-1} F(s) \Rightarrow_\alpha \pi_G^{-1} G(s) &= \end{aligned}$$

Given (the object part of a) functor:  $f : dS \rightarrow Y$ , we follow it around:  $KJ(f) = K(\coprod_S f(s), \pi_f) = \pi_f^{-1}$ .  $KJ(f)(s) = \{(s, y) | f(s) = y\}$ . We can call this set  $Y_s$  where

$Y \cong \coprod_S Y_s$  and the isomorphism is natural in **Set**. If we show this isomorphism, we will have that  $KJ$  takes  $F$  to something isomorphic to  $F$ . I will not go through the details of that—effectively, the tagged inverse images partition  $Y$  and we just take the functor that glues the pieces together. We glue the morphisms together in the same way. That is, that  $KJ$  is equivalent to the identity on elements. By the same logic,  $KJ$  is equivalent to the identity on morphisms (natural transformations).

Hence,  $KJ \cong 1$ , so  $[dS, \mathbf{Set}] \simeq \mathbf{Set}/S$ , and we are done!  $\square$

**Problem 38.** Let  $\mathcal{K}$  be a 2-category with horizontal composition given by a functor

$$\mathcal{K}(B, C) \times \mathcal{K}(A, B) \rightarrow \mathcal{K}(A, C)$$

Prove that the composition functor preserving vertical composition of 2-cells is the same equality as in the interchange law.

**Solution.** Vertical composition of 2-cells is composition of morphisms in the category of 1 and 2 cells (composing 2-cells along 1-cells). Horizontal composition of 2-cells is like composing 2-cells along 0-cells.

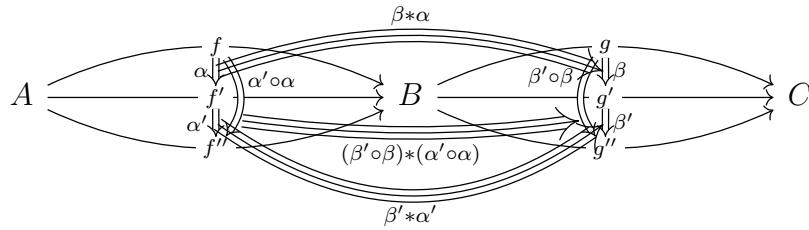
For horizontal composition to be a functor, it must respect identities and composition. In particular, respecting composition means that if  $f, f', f''$  are objects (1-cells) in  $\mathcal{K}(A, B)$  with morphisms (2-cells)  $\alpha : f \Rightarrow f'$  and  $\alpha' : f' \Rightarrow f''$  and similarly in  $\mathcal{K}(B, C)$  has morphisms  $\beta : g \Rightarrow g'$  and  $\beta' : g' \Rightarrow g''$ .

The functor  $*$  takes objects of the product category (pairs of 1-cells)  $(g, f)$  to the object (1-cell)  $g \circ f$  in the category  $\mathcal{K}(A, C)$ .

On morphisms,  $*$  takes morphisms in the product category (pairs of 2-cells)  $(\beta, \alpha)$  to the 2-cell  $\beta * \alpha$  in  $\mathcal{K}(A, C)$ ,

For  $*$  to respect vertical composition (which is to say, composition in the product and target categories), first notice that  $(\beta', \alpha') \circ (\beta, \alpha) = (\beta' \circ \beta, \alpha' \circ \alpha)$ . The functor  $*(a, b) = a * b$  must have that  $*((\beta', \alpha') \circ (\beta, \alpha)) = *((\beta', \alpha')) \circ *((\beta, \alpha))$ . This is precisely the interchange law.

We observe the abomination:



That is, the path from  $\alpha$  to  $\beta$  vertically composed with the path from  $\alpha'$  to  $\beta'$  should be the same as the path from  $\alpha' \circ \alpha$  to  $\beta' \circ \beta$ .

In symbols,  $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$

**Problem 41.** Write out the details for the 2-category of posets, order-preserving functions, and function domination

**Solution.**

**Problem 44.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor and  $G_1, G_2 : \mathcal{B} \rightarrow \mathcal{C}$  a pair of functors with a natural transformation  $\alpha : G_1 \Rightarrow G_2$ . We will write  $\alpha F = \alpha * 1_F$  and call it "the whiskering of  $\alpha$  by  $F$ ".

1. Prove that for transformations  $\beta, \alpha$  for which  $\beta * \alpha$  is defined, this composite can be derived from vertical composition and whiskering (on both sides).
2. Rewrite the definition of a 2-category using whiskering instead of a composition functor.

**Solution.**