Problem 27. Let G, H be groups. Prove that the functors $BG \to BH$ are in bijection with group homomorphisms $G \to H$. Build on this to understand the natural transformations between them, and therefore the category $[BG, BH]$.

Solution.

Let ϕ be a homomorphism from $G \to H$. Then, ϕ induces a functor $f_\phi : \mathbf{Grp} \to \mathbf{Grp}$ taking the one element $*_G$ of BG to the one element $*_H$ of BH and operating on morphisms $g : *_G \to *_G$ via $f_\phi(g) = \phi(g)$ in BH. Throughout, I will not explicate any more sources and targets of morphisms, as there is only one object and all morphisms are from that object to itself. That is, there is one hom-set and so I can refer to things like "the" identity morphism, etc.

I will show that f_ϕ is a functor. Firstly, it maps identities to identities, as homomorphisms map identity elements to identity elements. To write it in symbols, $f_\phi(e_G) = \phi(e_G) = e_H$ and e_H is the identity morphism in BH. It also respects morphism composition. Suppose g, g' are morphisms of BG with $\phi(g) = h, \phi(g') = h'$. Then $f_\phi(gg') = \phi(gg') = \phi(g)\phi(g') = f_\phi(g)f_\phi(g')$. Thus, f_ϕ is a functor.

Now, take a functor $F : BG \to BH$. I will show that this induces a group homomorphism from $G \to H$. Given g, g' morphisms in BH, $F(gg') = F(g)F(g')$. Interpreting the morphism g in BG as an element of G and the morphism $F(g)$ in BH as an element of H, this is precisely what it means to be a group homomorphism.

So, we have that functors between the $B-$ categories are in bijection with group homomorphisms.

Suppose F, F' are functors $BG \to BH$. A natural transformation $\alpha : F \Rightarrow F'$ would be a collection of morphisms in BH, which is to say, elements of H with a commutative diagram as follows. Notice that because BG only has one element, there is only one component of α, namely α_* and so we need to only consider one commutative diagram and we know its objects.

$$\begin{tikzcd}
\boxed{F*_G = *_H} \arrow[r, "Fg"] \arrow[d, "\alpha_*"'] & \boxed{F*_G = *_H} \arrow[d, "\alpha_*"] \\
\boxed{F'*_G = *_H} \arrow[r, "F'g"'] & \boxed{F'*_G = *_H}
\end{tikzcd}$$

For this to commute, we need that $Fg \circ \alpha_* = \alpha_* \circ F'g$ for all g. We also know that α_* is a morphism in BH, which we will denote h. That is, we need $F(g)h = hF'(g)$ for all g. H is a group, though, so we can take the inverse on the left on both sides and conclude that $F(g) = hF'(g)h^{-1}$ for all g. We conclude that the natural transformations between homomorphisms are only those given by conjugation by some element of H. □

Problem 30. Let $\alpha : F \Rightarrow G$ be a natural transformation, where $F, G : \mathcal{A} \to \mathcal{B}$. Prove that α is an isomorphism in the functor category $[\mathcal{A}, \mathcal{B}]$ if and only if every component $\alpha_A : FA \to GA$ is an isomorphism

Solution.

($\implies$) Suppose that α is an isomorphism in the functor category. Then, there exists a $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. I will prove that the components of β

are left inverses of the components of α. The proof of right inverses is near-identical, and I will elaborate on the right-inverse argument later.

Given a morphism $f : A \to A'$ in $\mathcal{A}$, because α and β are natural transformations the following diagram commutes (both the top and bottom square commute by definition, and hence the entire thing does).

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FA' \arrow[d, "\alpha_{A'}"] \\
GA \arrow[r, "Gf"] \arrow[d, "\beta_A"'] & GA' \arrow[d, "\beta_{A'}"] \\
FA \arrow[r, "Ff"] & FA'
\end{tikzcd}$$

In particular, the outer square commutes. Taking the outer square, we note the existence of one more arrow:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[ddr, "Ff"] & FA' \arrow[d, "\alpha_{A'}"] \\
GA \arrow[d, "\beta_A"'] & GA' \arrow[d, "\beta_{A'}"] \\
FA \arrow[r, "Ff"] & FA'
\end{tikzcd}$$

This is precisely the statement that $\beta_A \circ \alpha_A = 1_{FA}$. That is, β_A is a left inverse for α_A. The exact same argument but with the naturality square having (vertically) GA, FA, GA gives the same statement but for the right inverse. We conclude that β_A is the inverse of α_A, which is to say, that α_A is an isomorphism.

($\Longleftarrow$) Suppose each component of α is an isomorphism. Let $\beta_A : G \to F$ be the inverse of $\alpha_A : F \to G$ for each α_A. We will show that the β_A's form a natural transformation, and that the natural transformation in question is an inverse for α under vertical composition. Let $f : A \to A'$ be a morphism.

Our question, then, is if the first square commutes given that the second one does (I've drawn the second one upside down because it makes more sense to me for this problem):

$$\begin{tikzcd}
GA \arrow[r, "Gf"] \arrow[d, "\beta_A"'] & GA' \arrow[d, "\beta_{A'}"] \\
FA \arrow[r, "Ff"] & FA'
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[r, "Gf"] & GA' \\
FA \arrow[r, "Ff"] \arrow[u, "\alpha_A"] & FA' \arrow[u, "\alpha_{A'}"']
\end{tikzcd}$$

The second square tells us that $\alpha_{A'} \circ Ff = Gf \circ \alpha_A$. Taking $\beta_{A'}$ on the left, we see $Ff = \beta_{A'} \circ Gf \circ \alpha_A$. Morphisms are associative, so the statement on the right is sensible. We then take β_A on the right and see that $Ff \circ \beta_A = \beta_{A'} \circ Gf$. That means that the naturality square for β commutes! Hence, β is a natural transformation, and obviously, $\beta \circ \alpha = 1_F, \alpha \circ \beta = 1_G$. So, β is the inverse of α as a morphism in the functor category. Hence, α is an isomorphism in the functor category! $\square$

Problem 32. Let G be a group and define G^R to have the same elements but with multiplication given by $g \cdot_R h := hg$.

1. Prove that $G \mapsto G^R$ is a functor
2. Prove that the function $i : G \to G^R$ given by $i(g) = g^{-1}$ is a natural isomorphism between the identity functor and $(-)^R$.

Solution.

1. On objects, $(-)^R$ takes the group G to the group G^R. That's simple enough.

   On morphisms $\phi : G \to H$, I define $\phi^R : G^R \to H^R$ given by $\phi^R(g) = \phi(g)$ with composition given by function composition. These are homomorphisms: $\phi^R(g \cdot_R g') = \phi(g'g) = \phi(g')\phi(g) = \phi(g) \cdot_R \phi(g') = \phi^R(g) \cdot_R \phi^R(g')$.

   The identity morphisms are simply the identity homomorphisms $1_{G^R} = 1_G^R = 1_G$.

   Now, we check the axioms. We know $1_G^R(g) = 1_G(g) = g$. So, given $\phi^R : G^R \to H^R$, we can check $(\phi^R \circ 1_{G^R})(g) = (\phi \circ 1_G)(g) = \phi(g) = \phi^R(g)$. The left sided inverse follows identically.

   Moreover, $(f \circ g)^R = f \circ g = f^R \circ g^R$ by definition. Hence, $(-)^R$ is a functor. $\square$

2. Now, we need to show that there is a natural isomorphism between id and $(-)^R$ as given. We will first show that there is a natural transformation. That means the following square must commute for every group G.

$$\begin{tikzcd} id(G) \arrow[r, "id(\phi)"] \arrow[d, "i_G"] & id(H) \arrow[d, "i_H"] \\ G^R \arrow[r, "\phi^R"] & H^R \end{tikzcd}$$

   Where $i_G(g) = g^{-1}$ is a morphism in G^R.

   In symbols, we need $i_H \circ id(\phi) = \phi^R \circ i_G$.

   First, we ought to check that i_G is a morphism in G^R. $i_G(g \cdot g') = (g \cdot g')^{-1} = g'^{-1} \cdot g^{-1} = g^{-1} \cdot_R g^{-1} = i_G(g) \cdot_R i_G(g')$. Yep!

   $i_H \circ id(\phi) = i_H \circ \phi$. Let's apply this to some arbitrary $g \in G$. $i_H(\phi(g)) = \phi(g)^{-1} = \phi(g^{-1}) = \phi(i_G(g)) = \phi^R(i_G(g))$. Hence, the square commutes for every g, and thus commutes in general. Hence, the natural transformation i with components i_G is a natural transformation.

   To show that i is a natural isomorphism, we can, by problem 30, find morphisms i_G^{-1} for each component i_G of i. Luckily, every i_G is its own inverse. $i_G(i_G(g)) = g$, and so i_G is in fact, an isomorphism. Hence, i is a natural isomorphism. $\square$