Problem 46. Give a direct proof that $H_A \cong H_{A'} \implies A \cong A'$

Solution. So I don't need to put curly braces around all of my subscripts (thanks LaTeX), I'll replace A' with B. We know $H_A : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is a functor, and so isomorphisms are given by natural isomorphisms. Suppose we have such a natural isomorphism $\alpha : H_A \Rightarrow H_B$ with components $\alpha_C : H_A(C) \to H_B(C)$ for $C \in \mathcal{C}^{\text{op}}$.

Then, the following diagram commutes for every $f^{\text{op}} : C \to D$ derived from a $f : D \to C$

$$\begin{tikzcd}
H_A(C) = \mathcal{C}(C, A) \arrow[r, "H_A(f^{\text{op}}) := - \circ f"] \arrow[d, "\alpha_C"'] & H_A(D) = \mathcal{C}(D, A) \arrow[d, "\alpha_D"] \\
H_B(C) = \mathcal{C}(C, B) \arrow[r, "H_B(f^{\text{op}}) := - \circ f"'] & H_B(D) = \mathcal{C}(D, B)
\end{tikzcd}$$

In particular, for $C = A$, we must have the identity morphism 1_A. To show that $A \cong B$, we need an invertible morphism $g : A \to B$. Let's chase 1_A around this commuting diagram in the hope of getting one...

$$\begin{tikzcd}
H_A(A) = \mathcal{C}(A, A) \arrow[rrr, "H_A(f^{\text{op}}) := - \circ f"] \arrow[ddd, "\alpha_A"'] & & & H_A(D) = \mathcal{C}(D, A) \arrow[ddd, "\alpha_D"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& \alpha_A(1_A) \arrow[r, mapsto] & \alpha_A(1_A) \circ f = \alpha_D(f) & \\
H_B(A) = \mathcal{C}(A, B) \arrow[rrr, "H_B(f^{\text{op}}) := - \circ f"'] & & & H_B(D) = \mathcal{C}(D, B)
\end{tikzcd}$$

α_A is a morphism from $H_A(A) \to H_B(A)$, and so $g = \alpha_A(1_A)$ is an element of $\mathcal{C}(A, B)$. By the invertibility of α, we can find $h = \alpha_B^{-1}(1_B)$, a morphism $B \to A$. All that remains is to show that they are inverses. We choose $f = h$, and conclude that the following commutes, which is to say that $gh = 1_B$. Walking backwards along the diagram gives us the statement that $hg = 1_A$. Thus, $A \cong B$. $\square$

$$\begin{tikzcd}
H_A(A) = \mathcal{C}(A, A) \arrow[rrr, "H_A(h^{\text{op}}) := - \circ h"] \arrow[ddd, "\alpha_A"'] & & & H_A(B) = \mathcal{C}(B, A \arrow[ddd, "\alpha_B"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_B^{-1}(1_B) \arrow[d, mapsto] & \\
& \alpha_A(1_A) \arrow[r, mapsto] & \alpha_A(1_A) \circ \alpha_B(1_B) = \alpha_B(\alpha_B^{-1}(1_B)) = 1_B & \\
H_B(A) = \mathcal{C}(A, B) \arrow[rrr, "H_B(h^{\text{op}}) := - \circ h"'] & & & H_B(B) = \mathcal{C}(B, B
\end{tikzcd}$$

Problem 49. Using the methods in the proof of the Yoneda lemma (NO USING THE ACTUAL STATEMENT OF THE YONEDA LEMMA), prove that the Yoneda embedding is faithful.

Solution. For the Yoneda embedding $H_\bullet : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to be faithful, we need the morphism part of the functor to be injective. That is, $H_f = H_g \implies f = g$.

Let $f, g : A \to B$ in $\mathcal{C}(A, B)$. We write $H_f : H_A \Rightarrow H_B$, where components $(H_f)_A : H_A(A) \to H_B(A)$ for A are given by $- \circ f$, and similarly for g.

As in the proof of the Yoneda lemma, we abuse that natural transformations of representable functors are defined fully by their action on the identity of their representative.

Suppose that $H_f = H_g$. They must be equal at all components. In particular, for any A, $(H_f)_A(h) = h \circ f = (H_g)_A h = h \circ g$. We choose $h = 1_A$. Then, $f = g$. $\square$