Problem 5. Let $\mathcal{C}$ be a category and $\simeq$ an equivalence relation on the morphisms of $\mathcal{C}$ such that

1. $f \simeq g \implies f, g$ have the same source and target
2. Given f, f', g, g' where $f \circ f'$ and $g \circ g'$ are defined, $f \simeq g$ and $f' \simeq g' \implies f \circ f' \simeq g \circ g'$

Write a category $\bar{\mathcal{C}} = \mathcal{C}/\simeq$. Perform the construction explicitly for $\mathcal{C} = \mathbf{Grp}$ and $\simeq$ equivalence up to conjugation.

Solution. For notational convenience, I'll refer to the objects of $\bar{\mathcal{C}}$ in the same way I do the objects of $\mathcal{C}$, but the morphisms of $\bar{\mathcal{C}}$ with a bar above them.

Indeed, the objects of $\bar{\mathcal{C}}$ are the objects $\mathrm{ob}(\mathcal{C})$. The morphisms are $\bar{\mathcal{C}}(A, B) = \mathcal{C}(A, B)/\simeq$. The identity morphisms in $\bar{\mathcal{C}}$ are the conjugacy classes of the identity morphisms in $\mathcal{C}$.

To show that our identity behaves well, let $1_A, \iota_A$ be representatives of $\bar{1_A}$, where 1_A is the actual identity in $\mathcal{C}$ and f, f' representatives of $\bar{f} : A \to B$ where f is the actual morphism f in $\mathcal{C}$. Our goal is to show that $\bar{f} \circ \bar{1_A} = \bar{f}$. The other direction of the identity axiom follows similarly. By the first condition on $\simeq$, all of these statements are sensible. By the second condition on $\simeq$, we can conclude $f = 1_A \circ f \simeq \iota_A \circ g$. Hence, $\bar{1_A}$ is indeed the identity on A.

Associativity is also sensible, and carries much the same proof as identities: let $A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C$ with representatives as before. Then by the second condition on $\simeq$ and the fact that $\mathcal{C}$ is a category (and thus $f \circ g$ is defined), $\bar{\mathcal{C}}$ satisfies the associativity property. $\square$

As an example, in the category $\mathbf{Grp}$, $A \xrightarrow{f,f'} B \xrightarrow{g,g'} C$ are groups and group homomorphisms where $\exists b \in B$ such that $f(a) = bf'(a)b^{-1}$ for all $a \in A$ and similarly $\exists c \in C$ s.t. $g(b) = cg'(b)c^{-1} \forall b \in B$. Let's show that the equivalence preserves associativity with this setup: $(g \circ f)(a) = g(bf'(a)b^{-1}) = cg'(bf'(a)b^{-1})c^{-1} = cg'(b)g'(f'(a))g'(b^{-1})c^{-1} = (cg'(b))g'(f'(a))(cg'(b))^{-1} \simeq (g' \circ f')(a)$.

The identity has pretty much the same proof: $1_A \simeq \iota_A \iff 1_A(a) = a'\iota_A(a)a'^{-1} \iff a'^{-1}aa' = \iota_A(a)$. We will consider the left identity this time: let $f \simeq f' : A \to B$ under conjugation by b and $1_B \simeq \iota_B$ under conjugation by b'. Then, consider $f(a) = (1_B \circ f)(a) = 1_B(bf'(a)b^{-1}) = b'\iota_B(bf'(a)b^{-1})b'^{-1} = (b'b)\iota_B(f'(a))(b'b)^{-1} \simeq (\iota_B \circ f')(a)$.

Indeed, $\mathbf{Grp}$ can be taken mod homomorphism conjugation to yield a new category.

Problem 8. Given a set X, let $\mathcal{P}X$ be the power set of X. Build a category Γ with the following properties:

1. $\mathrm{ob}(\Gamma)$ consists of all finite sets
2. A morphism $S \rightsquigarrow T$ is a function $f : S \to \mathcal{P}T$ such that for any $s_1 \neq s_2 \in S, f(s_1) \cap f(s_2) = \emptyset$.

Solution. The objects and morphisms of our category have already been defined. So, I need to define composition and identities and check the axioms.

Given $R \xrightarrow{g} S \xrightarrow{f} T$, I define a function $f \circ g : R \to \mathcal{P}T$ such that $\forall r \in R, (f \circ g)(r) := \bigcup_{s \in g(r)} f(s)$. We need to check that such a morphism satisfies our rule above. Suppose $r_1 \neq r_2 \in R$. Let $t_1 \in (f \circ g)(r_1), t_2 \in (f \circ g)(r_2)$. If we can show that t_1 cannot equal t_2, that would mean the two sets they come from are disjoint. By our construction, t_1 must have come from one of the sets $f(s_1)$ for some $s_1 \in g(r_1)$. Similarly, t_2 comes from some $f(s_2)$ for some $s_2 \in g(r_2)$. But, by definition of g, $g(r_2) \cap g(r_1) = \emptyset$, and so $s_1 \neq s_2$. Similarly, by definition of f, we conclude that $f(s_1) \cap f(s_2) = \emptyset$, and so $t_1 \neq t_2$. Hence, $(f \circ g)(r_1) \cap (f \circ g)(r_2) = \emptyset$.

With our composition rule established, we'll show associativity first.

Given $Q \xrightarrow{h} R \xrightarrow{g} S \xrightarrow{f} T$

$(f \circ (g \circ h))(q) = \bigcup_{s \in (g \circ h)(q)} f(s) = \bigcup_{s \in \bigcup_{r \in h(q)} g(r)} f(s) = \bigcup_{r \in h(q)} \bigcup_{s \in g(r)} f(s) = \bigcup_{r \in h(q)} (f \circ g)(r) = ((f \circ g) \circ h)(q)$, as desired!

As identities, we will take the function $1_S : S \to \mathcal{P}S$ where $s \mapsto \{s\}$. Indeed, given $S \xrightarrow{f} T$, $(1_T \circ f)(s) = \bigcup_{t \in f(s)} 1_T(t) = f(s)$ and $(f \circ 1_S)(s) = \bigcup_{s' \in 1_S(s')} f(s') = f(s)$. Hence, our identities are indeed identities. Thus, Γ is a category! Yay! $\square$

Problem 12.

1. Prove if $\mathcal{S}_1 \subseteq \mathcal{C}_1$ and $\mathcal{S}_2 \subseteq \mathcal{C}_2$ are subcategories, that $\mathcal{S}_1 \times \mathcal{S}_2$ is a subcategory of $\mathcal{C}_1 \times \mathcal{C}_2$.

2. Does the same hold for disjoint unions?

Solution.

We inherit a notion of composition and identity from the ambient categories, so we really just need to check that $\mathcal{S}_1 \times \mathcal{S}_2$ is closed under composition and has identities.

There's a good bit of defining to do...Let $A_1, B_1 \in \mathcal{S}_1, A_2, B_2 \in \mathcal{S}_2$ with corresponding identities $1_{A_1}, 1_{A_2}$, etc. and morphisms $f, f' : A_1 \to B_1, g, g' : A_2 \to B_2$.

Let's take care of the identities first. I claim that $1_A = (1_{A_1}, 1_{A_2})$ is an identity for $A = (A_1, A_2)$. and similarly for $B = (B_1, B_2)$. Everything mentioned so far is in $\mathcal{S}_1 \times \mathcal{S}_2$ by construction and definition of subcategory. Consider any morphism $h = (f, g)$. Then, $h \circ 1_A = (f \circ 1_{A_1}, g \circ 1_{A_2}) = (f, g)$. The same occurs on the other side with the same logic.

Now, we ought to check composition. Within $\mathcal{S}_1 \times \mathcal{S}_2$, let $h : A \to B$ and $h' : B \to C$ in the product category. $h' \circ h = (f', g') \circ (f, g) = (f' \circ f, g' \circ g)$, which is in $\mathcal{S}_1 \times \mathcal{S}_2$.

Hence, $\mathcal{S}_1 \times \mathcal{S}_2$ is a subcategory of $\mathcal{C}_1 \times \mathcal{C}_2$. $\square$

I also claim that the same holds for disjoint unions, as any subcategory of a single one of the two categories is pretty clearly a subcategory of the disjoint union of the ambient. Given two subcategories, their disjoint union ought to be a subcategory of the disjoint union of the ambient categories. The toughest part ends up being notation, so please pardon the following abomination.

In the disjoint union case, we inherit our objects and morphisms, so composition and identities are well defined. I claim that the identity within $\mathcal{S}_1 \coprod \mathcal{S}_2(A, A)$ is the identity from whichever $\mathcal{S}A$ was from. WLOG A is in $\mathcal{S}_1$. Denote $1_A \in \mathcal{S}_1$ by $1'_A$ and $f : A \to B$, where $B \in \mathcal{S}_1$. "Tagging" the elements I'll write $(1, f)$ to denote f from $\mathcal{S}_1$, and similarly for other items. Then, $(1, f) \circ (1, 1'_A) = (1, f)$. Then, $1_A \in \mathcal{S}_1 \coprod \mathcal{S}_2(A, A) = (1, f)$. Hence, we are good with identities.

The only time composition of morphisms is sensible is when both objects are from the same category initially. Again, WLOG, suppose we have $A, B, C \in \mathcal{S}_1$ with morphisms $f : A \to B, g : B \to C$. Then, $(1, g) \circ (1, f) = (1, g \circ f)$, which is in the disjoint union because, again, $\mathcal{S}_1$ is a subcategory. Hence, composition is taken care of!

Thus, $\mathcal{S}_1 \coprod \mathcal{S}_2$ is a subcategory of $\mathcal{C}_1 \coprod \mathcal{C}_2$. $\square$