Problem 35. Let $\mathcal{C}$ be a category, and $X \in \mathcal{C}$. The slice category $\mathcal{C}/X$ has objects (Y, f) where $Y \in \mathcal{C}$ and $f \in \mathcal{C}(Y, X)$.

1. Construct the rest of the slice category
2. For a set S with corresponding discrete category dS, $[dS, \mathbf{Set}] \simeq \mathbf{Set}/S$

Solution.

The first part of this problem is relatively straightforward. The second is also straightforward, but takes a lot of checking. I use **Bold** text to break up the work into steps.

Motivating Part 1

The equivalence given in part 2 gives us motivation for the morphisms in the slice category: A Set mod S should look like the functor category between dS and Set, which has morphisms natural transformations between functors from $dS \to \mathbf{Set}$.

What are the morphisms of $\mathcal{C}/X$?

So, given $(Y, f), (Z, g) \in \mathcal{C}/X$, we can say that a morphism F must take Y to Z (a morphism in $\mathcal{C}$) and $f : Y \to X$ to $g : Z \to X$. We draw:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "?"'] & X \arrow[d, "1_X"] \\
Z \arrow[r, "g"'] & X
\end{tikzcd}$$

So, our ? ought to be the morphism $h : Y \to Z$ such that $g \circ h = f$.

That is, F is a morphism $F : Y \to Z$ with $g \circ F = f$. Morphism composition is given by morphism composition in $\mathcal{C}$, and the identity is the identity in the underlying category as well.

Showing that $\mathcal{C}/X$ is actually a category

We must show that $\mathcal{C}/X$ is a category with this construction. We will first check that morphism composition is sensible, then that identities function how we want. We inherit associativity from the underlying category.

Given $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2) \xrightarrow{G} (Y_3, f_3)$, $G \circ F$ has the right sources and targets (it maps from $Y_1 \to Y_3$). Then, using associativity of morphisms in the underlying category and the condition on morphisms in the slice category, $f_3 \circ (G \circ F) = f_2 \circ F = f_1$. Hence, the composition of two morphisms is a morphism!

Lastly, we check identities. Let $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2)$. We said that the identity on (Y_1, f_1) ought to be 1_{Y_1}. $f_2 \circ (F \circ 1_{Y_1}) = f_2 \circ F = f_1$. This proves right identities. Left identities follow identically.

We conclude that $\mathcal{C}/X$ is a category. □

Part 2 Roadmap

To show equivalence, we can construct two functors, one from $[dS, \mathbf{Set}]$ to $\mathbf{Set}/S$ and one the other way. To define such a functor, we must first show what it does on objects, then morphisms, check that it respects composition and identities, and lastly we will show that the functors compose in a way that is naturally isomorphic to the identity (both ways).

Building the functor $J : [dS, \mathbf{Set}] \to \mathbf{Set}/s$

Given a functor $F : dS \to \mathbf{Set}$, we must send it to a pair $(X, f : X \to S)$ in $\mathbf{Set}/S$. F has two parts: on objects, it maps $s \mapsto Y_s$. On morphisms (of which there are only identities), it maps $1_s \mapsto 1_Y$. We can map F to the pair $(\coprod_S F(s), \pi : \coprod_S F(s) \to S)$ where $\pi(k) = s$ if $k \in F(s)$. This is well defined, because even if F maps two elements s, s' of dS to the same set, we use the disjoint product to identify where each s, s' came from. Let's call the functor performing this operation J for "Jake has no better ideas for notation."

We must see what J does on morphisms. Morphisms in the functor category are natural transformations. Given $\alpha : F \Rightarrow G$, we define $J(\alpha) = \coprod_S \alpha_s$ where α_s are morphisms $p_s : F(s) \to G(s)$ in $\mathbf{Set}$. This has the proper source and target. dS is discrete, so the only $f : s \to s'$ is the identity on s. The naturality square of α just says that

$$\begin{tikzcd} Fs \arrow[r, "1_{Fs}"] \arrow[d, "\alpha_s"'] \arrow[dr, "\alpha_s"] & Fs \arrow[d, "\alpha_s"] \\ Gs \arrow[r, "1_{Gs}"'] & Gs \end{tikzcd}$$

This obviously commutes.

Let JF_π be the projection part of JF, and similarly for JG. We need to first show that $JG_\pi \circ J\alpha = JF_\pi$. Suppose $k \in F(s_0)$. Then, $p_s(k) \in G(s_0)$

Then $JF_\pi(k) = s_0$, and $JG_\pi \circ \coprod_S p_s(k) = s_0$.

Showing J is a functor

Now, we show J is a functor: The identity natural transformation 1_F has components identities $1_{Fs} = 1_{Fs}$. We need that $J(1_F) = 1_{JF}$. So, we write $J(1_F) = \coprod_S 1_{F(s)} = 1_{\coprod_S F(s)} = 1_{JF}$

Now, consider $\alpha, \beta : F \Rightarrow G$ where $F, G \in [dS, \mathbf{Set}]$ and their vertical composition $\alpha \circ \beta$. $J(\alpha \circ \beta) = \coprod_S \alpha_s \circ \beta_s = \coprod_S \alpha_s \circ \coprod_S \beta_s = J(\alpha) \circ J(\beta)$.

Thus, J is a functor.

Building the functor $K : \mathbf{Set}/S \to [dS, \mathbf{Set}]$

Given $(Y, f : Y \to S)$ in the slice category, we must map it to a functor F from $dS \to \mathbf{Set}$. First, we build the functor F, then show that it is a functor. We define F on objects to take $s \mapsto f^{-1}(s)$. On morphisms, the only morphisms of dS are the identities, and so they go to the corresponding identities. The only morphism composition that F needs to respect is repeated identity composition, which it clearly does. Hence, F is a functor. That is, $K(Y, f) = F$.

Now, K must take morphisms of $\mathbf{Set}/S$ to natural transformations between functors between $dS \to \mathbf{Set}$. Luckily, we had defined our morphisms of the slice category so that this would make sense! Given a morphism $H : (Y, f) \to (Z, g)$, where $K(Y, f) = F$ and $K(Z, g) = G$, we think of H as a morphism $h : Y \to Z$ with $g \circ h = f$.

So, $K(H)$ must be a natural transformation given by components $KH_s : F(s) \to G(s)$. By the same argument with J, the naturality square automatically commutes as there is only the identity morphism from each $s \to s$ in dS. We can let $KH_s = h|_{f^{-1}(s)}$. We know that $g \circ h = f$, so if $x \in f^{-1}(s)$, then $g(h(x)) = s$, so $h(x) \in g^{-1}$. Hence KH_s has the proper source $(f^{-1}(s))$ and target $(g^{-1}(s))$.

Showing K is a functor Identities: $K(1_X)_s = 1_X|_{f^{-1}(s)} = 1_{KX}$.

Composition: $K(X \xrightarrow{F} Y \xrightarrow{G} Z)_S = (g \circ f)_{(g \circ f)^{-1}(s)}$. $x \in (f \circ g)^{-1}(s) \iff fx \in g^{-1}(s)$. So, we can split the restriction to $g|_{g^{-1}(s)} \circ f|_{f^{-1}(s)}$. This is $KG \circ KF$.

Proving equivalence We need to show that KJ and JK are both naturally isomorphic to the identity, beginning with JK.

$$\begin{aligned} JK((Y, f : Y \to S) &\xrightarrow{F} (Z, g : Z \to S)) \\ = J(f^{-1}(S) &\xrightarrow{F|_{f^{-1}(S)}} g^{-1}(S)) \\ = (\coprod_S f^{-1}(s), \pi_f) &\xrightarrow{\coprod_S F|_{f^{-1}(s)}} (\coprod_S g^{-1}(s), \pi_g) \end{aligned}$$

We must find an isomorphism Φ from $\mathbf{Set}/S \Rightarrow \mathbf{Set}/S$ taking (Y, f) to $\coprod_S f^{-1}(s)$ such that the following diagram commutes.

$$\begin{tikzcd} (\coprod_S f^{-1}(s), \pi_f) \arrow[r, "\coprod_S h|_{f^{-1}(s)}"] \arrow[d, "\Phi_{Y,f}"'] & (\coprod_S g^{-1}(s), \pi_g) \arrow[d, "\Phi_{Z,g}"] \\ (Y, f) \arrow[r, "h"'] & (Z, g) \end{tikzcd}$$

Given some $(s, y) \in \coprod_S f^{-1}(s)$ (thinking about the disjoint union as a tagged union), we take $\Phi_{Y,f}((s, y), \pi_f) = (y, f)$.

Then, $(h \circ \Phi_{Y,f})((s, y), \pi_f) = (h(y), g)$ and $(\Phi_{Z,g} \circ \coprod_S h|_{f^{-1}(s)})((s, y), \pi_f) = \Phi_{Z,g}((h(s), h(y)), \pi_g) = (h(y), g)$, so the square commutes.

Moreover, Φ is invertible: we can define $\Phi^{-1}_{Y,f}(y, f) = ((s, y), \pi_f)$ where s is an element (choose one!) such that $y \in f^{-1}(s)$. Then, $\Phi \circ \Phi^{-1}$ is clearly the identity, and $\Phi^{-1} \circ \Phi$ is isomorphic to the identity (by choosing the "right" s). We can just redefine Φ to choose such s, and this gives us that the natural transformation Φ is in fact a natural isomorphism from JK to the identity. Hence, $JK \cong 1$.

Lastly, we show that $KJ \cong 1$.

$$\begin{aligned} &KJ(F \Rightarrow_\alpha G) = \\ K((\coprod_S F(s), \pi_F) &\xrightarrow{\coprod_S \alpha_s} (\coprod_S G(s), \pi_G)) = \\ &\pi_F^{-1} F(s) \Rightarrow_\alpha \pi_G^{-1} G(s) = \end{aligned}$$

Given (the object part of a) functor: $f : dS \to Y$, we follow it around: $KJ(f) = K(\coprod_S f(s), \pi_f) = \pi_f^{-1}$. $KJ(f)(s) = \{(s, y) | f(s) = y)\}$. We can call this set Y_s where

$Y \cong \coprod_S Y_s$ and the isomorphism is natural in **Set**. If we show this isomorphism, we will have that KJ takes F to something isomorphic to F. I will not go through the details of that– effectively, the tagged inverse images partition Y and we just take the functor that glues the pieces together. We glue the morphisms together in the same way. That is, that KJ is equivalent to the identity on elements. By the same logic, KJ is equivalent the identity on morphisms (natural transformations).

Hence, $KJ \cong 1$, so $[dS, \mathbf{Set}] \simeq \mathbf{Set}/S$, and we are done! □

Problem 38. Let $\mathcal{K}$ be a 2-category with horizontal composition given by a functor

$$\mathcal{K}(B, C) \times \mathcal{K}(A, B) \to \mathcal{K}(A, C)$$

Prove that the composition functor preserving vertical composition of 2-cells is the same equality as in the interchange law.

Solution. Vertical composition of 2-cells is composition of morphisms in the category of 1 and 2 cells (composing 2-cells along 1-cells). Horizontal composition of 2-cells is like composing 2-cells along 0-cells.

For horizontal composition to be a functor, it must respect identities and composition. In particular, respecting composition means that if f, f', f'' are objects (1-cells) in $\mathcal{K}(A, B)$ with morphisms (2-cells) $\alpha : f \Rightarrow f'$ and $\alpha' : f' \Rightarrow f''$ and similarly in $\mathcal{K}(B, C)$ has morphisms $\beta : g \Rightarrow g'$ and $\beta' : g' \Rightarrow g''$.

The functor $*$ takes objects of the product category (pairs of 1-cells) (g, f) to the object (1-cell) $g \circ f$ in the category $\mathcal{K}(A, C)$.

On morphisms, $*$ takes morphisms in the product category (pairs of 2-cells) (β, α) to the 2-cell $\beta * \alpha$ in $\mathcal{K}(A, C)$.

For $*$ to respect vertical composition (which is to say, composition in the product and target categories), first notice that $(\beta', \alpha') \circ (\beta, \alpha) = (\beta' \circ \beta, \alpha' \circ \alpha)$. The functor $*(a, b) = a * b$ must have that $*((\beta', \alpha') \circ (\beta, \alpha)) = *((\beta', \alpha')) \circ *((\beta, \alpha))$. This is precisely the interchange law. □

Problem 41. Write out the details for the 2-category of posets, order-preserving functions, and function domination

Solution.

The solution is effectively recognizing that order preserving functions are themselves a poset under function domination, and then using uniqueness of 2-cells to verify that all of the 2-category axioms hold.

The 0-cells of **Poset** are posets. Between any two posets $(A, \leq_A)$ and $(B, \leq_B)$, there is a category $\mathbf{Poset}(A, B)$ (I drop the $\leq_A$ and $\leq_B$—they should be clear from here on out.)

A 1-cell in $\mathbf{Poset}(A, B)$ is an order preserving function $f : A \to B$ where $a \leq_A a' \implies f(a) \leq_B f(a')$. A 2-cell between $f : A \to B$ and $g : A \to B$ is a unique morphism $\heartsuit_{f,g}$ if $f(a) \leq_B g(a) \forall a \in A$. Composition is given by $\heartsuit_{g,h} \circ \heartsuit_{f,g} = \heartsuit_{f,h}$ and is well-defined by the transitive property of inequalities (if $f(a) \leq_B g(a)$ and $g(a) \leq_B h(a)$ for all $a \in A$, then $f(a) \leq_B h(a)$ and so $\heartsuit_{f,h}$ exists)

First, we show that $\mathbf{Poset}(A, B)$ is a category. The identity morphism $\heartsuit_{f,f}$ exists and is clearly an identity. Associativity is also obvious. Hence, $\mathbf{Poset}(A, B)$ is a category.

Next, we will show that there is a functor from the one-object category to the identity 1 and 2-cells of an 0-cell A. This is effectively the question if there exists a 1-cell that serves as an identity for each 0-cell. The identity function 1_A on A is the relevant 1-cell. Then, the 2-cell in question is $\heartsuit_{1_A,1_A}$.

Now, we need to define the horizontal composition functor $*$. I'll write $f*g$ for $*(f,g)$. Given posets A,B,C and order preserving functions (1-cells) $f \in \mathbf{Poset}(A,B), g \in \mathbf{Poset}(B,C)$, we define $g*f = g \circ f$. Moreover, given a 2-cell $\heartsuit_{f,f'} \in \mathbf{Poset}(A,B)(f,f')$ and similarly for g, g', we define $\heartsuit_{g,g'} * \heartsuit_{f,f'} = \heartsuit_{g \circ f, g' \circ f'}$. This is well defined by uniqueness of 2-cells and exists by transitivity of the $\leq$ operation.

Horizontal composition is associative on objects by associativity of 1-morphisms, and satisfies the identity axioms similarly.

As there's really only one choice to make at each step of composition, and $\leq$ is a transitive operation, horizontal composition is associative on morphisms. Moreover, horizontally composing identities yields an identity, because the identity on $g*f$ is the unique morphism $\heartsuit_{g\circ f, g\circ f}$, which is $\heartsuit_{g,g} * \heartsuit_{f,f}$. The same logic tells us that $*$ is indeed a functor:

Let $\beta = \heartsuit_{g,g'}, \beta' = \heartsuit_{g',g''}$ in the category $\mathbf{Poset}(B,C)$ and similarly for α. Then

$(\beta' \circ \beta) * (\alpha' \circ \alpha) = \heartsuit_{g,g''} * \heartsuit_{f,f''} = \heartsuit_{g \circ f, g'' \circ f''} = \heartsuit_{g' \circ f', g'' \circ f''} \circ \heartsuit_{g \circ f, f' \circ g'} = (\beta' * \alpha') \circ (\beta * \alpha)$

So, horizontal composition is a functor (interchange law).

So, the 2-category **Poset** as defined is indeed a 2-category. Ta-da! $\square$