Problem 35. Let $\mathcal{C}$ be a category, and $X \in \mathcal{C}$. The slice category $\mathcal{C}/X$ has objects (Y, f) where $Y \in \mathcal{C}$ and $f \in \mathcal{C}(Y, X)$.

1. Construct the rest of the slice category
2. For a set S with corresponding discrete category dS, $[dS, \mathbf{Set}] \simeq \mathbf{Set}/S$

Solution. The equivalence given in part 2 gives us motivation for the morphisms in the slice category: A Set mod S should look like the functor category between dS and Set, which has morphisms natural transformations between functors from $dS \to \mathbf{Set}$.

What are the morphisms of $\mathcal{C}/X$?

So, given $(Y, f), (Z, g) \in \mathcal{C}/X$, we can say that a morphism F must take Y to Z (a morphism in $\mathcal{C}$) and $f : Y \to X$ to $g : Z \to X$. We draw:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "?"'] & X \arrow[d, "1_X"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

So, our ? ought to be the morphism $h : Y \to Z$ such that $g \circ h = f$.

That is, F is a morphism $F : Y \to Z$ with $g \circ F = f$. Morphism composition is given by morphism composition in $\mathcal{C}$, and the identity is the identity in the underlying category as well.

Showing that $\mathcal{C}/X$ is actually a category

We must show that $\mathcal{C}/X$ is a category with this construction. We will first check that morphism composition is sensible, then that identities function how we want. We inherit associativity from the underlying category.

Given $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2) \xrightarrow{G} (Y_3, f_3)$, $G \circ F$ has the right sources and targets (it maps from $Y_1 \to Y_3$). Then, using associativity of morphisms in the underlying category and the condition on morphisms in the slice category, $f_3 \circ (G \circ F) = f_2 \circ F = f_1$. Hence, the composition of two morphisms is a morphism!

Lastly, we check identities. Let $(Y_1, f_1) \xrightarrow{F} (Y_2, f_2)$. We said that the identity on (Y_1, f_1) ought to be 1_{Y_1}. $f_2 \circ (F \circ 1_{Y_1}) = f_2 \circ F = f_1$. This proves right identities. Left identities follow identically.

We conclude that $\mathcal{C}/X$ is a category. □

To show equivalence, we can construct two functors, one from $[dS, \mathbf{Set}]$ to $\mathbf{Set}/S$ and one the other way. To define such a functor, we must first show what it does on objects, then morphisms, check that it respects composition and identities, and lastly we will show that the functors compose in a way that is naturally isomorphic to the identity (both ways).

Building the functor $J : [dS, \mathbf{Set}] \to \mathbf{Set}/s$

Given a functor $F : dS \to \mathbf{Set}$, we must send it to a pair $(X, f : X \to S)$ in $\mathbf{Set}/S$. F has two parts: on objects, it maps $s \mapsto Y_s$. On morphisms (of which there are only identities), it maps $1_s \mapsto 1_Y$. We can map F to the pair $(\coprod_S F(s), \pi : \coprod_S F(s) \to S)$

where $\pi(k) = s$ if $k \in F(s)$. This is well defined, because even if F maps two elements s, s' of dS to the same set, we use the disjoint product to identify where each s, s' came from. Let's call the functor performing this operation J for "Jake has no better ideas for notation."

We must see what J does on morphisms. Morphisms in the functor category are natural transformations. Given $\alpha : F \Rightarrow G$, we define $J(\alpha) = \coprod_S \alpha_s$ where α_s are morphisms $p_s : F(s) \to G(s)$ in **Set**. This has the proper source and target. dS is discrete, so the only $f : s \to s'$ is the identity on s. The naturality square of α just says that

$$\begin{tikzcd}
Fs \arrow[r, "1_{Fs}"] \arrow[d, "\alpha_s"'] \arrow[dr, "\alpha_s"] & Fs \arrow[d, "\alpha_s"] \\
Gs \arrow[r, "1_{Gs}"'] & Gs
\end{tikzcd}$$

This obviously commutes.

Let JF_π be the projection part of JF, and similarly for JG. We need to first show that $JG_\pi \circ J\alpha = JF_\pi$. Suppose $k \in F(s_0)$. Then, $p_s(k) \in G(s_0)$

Then $JF_\pi(k) = s_0$, and $JG_\pi \circ \coprod_S p_s(k) = s_0$.

Showing J is a functor

Now, we show J is a functor: The identity natural transformation 1_F has components identities $1_{F_s} = 1_{Fs}$. We need that $J(1_F) = 1_{JF}$. So, we write $J(1_F) = \coprod_S 1_{F(s)} = 1_{\coprod_S F(s)} = 1_{JF}$

Now, consider $\alpha, \beta : F \Rightarrow G$ where $F, G \in [dS, \mathbf{Set}]$ and their vertical composition $\alpha \circ \beta$. $J(\alpha \circ \beta) = \coprod_S \alpha_s \circ \beta_s = \coprod_S \alpha_s \circ \coprod_S \beta_s = J(\alpha) \circ J(\beta)$.

Thus, J is a functor.

Building the functor $K : \mathbf{Set}/S \to [dS, \mathbf{Set}]$

Given $(Y, f : Y \to S)$ in the slice category, we must map it to a functor F from $dS \to$ **Set**. First, we build the functor F, then show that it is a functor. We define F on objects to take $s \mapsto f^{-1}(s)$. On morphisms, the only morphisms of dS are the identities, and so they go to the corresponding identities. The only morphism composition that F needs to respect is repeated identity composition, which it clearly does. Hence, F is a functor. That is, $K(Y, f) = F$.

Now, K must take morphisms of $\mathbf{Set}/S$ to natural transformations between functors between $dS \to$ **Set**. Luckily, we had defined our morphisms of the slice category so that this would make sense! Given a morphism $H : (Y, f) \to (Z, g)$, where $K(Y, f) = F$ and $K(Z, g) = G$, we think of H as a morphism $h : Y \to Z$ with $g \circ h = f$.

So, $K(H)$ must be a natural transformation given by components $KH_s : F(s) \to G(s)$. By the same argument with J, the naturality square automatically commutes as there is only the identity morphism from each $s \to s$ in dS. We can let $KH_s = h|_{f^{-1}(s)}$. We know that $g \circ h = f$, so if $x \in f^{-1}(s)$, then $g(h(x)) = s$, so $h(x) \in g^{-1}$. Hence KH_s has the proper source $(f^{-1}(s))$ and target $(g^{-1}(s))$.

Showing K is a functor Identities: $K(1_X)_s = 1_X|_{f^{-1}(s)} = 1_{KX}$.

Composition: $K(X \xrightarrow{F} Y \xrightarrow{G} Z)_S = (g \circ f)_{(g \circ f)^{-1}(s)}$. $x \in (f \circ g)^{-1}(s) \iff fx \in g^{-1}(s)$. So, we can split the restriction to $g|_{g^{-1}(s)} \circ f|_{f^{-1}(s)}$. This is $KG \circ KF$.

Proving equivalence We need to show that KJ and JK are both naturally isomorphic to the identity, beginning with JK.

$$\begin{aligned} JK((Y, f : Y \to S) \xrightarrow{F} (Z, g : Z \to S)) \\ = J(f^{-1}(S) \xrightarrow{F|_{f^{-1}(S)}} g^{-1}(S)) \\ = (\coprod_S f^{-1}(s), \pi_f) \xrightarrow{\coprod_S F|_{f^{-1}(s)}} (\coprod_S g^{-1}(s), \pi_g) \end{aligned}$$

We must find an isomorphism Φ from $\mathbf{Set}/S \Rightarrow \mathbf{Set}/S$ taking (Y, f) to $\coprod_S f^{-1}(s)$ such that the following diagram commutes.

$$\begin{tikzcd} (Y, f) \arrow[r, "h"] \arrow[d, "\Phi_{Y,f}"'] & (Z, g) \arrow[d, "\Phi_{Z,g}"] \\ (\coprod_S f^{-1}(s), \pi_f) \arrow[r, "\coprod_S h|_{f^{-1}(s)}"'] & (\coprod_S g^{-1}(s), \pi_g) \end{tikzcd}$$

That is, $\Phi_{Z,g} \circ \coprod_S h|_{f^{-1}(s)} = h \circ \Phi_{Y,f}$