Modelling: The Virus Awakens

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A long time ago in a galaxy far, far away, 5 Jemedi came across the planet Florrum where the Weequay species lived. This planet had hostile living conditions leading to this species developing a new organ (the maul-bladder) which secretes cells known as midiflorians. There is believed to be a direct correlation between these cells and the life expectancy of the Weequay.

The Jemedi decide to make a report on this expectancy by analysing the number of Midiflorians in a Weequay's bloodstream. Whilst investigating their findings they came across a dark lord Tenebrous who tried to harness these cells power to achieve life eternal. Ultimately failing, he decides to release a virus to kill all the Weequay.

In this report the Jemedi will investigate how to survive this attack by analysing the impact of the virus and the how the midiflorians will react.

1 Life Expectancy

In this section we will propose a suitable model for the midiflorian count in the bloodstream for an individual Weequay. The midiflorian count is denoted by $M(t)$ at time t and bloodstream s.
We start by identifying some assumptions:
1. There is a maximum number of midiflorians that can be in the blood stream of a Weequay. This is known as the carrying capacity and is dependent on the other factors.
2. There is a critical level of initial midiflorians where if a newborn has less then this value they will succumb to the harsh environment. As the Weequay haven't yet died out, we can assume this critical value is less than the carrying capacity of their blood.
3.The secretion rate of midiflorians depends on the current number of these cells and factors such as nutrition and genetics.
4. Midiflorian count is life expectancy's only dependency.
Using Occam's razor, the book-keeping principal and the phenomenological approach based on the idea of limited resources, given $t > t_0$ and the Midiflorian count $M(t_0) = m_0 > 0$ for each Weequay, we have:

$$M(t) = M(t_0)+(a-bM)\int_{t_0}^{t} M ds-(c)\int_{t_0}^{t} M ds \tag{1}$$

Where $a - bM$ is the rate of midiflorians secreted with $a > 0$ being the initial secretion rate, and c is the rate of other factors affecting this. c could be positive or negative depending on their effect, but is assumed to be constant since things such as their genetics don't change.

Differentiating by using the fundamental theorem of calculus we arrive at a logistic equation:

$$\frac{dM}{dt} = rM(1 - \frac{M}{k}) \tag{2}$$

Where M is the midiflorian count at time t, $r_1 = a - c$ and $k = \frac{r_1}{b}$ (where $k > 0$). Then r_1 is the secretion rate in the Maul-bladder and k is the maximum number of midiflorians in the blood stream.
Solving this model, we obtain the solution:

$$M(t) = \frac{m_0 k e^{rt}}{k - m_0 + m_0 k e^{rt}} \tag{3}$$

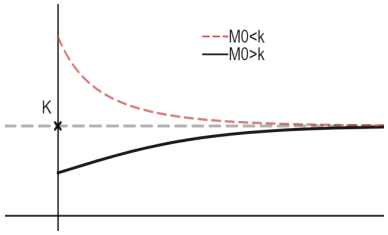

Figure 1: A graph showing how M behaves for various initial conditions m_0

m_0 increases monotonically to k for $0 < m_0 < k$ and decreases monotonically to k as $t \to \infty$ for $m_0 > k$. k is globally stable for all positive initial conditions, so $M^* = 0$ is an unstable steady state. $m_0 \neq k$ will move towards $M^* = k$ so is a stable steady state.

This only models Weequay that have an initial midiflorian count above the critical value. This should not be the case for our final model as below this critical value the Weequay cannot survive. This also suggests that there should be a stable steady state at $t = 0$.

To analyse this we can use a modified logistic model for the midiflorian count of a Weequay, which includes a strong Allee effect:

$$\frac{dM}{dt} = r_1 M(1 - \frac{M}{k})(\frac{M}{A} - 1) \qquad (4)$$

Where $A \in (0, k)$ is the critical value. When $M > A$ the midiflorian count will increase monotonically to k, and when $M < A$ the midiflorians will decrease until the Weequay eventually dies.

Finding the steady states of the model by setting $\frac{dM}{dt} = 0$ and solving for M, we find the trivial steady state $M^* = 0$, the carrying capacity steady state $M^* = k$, and the critical steady state $M^* = A$. We carry out stability analysis by letting

$$f(m) = \frac{dM}{dt} \qquad (5)$$

Differentiating (6) we get:

$$f'(m) = r(-\frac{3rm^2}{kA} + 2rm(\frac{1}{A} + \frac{1}{k}) - 1) \quad (6)$$

For $m^* = 0$, $f'(0) = -r < 0$ (assuming that $r > 0$) and so this steady state is stable. For $m^* = k$, $f'(k) = r(-\frac{rK}{A} + 2r - 1) < 0$ since $k > A$. For $m^* = A$, $f'(A) = r(-\frac{rA}{k} + 2r - 1) > 0$ since $k > A$, so this steady state must be unstable. This is confirmed by the graphical method:

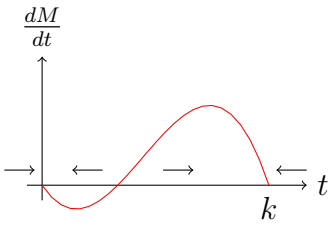

Figure 2: Graphical representation of the steady states of the Allee effect model (Courchamp et al., 2008).

2 Tenebrous' Drug

We now aim to model the number of Midiflorians (M) in Tenebrous' bloodstream at a time t, using the following assumptions:
1. The drug has two functions - to increase the rate at which midiflorians are secreted, and to increase the carrying capacity of his blood.
2. The drug has more of an impact on production as the the normal secretion rate drops off.
3. Environmental factors will always reduce the rate of secretion compared to the theoretical maximum
4. The body is a closed system. This assumption will apply for all analysis going forwards.

We take a similar approach to Q1, expecting a logistic model and only consider (2) as we are in the case where the Weequay have enough midiflorians at birth to survive.
Using the book-keeping principle and (1) we

have:

$$M(t) = M(t_0) + (a - bM)\int_{t_0}^{t} M ds \\ -(c)\int_{t_0}^{t} M ds + d_r \int_{t_0}^{t} M ds \tag{7}$$

where d_r is the rate the drug boosts midiflorian count and tries to override the maul bladder. Differentiating and using fundamental theorem of calculus we arrive at the modified logistic model:

$$\frac{dM}{dt} = r_1 M(1 - \frac{M}{k}) + d_r M \tag{8}$$

which we can find the steady state as $M^* = 0$ and $M^* = k + \frac{d_r k}{r}$ using linear stability analysis and the same approach as in question one we find that:

$$f'(m) = r - \frac{2}{k} Mr + d_r \tag{9}$$

Substituting in each steady state we find that $f'(0) > 0$ so is unstable and $f'(k + \frac{d_r k}{r}) < 0$ and is therefore stable.

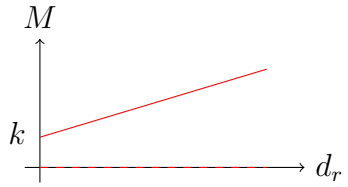

Figure 3: Bifurcation diagram with $x = d_r$ and $y = m$ shows that as the effectiveness of the drug increases Tenebrous' maximum midiflorian count increases linearly.

We know that $\frac{d_r k}{r} > 0$ due to each variable being > 0. This means that the new steady state is greater than k, so Tenebrous has achieved his aim of increasing his midiflorian count and achieving prolonged life. The bifurcation diagram emphasises that his drug is effective however as we have achieved a steady state his midiflorian count does not tend to infinity and has a maximum. This implies he needs an infinitely effective drug in order to achieve eternal life, which is not possible.

3 Impact of the virus

With the introduction of the airborne virus, any Weequay that gets infected now has all of their Midiflorians destroyed within hours, resulting in their eventual death. To model the population of both Midiflorians and the virus in an individual's system, let $M(t)$ denote the population of Midiflorians at time t and $V(t)$ denote the population of the virus at time t. The modelling assumptions for this system are as follows:

1. The virus cannot be destroyed
2. The population of the virus increases at a rate proportional to the population of Midiflorians
3. The Weequay will not die naturally, only to the virus
4. The production of Midiflorians in the system is still active provided the population has not already reached its maximum capacity

Using the book-keeping principal, the models for the Midiflorians and the virus population are as follows:

$$M(t) = M(t_0) + (a - bM)\int_{t_0}^{t} M(s) ds \\ -c\int_{t_0}^{t} M(s) ds - p\int_{t_0}^{t} M(s)V(s) ds \tag{10}$$

$$V(t) = V(t_0) + q\int_{t_0}^{t} M(s)V(s) ds \tag{11}$$

Differentiating (11) and (12) using the Fundamental Theorem of Calculus, we get:

$$\frac{dM}{dt} = (a - bM) - cM - pV \tag{12}$$

$$\frac{dV}{dt} = nMV \tag{13}$$

Let $f(M, V) =$ (13), and $g(M, V) =$ (14). Then $f(M, V) = r_1 M(1 - \frac{1-m}{k}) - pV$, from (1), where μ is the rate at which the virus kills midiflorians and q is the rate at which the virus

reproduces.

Here the steady states are $(M^*, V^*) = \{(0,0), (k,0)\}$

The Jacobian here is $\begin{pmatrix} r_1 - 2bM & -\mu \\ qV & qM \end{pmatrix}$. At the steady states, this becomes $\begin{pmatrix} r_1 & -\mu \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} r_1 - 2bk & -\mu \\ 0 & qk \end{pmatrix}$ for $(0,0)$ and $(k,0)$ respectively. For $(0,0)$ the eigenvalues are r_1 and 0, and since $r_1 > 0$, $(0,0)$ is non-hyperbolic. For $(k,0)$ the eigenvalues are $r_1 - 2bk$ and nk. $r_1 - 2bk = -r_1 < 0$ since $r_1 > 0$. qk is positive, so the eigenvalues have opposite signs, meaning that the solution forms an unstable saddle.

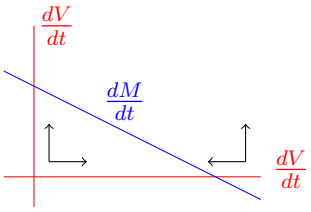

Figure 4: The phase plane shows that the steady states only exist when there is no virus in the Weequay and the number of Midiflorians is either zero, implying the Weequay is dead, or at its maximum level, k.

This means there is no steady state for when the virus enters the system.

4 Anti-virals strike back

When a Weequay is infected, antivirals will be produced by the maul-bladder, hindering the production of midiflorians. We now investigate the competition between the midiflorians, viruses and anti-virals in the bloodstream. We start by using the book-keeping principle, under the following assumptions:

1. The population of the virus increases at a rate proportional to the midiflorian count.
2. The Weequay will not die naturally, only to the virus.
3. When antivirals are produced, the rate of midiflorian production is reduced by a fixed rate E
4. Antivirals destroy viral particles at a fixed rate μ
5. Midiflorian mortality is only caused by viruses, and viral mortality is only caused by antivirals

$$M(t) = M(t_0) + (a - bM - E)\int_{t_0}^{t} M(s)ds - c\int_{t_0}^{t} M(s)ds - p\mu\int_{t_0}^{t} V(s)M(s)ds \tag{14}$$

$$V(t) = V(t_0) + q\int_{t_0}^{t} M(s)V(s)ds - \mu\int_{t_0}^{t} \tag{15}$$

We use the Law of Mass Action to show the interactions between the antiviral and the midiflorians for the reproduction and destruction of the virus particles and midiflorians respectively. Differentiating using the Fundamental Theorem of Calculus we get the following:

$$\frac{dM}{dt} = f(m,v) = (a - bM - E)M - cM - \mu pVM$$

$$\frac{dM}{dt} = (1 - \frac{M}{k})Mr_2 - \mu pVM \tag{16}$$

Where $r_2 = a - E - c > 0$, and $k = \frac{r_2}{b}$

$$\frac{dV}{dt} = g(m,v) = qMV - \mu V = V(qM - \mu) \tag{17}$$

By setting (17) and (18) to 0 we get the following steady state solutions by Hartman-Gobman:

$$(M^*, V^*) = \{(0,0), (k,0), (\frac{\mu}{q}, \frac{1-\mu}{qk}\} \tag{18}$$

Jacobian Matrix:

$$J = \begin{pmatrix} r_2\frac{1-2M}{k} - \mu pM & -\mu pM \\ qV & qM - \mu \end{pmatrix} \tag{19}$$

For $(0,0)$: $J_1 = \begin{pmatrix} r_2 & 0 \\ 0 & -\mu \end{pmatrix} \Rightarrow \lambda_1 = r_2, \lambda_2 = -\mu$

For $(k, 0)$: $J_2 = \begin{pmatrix} -r_2 & 0 \\ 0 & \mu \end{pmatrix} \Rightarrow \lambda_1 = -r_2, \lambda_2 = qk - \mu$

Therefore we have a stable node if $qk < \mu$, and an unstable node if $qk > \mu$

For $(\frac{\mu}{q}, \frac{r_2}{\mu p}(1 - \frac{\mu}{qk}))$: $J_3 = \begin{pmatrix} -\frac{r_2\mu}{qk} & -\frac{\mu^2 p}{q} \\ \frac{r_2 q}{\mu p}(1 - \frac{\mu}{qk}) & 0 \end{pmatrix}$

$\operatorname{tr} A = -\frac{r_2\mu}{qk} > 0$

$\det A = r_2 k(1 - \frac{\mu}{qk}) > 0$

This gives us Asymptotic Stability by Ruth-Huratz Criterion

From these results, we can see two cases:

Case 1: $\frac{\mu}{qk} > 1$ gives us two realistic solutions, as our third solution would imply a negative number of viruses. The first solution is just a saddle point, and the second solution is a stable node, meaning that no matter our initial conditions m_0 and v_0, the population of both will converge to this steady state. This implies a full recovery, and the midiflorian count eventually returning to k.

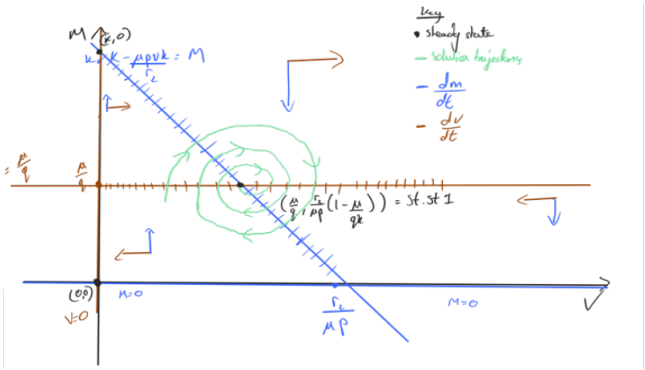

Figure 5: Phase plane analysis in case 1.

Case 2: $\frac{\mu}{qk} < 1$ gives 3 realistic solutions. The first steady state is still a steady state, whereas the second is now an unstable node, and the third is asymptotically stable, implying that any given initial conditions will eventually converge to this steady state. This implies that the virus will never be eradicated within the body, and that the midiflorians will never recover to their original count.

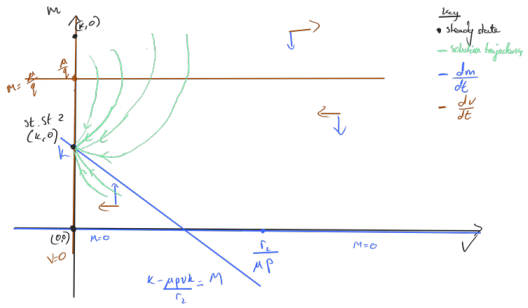

Figure 6: Phase plane analysis in case 2.

5 How to Survive

To achieve full recovery, we need $\frac{\mu}{qk} > 1$. We can either do this by increasing the virus destruction rate μ, or by decreasing virus production q. Decreasing q could be difficult, but increasing μ is possible.

Vaccination is the most effective way to increase μ, and it also creates herd immunity, limiting virus spread and protecting the most vulnerable Weequay. Anti-viral drugs are also a good option, as they can directly kill the virus or enhance the maul-bladder's function to produce anti-virals and midiflorians efficiently.

Increasing the maximum carrying capacity of Weequay's blood can theoretically improve survival chances, as there's a higher chance of being above the critical value if the full recovery condition isn't met. However, Tenebrous' research shows that this is impossible, so alternative survival methods must be considered.

Environmental factors can impact maul-bladder function, and while some loss is inevitable due to genetics, age, or health factors, diet and physical fitness are controllable factors that can improve the body's immune response. Weequay can mitigate these factors to minimize the impact on their maul-bladder function.

References

Courchamp, F., Berec, L. and Gascoigne, J., 2008 'Allee Effects in Ecology and Conservation' [Online]. Oxford, Oxford Academic. Available from: https://doi.org/10.1093/acprof:oso/9780198570301.001.0001, accessed 27 Apr. 2023.