Data Science Math Skills
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Sets: Basics and Vocabulary

Video companion

1 Set theory basics

- What is a set?
- Cardinality (size)
- Intersections
- Unions

2 What is a set?

Vocab: A *set* is made up of *elements*.

Example: $A = \{1, 2, -3, 7\}$ and $E = \{\text{apple, monkey, Daniel Egger}\}$

- $2 \in A$: "2 is an element of A"
- $8 \notin A$: "8 is NOT an element of A"

3 Cardinality

Vocab: The *cardinality* (size) of a set is the number of elements in it.

- $|A| = 4$ (there are 4 elements in A, so the cardinality is 4)
- $|E| = 3$ (there are 3 elements in E, so the cardinality is 3)

4 Intersections

The *intersection* is defined as elements that are in both sets.

Symbol $\cap$: "intersects" (and)

Example: $A = \{1, 2, -3, 7\}$ and $B = \{2, -3, 8, 10\}$ and $D = \{5, 10\}$

- $A \cap B = \{2, -3\}$
- $B \cap D = \{10\}$

In general, $A \cap B = \{x : x \in A \text{ } and \text{ } x \in B\}$

If there are no elements in common, the answer is the empty set $\emptyset$. The cardinality of the empty set $|\emptyset| = 0$.

- $A \cap D = \emptyset$

5 Unions

The *union* is defined as elements that are in either set.

Symbol $\cup$: "union" (or)

Example: $A = \{1, 2, -3, 7\}$ and $B = \{2, -3, 8, 10\}$ and $D = \{5, 10\}$

- $A \cup B = \{1, 2, -3, 7, 8, 10\}$
- $A \cup D = \{1, 2, -3, 7, 5, 10\}$

In general, $A \cup B = \{x \in A \text{ } or \text{ } x \in B\}$.

Sets: Medical Testing Example

Video companion

1 Example using set theory

VBS: "very bad syndrome"

X = set of people in a clinical trial

$S = \{x \in X : x \text{ has VBS}\}$
$H = \{x \in X : x \text{ does not have VBS}\}$

$X = S \cup H$ (you either have VBS or you don't)
$S \cap H = \emptyset$ (no one both has and doesn't have it)

Point of medical testing to figure out whether a person is in S or in H

2 Test

$P = \{x \in X : x \text{ tests positive for VBS}\}$
$N = \{x \in X : x \text{ tests negative for VBS}\}$

$P \cup N = X$ (you either test positive or negative)
$P \cap N = \emptyset$ (no one tests both positive and negative)

In a perfect world, S would equal P—the sick people would always test positive, and H would equal N—the healthy people would always test negative.

...but this is not always the case.

$S \cap P$	$H \cap N$	$S \cap N$	$H \cap P$
true positive	true negative	false negative	false positive

3 Cardinality

$\frac{|S|}{|X|}$ = proportion of people in the study who do genuinely have VBS
$\frac{|H|}{|X|}$ = proportion of people in the study without VBS
$\frac{|S|}{|X|} + \frac{|H|}{|X|} = 1$

$\frac{\lvert S \cap P \rvert}{\lvert S \rvert}$	true positive rate	would like to be close to 1
$\frac{\lvert H \cap P \rvert}{\lvert H \rvert}$	false positive rate	would like to be as small as possible
$\frac{\lvert S \cap N \rvert}{\lvert S \rvert}$	false negative rate	would like to be as small as possible
$\frac{\lvert H \cap N \rvert}{\lvert H \rvert}$	true negative rate	would like to be close to 1

Sets: Venn Diagrams

Video companion

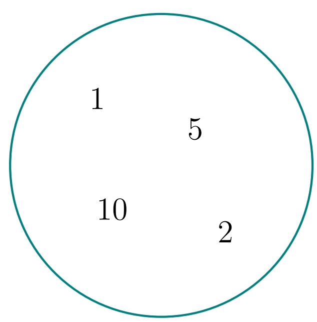
1 Visualizing sets

- Venn diagrams
- Inclusion-exclusion formula
- Medical testing example, re-visited

2 Single set

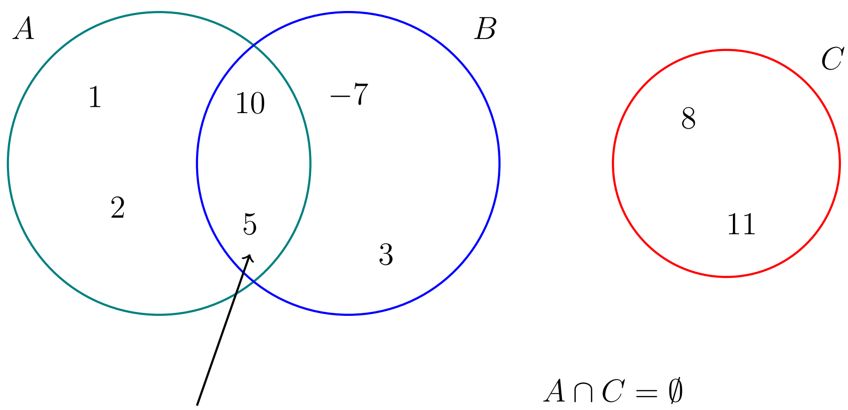
$A = \{1, 5, 10, 2\} \qquad |A| = 4$

$A =$



3 Multiple sets

$A = \{1, 5, 10, 2\}$
$B = \{5, -7, 10, 3\}$
$C = \{8, 11\}$

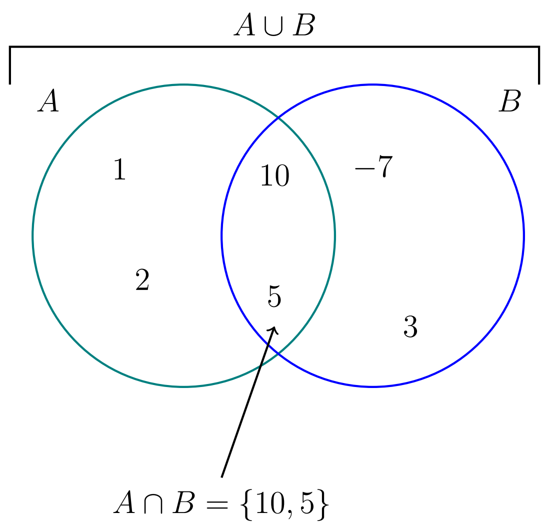


$A \cap B = \{10, 5\}$

$A \cap C = \emptyset$

$B \cap C = \emptyset$

4 Inclusion-exclusion formula



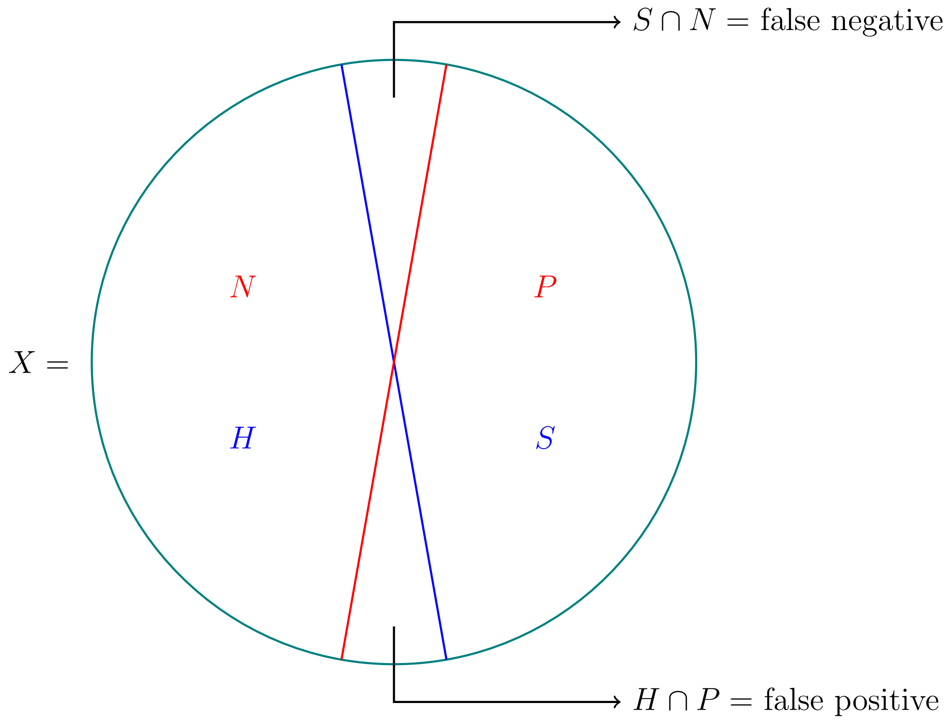
Inclusion-exclusion formula:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Check with this example:

$$6 \stackrel{?}{=} 4 + 4 - 2$$
$$6 = 6 \quad \checkmark$$

5 Medical testing example



$$X = H \cup S \qquad H \cap S = \emptyset$$
$$S = N \cup P \qquad N \cap P = \emptyset$$

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Numbers: The Real Number Line

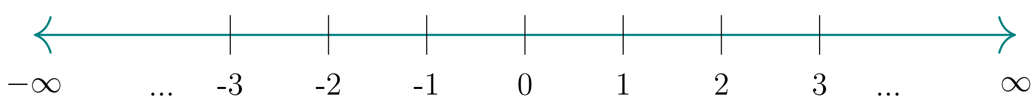
Video companion

1 Introduction

- What is $\mathbb{R}$?
- Positive, negative
- Absolute value

2 Integers and rational numbers

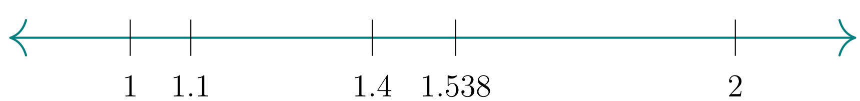
Graph of $\mathbb{R}$, the real numbers:



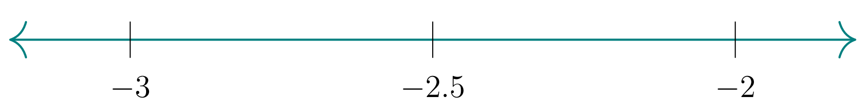
Subset of real numbers, integers:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Segment between 1 and 2:

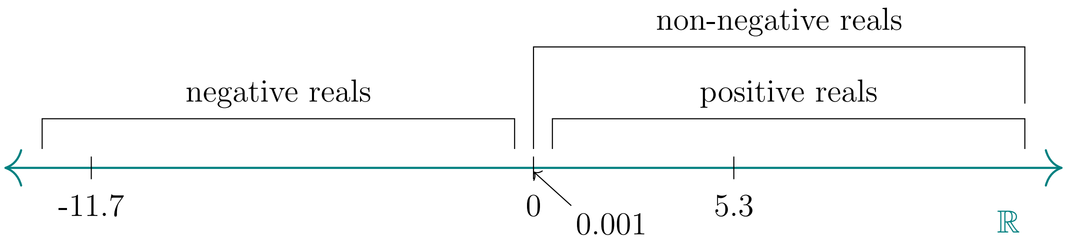


Segment between -3 and -2:

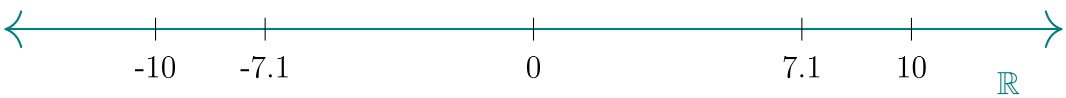


Some real numbers terminate, and some do not.
The number $\pi = 3.14159...$ is *irrational*, i.e. it does not repeat after the decimal point.

3 Sets of real numbers



4 Absolute value



The absolute value of a number x, $|x|$, is the distance from x to 0.

Example:

$$\begin{aligned} |7.1| &= 7.1 \\ |-7.1| &= 7.1 = -(-7.1) \end{aligned}$$

General rule:
For any $x \in \mathbb{R}$,

$$|x| = \begin{cases} x, & \text{if } x \text{ is non-negative} \\ -x, & \text{if } x \text{ is negative} \end{cases}$$

Check:

$$\begin{aligned} |8.7| &= 8.7 \\ |-10| &= -(-10) = 10 \end{aligned}$$

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Numbers: Greater-than and Less-than

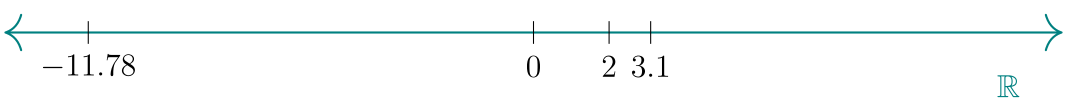
Video companion

1 Inequalities, basic idea

Introduction to symbols:

$a < b$	"a is less than b"
$x > y$	"x is greater than y"
$c \leq d$	"c is less than or equal to d"
$z \geq w$	"z is greater than or equal to w"
$e \ll f$	"e is much, much less than f"

2 Inequality on the real number line



$2 < 3.1$	"2 is to the left of 3.1 on the real number line"
$-11.78 < 3.1$	"−11.78 is to the left of 3.1 on the real number line"

For any $a < b$, a must be to the left of b on the real number line.

$3.1 > 2$	"3.1 is to the right of 2 on the real number line"

In general, a is less than b, if, and only if, b is greater than a:

$$\boxed{a < b \iff b > a}$$

3 Much, much less than

$x \ll y$ "x is much, much less than y"
(Not proper math, but used frequently in data science)

For example, $1 \ll 1,000,000$, which is reasonable but not possible to prove "true"

4 Less than or equal to

$a \leq b$ means $a < b$ or $a = b$

Examples:

Is $2 \leq 3.1$ true?

$$\left.\begin{array}{ll} 2 < 3.1 & \checkmark \\ 2 = 3.1 & \times \end{array}\right] \checkmark$$

Is $2 \leq 2$ true?

$$\left.\begin{array}{ll} 2 < 2 & \times \\ 2 = 2 & \checkmark \end{array}\right] \checkmark$$

Is $2 \leq 0.8$ true?

$$\left.\begin{array}{ll} 2 < 0.8 & \times \\ 2 = 0.8 & \times \end{array}\right] \times$$

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Numbers: Algebra with Inequalities

Video companion

1 Introduction

- Review algebra with equalities ($=$)
  - how?
  - why?
- Learn algebra with inequalities ($<, >, \leq, \geq$)
  - what works
  - A BIG WARNING

2 Algebra with equalities

$$\begin{aligned} 4 &= 4 \\ 4+3 &= 4+3 \\ 7 &= 7 \quad \checkmark \end{aligned}$$

Rule:
If $a = b$, then $a + c = b + c$.

Example:

$$\begin{aligned} x+3 &= 10 \\ (x+3)-3 &= 10-3 \\ x &= 7 \end{aligned}$$

Similarly with multiplication,

$$\begin{aligned} 4 &= 4 \\ 2 \cdot 4 &= 2 \cdot 4 \\ 8 &= 8 \quad \checkmark \end{aligned}$$

$$
\begin{aligned}
4 &= 4 \\
(-3) \cdot 4 &= (-3) \cdot 4 \\
-12 &= -12 \quad \checkmark
\end{aligned}
$$

Rule:

If a, b, and c are numbers, and $c \neq 0$, and $a = b$, then $c \cdot a = c \cdot b$.

Example:

$$
\begin{aligned}
-5x &= 15 \\
\left(-\frac{1}{5}\right) \cdot (-5x) &= \left(-\frac{1}{5}\right) \cdot 15 \\
x &= -3
\end{aligned}
$$

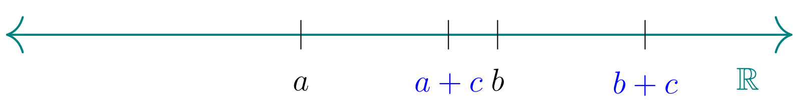
3 Algebra with inequalities

$$
\begin{aligned}
4 &< 7 \\
4 + 2 &\stackrel{?}{<} 7 + 2 \\
6 &\stackrel{?}{<} 9 \quad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
4 &< 7 \\
4 - 1 &\stackrel{?}{<} 7 - 1 \\
3 &\stackrel{?}{<} 6 \quad \checkmark
\end{aligned}
$$

Rule:

If $a < b$, then $a + c < b + c$.



Example:

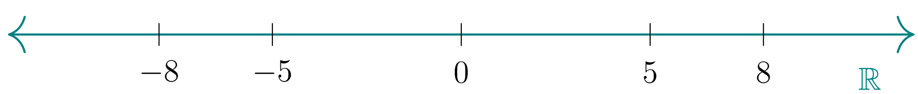
$$\begin{aligned} x+3 &< 10 \\ (x+3)-3 &< 10-3 \\ x &< 7 \end{aligned}$$

$x \in (-\infty, 7)$

Test cases with multiplication:

$$\begin{aligned} 5 &< 8 \\ 3 \cdot 5 &\overset{?}{<} 3 \cdot 8 \\ 15 &\overset{?}{<} 40 \quad \checkmark \end{aligned}$$

$$\begin{aligned} 5 &< 8 \\ (-1) \cdot 5 &\overset{?}{<} (-1) \cdot 8 \\ -5 &\overset{?}{<} -8 \quad \times \\ -5 &> -8 \quad ! \end{aligned}$$



Rule:

Suppose $a < b$.

If $c > 0$, then $a \cdot c < b \cdot c$.

If $c < 0$, then $a \cdot c > b \cdot c$.

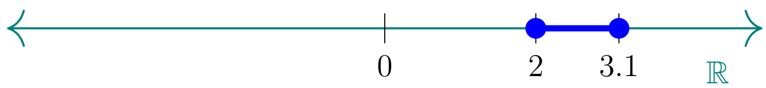
Example:

$$
\begin{aligned}
-2x &< 10 \\
\left(-\frac{1}{2}\right) \cdot (-2x) &> \left(-\frac{1}{2}\right) \cdot 10 \\
x &> -5
\end{aligned}
$$

Numbers: Intervals and Interval Notation

Video companion

1 Closed intervals

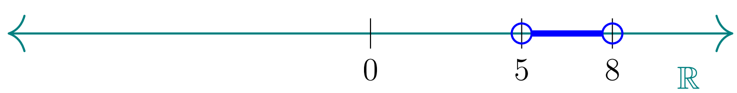


Real number line is an infinite set. There are also infinite subsets.

$[2, 3.1] = \{x \in \mathbb{R} : 2 \leq x \leq 3.1\}$

$$\begin{aligned}
2.3 &\in [2, 3.1] \qquad \text{because } 2 \leq 2.3 \leq 3.1 \\
3 &\in [2, 3.1] \\
3.1 &\in [2, 3.1] \\
1 &\notin [2, 3.1] \qquad \text{because } 2 \not\leq 1 \leq 3.1
\end{aligned}$$

2 Open intervals

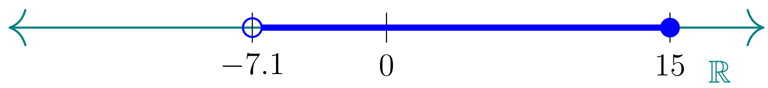


$(5, 8) = \{x \in \mathbb{R} : 5 < x < 8\}$

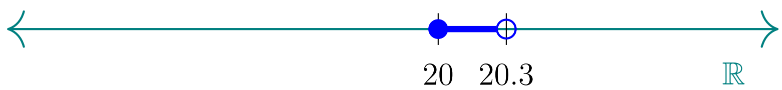
$$\begin{aligned}
5.5 &\in (5, 8) \qquad \text{because } 5 < 5.5 < 8 \\
5.0001 &\in (5, 8) \\
5 &\notin (5, 8) \qquad \text{because } 5 \not< 5 < 8
\end{aligned}$$

The intervals $[5, 8]$ and $(5, 8)$ differ at exactly two numbers: 5 and 8.

3 Half-open intervals



$(-7.1, 15] = \{x \in \mathbb{R} : -7.1 < x \leq 15\}$

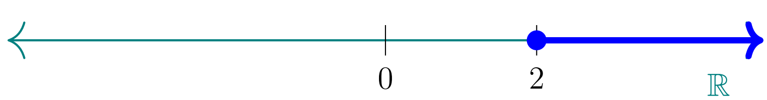


$[20, 20.3) = \{x \in \mathbb{R} : 20 \leq x < 20.3\}$

4 Recap vocabulary

- Closed intervals $[2, 3.1]$
- Open intervals $(5, 8)$
- Half-open intervals $(2, 3]$, $[20, 20.3)$

5 Rays



$[2, \infty) = \{x \in \mathbb{R} : x \geq 2\}$

Another example:
$(-\infty, 7.1) = \{x \in \mathbb{R} : x < 7.1\}$

6 What does an "answer" mean?

Solving an equality gives you a number:

$$\begin{aligned} x + 5 &= 10 \\ x &= 5 \end{aligned}$$

Solving an inequality give you an interval:

$$\begin{aligned} 1 \leq x + 5 &< 10 \\ -4 \leq x &< 5 \\ x &\in [-4, 5) \end{aligned}$$

Sigma Notation: Introduction to Summation

Video companion

1 Sigma notation (Σ)

Examples that will be seen in this video:

$$\sum_{i=1}^{4} i^2 = 30$$

$$\sum_{i=1}^{5} (2i+3) = 45$$

$$\sum_{j=3}^{7} \frac{j}{2} = \frac{25}{2}$$

2 First example

Example:

$$\begin{aligned}\sum_{i=1}^{4} i^2 &= 1^2 + 2^2 + 3^2 + 4^2 \\ &= 30\end{aligned}$$

$i = 1$ on bottom tells us to *start* with $i = 1$.
4 on top tells us to *finish* with $i = 4$.
Implicitly know that you increment by 1.

For each number i that you count,

$$\begin{aligned}i = 1 : i^2 = 1^2 \\ i = 2 : i^2 = 2^2 \\ i = 3 : i^2 = 3^2 \\ i = 4 : i^2 = 4^2\end{aligned}$$

then the Σ tells you to *sum* the results.

3 Second example

Example:

$$\begin{aligned}\sum_{i=1}^{5}(2i+3) &= (2(1)+3)+(2(2)+3)+(2(3)+3)+(2(4)+3)+(2(5)+3)\\ &= 45\end{aligned}$$

Work for problem:

$$\begin{aligned}&i=1 : 2i+3 = 2(1)+3\\ &i=2 : 2i+3 = 2(2)+3\\ &i=3 : 2i+3 = 2(3)+3\\ &i=4 : 2i+3 = 2(4)+3\\ &i=5 : 2i+3 = 2(5)+3\end{aligned}$$

4 Third example

Example:

$$\sum_{j=3}^{7}\frac{j}{2} = \frac{3}{2}+\frac{4}{2}+\frac{5}{2}+\frac{6}{2}+\frac{7}{2} = \frac{25}{2}$$

$$\sum_{r=3}^{7}\frac{r}{2} = \frac{25}{2}$$

j and r are "dummy indices," symbols for counters.

$$\sum_{\smiley=3}^{7}\frac{\smiley}{2} = \frac{25}{2}$$

Common choices for indices:
i, j, k, l, r, m, n

Sigma Notation: Simplification Rules

Video companion

1 Distributive property

Examples:

$$\sum_{i=1}^{4} i^2 = 30$$

$$\begin{aligned}\sum_{i=1}^{4} 3i^2 &= 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 \\ &= 3[1^2 + 2^2 + 3^2 + 4^2] \\ &= 3\left[\sum_{i=1}^{4} i^2\right]\end{aligned}$$

$$\sum_{r=4}^{25} 18r^3 = 18\left[\sum_{r=4}^{25} r^3\right]$$

This is due to the *distributive property*:

$$a(b+c) = ab + ac$$

In other words, constants inside the summed expression can be pulled outside.

2 Commutative property

$$
\begin{aligned}
\sum_{i=1}^{4}(i^2+2i) &= \big(1^2+2(1)\big)+\big(2^2+2(2)\big)+\big(3^2+2(3)\big)+\big(4^2+2(4)\big)\\
&= \big(1^2+2^2+3^2+4^2\big)+\big(2(1)+2(2)+2(3)+2(4)\big)\\
&= \left(\sum_{i=1}^{4} i^2\right)+\left(\sum_{i=1}^{4} 2i\right)
\end{aligned}
$$

This is due to the *commutative property*:

$$a+b=b+a$$

In other words, we can add the terms in any order.

3 Summation of constants

Examples:

$$
\begin{aligned}
\sum_{k=1}^{10} 5 &= 5+5+5+5+5+5+5+5+5+5\\
&= 10\cdot 5\\
&= 50
\end{aligned}
$$

$$
\begin{aligned}
\sum_{r=1}^{7} 8 &= 8+8+8+8+8+8+8\\
&= 7\cdot 8\\
&= 56
\end{aligned}
$$

When summing constants, you can multiply the constant by the number of indices you count.

Sigma Notation: Mean and Variance

Video companion

1 Introduction

Important equations for this video:

$$X = \{x_1, ..., x_n\}$$
$$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\sigma_x{}^2 = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \mu_x)^2\right]$$

The symbol μ_x is the "mean of x," and $\sigma_x{}^2$ is the "variance of x." The standard deviation is denoted σ_x.

2 Mean

Example:

$$Z = \{1, 5, 12\}$$
$$|Z| = 3$$
$$\mu_z = \frac{1+5+12}{3} = \frac{18}{3} = 6$$

The mean μ_z is also denoted $\mu(z)$ or simply μ.

Symbolic example:

$$Y = \{y_1, y_2, y_3, y_4\}$$
$$\mu_y = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)$$
$$= \frac{1}{4}\left(\sum_{i=1}^{4} y_i\right)$$

In general, suppose you have a set

$$X = \{x_1, x_2, ..., x_n\},$$

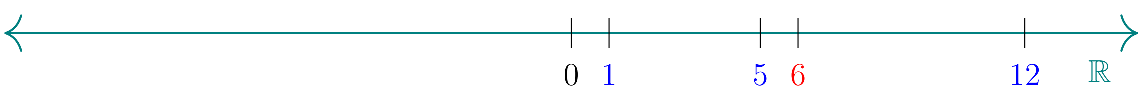
then the mean of X is

$$\mu_x = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right).$$

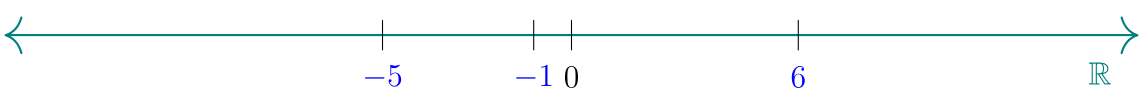
The variable i is a counter. The variable n is a number, which tells you when to stop counting.

3 Mean centering

$$Z = \{1, 5, 12\}$$
$$\mu_z = 6$$



$$\begin{aligned} Z' &= \{1-6, 5-6, 12-6\} \\ &= \{-5, -1, 6\} \\ \mu_{z'} &= 0 \end{aligned}$$

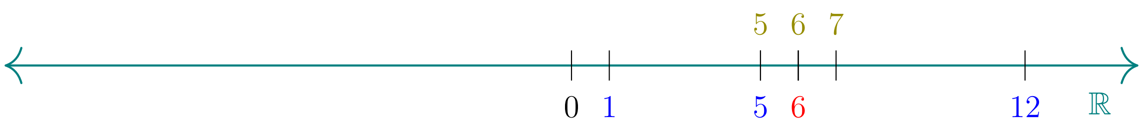


Mean centering data produces a new data set, which has the same relationships, but the mean is zero.

4 Variance

$Z = \{1, 5, 12\}$
$\mu_z = 6$

$W = \{5, 6, 7\}$
$\mu_w = 6$



Set Z (blue) is more "spread out" than set W (olive).

If $X = \{x_1, ..., x_n\}$, the variance of X is

$$\sigma_x{}^2 = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \mu_x)^2\right].$$

The standard deviation is given by

$$\sigma_x = \sqrt{\sigma_x{}^2}.$$

Z and W have the same mean, but Z is more spread out, so σ_z should be greater than σ_w.

$$\begin{aligned}
\sigma_w^2 &= \frac{1}{3}\left[\sum_{i=1}^{3}(w_i - \mu_w)^2\right] \\
&= \frac{1}{3}\left[(5-6)^2 + (6-6)^2 + (7-6)^2\right] \\
&= \frac{1}{3}\left[(-1)^2 + 0^2 + 1^2\right] \\
&= \frac{2}{3} \\
\sigma_w &= \sqrt{\frac{2}{3}}
\end{aligned}$$

$$\begin{aligned}
\sigma_z^2 &= \frac{1}{3}\left[\sum_{i=1}^{3}(z_i - \mu_z)^2\right] \\
&= \frac{1}{3}\left[(1-6)^2 + (5-6)^2 + (12-6)^2\right] \\
&= \frac{1}{3}\left[(-5)^2 + (-1)^2 + 6^2\right] \\
&= \frac{62}{3} \\
\sigma_z &= \sqrt{\frac{62}{3}}
\end{aligned}$$

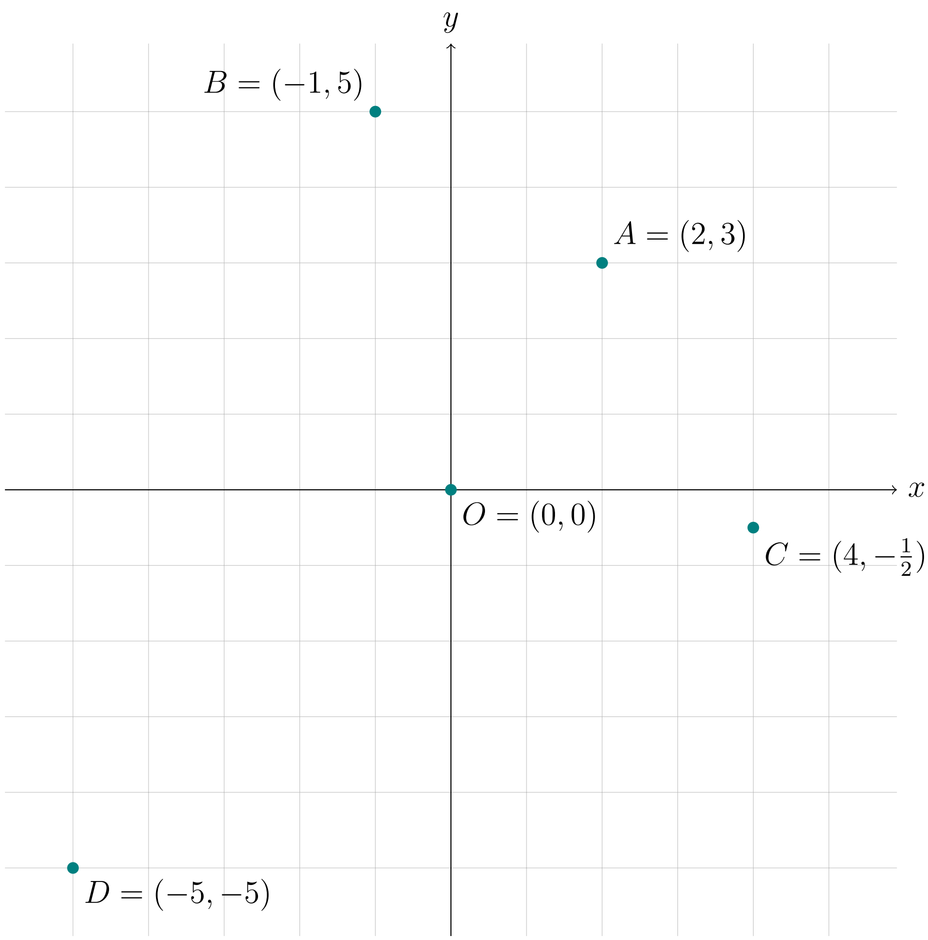
$\sigma_z^2 \gg \sigma_w^2$, so Z is much more spread out than W.

Cartesian Plane: Plotting Points

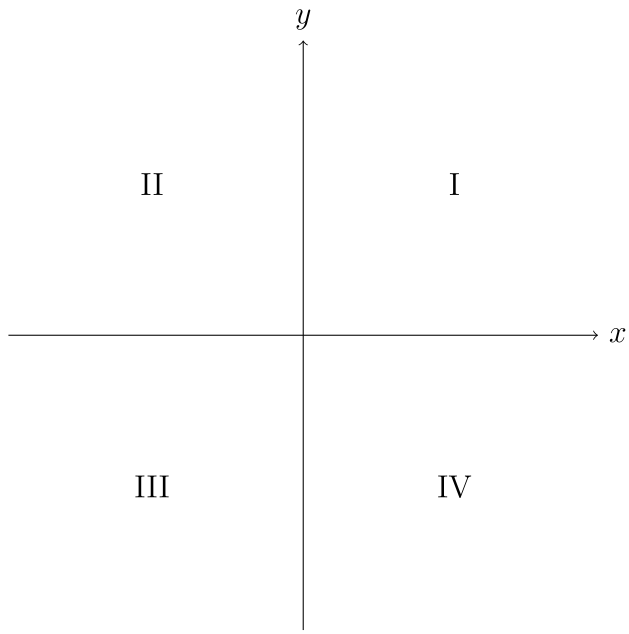
Video companion

1 Introduction

Cartesian plane denoted $\mathbb{R}^2$



2 Axes and quadrants



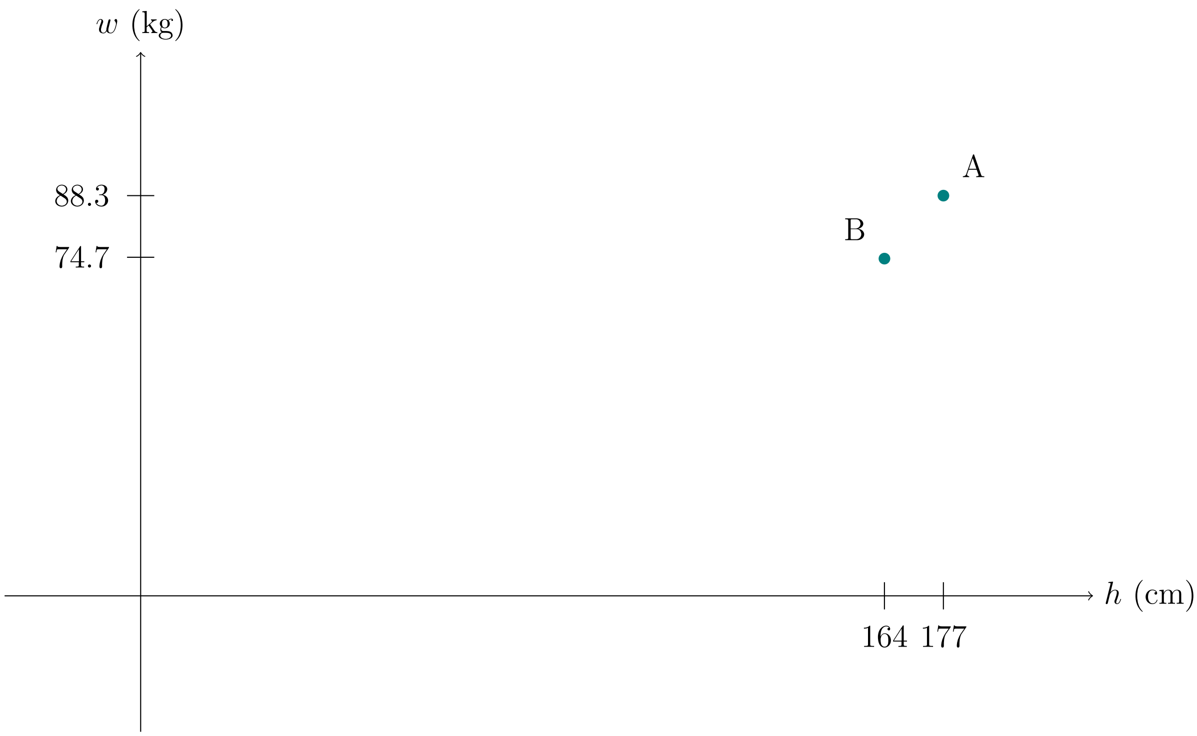
$$\begin{aligned} x\text{-axis} &= \left\{(x,y) \in \mathbb{R}^2 : y = 0\right\} \\ y\text{-axis} &= \left\{(x,y) \in \mathbb{R}^2 : x = 0\right\} \end{aligned}$$

$$\begin{aligned} \text{first quadrant} &= \left\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\right\} \\ \text{second quadrant} &= \left\{(x,y) \in \mathbb{R}^2 : x < 0, y > 0\right\} \\ \text{third quadrant} &= \left\{(x,y) \in \mathbb{R}^2 : \qquad\qquad\qquad \right\} \\ \text{fourth quadrant} &= \left\{(x,y) \in \mathbb{R}^2 : \qquad\qquad\qquad \right\} \end{aligned}$$

3 Real-world example

Table of height and weight:

	h (cm)	w (kg)
A	177	88.3
B	164	74.7



Cartesian Plane: Distance Formula

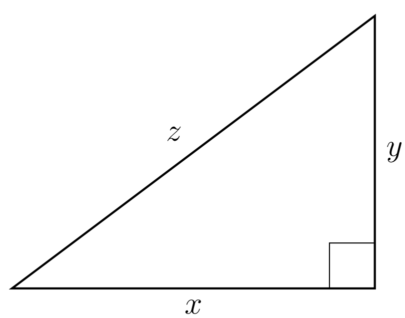
Video companion

1 Introduction

In this video:

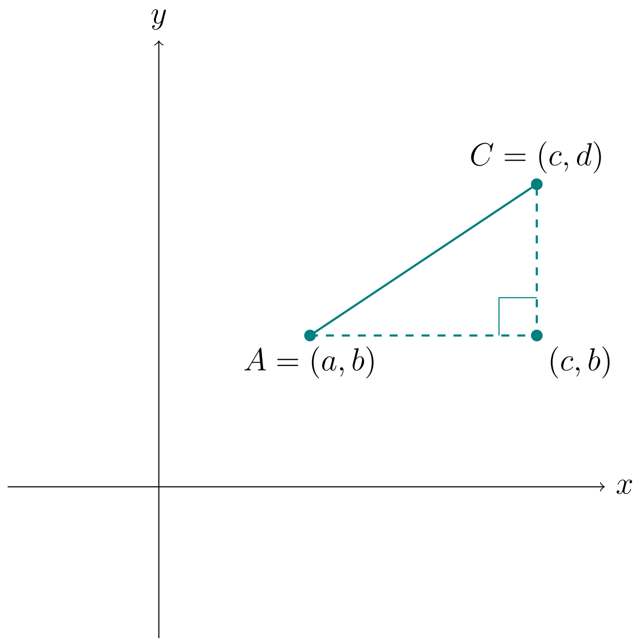
- The distance formula
- Nearest neighbors
- Clustering

2 Pythagorean theorem



$$z^2 = x^2 + y^2$$
$$z = \sqrt{x^2 + y^2}$$

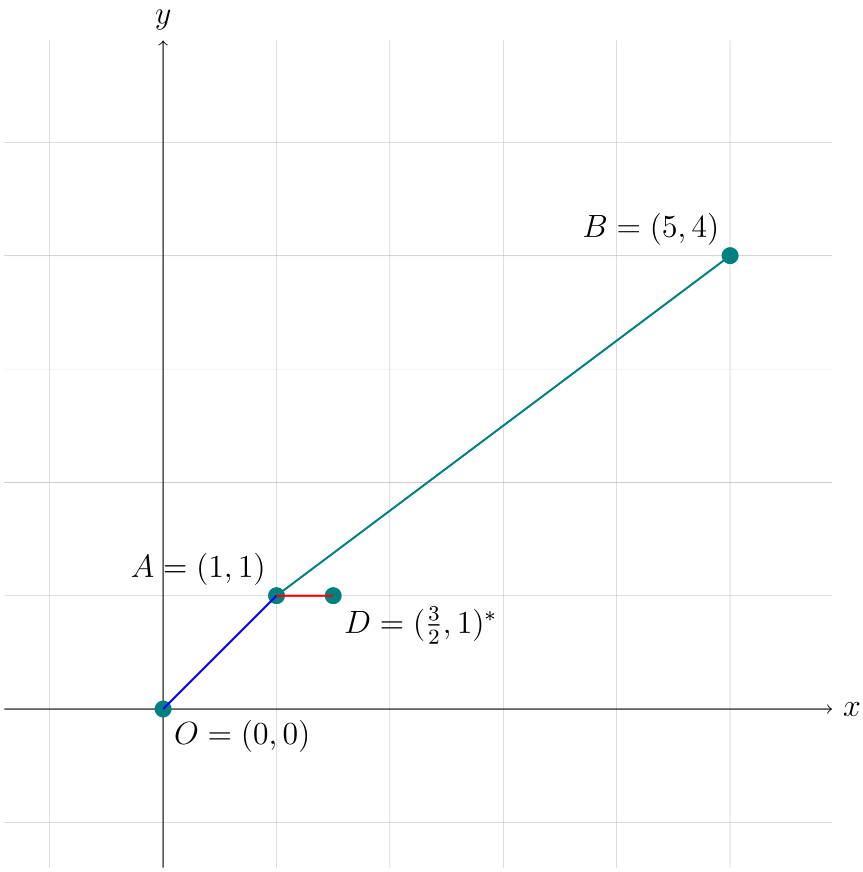
3 Graph of distance formula



Distance formula:

$$\text{dist}(A, C) = \sqrt{(c-a)^2 + (d-b)^2}$$

4 Example and nearest neighbors



$$\begin{aligned}\mathrm{dist}(A,B) &= \sqrt{(5-1)^2+(4-1)^2}\\ &= 5\end{aligned}$$

$$\begin{aligned}\mathrm{dist}(A,O) &= \sqrt{(1-0)^2+(1-0)^2}\\ &= \sqrt{2}\approx 1.4\end{aligned}$$

$$\begin{aligned}\mathrm{dist}(A,D) &= \sqrt{\left(\frac{3}{2}-1\right)^2+(1-1)^2}\\ &= \frac{1}{2}\end{aligned}$$

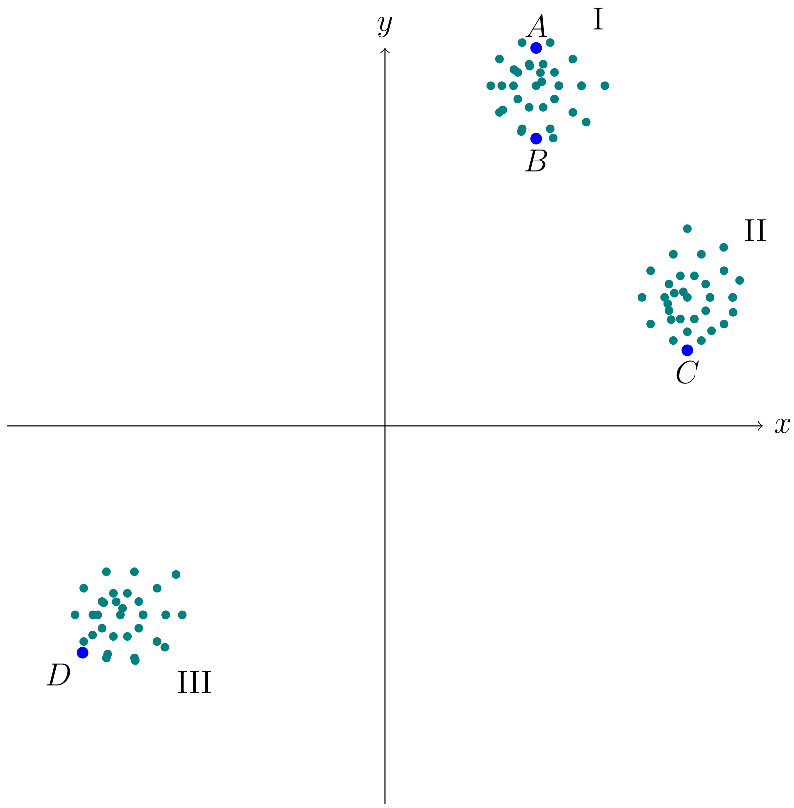
*Note that the x and y values of point D are reversed in the video, but it does not matter in calculating the distance from A.

Consider set S:

$$S = \{O, B, D\}$$

The *nearest neighbor* of A in S is D.
The second nearest neighbor of A in S is O.
The third nearest neighbor of A in S is B.

5 Clustering



Three clusters: I, II, and III

If A and B are in cluster I,
and C is in cluster II,
and D is in cluster III,

Then $\text{dist}(A, B) << \text{dist}(A, C)$,
$<< \text{dist}(A, D)$

Cartesian Plane: Point-Slope Formula for Lines

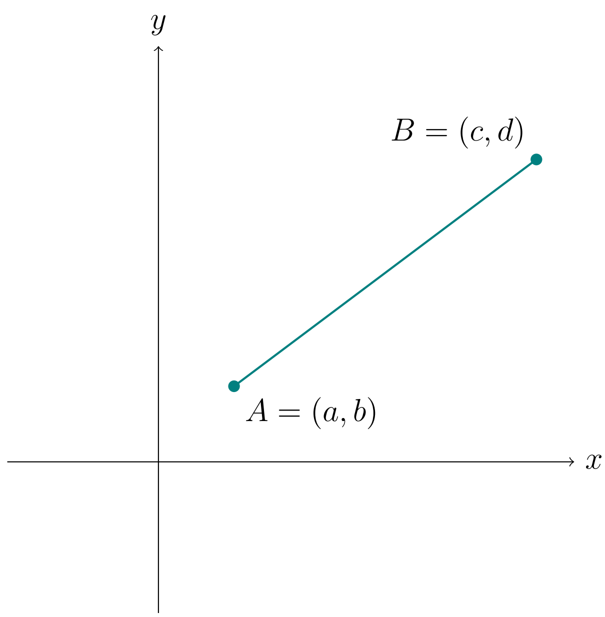
Video companion

1 Introduction

In this video: Demystify formulas for equations of lines

$$\begin{array}{ll} y - y_0 = m(x - x_0) & \text{Point-slope form} \\ y = mx + b & \text{Slope-intercept form} \end{array}$$

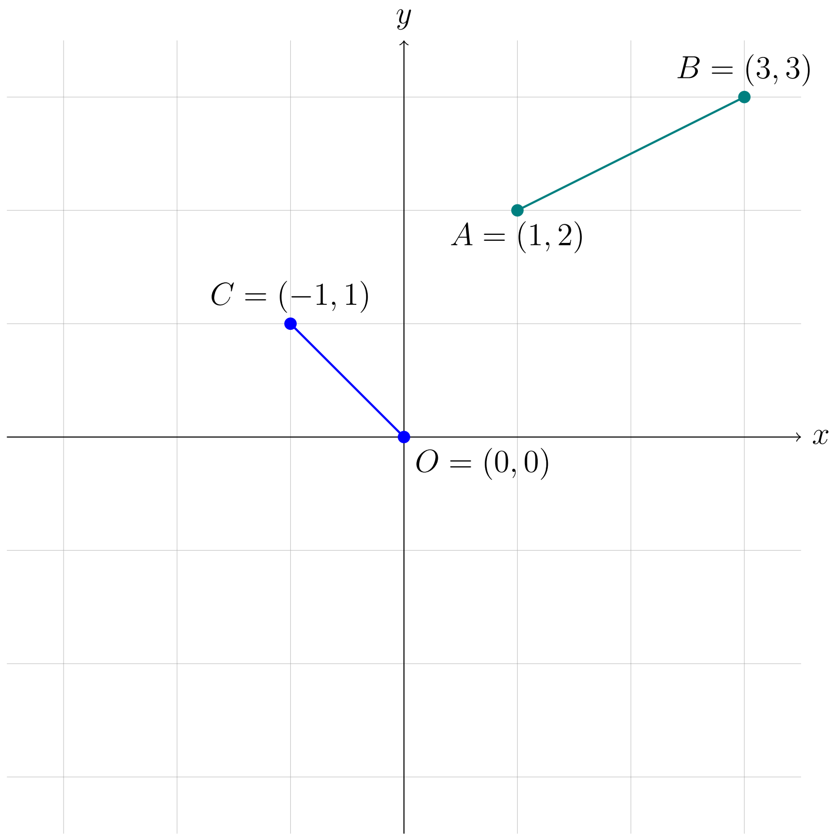
2 Slope of a line segment



Slope of $\overrightarrow{AB}$:

$$m = \frac{d-b}{c-a} = \frac{\text{“rise”}}{\text{“run”}}$$

3 Examples



Slope of $\overrightarrow{AB}$:

$$m = \frac{3-2}{3-1} = \frac{1}{2}$$

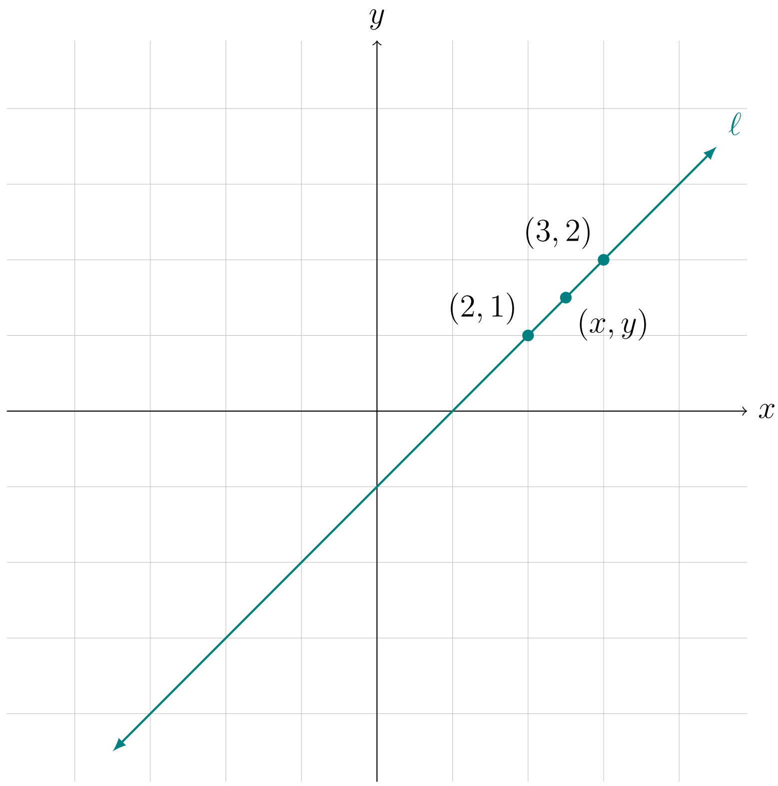
$m = \frac{1}{2}$ is a positive slope.

Slope of $\overrightarrow{CO}$:

$$m = \frac{0-1}{0-(-1)} = -1$$

$m = -1$ is a negative slope.

4 Equation of a line



For a point (x, y) to be on the line, the line segment from $(2, 1)$ to (x, y) need to have a slope of 1.

$$
\begin{aligned}
1 &= \frac{y-1}{x-2} \\
y-1 &= 1(x-2)
\end{aligned}
$$

The line is defined by this formula:

$$\ell = \left\{(x, y) \in \mathbb{R}^2 : y - 1 = 1(x - 2)\right\}$$

Check that $(3, 2)$ is on the line:

$$
\begin{aligned}
&(3,2) \in \ell\,? \\
2-1 &\stackrel{?}{=} 1(3-2) \\
1 &\stackrel{?}{=} 1 \quad \checkmark
\end{aligned}
$$

Check if $(5, 1)$ is on the line:

$$
\begin{aligned}
&(5,1) \in \ell\,?\\
&1 - 1 \stackrel{?}{=} 1(5-2)\\
&\quad\;\; 0 \stackrel{?}{=} 3 \quad \times
\end{aligned}
$$

5 Point-slope formula

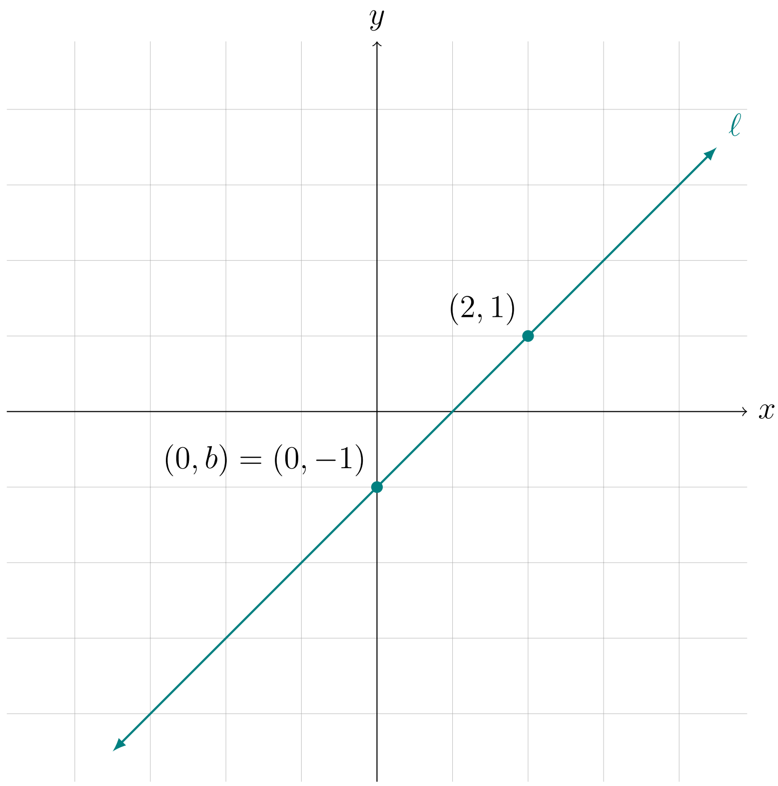
If a line ℓ has slope m, *and* if (x_0, y_0) is *any* point on ℓ, then ℓ has the equation

$$\boxed{y - y_0 = m(x - x_0).}$$

Cartesian Plane: Slope-Intercept Formula for Lines

Video companion

1 Derivation using point-slope form



From last video, the equation of a line in point-slope form that passes through $(2, 1)$ and has slope $m = 1$ is

$$y - 1 = 1(x - 2).$$

The y-intercept is at point $(0, b)$. To find b, we substitute that point into the definition of the line:

$$\begin{aligned}&(0, b) \in \ell, \text{ so}\\ &b - 1 = 1(0 - 2)\\ &\quad b = -1\end{aligned}$$

Using the y-intercept in the equation for the line in point-slope form:

$$\begin{aligned} y - (-1) &= 1(x - 0) \\ y + 1 &= x \\ y &= 1x - 1 \end{aligned}$$

2 Slope-intercept form

If ℓ has slope m, and ℓ hits the y-axis at $(0, b)$, then

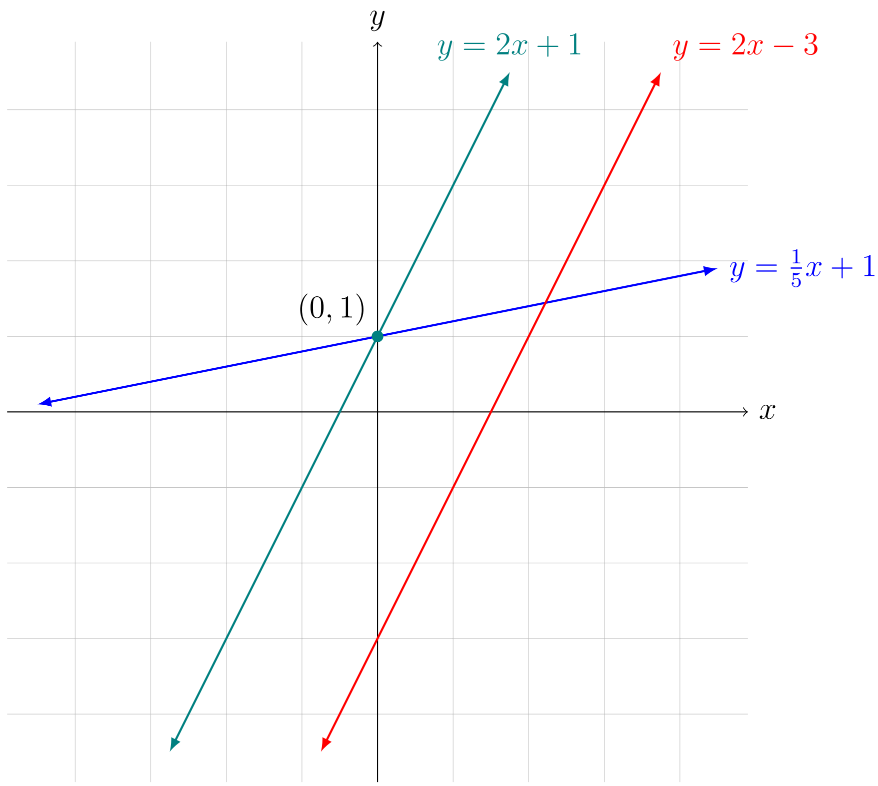
$$\boxed{y = mx + b}$$

is an equation for ℓ, where m is the slope and b is the y-intercept.

3 Drawing lines

Draw line with equation

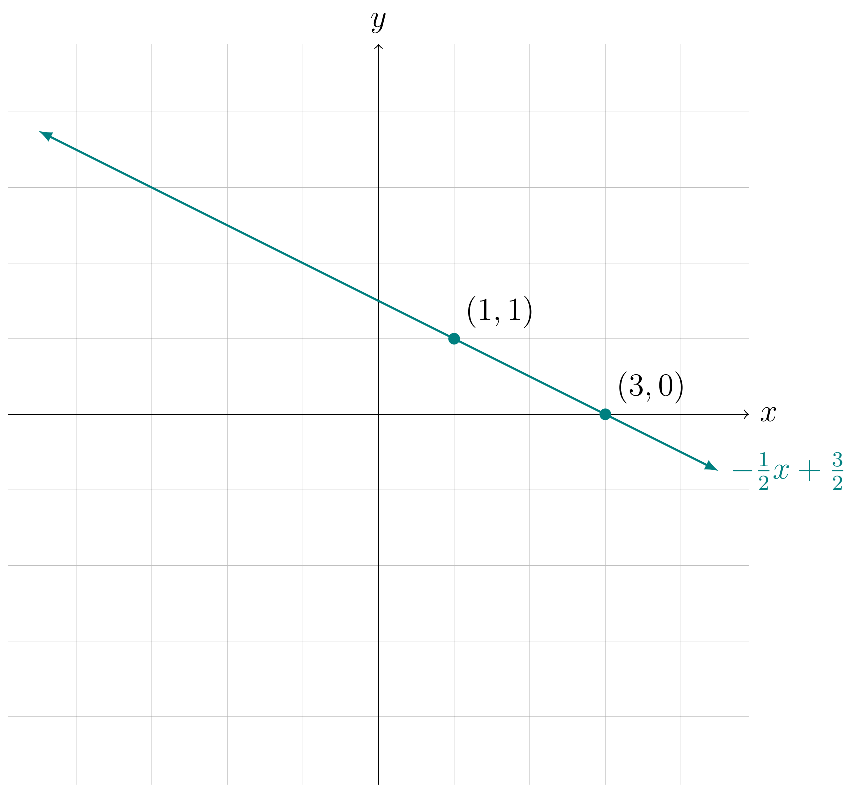
$$y = 2x + 1$$
$$y = \frac{1}{5}x + 1$$
$$y = 2x - 3$$



The slope tells you how to angle the line, and the y-intercept tells you where to anchor it on the y-axis.

4 Example

Problem: Line ℓ has points $(1,1)$ and $(3,0)$ on it. Find an equation for ℓ.



Find the slope:

$$m = \frac{0-1}{3-1} = -\frac{1}{2}$$

Some possible equations for the line in point-slope form:

$$y-1 = -\frac{1}{2}(x-1)$$
$$y-0 = -\frac{1}{2}(x-3)$$

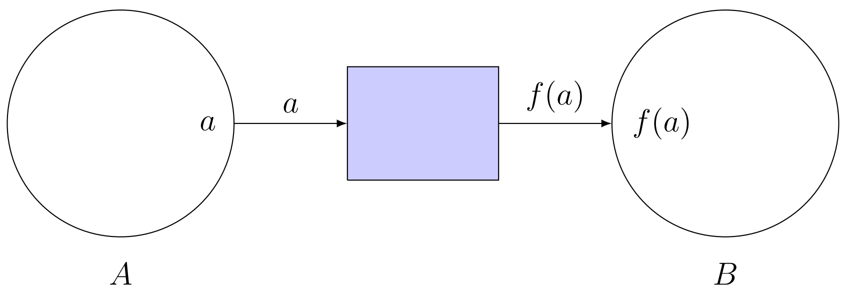
An equation for the line in slope-intercept form:

$$y = -\frac{1}{2}x + \frac{3}{2}$$

Functions: Mapping from Sets to Sets

Video companion

1 Function as a machine



A function $f : A \to B$ is a rule/formula/machine that transforms each element $a \in A$ into $f(a) \in B$.

$$\begin{aligned} a &: \text{input} \\ f(a) &: \text{output} \end{aligned}$$

2 Examples

Abstract example:

$$\begin{aligned} A &= \{1, 2, 10\} \qquad B = \{\text{apple}, \text{DE}, \text{monkey}\} \\ f &: A \to B \\ f(1) &= \text{apple} \\ f(2) &= \text{apple} \\ f(10) &= \text{monkey} \end{aligned}$$

Study participants test positive or negative:

$$X = \{\text{all people in VBS study}\} \qquad Y = \{+, -\}$$
$$\text{Test} : X \to Y$$
$$\text{Test(person)} = +$$
$$\text{Test(person)} = -$$

Profit by year:

$$Y = \{...2010, 2011, 2012, ...\} \qquad \mathbb{R}$$
$$\text{Profit} : Y \to \mathbb{R}$$
$$\text{Profit(year)} = \text{profit/loss in year}$$
$$\text{Profit}(2011) = 1,007$$
$$\text{Profit}(2012) = -10,000$$

3 Supervised learning

Given: some examples of inputs $a \in A$ and outputs $f(a) \in B$
Mission: figure out $f : A \to B$

Data Science Math Skills
Paul Bendich and Daniel Egger
Duke University

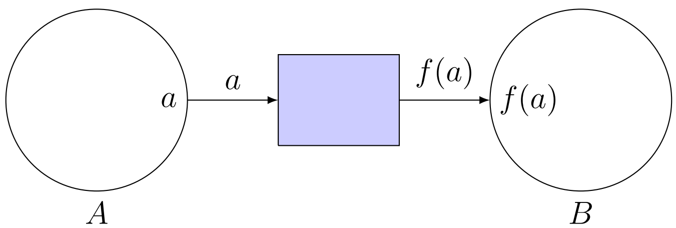
Functions: Graphing in the Cartesian Plane

Video companion

1 Introduction

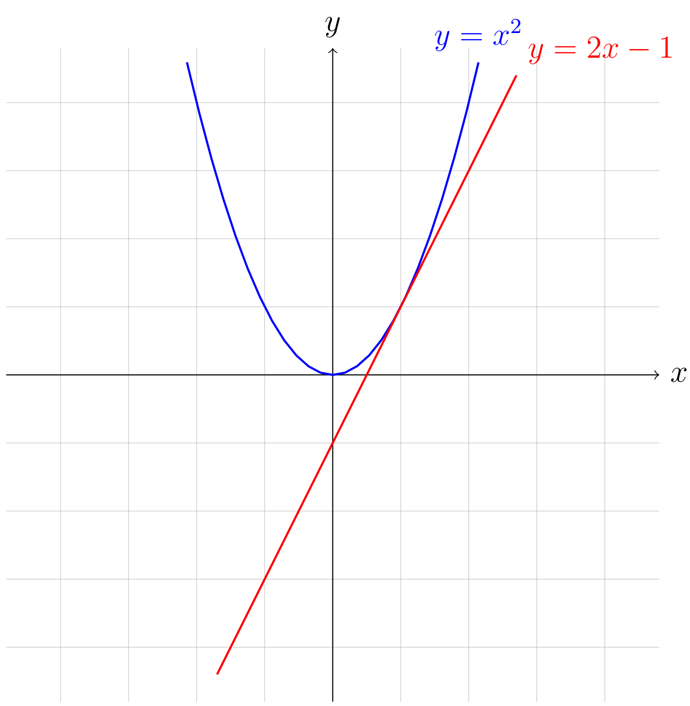
Last time: abstract depiction of a function as a machine

$f : A \to B$



This video: graphs of functions

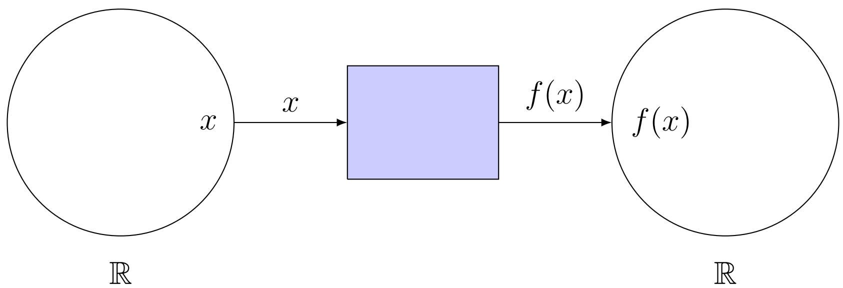
$f : \mathbb{R} \to \mathbb{R}$



You will learn the difference between a *graph* of a function and the function itself.

2 Map real line to real line

$f : \mathbb{R} \to \mathbb{R}$



A function is a formula, a rule for how to operate the machine.

$$\begin{aligned} f(x) &= 2x - 1 \\ f(1) &= 2(1) - 1 = 1 \\ f(0) &= 2(0) - 1 = -1 \\ f(5.1) &= 2(5.1) - 1 = 9.2 \end{aligned}$$

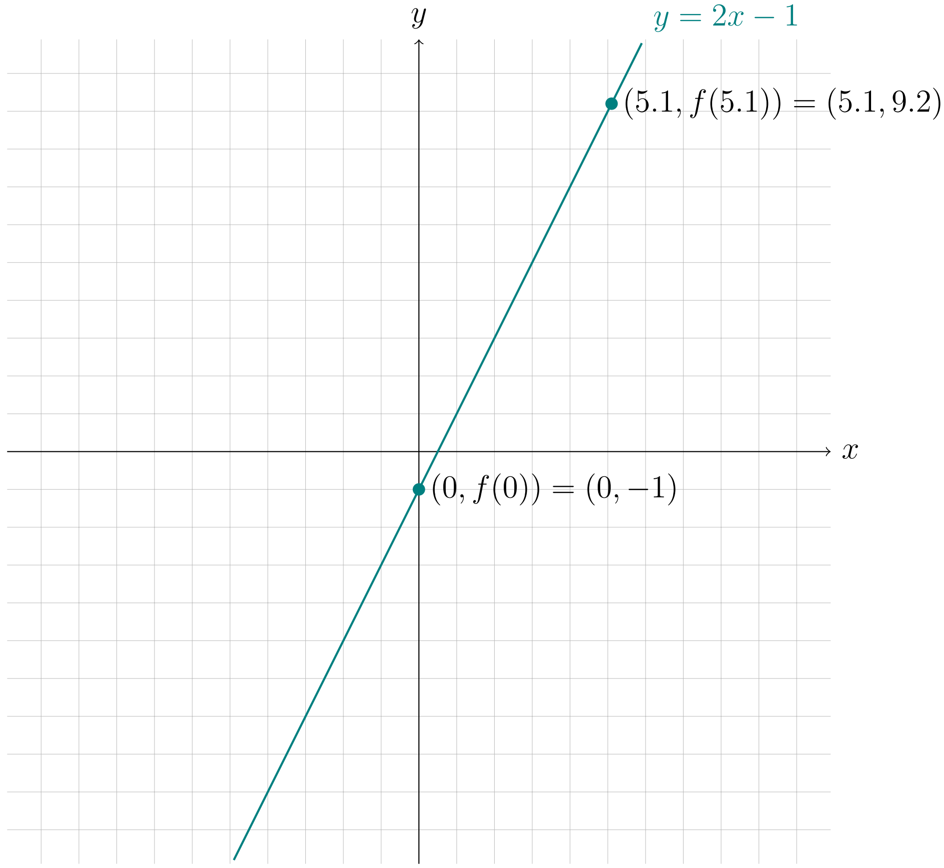
More complicated formulas, like absolute value:

$$\begin{aligned} g(x) &= |x| \\ &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \end{aligned}$$

Both f and g are functions, with a formula for how to compute the result.

3 What is a graph?

Graph of the function $f : \mathbb{R} \to \mathbb{R}$

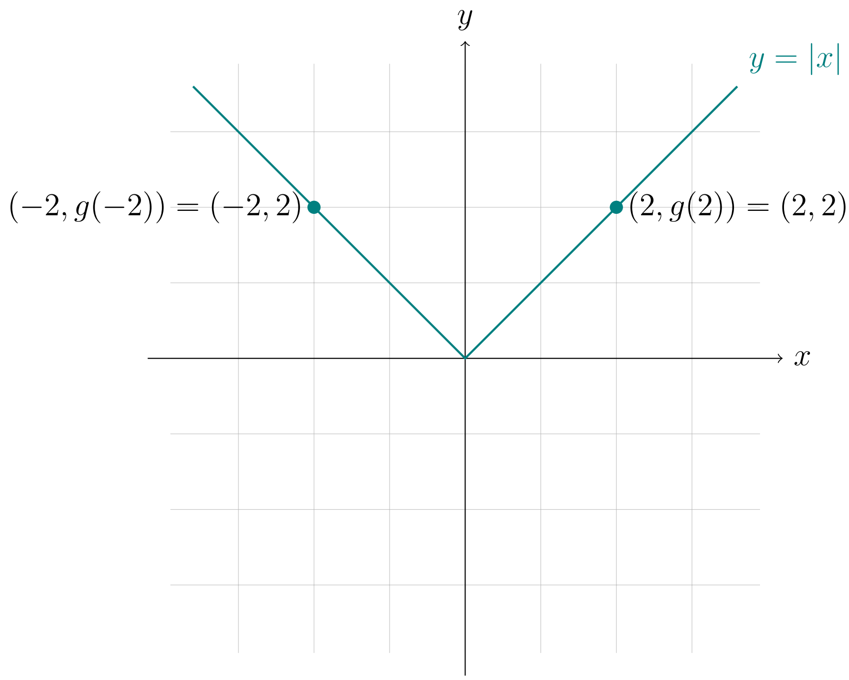


If g is a function $: \mathbb{R} \to \mathbb{R}$, the graph of $g = \{(x, y) \in \mathbb{R}^2 : y = g(x)\}$

4 Examples

Absolute value function

$$
\begin{aligned}
g(x) &= |x| \\
&= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}
\end{aligned}
$$



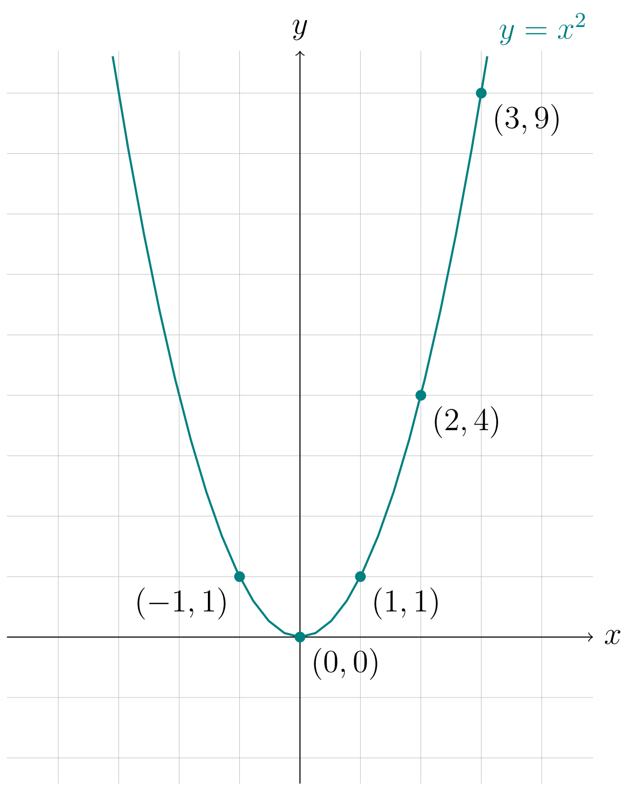
Quadratic function

$$h(x) = x^2$$

Graph a function by testing input and output pairs, see a pattern, and try to draw a curve through it. This is similar to *querying* in supervised learning.

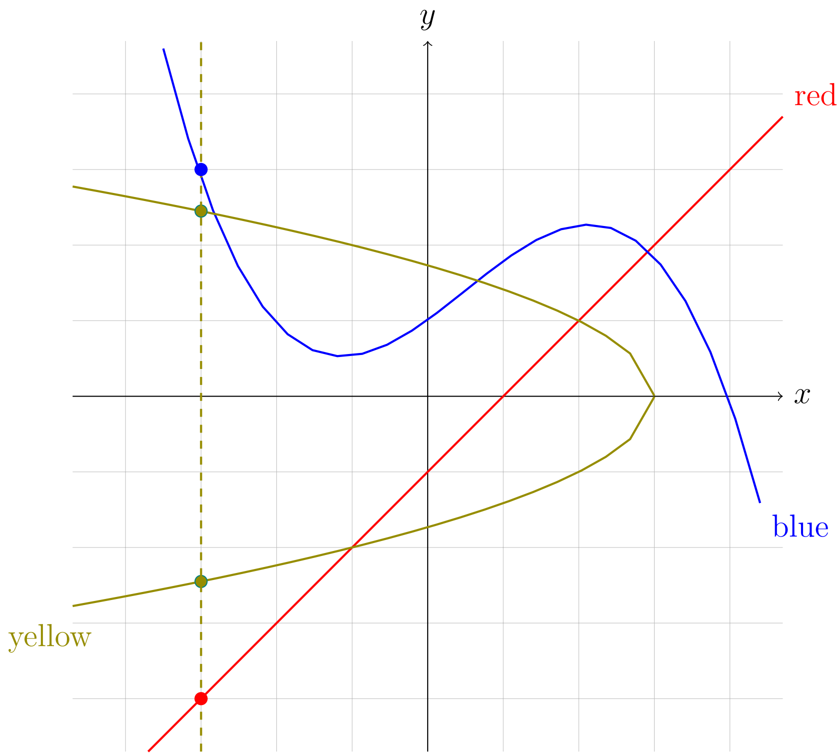
Table of values:

x	$h(x)$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
-1	$(-1)^2 = 1$



$h(x) = x^2$ is a *quadratic* function.

5 Vertical line test

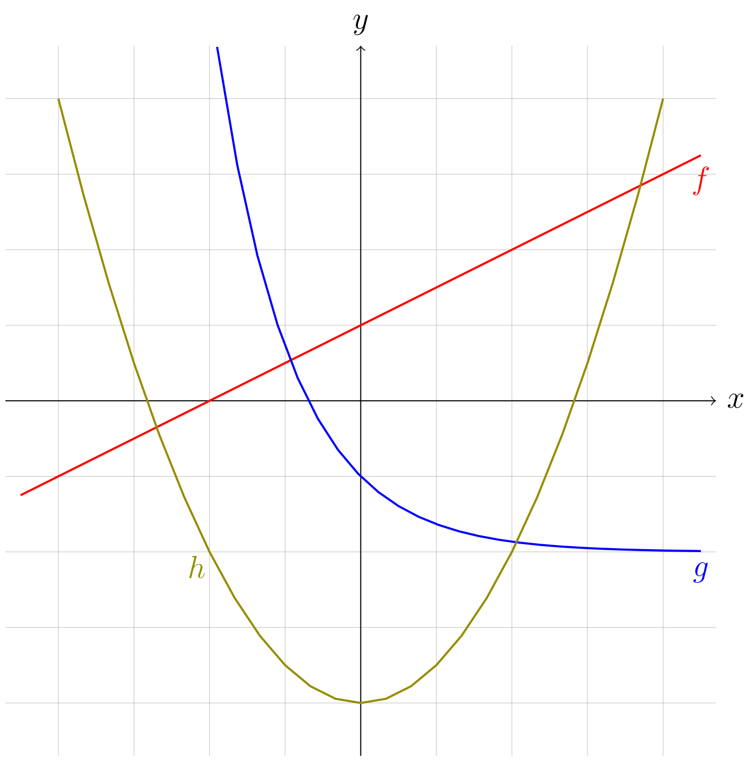


Red and blue could be graphs of functions. Yellow could not be the graph of a function because it violates the *vertical line test*, which states that *any vertical line intersects the graph of a function once.*

Functions: Increasing and Decreasing Functions

Video companion

1 Introduction



- f is *strictly increasing*
- g is *strictly decreasing*
- h is neither

Let $f : \mathbb{R} \to \mathbb{R}$,
f is strictly increasing if whenever $a < b$, we have $f(a) < f(b)$.
f is strictly decreasing if whenever $a < b$, we have $f(a) > f(b)$.

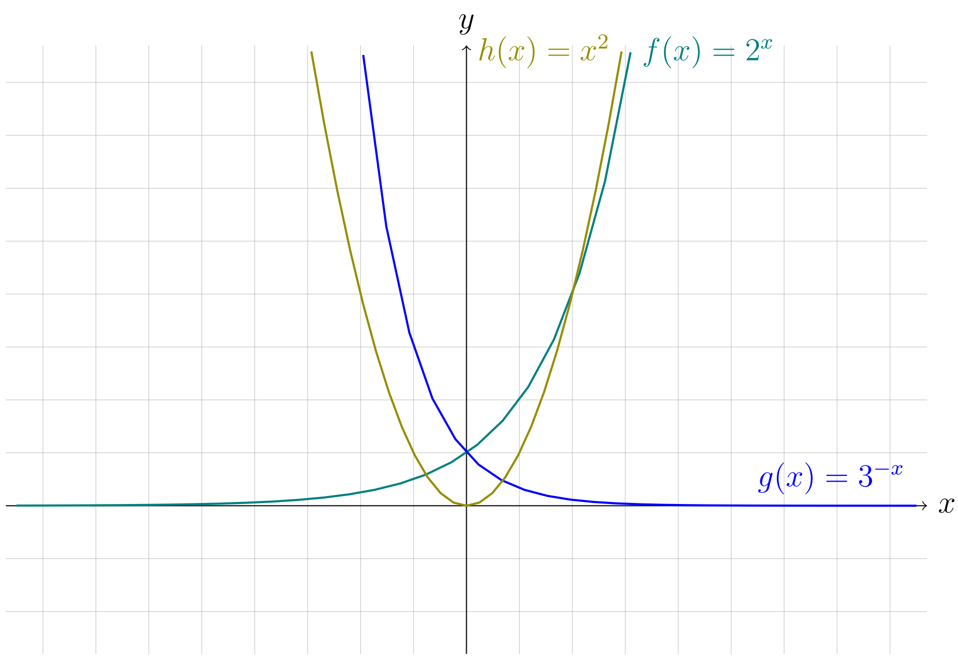
2 Examples

$f(x) = 2^x$ (exponential function)
$g(x) = 3^{-x}$
$h(x) = x^2$

x	$f(x)$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
-1	$2^{-1} = \frac{1}{2}$

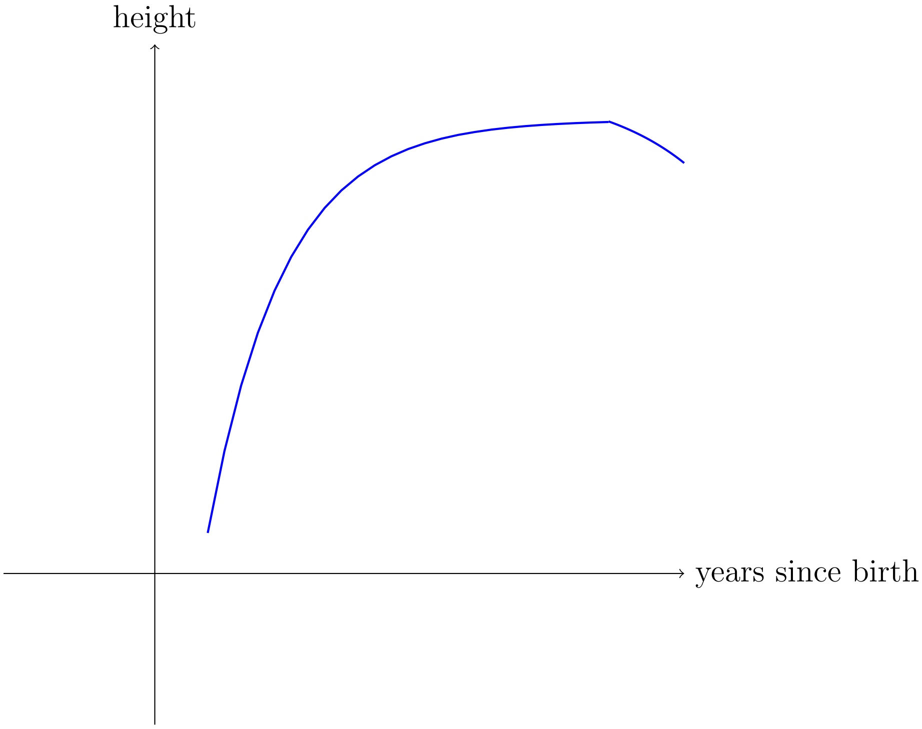
x	$g(x)$
0	$3^0 = 1$
1	$3^{-1} = \frac{1}{3}$
2	$3^{-2} = \frac{1}{9}$
3	$3^{-3} = \frac{1}{27}$
-1	$3^1 = 3$

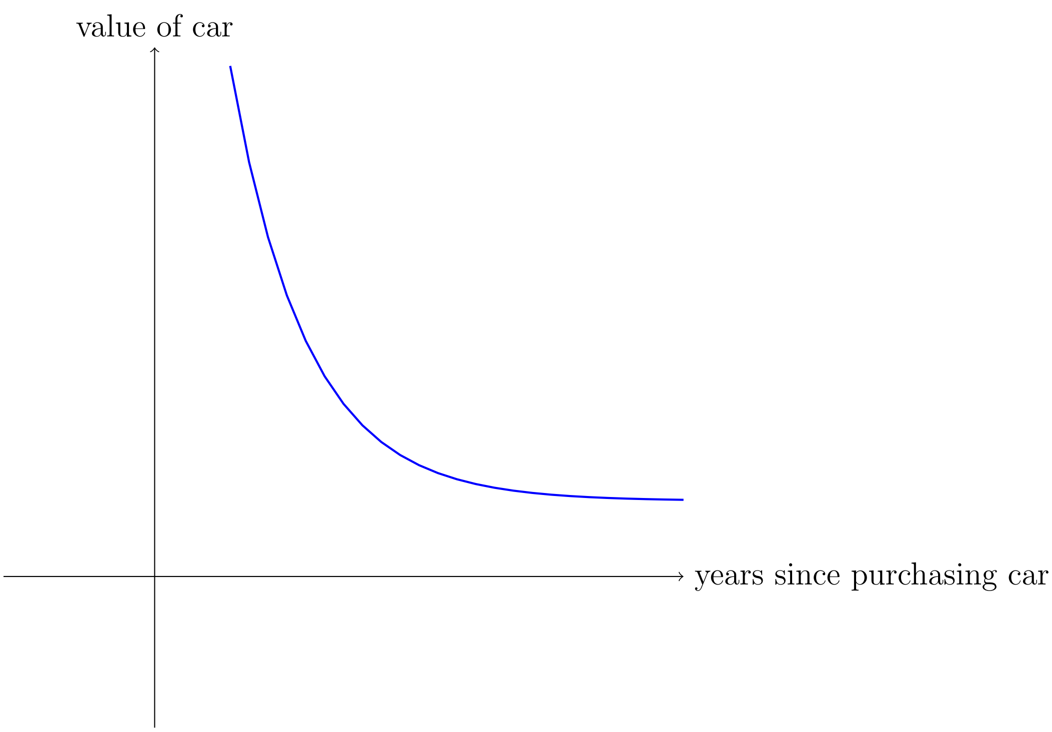
x	$h(x)$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
-1	$(-1)^2 = 1$



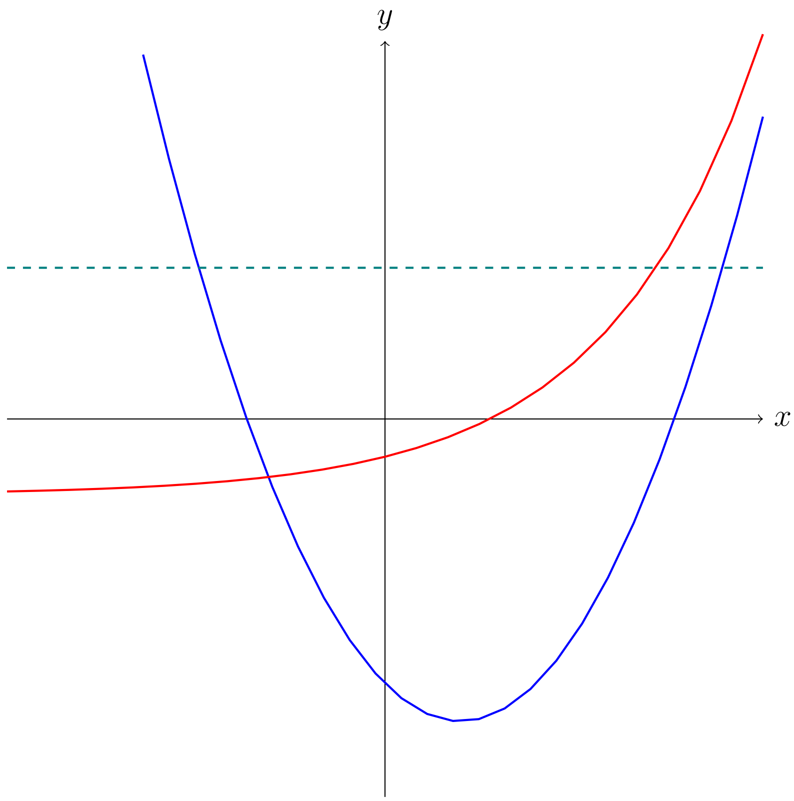
- f is strictly increasing
- g is strictly decreasing
- h is neither
  - h is strictly increasing on $[0, \infty)$
  - h is strictly decreasing on $(-\infty, 0]$

3 "Real-world" examples





4 Horizontal line test



A function is strictly increasing or strictly decreasing if a horizontal line crosses it only once.

Functions: Composition and Inverse

Video companion

1 Introduction

- Composing two functions
  - Basic identity
  - A warning
- Inverse functions
  - Basic identity
  - A neat picture
  - A warning

2 Composing functions

Definition: Given functions f and g, $(g \circ f)(x) = g(f(x))$, and $(f \circ g)(x) = f(g(x))$

Example:

$$f(x) = x^2$$
$$g(x) = x + 5$$

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^2) = x^2 + 5 \\
g(f(2)) &= g(2^2) = 2^2 + 5 = 9 \\
(f \circ g)(x) &= f(g(x)) \\
&= f(x+5) = (x+5)^2 \neq x^2 + 5
\end{aligned}$$

3 Inverse functions

Example:

$$f(x) = 2x$$
$$g(x) = \frac{1}{2}x$$

$$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2}(2x) = x$$

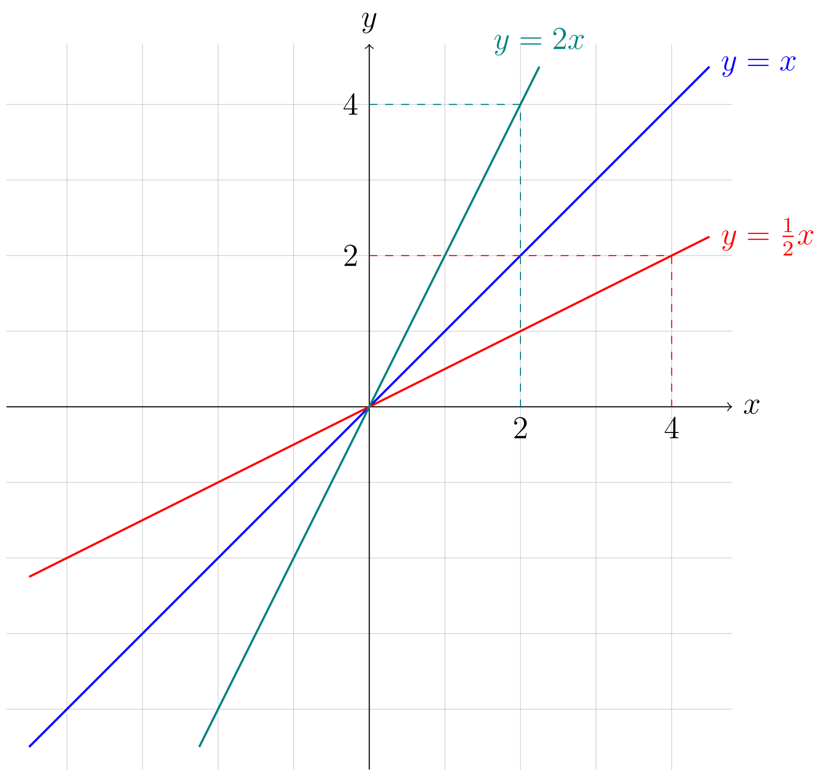
Notice: true for all x

$$(g \circ f)(3) = g(f(3)) = g(2 * 3) = \frac{1}{2}(2 * 3) = 3$$
$$(g \circ f)(\pi) = g(f(\pi)) = g(2 * \pi) = \frac{1}{2}(2 * \pi) = \pi$$

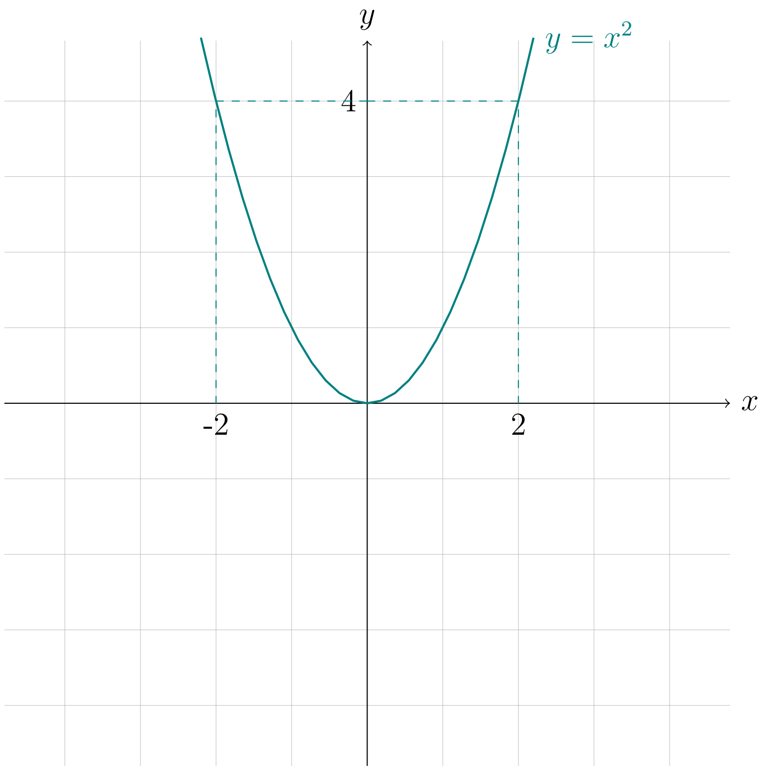
f and g are *inverses* of each other, i.e. f undoes what g does.

$$g = f^{-1}$$

4 Graphical depiction



Warning: not every function $f : \mathbb{R} \to \mathbb{R}$ has an inverse.



Warning: if the graph of f fails the horizontal line test, then f has no inverse. The only invertible functions are those that are either strictly increasing or strictly decreasing.