Data Science Math Skills
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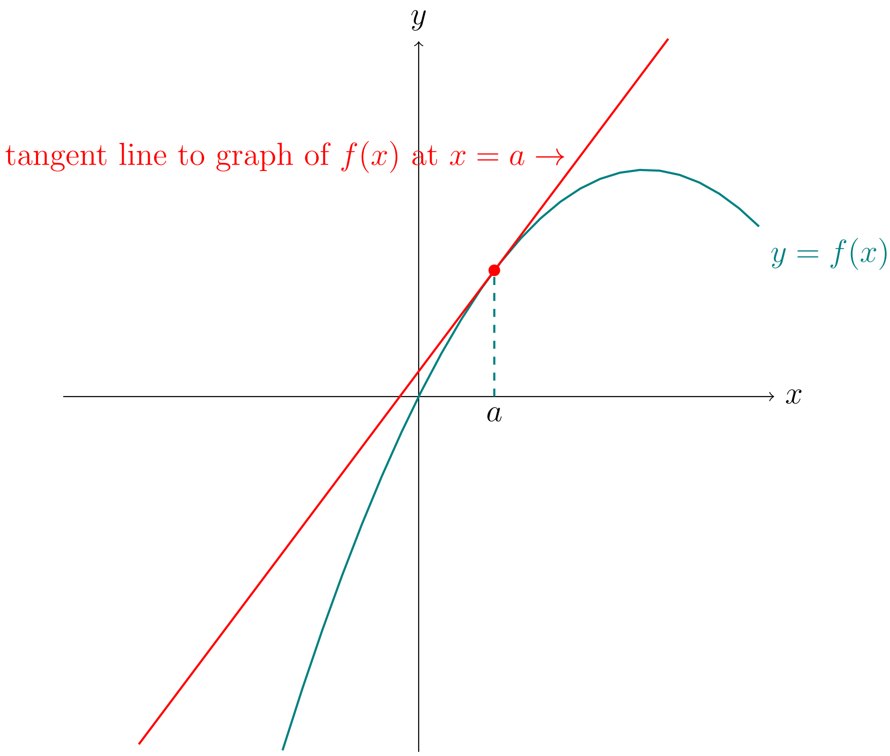
Tangent Lines: The Slope of a Graph at a Point

Video companion

1 Introduction



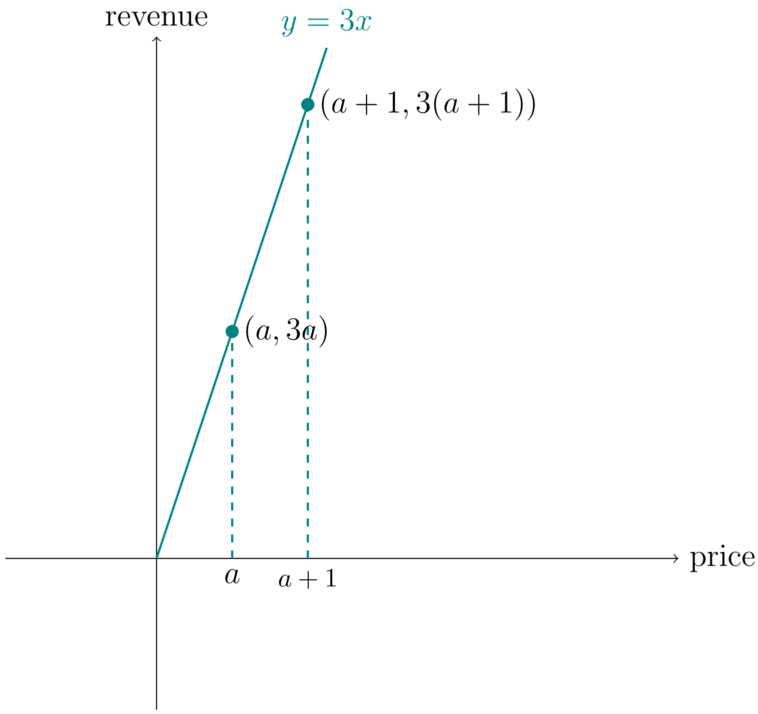
Question: How fast is $f(x)$ changing at $x = a$?

The slope of the *tangent line* gives the instantaneous rate of change. This is also called the *derivative* of the function at that point, or $f'(a)$.

Limit to find slope at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

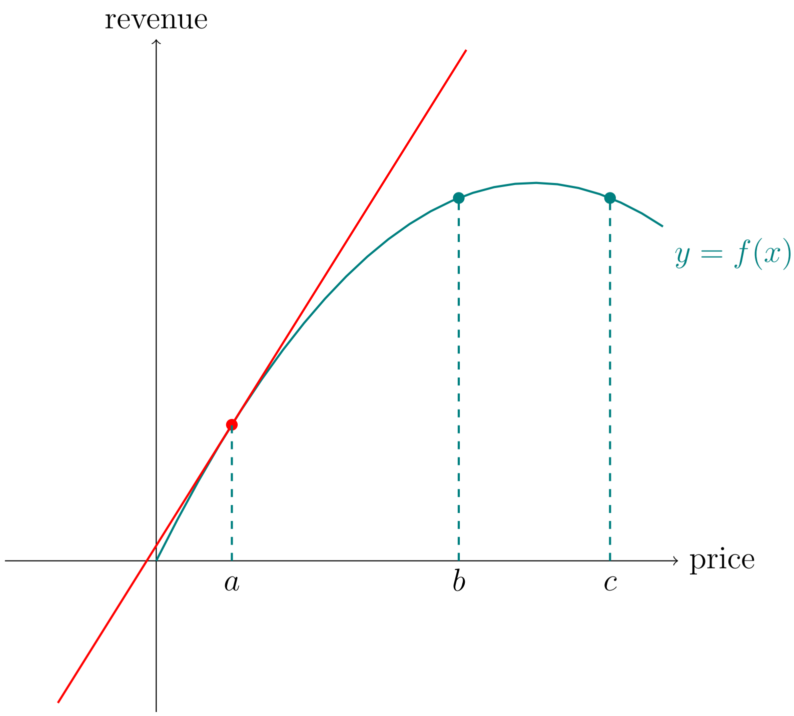
2 Simple example



Slope:

$$\frac{3(a+1) - 3a}{a+1-a} = 3$$

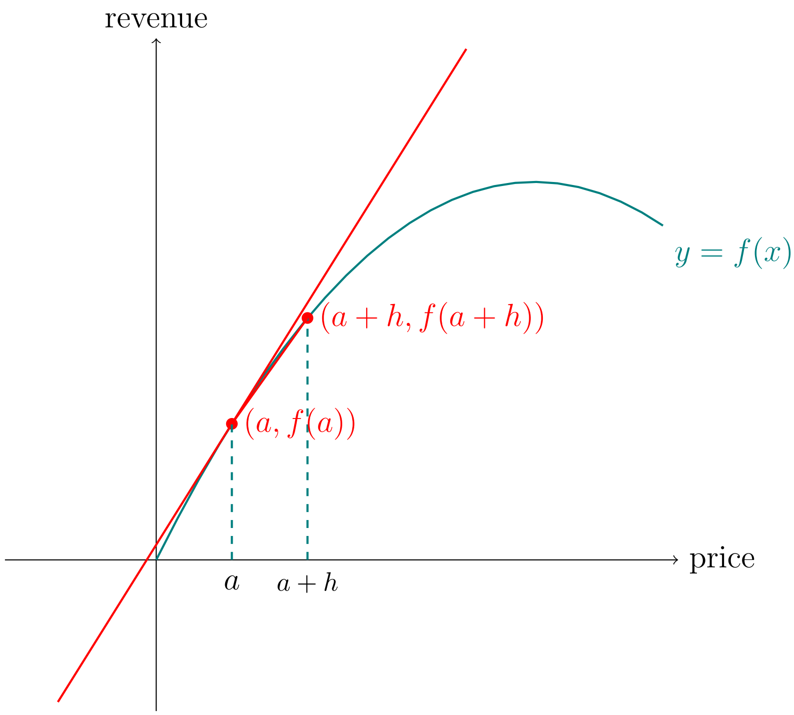
3 More realistic example



What is the instantaneous rate of change of revenue at a price point? It depends on the slope of the tangent line, which changes depending on the price point.

The answer is the slope, or derivative of the function at the price value $x = a$: $f'(a)$.

Related question: What is the slope of a line segment through a and another point on the line?



Slope of line from a to $a+h$:

$$\frac{f(a+h) - f(a)}{h}$$

Slope of tangent line at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is calculus.

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Tangent Lines: The Derivative Function

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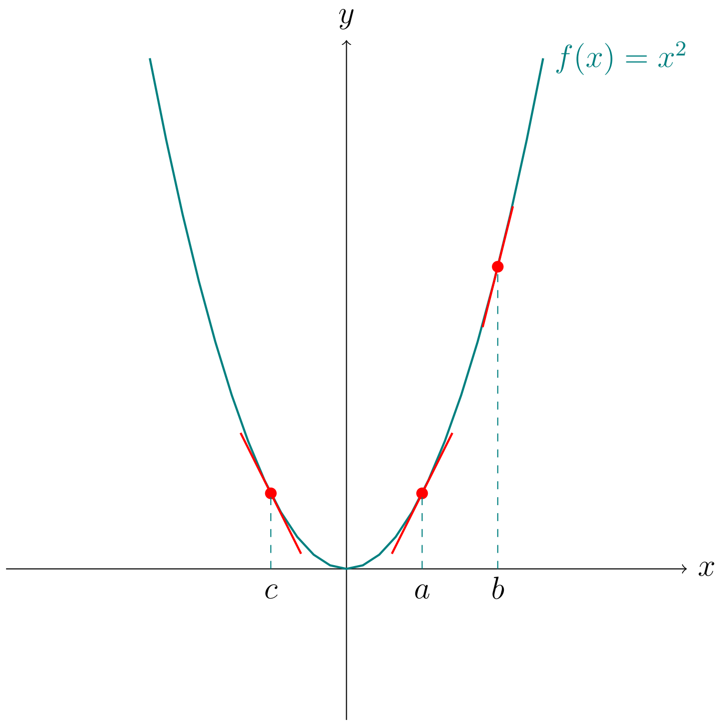
1 Introduction

Derivative formula:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



- Slope is positive at a: $f'(a) > 0$
- Slope is positive at b and greater than at a: $f'(b) > f'(a)$
- Slope is negative at c: $f'(c) < 0$

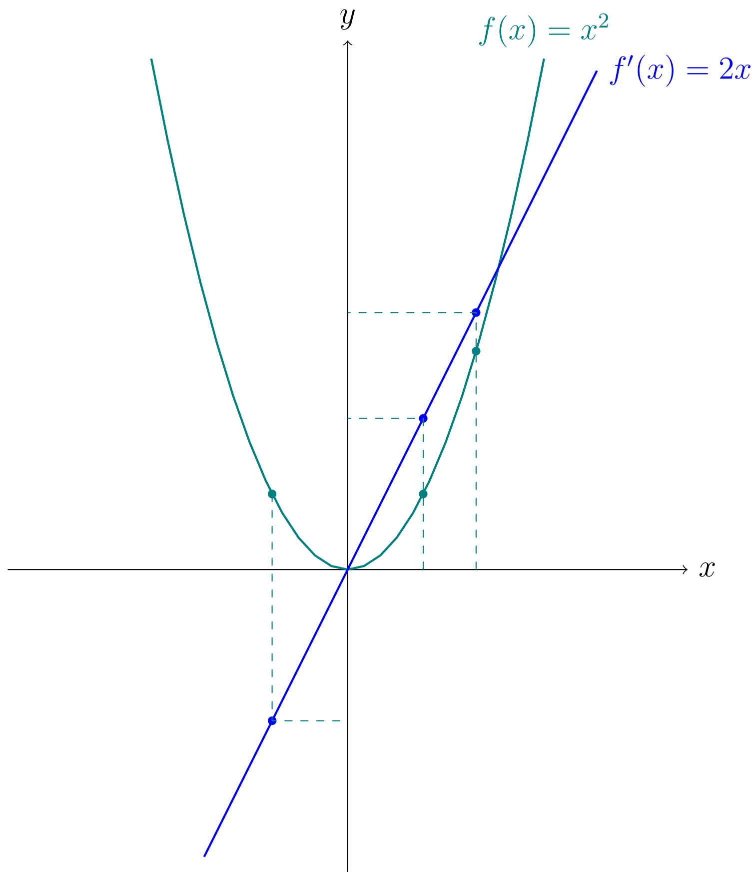
2 Calculate derivative

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\
&= \lim_{h \to 0} \frac{h(2a+h)}{h} \\
&= \lim_{h \to 0} (2a+h) \\
&= 2a
\end{aligned}
$$

$$
\begin{aligned}
f'(a) &= 2a \\
f'(b) &= 2b \\
f'(c) &= 2c
\end{aligned}
$$

Can verify $2a > 0$, $2b > 2a$, and $2c < 0$

3 Graph of derivative function



Next video: Finding where derivative is zero (where the tangent line to the function is horizontal) is important for optimization problems.

Fast Growth, Slow Growth: Using Integer Exponents

Video companion

1 Positive Integer Exponents

$$\begin{aligned} 9 &= 3 \cdot 3 &&= 3^2 \\ 27 &= 3 \cdot 3 \cdot 3 &&= 3^3 \\ 81 &= 3 \cdot 3 \cdot 3 \cdot 3 &&= 3^4 \end{aligned}$$

Exponents count how many times factors repeat in a number. 3^4 is pronounced "three to the fourth power" or "three to the fourth."

Example

$$248{,}832 = 12 \cdot 12 \cdot 12 \cdot 12 \cdot 12 = 12^5$$

A note on pronunciation $4 \cdot 4 = 4^2$ can be pronounced "four to the second"—but also "four squared." Similarly, $4 \cdot 4 \cdot 4 = 4^3$ can be pronounced "four to the third"—but also "four cubed."

2 Zero as an Exponent

$$1^0 = 1 \qquad (2\pi)^0 = 1$$

$$2^0 = 1 \qquad \left(\frac{1}{x^3}\right)^0 = 1$$

$$3^0 = 1$$

Definition By the definition of exponents, any number, except for zero, raised to the zeroth power is one. Note that 0^0 is undefined.

3 Negative Integer Exponents

$$2^{-1} = \frac{1}{2^1} = \frac{1}{2} \qquad \frac{1}{2^{-1}} = 2^1 = 2$$
$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} \qquad \frac{1}{2^{-2}} = 2^2 = 4$$
$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} \qquad \frac{1}{2^{-3}} = 2^3 = 8$$

General rule

$$x^{-n} = \frac{1}{x^n}$$
$$\frac{1}{x^{-n}} = x^n$$

4 Scientific Notation

Mass of the Earth (kg)

$$5,972,000,000,000,000,000,000,000$$
$$= 5.972 \times 10^{24}$$

Mass of an electron (kg)

$$0.0000000000000000000000000000009109$$
$$= 9.109 \times 10^{-31}$$

Keep the significant digits, and there is always one digit to the left of the decimal.

Fast Growth, Slow Growth: Simplification Rules for Algebra Using Exponents

Video companion

1 Exponent simplification rules

Five rules for simplifying algebraic expressions with exponents

1. Multiplication rule
   $x^n x^m = x^{(n+m)}$
2. Power to a power
   $(x^n)^m = x^{nm}$
3. Product to a power
   $(xy)^n = x^n y^n$
4. Fraction to a power
   $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
5. Division and negative powers
   $\frac{x^n}{x^m} = x^{(n-m)}$

2 Examples

Simple examples

$$\left(7^3\right)\left(7^7\right) = 7^{(3+7)} = 7^{10}$$

$$\left(4^3\right)^5 = 4^{(3\cdot 5)} = 4^{15}$$

$$(8\cdot 9)^7 = \left(8^7\right)\left(9^7\right) = 1.00306 \times 10^{13}$$

$$\left(\frac{2}{7}\right)^3 = \frac{2^3}{7^3} = 0.023323615$$

$$\frac{10^5}{10^3} = 10^{(5-3)} = 10^2 = 100$$

Complex examples

$$\frac{x^3y^4z^5}{x^3y^5z^2} = \frac{x^3}{x^3}\frac{y^4}{y^5}\frac{z^5}{z^2} = x^{(3-3)}y^{(4-5)}z^{(5-2)} = y^{-1}z^3 = \frac{z^3}{y}$$

$$\left[\frac{(xy)^2}{x^{-3}y^2}\right]^{-1} = \left[\frac{x^2y^2}{x^{-3}y^2}\right]^{-1} = \left[x^{(2--3)}y^{(2-2)}\right]^{-1} = \left[x^5\right]^{-1} = x^{-5} = \frac{1}{x^5}$$

3 Fractional exponents

In general

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

Examples

$$\begin{aligned} 8^{\frac{2}{3}} &= \left[\sqrt[3]{8}\right]^2 \\ &= \left[\sqrt[3]{2\cdot 2\cdot 2}\right]^2 = 2^2 = 4 \end{aligned}$$

or

$$\begin{aligned} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} = \sqrt[3]{4\cdot 4\cdot 4} = 4 \end{aligned}$$

$$\begin{aligned} 125^{\frac{4}{3}} &= \left[\sqrt[3]{125}\right]^4 \\ &= \left[\sqrt[3]{5\cdot 5\cdot 5}\right]^4 = 5^4 = 625 \end{aligned}$$

Fast Growth, Slow Growth: How Logarithms and Exponents Are Related

Video companion

1 Introduction

Logarithm means "raised to what power?"

If the question is "what power of two is $2 \cdot 2 \cdot 2 = 8$?" then the answer is the logarithm to the base two of eight, which is $\log_2(8) = 3$.

Two general forms

$b^x = N$	"exponential form"
$x = \log_b(N)$	"logarithmic form"

Examples

If $b = 2$, $x = 3$, and $N = 8$:

$$2^3 = 8$$
$$3 = \log_2(8)$$

If $b = 2$, $x = 4$, and $N = 16$:

$$2^4 = 16$$
$$4 = \log_2(16)$$

2 Logs of one

Recall that raising any number to the power of zero is one, $b^0 = 1$. Therefore, the log, to any base, of one is zero.

$\log_2(1) = 0$	$2^0 = 1$
$\log_{10}(1) = 0$	$10^0 = 1$
$\log_{20}(1) = 0$	$20^0 = 1$

3 General rules

1. Product rule
   $\log(xy) = \log(x) + \log(y)$
2. Quotient rule
   $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$
3. Power and root rule
   $\log(x^n) = n\log(x)$

Examples

$$\begin{aligned}
\log_b(35) &= \log_b(5) + \log_b(7) \\
&= \log_b(70) - \log_b(2) \\
\log_2\left(\frac{16}{4}\right) &= \log_2(16) - \log_2(4) = 4 - 2 = 2 \\
\log_2(1000)^{\frac{1}{3}} &= \frac{1}{3}\log_2(1000) \\
\log_{10}(7)^5 &= 5\log_{10} 7 \\
\log_b(x)^{-1} &= -\log_b(x)
\end{aligned}$$

$$\begin{aligned}
\log_b x^2 y^{-3} &= \log_b x^2 + \log_b y^{-3} \\
&= 2\log_b x - 3\log_b y \\
\log_b \frac{x^2}{y^{-\frac{1}{2}}} &= \log_b x^2 - \log_b y^{-\frac{1}{2}} \\
&= 2\log_b x + \frac{1}{2}\log_b y
\end{aligned}$$

4 Problem-solving technique

Problem-solving technique: Treat both sides of an equation as though they were exponents.

$$x = y \qquad\qquad z^x = z^y$$

Example

$$\begin{aligned}
\log_2\left(\frac{39x}{(x-5)}\right) &= 4 \\
2^{\log_2\left(\frac{39x}{(x-5)}\right)} &= 2^4 \\
\frac{39x}{(x-5)} &= 16 \\
39x &= 16x - 80 \\
23x &= -80 \\
x &= -\frac{80}{23}
\end{aligned}$$

Fast Growth, Slow Growth: The Change of Base Formula

Video companion

1 Introduction

Generally use base 10, base 2, and natural log (base e) in data science.

$$\log_2(12) = 3.585 \qquad \log_2(7) = 2.807$$
$$\log_{10}(12) = 1.079 \qquad \log_{10}(7) = 0.8451$$

$$2^{3.585} = 12 \qquad 2^{2.807} = 7$$
$$10^{1.079} = 12 \qquad 10^{0.8451} = 7$$

The change of base formula: "Old" base is x, "new" base is a,

$$\log_a(b) = \frac{\log_x(b)}{\log_x(a)}$$

Examples
Want to convert $\log_{10}(12)$ to base $a = 2$:

$$\log_2(12) = \frac{\log_{10}(12)}{\log_{10}(2)} = \frac{1.079}{0.30103} = 3.585$$

Want to convert $\log_2(7)$ to base $a = 10$:

$$\log_{10}(7) = \frac{\log_2(7)}{\log_2(10)} = \frac{2.8073}{3.3219} = 0.8540$$

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Fast Growth, Slow Growth: The Rate of Growth of Continuous Processes

Video companion

1 Introduction

"Exponential rate of growth" can be a *discrete* exponential rate of growth or a *continuous* exponential rate of growth

Discrete rate of growth

$\$1.00(1+r)^t$

How much money would grow in discrete intervals of time t

If $r = 100\%/\text{year}$ and $t = 1$, then would have \$2.00 after one year,
After 2 years, would have \$4.00
After 3 years, would have \$8.00...

2 Continuous exponential growth

Euler's constant e

100% interest per year (discrete)
50% interest for 6 months, then interest on interest for another 6 months.

Interval	Factor	Repeats		Result
1 year	$1+1$	1	$(2)^1 =$	2
6 months	1.5	2	$(1.5)^2 =$	2.25
3 months	1.25	4	$(1.25)^4 =$	2.44

As time intervals decrease, does result increase in an unlimited way?

No...

Interval	Factor	Repeats		Result
1 month	1.08	12	$(1.08)^{12} =$	2.613
1 week	1.019	52	$(1.019)^{52} =$	2.693
1 day	1.002739	365	$(1.002739)^{365} =$	2.7146
1 hour	1.000114	8760	$(1.000114)^{8760} =$	2.71813
1 minute	1.0000019	525,600	$(1.0000019)^{525,600} =$	2.71828
1 second	1.0000000317	31,536,000	$(1.0000000317)^{31,536,000} =$	2.71828

$e = 2.71828$, Euler's constant

Problem A baby elephant weighing 200 kg grows at a continuously compounded rate of 5%/year. How much does it weigh in 3 years?

$$(200 \text{ kg})e^{(0.05)(3)} = 232.4 \text{ kg}$$

3 Continuous rate of return

"Log to the base e of x" is given by the symbol $\ln(x)$, where ln stands for *natural logarithm.*

Problem Rabbit population increases in mass at a rate of 200% per year. Population starts at 10 kg. If they increase at a continuously compounded rate, how many years is it until they weigh as much as the Earth (5.972×10^{24} kg)?

$$\begin{aligned} 5.972 \times 10^{24} \text{ kg} &= (10 \text{ kg})e^{2t} \\ 5.972 \times 10^{23} &= e^{2t} \\ \ln(5.972 \times 10^{23}) &= \ln(e^{2t}) = 2t \\ \frac{\ln(5.972 \times 10^{23})}{2} &= t = 27.37 \text{ years} \end{aligned}$$