Discrete Mathematics and Logic (UE14CS205)

Unit III - Counting

Mr. Channa Bankapur
channabankapur {@pes.edu, @gmail.com}

Channa Bankapur @ PES UNIVERSITY

- Counting is a basic human trait.
  - There is archeological evidence suggesting that humans have been counting for at least 50,000 years.
  - Learning to count is a child's first step into Mathematics.
- Peano Axioms (http://mathworld.wolfram.com/PeanosAxioms.html)
  - 0 is a natural number.
  - For every natural number n, successor S(n) is a natural number. It’s the basic form of Counting.

Essentially, Counting is at the root of Mathematics.

Combinatorics is the study of arrangement of objects.

Counting is the signification part of Combinatorics. It is the action of finding the number of elements of a finite set of objects.

In mathematics, the essence of counting a set and finding a result n, is that it establishes a one-to-one correspondence (aka bijection) of the set with the set of numbers {1, 2, ..., n}.

Applications of Counting:

There are uncountable (sorry, countable, but large) number of applications of counting. Here are the ones you need in the near future.

- Probability
- Graph Theory
- Analysis of algorithms.
- Algorithms which inherently need counting methods like in Analytics.
- …

The kind of problems dealt in this Unit:
Suppose that a password for an encrypted network packet is 8 to 15 characters long. The password is alphanumeric and case-sensitive. The password has at least one digit and one alphabet.
How many passwords a hacker needs to loop through to decode a network packet?

<Don’t try to solve this for now!>

Topics to be covered in Counting:

- Product Rule and the Sum Rule
- Pigeonhole Principle
- Permutation and Combination
- Binomial Theorem and Coefficients
- Permutations and Combinations with Repetition
- Arrangement of (Un)labelled objects in (un) labelled boxes
- Recurrence Relations - Solving “linear homogeneous recurrence relations with constant coefficients”

The Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are $\mathbf{n_1}$ ways to do the first task and for each of these ways of the first task, there are $\mathbf{n_2}$ ways to do the second task, then there are $\mathbf{n_1 * n_2}$ ways to do the procedure.

Eg: To order a pizza, you first choose the type of crust: thin or deep dish **(2 choices)**. Next, you choose the topping: cheese, pepperoni, or sausage **(3 choices)**. Therefore, there are **$2 \times 3 = 6$** possible ways of ordering a pizza.

Eg: Suppose, there are 7 trains from Mumbai to Ahmedabad and 3 trains from Ahmedabad to Kutch. In how many ways our Pappu can reach Kutch from Mumbai?

Ans: From Mumbai to Ahmedabad, Pappu has 7 choices (because there are 7 trains, he can pick anyone). Similarly from Ahmedabad to Kutch, he has 3 choices.
Total # of ways = **7** ways x **3** ways = **21** ways.
Pappu can reach Kutch in 21 different ways.

Eg: Suppose there are 10 contestants for a race. There will be a 1st and 2nd prize winners. How many ways the prizes can be awarded?

Ans: There are 10 ways to chose the 1st prize winner. For each of the 10 ways, there are 9 ways to choose the 2nd prize winner.

Therefore, $10 * 9 = 90$ ways.

Eg: Suppose there are five empty chairs. A couple (two people) is allowed to occupy any of the seats. How many unique ways are there for the them to sit when the ordering matters?

Ans: 5 * 4 = 20 ways

Generalization of the Product Rule

Suppose that a procedure is carried out by performing the tasks $\mathbf{T_1, T_2, \ldots, T_m}$ in sequence. If each task $\mathbf{T_i}$ can be done in $\mathbf{n_i}$ ways, regardless of how the previous tasks were done, then there are $\mathbf{n_1 * n_2 * \ldots * n_m}$ ways to do the procedure.

Eg: When ordering a certain type of computer, there are **3** choices of hard drive, **4** choices for the amount of memory, **2** choices of video card, and **3** choices of monitor. The number of ways a computer can be ordered is **3*4*2*3 = 72** ways.

Eg: Suppose there are 10 contestants for a race. There will be a 1st, 2nd and 3rd prize winners. How many ways the prizes can be awarded?

Ans: There are 10 ways to chose the 1st prize winner. For each of the 10 ways, there are 9 ways for the 2nd prize winner. For each of the 10 * 9 ways, there are 8 ways for the 3rd prize winner.

Therefore, 10 * 9 * 8 = 720 ways.

Eg: Suppose three people sit on three of the available five empty chairs. In how many unique ways they can sit when the order of their seating matters?

Ans: There are 5 ways to chose a chair for the first person. For each of the 5 ways, there are 4 ways for the second person to chose a chair. For each of the 5*4 ways, there are 3 ways for the third person to chose a chair.
Therefore, 5 * 4 * 3 = 60 ways.

Eg: Suppose that a password for encrypted network packet is eight characters long, alphanumeric and case-sensitive. How many passwords a hacker needs to loop through to decode the network packet?

Ans: $62*62*62*62*62*62*62*62 = 62^8$ different possibilities of the password.

What is the value of **`ctr`** printed?

```
ctr = 0
for i = 1 to 100
  for j = 1 to 500
    for k = 1 to 1000
      ctr = ctr + 1
print ctr
```

Output: 50,000,000
(= 100*500*1000)

What is the value of k printed?

```
k = 0
for i_1 = 1 to n_1
   for i_2 = 1 to n_2
      .
      .
      for i_m = 1 to n_m
         k = k + 1
print k
```

Output: $\mathbf{n_1 * n_2 * \ldots * n_m}$

Eg: Suppose a thief steals an ATM card and it takes a 4-digit secret pin code to draw money from the ATM. How many combinations the thief has to try at worst to steal?
Ans: $10 * 10 * 10 * 10 = 10^4$ different possibilities of the pin code.

Eg: Suppose the thief knows that the pin code has unique digits in it (no digit is repeated). Then?

Ans: $10 * 9 * 8 * 7 = 5040$

The Sum Rule

If a task can be done in one of $\mathbf{n_1}$ ways or in one of $\mathbf{n_2}$ ways, where none of the set of $\mathbf{n_1}$ ways is the same as any of the set of $\mathbf{n_2}$ ways, then there are $\mathbf{n_1 + n_2}$ ways to do the task.

Eg: CSE department offers seven elective courses and ISE offers five. How many different ways a student can pick if he/she is allowed to pick only one elective?

Ans: 7 + 5 = 12 choices

Generalization of the Sum Rule

If a task can be done in one of $\mathbf{n_1}$ ways or in one of $\mathbf{n_2}$ ways or … or in one of $\mathbf{n_m}$ ways, then there are $\mathbf{n_1+n_2+\ldots+n_m}$ ways to do the task.

Eg: CSE department offers **seven** elective courses, ISE offers **five** and ECE offers **eight**. How many different ways a student can pick if he/she is allowed to pick only one elective?

Ans: 7 + 5 + 8 = 20 choices

Eg: Suppose that a password for encrypted network packet is 3 to 5 characters long, alphanumeric and case-sensitive. How many passwords a hacker needs to loop through in the worst case to decode the network packet?

Ans: Number of passwords of length 3:
$62*62*62 = 62^3$ ---using the product rule

Number of passwords of length 3, 4 or 5:
$62^3 + 62^4 + 62^5$ ---using the sum rule

What is the value of k printed?

$$
\begin{aligned}
&\texttt{k=0} \\
&\texttt{for } i_1\texttt{=1 to } n_1 \\
&\quad \texttt{k = k+1} \\
&\texttt{for } i_2\texttt{=1 to } n_2 \\
&\quad \texttt{k = k+1} \\
&\qquad \vdots \\
&\texttt{for } i_m\texttt{=1 to } n_m \\
&\quad \texttt{k = k+1} \\
&\texttt{print k}
\end{aligned}
$$

Output: $\mathbf{n_1 + n_2 + \ldots + n_m}$

What is the value of ctr printed?

```
ctr = 0
for i = 1 to 100
   for j = 1 to 500
      for k = 1 to 1000
         ctr = ctr + 1
for i = 1 to 200
   for j = 1 to 600
      for k = 1 to 1000
         ctr = ctr + 1
print ctr
```

Output: 170,000,000
(= 100*500*1000 + 200*600*1000)

Q: In an earlier version of the computer programming language BASIC, the name of a variable is a string of one or two alphanumeric case-insensitive characters. Moreover, a variable name must begin with a letter and must be different from the five keywords of two character length. How many different variable names are there in this version of BASIC?

Ans: $26 + (26 * 36 - 5)$
$= 26 + 931 = 957$

Q: Suppose the password for a system is 6 to 8 characters long where each character is a lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Ans: $\mathbf{(36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)}$

For six char passwd, why $\mathbf{(36^6 - 26^6)}$ is correct,
but **not** $\mathbf{36^5 * 10 * 6}$?

Explanation: (Courtesy: **Mohit Surana**, PES-2013-17-CSE)

$36^6 - 26^6 = 1{,}867{,}866{,}560$

$36^5 * 10 * 6 = 3{,}627{,}970{,}560$

The latter counts everything the former counts, but some passwords are counted more than once. When a password has the same digit more than once, the password is counted that many times. For example, a7b77c is counted thrice.

Eg: An RTO in India has an unique code. For example, the code for Jayanagar-Bangalore is KA05. Suppose the licence number plates issued by the Jayanagar RTO are in the format KA05-XX-YYYY, where an ‘X’ represents an uppercase English letter and a ‘Y’ represents a digit. How many unique licence numbers the Jayanagar RTO can issue?

Ans: $26 * 26 * 10 * 10 * 10 * 10$

$= 26^2 * 10^4$

$= 6{,}760{,}000$

Q: How many different bit strings of length eight are there?

Ans: 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2

$= 2^8$

$= 256$

Q: How many bit strings of length eight start with a 1 bit **and** end with 00?

Ans: $1 * 2 * 2 * 2 * 2 * 2 * 1 * 1$

$= 2^5$

$= 32$

Q: How many bit strings of length eight either start with a 1 bit **or** end with 00?

Ans: …

The Inclusion-Exclusion Principle (aka Subtraction Principle)

- It's essentially subtracting some cases out of the Sum rule.
- In the Sum rule, we add the possibilities of two stages of the procedure because there is no overlap among the stages.
- In cases where there is a partial overlap between the stages of the procedure, we need to subtract the common cases after applying the Sum rule.
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Q: How many bit strings of length eight either start with a 1 bit or end with 00?

Ans: $2^7 + 2^6 - 2^5$
$= 128 + 64 - 32$
$= 160$

Alternate method:
Ans: $2^8 - (1 * 2^5 * 3)$
$= 256 - 96$
$= 160$

Q: How many functions are there from a set with **m** elements to a set with **n** elements?

Ans: n * n * … n (m times) = $\mathbf{n^m}$

Q: How many **one-to-one** functions are there from a set with **m** elements to a set with **n** elements?

Ans: **n * (n - 1) * (n - 2) * … * (n - m + 1),**
where m ≤ n

Q: How many **onto** functions …
Its solution is out of scope for now.

Q38: How many subsets of a set with 100 elements have more than one element?

Ans: 2^{100} - (1+100)

Q39: A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

Ans: $2^{ceil(n/2)}$

Q40: In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a. the bride must be in the picture?
b. both bride and groom must be in the picture?
c. exactly one of the bride and the groom is in the picture?

Ans: 10*9*8*7*6*5 = 151200

a. (10*9*8*7*6*5) - (9*8*7*6*5*4) = **90720** OR
   9*8*7*6*5*(6) = **90720**

b. 8*7*6*5*(5*6) = **50400**

c. 8*7*6*5*4*(6+6) = **80640**

Q41: In how many ways can a photographer at a wedding arrange 6 people in a row, including the bride and the groom, if

a. the bride must be next to the groom?
b. the bride is not next to the groom?
c. the bride is positioned somewhere to the left of the groom?

Ans: 6*5*4*3*2*1 = 720

a. 5*4*3*2*1*(2) = **240**

b. 720 - 240 = **480**

c. 720/2 = **360**

Q42: How many bit strings of length seven either begin with two 0s or end with three 1s?

Ans: $2^5 + 2^4 - 2^2$

Q43: How many bit strings of length ten either begin with three 0s or end with two 0s?

Ans: $2^7 + 2^8 - 2^5$

Suppose 11 pigeons flies into a set of 10 pigeonholes to roost. At least one of these 10 pigeonholes must have at least 2 pigeons in it.

In general, if k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Theorem: **The Pigeonhole Principle**
If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof: by contraposition.
If none of the k boxes contains more than one object, then total number of objects cannot be more than k.

Corollary: A function f from a set with $k+1$ or more elements to a set with k elements cannot be one-to-one.

Examples of the pigeonhole principle:

- Among a group of 367 people, there must be at least two with the same birthday.
- In a random collection of 27 letters, there must be at two similar letters.
- How many students must be in a class to guarantee that at least two students receive the same score in the test, if the test is graded on a scale from 0 to 40 marks?

Eg: Show that for every integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion.

Ans:
Consider $|n| + 1$ integers 1, 11, 111, …, 11...1

At least two of them has the same remainder when divided by n.

Difference of the two remainders is a multiple of n and has only 0s and 1s.

Eg: Suppose 21 pigeons flies into a set of 10 pigeonholes to roost. Then at least one of the 10 pigeonholes must have at least 3 pigeons in it.

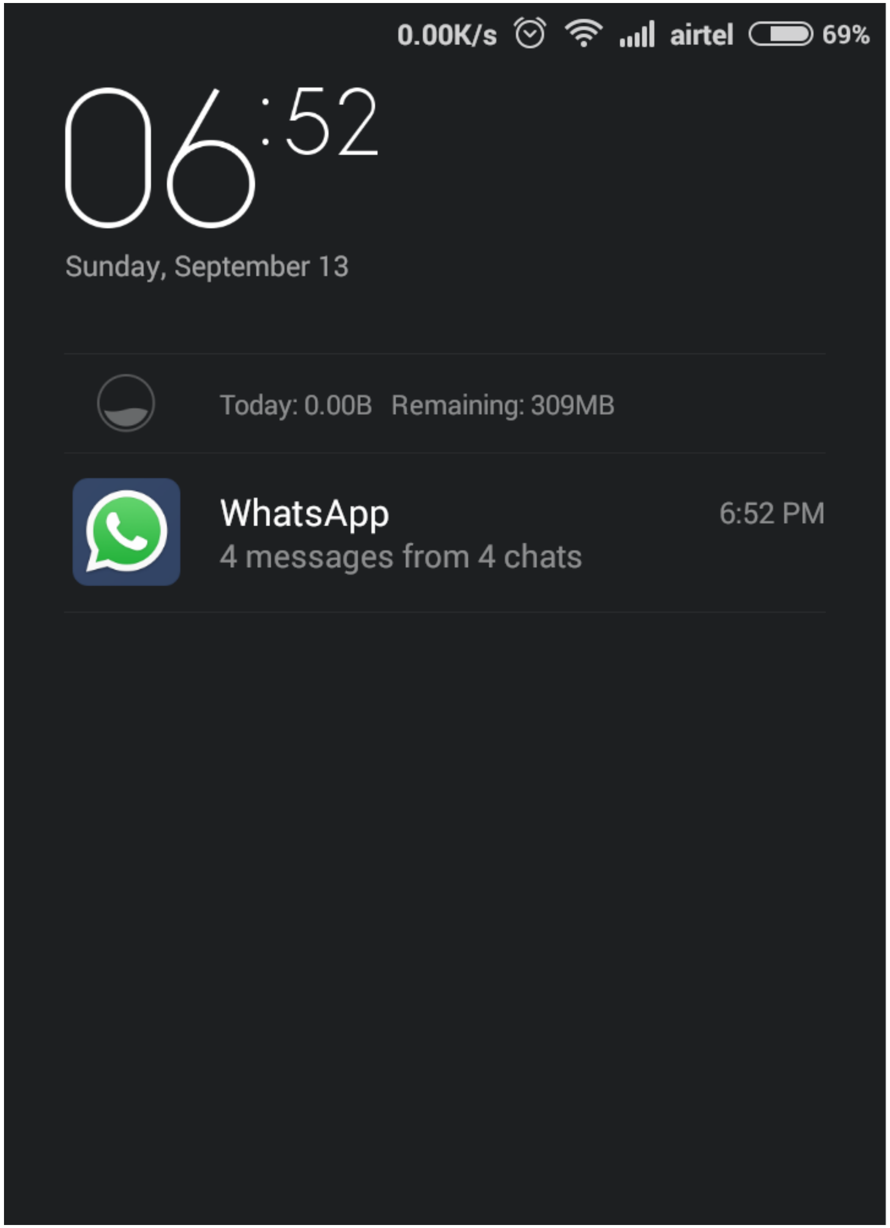
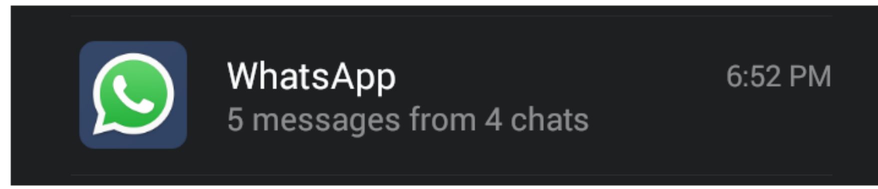
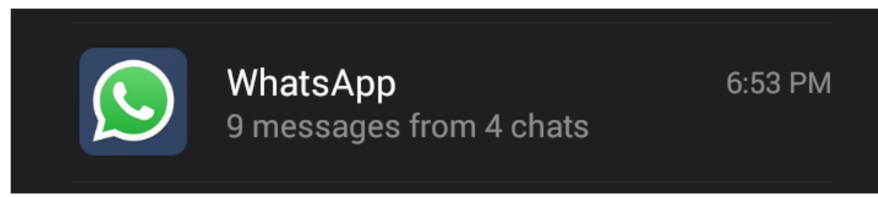
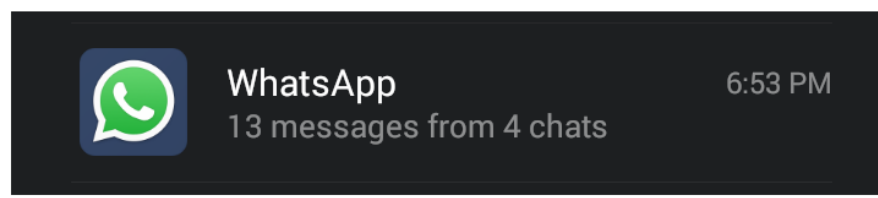
Eg: Suppose an instructor takes 11 lectures in a week of five days. Then at least on one of the days, the instructor takes three lectures in a day.

Eg: Suppose 21 pigeons flies into a set of 10 pigeonholes to roost. Then at least one of the 10 pigeonholes must have at least 3 pigeons in it.

The Generalized Pigeonhole Principle:

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Eg: Among 60 students in a class, there are at least **$\lceil 60/7 \rceil = 9$** who were born in the same day of the week.


0.00K/s
airtel
69%
06:52
Sunday, September 13
Today: 0.00B Remaining: 309MB
WhatsApp
6:52 PM
4 messages from 4 chats

WhatsApp
6:52 PM
5 messages from 4 chats

WhatsApp
6:53 PM
9 messages from 4 chats

WhatsApp
6:53 PM
13 messages from 4 chats

Q: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
Ans: min of x for which ceil(x/4) = 3
x=9

Q: How many must be selected to guarantee that at least three hearts are selected?
Ans: 39+3

Harder problems (pigeonhole principle):

Q: During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games (in the month). Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Eg:
1,1,2,1,3,2,1,1,1,4,5,1,1,1,1,1,2,1,1,1,1,1,2,3,1,1,1,2,1,1

a_j be the number of games played till j^{th} day.
a_1, a_2, …, a_{30}.

Eg: 1,2,4,5,8,10,11,12,13,17,22,23, …, 41,43,44,45
This is a strictly increasing sequence of distinct integers in the range [1,45].

a_1+14, a_2+14, …, a_{30}+14 is a strictly increasing sequence of distinct integers in the range [15,59]
Eg: 15,16,18,19,22, …, 55,57,58,59

$\mathbf{a_1, a_2, \ldots, a_{30}}$ is a strictly increasing sequence of distinct integers in the range [1,45].

$\mathbf{a_1+14, a_2+14, \ldots, a_{30}+14}$ is a strictly increasing sequence of distinct integers in the range [15,59].

Put together, 60 integers in the range [1,59]

By Pigeonhole principle, at least two of these integers are equal.

$\exists i \exists j \; a_i = a_j + 14$ and $i>j$

1,1,2,1,3,**2,1,1,1,4,5**, … (games played on each day)

1,2,4,5,8,10,11,12,13,17,**22**, … ($a_1, a_2, a_3, \ldots$)

15,16,18,19,**22**, … ($a_1+14, a_2+14, a_3+14, \ldots$)

By Pigeonhole principle, at least two of these integers are equal.

$\exists i \exists j\ a_i = a_j + 14$ and $i>j$

01,01,02,01,03,**02**,**01**,**01**,**01**,**04**,**05**,...
a_i: 01,02,04,05,08,10,11,12,13,17,**22**,...
a_i+14: 15,16,18,19,**22**,...

In the example, $a_{11} = a_5+14 = 22$. Until day 5, they've played 08 games and until day 11, they've played another 14 games (from day 5+1 to day 11).

That is, 14 games were played from day **j+1** to day **i**.

Q: During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games (in the month). Show that there must be a period of some number of consecutive days during which the team must play **exactly 14 games.**

Q: … exactly 13 games?
Yes, there'll be such a period for sure.

Q: … exactly 15 games?
No, we cannot be sure of that.

Q: Show that among any n+1 positive integers not exceeding 2*n, there must be an integer that divides one of the other integers.

Ans: $a_1, a_2, \ldots, a_{n+1}$
Write these in the form a_j = Power(2,k_j)* q_j where q_j is odd.

Eg: **n = 4**. Therefore, **2n = 8.**
Let **n+1** numbers in [1,2n]: **2,3,4,5,7**
We can write them as: $2^{\mathbf{1}}*1$, $2^{\mathbf{0}}*3$, $2^{\mathbf{2}}*1$, $2^{\mathbf{0}}*5$, $2^{\mathbf{0}}*7$

Ans: $a_1, a_2, \ldots, a_{n+1}$
Write these in the form $a_j = q_j * \text{Power}(2,k_j)$ where q_j is odd.
Eg: **n = 4**. Therefore, **2n = 8.**
Let **n+1** numbers in [1,2n]: **2,3,4,5,7**
We can write them as: $1*2^{\mathbf{1}}, 3*2^{\mathbf{0}}, 1*2^{\mathbf{2}}, 5*2^{\mathbf{0}}, 7*2^{\mathbf{0}}$

There are **n+1** q_j odd numbers.
There can be only **n** unique odd numbers in [1,2n].
∴ By pigeonhole principle, there are at least two q_j numbers (say q_a and q_b) which are equal ($q_a = q_b$).

The two numbers of the above n+1 numbers having $q_a = q_b$ are $\mathbf{Power(2,k_a)* q_a}$ and $\mathbf{Power(2,k_b) * q_b}$
Because $q_a = q_b$, $\mathbf{Power(2,k_a)*q_a}$ divides $\mathbf{Power(2,k_b)*q_b}$ if $k_a < k_b$, otherwise $\mathbf{Power(2,k_b)*q_b}$ divides $\mathbf{Power(2,k_a)*q_a}$.

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length n+1 that is either strictly increasing or strictly decreasing.

Eg: **for n=3;** 8,11,9,1,4,6,12,10,5,7
There is a subsequence of 4 (=3+1) numbers with strictly increasing or strictly decreasing.
11,9,6,5

Eg: 2,1,5,9,3,10,7,4,8,6
1,3,4,6

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length n+1 that is either strictly increasing or strictly decreasing.

Proof: $a_1, a_2, \ldots, a_{n*n+1}$

Ordered pairs (longest strictly increasing subsequence starting at a_k,
longest strictly decreasing subsequence starting at a_k)
$(i_1,d_1), (i_2,d_2), \ldots, (i_k,d_k), \ldots, (i_{n*n+1}, d_{n*n+1})$

$(i_1,d_1), (i_2,d_2), \ldots, (i_k,d_k), \ldots, (i_{n*n+1}, d_{n*n+1})$
Suppose i_k and d_k are in [1,n] for all k in [1, n^2+1]. We'll prove this is a contradiction and hence there exists a k for which i_k or d_k is at least n+1.

By product rule, there are n^2 possible values for the ordered pairs (i_k, d_k) where k is in [1, n^2+1].
By pigeonhole principle, there are at least two ordered pairs (i_p, d_p) and (i_q, d_q), which are identical ($p<q$ and $i_p= i_q$ and $d_p= d_q$). Let's call it (i,d) because they are same.
So what?

That means, starting at a_p , the longest subsequences are (i,d) and starting from a_q, the longest subsequences are (i,d).

We can prove that this is impossible and hence contradicting the assumption.

Case 1: $a_p < a_q$
Because there is an increasing sequence from a_q of length i and a_p appears before a_q , we can build an increasing sequence of length (i + 1) starting from a_p . That's a contradiction.

Case 2: $a_p > a_q$
Similar to the above case, we can build a decreasing sequence of length (d + 1) starting from a_p . That's a contradiction.

Therefore, there exists a k for which i_k or d_k is at least n+1. Hence the proof.

Q: Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Let A be one of the six people.
Using pigeonhole principle, there are at least 3 friends or 3 enemies of A.

Case 1: A has at least 3 friends.
If at least 2 of them are friends of each other, then including A, we've a friends group of at least 3.
If there are all enemies of each other, then excluding A we've enemies group of at least 3.

Case 2: A has at least 3 enemies.
If at least 2 of them are enemies of each other, then including A, we've a enemies group of at least 3.
If there are all friends of each other, then excluding A we've friends group of at least 3.

Ramsey theory

The Ramsey number **R(m, n)**, where **m** and **n** are positive integers greater than or equal to **2**, denotes the minimum number of people at a party such that there are either **m** mutual friends or **n** mutual enemies, assuming every pair of people at the party are friends or enemies.

$R(3, 3) = 6$

$R(m, n) = R(n, m)$

$R(2, n) = n$

$R(4, 4) = 18$

$43 \leq R(5, 5) \leq 49$

Example 9 *Suppose that a computer science laboratory has* 15 *workstations and* 10 *servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of* 10 *or fewer workstations can simultaneously access different servers via direct connections. Although we could do this by connecting every workstation directly to every server (using* 150 *connections), what is the minimum number of direct connections needed to achieve this goal?*

Solution Suppose that we label the workstations $W_1, W_2, \ldots, W_{15}$ and the servers $S_1, S_2, \ldots, S_{10}$. Furthermore, suppose that we connect W_k to S_k for $k = 1, 2, \ldots, 10$ and each of $W_{11}, W_{12}, W_{13}, W_{14}$, and W_{15} to all 10 servers. We have a total of 60 direct connections. Clearly any set of 10 or fewer workstations can simultaneously access different servers. We see this by noting that if workstation W_j is included with $1 \le j \le 10$, it can access server S_j, and for each workstation W_k with $k \ge 11$ included, there must be a corresponding workstation W_j with $1 \le j \le 10$ not included, so W_k can access server S_j. (This follows because there are at least as many available servers S_j as there are workstations W_j with $1 \le j \le 10$ not included.)

Now suppose there are fewer than 60 direct connections between workstations and servers. Then some server would be connected to at most $\lfloor 59/10 \rfloor = 5$ workstations. (If all servers were connected to at least six workstations, there would be at least $6 \cdot 10 = 60$ direct connections.) This means that the remaining nine servers are not enough to allow the other 10 workstations to simultaneously access different servers. Consequently, at least 60 direct connections are needed. It follows that 60 is the answer. ◀

Q3: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

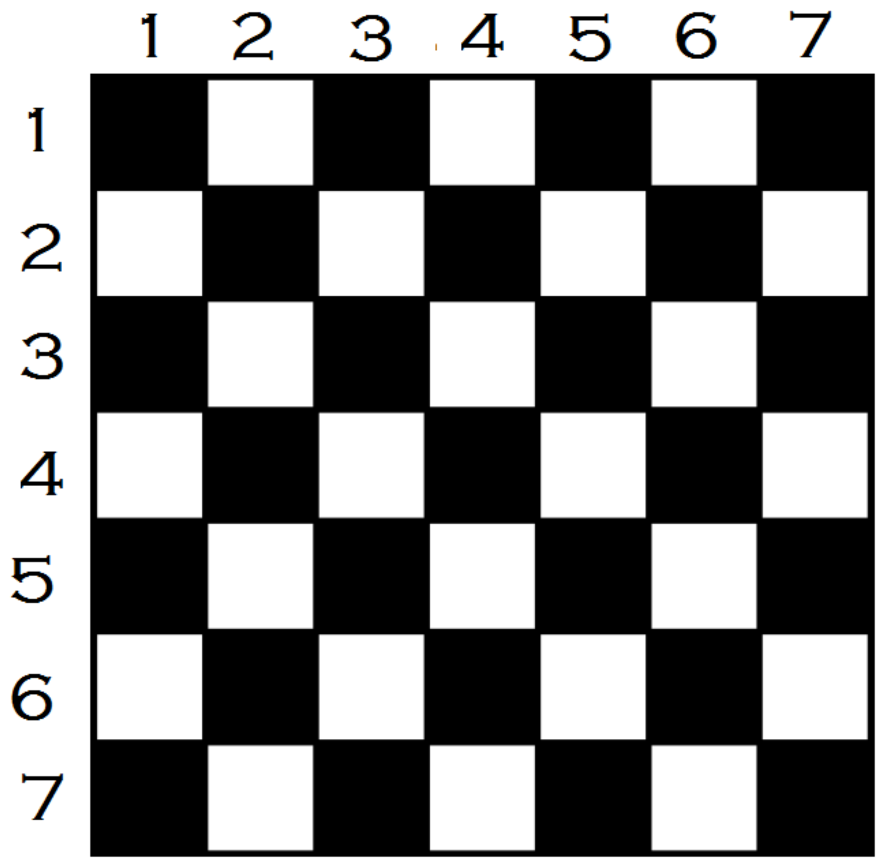
a. How many socks must he take out to be sure that he has at least two socks of the same color?
b. How many socks must he take out to be sure that he has at least five socks of the same color?
c. How many socks must he take out to be sure that he has at least two black socks?

Q6: Let d be a positive integer. Show that among any group of d+1 (not necessarily consecutive) integers there are two with exactly same remainder when they are divided by d.

Q: A closet has 3 red, 7 blue and 10 black shirts. What is the minimum number of shirts you've to blindfoldedly pick to ensure at least 4 of the same color?

Q: at least 5 of the same color?

Q: How many ways are there to select an ordered pair of numbers from 1 to 7 so that the sum is even?



Eg: Suppose a thief steals an ATM card and it takes a four digit secret pin code to draw money from the ATM. How many combinations the thief has to try at worst to steal money?

Ans: 10*10*10*10 = 10^4 different possibilities of the pin code.

Eg: Suppose the thief knows that the pin code has unique digits in it (no digit is repeated). Then?

Ans: 10*9*8*7

Déjà vu!

Eg: Suppose the thief knows that the pin code has unique digits in it (no digit is repeated). Then?
Ans: 10*9*8*7

Eg: Suppose a code is 30-character long and it takes case-sensitive alphanumeric characters. And, the thief knows that the pin code has unique characters in it. Then?
Ans: 62*61*60* … *34*33.
= 62! / 32!

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

r-permutation of a set of distinct objects is an ordered arrangement of some of the elements (say, r) of the set.

P(n,r) represents the number of r-permutations of a set with n elements.

Theorem: **r-permutations**
If n is a positive integer and r is an integer with $0 \leq r \leq n$, then there are $P(n,r) = n(n-1)...(n-r+1)$

Proof: Prove using the product rule

Corollary: If n and r are integers with $0 \leq r \leq n$, then $P(n,r) = n! / (n-r)!$

Q: Suppose a classroom bench can accommodate four students. In how many ways a class of 80 students can sit 4 at a time on the bench.

$= P(80, 4)$
$= 80 * 79 * 78 * 77$
$= 80! / 76!$

Q: How many permutations of the letter ABCDEFGHIJ contain the string ABC?

Ans: 8!

Considering ABC as one object, there are 8 objects to permutate.

Imagine listing out all the 10! permutations of the sequence and then searching for the substring "ABC". The search will have 8! results.

An **r-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Q: How many 2-combinations of {a,b,c,d} are there?

Ans: {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}

Q: How many 4-combinations of {a,b,c,d,e,f,g,h,i, j} are there?

Q. How many 3-permutations of A,B,C,D,E are there?

Q: How many 3-combinations of A,B,C,D,E are there?

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Q: How many 4-combinations of {a,b,c,d,e,f,g,h,i, j} are there?
Ans: 10! / ((10-4)! * 4!)

Theorem: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with 0 <= r <= n, equals

$$C(n,r) = \frac{n!}{(n-r)!\, r!}$$

In other words, P(n,r) = C(n,r) * P(r,r)

Q: How many poker hands of five cards can be shown from a standard deck of 52 cards?

Ans: $52! / (5! * 47!)$

Q: From a group of 10 students, in how many ways a committee of 4 students can selected?

Ans: 10! / (6! * 4!)

Q: From a group of 10 students, in how many ways a committee of 6 students can selected?
Ans: 10! / (4! * 6!)

Corollary: Let n and r be nonnegative integers with r<=n. Then C(n,r) = C(n, n-r)

Q: How many bit strings of length 8 contain exactly three 1s?
Ans: 8! / (3! * 5!)

Note: The above result is same as the number of bit strings of length 8 containing exactly five 0s: 8! / (5! * 3!)

Q: How many bit strings of length **n** contain exactly **r** 1s?
Ans: n! / ((n-r)! * r!)

r-permutation of objects from a set of n objects

Ordered r-selection

$\mathbf{n^{\underline{r}}}$ or $\mathbf{{}_rP_n}$

P(n,r) = n*(n-1)* … *(n-r+1) = n! / (n-r)!

r-combination of objects from of a set of n obts

Unordered r-selection

$\mathbf{{}_rC_n}$

C(n,r) = P(n,r)/r! = n! / ((n-r)! * r!)

C(n,r) = C(n, n-r)

1. List all the permutations of $\{a, b, c\}$.
2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?
3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
4. Let $S = \{1, 2, 3, 4, 5\}$.
   a) List all the 3-permutations of S.
   b) List all the 3-combinations of S.
5. Find the value of each of these quantities.
   a) $P(6, 3)$ b) $P(6, 5)$ c) $P(8, 1)$
   d) $P(8, 5)$ e) $P(8, 8)$ f) $P(10, 9)$
6. Find the value of each of these quantities.
   a) $C(5, 1)$ b) $C(5, 3)$ c) $C(8, 4)$
   d) $C(8, 8)$ e) $C(8, 0)$ f) $C(12, 6)$
7. Find the number of 5-permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

11. How many bit strings of length 10 contain

a) exactly four 1s? **b)** at most four 1s?

c) at least four 1s?

d) an equal number of 0s and 1s?

12. How many bit strings of length 12 contain

a) exactly three 1s? **b)** at most three 1s?

c) at least three 1s?

d) an equal number of 0s and 1s?

13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

14. In how many ways can a set of two positive integers less than 100 be chosen?

15. In how many ways can a set of five letters be selected from the English alphabet?

16. How many subsets with an odd number of elements does a set with 10 elements have?

17. How many subsets with more than two elements does a set with 100 elements have?

18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?

b) contain exactly three heads?

c) contain at least three heads?

d) contain the same number of heads and tails?

19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?

b) contain exactly two heads?

c) contain at most three tails?

d) contain the same number of heads and tails?

20. How many bit strings of length 10 have

a) exactly three 0s?

b) more 0s than 1s?

c) at least seven 1s?

d) at least three 1s?

21. How many permutations of the letters *ABCDEFG* contain

a) the string *BCD*?

b) the string *CFGA*?

c) the strings *BA* and *GF*?

d) the strings *ABC* and *DE*?

e) the strings *ABC* and *CDE*?

f) the strings *CBA* and *BED*?

22. How many permutations of the letters *ABCDEFGH* contain

a) the string *ED*?

b) the string *CDE*?

c) the strings *BA* and *FGH*?

d) the strings *AB*, *DE*, and *GH*?

e) the strings *CAB* and *BED*?

f) the strings *BCA* and *ABF*?

23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]

25. One hundred tickets, numbered 1, 2, 3, . . ., 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a) there are no restrictions?

b) the person holding ticket 47 wins the grand prize?

c) the person holding ticket 47 wins one of the prizes?

d) the person holding ticket 47 does not win a prize?

e) the people holding tickets 19 and 47 both win prizes?

f) the people holding tickets 19, 47, and 73 all win prizes?

g) the people holding tickets 19, 47, 73, and 97 all win prizes?

h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?

i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?

j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

Pascal’s Triangle:

```
1
1   1
1   2   1
1   3   3   1
1   4   6   4   1
1   5   10  10  5   1
1   6   15  20  15  6   1
…
```

```
                  1
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5   10  10  5   1
      1   6   15  20  15  6   1
      …
```

Some of you might have written a computer program to print Pascal’s triangle upto n rows in the above formats.

Binomial Expansion:

$(x+1)^0 = 1$

$(x+1)^1 = 1 + x$

$(x+1)^2 = 1 + 2x + x^2$

$(x+1)^3 = 1 + 3x + 3x^2 + x^3$

$(x+1)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

1 5 10 10 5 1

1 6 15 20 15 6 1

The Binomial Theorem gives the coefficients of the expansion of powers of binomial expressions.

Binomial Theorem: In the expansion of $(x+y)^n$, the coefficient of $x^{n-r}y^r$ equals $\binom{n}{r}$.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Proof:

…

Examples:

$(x+y)^3 = (x+y)(x+y)(x+y)$

$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

$= C(4,0)\, x^4 +$

$C(4,1)\, x^3y +$

$C(4,2)\, x^2y^2 +$

$C(4,3)\, xy^3 +$

$C(4,4)\, y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Theorem: In the expansion of $(x+y)^n$, the coefficient of $x^{n-r}y^r$ equals $\binom{n}{r}$.

Proof: We can prove it by a combinatorial method.

The terms in the product when it is expanded are of the form $x^{n-r}y^r$ for r=0,1,2, …, n.

To count the number of terms of the form $x^{n-r}y^r$, note that to obtain such a term it is necessary to choose **n-r x**s from the **n** sums and the other **r** terms in the product are obviously **y**s.

Therefore, the coefficient of $x^{n-r}y^r$ is $\binom{n}{n-r}$, which is equal to $\binom{n}{r}$. Hence the proof.

Q: What is the expansion of $(x+y)^5$?

Ans: …

Q: What is the coefficient of x^3y^2 in the expansion of $(x+y)^5$?

Ans: C(5,3)

Q: What is the coefficient of $x^{13}y^{14}$ in the expansion of $(x+y)^{27}$?

Ans: C(27, 13)

Q: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Ans: $C(25, 12) * 2^{12} * (-3)^{13}$

Q: … $x^{13}y^{12}$ …
Ans: $C(25, 12) * 2^{13} * (3)^{12}$

Q: Let n be a nonnegative integer. Then

$$\sum_{r=0}^{n} \binom{n}{r} = ?$$

Hint:

$(x+y)^3 = (x+y)(x+y)(x+y)$

$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

= number of n-bit strings having **0** 1s

+ number of n-bit strings having **1** 1s

+ number of n-bit strings having **2** 1s

…

+ number of n-bit strings having **n** 1s

= all possible n-bit strings counted exactly once

= $\mathbf{2^n}$

We could also explain using subsets of a set adding up to power set.

Corollary: Let n be a nonnegative integer. Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

Proof:

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r}$$

Q: For a positive integer n, prove that

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$$

Ans:

$$0 = 0^n = ((-1)+1)^{\mathbf{n}}$$

$$= \sum_{r=0}^{n} \binom{n}{r}(-1)^r 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r}(-1)^r$$

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots = 0$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

1

1 - 1 = 0

1 - 2 +1 = 0

1 - 3 + 3 - 1 = 0

1 - 4 + 6 - 4 + 1 = 0

1 - 5 + 10 -10 + 5 - 1 = 0

1 - 6 + 15 - 20 + 15 - 6 + 1 = 0

…

Corollary: For a positive integer n, prove that

$$\sum_{r=0}^{n} \binom{n}{r} 2^r = 3^n$$

Proof:

$$3^n = (1+2)^n = \ldots$$

Pascal's Identity:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

…

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

…

Theorem: Let n and k be positive integers with n >= k. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: Let n and k be positive integers with n >= k. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: (Combinatorial method)
Suppose that T is a set containing n+1 elements.
Let **‘a’** be an element in T.

There are $\binom{n+1}{k}$ subsets of T containing k elements.

A subset of T either contains **a** with k-1 elements of the rest of the elements in T, **or** doesn’t contain **a** and contains k elements of the rest of the elements in T.

$\binom{n+1}{k}$ subsets of T containing k elements

=

Subsets containing **a** with k-1 elements of the rest of the elements in T

+

Subsets not containing **a** and containing k elements of the rest of the elements in T.

Subsets containing **a** with k-1 elements of the rest of the elements in T $= \binom{n}{k-1}$

Subsets not containing **a** and containing k elements of the rest of the elements in T $= \binom{n}{k}$

$$\therefore \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

PASCAL'S RECURSION

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof:

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} \\
&= \frac{(n-1)!r}{(n-r)!r!} + \frac{(n-1)!(n-r)}{(n-r)!r!} \\
&= \frac{(n-1)![r+(n-r)]}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!r!} \\
&= \frac{n!}{(n-r)!r!} = \binom{n}{r}
\end{aligned}$$

```
1
1   1
1   2   1
1   3   3   1
1   4   6   4   1
1   5   10  10  5   1
1   6   15  20  15  6   1
…
```

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
…
```

Pascal's Triangle generation:

(Optional) Write a computer program to print Pascal's triangle upto n rows using Pascal's identity.

Initial conditions: $\binom{n}{0} = \binom{n}{n} = 1$

You won't need multiplications, only additions are good enough.

Vandermonde Identity
Let m, n, and r be nonnegative integers such that $r \leq m$ and $r \leq n$. Then
$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$
Proof: Partition a set S of size $m+n$ into subsets T and U of sizes m and n. To choose r objects from S, one may choose k objects from T and the remaining $r-k$ objects from U.

Corollary: For a nonnegative integer n, prove that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$$

Proof:

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r} = \sum_{r=0}^{n} \binom{n}{r}^2$$

Corollary: For nonnegative integers n and r and r<= n, prove that

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof: …

Binomial Coefficients: are the coefficients of the terms $\mathbf{x^r}$ in the expansion of the binomial $\mathbf{(x+1)^n}$

NOTATION: for the coefficient of x^r: $\binom{n}{r}$

Thus,

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n} \binom{n}{r} x^r \\ &= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n\end{aligned}$$

Binomial Theorem: In the expansion of $(x+1)^n$,

the coefficient of $\mathbf{x}^r$, denoted by $\binom{n}{r}$, equals C(n, r).

Proof: $(x+1)^n = (x+1)_1(x+1)_2 \ldots (x+1)_n$

Number of x^r terms is same as the number of ways to select r factors from n factors. Adding all of them results in $C(n,r) * x^r$.

$\therefore$ the coefficient of x^r equals C(n,r)

r-permutation of objects from a set of n objects

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$
$$P(n, r) = \frac{n!}{(n-r)!}$$

r-combination of objects from of a set of n obts

$$\binom{n}{r} = C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!\, r!}$$

Permutations with repetitions:
Eg: There are 26^4 four-letter alphabetic strings in the English alphabet: AAAA, AAAB, ..., ZZZZ.

Theorem:
The number of r-permutations of a set of n objects with repetition allowed is $\mathbf{n^r}$.

Proof: Using the product rule.

Combinations with Repetitions:

Eg: How many ways are there to select 2 coins from a cash box containing 1, 2, 5 and 10 rupees coins?

- Order of selection doesn’t matter
- Coins of a denomination are indistinguishable.
- Practically, there are infinite coins in the box.

Q: How many ways are there to select 2 coins from a cash box containing 1, 2, 5 and 10 rupees coins?

Soln:

1 1,	1 2,	1 5,	1 10,
2 2,	2 5,	2 10,	
5 5,	5 10,		
10 10			

Eg: How many ways are there to select 4 coins from a cash box containing 25p, 50p, Re 1, Rs 2, Rs 5 and Rs 10 coins?

6 types of coins makes 5 separators and 4 coins to chose. With five bars (separators), 4 stars (coins) could be placed in all possible ways. It can be mapped to number bit strings having 5 1s and 4 0s in a bit string of length 9 (= 5 + 4).

$C(6 - 1 + 4, 4) = C(9, 4)$
$= 9*8*7*6 / (2*3*4)$
$= 126$ ways

Combinations with Repetitions:
Eg: There are 3 piles of Apples, Blackberries and Androids. All the phones in a pile are identical. How many ways are there to select 5 of them for your team?

Ans: 3-1 bars and 5 stars…
$C(3-1+5, 5)$

Combinations with Repetitions:
Theorem: There are **C(n-1+r, r)** = C(n-1+r, n-1) **r**-combinations from a set with **n** elements when repetition of elements is allowed.

Proof: Prove using **n-1** bars making space for **n** possible choices and **r** stars representing the possible positions of **r** objects.

PERMUTATIONS

Order Matters
Repetition Allowed

$$Possibilities = n^r$$

Order Matters
Repetition Not Allowed

$$Possibilities = \frac{n!}{(n-r)!}$$

COMBINATIONS

Order Doesn't Matter
Repetition Not Allowed

$$Possibilities = \frac{n!}{r!\,(n-r)!}$$

Order Doesn't Matter
Repetition Allowed

$$Possibilities = \frac{(n+r-1)!}{r!\,(n-1)!}$$

Eg: How many solutions does the equation $x_1+x_2+x_3+x_4 = 2$ have, where the variables are nonnegative integers?

It's same as choosing 2 coins from piles of 4 denominations of coins.

Using the formula for 2-combinations with repetition allowed from a set of 4 elements,
$C(n-1+r, r) = C(4-1+2, 2) = C(5,2)$
$= 10$ is the number of solutions the equation has.

Eg: How many solutions does the equation $x_1+x_2+x_3 = 11$ have, where the variables are nonnegative integers?

To find the number of solutions, we can use r-combinations with repetition allowed from a set with n elements, $C(n-1+r, r)$ where $n=3$ and $r=11$.

$= C(3-1+11, 11) = C(13,11) = 13*12/2$
$= 78$ is the number of solutions the equation has.

Eg: How many solutions does the equation $x_1+x_2+x_3 = 11$ have, where the variables are nonnegative integers and $\mathbf{x_1 \geq 1, x_2 \geq 2, x_3 \geq 3}$?

Minimum of 1, 2 and 3, in total 6 allocated to the variables. For the remaining 5, we have a case of 5-combinations from a set of 3 elements when repetition is allowed.
C(n-1+r, r) where n=3 and r=5.

= **C(3-1+5, 5)** = C(7, 5) = 7*6/2
= **21** is the number of solutions the equation has.

Permutations with Indistinguishable Objects:

Eg: How many different strings can be made by reordering the letters of the word PESU?

= 4!

Eg: How many different strings can be made by reordering the letters of the word LOL?

< 3! (LLO, LLO, LOL, LOL, OLL, OLL)

= 3 (LLO, LOL, OLL)

Eg: How many different strings can be made by reordering the letters of the word LOLL?

< 4! (LLLO, LLLO, , LLLO, LLLO, LLLO, , LLLO)

= 4!/3! = 4 (LLLO, LOLL, LLOL, LLOL)

Permutations with Indistinguishable Objects:

Eg: How many different strings can be made by reordering the letters of the word LOLL?

< 4! (LLLO, LLLO, , LLLO, LLLO, LLLO, , LLLO)

= 4!/3! = 4 (LLLO, LOLL, LLOL, LLOL)

Eg: How many different strings can be made by reordering the letters of the word LOLOL?

= 5!/(3!*2!) = 10

Eg: How many different strings can be made by reordering the letters of the word SUCCESS?

< 7!

3 **S**s can be placed among 7 positions: C(7,3)
2 **C**s can be placed among 4 positions: C(4,2)
1 **U** can be placed among 2 positions: C(2,1)
1 **E** can be placed among 1 position: C(1,1)

$$= C(7,3) * C(4,2) * C(2,1) * C(1,1)$$
$$= 7! / (3!*4!) * 4!/ (2!*2!) *2!/(1!*1!) *1!/(1!*0!)$$
$$= 7! / (3!*2!*1*1!)$$
$$= 7! / (3! * 2!)$$

Permutations with Indistinguishable Objects:

Theorem *Let* $S = \{a_1, a_2, \ldots, a_k\}$, *and let* $n = n_1 + n_2 + \cdots + n_k$. *The number of length-*n *sequences in* S *with* n_j *occurrences of object* a_j, *for* $j = 1, \ldots, k$ *is*

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

Proof: Use the explanation of the example.

n_1 $\mathbf{a_1}$s can be placed among $\mathbf{n}$ positions in C($\mathbf{n,n_1}$) ways

n_2 $\mathbf{a_2}$s can be placed among ($\mathbf{n-n_1}$) positions in C($\mathbf{n-n_1,n_2}$) ways

By the product rule, the total number of different permutations is

$C(\mathbf{n,n_1})C(\mathbf{n-n_1,n_2})C(\mathbf{n-n_1-n_2,n_3}) \ldots C(\mathbf{n-n_1-n_2- \ldots -n_{k-1}, n_k})$

$$= \frac{n!}{n_1!(n-n_1)!} * \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} * \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)} * \ldots * \frac{(n-n_1-n_2- \ldots -n_{k-1})}{n_k!0!}$$

$$= \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

Distributing Objects into Boxes:

Objects can be distinguishable / indistinguishable (aka labeled / unlabeled)

Boxes can be distinguishable / indistinguishable (aka labeled / unlabeled)

Distributing **n** distinguishable / indistinguishable objects into **r** distinguishable / indistinguishable boxes.

Distinguishable Objects and Distinguishable Boxes:

Eg: How many ways are there to distribute hands of 5 cards to each of the 4 players from the standard deck of 52 cards?

Cards are the Objects.
Players are the Boxes.

$= C(52,5) * C(47,5) * C(42,5) * C(37,5)$

$= (52!/(47!*5!)) * (47!/(42!*5!)) * (42!/(37!*5!)) * (37!/(32!*5!))$

$= 52! / (5! * 5! * 5! * 5! * 32!)$

Distinguishable Objects & Distinguishable Boxes:

Number of ways to distribute **n** distinguishable objects into **k** distinguishable boxes so that $\mathbf{n_i}$ objects are placed into box **i**, **i** = 1,2, …,k, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Indistinguishable Objects & Distinguishable Boxes:

Eg: How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

This is same as choosing 10 coins from 8 denominations of coins where there are at least 10 of each denominations are available.

It’s 10-combinations with repetition allowed from a set of 8 elements.

Indistinguishable Objects & Distinguishable Boxes:

Eg: How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

= C(8+10-1, 10) = 17! / (10! * 7!)

Therefore,

There are C(n-1+r, r) = C(n-1+r, n-1) ways to place **r** indistinguishable objects into **n** distinguishable boxes.

Distinguishable Objects & Indistinguishable Boxes:
Eg: How many ways are there to distribute 80 different students into 25 indistinguishable team?

Eg: How many ways are there to put 4 different students into 3 indistinguishable team (not more than 3 teams)?

4 students **in 1 team:** **{**{A,B,C,D}**}**
in 2 teams: **{**{A,B,C},{D}**}**, **{**{A,B,D},C**}**,
{{A,C,D},B**}**, **{**{B,C,D},A**}**,
{{A,B},{C,D}**}**, **{**{A,C},{B,D}**}**, **{**{A,D},{B,C}**}**
in 3 teams: **{**{A,B},{C},{D}**}**, **{**{A,C},{B},{D}**}**,
{{A,D},{B},{C}**}**, **{**{B,C},{A},{D}**}**,
{{B,D},{A},{C}**}**, **{**{C,D},{A},{B}**}**

1+7+6 = 14 ways

Distinguishable Objects & Indistinguishable Boxes:

n distinguishable objects into **r** indistinguishable boxes:

$$\sum_{j=1}^{r} \mathbf{S(n, j)}, \text{ where } \mathbf{S(n, j) = (1/j!)} \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j-i)^n$$

S(n, j): **Stirling numbers of the second order.** The number of ways of partitioning a set of **n** elements into **j** nonempty sets.

Indistinguishable Objects & Indistinguishable Boxes:

Eg: How many ways are there to pack 6 copies of the same book into 4 identical boxes, where a box can contain at least six books?

6

5,1

4,2

4,1,1

3,3

3,2,1

3,1,1,1

2,2,2

2,2,1,1

= 9 ways

n Indistinguishable Objects &
r Indistinguishable Boxes:

n as the sum of at most **r** positive integers in non-decreasing order.

$a_1+a_2+ \dots +a_j = n$
$0 < a_1 <= a_2 <= \dots <= a_j$, and $j <= r$
$a_1,a_2, \dots , a_j$ is a partition of **n** into **j** positive ints.

There are $\mathbf{P_r(n)}$ ways to distribute **n** indistinguishable objects into **r** indistinguishable boxes, where $\mathbf{P_r(n)}$ is the number of partitions of **n** into at most **r** positive integers. No simple closed formula exists for this number.

Eg: There are 3 piles of Apples, Blackberries and Androids. All the phones in a pile are identical. How many ways are there to select 5 of them for your team?

Soln: $C(3-1+5, 5)$

Eg: How many ways are there to distribute 80 different students into not more than 12 indistinguishable team?

Soln:

$$\sum_{j=1}^{12} \mathbf{S(80, j)}, \text{ where } \mathbf{S(n, j) = (1/j!)\sum_{i=0}^{j} (-1)^i \binom{j}{i} (j-i)^n}$$

Eg: In how many different ways can five elements be selected in order from a set with four elements when repetition is allowed?

Soln: $\mathbf{4^5}$

Eg: Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are **3** kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the **5** working days of a week if the **order** in which the sandwiches are chosen **matters**?

Soln: $\mathbf{3^5}$

Eg: Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are **3** kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the **5** working days of a week if the **order** in which the sandwiches are chosen **does not matters**?

Soln: C(**3-1+5, 5**) = C(7, 5) = 21

Eg: How many ways are there to assign **3** jobs to **5** employees if each employee can be given **at most one job**?

Soln: …

Eg: How many ways are there to assign **3** jobs to **5** employees if each employee can be given **more than one job**?

Soln: …

Eg: How many different combinations of **1, 2, 5 and 10 rupee coins** can a piggy bank contain if it has **10 coins** in it?

Soln: …

Eg: A mobile store has **3000 identical** Android phones. How many ways are there to store these phones in their **3 storerooms**?

Soln: …

Recurrence Relations

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n>=n_0$, where n_0 is a nonnegative integer.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Q: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_5?

Ans:

$a_2 = 5 - 3 = 2$

$a_3 = 2 - 5 = -3$

$a_4 = -3 - 2 = -5$

$a_5 = -5 - (-3) = -2$

Q: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Ans:

(a) True because $2(3(n-1))-3(n-2) = 3n$

(b) False because $2(2^{n-1})-2^{n-2} = 2^n$ is false.

(c) True because $2(5) - 5 = 5$.

Fibonacci numbers:

$f_n = f_{n-1} + f_{n-2}$, where $f_1 = 1$ and $f_2 = 1$.

Eg: Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length eight.

Ans: $a_n = a_{n-1} + a_{n-2}$, where $a_1 = 2$ and $a_2 = 3$.

2, 3, 5, 8, 13, 21, 34, 55.

55 bit strings of length 8 don't have 2 consecutive 0s.

The Tower of Hanoi puzzle:

Move n disks from peg A to peg B using peg C.

Move top n-1 disk from peg A to C.

Move disk# n from peg A to B.

Move n-1 disks from peg C to B

# of moves, $H_n=2H_{n-1}+1$ with the initial condition $H_1=1$.

Power(x, y) = **x * Power(x, y-1)**, where $x \in R$, $y \in Z^+$ with the initial condition Power(x, 0) = 1.

Factorial(n) = **n * Factorial(n-1)**,
where $n \in Z^+$ and Factorial(0) = 1.

Solving Recurrence Relations

Finding solution to a recurrence relation means finding "**closed form expressions**".

Often in computer science, it means to find a way of evaluating an expression in constant time instead evaluating recursively in the order of a non-constant function of 'n'.

Q: Solve $a_n = a_{n-1} + C$, where a_0 is a constant
Solution: $\mathbf{a_n = C*n + a_0}$

Q: Solve $a_n = Ca_{n-1}$, where a_0 is a constant
Solution: $\mathbf{a_n = C^n * a_0}$

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, c_3, \ldots, c_k$ are reals and $c_k \neq 0$.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

Homogeneous when all terms are multiples of a_j.

- $\mathbf{a_n = 2a_{n-1} + 3}$ is not homogeneous (not all terms are multiples of a_j)
- $\mathbf{a_n = na_{n-1}}$ coefficient not constant
- $\mathbf{a_n = 3a^2_{n-1}}$ is not linear
- $\mathbf{a_n = a_{n-1} + a_{n-2}}$ is of degree 2
- $\mathbf{a_n = 3a_{n-5}}$ is of degree 5

Linear Homogenous Recurrence Relation with constant coefficients occur often in modeling problems and can be systematically solved.

Theorem 1 - For degree 2 and distinct roots r_1 and r_2

$\{a_n\}$ is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, ...,$

where $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2,

with constants α_1 and α_2.

$\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$

where $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2,

with constants α_1 and α_2

LHRRwCC: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Characteristic equation: $r^2 - c_1 r - c_2 = 0$

Characteristic roots: r_1 and r_2 (distinct roots)

Solution: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$

Constants: α_1 and α_1 are constants, which can be

found using initial conditions a_0 and a_1.

Q: What is the solution (closed form) of
$a_n = a_{n-1} + 2a_{n-2}$ where $a_0 = 2$, $a_1 = 7$

Soln: $c_1 = 1$, $c_2 = 2$

$\mathbf{r^2 - c_1 r - c_2 = 0}$ is the characteristic equation

$r^2 - r - 2 = (r+1)(r-2) = 0$
$\mathbf{r_1 = 2, r_2 = -1}$ are the characteristic roots

Using initial conditions and roots, solve
$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for α_1 and α_2

$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$
for constants α_1 and α_2

$a_0 = 2$

$= \alpha_1 r_1{}^0 + \alpha_2 r_2{}^0$

$= \alpha_1 2^0 + \alpha_2(-1)^0$

$= \alpha_1 + \alpha_2$
$\boldsymbol{\alpha_1 + \alpha_2 = 2}$

$a_1 = 7$

$= \alpha_1 r_1{}^1 + \alpha_2 r_2{}^1$

$= \alpha_1 2^1 + \alpha_2(-1)^1$

$= 2\alpha_1 - \alpha_2$
$\boldsymbol{2\alpha_1 - \alpha_2 = 7}$

Solving simultaneous equations

$\alpha_1 + \alpha_2 = 2$
$2\alpha_1 - \alpha_2 = 7$

$\alpha_1 = 3$, $\alpha_2 = -1$

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$

$\mathbf{a_n = 3*2^n + (-1)^{n+1}}$

is the solution (closed form) for
$a_n = a_{n-1} + 2a_{n-2}$ where $a_0 = 2$, $a_1 = 7$

Q: Find the solution (closed form) of $a_n = 7a_{n-1} - 10a_{n-2}$ where $a_0 = 2$, $a_1 = 1$

Soln: ...

Q: Find the solution (closed form) of $f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$, $f_1 = 1$

Soln: ...

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: …

(Hint: it's of degree 1, and it's of a known form)

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: This is a known form.

The solution of $a_n = Ca_{n-1}$ would be $\mathbf{a_0 * C^n}$

Therefore, solution of $a_n = 2a_{n-1}$ is $\mathbf{3 * 2^n}$

Alternate Soln: By substitution.

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: By substitution

$a_n = 2a_{n-1}$ where $a_0 = 3$

$a_n = 2a_{n-1}$

$= 2(2a_{n-2}) = 2^2 a_{n-2}$

$= 2^2(2a_{n-3}) = 2^3 a_{n-3}$

$= 2^i a_{n-i}$

When n-i=0, i=n

$= 2^n a_0$

$= 3*2^n$

Soln: By the theorem 1 with degree 2.

The recurrence relation and initial conditions

$a_n = 2a_{n-1}$

$\quad = 2a_{n-1} + 0a_{n-2}$ for $n \geq 1$

It's of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ as in the theorem for degree 2.

$a_0 = 3$

$c_1 = 2,\ c_2 = 0$

Determine if there are two distinct roots, r_1 and r_2.

$r^2 - c_1 r - c_2 = 0$

$r^2 - 2r - 0 = (r-2)(r+0) = 0$

$r_1 = 2$

$r_2 = 0$

Using initial conditions and roots, solve

$a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$ for constants α_1 and α_2

$a_n = \alpha_1 2^n + \alpha_2 0^n$

$\alpha_1 + \alpha_2 = 3$

$2\alpha_1 = 6$

$\boldsymbol{\alpha_1} = 3$, $\boldsymbol{\alpha_2} = 0$

$a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$

$= 3*2^n + (0)0^n$

$= 3*2^n$ is the solution for $a_n = 2a_{n-1}$ where $a_0 = 3$

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$, $a_1 = 6$

Soln: $c_1 = 6$, $c_2 = -9$

$r^2 - 6r + 9 = (r-3)(r-3) = 0$

$r_1 = 3$, $r_2 = 3$

but the roots are not distinct…

Theorem 2 - For degree 2 and 1 root (multiplicity 2)

$\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

iff $a_n = \alpha_1 r_0^n + \alpha_2 \mathbf{n} r_0^n$ for n=0,1,2,...,

where $r^2 - c_1 r - c_2 = 0$ has one root r_0,

with constants α_1 and α_2.

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$, $a_1 = 6$

Soln: $c_1 = 6$, $c_2 = -9$

$r^2 - 6r + 9 = (r-3)(r-3) = 0$

$r_0 = 3$

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n = \alpha_1 3^n + \alpha_2 n3^n$

for constants α_1 and α_2

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$

$= \alpha_1 3^n + \alpha_2 n3^n$

$= 3^n + n3^n$

Theorem 3 - For degree k and distinct roots

$\{a_n\}$ is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + .. + c_ka_{n-k}$

iff $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n} + .. + \alpha_k r_k^{\ n}$ for n=0, 1, 2, ...,

where $r^k - c_1r^{k-1} - .. - c_k = 0$ has k distinct roots

$r_{1,}\ r_{2,}$ .. r_k, with constants $\alpha_1, \alpha_2, .. , \alpha_k$

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ where $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

Soln: …

Theorem 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Q: What is the solution (closed form) of

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ where $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Soln: …

<Thank You />

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Channa Bankapur @ PES UNIVERSITY