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Einstein's Field Equations: derivations and solutions

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Abstract

In 1916 Einstein published his General Relativity paper, the completion of the previous 1905 article on Special Relativity. General Relativity is not only a beautiful theory of gravity, but it has also opened a large branch of Mathematics.

In this thesis, the so called Einstein's Field Equations (EFE) are treated. In chapter 1 the mathematical setting is developed: the standard Euclidean space is no longer sufficient, but one needs to work on manifolds and tensors defined on them.

In chapter 2, after a short introduction on Special Relativity, the Einstein's Field Equations are derived in two different ways: generalizing the standard Newton's gravitational law, and using the principle of least action applied to the Einstein-Hilbert action.

Chapter 3 is devoted to solutions of the EFE. First of all we try to model the expansion of our universe, assuming homogeneity and isotropy, with three models of increasing complexity. Then we analyse the solution outside a static and spherically symmetric body, known as Schwarzschild metric.

Finally, in the last chapter, we will see that, if the metric satisfy some constraint equations, the EFE not only admits a solution, but it is also unique.

Sommario

Nel 1916 Einstein pubblicò il suo articolo sulla Relatività Generale che rappresenta la naturale conclusione del suo precedente lavoro, pubblicato nel 1905, sulla Relatività Speciale. La Relatività Generale non è solo una straordinaria teoria sulla gravità, ma ha anche aperto la strada a nuove branche della Matematica.

In questa tesi verranno studiate le equazioni di campo di Einstein, il cuore di tutta la teoria. Nel capitolo 1 verranno sviluppati gli strumenti matematici necessari: il tradizionale spazio euclideo non è più sufficiente, ma vi è la necessità di lavorare su varietà e con tensori definiti su di esse.

Nel capitolo 2, dopo una breve introduzione alla Relatività Speciale, si procede con la derivazione delle equazioni di campo, con due approcci differenti: generalizzando la legge di gravitazione di Newton e applicando il principio di minima azione all'azione di Einstein-Hilbert.

Il capitolo 3 è dedicato alle soluzioni delle equazioni. Verrà analizzato il problema dell'espansione del nostro universo, assumendo omogeneità e isotropia, mediante tre modelli di complessità crescente. Successivamente verrà presentata la soluzione all'esterno di un corpo statico e a simmetria sferica; tale soluzione è nota come metrica di Schwarzschild.

Infine, nell'ultimo capitolo, si mostrerà che, se la metrica soddisfa ulteriori vincoli, le equazioni di Einstein non solo ammettono soluzione, ma essa è anche unica.

Chapter 1

Riemannian Geometry

In this first chapter we will develop all the mathematical tools needed in order to properly derive and state the Einstein Field Equations (EFE). In particular we recall some basic definitions and results from Riemannian geometry such as the notion of manifold, connections and curvature tensors. As we will discuss, in General Relativity we will work with *Lorentzian* manifolds; they differ from Riemannian manifolds by the *signature* of the metric tensor defined on them. However this difference is relevant especially with those theorems regarding compactness of the space, but we are not going into these details since they are not used in this work. Finally we will introduce the variations of some quantities, useful in the following chapters [CM20].

1.1 Manifolds

The notion of manifold is necessary to extend the methods of differential calculus to spaces more general than \mathbb{R}^n .

Definition 1.1. A manifold of dimension n is a set \mathcal{M} and a family of injective mappings $\varphi_j : O_j \subset \mathbb{R}^n \rightarrow \mathcal{M}$ of open sets O_j of \mathbb{R}^n , $j \in J$, such that:

i) $\bigcup_j O_j = \mathcal{M}$;

ii) For any pair (i, j) , if $A_{ij} = O_i \cap O_j \neq \emptyset$, the sets $U_i = \varphi_i^{-1}(A_{ij})$ and $U_j = \varphi_j^{-1}(A_{ij})$ are open sets in \mathbb{R}^n , and we can define a transition map $F_{ij} : U_i \rightarrow U_j$, $F_{ij} = \varphi_j^{-1} \circ \varphi_i$.

Each map φ_j is called *chart*, by mathematicians, or *coordinate system*, by physicists. The pairs $\{U_j, \varphi_j\}_j$ are called *atlas* and \mathcal{M} is said to be *differentiable* if all the F_{ij} are differentiable.

The manifold structure, as we have defined, induces a topology on \mathcal{M} ; indeed we have the following proposition [Aba11].

Proposition 1.1. An atlas $\mathcal{A} = \{(U_j, \varphi_j)\}$ of dimension n on a set \mathcal{M} induces a topology on \mathcal{M} setting $A \subseteq \mathcal{M}$ open if and only if $\varphi_j(A \cap U_j)$ is an open set of \mathbb{R}^n for every chart $(U_j, \varphi_j) \in \mathcal{A}$. Moreover this is the only topology on \mathcal{M} such that every U_j are open sets and every φ_j are homeomorphism.

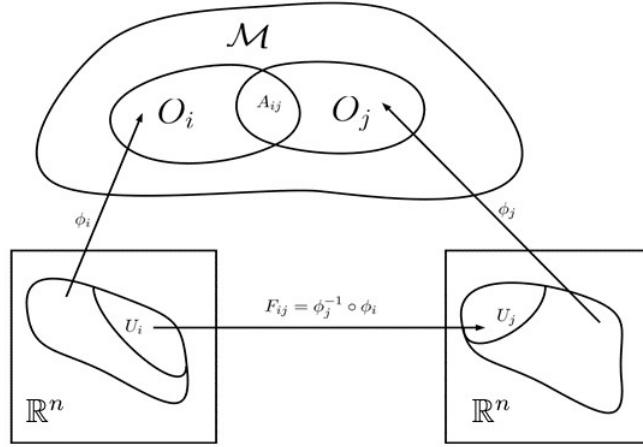


Figure 1.1: An illustration of the map F_{ij} arising when two systems of coordinates overlap.

With the structure on \mathcal{M} given by the coordinate systems, we may now define the notion of differentiability and smoothness of maps between manifolds.

Definition 1.2. Let \mathcal{M} and \mathcal{N} be two manifolds. A map $F : \mathcal{M} \rightarrow \mathcal{N}$ is said to be differentiable (or C^∞) at a point $p \in \mathcal{M}$ if there exists a chart (U, φ) at p and a chart (V, ψ) at $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is C^∞ in a neighborhood of $\varphi(p)$.

If F is differentiable at every point of \mathcal{M} we say that it is differentiable. If $F : \mathcal{M} \rightarrow \mathcal{N}$ is C^∞ , one-to-one, onto and has C^∞ inverse then it is called *diffeomorphism*.

1.2 Vectors

Having defined the concept of (differentiable) maps between manifolds, we want now to extend the notion of tangent vector to differentiable manifolds. In pre-relativistic physics it is assumed that the space has the structure of a three dimensional vector space like \mathbb{R}^3 , once one has identified a point to serve as the origin.

In Special Relativity spacetime has still the structure of vector space, but with four dimensions. However, as soon as one considers curved geometries, and in General Relativity it is mandatory, this vector space structure is lost. The simplest example to realize this concept may be the one of a three dimensional sphere: there is no notion of “adding” two points on a sphere and end up with a third point on the sphere. We can recover the vector space structure if we consider the limits of infinitesimal displacement. For manifolds like the sphere it is natural to think about a tangent vector at p as a vector lying in the tangent plane (fig. 1.2).

This notion is peculiar to all the manifolds that arise naturally as surfaces embedded in \mathbb{R}^3 . However, when this embedding is not possible, we need a definition of tangent vector that refers only to the intrinsic structure of the manifold.

Such a definition is provided by the notion of *derivation*.

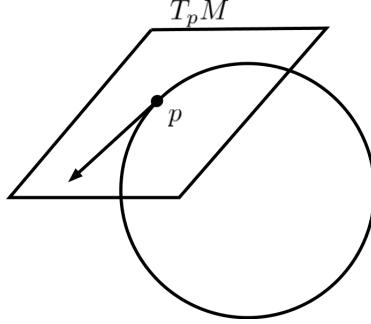


Figure 1.2: Tangent plane and vector to a sphere.

Definition 1.3. Let $\mathcal{F} = \{f : \mathcal{M} \rightarrow \mathbb{R}, f \in C^\infty(\mathcal{M})\}$. A **derivation** at $p \in \mathcal{M}$ is a map $X : \mathcal{F} \rightarrow \mathbb{R}$ such that:

1. $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g) \quad \forall \alpha, \beta \in \mathbb{R}, \forall f, g \in \mathcal{F};$
2. $X(fg)(p) = f(p)X(g) + g(p)X(f).$

The collection of all the derivation at p has the structure of vector space and is called **tangent space** of \mathcal{M} at p . We will denote it by $T_p M$. A crucial property of $T_p M$ is given by the following theorem (for a proof see e.g. [Aba11]).

Theorem 1.1. Let \mathcal{M} be an n -dimensional manifold, $p \in \mathcal{M}$ and $T_p M$ be the tangent space at p . Then, for a given chart φ ,

1. $\dim T_p M = n;$
2. $X_i : \mathcal{F} \rightarrow \mathbb{R}$, $X_i(f) := \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})|_{\varphi(p)}$, $i = 1 \dots n$, are a basis of $T_p M$. Sometimes we refer to it with the notation ∂_i or $\frac{\partial}{\partial x^i}$.

Remark 1.1. One can obviously choose another chart ψ in the previous theorem and obtain a new basis $\{Y_j\}_{j=1}^n$. We can express X_i in terms of the new basis Y_j using the chain rule

$$X_i = \sum_{j=1}^n \frac{\partial x^j}{\partial x^i} \Big|_{\psi(p)} Y_j. \quad (1.1)$$

Now, having the notion of tangent vector we can define a vector field on \mathcal{M} .

Definition 1.4. A **vector field** \mathcal{V} on a differentiable manifold \mathcal{M} is a correspondence that associates to each point $p \in \mathcal{M}$ a vector $v(p) \in T_p M$.

Considering a parametrization $\mathbf{x} : U \subset \mathbb{R}^n \rightarrow \mathcal{M}$, we can write

$$v(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i}, \quad (1.2)$$

where each $f_i : U \rightarrow \mathbb{R}$ is a function on U and $\left\{ \frac{\partial}{\partial x_i} \right\}$ is the basis associated to \mathbf{x} .

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Definition 1.5. The dual (cotangent) space $(T_p M)^*$ is the set of linear functions (or dual vectors) $w : T_p M \rightarrow \mathbb{R}$.

$(T_p M)^*$ is itself a vector space but we can say more. We can indeed consider $v \in T_p M$ as a function acting on w , rather than vice versa, defining $v(\alpha w_1 + \beta w_2) = \alpha v(w_1) + \beta v(w_2)$. It follows that v is a linear function

$$v : (T_p M)^* \rightarrow \mathbb{R}. \quad (1.3)$$

The tangent space form the dual space of its own dual space or $T_p M \simeq (T_p M)^{**}$. This identification is crucial wherever we consider the *cobasis* $\{dx^j\}$ such that $dx^j(\partial_i) = \partial_i(dx^j) = \delta_i^j$.

By analogy with eq. (1.1) we find the dual transformation law: if $\{x^j\}$ and $\{x^k\}$ are two coordinate systems and $w = \sum_j w_j dx^j$, then

$$w_k = \sum_j w_j \frac{\partial x^j}{\partial x^k}. \quad (1.4)$$

1.3 Tensors

So far we have defined (differentiable) manifolds and vector fields on them. Now we wish to operate upon our constructions. In particular we want to extend the notion of distance and length of vectors. This will be possible with tensors.

Definition 1.6. Let \mathcal{M} be a smooth manifold and $p \in \mathcal{M}$. An (n, m) tensor T on \mathcal{M} at p is a multilinear function

$$T : \underbrace{(T_p M)^* \times \cdots \times (T_p M)^*}_n \times \underbrace{T_p M \times \cdots \times T_p M}_m \rightarrow \mathbb{R}.$$

It is clear that vectors are $(1, 0)$ tensors, scalars are $(0, 0)$ tensors and the set of all (n, m) tensors \mathcal{T}_m^n is a vector space.

With the same procedure we used to derive the vector transformation law we can find the *tensor transformation law*

$$T_{j_1, \dots, j_m}^{i_1, \dots, i_n} = \frac{\partial x^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{k_n}}{\partial x^{i_n}} \frac{\partial x^{h_1}}{\partial x^{j_1}} \cdots \frac{\partial x^{h_m}}{\partial x^{j_m}} T_{h_1, \dots, h_m}^{k_1, \dots, k_n} \quad (1.5)$$

Definition 1.7. The **contraction** with respect to the i^{th} and j^{th} indices, $1 \leq i \leq n$, $1 \leq j \leq m$, is the linear map

$$C_j^i : \mathcal{T}_m^n \rightarrow \mathcal{T}_{m-1}^{n-1}$$

defined as

$$\begin{aligned} C_j^i(\partial_1 \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes dx^m) = \\ dx^j(\partial_i) \partial_1 \otimes \cdots \otimes \hat{\partial}_i \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes \hat{dx}^j \otimes \cdots \otimes dx^m, \end{aligned} \quad (1.6)$$

where $\{\partial_k\}_{k=1}^n$ and $\{dx^k\}_{k=1}^m$ are basis of $T_p M$ and $(T_p M)^*$ respectively, and \otimes denotes the tensor product. The hatted basis vectors are those to be removed in the tensor product.

Remark 1.2. The order of the entries of C_j^i is not relevant.

By linearity, we can extend this definition to a general element $T \in \mathcal{T}_m^n$:

$$\begin{aligned} C_j^i(T) &= T_{1,\dots,m}^{1,\dots,n} dx^j(\partial_i) \partial_1 \otimes \cdots \otimes \hat{\partial}_i \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes dx^j \otimes \cdots \otimes dx^m \\ &= T_{1,\dots,j,\dots,m}^{1,\dots,i,\dots,n} \delta_j^i \partial_1 \otimes \cdots \otimes \hat{\partial}_i \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes dx^j \otimes \cdots \otimes dx^m \\ &= T_{1,\dots,j-1,k,j+1,\dots,m}^{1,\dots,i-1,k,i+1,\dots,n} \partial_1 \otimes \cdots \otimes \partial_{i-1} \otimes \partial_{i+1} \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes dx^{j-1} \\ &\quad \otimes dx^{j+1} \otimes \cdots \otimes dx^m \\ &= \tilde{T}_{1,\dots,j-1,j+1,\dots,m}^{1,\dots,i-1,i+1,\dots,n} \partial_1 \otimes \cdots \otimes \partial_{i-1} \otimes \partial_{i+1} \otimes \cdots \otimes \partial_n \otimes dx^1 \otimes \cdots \otimes dx^{j-1} \\ &\quad \otimes dx^{j+1} \otimes \cdots \otimes dx^m. \end{aligned}$$

For example if T is a $(1, 1)$ tensor, the contraction

$$T_i^i = \text{tr}(T)$$

is the *trace* of T .

Now we are ready to define the most important tensor in the theory of General Relativity: the metric tensor.

Definition 1.8. A **metric** g on a manifold is a smooth, $(0, 2)$, symmetric and non degenerate tensor field. That is:

1. $g(v_1, v_2) = g(v_2, v_1) \quad \forall v_1, v_2 \in T_p M$;
2. $g(v_1, v) = 0 \quad \forall v \iff v_1 = 0$.

The metric can be interpreted as a measure of the infinitesimal squared distance between two points. We emphasize that two vectors v_1 and v_2 still must belong to the same tangent space. In fact, despite the case of \mathbb{R}^n , we have not the notion of *parallel transport* in a curved manifold. In order to give a meaning to the distance between $v \in T_p M$ and $w \in T_q M$ we need more theory, which will be developed in the next sections.

Example 1.1. Let $\mathcal{M} = \mathbb{R}^4$ and the metric given by

$$\eta_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

η is called *spacetime* metric and (\mathbb{R}^4, η) is the *Minkowski's space*. This is the frame in which Special Relativity is developed.

Once we have endowed a manifold with a metric, we can associate an inner product $\langle v_1, v_2 \rangle$ on the tangent space. More precisely: if g is a metric on \mathcal{M} and $v_1, v_2 \in T_p M$ then

$$\langle v_1, v_2 \rangle_g := g_{ij} v_1^i v_2^j. \tag{1.7}$$

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Given the metric g , and since we are in a finite dimensional space, it is always possible to find an orthonormal basis $\{v_i\}_{i=1}^n$ of the tangent space at each point p , i.e.

$$g(v_i, v_j) = \begin{cases} \pm 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \quad (1.8)$$

Since we are not requiring g to be positive definite in general, there will be k basis vectors such that $g(v_i, v_i) = +1$ and h basis vectors such that $g(v_i, v_i) = -1$, with $k + h = n$. The pair (k, h) is called **signature** of the metric. A positive definite metric has signature $(n, 0)$ and is called *Riemannian metric*. Metrics with $h = 1$ are called *Lorentzian metrics*. Those metrics are the ones we are interested in when we speak about spacetime metrics.

We saw that g allows us to define an inner product which in turn establishes a natural isomorphism between vectors and dual vectors. We associate a unique ¹ dual vector w to a vector v if for any $u \in T_p M$

$$w(u) = \langle v, u \rangle_g$$

or in coordinates

$$w_i u^i = w(u) = g_{ij} v^i u^j. \quad (1.9)$$

By the arbitrariness of u the components of w are given by $g_{ij} v^i$. This process is called *lowering the index of v* . The inverse process pass through the definition of the inverse of the metric g^{ij} and $v^i = g^{ij} w_j$; this is *raising the index of w* .

1.4 Curvature

Our intuitive notion of curvature arises mainly from two-dimensional surfaces which are embedded in an ordinary three dimensional Euclidean space. We usually think of a surface as curved because of the way it bends in \mathbb{R}^3 . As we will see better in chapter 2, our interest is to investigate curvature of spacetime, represented by a manifold \mathcal{M} endowed with a metric g (we will shortly denote it by (\mathcal{M}, g) from now on) which is not in general embedded in a higher dimensional space.

Such a notion of curvature can be defined in terms of parallel transport. However, as already pointed out, we do not have a natural notion of parallel transport on a generic manifold. The reason is that the tangent spaces $T_p M$ and $T_q M$ are *different* vector spaces and there is no way of saying that a tangent vector at p is the same as a tangent vector at q .

In order to generalize the concept of parallel transport we need firstly to generalize the notion of derivative. Let $\mathcal{X}(\mathcal{M})$ the set of smooth vector fields on \mathcal{M} , $f \in C^\infty(\mathcal{M})$, $X, Y, X_1, X_2, Y_1, Y_2 \in \mathcal{X}(\mathcal{M})$.

Definition 1.9. A *linear connection* (or *covariant derivative*) on a differentiable manifold is a map $\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M})$

$$\nabla : (X, Y) \mapsto \nabla_X Y$$

satisfying the following properties

¹Riesz representation theorem.

1. $\nabla_{X_1+X_2}Y = \nabla_{X_1}Y + \nabla_{X_2}Y;$
2. $\nabla_X(Y_1 + Y_2) = \nabla_XY_1 + \nabla_XY_2;$
3. $\nabla_{fX} = f\nabla_XY;$
4. $\nabla_X(fY) = f\nabla_XY + X(f)Y$

Remark 1.3. When the vector field X is the coordinate basis ∂_i we will use the compact notation

$$\nabla_i := \nabla_{\partial_i}.$$

Given a chart (U, φ) on a manifold with bases $\{\partial_i\}$ for T_pM and $\{dx^i\}$ for $(T_pM)^*$ the action of the covariant derivative on a vector $v \in T_pM$ is uniquely determined [Aba11; Tol] by the so called *Christoffel symbols* Γ_{ij}^k :

$$\nabla_i v^k = \partial_i v^k + \Gamma_{ij}^k v^j. \quad (1.10)$$

Example 1.2. The Christoffel symbol of $(\mathbb{R}^n, \text{I}_n)$ are all identically zero. If a connection has vanishing Christoffel symbols is said to be *flat*.

Remark 1.4. Christoffel symbols are not tensors. Indeed let (x^1, \dots, x^n) and $(\tilde{x}^1, \dots, \tilde{x}^n)$ be two systems of coordinate, then we have the following transformation law:

$$\tilde{\Gamma}_{kl}^i = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^k} \frac{\partial x^p}{\partial \tilde{x}^l} \Gamma_{np}^m + \frac{\partial^2 x^m}{\partial \tilde{x}^k \partial \tilde{x}^l} \frac{\partial \tilde{x}^i}{\partial x^m}.$$

Thus we see that Christoffel symbols depend on the choice of coordinate system which, in turn, make the definition of the connection coordinate-dependent too.

The connection allows us to generalize the concept of parallel transport of a vector along a curve γ with tangent t^i .

Definition 1.10. A vector v^i is said to be *parallelly transported* as one moves along the curve γ if the following equation is satisfied

$$t^i \nabla_i v^j = 0 \quad (1.11)$$

Choosing a coordinate system and using eq. (1.10) we can express eq. (1.11) as

$$t^i \partial_i v^k + t^i \Gamma_{ij}^k v^j = 0 \quad (1.12)$$

or, in terms of components in the coordinate basis and a parameter λ along the curve

$$\frac{dv^k}{d\lambda} + t^i \Gamma_{ij}^k v^j = 0. \quad (1.13)$$

This is a set of ordinary differential equations which admit a unique solution provided initial values of v^k are given [PS10]. Once again we notice that everything depends on the choice of the curve and on the coordinate system. Hence we may wonder if we can establish some uniqueness results on the connection ∇ , and if there is a generalization of the notion of “shortest path” between two points $p, q \in \mathcal{M}$. The former is given by the following theorem [Wal84; Aba11]

1 Riemannian Geometry

Theorem 1.2. Let (\mathcal{M}, g) a Riemannian manifold. There exists a unique connection ∇ , the Levi-Civita derivative, such that

$$\nabla_k g_{ij} = 0 \quad (1.14)$$

and is torsion-free.

As a result we have an expression for the Christoffel symbols in terms of the metric coefficients

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{ik} - \partial_k g_{ij}) \quad (1.15)$$

The *torsion-free* condition can be formulated as follows: let X, Y be smooth vector fields on \mathcal{M} , we require the Levi-Civita connection to satisfy

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad (1.16)$$

where $[,]$ are the Lie brackets on \mathcal{M} . Since the torsion of a generic connection

$$\text{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad (1.17)$$

is a $(0, 2)$ tensor, it follows that the difference of two connection is a $(0, 2)$ tensor. Using the Liebniz rule we can show that for any smooth scalar field f , dual field w and basis vector fields ∂_i and ∂_j :

$$[\nabla_i, \nabla_j](fw) = f(\nabla_i \nabla_j - \nabla_j \nabla_i)(w).$$

Since $[\nabla_i, \nabla_j]w$ depends only on w , it is a linear map between dual vectors to $(0, 3)$ tensors. Hence

$$[\nabla_i, \nabla_j] : (T_p M)^* \times T_p M \times T_p M \times T_p M \rightarrow \mathbb{R}$$

is a $(1, 3)$ tensor.

Definition 1.11. The **Riemann tensor** on \mathcal{M} is the $(1, 3)$ tensor R_{ijk}^l such that for any dual vector w

$$R_{ijk}^l w_l = \nabla_i \nabla_j w_k - \nabla_j \nabla_i w_k. \quad (1.18)$$

The Riemann tensor is obviously antisymmetric in the first two lower indexes. If the Riemann tensor is identically zero at every point of a manifold, then \mathcal{M} is said to be *flat*.

Substituting eq. (1.10) in R_{ijk}^l we get an explicit equation for the Riemann tensor:

$$R_{ijk}^l = \partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l + \Gamma_{ik}^h \Gamma_{hk}^l - \Gamma_{jk}^h \Gamma_{hi}^l. \quad (1.19)$$

Some useful properties of the Riemann tensor are the following ((0, 4) form used here):

1. Skew symmetry

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

2. Interchange symmetry

$$R_{ijkl} = R_{klji}$$

3. First Bianchi identity

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad (1.20)$$

4. Second Bianchi identity

$$\nabla_h R_{ijkl} + \nabla_k R_{ijlh} + \nabla_l R_{ijhk} = 0 \quad (1.21)$$

with ∇ the Levi-Civita connection.

We now introduce a few new fields closely associated with the Riemann tensor.

Definition 1.12. *The **Ricci tensor** R_{ij} is the contraction of the Riemann tensor with respect to the second lower and the upper index:*

$$R_{ij} = R^k_{ikj}. \quad (1.22)$$

*The **scalar curvature** R is the contraction of the Ricci tensor*

$$R = R_{ii} = \delta^{ij} R_{ij}. \quad (1.23)$$

Now that we have a way to describe the global properties of a manifold \mathcal{M} , we are left with the last point: generalize the concept of shortest path. This can be done requiring that tangent vectors have zero derivative with respect to themselves.

Definition 1.13. *Let (\mathcal{M}, g) a Riemannian manifold. A **geodesics** on \mathcal{M} is a curve $\gamma(\lambda)$ whose tangent vectors $v(\lambda)$ satisfy the equation $v^i \nabla_i v^j = 0$ for all λ .*

In order to better understand this definition let us write its components in a coordinate basis. In a coordinate system φ the geodesics is mapped into a curve $x^i(\lambda)$ in \mathbb{R}^n . From eq. (1.13) the components of v satisfy

$$\frac{dv^k}{d\lambda} + \sum_{i,j} \Gamma_{ij}^k v^i v^j = 0.$$

Since the components of v are given by

$$v^i = \frac{dx^i}{d\lambda},$$

the geodesics equation becomes

$$\frac{d^2 x^k}{d\lambda^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0. \quad (1.24)$$

This is a system of n coupled ordinary differential equations which admits a unique solution for any initial value of x^i and $\frac{dx^i}{d\lambda}$.

1 Riemannian Geometry

This means that given $p \in \mathcal{M}$ and a tangent vector $v_p \in T_p M$, there always exists a unique geodesics $\gamma(t; p, v_p)$ such that [PS10]

$$\gamma(0) = p, \quad \dot{\gamma}(0) = v_p.$$

Moreover the geodesics $\gamma(t)$ depends smoothly on p and v_p . We let

$$\mathcal{E} = \{(p, v_p) | \gamma(t; p, v_p) \text{ is defined on an interval containing } [0, 1]\}.$$

By definition $\mathcal{E} = TM := \bigcup_p T_p M$ if and only if (\mathcal{M}, g) is geodesically complete, i.e. each geodesics can be defined on \mathbb{R} .

Note that a linear reparametrization of a geodesics is again a geodesics [Do 92]. So, for any $\gamma(t; p, v_p)$ and any $\lambda > 0$, the curve

$$\tilde{\gamma}(t) = \gamma(\lambda t; p, v_p)$$

is the geodesics with $\tilde{\gamma}(0) = p$ and $\dot{\tilde{\gamma}}(0) = \lambda v_p$. This fact implies

1. If $(p, v_p) \in \mathcal{E}$ then for any $0 < \lambda < 1$, $(p, \lambda v_p) \in \mathcal{E}$.
2. If $(p, v_p) \notin \mathcal{E}$, then one can find $\epsilon > 0$ such that $(p, \epsilon v_p) \in \mathcal{E}$.

On the other hand, the maximal existence time of geodesics is continuous with respect to p and is lower semi-continuous with respect to v_p . It follows that for each $p \in \mathcal{M}$, $\mathcal{E} \cap T_p M$ is star shaped around $0 \in T_p M$. In particular, \mathcal{E} contains a neighborhood of the zero section in TM .

Definition 1.14. *The exponential map is defined to be*

$$\exp : \mathcal{E} \rightarrow \mathcal{M}, \quad (p, v_p) \mapsto \exp_p(v_p) := \gamma(1; p, v_p) \tag{1.25}$$

By definition the point $\exp_p(v_p)$ is the end point of the geodesics segment that starts at p in the direction of v_p whose length equals $|v_p|$.

According to the smooth dependence in ODE theory [PS10], the exponential map is smooth. In particular, for each $p \in \mathcal{M}$, the map

$$\exp_p : T_p M \cap \mathcal{E} \rightarrow \mathcal{M}$$

is smooth. By definition \exp_p maps $0 \in T_p M$ to $p \in \mathcal{M}$. As a consequence of the inverse function theorem we have

Proposition 1.2. *For any $p \in \mathcal{M}$, there exists a neighborhood V of 0 in $T_p M$ and a neighborhood U of p in \mathcal{M} so that $\exp_p : V \rightarrow U$ is a diffeomorphism. The inverse map is called logarithmic map*

$$\log(\cdot) := \exp^{-1}(\cdot)$$

U is called **normal neighborhood** centered at p .

Recall that we have not required the metric g to be positive definite. This could result in a negative value of the length of a curve. This fact doesn't mean that our constructions are meaningless. Remember that from special relativity space and time are merged in a four dimensional space, hence the "length" of a curve is not the one we are familiar with in the standard Euclidean space.

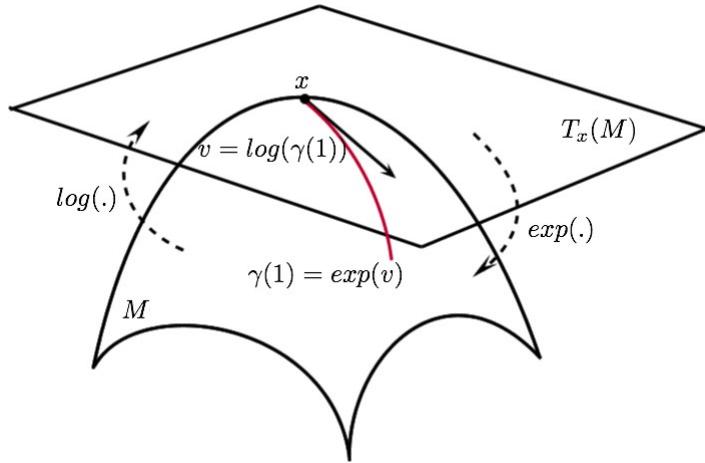


Figure 1.3: Exponential map applied to a vector

Definition 1.15. Let (\mathcal{M}, g) be a Riemannian metric. The **proper time** τ of a curve $\gamma(\lambda)$ with tangent vector $v(\lambda)$ is given by the following integral

$$\tau = \int_{\gamma} \sqrt{g_{ij}v^i(\lambda)v^j(\lambda)} d\lambda. \quad (1.26)$$

The proper time is essential to define relativistic quantities like velocity and momentum, as we will see in the following chapter.

1.5 Variations of geometric quantities

This is a technical section where we derive the variations of some quantities which will be used to derive Einstein's Equation in section 2.2. First of all we define what a variation is.

Definition 1.16. Let (\mathcal{M}, g) be a Riemannian manifold and $Q = Q(g)$ a geometric quantity (i.e. depending on g). The **variation** $\delta Q[h]$ of Q in the direction of $h \in S_0^2(\mathcal{M})$ (where $S_0^2(\mathcal{M})$ is the space of symmetric $(0, 2)$ tensors) is:

$$\delta Q[h] = \lim_{t \rightarrow 0} \frac{Q(g + th) - Q(g)}{t} \quad (1.27)$$

Clearly $\delta g[h] = h$.

- The inverse of the metric tensor g^{ij}

$$(\delta g^{-1}[h]) =: \delta g^{ij} = -g^{ip}g^{jq}h_{pq} \quad (1.28)$$

Proof. Take the variation of $g^{ip}g_{pj} = \delta_j^i$ to get

$$(\delta g^{ip})g_{pj} + g^{ip}(\delta g_{pj}) = (\delta g^{ip})g_{pj} + g^{ip}h_{pj} = 0$$

which implies (1.28). □

1 Riemannian Geometry

- The volume form $dV_g = \sqrt{-g}d^n x$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ij}\delta g^{ij} \quad (1.29)$$

where $-g = \det g_{ij}$.

Proof. We use the following identity which is true for every matrix M

$$Tr(\ln M) = \ln(\det M)$$

Taking the variation of this identity gives

$$Tr(M^{-1}\delta M) = \frac{1}{\det M}\delta(\det M)$$

Now we want to apply it to $M = g^{ij}$. Then $\det M = (\det g_{ij})^{-1}$, and

$$\begin{aligned} \delta\sqrt{-g} &= \delta[(-g)^{-1/2}] \\ &= -\frac{1}{2}(-g)^{-3/2}\delta(-g^{-1}) \\ &= -\frac{1}{2}\sqrt{-g}g_{ij}\delta g^{ij} \end{aligned}$$

□

- Christoffel Symbols.

$$(\delta\Gamma[h])_{ij}^k = \frac{1}{2}g^{kp}(\nabla_i h_{pj} + \nabla_j h_{ip} - \nabla_p h_{ij}) \quad (1.30)$$

Proof. From eq. (1.15) we have

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{ik} - \partial_k g_{ij})$$

using eq. (1.28) and the fact that variations commute with partial derivatives, we deduce

$$\begin{aligned} (\delta\Gamma[h])_{ij}^k &= \frac{1}{2}\{\delta g^{kp}(\partial_i g_{pj} + \partial_j g_{ip} - \partial_p g_{ij}) + g^{kp}(\partial_i(\delta g_{ip}) + \partial_j(\delta g_{ip}) - \partial_p(\delta g_{ij}))\} \\ &= -g^{kp}h_{pq}\Gamma_{ij}^q + \frac{1}{2}g^{kp}(\partial_i h_{pj} + \partial_j h_{ip} - \partial_p h_{ij}) \end{aligned}$$

By definition of covariant derivative we have

$$\nabla_p h_{ij} = \partial_p h_{ij} - \Gamma_{ip}^q h_{qj} - \Gamma_{jp}^q h_{iq}$$

and substituting in the previous expression we obtain (1.30). □

- **Riemann curvature tensor** ((0, 4) version).

$$(\delta \text{Riem}[h])_{ijkl} = -\frac{1}{2}(\nabla_l \nabla_j h_{ik} - \nabla_k \nabla_j h_{il} + \nabla_k \nabla_i h_{jl} - \nabla_l \nabla_i h_{jk} - R_{ijkp} h_{pl} + R_{ijlp} h_{pk}). \quad (1.31)$$

Proof. By definition

$$R_{ijkl} = g_{ip} R_{jkl}^p = g_{ip} (\partial_k \Gamma_{lj}^p - \partial_l \Gamma_{kj}^p + \Gamma_{lj}^q \Gamma_{kq}^p - \Gamma_{kj}^q \Gamma_{lq}^p),$$

and thus we have

$$\delta R_{ijkl} = h_{ip} R_{jkl}^p + g_{ip} [\partial_k (\delta \Gamma_{lj}^p) - \partial_l (\Gamma_{kj}^p)] \quad (1.32)$$

$$+ g_{ip} [(\delta \Gamma_{lj}^q) \Gamma_{kq}^p - \Gamma_{lj}^q (\delta \Gamma_{kq}^p) - (\delta \Gamma_{kj}^q) \Gamma_{lq}^p - \Gamma_{kj}^q (\delta \Gamma_{lq}^p)]. \quad (1.33)$$

Note that, by definition of covariant derivative,

$$\nabla_l \nabla_k h_{ij} = \partial_l \nabla_k h_{ij} - \Gamma_{il}^p \nabla_k h_{pj} - \Gamma_{jl}^p \nabla_k h_{ip} - \Gamma_{kl}^p \nabla_p h_{ij};$$

moreover, the compatibility of the connection with the metric implies

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ik}^p g_{pj} - \Gamma_{jk}^p g_{ip} = 0$$

and

$$\nabla_k g^{ij} = \partial_k g^{ij} + \Gamma_{kp}^i g^{pj} + \Gamma_{kp}^j g^{ip} = 0.$$

It can be shown [CM20] that

$$\nabla_l \nabla_k h_{ij} - \nabla_k \nabla_l h_{ij} = h_{pj} g^{pq} R_{qikl} + h_{ip} g^{pq} R_{qjkl}.$$

Putting all these relations in eq. (1.33) we get eq. (1.31). \square

- **Ricci tensor**

$$(\delta \text{Ric}[h])_{ij} = -\frac{1}{2}(\Delta h_{ij} + 2R_{ijkl} h_{kl} - R_{ip} h_{pj} - R_{jp} h_{pi} + H_{ij} - \nabla_j \nabla_t h_{it} - \nabla_i \nabla_t h_{jt}), \quad (1.34)$$

where $H_{ij} = [g^{-1}h]_{ij}$.

Proof. Since $R_{ij} = g^{pq} R_{ipjq}$, we have

$$(\delta \text{Ric}[h])_{ij} = (\delta g^{pq}) R_{ipjq} + g^{pq} (\delta R_{ipjq});$$

substituting the variations of the Riemann tensor and of the inverse of the metric we obtain eq. (1.34). \square

- **Scalar curvature**

$$\delta R[h] = -\Delta H + \nabla_q \nabla_p h_{pq} - R_{ts} g^{tp} g^{sq} h_{pq}, \quad (1.35)$$

or, equivalently,

$$\delta R[h] = -\Delta H + \nabla_j \nabla_i h_{ij} - R_{ij} h_{ij}.$$

Proof. It follows immediately tracing eq. (1.34). \square

Chapter 2

Einstein's Equations

In chapter 1 we developed all the mathematical objects we need in General Relativity. This chapter is devoted to develop the physics which allow us to understand what General Relativity is and why we need it. As a result the style will be less rigorous than previous chapter and richer in physical considerations. The logical path we are going to follow starts from few notions on Special Relativity, in order to familiarize with the notations and how the laws of classical physics change in a relativistic framework. Then we will move on to General Relativity and the derivation of the fundamental equations of the all theory: Einstein's equations.

2.1 Special Relativity

Special Relativity is based essentially on two axioms:

- All physical laws must be invariant in form in all inertial frames.
- There exists a finite, constant, maximum velocity in every possible inertial frame. This velocity is the speed of light in vacuum which is $c = 299792458$ m/s.

First axiom states the invariance in form of physical laws in every possible inertial frame. An inertial frame is characterized by the fact that, in absence of external forces, acceleration vanishes. As a consequence, in order to change from one inertial frame to another, we need *linear* transformations. In classical mechanics time is absolute; in other words the properties of time are assumed to be independent from the reference system. There is one time for all the reference systems. This means that if any two phenomena occur simultaneously for one observer then they occur simultaneously also for any other observer.

Space is already relative thanks to *Galileo transformations*. However, if we combine them with the idea of an absolute time, we easily get a contradiction of the second postulate. For this reason Galileo transformations can not be correct in Special Relativity.

Another consequence of the two axioms is that space and time can not be considered two different quantities any more. They should be considered as the components of a *four*

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dimensional manifold called **spacetime**. The structure of such manifold was proposed for the first time by Minkowski and, as already presented, $\mathbb{M}_4 := (\mathbb{R}^4, \eta)$ is known as Minkowski's space. As a consequence we have to introduce the notion of *event*: an event is described by the place and the time it occurred. Hence \mathbb{M}_4 is characterized by events $\mathbf{x} = \{x^i\}$, with $i = 0, 1, 2, 3$, that in coordinates reads

$$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z \quad (2.1)$$

where x^1, x^2, x^3 are coordinates describing a three dimensional Euclidean space (they could be cartesian, cylindrical, spherical and so on).

Considering two different systems of coordinates O and O' , with O' moving on the x axis with relative velocity v , a set of transformations coherent with the axioms was provided by Lorentz (for the sake of simplicity we use cartesian coordinates)

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases} \quad (2.2)$$

where $\gamma(1 - \frac{v^2}{c^2})^{-1/2}$. Note that in the limiting case of $v \ll c$ we recover Galileo transformations. Thanks to Lorentz transformations we have a way to link one inertial frame to another. If we consider an event in two different systems $\mathbf{x} = (ct, x, y, z)$ and $\mathbf{x}' = (ct', x', y', z')$, connected through Lorentz transformations, we can prove that the quantity

$$s^2 = \eta_{ij}x^i x^j \quad (2.3)$$

is invariant. Indeed

$$s^2 = -c^2t^2 + x^2 + y^2 + z^2$$

and (c is constant in all inertial frames)

$$\begin{aligned} s'^2 &= -c^2t'^2 + x'^2 + y'^2 + z'^2 \\ &= -c^2\gamma^2\left(t - \frac{v}{c^2}x\right)^2 + \gamma^2(x - vt)^2 + y^2 + z^2 \\ &= -c^2\gamma^2\left(t^2 + \frac{v^2}{c^4}x^2 - 2\frac{vxt}{c^2}\right) + \gamma^2(x^2 + v^2t^2 - 2vxt) + y^2 + z^2 \\ &= t^2\frac{c^2}{c^2 - v^2}(v^2 - c^2) + x^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) + y^2 + z^2 \\ &= -c^2t^2 + x^2 + y^2 + z^2 = s^2. \end{aligned}$$

In Riemannian manifolds, with positive definite metrics, s^2 would be the squared length of a vector; however in Special (and General) Relativity we can not refer to it as a length, as already pointed out in previous chapter.

Consider now two events \mathbf{x}_1 and \mathbf{x}_2 . We are interested in the “distance”

$$s_{12} = [-c^2(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}$$

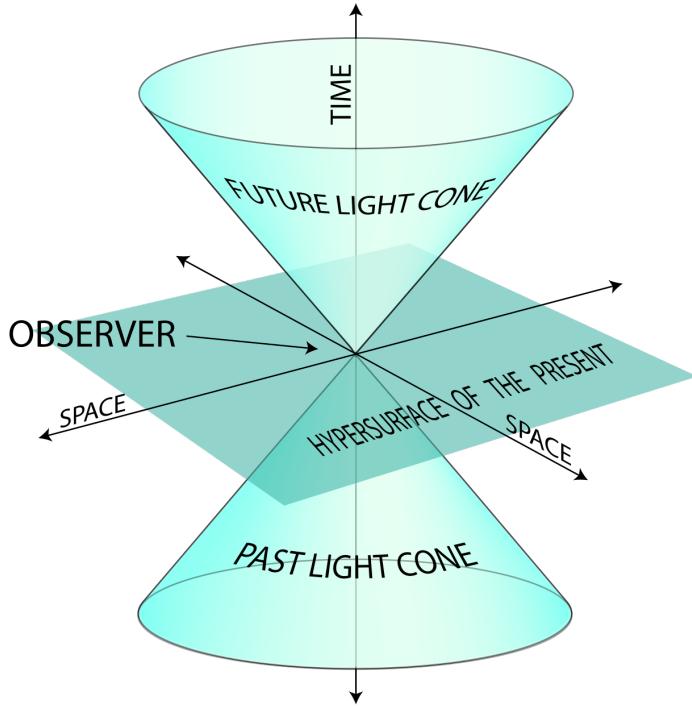


Figure 2.1: Light cone. No events can be causally related if one of them lies outside the cone.

between them. In a relativistic framework, s_{12} is called *interval* between the two events \mathbf{x}_1 and \mathbf{x}_2 . We have two possible scenarios:

1. $s_{12} \in \mathbb{R}$: in this case the interval is said to be *timelike*;
2. $s_{12} \in \mathbb{C}$: in this case the interval is said to be *spacelike*.

The division of intervals between spacelike and timelike is, because of their invariance, an absolute concept. This means that the timelike or spacelike character of an interval is independent of the reference system. The timelike nature of an interval is related to the concept of causality. Indeed, let us consider an event \mathbf{x} which can be thought, without loss of generality, placed in the origin of the space. We now consider what relation other events bear to the given event \mathbf{x} . Two events can be related causally to each other only if the interval between them is timelike; this follows from the second axioms, no interaction can propagate with a velocity grater than the speed of light. Hence the only events that can be causally connected lies in the region of space delimited by the so called *light cone* $x^2 + y^2 + z^2 - c^2 t^2 = 0$. For visualization, let us consider only two spatial coordinates and time. The origin corresponds to “here and now”, while regions with $t > 0$ or $t < 0$ represents the future or the past respectively (fig. 2.1).

So far we have not taken particularly care about the position of the index i of vectors or tensors. Now we need to make a distinction between the two possible ways to write down equations involving these quantities: if we choose the subscript notation (x_i, T_{ij}) we are

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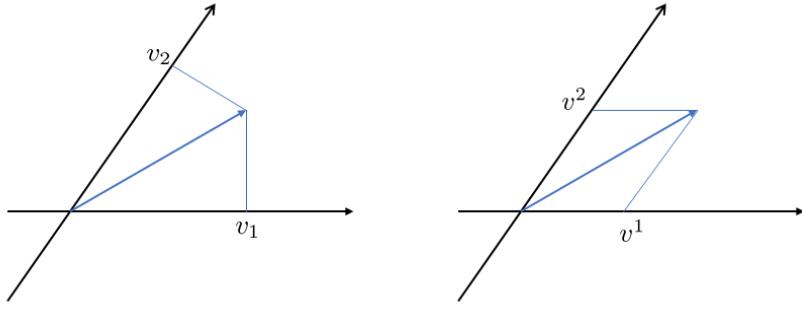


Figure 2.2: Covariant coordinates (left) and contravariant coordinates (right) of a vector.

dealing with the *covariant* formulation, otherwise if we choose the superscript notation (x^i, T^{ij}) we are dealing with *contravariant* formulation. For a geometrical interpretation see fig. 2.2. The first axiom states the invariance in form of all physical laws. Since physical laws are relations between tensors it turns out that the invariance in form is reached whenever they are written as relations between tensors. To change from one formulation to the other it is sufficient to raise/lower through the metric tensor.

One of the most famous and simple physical law we can think about is the Newton's law

$$\mathbf{F} = m\mathbf{a} \quad (2.4)$$

where \mathbf{F} and \mathbf{a} are three dimensional vectors. This equation can be written, equivalently, introducing the linear momentum $\mathbf{p} = m\mathbf{v}$, as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}. \quad (2.5)$$

Unfortunately these equations are not relativistically correct if thought as relations between tensors. So we need to extend the definition of velocity, momentum and so on in order to be able to write down covariant expressions that generalize the Newton's law.

We have already introduced a quantity which may help us in our goal: the proper time τ defined in eq. (1.26) which is a so called *relativistic scalar*. Indeed

$$\underbrace{\frac{ds^2}{c^2}}_{\text{scalar}} = dt^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{dt^2}{\gamma^2}$$

which implies

$$\frac{ds}{dv} = \frac{dt}{\gamma} := d\tau. \quad (2.6)$$

The proper time is then the time elapsing when a body is at rest.

Definition 2.1. *The relativistic velocity u^i of a body with respect to a coordinate system $\{x^i\}$ is the vector whose components are*

$$u^i := \frac{dx^i}{d\tau} = (\gamma c, \gamma v^1, \gamma v^2, \gamma v^3) = (\gamma c, \gamma \mathbf{v}). \quad (2.7)$$

Analogously the relativistic acceleration is defined as

$$w^i := \frac{du^i}{d\tau}. \quad (2.8)$$

Definition 2.2. The relativistic momentum p^a of a body with respect to a coordinate system $\{x^i\}$ is the vector whose components are

$$p^i = mu^i = (m\gamma c, m\gamma \mathbf{v}) \quad (2.9)$$

where m is the body's mass.

The next step is finding a relativistically correct expression equivalent to eq. (2.4),

$$\frac{dp^i}{d\tau} = mw^i =: K^i. \quad (2.10)$$

In components K^i reads

$$K^i = \gamma \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}, \mathbf{F} \right).$$

Equation (2.4), using eq. (2.6), becomes

$$\begin{cases} \frac{d(m\gamma c^2)}{dt} = \mathbf{F} \cdot \mathbf{v} \\ \frac{d(m\gamma \mathbf{v})}{dt} = \mathbf{F} \end{cases}. \quad (2.11)$$

Since $\mathbf{F} \cdot \mathbf{v}$ has the dimension of a power, $m\gamma c^2$ must have the dimension of an energy. When the velocity of a body is zero, or $v \ll c$, we find the most famous equation of Special Relativity

$$E = m_0 c^2, \quad (2.12)$$

with m_0 denoting the *rest mass*, which states that mass is a manifestation of energy and momentum.

The last quantity we need to develop a theory of gravitation is the *stress-energy tensor*.

Definition 2.3. The stress-energy tensor T_{ab} is the $(0, 2)$ tensor whose (i, j) component T_{ij} is given by the flux of the i^{th} component of the relativistic momentum across a surface of constant x^j coordinate. Thus T^{00} is the flow of relativistic mass through time; T^{i0} is the flow of the i^{th} component of spatial momentum through time; T^{0j} is the flow of relativistic mass through a surface of constant coordinate x^j ; the other components are the mechanical stresses.

Example 2.1. For a perfect fluid in thermodynamic equilibrium the stress-energy tensor takes the simple form

$$T_{ij} = \left(\rho + \frac{p}{c^2} \right) u_i u_j + p g_{ij}$$

where ρ is the mass-energy density, p the pressure and g_{ij} the metric tensor.

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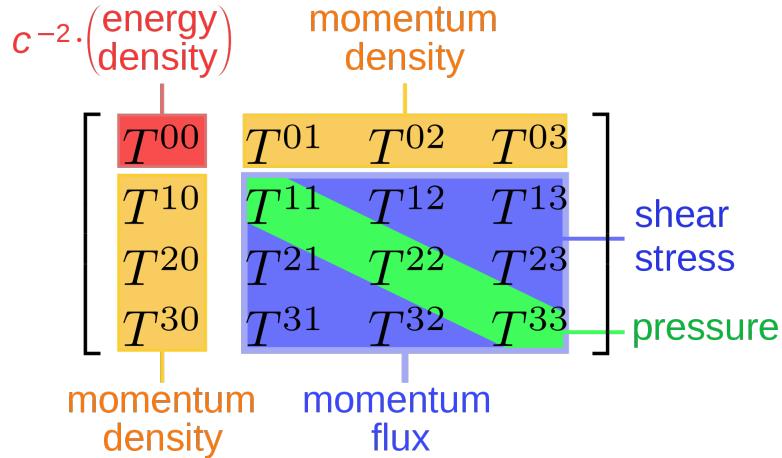


Figure 2.3: Stress-energy tensor in contravariant form

2.2 General Relativity

One of the main consequences of the two axioms of Special Relativity was the flatness of the space as we can immediately see from the metric tensor η_{ij} . However it turns out that this is a too strong assumption which does not allow us to consider, for example, non inertial reference systems, such as uniformly accelerating systems. Hence the structure of spacetime can not be the one of Minkowski's space but we need the more general structure of a curved manifold, as described in chapter 1. Before developing the formalism to write all the equations of such a spacetime, let us analyse some physical considerations.

Remark 2.1. From now on we set $c = 1$.

Consider Newton's Second law as seen in eq. (2.4): to be precise the constant m should be replaced by m_i standing for *inertial mass*

$$\mathbf{F} = m_i \mathbf{a}. \quad (2.13)$$

We also have the law of gravitation which states that the gravitational force exerted on an object is proportional to the gradient of a scalar field Φ , known as the gravitational potential. The constant of proportionality in this case is called *gravitational mass* m_g :

$$\mathbf{F}_g = -m_g \nabla \Phi. \quad (2.14)$$

Galileo showed that every object falls at the same rate in gravitational field, independent of the composition of the object. In Newtonian mechanics this translates in the **weak equivalence principle (WEP)**

$$m_i = m_a. \quad (2.15)$$

As a consequence we have that the behaviour of free falling particles is universal and

$$\mathbf{a} = -\nabla \Phi. \quad (2.16)$$

The WEP implies that there is no way to disentangle the effects of a gravitational field from those of being in a uniformly accelerating frame, simply observing the motion of a free falling particle. Actually it is true only in sufficiently small region of spacetime. If we put ourselves in a big enough box the gravitational field would change from place to place in an observable way, while the effect of acceleration is always in the same direction. This is what happen if we could place a very big box in the Earth's gravitational field: particles will move toward the center of the Earth, which might be a different direction from place to place.

After the event of Special Relativity, the concept of mass lost its centrality since it is no more than manifestation of energy (cfr eq. (2.12)). It was natural to Einstein to think about generalizing the WEP to something more inclusive. His idea was simple: he argued that there should be no way for the physicist in the box to distinguish between uniform acceleration and an external gravitational field. This reasonable extrapolation became what is now known as the **Einstein Equivalence Principle** or EEP: “*In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field*”.

In General Relativity therefore we can no more speak about inertial frames. The best we can do is to consider *locally inertial frame* in which laws of Special Relativity apply. We can no more compare velocities of widely separated region of spacetime. All these physical considerations have their mathematical counterpart. Starting from the second one we recognize the path dependence of the parallel transport on a curved manifold. The first one is more involved and correspond to the possibility to construct a special reference system at any point of a manifold: the so called *Riemannian normal coordinates*.

Definition 2.4. *Geodesics normal coordinates on a manifold \mathcal{M} , endowed with an affine and symmetric connection ∇ , are local coordinates afforded by the exponential map*

$$\exp_p : T_p M \supset V \rightarrow \mathcal{M}$$

and an isomorphism

$$E : \mathbb{R}^n \rightarrow T_p M$$

given by any basis of the tangent space at $p \in \mathcal{M}$. If the additional structure of a (pseudo) Riemannian metric is imposed, then the basis defined by E may be required to be orthonormal and the resulting coordinate system is known as **Riemannian normal coordinate system**.

Properties of Riemannian normal coordinates often simplify computations. Let U be a normal neighborhood centered at $p \in \mathcal{M}$ and x^i normal coordinates. Then:

1. Let $v_p \in T_p M$ with components v_p^i , $\gamma(0; p, v_p)$ the geodesic passing through p with velocity v_p at $t = 0$. Then γ is represented in normal coordinates by

$$\gamma(t) = (t v_p^1, \dots, t v_p^n)$$

as long as it is in U .

2. The coordinates at p are $(0, \dots, 0)$.

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3. The metric tensor at p reduces to the Kronecker delta: $g_{ij}(p) = \delta_{ij}$.
4. The Christoffel symbols vanishes at p : $\Gamma_{ij}^k(p) = 0$. As a consequence, the covariant derivative reduce to the standard partial derivative

$$\nabla_i \rightarrow \frac{\partial}{\partial x^i}.$$

These motivations led to the conclusion that, in the presence of gravity, spacetime should be thought of as a curved manifold. This is an hypothesis of the theory so it is impossible to prove it since scientific hypothesis can only be falsified. So we take it as true and begin to set up how physics works in curved spacetime.

The principle of equivalence tells us that the laws of physics, in small enough regions of spacetime, look like those of special relativity. In the language of manifolds it is equivalent to say that those laws, when written in Riemannian normal coordinates x^i centered at some point p , are described by equations which take the same form as they would in flat space. The simplest example is given by free falling particles. In flat space such particles move in straight lines, *i.e.* acceleration should vanish

$$\frac{d^2x^i}{d\lambda^2} = 0 \quad (2.17)$$

for a parametrized path $x(\lambda)$. According to EEP this equation should hold also in curved space, as long as x^i are Riemannian normal coordinates. There is a unique tensorial equation that reduces to (2.17) when Christoffel symbols vanish; it is

$$\frac{d^2x^k}{d\lambda^2} + \Gamma_{ij}^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0 \quad (2.18)$$

which is simply the geodesics equation. In General Relativity free particles move along geodesics.

Considering as affine parameter the proper time τ we now introduce the so called **Newtonian limit** by three requirements:

- Particles are moving slowly with respect to the speed of light.
- The gravitational field is weak and can be considered as a perturbation of flat space.
- The field is static.

A general theory of gravitation should include the Newtonian gravity described by the previous requirements. Let us see what are the implications on the geodesic equation. “Moving slowly” means that

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \quad i = 1, 2, 3.$$

So the geodesic equation becomes

$$\frac{d^2x^k}{d\tau^2} + \Gamma_{00}^k \left(\frac{dt}{d\tau} \right)^2 = 0. \quad (2.19)$$

The static of the field is simply

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial t} = 0,$$

which implies

$$\Gamma_{00}^k = \frac{1}{2}g^{kl}(\partial_0 g_{l0} + \partial_0 g_{0l} - \partial_l g_{00}) = -\frac{1}{2}g^{kl}\partial_l g_{00}. \quad (2.20)$$

Finally the weakness of the field allow us to decompose the metric into the Minkowski one plus a perturbation

$$g_{ij} = \eta_{ij} + h_{ij}, \quad |h_{ij}| \ll 1. \quad (2.21)$$

Starting from $g^{ij}g_{jk} = \delta_k^i$ we can compute the form of g^{ij} .

Proposition 2.1. *We have*

$$g^{ij} = \eta^{ij} - h^{ij} \quad (2.22)$$

Proof. Let

$$g^{ij} = X^{ij} + Y^{ij}, \quad |X^{ij}| \sim O(1), \quad |Y^{ij}| \sim \varepsilon.$$

Then

$$\delta_k^i = (X^{ij} + Y^{ij})(\eta_{jk} + h_{jk}) = X^{ij}\eta_{jk} + X^{ij}h_{jk} + Y^{ij}h_{jk} + O(\varepsilon^2)$$

Since

$$\underbrace{\delta_k^i}_{\sim O(1)} = \underbrace{X^{ij}\eta_{jk}}_{\sim O(1)} + \underbrace{X^{ij}h_{jk}}_{\sim O(1)} + \underbrace{Y^{ij}h_{jk}}_{\sim O(\varepsilon)},$$

we deduce that

$$\delta_k^i = (X^{ij}) \Rightarrow X^{ij} = \eta^{ij} \quad (2.23)$$

and

$$X^{ij}h_{jk} + Y^{ij}h_{jk} = 0 \Rightarrow h^{im} := Y^{im} = -\eta^{ij}\eta^{km}h_{jk} \quad (2.24)$$

which implies (2.22) after relabelling some indices. \square

Putting all together we find

$$\begin{aligned} \Gamma_{00}^k &= -\frac{1}{2}g^{km}\partial_m g_{00} = -\frac{1}{2}(\eta^{km} - h^{km})\partial_m(\eta_{00} + h_{00}) \\ &= -\frac{1}{2}\eta^{km}\partial_m h_{00} \end{aligned} \quad (2.25)$$

and the geodesic equation reduces to

$$\frac{d^2x^k}{d\tau^2} = \frac{1}{2}\eta^{km}\partial_m h_{00} \left(\frac{dt}{d\tau} \right)^2. \quad (2.26)$$

Using the hypothesis of static metric the $k = 0$ component is simply

$$\frac{d^2t}{d\tau^2} = 0. \quad (2.27)$$

2 Einstein's Equations

Hence $\frac{dt}{d\tau}$ is constant. The space-like components instead obeys to

$$\frac{d^2x^k}{d\tau^2} = \frac{1}{2} \left(\frac{dt}{d\tau} \right)^2 \partial_k h_{00}, \quad k = 1, 2, 3. \quad (2.28)$$

Dividing both sides by $(\frac{dt}{d\tau})^2$ led to

$$\frac{d^2x^k}{dt^2} = \frac{1}{2} \partial_k h_{00}, \quad k = 1, 2, 3 \quad (2.29)$$

which looks like Newton's equation for gravitational fields. Comparing to eq. (2.16) we see that they are the same equation once we identify

$$h_{00} = -2\Phi$$

or, in other words,

$$g_{00} = -(1 + 2\Phi). \quad (2.30)$$

We have shown that the curvature of spacetime is sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form of (2.30). It remains to find field equations for the metric which imply this is the form taken, and for a single gravitating body we recover the Newtonian formula

$$\Phi = -\frac{Gm}{r} \quad (2.31)$$

where G is the gravitational constant.

Our next task, therefore, will be to derive equations that govern how the metric responds to energy and momentum. We will do it in two different ways: first with an informal, physical argument which attempts to generalise Newton's law, and then starting with an action and deriving the corresponding equation of motion. The latter could be considered more formal but has the weakness of being less physical transparent. We will mainly follow [Car97], [LL51], [Tol].

2.2.1 Generalization of Newton's law

The first derivation begins with the aim of finding an equation that supersedes the Poisson equation for the Newtonian potential:

$$\Delta\Phi = 4\pi G\rho \quad (2.32)$$

where $\Delta = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ the mass density. On the left hand side of this equation we have a linear, second order, differential operator acting on the potential, on the right hand side a measure of the mass distribution. We are looking for an equation between tensors. The relativistic generalization of the mass density is the easy part: ρ should be replaced by the energy-momentum tensor T_{ij} . On the other hand the gravitational potential should be replaced by the metric tensor, hence our equation

will have the energy-momentum tensor set proportional to some tensor which is second order in derivatives of the metric. Using eq. (2.30) for the metric in the Newtonian limit and setting $T_{00} = \rho$ we are looking for an equation that predicts

$$\Delta h_{00} = -8\pi G T_{00} \quad (2.33)$$

but of course we want it to be completely tensorial.

The first choice might be to act the D'Alambertian $\square = \nabla^i \nabla_i$ on the metric g_{ij} , but this is automatically zero by metric compatibility. Another quantity which is second order in derivatives of g_{ij} and not identically zero is the Riemann tensor R_{ijk}^l . Contracting it, in order to have the right number of indices, it is reasonable to guess

$$R_{ij} = kT_{ij} \quad (2.34)$$

for a suitable constant k .

Conservation of energy in curved spacetime is

$$\nabla^i T_{ij} = 0 \quad (2.35)$$

which implies

$$\nabla^i R_{ij} = 0. \quad (2.36)$$

However this is not true in general. Exploiting Bianchi identities (1.20) and (1.21) we have

$$\begin{aligned} 0 &= g^{ij} g^{kl} (\nabla_l R_{mjk} - \nabla_m R_{jlk} + \nabla_j R_{lmk}) \\ &= \nabla^k R_{mk} - \nabla_m R + \nabla^i R_{mi} \\ &\Rightarrow \nabla^j R_{ij} = \frac{1}{2} \nabla_i R. \end{aligned}$$

Define the **Einstein tensor** G_{ij} as

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \quad (2.37)$$

which is symmetric and automatically conserved

$$\nabla^i G_{ij} = 0. \quad (2.38)$$

So it is reasonable to propose

$$G_{ij} = kT_{ij} \quad (2.39)$$

as field equation for the metric. It has all the properties required:

1. The right hand side is a covariant expression of energy and momentum density in a form of a $(0, 2)$ symmetric and conserved tensor.
2. The left hand side is a $(0, 2)$ symmetric and conserved tensor constructed from the metric and its first and second derivatives.

2 Einstein's Equations

It remains to verify that it predicts gravity as we know it. Contracting eq. (2.39) we get

$$R = -kT,$$

where T is the trace of T_{ij} , so we can rewrite (2.39) as

$$R_{ij} = k \left(T_{ij} - \frac{1}{2} T g_{ij} \right). \quad (2.40)$$

In the weak field, time independent, slowly moving particles limit the rest energy $\rho = T_{00}$ will be much larger than the other components of T_{ij} . We saw that in Newtonian limit

$$g_{00} = -1 + h_{00} \quad g^{00} = -1 - h_{00}.$$

The trace of T_{ij} , at lowest order, is

$$T = g^{00} T_{00} = -T_{00}.$$

Plugging into eq. (2.40), we get

$$R_{00} = \frac{1}{2} k T_{00}. \quad (2.41)$$

To find the explicit expression of it we need to evaluate $R_{00} = R_{0j0}^i$:

$$R_{0j0}^i = \partial_j \Gamma_{00}^i - \underbrace{\partial_0 \Gamma_{j0}^i}_{=0} + \underbrace{\Gamma_{jk}^i \Gamma_{00}^k - \Gamma_{0k}^i \Gamma_{j0}^k}_{\sim \varepsilon^2}. \quad (2.42)$$

At lowest order we are left with $R_{0j0}^i = \partial_j \Gamma_{00}^i$. From this we get

$$\begin{aligned} R_{00} &= R_{0i0}^i \\ &= \partial_i \left(\underbrace{\frac{1}{2} g^{ik} (\partial_0 g_{k0} + \underbrace{\partial_0 g_{0k}}_{=0} - \partial_k g_{00})}_{=0} \right) \stackrel{\text{eq. (2.25)}}{=} -\frac{1}{2} \eta^{ij} \partial_i \partial_j h_{00} = -\frac{1}{2} \Delta h_{00}. \end{aligned} \quad (2.43)$$

Comparing to eq. (2.41), we see that in the Newtonian limit

$$\Delta h_{00} = -kT_{00}, \quad (2.44)$$

which is exactly eq. (2.33) if we set $k = 8\pi G$. We are now in the position to present **Einstein's field equations** for general relativity

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}. \quad (2.45)$$

To convince ourselves that these are the correct equations we now proceed deriving them in a different way exploiting the *principle of least action*.

2.2.2 Einstein-Hilbert action

The principle of least action is defined by the statement that for each mechanical system there exists a certain integral \mathcal{S} , the *action*, which has a minimum value for the actual motion, so that its variation $\delta\mathcal{S}$ is zero [LL51].

We notice that this integral must not depend upon our choice of reference system, that is, it must be invariant under Lorentz transformations. It follows that it must depend on a *scalar* quantity. Thus the action integral can be represented as an integral over a suitable domain Ω

$$\mathcal{S} = \int_{\Omega} \mathcal{L} d\Omega. \quad (2.46)$$

In classical mechanics the domain Ω is a time interval $[t_1, t_2]$. As we saw, time is not a relativistic scalar so we can not integrate over a time interval; the simplest choice is to integrate over the whole spacetime so that $d\Omega = dx^0 dx^1 dx^2 dx^3$. The coefficient \mathcal{L} of $d\Omega$ represents the *Lagrange density* of our mechanical system.

The action we are interested in is the one proposed by Hilbert (which first derived the equations based on Einstein's previous papers on the subject):

$$\mathcal{S}_H = \int \mathcal{L}_H d\Omega, \quad \mathcal{L}_H = \sqrt{-g} R \quad (2.47)$$

where $g = \det(g_{ij})$ is the determinant of the metric tensor and R the scalar curvature. This expression is justified by the following proposition [Car97]:

Proposition 2.2. *The simplest (i.e. linear in the second derivatives of the metric) independent scalar we could construct from the Riemann tensor is the scalar curvature. Moreover, the only independent non trivial tensor, made from the metric and its first and second derivatives, is the Riemann tensor.*

The equation of motion should come from varying the action with respect to the metric. We have

$$\delta\mathcal{S}_H = \delta \int \sqrt{-g} R d\Omega = \int [R \delta \sqrt{-g} + \sqrt{-g} \delta R] d\Omega. \quad (2.48)$$

Substituting in this equation the variations of volume form and scalar curvature derived in section 1.5, we obtain the relation

$$\delta\mathcal{S} = \int \left(R_{ij} - \frac{1}{2} g_{ij} R \right) \delta g^{ij} \sqrt{-g} d\Omega + \int \frac{\partial(\sqrt{-g} w^i)}{\partial x^i} d\Omega. \quad (2.49)$$

The second integral is the divergence of a vector, so we can apply the divergence theorem to get

$$\int \frac{\partial(\sqrt{-g} w^i)}{\partial x^i} d\Omega = \int \sqrt{-g} w^i d\sigma = 0, \quad (2.50)$$

since the variations of the field are zero at the boundaries.

Thus the variation $\delta\mathcal{S}_H$ is equal to

$$\delta\mathcal{S} = \int \left(R_{ij} - \frac{1}{2} g_{ij} R \right) \delta g^{ij} \sqrt{-g} d\Omega. \quad (2.51)$$

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If we consider also the presence of matter (or in general external forces), described by the energy momentum tensor T_{ij} , its variation is simply given by

$$\delta\mathcal{S}_m = \int T_{ij} \delta g^{ij} \sqrt{-g} d\Omega. \quad (2.52)$$

Then the principle of least action prescribes

$$\delta\mathcal{S}_H + \delta\mathcal{S}_m = 0 \quad (2.53)$$

which is

$$\int \left(R_{ij} - \frac{1}{2}g_{ij}R - T_{ij} \right) \delta g^{ij} \sqrt{-g} d\Omega = 0 \quad (2.54)$$

and from the arbitrariness of δg^{ij}

$$R_{ij} - \frac{1}{2}g_{ij}R = T_{ij}. \quad (2.55)$$

Note that in all the calculations above we have set the normalization constants equal to 1.

The next chapter will be devoted to analyse some possible solutions to Einstein's equations. Before moving in that direction let us consider some general aspects and characterizations of such equations.

2.2.3 On the nature of Einstein's equations

The first thing one can notice is that if we choose a coordinate system and express the coordinate basis components R_{ij} in terms of the metric g_{ij} , we see that R_{ij} depends on derivatives of the metric up to second order and is highly non linear in g_{ij} . There are ten independent equations (both sides are $(0, 2)$ symmetric tensors) for ten unknown functions of the metric components. However, the Bianchi identity $\nabla^i G_{ij} = 0$ represents four constraints on the functions R_{ij} , so there are only six truly independent equations. This is representative of the physical consideration that, if a metric is a solution in one coordinate system, it must be a solution also in another one. Furthermore, the four components of the relativistic velocity u^i , which appear in the energy momentum tensor of the matter, are related to one another by the $u^i u_i = c^2$, so only three of them are independent. The last unknown is the mass density ρ .

Due to the non linearity of the equations we are not guaranteed that a solution exists in general. For a metric with Lorentz signature it can be shown that they are equivalent to a system of coupled hyperbolic partial differential equations so a good initial value formulation can be worked out. We will discuss such formulation in chapter 4.

The final remark concerns the energy momentum tensor. It is clear that we can take an arbitrary metric g_{ij} , compute the Einstein tensor G_{ij} and demand that T_{ij} be equal to G_{ij} . This is not a very interesting procedure to carry on from the physical point of view since we are interested in solutions in the presence of "realistic" sources of energy and momentum. The most natural property to demand is that T_{ij} represent a positive energy

density (no negative masses are allowed). In a locally inertial frame this requirement can be stated as $\rho = T_{00} \geq 0$. The correspondent coordinate-independent statement is

$$T_{ij}v^i v^j \geq 0$$

for all timelike vectors v^i .

Chapter 3

Solutions of Einstein's equations

In the previous chapter we analysed the theory of General Relativity deriving its fundamental equations. The goal of this chapter is trying to solve these equations. As already pointed out, they are very complicated differential equations and there is no hope to solve them in general. For this reason we will make some simplifying (but reasonable) assumptions on symmetries of g_{ij} , rather than on the form of the energy momentum tensor.

We will start with one of the question that arise naturally: “Which solution of Einstein’s equation describes the spacetime we observe, *i.e.* , which solution correspond to our universe or, at least, an idealized model of our universe?” With some assumption on the nature of our universe we will be able to solve exactly the equations and make predictions on its dynamical evolution. In particular we will model the energy momentum tensor with a perfect fluid structure and we will assume that the metric tensor has no time-space components. In section 3.1 we will introduce three possible geometries for our universe, all characterized by a *scaling factor* $a(t)$, which will be a *positive* function of the proper time, here denoted by t . Unfortunately this is not enough to close the system of equation, since we have three unknowns (ρ , $a(t)$, and pressure P) but only two equations. We need an *equation of state* which links the presence of matter with the pressure. We will present three models of increasing generalities: in section 3.1 the equation will be $P = 0$, for the so called *dust* model, or $P = \rho/3$, for the so called *radiation filled* model; in section 3.2 we generalize the model with a *barotropic* description $P = (\gamma - 1)\rho$ with γ constant; finally in section 3.3 we present a *politropic* model $P = (\alpha + k\rho^{1/n})\rho$ from which we will be able to reconstruct all the history of our universe.

The last part of this chapter is devoted to derive and analyse the *Schwazschild metric*. Derived for the first time in 1916, this solution of the Einstein’s equations describes the behaviour of spacetime around a static and spherically symmetric body. The Schwazschild metric allow us to explain one of the weakness of Newtonian gravity: the precession of Mercury’s perihelion.

3 Solutions of Einstein's equations

3.1 Homogeneous, isotropic universe

Two hypothesis on the nature of our universe which seems reasonable, and are supported by observational data, are *homogeneity* and *isotropy*.

Homogeneity, loosely speaking, means that at any given “instant of time” each point of “space” should “look like” any other point. This last sentence have its mathematical counter part.

Definition 3.1. *Spacetime is said to be (spatially) homogeneous if there exists a one-parameter family of spacelike hypersurfaces Σ_t such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric g_{ij} which takes p into q .*

As regard isotropy a precise formulation is given as follows.

Definition 3.2. *Spacetime is said to be (spatially) isotropic at each point if there exists a congruence of timelike curves (i.e. observers) with tangents u^i satisfying the following property. Given any point p and any two unit “spatial” tangent vectors $w_1^i, w_2^i \in V_p$ (i.e. vectors at p orthogonal to u^i), there exists an isometry of g_{ij} which leaves p and u^i at p fixed but rotates w_1^i into w_2^i .*

Thus in an isotropic universe is impossible to construct a geometrically preferred tangent vector orthogonal to u^i . Since the isotropic observers are orthogonal to the homogeneous surfaces, we may express the four-dimensional spacetime metric g_{ij} as

$$g_{ij} = -u_i u_j + h_{ij}(t) \quad (3.1)$$

where for each t , $h_{ij}(t)$ is the metric of either a sphere, flat Euclidean space or an hyperboloid on Σ_t [Wal84]. These possibilities correspond to the so called Robertson Walker cosmological model in which the metric takes the form

$$ds^2 = -dt^2 + a^2(t) \begin{cases} d\psi^2 + \sin^2 \psi(d\theta^2 + \sin^2 \theta d\varphi^2) & \text{sphere} \\ dx^2 + dy^2 + dz^2 & \text{Euclidean} \\ d\psi^2 + \sinh^2 \psi(d\theta^2 + \sin^2 \theta d\varphi^2) & \text{hyperboloid} \end{cases} \quad (3.2)$$

The last ingredient we need is the form of the energy momentum tensor which appear on the right hand side of eq. (2.45). The most general form of T_{ij} consistent with homogeneity and isotropy is the one of a perfect fluid

$$T_{ij} = \rho u_i u_j + P(g_{ij} + u_i u_j). \quad (3.3)$$

This form is justified by two facts:

1. At the scale we are interested in all the matter present in the universe can be considered as a “grain of dust” with density ρ and negligible pressure.
2. A constant thermal distribution of radiation at a temperature of about 3K fills the universe. For massless thermal radiation we have $P = \rho/3$.

3.1 Homogeneous, isotropic universe

From the ten equations that a priori we need to solve, only two are truly independent. Indeed the vector $G^{ij}u_j$ (and also $T^{ij}u_j$) can not have spatial components otherwise isotropy will be violated. Thus, the time-space components of Einstein's equation are identically zero. Similarly if we project both indices of G_{ij} into the homogeneous hypersurface led us to the conclusion that the space-space part of T_{ij} is multiple of the identity matrix. Hence we are left with only two equations

$$\begin{aligned} G_{tt} &= 8\pi GT_{tt} = 8\pi G\rho \\ G_{**} &= 8\pi GT_{**} = 8\pi GP \end{aligned} \quad (3.4)$$

where $G_{**} = G_{ij}w^i w^j$ with w^i any unit tangent vector to the homogeneous hypersurface. We can now proceed in the computation of G_{tt} and G_{**} . We will do it explicitly for the case of flat spatial geometry, *i.e.* for a metric with the form

$$g_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix}. \quad (3.5)$$

The Christoffel's symbols are easily computed through eq. (1.15); note that now (t, x, y, z) are the coordinate basis whose components are labelled in general as α^i

$$\begin{aligned} \Gamma_{xx}^t &= \frac{1}{2} g^{tk} (\underbrace{\partial_x g_{xk}}_{=0 \forall k} + \underbrace{\partial_x g_{xk}}_{=0 \forall k} - \partial_{\alpha^k} g_{xx}) \\ &= -\frac{1}{2} \left(-\frac{\partial a^2(t)}{\partial t} \right) = a\dot{a} \end{aligned} \quad (3.6)$$

where the dot means derivative with respect to time.

$$\begin{aligned} \Gamma_{xt}^t &= \frac{1}{2} g^{xk} (\underbrace{\partial_x g_{tk}}_{=0 \forall k} + \partial_t g_{xk} - \underbrace{\partial_{\alpha^k} g_{xt}}_{=0 \forall k}) \\ &= \frac{1}{2} \frac{1}{a^2} 2a\dot{a} = \frac{\dot{a}}{a}. \end{aligned} \quad (3.7)$$

From symmetry we have

$$\Gamma_{xx}^t = \Gamma_{yy}^t = \Gamma_{zz}^t = a\dot{a}$$

and

$$\Gamma_{xt}^t = \Gamma_{tx}^x = \Gamma_{yt}^y = \Gamma_{ty}^y = \Gamma_{zt}^z = \Gamma_{tz}^z = \frac{\dot{a}}{a}.$$

The components of the Ricci tensor are given by eq. (1.22) which is explicitly

$$R_{ij} = \sum_k \frac{\partial}{\partial \alpha^k} \Gamma_{ij}^k - \frac{\partial}{\partial \alpha^i} \left(\sum_k \Gamma_{kj}^k \right) + \sum_{n,k} (\Gamma_{ij}^n \Gamma_{nk}^k - \Gamma_{kj}^n \Gamma_{ni}^k). \quad (3.8)$$

3 Solutions of Einstein's equations

Hence

$$\begin{aligned}
R_{tt} &= \underbrace{\sum_k \frac{\partial}{\partial \alpha^k} \Gamma_{tt}^k}_{\Gamma_{tt}^{(\cdot)} = 0 \forall k} - \frac{\partial}{\partial t} \left(\sum_k \Gamma_{kt}^k \right) + \sum_{n,k} \left(\underbrace{\Gamma_{tt}^n}_{=0 \forall n} \Gamma_{nk}^k - \underbrace{\Gamma_{kt}^n \Gamma_{nt}^k}_{\neq 0 \iff n=k} \right) \\
&= -\frac{\partial}{\partial t} \left(3 \frac{\dot{a}}{a} \right) - 3 \left(\frac{\dot{a}}{a} \right)^2 = -3 \frac{\ddot{a}}{a} \\
R_{xx} &= \underbrace{\sum_k \frac{\partial}{\partial \alpha^k} \Gamma_{xx}^k}_{=0} - \frac{\partial}{\partial x} \left(\sum_k \Gamma_{xx}^k \right) + \sum_{k,n} \left(\Gamma_{xx}^n \Gamma_{nk}^k - \Gamma_{kx}^n \Gamma_{nx}^k \right) \\
&= \frac{\partial}{\partial t} \Gamma_{xx}^t + \left[\Gamma_{xx}^t (\Gamma_{tx}^x + \Gamma_{ty}^y + \Gamma_{tz}^z) - \Gamma_{xx}^t \Gamma_{tx}^x - \Gamma_{tx}^x \Gamma_{xx}^t \right] \\
&= \frac{\partial}{\partial t} (a \dot{a}) + \left[a \dot{a} \left(3 \frac{\dot{a}}{a} \right) - 2 \dot{a}^2 \right] \\
&= a \ddot{a} + 2 \dot{a}^2.
\end{aligned}$$

Finally we obtain

$$R_{**} = a^{-2} R_{xx} = \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2}. \quad (3.9)$$

The last quantity we need to compute the left hand side of th Einstein's equations is the Ricci scalar

$$R = -R_{tt} + 3R_{**} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad (3.10)$$

thus the resulting equations are

$$G_{tt} = 3 \frac{\dot{a}^2}{a^2} = 8\pi G \rho \quad (3.11)$$

$$G_{**} = -2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi G P. \quad (3.12)$$

Using the first equation we can rewrite the second one as

$$3 \frac{\ddot{a}}{a} = -4\pi G (\rho + 3P). \quad (3.13)$$

Repeating the above procedure also for the sphere and hyperboloid we get the evolution equation for homogeneous and isotropic universe

$$G_{tt} = 3 \frac{\dot{a}^2}{a^2} = 8\pi G \rho - 3 \frac{k}{a^2} \quad (3.14)$$

$$3 \frac{\ddot{a}}{a} = -4\pi G (\rho + 3P), \quad (3.15)$$

where $k = 1$ for the sphere, $k = 0$ for flat space and $k = -1$ for the hyperboloid.

Before deriving the exact solutions of these equations let us make few qualitative considerations.

First of all from eq. (3.15) we have that universe can not be static provided only that $\rho > 0$ and $P \geq 0$, since $\ddot{a} < 0$. Thus the universe can be either expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$). Suppose the universe is expanding, eq. (3.15) tells us that it must have been expanding at a faster and faster rate as one goes backward in time. If the universe has always expanded at its present rate, then at the time $T = a/\dot{a}$ we would have had $a = 0$. This means that at time less than T the universe was in a singular state: the distance between all points was zero; the density of matter and curvature of spacetime was infinity. This singular state of the universe is referred to as the **Big Bang**. Since spacetime structure is singular at the Big Bang, it does not make any sense, either physically or mathematically, to ask about the state of the universe *before* the Big Bang: there is no way to extend spacetime manifold and metric beyond the Big Bang singularity.

Another interesting fact regards the case of spherically universe. The first term on the right hand side of eq. (3.14) decreases with a more rapidly than the second one; since the left hand side must be positive, we conclude that there exists a critical value a_c such that $a \leq a_c$. Furthermore \ddot{a} is bounded from below then a can not reach asymptotically a_c as $t \rightarrow +\infty$. As a result we conclude that, in case of $k = 1$, the universe will expand until a finite time t , it reaches the maximum expansion a_c , then recontraction begins and a **big crunch** end of universe will occur.

On the other hand, if $k = -1$ or $k = 0$ eq. (3.14) shows that \dot{a} can not become zero. Thus if the universe is presently expanding, then it will expand forever.

Before solving eq. (3.14) and (3.15) we can simplify them as follows: multiply eq. (3.14) by a^2 , differentiate with respect to t and then eliminate \ddot{a} by eq. (3.15). We obtain the energy conservation equation

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0 \quad (3.16)$$

which is

$$\rho a^3 = \text{constant} \quad (3.17)$$

for dust, and

$$\rho a^4 = \text{constant} \quad (3.18)$$

for radiation. These two last expression led us the following *first order, non linear, ordinary differential equations*

$$\dot{a}^2 - \frac{C_d}{a} + k = 0 \quad (3.19)$$

in case of dust ($C_d = 8\pi G\rho a^3/3$),

$$\dot{a}^2 - \frac{C_r}{a^2} + k = 0 \quad (3.20)$$

in case of radiation ($C_r = 8\pi G\rho a^4/3$). The initial condition is set to be $a(0) = 0$.

We can now proceed solving them for all the three spatial geometries and for the two different models. All these equations will be solved by separation of variables method [PS10].

3 Solutions of Einstein's equations

3.1.1 Sphere, $k = 1$

Dust

The equation to solve is

$$\dot{a}^2 - \frac{C_d}{a} + 1 = 0$$

which reduces to

$$\int \sqrt{\frac{a}{C_d}} \frac{1}{\sqrt{1 - \frac{a}{C_d}}} da = t + c.$$

with c constant of integration. The integral on the left hand side can be computed by the change of variable $a = C_d \sin^2 x$ which yields to the following solution

$$\begin{aligned} t &= \frac{1}{2} C_d (2x - \sin(2x)) \\ a &= \frac{1}{2} C_d (1 - \cos(2x)) \end{aligned}$$

Radiation

The equation to solve is

$$\dot{a}^2 - \frac{C_r}{a^2} + 1 = 0$$

which reduces to

$$\int \frac{a}{\sqrt{C_r - a^2}} da = t + c$$

with c constant of integration. This is an immediate integral and the solution is

$$a = \sqrt{C_r} \left[1 - \left(1 - \frac{t}{\sqrt{C_r}} \right)^2 \right]^{1/2}$$

3.1.2 Flat space, $k = 0$

Dust

The equation to solve is

$$\dot{a}^2 - \frac{C_d}{a} = 0$$

whose solution is simply

$$a = \left(\frac{9C_d}{4} \right)^{1/3} t^{2/3}$$

Radiation

The equation to solve is

$$\dot{a} - \frac{C_r}{a^2} = 0.$$

Once again the integral is immediate and the solution is

$$a = \sqrt[4]{4C_r t^{\frac{1}{2}}}$$

3.1.3 Hyperboloid, $k = -1$

Dust

The equation to solve is

$$\dot{a}^2 - \frac{C_d}{a} - 1 = 0$$

which reduces to

$$\int \sqrt{\frac{a}{C_d}} \frac{1}{\sqrt{1 + \frac{a}{C_d}}} da = t + c$$

with c constant of integration. The integral on the left hand side can be computed by the substitution $a = c \sinh^2 x$ which yields

$$\begin{aligned} t &= \frac{1}{2} C_d (\sinh(2x) - 2x) \\ a &= \frac{1}{2} C_d (\cosh(2x) - 1) \end{aligned}$$

Radiation

The equation to solve is

$$\dot{a}^2 - \frac{C_r}{a^2} - 1 = 0$$

which reduces to

$$\int \frac{a}{\sqrt{C_r + a^2}} da = t + c$$

with c constant of integration. The integral is an immediate one and the solution is

$$a = \sqrt{C_r} \left[\left(\frac{t}{C_r} + 1 \right)^2 - 1 \right]^{1/2}$$

3 Solutions of Einstein's equations

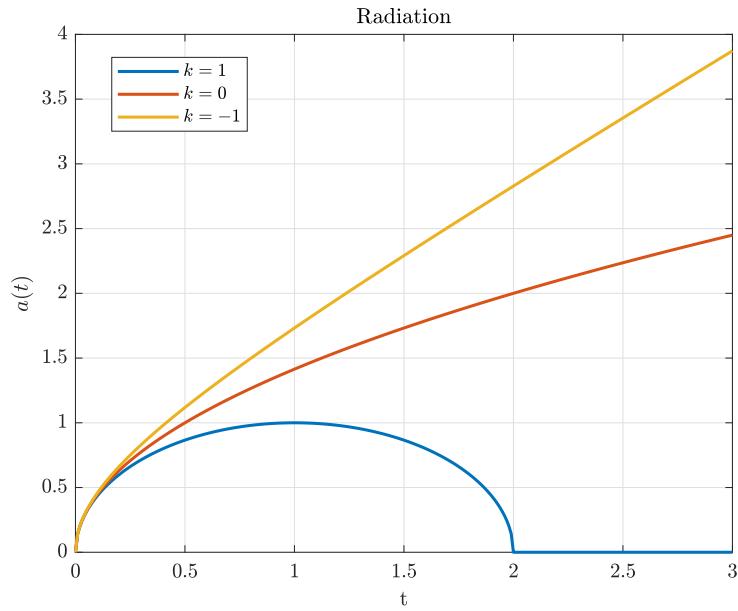


Figure 3.1: Solutions to Einstein equations in case of radiation filled universe with $C_r = 1$.

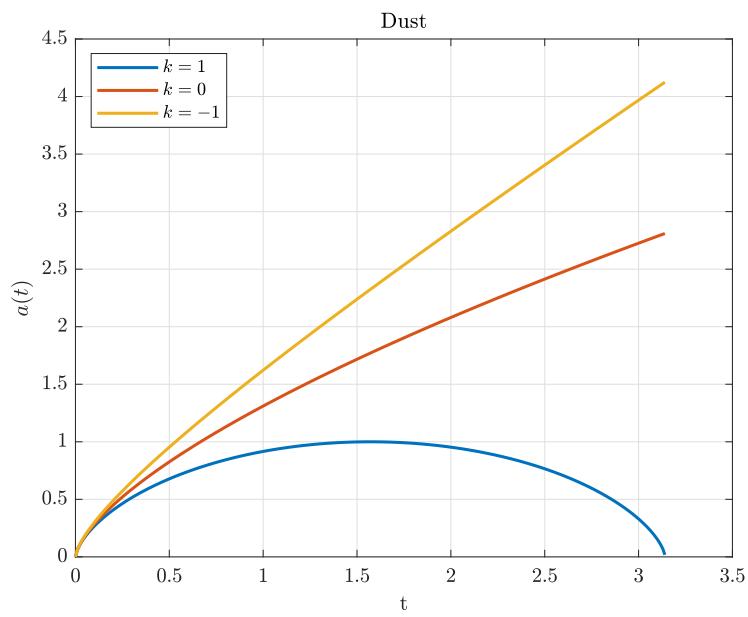


Figure 3.2: Solution to Einstein equations in case of dust universe with $C_d = 1$.

3.1.4 Evolution of our universe

The dust filled and the radiation models presented above are widely used to make predictions, especially on the nature of early universe. As one goes backward in time the scaling of the factor a on the matter has the same effects as if the matter was placed in a box whose walls contract at the same rate. Thus the contribution of radiation compared to ordinary matter increases in the past while it becomes negligible in the present. Thus, one would expect the radiation model of the universe to be a good approximation in the past, while the dust filled model should be a good approximation afterwards. If the early universe is radiation dominated, then for all models ($k = -1, 0, 1$) the dependence of a and ρ on t goes over the flat solution

$$a(t) = \sqrt[4]{4C_R} t^{\frac{1}{2}} \quad (3.21)$$

$$\rho = \frac{3}{32\pi G t^2}. \quad (3.22)$$

As regards the future evolution of the universe, the most important issue is whether our universe is “open” (*i.e.* solutions with $k = 0, -1$) or closed (*i.e.* solution with $k = 1$). We can rewrite equations (3.14) and (3.15) in terms of two parameters: the *Hubble’s constant* $H = \frac{a}{\dot{a}}$ and the *decreasing factor* $q = -\frac{\ddot{a}a}{\dot{a}^2}$. Since radiation is negligible in present universe we have

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \quad (3.23)$$

$$q = \frac{4\pi G\rho}{3H^2}. \quad (3.24)$$

Defining Ω by

$$\Omega = \frac{8\pi G\rho}{3H^2} \quad (3.25)$$

we see that the universe is close if and only if $\Omega > 1$, *i.e.* $\rho > \rho_c := 3H^2/8\pi G$.

The uncertainty in calibration of distances leads considerable uncertainty on the measure of H and ρ . However their ratio is not affected by this uncertainties, hence the value of $\Omega \propto \rho/H^2$ can be considered quite accurate. The estimated values of H and ρ are

$$H \sim 50 \text{ km} \cdot \text{s}^{-1} \text{Mpc}^{-1} \quad \rho \sim 2 \cdot 10^{-31} \text{ g/cm}^3$$

As a result $\Omega \sim 0.04$ and this provides evidence for an open universe.

3.2 General barotropic fluid

So far we have considered only two possible scenarios regarding the model of the spacelike part of the energy momentum tensor. We now move to a more general framework, always sticking on a perfect *barotropic* fluid description *i.e.* the equation of state is given by

$$P = (\gamma - 1)\rho, \quad \gamma = \text{constant.} \quad (3.26)$$

3 Solutions of Einstein's equations

For $\gamma = 1$ we obtain the dust model while for $\gamma = 4/3$ we obtain the radiation model. Other physical interesting values are $\gamma = 0$, from which we obtain the vacuum equation of state $P = -\rho$ corresponding to *inflation*, and $\gamma = 2/3$, which correspond to the *curvature dominated* coasting universe. In this last case eq. (3.15) reduces to $\ddot{a} = 0$ whose solution is $a(t) = a_0 t$. We now proceed with a derivation of a solution for any value of the parameter γ as proposed in [Far99].

From equations (3.14), (3.15) and (3.26) we have

$$\frac{\ddot{a}}{a} + c \left(\frac{\dot{a}}{a} \right)^2 + \frac{ck}{a} = 0 \quad (3.27)$$

where $c = \frac{3}{2}\gamma - 1$. The case $k = 0$ is straightforward and gives

$$a(t) = a_0 t^{\frac{2}{3\gamma}}, \quad \gamma \neq 0 \quad (3.28)$$

$$a(t) = a_0 e^{Ht}, \quad \dot{H} = 0, \quad \gamma = 0. \quad (3.29)$$

In order to solve the cases $k = \pm 1$ let us introduce the *conformal time* s as [LL51]:

$$dt = a(s)ds \quad (3.30)$$

so that eq. (3.27) becomes

$$\frac{a''}{a} + (c - 1) \left(\frac{a'}{a} \right)' + ck = 0 \quad (3.31)$$

where primes denotes differentiation with respect to s . By employing the variable

$$u := \frac{a'}{a} \quad (3.32)$$

we find

$$u' + cu^2 + ck = 0 \quad (3.33)$$

which is a Riccati equation. The general Riccati equation has the form

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x),$$

where $y = y(x)$, and can be solved explicitly [BL91]. We can find a solution introducing the variable w such that

$$u = \frac{1}{c} \frac{w'}{w} \quad (3.34)$$

which led us to

$$w'' + kc^2 w = 0. \quad (3.35)$$

This is a second order linear ordinary differential equation which admits a unique solution by simple integration methods [PS10]. For $k = 1$, one find the solution

$$\begin{aligned} a(s) &= a_0 [\cos(cs + d)]^{1/c} \\ t(s) &= a_0 \int_0^s [\cos(cx + d)]^{1/c} dx, \end{aligned}$$

while for $k = -1$

$$\begin{aligned} a(s) &= a_0 \sinh^{1/c}(cs) \\ t(s) &= a_0 \int_0^s \sinh^{1/c}(cx) dx. \end{aligned}$$

It is clear now that if $\gamma = 4/3$ or $\gamma = 1$ we obtain the solutions derived in the previous section.

As final remark we point out that the above procedure works *only* if $\gamma = \text{constant}$. A time dependent $\gamma(t)$ can be used to describe a non interacting mixture of dust and radiation; however in this case all the procedure can not be applied.

3.3 Politropic model

Both the previous models provide that at $t = 0$ the universe should have a singularity where mass density and temperature becomes infinity. This peculiar behaviour arises since we are not considering quantum effects that should be taken into account as soon as the “radius” of the universe reaches the Planck length $l_P = 1.32 \cdot 10^{-35}$ m. Under the Planck length the predictions of General Relativity become meaningless and it should be replaced by a quantum gravity theory [RV14], which is still an open subject. Present energy content of the universe is composed of approximately 5% baryonic matter, 20% dark matter, and 75% dark energy. The model proposed in [Cha18] takes into account the effect of vacuum energy, radiation, dark matter and dark energy through a generalized equation of state of the form (we now restore the c dependence)

$$P = (\gamma\rho + \alpha\rho^{1+1/n})c^2. \quad (3.36)$$

This is the sum of a standard barotropic equation of state $P = \gamma\rho c^2$ and a polytropic equation of state $P = \alpha\rho^\beta c^2$, where α is the polytropic constant and $\beta = 1 + 1/n$ is the polytropic index. Positive indices $n > 0$ describe the early universe where the polytropic component dominates the linear component because the density is high. Negative indices $n < 0$ describe the late universe where the polytropic component dominates the linear component because the density is low. On the other hand, a positive polytropic pressure ($\alpha > 0$) leads to past or future singularities (or peculiarities) while a negative polytropic pressure ($\alpha < 0$) leads to a phase of exponential expansion (inflation) in the past or in the future [Cha12].

We assume a flat description of the universe, hence the cosmological equations are eq. (3.14) with $k = 0$, together with the conservation of energy (3.16):

$$\dot{\rho} + 3\left(\rho + \frac{P}{c^2}\right)\frac{\dot{a}}{a} = 0 \quad (3.37)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \quad (3.38)$$

To close the system we need again to specify an equation of state fixing the parameters in eq. (3.36).

3 Solutions of Einstein's equations

3.3.1 Early universe

We begin considering the early universe which corresponds to the following choice of parameters in eq. (3.36):

$$\gamma = \frac{1}{3}, \quad n = 1, \quad \alpha = -\frac{4}{3\rho_P}$$

where $\rho_P = c^5/G^2\hbar = 5.16 \cdot 10^{99} g/m^3$ is the Planck density, corresponding to a dimension of the universe equal to l_P [Lin90]. The equation of state can, thus, be rewritten as [Cha14]

$$P = \frac{1}{3} \left(1 - \frac{4\rho}{\rho_P} \right) \rho c^2. \quad (3.39)$$

Substituting in eq. (3.37), we can integrate as follows:

$$\begin{aligned} \dot{\rho} + \left(4\rho - \frac{4\rho^2}{\rho_P} \right) \frac{\dot{a}}{a} &= 0 \\ -\frac{\dot{\rho}}{4\rho(1 - \rho/\rho_P)} &= \frac{\dot{a}}{a} \\ \rho_P \int \frac{\dot{\rho}}{\rho(\rho_P - \rho)} dt &= \ln \left(\frac{a}{a_1} \right)^4 \\ \ln \frac{\rho_P - \rho}{\rho} &= \ln \left(\frac{a}{a_1} \right)^4 \end{aligned}$$

from which we find

$$\rho = \frac{\rho_P}{(a/a_1)^4 + 1} \quad (3.40)$$

where a_1 is a constant of integration. From this expression we immediately see that no singularities on density arise: indeed, whenever $a \rightarrow 0$, $\rho \rightarrow \rho_P$ which represent an upper bound for the density. When $a \ll a_1$ (*inflation era*) the density is approximately constant and the pressure tends to $P = -\rho_P c^2$. Hence we can integrate eq. (3.38) with $\rho = \rho_P$, $a(0) = l_P$ as initial condition, to obtain an exponential increasing scale factor a

$$a(t) = l_P \exp \left(\sqrt{\frac{8\pi}{3}} \frac{t}{t_P} \right) \quad (3.41)$$

where $t_P = 1/(\rho_P G)^{1/2} = 2.39 \cdot 10^{-44} s$ is the Planck time which gives the timescale of the exponential growth. This universe exists at any time in the past ($t \rightarrow -\infty$) without any singularity.

When $a \gg a_1$ (*radiation era*), we get $\rho/\rho_P \sim (a_1/a)^4$. When the density is “low”, the equation of state reduces to the linear equation of state $p = \rho c^2/3$ we found in section 3.1. From eq. (3.18) we get $\rho_P a_1^4 = \rho_{rad,0} a_0^4$, where $\rho_{rad,0}$ is the present density of radiation and $a_0 = c/H_0 = 1.32 \cdot 10^{26} m$ the present distance of cosmological horizon determined by the Hubble constant $H_0 = 2.27 \cdot 10^{-18} s^{-1}$ (the Hubble time is $H_0^{-1} = 4.41 \cdot 10^{17} s$). Writing $\rho_{rad,0} = \Omega_{rad,0} \rho_0$, where $\rho_0 = 9.20 \cdot 10^{-24} g/m^3$ is the estimated present density of the universe and $\Omega_{rad,0} = 8.48 \cdot 10^{-5}$ is the present fraction of radiation in the universe,

we obtain $a_1/a_0 = (\Omega_{rad,0}\rho_0/\rho_P)^{1/4}$, thus $a_1 = 2.61 \cdot 10^{-6}$. This size marks the transition between the vacuum energy era and the radiation era [Cha12; Cha14].

The general solution of eq. (3.38) with mass density given by eq. (3.40) can be computed exactly as

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}} \frac{1}{t_p} \frac{1}{\sqrt{(a/a_1)^4 + 1}}$$

$$\int \frac{\dot{a}}{a} \sqrt{\frac{a^4}{a_1^4} + 1} dt = \sqrt{\frac{8\pi}{3}} \frac{t}{t_P} + C$$

The integral on the left hand side can be computed explicitly by the substitution $x = \sqrt{(a/a_1)^4 + 1}$ which gives

$$\sqrt{(a/a_1)^4 + 1} - \ln \left(\frac{1 + \sqrt{(a/a_1)^4 + 1}}{(a/a_1)^2} \right) = 2\sqrt{\frac{8\pi}{3}} \frac{t}{t_P} + C \quad (3.42)$$

where $C \simeq -134$ is a constant of integration determined in order to verify the condition $a(0) = l_P$. The transition between the vacuum energy era and the radiation era ($a = a_1$) corresponds to a density $\rho_1 = \rho_P/2$ and a time $t_1 = (3/32\pi)^{1/2}[\sqrt{2} - \ln(1 + \sqrt{2}) - C]t_P$. The universe is accelerating when $a < a_1$ and decelerating when $a > a_1$ from time t_1 .

3.3.2 Late universe

We now proceed considering the equation of state (3.36) with the following parameters

$$\gamma = 0, \quad n = -1, \quad \alpha = -\rho_\Lambda$$

where $\rho_\Lambda = \Omega_{\Lambda,0}\rho_0$ with $\Omega_{\Lambda,0} = 0.763$, according to observations; thus $\rho_\Lambda = 7.02 \cdot 10^{-24} g/m^3$. This is a model of the late universe which is dominated by dark matter and dark energy [CST06]. The resulting equation of state is

$$P = -\rho_\Lambda c^2. \quad (3.43)$$

We can, then, integrate eq. (3.37) to obtain

$$\rho = \rho_\Lambda \left[\left(\frac{a_2}{a} \right)^3 + 1 \right] \quad (3.44)$$

where a_2 is a constant of integration.

When $a \gg a_2$ (*dark energy era*), the density is approximately constant: $\rho \simeq \rho_\Lambda$. A constant value of the density gives rise to a phase of late inflation. It is convenient to define a *cosmological time* $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = 1.46 \cdot 10^{18} s$, a *cosmological length* $l_\Lambda = ct_\Lambda = 4.38 \cdot 10^{26} m$ and a cosmological mass $M_\Lambda = \rho_\Lambda l_\Lambda^3 = 5.90 \cdot 10^{53} kg$. These are the counterpart of the Planck scales for the late universe [Cha12]. As a result the scale factor has an exponential behaviour

$$a(t) = l_\Lambda \exp \left(\sqrt{\frac{8\pi}{3}} \frac{t - t_f}{t_\Lambda} \right) \quad (3.45)$$

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with t_f defined as the time at which $a = l_\Lambda$ and $\rho_f = [(a_2/l_\Lambda)^3 + 1]\rho_\Lambda$. It turns out to be $t_f = 0.708t_\Lambda$. The solution (3.45) is known as **de Sitter solution**.

When $a \ll a_2$ (*matter era*), we get $\rho/\rho_\Lambda \sim (a_2/a)^3$. Since $\rho_\Lambda \ll 1$, then $p \simeq 0$. From the conservation equation (3.17) we have $\rho_\Lambda a_2^3 = \rho_{m,0} a_0^3$. Using the value of ρ_Λ defined above and $\rho_{m,0} = \Omega_{m,0}\rho_0$ with $\Omega_{m,0} = 0.237$, we obtain $a_2 = 8.95 \cdot 10^{25} m$; the scale a_2 marks the transition between matter era and dark energy era.

As we have done in the previous case we can integrate exactly eq. (3.38):

$$\begin{aligned} \frac{\dot{a}}{a} &= \sqrt{\frac{8\pi}{3}} \frac{1}{t_\Lambda} \left[\left(\frac{a_2}{a} \right)^3 + 1 \right] \\ \int \frac{\dot{a}}{a} \frac{1}{(a_2/a)^3 + 1} dt &= \sqrt{\frac{8\pi}{3}} \frac{t}{t_\Lambda} \\ \int \dot{a} \frac{1}{\sqrt{a_2^3}} \frac{\sqrt{a}}{\sqrt{1 + (a/a_2)^3}} dt &= \sqrt{\frac{8\pi}{3}} \frac{t}{t_\Lambda} \end{aligned}$$

The integral on the left hand side can be computed with the following substitution

$$\left(\frac{a}{a_2} \right)^{3/2} = \sinh x$$

so that

$$\frac{2}{3}x = \sqrt{\frac{8\pi}{3}} \frac{t}{t_\Lambda}$$

from which we have

$$x = \sqrt{6\pi} \frac{t}{t_\Lambda}.$$

The resulting form of $a(t)$, thus, is

$$\frac{a}{a_2} = \sinh^{2/3} \left(\sqrt{6\pi} \frac{t}{t_\Lambda} \right). \quad (3.46)$$

The expression for the density is, then, obtained from eq. (3.44)

$$\frac{\rho}{\rho_\Lambda} = \frac{1}{\tanh^2 \left(\sqrt{6\pi} \frac{t}{t_\Lambda} \right)}. \quad (3.47)$$

The universe is accelerating when $a > a_c = (1/2)^{1/3}a_2$ and decelerating when $a < a_c$. The transition between the matter era and the dark energy era ($a = a_2$) occurs at time $t_2 = (1/\sqrt{6\pi}) \arg \sinh(1)t_\Lambda$.

3.3.3 The whole evolution of the universe

In section 3.3.1 we have described the transition between the vacuum energy era and the radiation era in the early universe by a single equation of state (3.39). This equation interpolates smoothly between the equation of state of vacuum energy ($P = -\rho c^2$) and

the equation of state of radiation $P = \rho c^2/3$. It provides, therefore, a unified description of vacuum energy and radiation; we shall consider that it describes a fluid of “generalized radiation”. We can combine them in a single equation as

$$\frac{\rho}{\rho_0} = \frac{\Omega_{rad,0}}{\frac{\Omega_{rad,0}\rho_0}{\rho_P} + (a/a_0)^4} \quad (3.48)$$

where $\Omega_{rad,0}\rho_0/\rho_P = 1.51 \cdot 10^{-127}$ [Cha12].

In section 3.3.2 we have described the transition between the dark matter era and the dark energy era in the universe by a single equation of state (3.44). This equation interpolates smoothly between the equation of state of pressureless matter ($P = 0$) and the equation of state of dark energy ($P = -\rho c^2$). We shall consider that it describes a “dark fluid” whose relation between energy density and the scale factor is

$$\frac{\rho}{\rho_0} = \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (3.49)$$

We can combine these two models by assuming that the universe is filled with a fluid of “generalized radiation” and a “dark fluid”. The resulting relation is

$$\frac{\rho}{\rho_0} = \frac{\Omega_{rad,0}}{\frac{\Omega_{rad,0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (3.50)$$

Inserting this last expression into eq. (3.38) we obtain a unified cosmic equation [Cha14; Cha13b; Cha13a]:

$$\frac{1}{H_0^2} \left(\frac{\dot{a}}{a} \right)^2 = \frac{\Omega_{rad,0}}{\frac{\Omega_{rad,0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (3.51)$$

Integrating this equation we get the whole evolution of the scale factor $a(t)$ and the energy density $\rho(t)$ reported in figs. 3.3 and 3.4.

3.4 Schwarzschild Metric

As described in the previous sections, the theory of general relativity has made a number of strikingly successful predictions concerning the spacetime structure of our universe. However, cosmological observations presently are not good enough to provide stringent quantitative test of general relativity. Such quantitative tests are provided by the gravitational fields occurring in our solar system, where precise measurements can be made. Thus, it is of great interest to determine the solution of Einstein’s equation corresponding to the exterior gravitational field of a **static and spherically symmetric** body (such as the Sun and many other bodies). This problem was solved by Karl Schwarzschild in 1916, only few months Einstein published his vacuum field equations [Wal84]. The derivation of the solution mainly follows [Pee14]; equivalent procedures can be found, *e.g.*, in [Wal84] and [Oas14].

3 Solutions of Einstein's equations

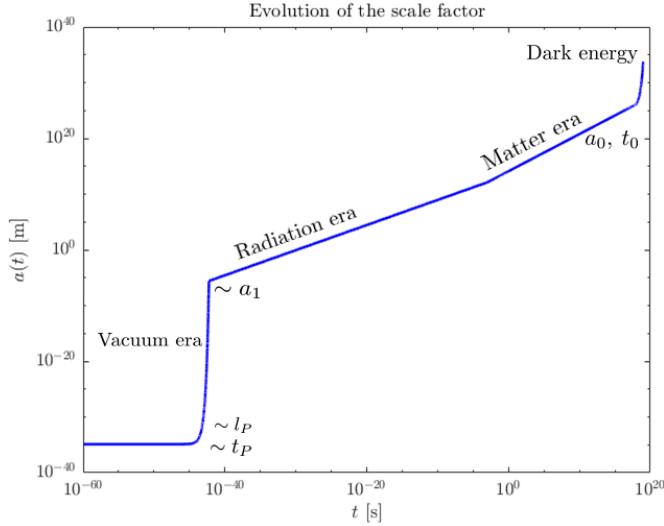


Figure 3.3: Evolution of the scale factor as a function of time. We can distinguish the four eras we described above; furthermore it shows that with this model the solution is also symmetric in time (inflation occurs both in the early and in the late universe.)

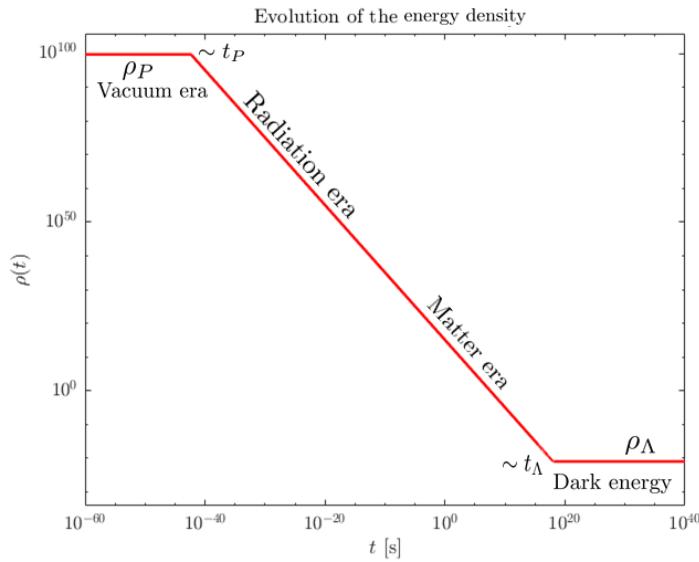


Figure 3.4: Evolution of the energy density ρ as a function of time. It ranges from a maximum value $\rho = \rho_P$ to a minimum value $\rho = \rho_\Lambda$. These two bounds are responsible for the early and late inflation. In the radiation and matter era the relation goes as t^{-2} .

3.4.1 Derivation of the Schwarzschild solution

With the hypothesis of isotropic field, the best way to describe it is based on employing polar coordinates

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (3.52)$$

Since our setting is a four dimensional spacetime, we need two more coordinates. The most general form is to choose arbitrary coordinates $t' = a$ and $r' = b$, and write the metric as

$$ds^2 = g_{aa}(a, b)da^2 + g_{ab}(a, b)(dadb + dbda) + g_{bb}(a, b) + r^2(a, b)d\Omega^2 \quad (3.53)$$

for a yet undetermined function $r(a, b)$, that will be identified with the usual definition of r later on.

We can change coordinates from (a, b) to (a, r) inverting $r(a, b)$ ¹, so that the metric becomes

$$ds^2 = g_{aa}(a, r)da^2 + g_{ar}(a, r)(dadr + drda) + g_{rr}(a, r)dr^2 + r^2d\Omega^2 \quad (3.54)$$

Now we want to find a function $t(a, r)$ such that, in the (t, r) coordinate system, there are no cross terms $dtdr + drdt$ in the metric. Since

$$dt = \frac{\partial t}{\partial a}da + \frac{\partial t}{\partial r}dr,$$

we find

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(dadr + drda) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2. \quad (3.55)$$

The metric we would like to have has no off diagonal terms, namely we would like to replace the first three terms in eq. (3.54) by

$$mdt^2 + ndr^2$$

for some functions m and n . This is equivalent to require

$$m\left(\frac{\partial t}{\partial a}\right)^2 = g_{aa} \quad (3.56)$$

$$n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr} \quad (3.57)$$

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar}. \quad (3.58)$$

We have three equations for three unknown functions $m(a, r)$, $n(a, r)$ and $t(a, r)$; we can therefore formally determine them (up to initial conditions) in terms of the functions

¹We are implicitly assuming that is possible to invert r .

3 Solutions of Einstein's equations

g_{aa} , g_{ar} and g_{rr} (which are still not known). We have shown that we can express the general static and isotropic metric as

$$ds^2 = m(t, r)dt^2 + n(t, r)dr^2 + r^2d\Omega^2. \quad (3.59)$$

Asymptotically, *i.e.* far from any object, we expect the metric to approach Minkowski metric; thus, the metric must have $(3, 1)$ signature. Therefore we choose $m(t, r)$ to be negative, and we can rewrite the functions m and n in terms of two new functions α and β as

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2d\Omega^2. \quad (3.60)$$

In order to determine α and β we need to actually solve Einstein's equations. Using the labels $(0, 1, 2, 3)$ for (t, r, θ, φ) respectively, the only non vanishing Christoffel symbols are given by²:

$\bullet \Gamma_{00}^0 = \partial_0\alpha$	$\bullet \Gamma_{01}^1 = \partial_0\beta$	$\bullet \Gamma_{13}^3 = \frac{1}{r}$
$\bullet \Gamma_{01}^0 = \partial_1\alpha$	$\bullet \Gamma_{11}^1 = \partial_1\beta$	$\bullet \Gamma_{33}^1 = -re^{-2\beta}\sin^2\theta$
$\bullet \Gamma_{11}^0 = e^{2(\beta-\alpha)}\partial_0\beta$	$\bullet \Gamma_{12}^2 = \frac{1}{r}$	$\bullet \Gamma_{33}^2 = -\sin\theta\cos\theta$
$\bullet \Gamma_{00}^1 = e^{2(\beta-\alpha)}\partial_1\alpha$	$\bullet \Gamma_{22}^1 = -re^{-2\beta}$	$\bullet \Gamma_{23}^3 = \frac{\cos\theta}{\sin\theta}$

From these we get the non vanishing components of the Riemann tensor:

$$\begin{aligned} R_{101}^0 &= e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] + [\partial_1\alpha\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2] \\ R_{202}^0 &= -re^{-2\beta}\partial_1\alpha \\ R_{303}^0 &= -re^{-2\beta}\sin^2\theta\partial_1\alpha \\ R_{212}^0 &= -re^{-2\alpha}\partial_0\beta \\ R_{313}^0 &= -re^{-2\alpha}\sin^2\theta\partial_0\beta \\ R_{212}^1 &= re^{-2\beta}\partial_1\beta \\ R_{313}^1 &= re^{-2\beta}\sin^2\theta\partial_1\beta \\ R_{323}^2 &= (1 - e^{-2\beta})\sin^2\theta \end{aligned}$$

Contracting them yields to the component of the Ricci tensor:

$$\begin{aligned} R_{00} &= [\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] + e^{2(\alpha-\beta)}\left[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\partial_1\beta + \frac{2}{r}\partial_1\alpha\right] \\ R_{11} &= -\left[\partial_1\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\partial_1\beta - \frac{2}{r}\partial_1\beta\right] + e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] \\ R_{01} &= \frac{2}{r}\partial_0\beta \\ R_{22} &= e^{-2\beta}[r(\partial_1\beta - \partial_1\alpha) - 1] + 1 \\ R_{33} &= R_{22}\sin^2\theta. \end{aligned}$$

²All the other symbols are related to the ones presented by symmetries

In vacuum, $R_{ij} = 0$. From $R_{01} = 0$ we get

$$\partial_0 \beta = 0.$$

Taking the time derivative of $R_{22} = 0$ and using $\partial_0 \beta = 0$, we get

$$\partial_0 \partial_1 \alpha = 0$$

so that

$$\beta = \beta(r) \quad (3.61)$$

$$\alpha = f(r) + g(t) \quad (3.62)$$

We are free to choose t such that $g(t) = 0$ to obtain

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \quad (3.63)$$

All the metric components are independent of the coordinate t .

Since both R_{00} and R_{11} vanish,

$$0 = e^{2(\beta-\alpha)} R_{00} + R_{11} = \frac{2}{r} (\partial_1 \alpha + \partial_1 \beta),$$

which implies $\alpha = -\beta + const.$. Once again, we can get rid of the constant by scaling our coordinates, so we have $\alpha = -\beta$. From the condition R_{22} we obtain the following differential equation

$$\partial_1(re^{2\alpha}) = 1$$

whose solution is immediate:

$$e^{2\alpha} = 1 + \frac{\mu}{r}, \quad (3.64)$$

with μ constant of integration. The final form of the metric is, then,

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (3.65)$$

We are left with the task of assign a physical meaning to μ . Recalling that in the weak field limit

$$g_{00} = -(1 + 2\Phi)$$

$$g_{rr} = (1 - 2\Phi)$$

with $\Phi = -GM/r$ the Newtonian potential. In the limit of $r \rightarrow \infty$ the g_{00} and g_{rr} components of our metric becomes

$$g_{00} = -\left(1 + \frac{\mu}{r}\right)$$

$$g_{rr} = \left(1 - \frac{\mu}{r}\right).$$

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Setting $\mu = -2GM$, we finally get the **Schwarzschild metric**

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)(cdt)^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\Omega^2 \quad (3.66)$$

where we have added explicitly c^2 .

The first thing one can notice is the presence of two interesting points in which the metric becomes singular: $r = 0$ and $r = 2GM/c^2$. However the metric coefficients are coordinate-dependent, hence we may ask ourselves if these singularities are points in which the curvature actually becomes infinity, or they are simply consequences of a bad choice of coordinate system. This is a very involved subject, thus we give only a simple argument, which is enough to convince ourselves that $r = 0$ is a real **singularity of the curvature**. In order to realize that, we need a quantity, related to curvature, which is independent on the choice of the coordinates. We already shown that this quantity is the scalar curvature. In the case of Schwarzschild metric, a direct calculation gives

$$R = R^{ijkl}R_{ijkl} = \frac{12G^2M^2}{r^6}.$$

We see that $r = 0$ is still a singularity, while $r = 2GM/c^2$ is no more a singular point. Indeed, with a suitable transformation of the coordinates, this surface is well behaved. Such coordinates are known as **Kruskal coordinates** [Oas14; Wal84].

Nonetheless, the surface $r = 2GM/c^2$ is of interest, and is known as **Schwarzschild radius**. It occurs for a numerical value of the radial coordinate

$$r_S = \frac{2GM}{c^2} \approx 3 \frac{M}{M_\odot} \text{km},$$

where $M_\odot \approx 2 \cdot 10^{30} \text{kg}$ is the mass of the Sun. Thus for an “ordinary object”, such as the Sun or the Earth, the Schwarzschild radius r_S is well inside the radius of the body, where the vacuum solution we found is, of course, no longer valid. There exists, however, some strange objects in our universe such that the Schwarzschild radius is outside their radius: these objects are what we call **Black holes**, and r_S is known as **event horizon**. In fig. 3.5 we report a qualitative plot of the curvature of spacetime around the Sun.

In a system of units in which $G/c^2 = 1$, the condition on the Schwarzschild radius becomes

$$r_S = 2M,$$

then we will have singularities when

$$\rho := \frac{M}{r} \geq \frac{1}{2}.$$

In fig. 3.6 we can observe graphically this condition.

3.4.2 Schwarzschild geodesics

The easiest way to explore the consequences of the Schwarzschild metric is to explore particles trajectories, along geodesics, in this metric [Pee14]. Recall that the geodesic

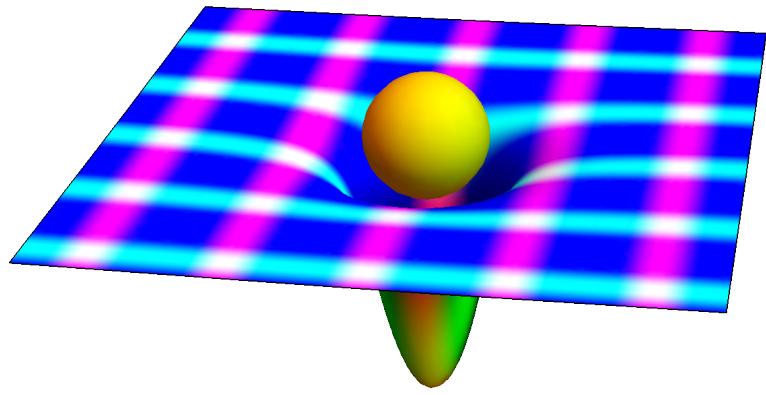


Figure 3.5: Qualitative plot of the curvature of spacetime around the Sun.

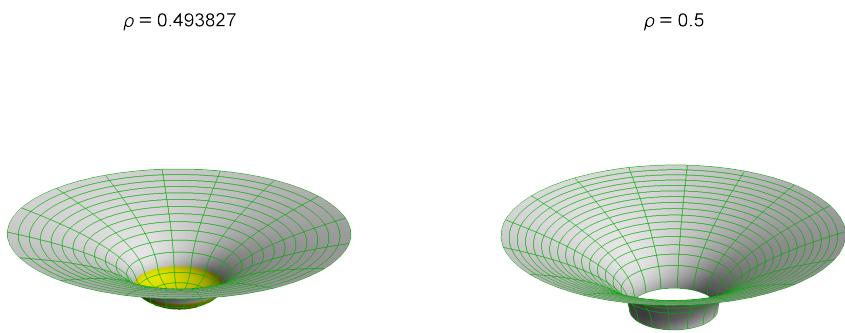


Figure 3.6: Graphical representation of the Schwarzschild radius: when $\rho < 1/2$ no singularities occurs (left) and the yellow region is the one with $r \leq r_s$. As soon $\rho = 1/2$ a black hole forms, here represented with an empty space (right).

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equation is

$$\frac{d^2x^k}{d\lambda^2} + \Gamma_{ij}^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0, \quad (3.67)$$

hence the first step is to find the non zero Christoffel symbols. They are

- $\Gamma_{00}^1 = \frac{GM}{r^3}(r - 2GM)$
- $\Gamma_{11}^1 = \frac{-GM}{r(r-2GM)}$
- $\Gamma_{01}^0 = \frac{GM}{r(r-2GM)}$
- $\Gamma_{33}^1 = \frac{(-r + 2GM)}{\sin^2 \theta}$
- $\Gamma_{22}^1 = -(r - 2GM)$
- $\Gamma_{13}^3 = \frac{1}{r}$
- $\Gamma_{12}^2 = \frac{1}{r}$
- $\Gamma_{33}^2 = -\sin \theta \cos \theta$
- $\Gamma_{23}^3 = \cot \theta$

The geodesic equation therefore turns into the following four equations:

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0 \quad (3.68)$$

$$\begin{aligned} \frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r - 2GM) \left(\frac{dt}{d\lambda} \right)^2 - \frac{GM}{r(r-2GM)} \left(\frac{dr}{d\lambda} \right)^2 \\ - (r - 2GM) \left[\left(\frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left(\frac{d\varphi}{d\lambda} \right)^2 \right] = 0 \end{aligned} \quad (3.69)$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} - \sin \theta \cos \theta \left(\frac{d\varphi}{d\lambda} \right)^2 = 0 \quad (3.70)$$

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r} \frac{d\varphi}{d\lambda} \frac{dr}{d\lambda} + 2 \cot \theta \frac{d\theta}{d\lambda} \frac{d\varphi}{d\lambda} = 0. \quad (3.71)$$

Unfortunately, there is no hope to solve this system of coupled equations exactly, but we can rely on some symmetries in order to find two constants of motion. Indeed the Schwarzschild metric is symmetrical about $\theta = \pi/2$ and it is actually a solution of the third equation by direct substitution. As a result the orbit of a test particle must be planar, and we can arrange the coordinates frame so that the equatorial plane is the plane of orbit. If we denote by $w(r) := 1 - r_s/r$, from the first and fourth equation with $\theta = \pi/2$, we get the two constant of motion

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \left[\ln \frac{d\varphi}{d\lambda} + \ln r^2 \right] \\ 0 &= \frac{d}{d\lambda} \left[\ln \frac{dt}{d\lambda} + \ln w \right]. \end{aligned}$$

3.4.3 Orbits of test particles

Assume we have a body of mass M which is attracting a test particle with mass m . We want to study the orbits of m in the equatorial plane. Choosing as affine parameter the proper time τ , the two constants derived above can be identified with the *total energy* E

$$\left(1 - \frac{r_s}{r} \right) \frac{dt}{d\tau} = \frac{E}{mc^2}$$

and the *specific angular momentum* h

$$h = \frac{L}{\mu} = r^2 \frac{d\varphi}{d\tau},$$

where L is the total angular momentum of the two bodies, and μ the reduced mass

$$\mu = \frac{Mm}{M+m}.$$

Substituting these constants into the definition of the Schwarzschild metric

$$c^2 = \left(1 - \frac{r_S}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{1 - \frac{r_S}{r}} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\varphi}{d\tau}\right)^2$$

yields an equation of motion for the radius as a function of the proper time τ :

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{r_S}{r}\right) \left(c^2 + \frac{h^2}{r^2}\right). \quad (3.72)$$

When the angular momentum is not zero we can replace the dependence on proper time by a dependence on the angle φ using the definition of h

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{dr}{d\tau} \frac{d\tau}{d\varphi}\right)^2 = \left(\frac{r^2}{h}\right)^2 \left(\frac{dr}{d\tau}\right)^2$$

which yields the equation for the orbit

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^2}{b^2} - \left(1 - \frac{r_S}{r}\right) \left(\frac{r^4}{a^2} + r^2\right) \quad (3.73)$$

where, for brevity, two length scales, $a = h/c$ and $b = Lc/E$, have been introduced. They are constant of motion and depend on the initial conditions (position and velocity) of the test particles. The solution of the orbit equation is

$$\varphi = \int \left\{ \pm r^2 \left[\frac{1}{b^2} - \left(1 - \frac{r_S}{r}\right) \left(\frac{1}{a^2} + \frac{1}{r^2}\right) \right] \right\}^{-1/2} dr \quad (3.74)$$

which can be expressed in terms of the Weierstrass elliptic functions [GV12; Sch11].

3.4.4 Effective potential

Eq. (3.72) can be rewritten, equivalently, using the definition of Schwarzschild radius, as

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \left[\frac{E^2}{2mc^2} - \frac{1}{2} mc^2\right] + \frac{GMm}{r} - \frac{L^2}{2\mu r^2} + \frac{G(M+m)L^2}{c^2 \mu r^3}, \quad (3.75)$$

which is equivalent to a particle moving in a one dimensional effective potential [Pee14; Wal84]

$$V(r) = -\frac{GMm}{r} + \frac{L^2}{2\mu r^2} - \frac{G(M+m)L^2}{c^2 \mu r^3} + \frac{1}{2} mc^2. \quad (3.76)$$

3 Solutions of Einstein's equations

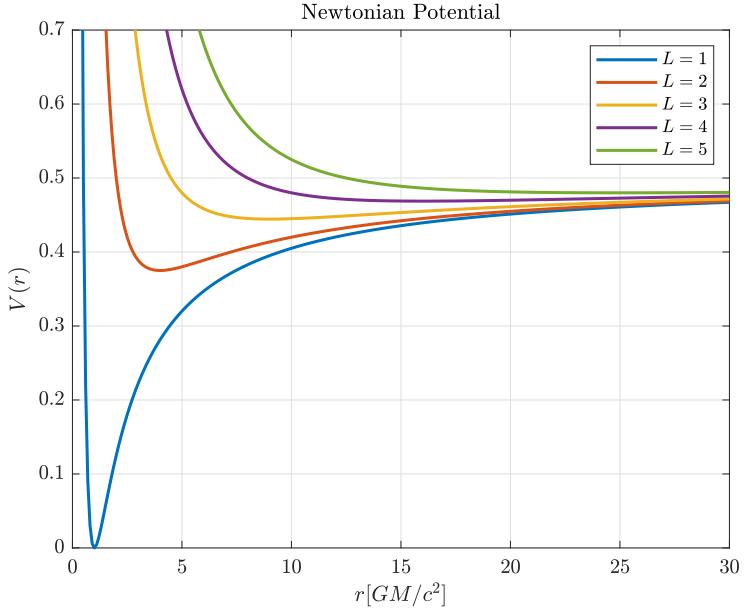


Figure 3.7: Gravitational potential for a unit mass: Newtonian case ($GM = 1$).

The first two terms in the expression of $V(r)$ are well known classical energies [LL51]: the first is the Newtonian gravitational potential and the second is a contribution from angular momentum (sometimes denoted by “centrifugal” potential). It is the third term, though, which contains the General Relativity contribution, which turns out to make a great deal of difference, especially for small values of r , as shown in figs. 3.7 and 3.8. Of course, for large r , the two potentials match. The last term is just a constant.

Among all possible orbits, circular orbits are of particular interest. They are possible when the effective force is zero

$$F = -\frac{dV}{dr} = -\frac{\mu c^2}{2r^4}[r_S r - 2a^2 r + 3r_S a^2] = 0 \quad (3.77)$$

i.e. , when the two attractive forces-Newtonian gravity and the third term unique to general relativity-are exactly balanced by the repulsive “centrifugal” force. We can solve easily the expression in the brackets to obtain two values for the radius which will be the radius of the circular orbit. In particular an orbit will be **stable** if it corresponds to a minimum of the potential, and unstable if it corresponds to a maximum. Bound orbits which are not circular will oscillate around the radius of the stable circular orbit. In case of a unit mass, and in a system of units with $c = 1$, eq. (3.77) becomes

$$GMr^2 - L^2 r + 3GML^2 \epsilon = 0 \quad (3.78)$$

where $\epsilon = 0$ in Newtonian gravity and $\epsilon = 1$ in general relativity.

In Newtonian gravity circular orbits appear at

$$r_c = \frac{L^2}{GM} \quad (3.79)$$

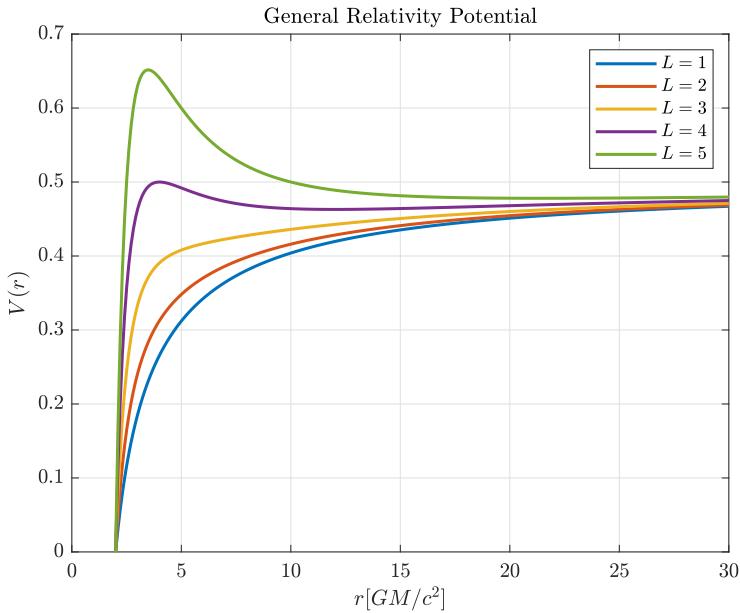


Figure 3.8: Gravitational potential for a unit mass: full GR potential ($c = GM = 1$).

and are always stable. We know that the orbits in Newton's theory are conic sections: bound orbits are either circles or ellipses, while unbound ones are either parabolas or hyperbolas [MN98].

In general relativity the situation is different: the term $-GML/r^3$ forces the potential to go to $-\infty$ when $r \rightarrow 0$. Another difference is that, in the GR case, the potential is always zero at $r = r_S$. Below that limit we enter in the region beyond the event horizon, in the black hole singularity. Solving for the radius eq. (3.78) we obtain

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}. \quad (3.80)$$

For large values of L there will be two circular orbits (one stable and one unstable) given by³

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM}, 3GM \right). \quad (3.81)$$

As L decreases, the two orbits come close together; they coincide when the discriminant of eq. (3.78) vanishes, at

$$L = \sqrt{12}GM,$$

for which

$$r_c = r_{ISCO} = 6GM, \quad (3.82)$$

and disappear for smaller L . Thus $6GM$ is the smallest possible radius of a stable circular orbit in the Schwarzschild metric [Wal84; Pee14]. Hence the name “inner most stable circular orbit”, and the sign r_{ISCO} . We have therefore found that the Schwarzschild

³ $(1+x)^\alpha \sim 1 + \alpha x$ as $x \rightarrow 0$.

3 Solutions of Einstein's equations

solution possesses stable circular orbits for $r > 6GM$ and unstable circular orbits for $3GM < r < 6GM$. It's important to remember that these are only the geodesics; there is nothing to stop an accelerating particle from dipping below $r = 3GM$ and emerging, as long as it stays beyond $r = r_s$.

3.4.5 A full example

We now present a full example of motion of a mass m around a central body of mass M , with $m \ll M$. We examine first the classical Newtonian case and then the relativistic one.

Newton

The starting point is the well known second law of dynamics

$$\mathbf{F} = m\mathbf{a}$$

where \mathbf{F} is the gravitational force. If we denote by x and y the coordinates in the plane of motion, we have the following equations

$$\begin{cases} \ddot{x} = -GM\frac{x}{r^3} \\ \ddot{y} = -GM\frac{y}{r^3}. \end{cases} \quad (3.83)$$

with $r = \sqrt{x^2 + y^2}$, and the dot stands for time derivative. In order to try to simplify these equations, we switch to polar coordinates (r, θ) , so that

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \end{cases} \quad (3.84)$$

We know that the angular momentum $\mathbf{L} = m\mathbf{r} \wedge \mathbf{v}$ is a conserved quantity and, from the decomposition $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta$, $L = mr v_\theta$. Now, since $v_\theta = r\dot{\theta}$,

$$0 = \frac{dL}{dt} = m \frac{d(r^2\dot{\theta})}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta}.$$

We have obtained the second equation of (3.84). So, we can rewrite that system as

$$\begin{cases} \ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{GM}{r^2} \\ \dot{\theta} = \frac{L}{mr^2}. \end{cases} \quad (3.85)$$

These are the equations that describe the trajectory of the mass m around M . To eliminate time and express r as a function of the angle θ , we proceed as follows: first we perform the substitution

$$u = \frac{1}{r}, \quad (3.86)$$

so that

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \frac{d\theta}{dt}.$$

We know that $\dot{\theta} = L/mr^2$, hence

$$\frac{dr}{dt} = \frac{L}{m} \frac{du}{d\theta}.$$

Differentiating this last expression gives

$$\frac{d^2r}{dt^2} = \left(\frac{L}{m}\right)^2 u^2 \frac{d^2u}{d\theta^2},$$

and finally

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}. \quad (3.87)$$

The solution of this equation, if r_0 and v_0 are initial radius and velocity, is [MN98]

$$\frac{1}{r} = \frac{GMm^2}{L^2} (1 + e \cos \theta), \quad (3.88)$$

where

$$e = \left(1 + \frac{2EL^2}{m^3 G^2 M^2}\right)^{\frac{1}{2}} \quad (3.89)$$

is the **eccentricity**. If e is less than 1, the orbit is an ellipse; if it is equal to 1, the orbit is a parabola; if it is greater than 1, the orbit is an hyperbola. This is exactly what we know from standard theory of gravitation.

Schwarzschild

The starting point to derive an equation similar to eq. (3.87) is eq. (3.72) which becomes, with the hypothesis of $m \ll M$ and setting $c = 1$,

$$\frac{E^2}{1 - 2GM/r} - \frac{\dot{r}^2}{1 - 2GM/r} - \frac{L^2}{r^2} = 1.$$

Performing the same substitution as above and repeating the same steps, we get the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} + 3GMm^2u^2. \quad (3.90)$$

The last term is the relativistic correction to eq. (3.87). It is responsible for one of the classical tests of general relativity: the *perihelion procession* of Mercury. This phenomenon was already known before Einstein formulated his theory. Indeed Urbain Jean Joseph Le Verrier, a French mathematician and astronomer, in 1846 discovered the planet Neptune based on anomalies in the orbits of the other planets in the solar system. In 1859 he also noticed that Mercury had such an anomaly, so he thought that there could be an unobserved planet, Vulcano, near Mercury, that influences its orbit. Of course Vulcano was never found, and only in 1915 when Einstein published his work, the perihelion precession

3 Solutions of Einstein's equations

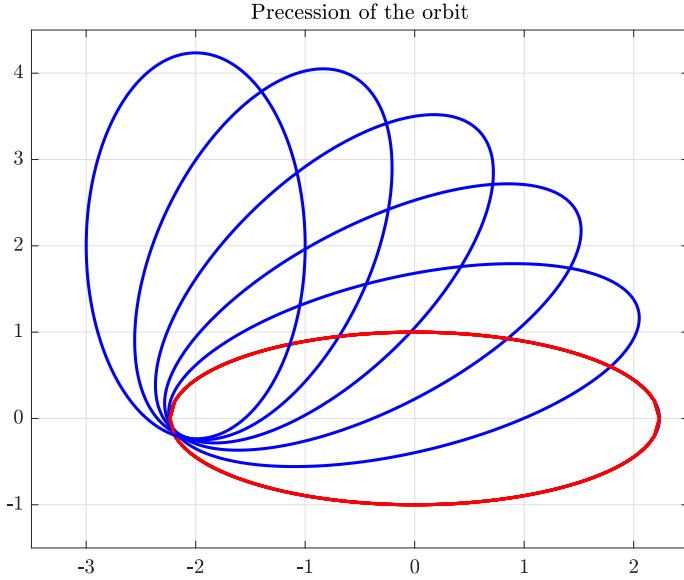


Figure 3.9: Precession of Mercury's orbit (not in scale). Red line is the Newtonian one, blue lines represent the relativistic solution. The Sun is placed at coordinates $(-2, 0)$. Note that, in order to have a precession like the one shown, it takes approximately 757000 years.

could be explained. It can be shown [Mag08] that the perihelion is not fixed but it precesses gradually by an angle

$$\delta\varphi = \frac{6\pi GM}{c^2 a(1 - e^2)}, \quad (3.91)$$

where a is semi-major axis of the orbit. The numerical values of the parameters, in case of Mercury, are:

$$a = 5.8 \cdot 10^{12} \text{ cm} \quad e = -0.206 \quad M = 2 \cdot 10^{33} \text{ g}.$$

The resulting angle of precession is then

$$\delta\varphi = 5 \cdot 10^{-7} \text{ radian/revolution} = 0.103''/\text{revolution}.$$

Knowing that there are 415 revolutions per century we conclude that the advance of the perihelion amounts to

$$\delta\varphi = 43'' \text{ per century.}$$

Chapter **4**

Initial Value Formulation

In the previous chapter we have been able to solve exactly Einstein's equation for a simple form of the energy-momentum tensor T_{ij} . However, some peculiarities already arose: the simple dust and radiation models led to singularities in spacetime, unless we modify the model as we did in section 3.3. Furthermore, these singularities are physical or a consequence of a high degree of symmetry of the solutions? Einstein himself faced this problem and though that singularities would not appear in less symmetric solutions. Thus, he proposed a model of a static, spherically symmetric universe, which is slightly different from the models discussed above, since Einstein had to introduce a *cosmological constant* Λ in order to ensure a static solution. The governing equation, therefore, are

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = \frac{8\pi G}{c^4}T_{ij}. \quad (4.1)$$

Unfortunately, the solution he advocated was unstable, hence not a reasonable model of the universe [Soa12].

In the 50's and 60's, Hawking and Penrose shed important light on the issue of singularities: they proved that spacetimes generally exhibit singularities, but in the sense of *geodesic incompleteness* [HP70]. There are, in fact, solutions which have singularities in the sense of geodesic incompleteness, but which do not have curvature singularities like the ones arising from eqs. (3.14) and (3.15). An example of this kind of a solution is

$$g = -dt^2 + t^2dx^2 + dy^2 + dz^2$$

on $(0, \infty) \times \mathbb{R}^3$. This solution is geodesic incomplete at $t = 0$, but has Riemann tensor that is identically zero.

In parallel with the studies of singularities, the idea of formulating Einstein's equation as an initial value problem was developed. The groundbreaking result in those questions was made by Yvonne Choquet- Bruhat in [Cho52], who proved local existence and uniqueness for Einstein Field Equations in the vacuum, when initial data was given as $(\Sigma; \gamma; K)$, Σ a spacelike hypersurface of \mathcal{M} , γ a Riemannian metric over Σ and K a symmetric bilinear form over Σ that plays the role of the second fundamental form in the final solution. The result holds only when γ and K together satisfy the so called constraint equations, a

4 Initial Value Formulation

geometric condition between those quantities and T that emerge from the Gauss-Codazzi equations. In the following we are going to develop these ideas based on the works in [CGP10] and [Han14].

4.1 The Cauchy problem

Let us consider vacuum Einstein's equations without any symmetry assumption with cosmological constant Λ :

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = 0 \quad (4.2)$$

regardless of whether or not Λ vanishes.

Given a manifold \mathcal{M} , eq. (4.2) forms a system of *second order, quasi-linear partial differential equations* for the metric g . In the evolutionary point of view, which is the one adopted in this work, all spacetimes of main interest have topology $I \times \Sigma$, where Σ is a 3-dimensional manifold carrying the initial data and $I \subset \mathbb{R}$; namely $\mathcal{M} = \mathbb{R} \times \Sigma$. Such spacetimes are called **globally hyperbolic spacetimes**.

There exist standard classes of PDEs which admit existence and uniqueness results, such as elliptic, hyperbolic or parabolic equations. Unfortunately, eq. (4.2) does not fall in any of these. This is not surprising, since, if we are able to find a solution $g(x)$ in some coordinate system x^i , then we can perform a change of variables $x^i \rightarrow y^j(x^i)$ and

$$\bar{g}_{kl}(y) = g_{ij}(x(y)) \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l}$$

is also a solution. This property is known as *diffeomorphism invariance*. As mentioned above a local existence result was proven by Chouquet-Bruhat, which proceeds by the introduction of so called *harmonic coordinates*.

4.1.1 Local evolution problem

Definition 4.1. A set of coordinates $\{y^i\}$ on an n -dimensional manifold \mathcal{M} is called **harmonic** if each of the functions y^i satisfies

$$\square y^i = 0. \quad (4.3)$$

Assuming eq. (4.3) holds we can rewrite eq. (4.2), taking its trace and then substituting $R = \frac{2(n+1)}{n-1}\Lambda$, as

$$0 = \hat{E}^{ij} := \square g^{ij} - g^{pq} \left(2g^{kl} \Gamma_{kp}^i \Gamma_{lq}^j + (g^{ik} \Gamma_{kl}^j + g^{jk} \Gamma_{kl}^i) \Gamma_{pq}^l \right) - \frac{4\Lambda}{n-1} g^{ij} \quad (4.4)$$

which is a system of quasi linear wave equations. We have established the hyperbolic evolutionary character of the Einstein equations, if not for the following problem: given an initial data for an equation as in eq. (4.4), there exists a unique solution, at least for some short time; this is a non trivial generalization of the following existence and uniqueness theorem for hyperbolic PDEs [Eva98].

Theorem 4.1. *For any initial data*

$$g^{ij}(0, y^k) \in H_{loc}^{s+1}, \quad \partial_0 g^{ij}(0, y^k) \in H_{loc}^s, \quad s > n/2, \quad (4.5)$$

prescribed on an open subset $\mathcal{O} \subset \{0\} \times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution g^{ij} of eq. (4.4) defined on an open neighborhood \mathcal{U} of \mathcal{O} .

Remark 4.1. The Sobolev space H_{loc}^k is defined, if $k \in \mathbb{N}$, as the space of functions which are in $L^2(K)$ for any compact set K , with their distributional derivatives also in $L^2(K)$ up to order k . For non integer k 's we define $H^k(\Omega)$ as the space of functions u such that $(1 + |\xi|^2)^{k/2}\hat{u} \in L^2(\Omega)$, where \hat{u} is the Fourier transform of u .

However, there is no *a priori* reason to expect that the solution will satisfy also eq. (4.3); if it does not, then a solution of (4.4) will *not* solve the Einstein equations. Indeed, if we set

$$\lambda^i := \square y^i, \quad (4.6)$$

then

$$R^{ij} = \frac{1}{2}(\hat{E}^{ij} - \nabla^i \lambda^j - \nabla^j \lambda^i) + \frac{2\Lambda}{n-1}g^{ij}, \quad (4.7)$$

so that is precisely the vanishing, or not, of λ which decides whether or not a solution of eq. (4.4) is a solution of the vacuum Einstein equations. This is exactly the problem solved by Chouquet-Bruhat in [Cho52]. The key observation is that (4.7) and the Bianchi identities imply a wave equation for λ^i . Indeed, contracting twice the Bianchi identity (1.21), we get

$$\nabla_i \left(R^{ij} - \frac{R}{2}g^{ij} \right) = 0. \quad (4.8)$$

Assuming eq. (4.7) holds, one finds

$$0 = -\nabla_i(\nabla^i \lambda^j + \nabla^j \lambda_i - \nabla_k \lambda^k g^{ij}) \quad (4.9)$$

$$= -(\square \lambda^j + R_i^j \lambda^i). \quad (4.10)$$

This shows that λ^i necessarily satisfies the second order hyperbolic system of equations

$$\square \lambda^j + R_i^j \lambda^i = 0. \quad (4.11)$$

It is well known that we will have $\lambda^i = 0$ provided that both λ^i and its derivatives vanishes on \mathcal{O} . In order to prove that both conditions are actually satisfied we need to develop some more geometry.

4.2 Geometric Setting

Remark 4.2. In this section we will use the notation \mathcal{R} for shortly denote the Riemann tensor.

4 Initial Value Formulation

4.2.1 Geometry of global hyperbolic spacetimes

Definition 4.2. Let (\mathcal{M}, g) be a spacetime, Σ an embedded spacelike hypersurface with normal vector field n , then the **second fundamental form** is defined as

$$K(X, Y) := g(\nabla_X n, Y) \quad X, Y \in \mathcal{X}(\Sigma)$$

where ∇ is the affine connection over \mathcal{M} .

Since Σ is embedded in \mathcal{M} , it has an induced Riemannian metric h (i.e. with positive definite signature) and an induced connection ∇^Σ such that for every $X, Y \in T\Sigma$

$$\nabla_X Y = \nabla_X^\Sigma Y + K(X, Y)N.$$

Remark 4.3. The geometric quantities related to ∇^Σ will be denoted by the superscript, while quantities without superscript refers to the connection on \mathcal{M} .

Now, we are interested in spacetime manifolds that can be written as $\mathcal{M} = \Sigma \times I$, $I \subset \mathbb{R}$, so if we define local coordinates $\{\mathbf{x}\}$ over $U \subset \Sigma$, they can be extended to (\mathbf{x}, t) over $U \times I \subset \mathcal{M}$ as local coordinates on \mathcal{M} . We denote by $\{\partial_t, \partial_i\}_{i \in \{1, 2, 3\}}$ the corresponding coordinate basis. ∂_t can be decomposed over Σ_t as [CGP10]

$$\partial_t = N_t n_t + X_t$$

with N_t a scalar field and X_t a tangential vector field. N_t is called *lapse function* and X_t is called *shift vector*.

Proposition 4.1. In terms of h, N and X the metric g can be expressed as

$$g = -N^2 dt^2 + h_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)$$

and the second fundamental form is given by

$$k_{ij} = K(\partial_i, \partial_j) = \frac{1}{2} N^{-1} \left(\frac{\partial h_{ij}}{\partial t} - \mathcal{L}_X h_{ij} \right)$$

where $\mathcal{L}_X h_{ij}$ is the Lie derivative of the spatial metric h along the direction X .

Remark 4.4. The Lie derivative is defined as follows [Aba11]: let \mathcal{M} be a differential manifold, $X \in T\mathcal{M}$ a vector field on \mathcal{M} and $\tau \in T_p M$. Then

$$(\mathcal{L}_X \tau)(p) = \left. \frac{d}{dt} \theta_t^*(p) \right|_{t=0},$$

for every $p \in \mathcal{M}$ and θ is the local flux of X .

The last proposition gives us a formula for the time derivative of the spatial metric

$$\partial_t h_{ij} = 2N k_{ij} + \mathcal{L}_X h_{ij} \tag{4.12}$$

so, in the special case when $N = 1$ and $X = 0$ (i.e. $\partial_t = n$), we have

$$\partial_t h_{ij} = 2k_{ij}. \tag{4.13}$$

This last relation will be used in order to determine the initial data of the Cauchy problem.

4.2.2 Geometric constraints: Gauss-Codazzi equations

The second fundamental form presents a useful way to relate \mathcal{R} restricted to $T\Sigma$ to \mathcal{R}^Σ . This link is made through the Gauss-Codazzi equations [Wal84]. The Gauss equation (tangent part) is:

$$\mathcal{R}(X, Y, Z, W) = \mathcal{R}^\Sigma(X, Y, Z, W) + K(X, W)K(Y, Z) - K(X, Z)K(Y, W) \quad (4.14)$$

and the Codazzi equation (normal part):

$$\mathcal{R}(X, Y, n, Z) = \nabla_X(K(Y, Z)) - \nabla_Y(K(X, Z)) \quad (4.15)$$

where in both cases, $X, Y, Z, W \in T\Sigma$.

If we choose a local chart around a point $p \in \Sigma$ such that the 0^{th} coordinate vector is normal to Σ and the others are tangent to Σ , we can write the two equations in coordinates:

$$R_{ijpl} = R_{ijpl}^\Sigma + k_{il}k_{jp} - k_{ip}k_{jl} \quad (4.16)$$

$$R_{ij0p}^\Sigma = \nabla_i k_{jp} - \nabla_j k_{ip}. \quad (4.17)$$

With those two equations it is possible to find an expression for the Einstein tensor over Σ as a function of K , and thus to relate the matter information (T) and the geometric information (metric and second fundamental form) over Σ . Those will be *constraint equations*. We only need to compute the Ricci tensor and the scalar curvature over Σ . In the following calculation latin indexes range from 1 to 3 while greek indexes range from 0 to 3:

$$\begin{aligned} R_{ij} &= R_{i\alpha j}^\alpha = R_{i0j}^0 + (R_{ikj}^k)^\Sigma + k_p^p k_{ij} - k_j^p k_{ip} \\ &= R_{i0j}^0 + R_{ij}^\Sigma + k_p^p k_{ij} - k_j^p k_{ip} \\ R_{0i} &= R_{i\alpha 0}^\alpha = R_{i00}^0 + \nabla_j k_i^j - \nabla_i k_j^j = \nabla_j k_i^j - \nabla_i k_j^j \end{aligned}$$

thus

$$\begin{aligned} R &= R_\alpha^\alpha = R_0^0 + R_{0i}^{0i} + (R_i^i)^\Sigma + k_p^p K_i^i - K_i^p K_p^i \\ &= R_0^0 + R_{0i}^{0i} + R^\Sigma + (k_i^i)^2 - k_i^p k_p^i \\ &= g^{0\beta} (R_{\beta 0} + R_{\beta 0i}^i) + R^\Sigma + (k_i^i)^2 - k_i^p k_p^i \end{aligned}$$

Now, with our choice of chart, $g_{0i} = 0$ if $i \neq 0$ and hence $g_{00}g^{00} = 1$. So

$$\begin{aligned} R &= g_{00}^{-1} (R_{00} + R_{00i}^i) + R^\Sigma + (k_i^i)^2 - k_i^p k_p^i \\ &= 2g_{00}^{-1} R_{00} + R^\Sigma + (k_i^i)^2 - k_i^p k_p^i. \end{aligned}$$

We can, therefore, replace these expression in the Einstein tensor to get

$$G_{00} = -\frac{1}{2} g_{00} (R^\Sigma + (k_i^i)^2 - k_i^p k_p^i)$$

and

$$G_{0i} = \nabla_j k_i^j - \nabla_i k_j^j.$$

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Although these expression are quite general, we are interested in manifolds describing spacetimes, hence we can choose $g_{00} = -1$ and demand that the metric satisfies the Einstein equations. We get the so called **Einstein constraint equations**

$$\begin{cases} 2G_{00} = R^\Sigma + (trK)^2 - \|K\|^2 = 2T_{00} \\ G_{0i} = \nabla_j k_i^j - \nabla_i k_j^j = T_{0i} \end{cases} \quad (4.18)$$

where we have normalized all constants to 1. The first equation is usually called *Hamiltonian constraint*, while the vector relation given by the second expression is usually called *Momentum constraint*.

4.3 Well posedness of the Initial Value Problem

Definition 4.3. An *initial data set* for the $(n+1)$ -dimensional vacuum Einstein's equations is a set (Σ, h, K) where (Σ, h) is a n -dimensional Riemannian manifold and K is a symmetric $(0, 2)$ tensor on Σ .

The symmetric $(0, 2)$ tensor we are looking for is exactly the second fundamental form, so we need to specify the Cauchy data for eq. (4.7) in terms of g_{ij} . First we define¹

$$g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & h_{ij} \end{bmatrix} \quad \text{at} \quad t = 0$$

which forces, from eq. (4.13),

$$\partial_t g_{ij} = 2k_{ij}.$$

The knowledge of g_{ij} and k_{ij} at $t = 0$ allows us to calculate $\partial_t g_{ij}$; thus k_{ij} can be seen as the geometric counterpart of $\partial_t g_{ij}$.

We are still free to choose $\partial_t g_{0\alpha}$. We will do this so that

$$\lambda_\alpha = 0 \quad \text{initially on } \Sigma.$$

The last thing we need to ensure $\lambda_\alpha = 0$ identically is $\partial_t \lambda_\alpha = 0$ at $t = 0$.

Proposition 4.2. The vacuum constraint equations for (Σ, h, K) imply that

$$\frac{\partial \lambda_\alpha}{\partial t} = 0.$$

Proof. The momentum constraint equation

$$G_{0i} = \text{div}K - \nabla(trK) = 0$$

implies that

$$-\frac{1}{2}\nabla_i \lambda_0 - \frac{1}{2}\frac{\partial \lambda_i}{\partial t} = 0.$$

¹Once again latin indexes range from 1 to 3 and greek indexes range from 0 to 3.

4.3 Well posedness of the Initial Value Problem

However we have $\nabla_i \lambda_0 = 0$ on Σ so that

$$\frac{\partial \lambda_i}{\partial t} = 0.$$

The Hamiltonian constraint equation gives

$$G_{00} = R^\Sigma - \|K\|^2 + (tr K)^2 = 0$$

and this shows that

$$\begin{aligned} G_{00} &= -\frac{\partial \lambda_0}{\partial t} - \frac{1}{2} \frac{\partial \lambda_0}{\partial t} g_{00} \\ &= -\frac{\partial \lambda_0}{\partial t} + \frac{1}{2} \frac{\partial \lambda_0}{\partial t} \\ &= -\frac{\partial \lambda_0}{\partial t} = 0 \end{aligned}$$

as desired. \square

Therefore $\partial_t \lambda_\alpha = 0$ so that $\lambda_\alpha = 0$ identically. We can now state the main result.

Theorem 4.2 (Choquet-Bruhat, 1952). *Given an initial data set (Σ, h, K) satisfying the vacuum constraint equations, there exists a spacetime (\mathcal{M}, g) satisfying the vacuum Einstein equations where Σ is an embedded spacelike surface with induced metric h and second fundamental form K .*

This is a local existence result. Few years later this theorem was extended.

Theorem 4.3 (Choquet-Bruhat-Georgi, 1969). *Given an initial data set (Σ, h, K) satisfying the vacuum constraint equations, there exist a unique, globally hyperbolic, maximal spacetime (\mathcal{M}, g) satisfying the vacuum Einstein equations where Σ is an embedded spacelike surface with induced metric h and second fundamental form K . Moreover, any other such solution is a subset of (\mathcal{M}, g) .*

This is a much more satisfactory result, but it still leaves open the most difficult questions concerning global existence.

As pointed already, in both of these results, a central role is played by the existence of initial data sets solving the Einstein constraint equations.

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