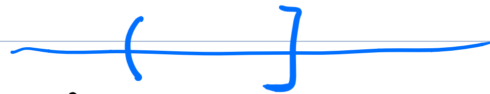
Define notions of certain sets.

$P(\Omega) \supseteq \mathcal{J}$

$\mathcal{J}$ is a semi-algebra if

(1) $\Omega \in \mathcal{J}$



(2) $A, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$

(3) $\forall A \in \mathcal{J} \Rightarrow A^c = \sum E_j$ (finite disjoint)

$\exists\ E_1, \ldots, E_n \in \mathcal{J}$

Example:

$\Omega = \mathbb{R}$. , $\{(a, b],\ a<b,\ a, b \in \mathbb{R}\}$

$\{(-\infty, b],\ b \in \mathbb{R}\}$

$\{(a, +\infty),\ a \in \mathbb{R}\}$, ϕ.

$\mathcal{J}$ is semi-algbra.

Def. $\mathcal{A} \subseteq P(\Omega)$ is an algebra if:

(1) $\Omega \in \mathcal{A}$

(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

(3) If $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

Obs if $\mathcal{A}$ is an algebra $\Rightarrow \mathcal{A}$ is a semi-algebra

Def. σ-algebra $\mathcal{F} \subseteq P(\Omega)$

(1) $\Omega \in \mathcal{F}$

(2) $A_j \in \mathcal{F} \Rightarrow \bigcap A_j \in \mathcal{F}$

(3) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

obs 1 Ω, $\mathcal{A}_\alpha \subseteq P(\Omega)$ $\mathcal{A}_\alpha$ is algebra
$\alpha \in I$

$\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is an algebra.

Pf: 1. $\Omega \in \mathcal{A}$

$\Omega \in \mathcal{A}_\alpha \in \mathcal{A}$ ✓

2. $A, B \in \mathcal{A}$

$A \in \mathcal{A}$, $B \in \mathcal{A}$, $(A \cap B) \in \mathcal{A}$

3. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

obs 2 $\mathcal{A}_\alpha$ σ-algebra

$\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

(1) ✓

(3) ✓

(2) $A_j \in \mathcal{A}$, $\bigcap A_j \in \mathcal{A}$

Ω $\quad \mathcal{C} \subseteq \mathcal{P}(\Omega)$

$\mathcal{A}(\mathcal{C})$ $\quad$ * $\mathcal{C} \subseteq \mathcal{A}$

$$* \left\{ \begin{array}{l} B \supseteq C \\ B \text{ algebra} \end{array} \right. \Rightarrow B \supseteq \mathcal{A}$$

algebra generated by class $\mathcal{C}$

$\mathcal{A}(\mathcal{C})$ smallest algebra.

Show existence.

$\mathcal{A} = \bigcap_\alpha \mathcal{A}_\alpha$ is an algebra.

Lemma

Ω, $\mathcal{S}$ is a semi-algebra.

$\mathcal{S} \subseteq \mathcal{P}(\Omega)$

$\mathcal{A}(\mathcal{S})$ algebra gen by $\mathcal{S}$

$A \in \mathcal{A}(\mathcal{S}) \Leftrightarrow \exists E_j \quad 1 \le j \le n \quad E_j \in \mathcal{S}$

$A = \sum E_j$.

(1) $\Leftarrow$

$A = \sum E_j \quad E_j \in \mathcal{S}$

$\cdot)$ $E, F \in \mathcal{A} \Rightarrow E \cup F \in \mathcal{A}$

$E \cup F = (E^c \cap F^c)^c$

DEF $\mathcal{C} \subseteq P(\Omega)$

$\emptyset \in \mathcal{C}$

Finite

$\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$

μ is <u>additive</u> if

1) $\mu(\phi) = 0$

2) $E_1, \dots, E_n \in \mathcal{C}$
$E = \sum E_j \in \mathcal{C}$ $\Rightarrow \mu(E) = \sum \mu(E_j)$

obs $\exists A \in \mathcal{C}$ s.t. $\mu(A) < \infty$.

$A = A \cup \phi$

$\mu(A) = \mu(A) + \mu(\phi)$

$\therefore \mu(\phi) = 0$

if $\exists \mu(A) < \infty$, A finite. 1) is trival

$\mathcal{C}$, $\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$ add.

obs 2 $E \subseteq F$. , $F \setminus E \in \mathcal{C}$, $E, F \in \mathcal{C}$.

1) $\mu(E) = +\infty$ $F = E \cup (F \setminus E)$

$\Rightarrow \mu(F) = +\infty$ $\mu(F) = \mu(E) + \mu(F \setminus E)$

2) $\mu(E) < \infty \Rightarrow \mu(F \setminus E) = \mu(F) - \mu(E)$

$\mu(E) \leq \mu(F)$

Ex 1. Discrete Measure

$\Omega \quad \{x_j \quad j\geq 1\} \quad x_j \in \Omega.$

$\mathcal{C} \subseteq \mathcal{P}(\Omega) \quad \{P_j \quad j \geq 1\} \quad P_j \geq 0$

$$\mu(A) = \sum_j P_j \mathbb{1}\{x_j \in A\}$$

μ is additive.

<u>DEF</u> $\mathcal{C} \subseteq \mathcal{P}(\Omega)$

$\phi \in \mathcal{C}$ Countable

$\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$

μ σ-additive if:

1) $\mu(\phi) = 0$

2) $E_j \in \mathcal{C} \quad E_j \cap E_k = \phi \ (j \neq k)$

$E = \sum_{j\geq 1} E_j \in \mathcal{C} \qquad \mu(E) = \sum_{j\geq 1} \mu(E_j)$

Ex 2. $\Omega = (0,1)$

$\mathcal{C} = \{(a,b] \quad 0 \le a < b < 1\}$

$\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$

$$\mu(a,b] = \begin{cases} +\infty & , \quad a = 0 \\ b-a & , \quad a > 0 \end{cases}$$