Goal of Measure Theory:

(0) $\lambda : P(\mathbb{R}) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$

(1) $\lambda\big((a,b]\big) = b-a$

(2) $A \subseteq \mathbb{R}$ $\qquad A+x = \{x+y, \ y \in A\}$

$\forall A \subseteq \mathbb{R}, \forall x \in \mathbb{R}, \ \lambda(A) = \lambda(A+x)$

(3) $A = \bigcup A_j$, $\quad A_j \cap A_k = \phi$.

$\lambda(A) = \sum \lambda(A_j)$

$x \sim y \quad x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$

$[x] = \{ y \in \mathbb{R} \text{ s.t. } y - x \in \mathbb{Q} \}$

$\Lambda = \mathbb{R}/\sim \qquad \alpha, \beta.$ $\quad$ Λ is not countable

$\Omega \subseteq \mathbb{R} \qquad \Omega \subseteq (0,1)$

Claim $\begin{cases} \Omega + q = \Omega + p, & q, p \in \mathbb{Q} \\ (\Omega + q) \cap (\Omega + p) = \phi. \end{cases}$

①

①

Assume $(\Omega + p) \cap (\Omega + q) \neq \emptyset$.

$x = \alpha + p, \quad \alpha \in \Omega$.

$\quad = \beta + q, \quad \beta \in \Omega$.

$\alpha - \beta = q - p \Rightarrow \alpha = \beta. \Rightarrow q = p$.

② $q \neq p \Rightarrow (\Omega + q) \cap (\Omega + p) = \emptyset$

$$\lambda\Big(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\Big)$$

Remark:

$\Omega + q \subseteq (-1, 2) \Rightarrow \sum (\Omega + q) \subseteq (-1, 2) = 3$

$$\lambda\Big(\sum (\Omega + q)\Big) \leq \lambda\big((-1, 2)\big)$$

$E \subseteq F \Rightarrow \lambda(E) \leq \lambda(F)$

Pf. $F = E \cup (F \setminus E)$

$\lambda(F) = \lambda(E \cup (F \setminus E))$

$\qquad = \lambda(E) + \lambda(F \setminus E)$

$\therefore \lambda(F) \geq \lambda(E)$

By (3).

$$\lambda\Big(\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}}(\Omega+q)\Big)=\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}}\lambda(q+\Omega)\leq 3$$

By (2) $\lambda((q+\Omega))=\lambda(\Omega)=0$

Claim $(0,1)\subseteq\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}}(\Omega+q)$

$$1\leq\lambda\Big(\sum_{q}(\Omega+q)\Big)$$

Some subsets of $\mathbb{R}$ are non-measurable.