Announcements

- HW1 due this Wed by midnight, submit on Gradescope
- Update in HW due date policy (see Canvas) — two 24-hr bonus extensions through the quarter (but lowest score no longer dropped)
- Midterm date: Fri Feb 16th, in-class, closed-book (but w/ cheat sheet)
- Maryam's OH today 1:30–2:50 pm, CSE 230
- Natalia's OH Tuesdays 4–5:30 pm, AERB 130
- Read along from book, ask Q's, start HW early!

Discussion Board

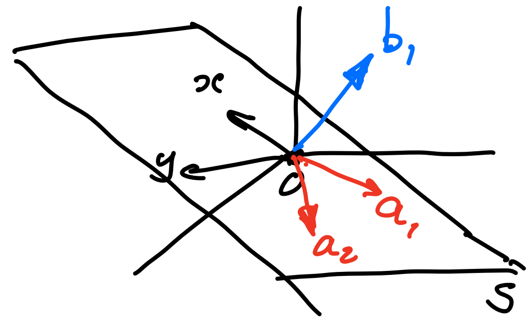
2. Convex sets (chapter 2 of B& V)

- subspaces, affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Subspaces

$S \subseteq \mathbf{R}^n$ is a *subspace* if for $x, y \in S, \quad \lambda, \mu \in \mathbf{R} \quad \Longrightarrow \lambda x + \mu y \in S$

geometrically: $x, y \in S \Rightarrow$ plane through $0, x, y \subseteq S$



representations

$$\begin{aligned} \text{range}(A) &= \{Aw \mid w \in \mathbf{R}^q\} \\ &= \{w_1 a_1 + \cdots + w_q a_q \mid w_i \in \mathbf{R}\} \\ &= \text{span}(a_1, a_2, \ldots, a_q) \end{aligned}$$

$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

where $A = \begin{bmatrix} a_1 & \cdots & a_q \end{bmatrix}$; and

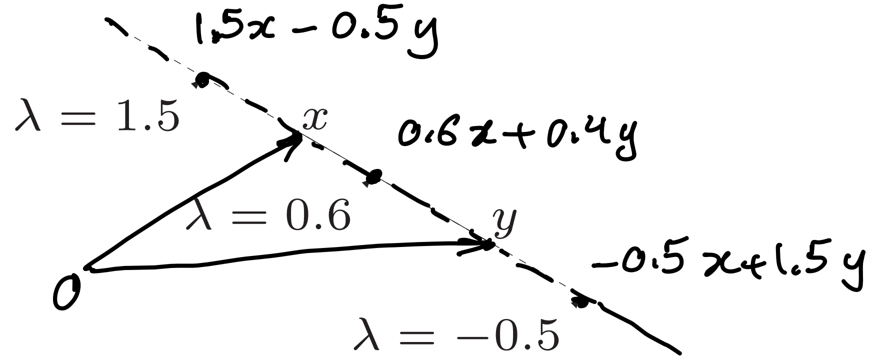
$$\begin{aligned} \text{nullspace}(B) &= \{x \mid Bx = 0\} \\ &= \{x \mid b_1^T x = 0, \ldots, b_p^T x = 0\} \end{aligned}$$

where $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}$

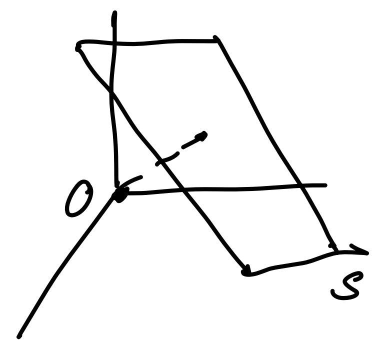
Affine sets

$S \subseteq \mathbf{R}^n$ is *affine* if for $x, y \in S$, $\lambda, \mu \in \mathbf{R}$, $\lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$

geometrically: $x, y \in S \Rightarrow$ line through $x, y \subseteq S$



representations: range of affine function



$$S = \{Az + b \mid z \in \mathbf{R}^q\}$$

via linear equalities

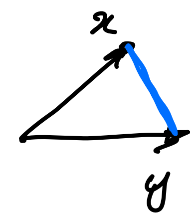
$$\begin{aligned} S &= \{x \mid b_1^T x = d_1, \ldots, b_p^T x = d_p\} \\ &= \{x \mid Bx = d\} \end{aligned}$$

Convex sets

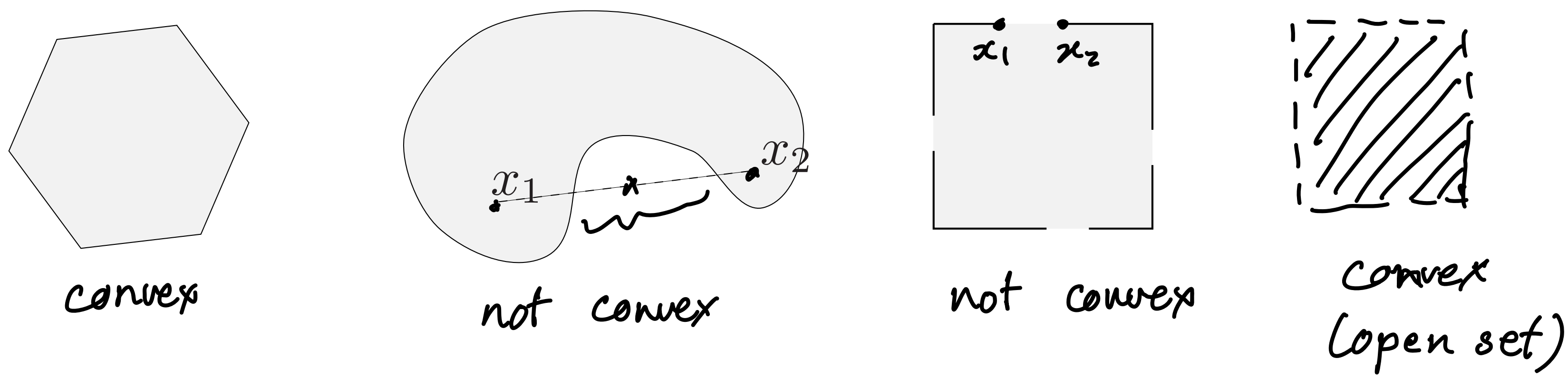
$S \subseteq \mathbf{R}^n$ is a **convex set** if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1 \Longrightarrow \overbrace{\lambda x + \mu y}^{\text{convex combination}} \in S$$

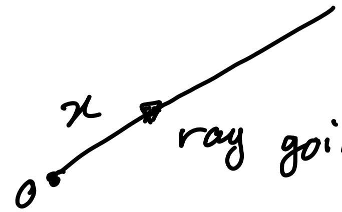
geometrically: $x, y \in S \Rightarrow$ segment $[x, y] \subseteq S$



examples (one convex, two nonconvex sets)



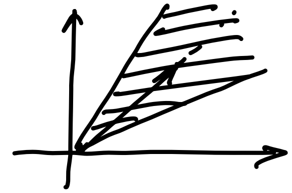
Convex cone



ray going through x

$S \subseteq \mathbf{R}^n$ is a **cone** if

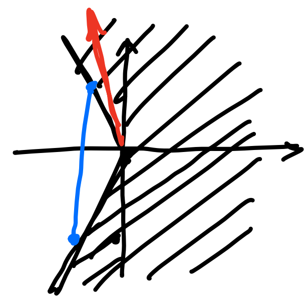
$$x \in S, \quad \lambda \geq 0, \quad \Longrightarrow \lambda x \in S$$



cone

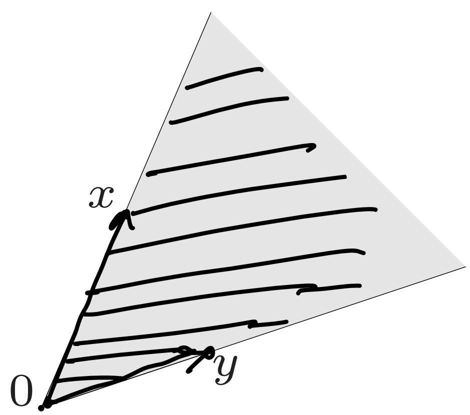
$S \subseteq \mathbf{R}^n$ is a **convex cone** if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \Longrightarrow \lambda x + \mu y \in S$$



cone
not convex

geometrically: $x, y \in S \Rightarrow$ 'pie slice' between $x, y \subseteq S$



conic optimization

Combinations and hulls

$y = \theta_1 x_1 + \cdots + \theta_k x_k$ is a

$x_1, \dots, x_k \in \mathbb{R}^n$ $\quad\theta_1, \dots, \theta_k \in \mathbb{R}$

- *linear* combination of $x_1, \dots, x_k$
- *affine* combination if $\sum_i \theta_i = 1$
- *convex* combination if $\sum_i \theta_i = 1$, $\theta_i \geq 0$
- *conic* combination if $\theta_i \geq 0$

• extends to ∞ number of points:

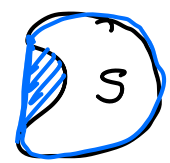
$y = \sum_{i=1}^{\infty} \theta_i x_i$, $\theta_i \geq 0$, $\sum \theta_i = 1$

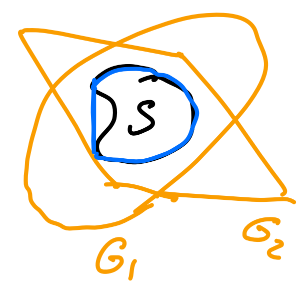
• $y = \int_{x \in K} p(x)\, x \, dx$, $\quad p(x) \geq 0$, $\quad \int_{x \in K} p(x) dx = 1$

• $y = \mathbb{E}_p(x)$

(linear,. . .) **hull** of S:
set of all (linear, . . .) combinations from S

linear hull:	$\mathrm{span}(S)$
affine hull:	$\mathbf{Aff}(S)$
convex hull:	$\mathbf{conv}(S)$
conic hull:	$\mathbf{Cone}(S)$





Theorem:

$$\underbrace{\mathbf{conv}(S)}_{LHS} = \underbrace{\bigcap \{G \mid S \subseteq G,\ G \text{ convex}\}}_{RHS}, \dots$$

proof:

show:

LHS $\subseteq$ RHS & RHS $\subseteq$ LHS

- conv(S) by def is convex & covers S
  conv(S) is one of G's
  $\Rightarrow$ RHS $\subseteq$ conv(S)

- $x, y \in S$ $\quad \lambda x + (1-\lambda) y \in G$ $\quad \forall$ G defined above $\quad \forall \quad 0 \leq \lambda \leq 1$

  $\underbrace{\lambda x + (1-\lambda) y}_{\in \text{conv}(S)}$ $\in$ intersection of all G's $\quad \forall \quad 0 \leq \lambda \leq 1$

  $\Rightarrow$ conv(S) $\subseteq$ RHS. DONE

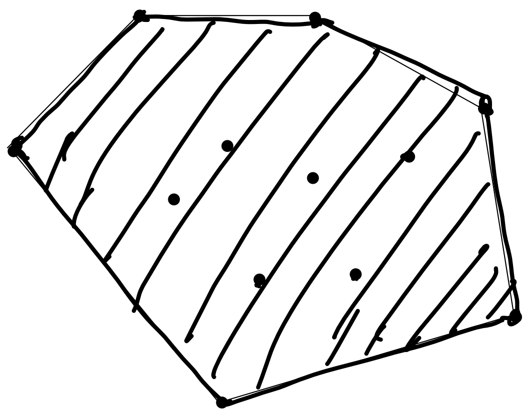
Convex combination and convex hull

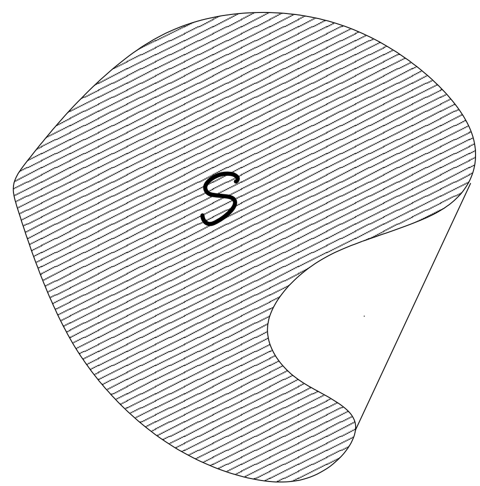
convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S

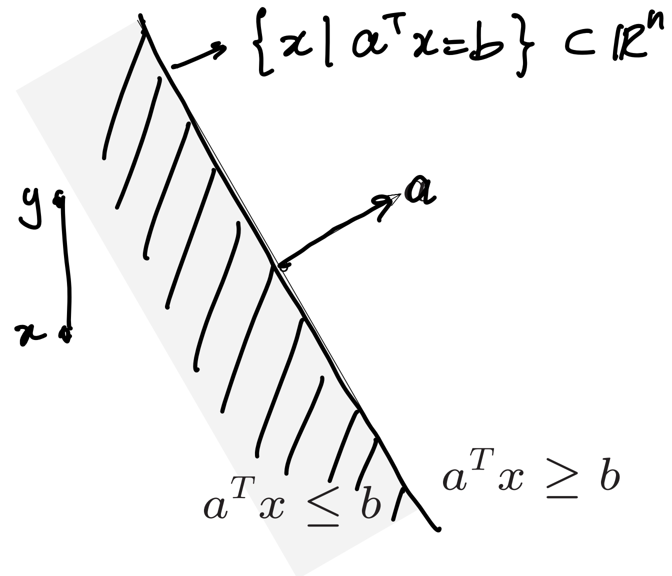




Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$

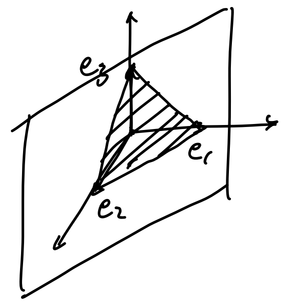


- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

is halfspace an affine set? no.

example:

$S = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$



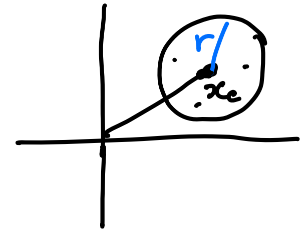
$\text{span}(S) = \mathbb{R}^3$

$\text{aff}(S) =$ plane through e_1, e_2, e_3

$\text{conv}(S) =$ unit simplex (probability simplex)

$\text{cone}(S) = \mathbb{R}^3_+$

Euclidean balls and ellipsoids



(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

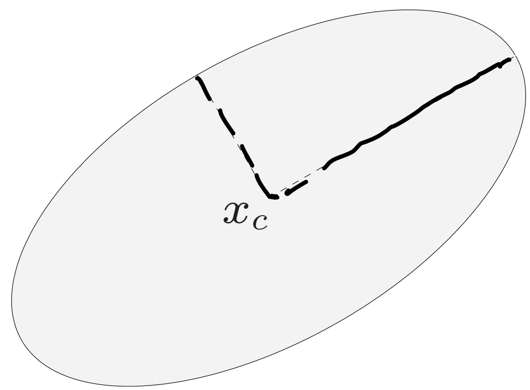
ellipsoid: set of the form $P = r^2 I$ $\quad \{x \mid \frac{1}{r^2}(x - x_c)^T (x - x_c) \le 1\}$

$\|x - x_c\|^2 \le r^2$

one parameterization of ellipsoid $P \in S^n_{++}$, $x_c \in \mathbb{R}^n$ $\rightarrow$

$$\{x \mid (x - x_c)^T \underbrace{P^{-1}}_{\text{eigendecomposition}} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite) $\quad \lambda_i(P) > 0$



$\longrightarrow$ other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle ineq.)

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

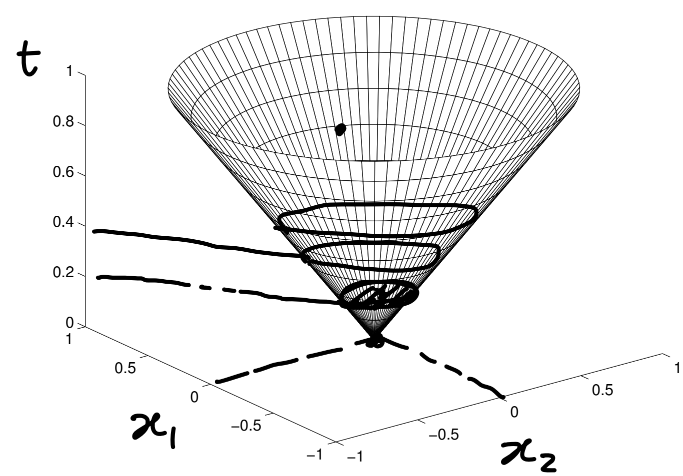
$(x,t) \in \mathbb{R}^{n+1}$

norm cone: $\{(x,t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone $\{(x,t) \mid \|x\|_2 \leq t\}$

ice-cream cone

Lorentz cone



→ norm balls and cones are convex (prove on your own!)

ℓ_p norms

ℓ_p norms on $\mathbf{R}^n$: for $p \geq 1$, $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$,, for $p = \infty$, $\|x\|_\infty = \max_i |x_i|$

- ℓ_2 norm is Euclidean norm $\|x\|_2 = \sqrt{\sum_i x_i^2}$
- ℓ_1 norm is sum-abs-values $\|x\|_1 = \sum_i |x_i|$
- ℓ_∞ norm is max-abs-value $\|x\|_\infty = \max_i |x_i|$

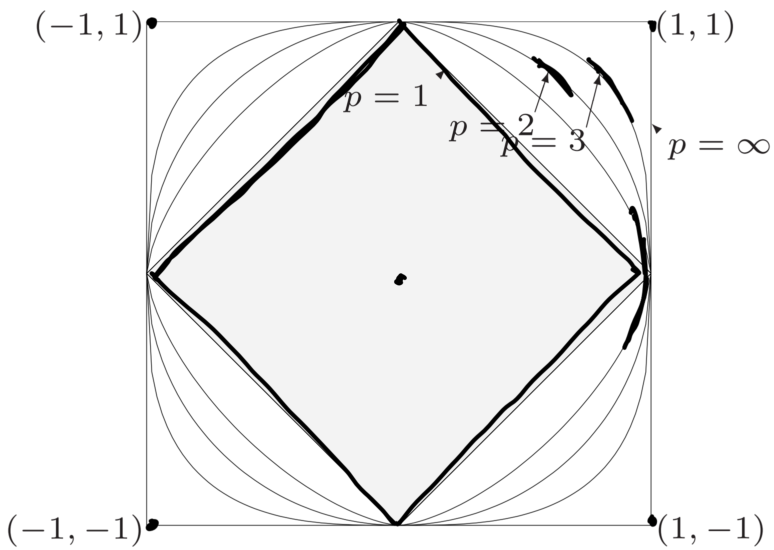
corresponding norm balls (in $\mathbf{R}^2$):

ℓ_1 ball

$\{(x_1, x_2) \mid |x_1| + |x_2| \leq 1\}$

$\{(x_1, x_2) \mid \max_{i=1,2} |x_i| \leq 1\}$

ℓ_∞ ball



Polyhedra

solution set of finitely many linear inequalities and equalities

entrywise

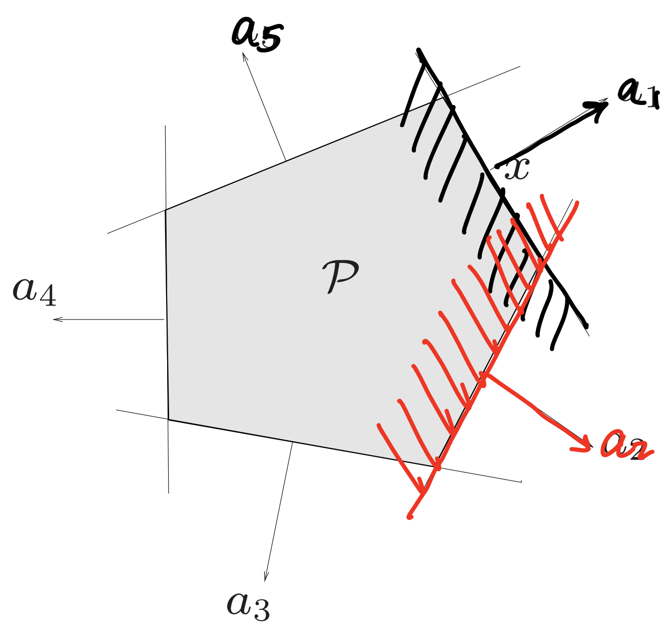
$$Ax \preceq b, \qquad Cx = d$$

$$\begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix} x = \begin{bmatrix} d_1 \\ \vdots \\ d_p \end{bmatrix}$$

$$\begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} x \leq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$a_i^T x \leq b_i$$

($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

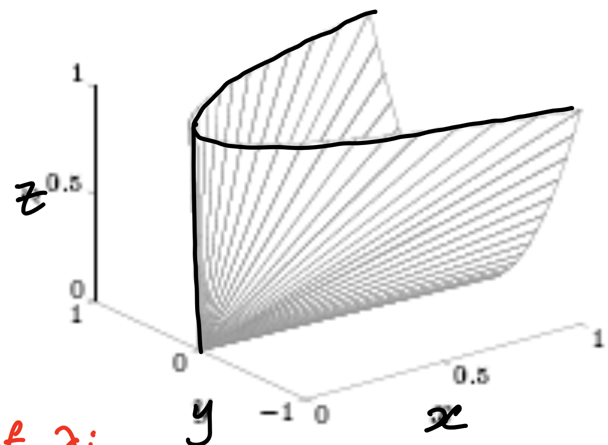
$$X \in \mathbf{S}^n_+ \iff \underbrace{z^T X z}_{\text{a quadratic form with matrix } X} \geq 0 \text{ for all } z \in \mathbb{R}^n \iff \lambda_i(X) \geq 0, \; i=1,\dots,n$$

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices $\iff \lambda_i(X) > 0$

$z^T X z > 0$ for all $z \neq 0 \in \mathbb{R}^n$

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



$$\lambda_i\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) \geq 0 \qquad \begin{cases} x+z \geq 0 \rightarrow \text{sum of } \lambda_i \\ xz - y^2 \geq 0 \rightarrow \text{product of } \lambda_i \\ x \geq 0, \; z \geq 0 \end{cases}$$

- S_+^n is a convex set:

$$\left.\begin{array}{llll} X_1 \in S_+^n & z^T X_1 z \geq 0 & \forall\ z \in \mathbb{R}^n & \times\ \theta \\ X_2 \in S_+^n & z^T X_2 z \geq 0 & \forall\ z \in \mathbb{R}^n & \times\ (1-\theta) \end{array}\right\} z^T \underbrace{\left(\theta X_1 + (1-\theta) X_2\right)}_{\in S_+^n} z \geq 0$$

$$\theta X_1 + (1-\theta) X_2 \overset{?}{\in} S_+^n \quad \forall \quad 0 \leq \theta \leq 1$$

Operations that preserve convexity

practical methods for establishing convexity of a set C

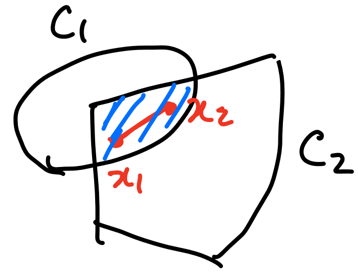
1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \underbrace{\theta x_1 + (1-\theta)x_2} \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection



$C_1 \cap C_2$

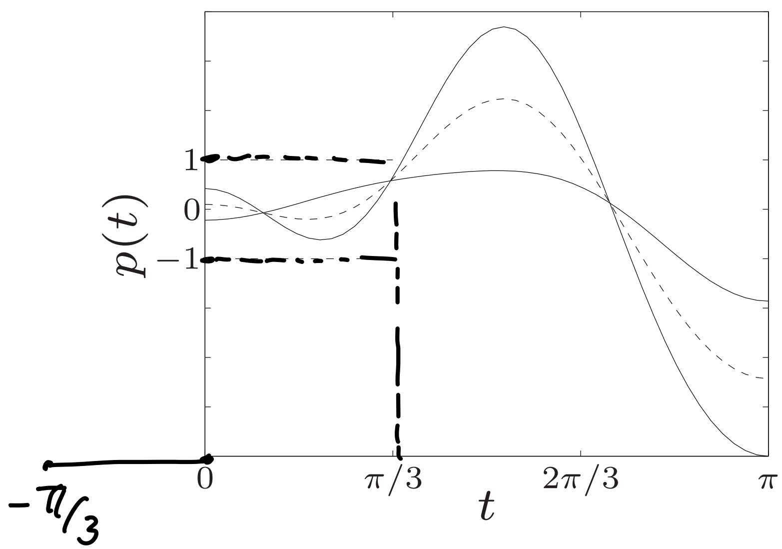
thm: the intersection of (any number of, even infinite) convex sets is convex

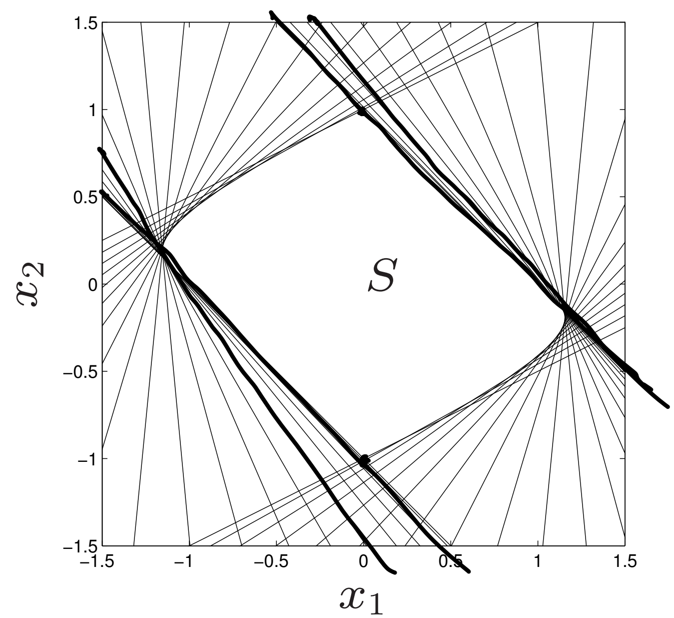
example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





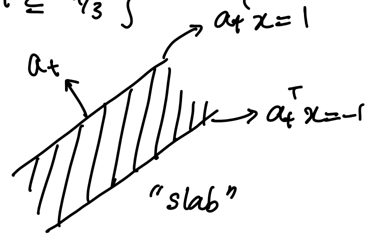
$m=2$:

$$p(t) = x_1 \cos t + x_2 \cos 2t = \begin{bmatrix} \cos t & \cos 2t \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad -\pi/3 \le t \le \pi/3$$

$$S = \left\{ x \in \mathbb{R}^2 \;\middle|\; -1 \le \underbrace{\begin{bmatrix} \cos t & \cos 2t \end{bmatrix}}_{a_t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le 1, \qquad -\pi/3 \le t \le \pi/3 \right\}$$

every fixed $t \Rightarrow$ fixed a_t,

$$\{ x \mid \underbrace{-1 \le a_t^T x \le 1} \quad \text{for } -\pi/3 \le t \le \pi/3 \}$$



$$S = \bigcap_{-\pi/3 \le t \le \pi/3} \underbrace{\{ x \in \mathbb{R}^2 \mid -1 \le a_t^T x \le 1 \}}_{\text{convex}}$$

S is convex.

Affine function (of a set)

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$)

thm:
- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

thm:
- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

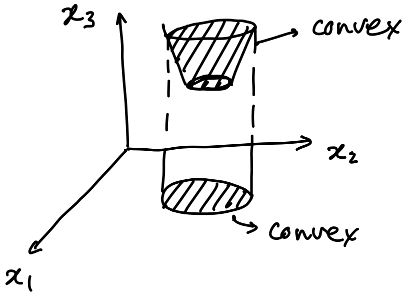
examples

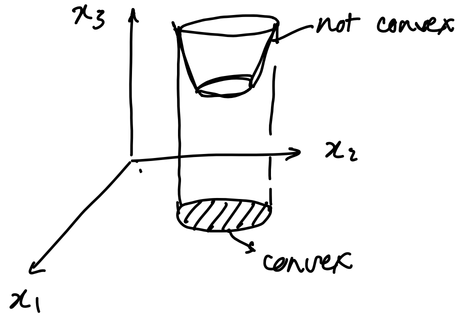
$x \to 2x$ $\quad$ $x+b$

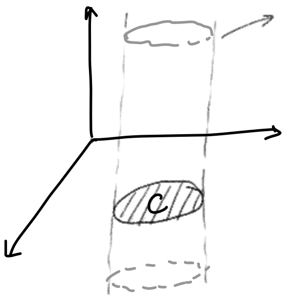
- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1A_1 + \cdots + x_mA_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^TPx \le (c^Tx)^2,\ c^Tx \ge 0\}$ (with $P \in \mathbf{S}^n_+$)

→ ice-cream cone: $\{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \le t\}$

Euclidean proj. in $\mathbb{R}^n$:







$f^{-1}(C) = \{x \in \mathbb{R}^3 \mid \text{proj}(x) \in C\}$ is convex

$$\{x \in \mathbb{R}^m \mid \overbrace{x_1 A_1 + \cdots + x_m A_m}^{A(x)} \preceq B\} \quad \text{LMI} \qquad A_i, B \in S^p$$

let $A(x) = \sum x_i A_i$ weighted sum of symm. matrices

S^p_+ : PSD cone (convex) $\qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad f: x \longrightarrow \underbrace{B - A(x)}$

$$\{x \in \mathbb{R}^m \mid A(x) \preceq B\}$$

$$= \{x \in \mathbb{R}^m \mid \underbrace{B - A(x) \succeq 0}_{\equiv\ B - A(x) \in S^p_+}\}$$

$$= f^{-1}(S^p_+)$$

S^p_+ convex $\Rightarrow$ $f^{-1}(S^p_+)$ is convex hyperbolic cone:

$$\{x \mid \underbrace{x^T P x}_{} \le (c^T x)^2, \; c^T x \geq 0\} \qquad P \succeq 0 \qquad P = Q\Lambda Q^T$$

$$= Q\Lambda^{1/2}(Q\Lambda^{1/2})^T$$

$$= P^{1/2} P^{T/2}$$

$$\underbrace{(x^T P^{1/2})}_{z^T}\underbrace{(P^{T/2} x)}_{z} \le (c^T x)^2$$

$$\|z\|_2^2 \le (c^T x)^2 \iff \|z\| \le \overbrace{c^T x}^{t}, \; c^T x \geq 0$$

$(z,t) \in$ ice-cream cone

$$f: x \to \begin{bmatrix} z \\ t \end{bmatrix} \qquad f(x) = \begin{bmatrix} P^{1/2} x \\ c^T x \end{bmatrix} = \begin{bmatrix} P^{1/2} \\ c^T \end{bmatrix} x \rightsquigarrow \text{affine map in } x$$

$\{(z,t) \mid z^T z \le t^2, \; t \geq 0\}$ is convex $\Rightarrow$ hyperbolic cone convex!

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$\begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix} \xrightarrow{P} \begin{bmatrix} x_1/t \\ x_2/t \end{bmatrix}$$

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

thm: images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

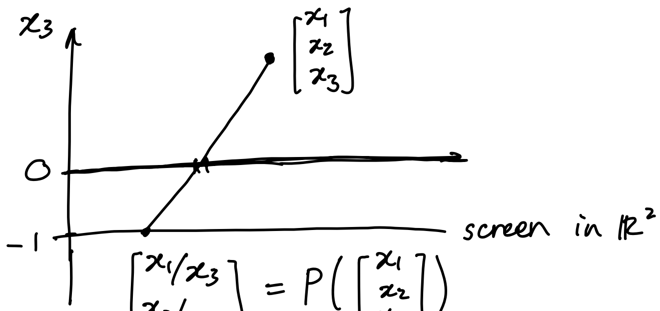
$$x \to \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} Ax+b \\ c^Tx+d \end{bmatrix}$$

$$f(x) = \frac{Ax+b}{c^Tx+d}, \qquad \mathbf{dom}\, f = \{x \mid c^Tx + d > 0\}$$

thm: images and inverse images of convex sets under linear-fractional functions are convex

comes up in : conditional probability
signal to noise ratio
⋮

just a side point: pin-hole camera



$$\begin{bmatrix} x_1/x_3 \\ x_2/x_3 \end{bmatrix} = P\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$$x = \begin{bmatrix} \tilde{x} \\ x_{n+1} \end{bmatrix} \qquad y = \begin{bmatrix} \tilde{y} \\ y_{n+1} \end{bmatrix}$$

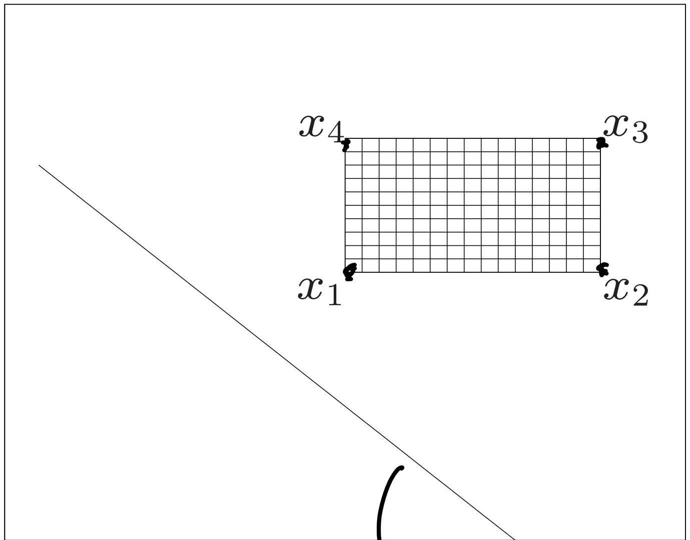
$$x_{n+1} > 0 \qquad y_{n+1} > 0$$

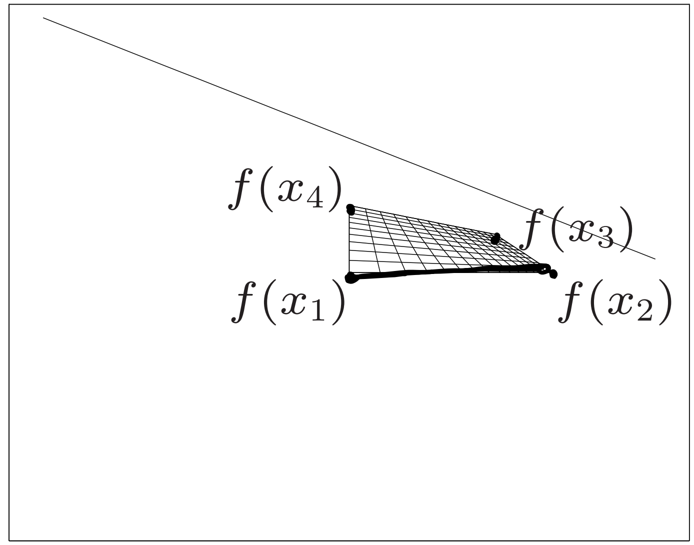
$$0 \le \theta \le 1: \quad P(\theta x + (1-\theta) y) = \frac{\theta \tilde{x} + (1-\theta)\tilde{y}}{\theta x_{n+1} + (1-\theta) y_{n+1}} = \mu P(x) + (1-\mu) P(y)$$

$$\mu = \frac{\theta x_{n+1}}{\theta x_{n+1} + (1-\theta) y_{n+1}} \in [0,1]$$

- image of convex set under P is convex.

example:





$x_1+x_2+1=0$

$f(x) = \frac{1}{x_1+x_2+1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{x_1+x_2+1}x$

line segments preserved: for x, $y \in \mathbf{dom}\, f$,

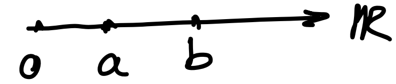
$\mathbf{dom}\, f = \{x \mid x_1+x_2+1>0\}$

$$f(\underbrace{[x,y]}_{\text{line segment}}) = [f(x), f(y)]$$

hence, if C convex, $C \subseteq \mathbf{dom}\, f$, then $f(C)$ convex

Generalized inequalities

complete/linear ordering

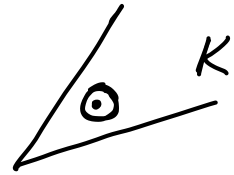


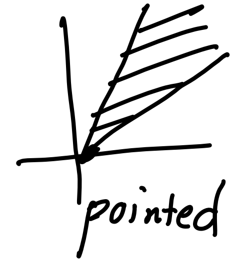
$b \geq a$ or $b \leq a$

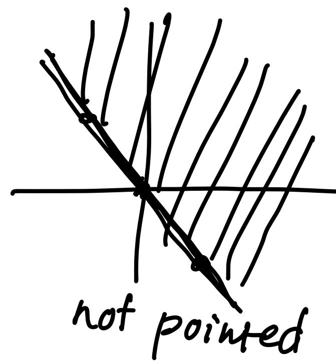
a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior) appendix A
- K is pointed (contains no line)

cone is not pointed if $\exists x \in K, -x \in K$





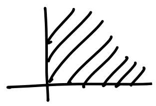


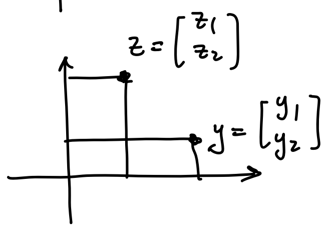
examples

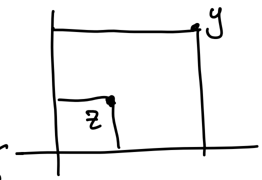
- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0,1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\}$$

• cone $K = \mathbb{R}^2_+$







z, y compared entrywise: $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff \begin{cases} z_1 \leq y_1 \\ z_2 \leq y_2 \end{cases}$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \geq \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \xleftrightarrow[\text{relation}]{\text{no}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

→ partial ordering

generalized inequality defined by a proper cone K:

$$x \preceq_K y \iff y - x \in K, \qquad x \prec_K y \iff y - x \in \mathbf{int}\, K$$

$y - x \succeq_K 0$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \iff x_i \le y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\le$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$

Announcements

- HW2 due tonight ; HW3 will be assigned Thurs morning.
- Reminder: in-class short midterm on Fri Feb 16 (more info as we get closer)
- Read the textbook as we go along:
  - sometimes proof are not fully detailed, feel free to ask us!

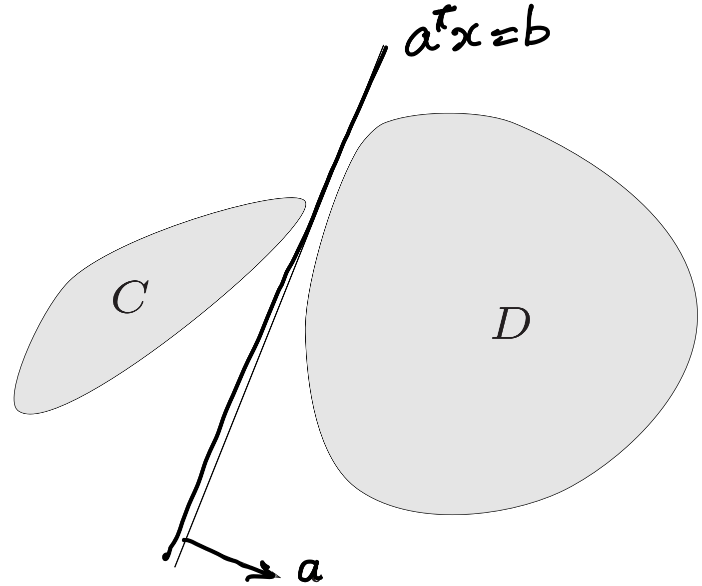
- A great talk next week: Prof. Yann LeCun (Meta & NYU) will give ECE Lytle Lecture : Wed Jan 24, 3:30pm
- Today: finish chap 2, start chap 3

Separating hyperplane theorem

$C \cap D = \phi$

thm: if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



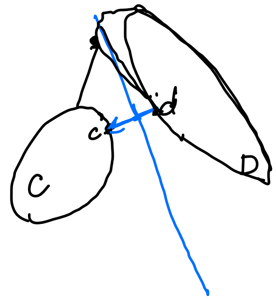
the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions (*e.g.*, C is closed, D is a singleton) $a^T x < b$, $\forall x \in C$ & $a^T x > b$, $\forall x \in D$

- proof sketch for a special case:

  C & D are convex and $C \cap D = \emptyset$

  $$\mathrm{dist}(C,D) = \inf_{\substack{x \in C \\ y \in D}} \|x-y\|_2$$



- assume dist (C,D) is achieved at points c, d, prove a, b exist:

  let $a = c-d$

  $b = -\frac{1}{2}\left(\|d\|_2^2 - \|c\|_2^2\right)$

  $$f(x) = a^T x - b = (c-d)^T x - \tfrac{1}{2}\left(\|d\|_2^2 - \|c\|_2^2\right)$$
  $$= \underbrace{(c-d)^T(x-d)}_{<0} + \underbrace{\tfrac{1}{2}\|d-c\|_2^2}$$

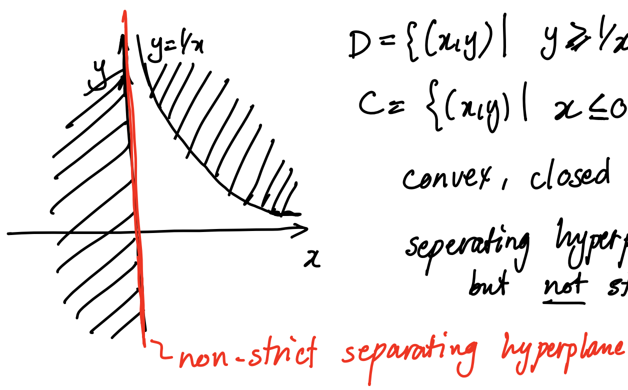
  $$\rightarrow \begin{cases} f(x) \le 0, \ \forall\, x \in C \\ f(x) \ge 0, \ \forall\, x \in D \end{cases}$$

  if not: suppose $\exists\, x \in D$ s.t. $f(x) < 0$ $\Rightarrow$ contradiction.

  (exercise 2.20)

- strict separation:

  example in $\mathbb{R}^2$:



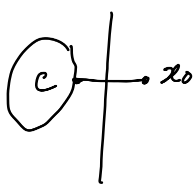
  $D = \{(x,y) \mid y \ge 1/x, x > 0\}$

  $C = \{(x,y) \mid x \le 0\}$

  convex, closed

  seperating hyperplane exists but <u>not</u> strict

- if C is convex & closed and $D = \{x_0\}$ singleton, then $a^T x < b, x \in C$
  $C \cap D = \emptyset$ $a^T x_0 > b$

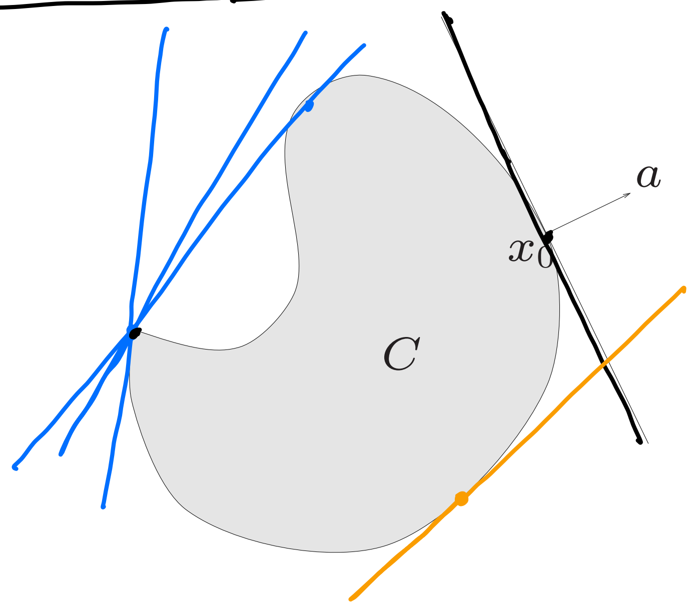


Supporting hyperplane theorem

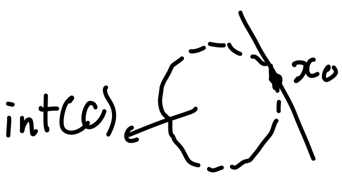
supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



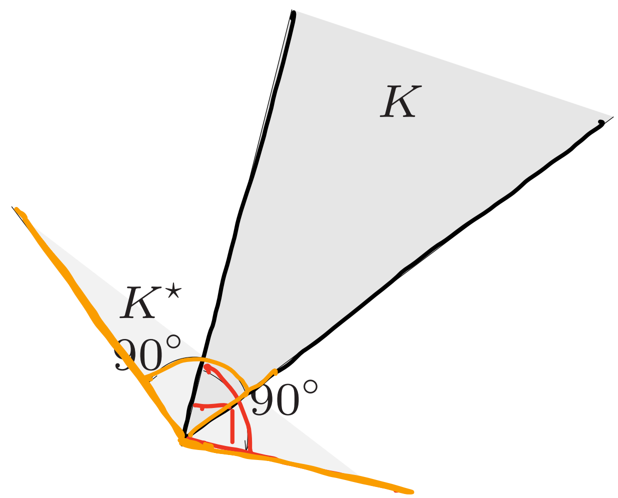
thm: **supporting hyperplane theorem:** if C is convex, then there exists a supporting hyperplane at every boundary point of C

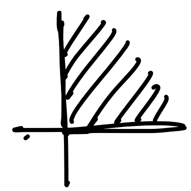


$\{x_0\} \cap \mathrm{int}(C) = \emptyset$

Dual cones and generalized inequalities

dual cone of a cone K: $\quad K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$





$\{y \mid y^T x \geq 0,\ \forall\ x \geq 0\} = \{y \geq 0\}$

examples

self-dual
- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$ $\qquad \{X \mid X \succeq 0\}$ $\qquad K^* = \{Y \mid \langle Y, X\rangle = \mathrm{Tr}\, Y^T X \geq 0 \ \ \forall\ X \succeq 0\}$ $= \{Y \succeq 0\}$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \in K^* \equiv \underline{y \succeq_{K^*} 0} \iff y^T x \geq 0 \text{ for all } \underbrace{x \succeq_K 0}_{x \in K}$$

recall def of dual norm of $\|\cdot\|$:

$$\|y\|_* = \sup_{\|x\| \leq 1 \Rightarrow \text{unit ball}} y^T x$$

Holder's ineq. : $|x^T y| \leq \|x\| \, \|y\|_*$

Cauchy-Schwartz: $|x^T y| \leq \|x\|_2 \|y\|_2$

$|x^T y| \leq \|x\|_p \, \|y\|_q \quad 1/p + 1/q = 1 \qquad p=1, \; q=\infty$