Problem 1

Compute the dual problem for the following problem, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. Take $y = b - Ax$

$$\begin{aligned} \min \quad & \|y\|_2 \\ \text{subject to} \quad & \|x\|_1 \leq \tau \\ & y = b - Ax \end{aligned}$$

The Lagrange is then

$$L((x,y),\lambda,\nu) = \|y\|_2 + \lambda(\|x\|_1 - \tau) + \nu(y - b - Ax)$$

Lagrange dual function:

$$g(\lambda,\nu) = \inf_{x,y} L((x,y),\lambda,\nu)$$

$$\inf_{x,y} \|y\|_2 + \lambda(\|x\|_1 - \tau) + \nu(y - b - Ax)$$

$$= \inf_{x,y} \|y\|_2 + \lambda\|x\|_1 - \tau\lambda + \nu y - \nu b - (A^T\nu)^T x$$

We first consider the infimum over x. Using the definition of the conjugate function to the l_1 norm we get

$$\inf_x \; \lambda\|x\|_1 - (A^T\nu)x = \begin{cases} 0 & \|A^T\nu\|_\infty \leq \lambda, \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Now consider the infimum over y

$$\inf_y \; \|y\|_2 - \tau\lambda + \nu y - \nu b$$

$$= \begin{cases} -\tau\lambda - b^T\nu & \|\nu\|_2 \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

therefore

$$g(\lambda, \nu) = \begin{cases} -\tau\lambda - b^T\nu & \|A^T\nu\|_\infty \leq \lambda, \lambda \geq 0, \\ & \|\nu\|_2 \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

and as a program

$$\max \; -\tau\lambda - b^T\nu$$

s.t.

$$\|A^T\nu\|_\infty \leq \lambda$$
$$\lambda \geq 0$$
$$\|\nu\|_2 \leq 1 .$$

Problem 2

(a) Give the optimal value and the optimal solution of the optimization problem.

$$\text{minimize} \quad x^T A x$$
$$\text{subject to} \quad c^T x \geq 1$$

dual function:

$$g(\lambda) = \inf_x \; x^T A x - \lambda(1 - c^T x)$$

$$\nabla_x L = 2Ax - \lambda(-c)$$
$$= 2Ax + \lambda c = 0$$

$$x^* = (1/2)\lambda A c$$

A^+ = pseudo inverse

Plugging in x^*

$$g(\lambda) = (\tfrac{1}{2}\lambda Ac)^T A (\tfrac{1}{2}\lambda Ac) + \lambda(1 - c(\tfrac{1}{2}\lambda Ac)$$

$$= \tfrac{1}{4}\lambda^2 c^T \underbrace{A^+ A A^+}c + \lambda - \tfrac{1}{2}\lambda^2 c A^+ c$$

$A^+ A A^+ = A^+$

$$\Rightarrow \quad g(\lambda) = -\frac{1}{4}\lambda^2 \underbrace{c^T A^+ c}_{K} + \lambda$$

$$g(\lambda) = -\frac{1}{4}\lambda^2 K + \lambda$$

$$g'(\lambda) = (-1/2) K \lambda + 1 = 0$$

$$\Rightarrow \quad \lambda^* = \frac{2}{K} = \frac{2}{c^T A^+ c}$$

$\downarrow$

optimal value $\quad x^* = \frac{1}{2}\lambda A^+ c = \dfrac{A^+ c}{c^T A^+ c}$

optimal solution

$$g(\lambda) = -\frac{1}{4}\left(\frac{2}{K}\right)^2 K + \frac{2}{K}$$

$$= -\frac{1}{K} + \frac{2}{K}$$

$$= \frac{1}{K}$$

$$= (1/2)\, c^T A^+ c .$$

We want to check the first order optimality conditions, which are necessary and sufficient.

First order optimality comes from checking the KKT conditions.

1. primal feasability $x^* \geq 0$
2. dual feasable $\lambda^* \geq 0$
3. complementary slackness $\lambda_i f_i(x) = 0,\ i = 1, \dots, m$ $\quad x^* \lambda^* = 0$
4. gradient of the Lagrange with respect to x vanishes.

$$L(x^*, \lambda^*) = x^{*T} A x^* - \lambda^*(1 - c^T)$$

$$\nabla_x L = 2Ax^* = 0$$

$$\text{if } x^* = 0 \Rightarrow \nabla_x L = 0$$

therefore our solution satisfies first-order optimality conditions.

(b) Formulate the following robust extension as a second order cone program (SOCP)

$$\text{minimize} \sup_{\|E\|_2 \le \delta} x^T(A+E)x$$

$$\text{s.t.} \quad (c+v)^T x \ge 1, \quad \forall v \text{ with } \|v\|_2 \le \alpha$$

we reformulate the objective as

$$\sup_{\|E\|_2 \le \delta} x^T(A+E)x$$

max
$E = \delta I$

$$= x^T A x + \sup_{\|E\|_2 \le \delta} x^T E x$$

$$= x^T A x + x^T \delta I x$$

$$= x^T(A + \delta I)x$$

and reformulate the constraint for the SOCP as restricting an affine function to the Lorentz cone in $\mathbb{R}^{k+1}$, so we have

$$(c+v)^T \ge 1, \quad \|v\|_2 \le \alpha$$

$$= \quad c^T x + v^T x \geq 1$$

applying Cauchy-Schwarz

$$c^T x - \|v\|_2 \|x\|_2$$

Choose worst-case v as α

$$\Rightarrow \quad c^T x - \alpha \|x\|_2 \geq 1$$

$$\|x\|_2 \leq \frac{c^T x - 1}{\alpha}$$

SOCP constraint form

Then our second order cone program becomes

$$\text{minimize} \quad x^T (A + \delta I) x$$

$$\text{subject to} \quad \|x\|_2 \leq \frac{c^T x - 1}{\alpha} .$$

Problem 3

Consider the optimization problem

$$\text{minimize } f(x) = E(z^T A(x)^{-1} z)$$

where $x \in \mathbb{R}^n$ and $A(x) = x_1 A_1 + \cdots + x_n A_n$, $A_i \in S^n$.
The domain of f is given $\text{dom} f = \{x \mid A(x) \succ 0\}$.
The vector $z \in \mathbb{R}^n$ is a random vector with mean $Ez = \bar{z}$
and covariance matrix $E(z-\bar{z})(z-\bar{z})^T = \Sigma$.

(a) Show that $E[zz^T] = \Sigma + \bar{z}\bar{z}^T$.

$$E[zz^T] = E[(z-\bar{z})(z-\bar{z})^T] + \bar{z}\bar{z}^T$$
$$= E[zz^T] - E[z]E[\bar{z}]^T + \bar{z}\bar{z}^T$$
$$= E[zz^T] - \bar{z}\bar{z}^T + \bar{z}\bar{z}^T$$
$$= E[zz^T]. \text{ QED.}$$

(b) Formulate problem (1) as a semidefinite program.

For simplicity and understanding we will formulate the simpler problem as an SDP without the expectation:

$$\min \quad f(x) = z^T A(x)^{-1} z$$

$$\text{s.t.} \quad A(x) \succ 0$$

take a slack variable t, then,

$$\min_{x,t} \quad t$$

$$\text{s.t.} \quad A(x) \succ 0$$

$$z^T A(x)^{-1} z^T \leq t$$

Applying the Shur complement to formulate the constraints as a LMI

$$\begin{matrix} A(x) \succ 0 \\ z^T A(x)^{-1} z - t \leq 0 \end{matrix} \Rightarrow \begin{bmatrix} A(x) & z \\ z^T & t \end{bmatrix} \succ 0$$

Therefore the SDP can be written as

$$\min_{x,t} \quad t$$

$$\text{s.t.} \quad \begin{bmatrix} A(x) & z \\ z^T & t \end{bmatrix} \succ 0 .$$

Lets now consider our objective function

$$\min \quad f(x) = E(z^T A(x)^{-1} z)$$

we showed in part (a) this can be written as

$$E[z A(x)^{-1} z]$$

$$= E[\mathrm{Tr}(z z A(x)^{-1})]$$

$$= \mathrm{Tr}(E[z z^T] A(x)^{-1}) \geq t$$

$$= \mathrm{Tr}((\Sigma + \bar{z}\bar{z}^T) A(x)^{-1}) \geq t$$

$$= \mathrm{Tr}(\bar{z}^T A(x)^{-1} \bar{z} + A(x)^{-1} \Sigma) \geq t$$

$$= \bar{z}^T A(x)^{-1} \bar{z} + \mathrm{Tr}(A(x)^{-1} \underbrace{\Sigma}_{\substack{\text{factor } \Sigma \text{ as sum of } z_i \\ \sum_{i=1}^{n} (z-\bar{z})(z-\bar{z})^T}}) \geq t$$

$$= \bar{z}^T A(x)^{-1} \bar{z} + \sum_{i=1}^{n} \underbrace{(z-\bar{z})^T}_{z_i} A(x)^{-1} \underbrace{(z-\bar{z})}_{z_i} \leq t$$

The problem is equivalent to

$$\text{minimize} \quad \bar{z}^T A(x)^{-1} \bar{z} + \sum_{i=1}^{n} z_i A(x)^{-1} z_i$$

Now similar to the previous (simpler) formulation we apply the Shur complement and write the SDP formulation as

$$\underset{x,t}{\text{minimize}} \quad t_0 + \sum_i^n t_i$$

$$\text{subject to} \quad \begin{bmatrix} A(x) & z \\ z^T & t_0 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} A(x) & z_i \\ z_i & t_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n.$$

Problem 4

formulate $f(x,y) = \sqrt{1 + x^4/y}$ as a DCP.

```
import cvxpy as cp

# create variables
x = cp.Variable()
y = cp.Variable()

f = cp.norm2(1 + cp.quad_over_lin(x,cp.sqrt(y)))
print(f)

obj = cp.Problem(cp.Minimize(f))

print(f"The problem is a DCP: {obj.is_dcp()}")
```

```
norm1(1.0 + quad_over_lin(var67, power(var68, 0.5)))
The problem is a DCP: True
```