Problem 1

1. Find the conjugates of the following functions:

a) $f(x) = \sum_{i=1}^{n} x_i \log(x_i / \mathbf{1}^T x)$ with $\text{dom} f = \mathbb{R}^n_{++}$.

$$f^*(y) = \sup_{x \in \text{dom} f} (y^T x - f(x))$$

$$= \sup_{x \in \mathbb{R}^n_{++}} \left(y^T x - \sum_{i=1}^{n} x_i \log(x_i / \mathbf{1}^T x)\right)$$

$$= \nabla_x \left[\sup_{x \in \mathbb{R}^n_{++}} \left(y^T x - \sum_{i=1}^{n} x_i \log(x_i / \mathbf{1}^T x)\right)\right]$$

$$= \nabla_x y^T x - \nabla_x \left(\sum_{i=1}^{n} x_i \log(x_i / \mathbf{1}^T x)\right)$$

$$\Rightarrow y_i - \log(x_i / \mathbf{1}^T x) = 0$$

$$y_i = \log(x_i / \mathbf{1}x)$$

$$\Rightarrow e^{y_i} = x_i / \mathbf{1}^T x \quad \text{and} \quad \sum_{i=1}^{n} e^{y_i} = 1.$$

For $\sum_{i=1}^{n} e^{y_i} > 1$

$$\sum_{i=1}^{n} \underbrace{y_i x_i}_{>0} - \underbrace{x_i \underbrace{\log(x_i / \mathbf{1}^T x)}_{<0}}_{>0}$$

as $x_i \to \infty$

then $f^*(y) \to \infty$

For $\sum_{i=1}^{n} e^{y_i} < 1$ take $\hat{y} > y$ with $\sum_{i=1}^{n} e^{\hat{y}_i} = 1$.
for any $x > 0$ the term $\sum_i y_i x_i - x_i \log(x_i / 1^T x)$
increases in y_i. Therefore, $f^*(y) \leq f^*(\hat{y}) = 0$, so
as $x_i \to 0$, $f^*(y) \to 0$.

We can write the consugates as

$$f^*(y) = \begin{cases} 0 & \sum_{i=1}^{n} e^{y_i} \leq 1 \\ \infty & \text{otherwise} \end{cases}.$$

b) $f(X) = \operatorname{Tr}(X^{-1})$ with $\operatorname{dom} f = S_{++}^{n}$

First we look at the domf. Suppose Y has the eigenvalue decomposition

$$Y = Q \Lambda Q^T = \sum_{i}^{n} \lambda_i q_i q_i^T$$

Then let $X = Q \operatorname{diag}(t, 1, \dots, 1) Q^T = t q_1 q_1^T + \sum_{i=2}^{n} q_i q_i^T$.

Then, $\operatorname{Tr}(XY) - \operatorname{Tr}(X^{-1}) = t\lambda_1 + \sum_{i=2}^{n} \lambda_i - 1/t - (n-1)$

with $\lambda_1 > 0$. This grows unbounded as $t \to \infty$.
Therefore $Y \in \operatorname{dom} f^*$.

Now assume $Y \prec 0$ then the conjugate is

$$f^*(y) = \sup_{X \in S^n_{++}} \left(y^T X - f(X) \right)$$

$$= \sup_{X \in S^n_{++}} \left(\mathrm{Tr}(XY) - \mathrm{Tr}(X^{-1}) \right)$$

$$\Rightarrow \nabla_X = Y + X^{-2} = 0$$

$$\Rightarrow Y^* = -X^{-2}$$

$$Y^* \in S^n_{--}$$

negative definite

$$\Rightarrow -Y^{-1/2} = X^*$$

Therefore, we have that the conjugate is

$$f^*(y) = -2\,\mathrm{Tr}(-Y)^{1/2}$$

when $Y \prec 0$.

Problem 2

a) Show that $f(x)$ is a quasiconvex function of x over $\{x \mid x \succeq 0, x \neq 0\}$.

Preliminary: $x_i \geq 0, \; i=1,\ldots,m$ and $\sum_{i=1}^{n} x_i > 0$.

$$g_i = \frac{1}{1^T x} \sum_{j=1}^{i} x_{(j)}, \quad i=1,\ldots,n.$$

$$f(x) = \frac{2}{n} \sum_{i=1}^{n} \left(\frac{i}{n} - g_i\right).$$

Also by pointwise maximum and sum of largest components we know

$$\sum_{j=1}^{i} x_{(j)} = \sum_{j=1}^{n} x_j - \sum_{j=1}^{n-i} x_{[j]}$$

So we can write

$$g_i = \frac{1}{1^T x}\left[1^T x - \sum_{j=1}^{n-i} x_{[j]}\right]$$

$$= 1 - \frac{1}{1^T x} \sum_{j=1}^{n-i} x_{[j]}$$

and so,

$$f(x) = \frac{2}{n}\left[\sum_{i=1}^{n} \frac{i}{n} - \sum_{i=1}^{n}\left(1 - \frac{1}{1^T x}\sum_{j=1}^{n-i} x_{[j]}\right)\right]$$

$$= \frac{2}{n}\left[\frac{1}{2}(n+1) - \left(n - \frac{1}{1^T x}\sum_{i=1}^{n}\sum_{j=1}^{n-i} x[j]\right)\right]$$

$$= \frac{2}{n}\left[\frac{1}{2}(1-n) + \frac{1}{1^T x}\sum_{i=1}^{n}\sum_{j=1}^{n-i} x[j]\right].$$

<u>def</u>

f is quasiconvex if $\mathrm{dom} f$ is convex and the sublevel sets $S_\alpha = \{x \in \mathrm{dom} f \mid f(x) \le \alpha\}$ are convex for all α.

$$f(x) = \frac{2}{n}\left[\underbrace{\frac{1}{2}(1-n)}_{\text{const.}} + \frac{1}{1x}\sum_{i=1}^{n}\sum_{j=1}^{n-i} x[j]\right] \le \alpha$$

$$\Rightarrow \frac{1}{1^T x}\sum_{i=1}^{n}\sum_{j=1}^{n-i} x[j] \le \alpha$$

$$\Rightarrow \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n-i} x[j]}_{\text{convex}} \le \underbrace{\alpha \cdot \frac{1}{1^T x}}_{\text{linear}}$$

hence, f is quasiconvex for $x \succeq 0$ and $x \neq 0$.

b) suppose individuals i and j get married and merge incomes. The entries x_i and x_j in the vector x are both replaced by $(x_i + x_j)/2$. What can we say about the change in $f(x)$.

recall the modified Jensen inequality for quasiconvex f

$$0 \leq \theta \leq 1 \Rightarrow f(\theta x + (1-\theta) y) \leq \max\{f(x), f(y)\}.$$

If we take and arbitrary x vectors as

$$X_i = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad X_j = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \theta = \frac{1}{2}$$

then Its clear to see the left hand of the Jensen's modified inequality will represent the joint marriage with $\theta = 1/2$ as $(X_i + X_j)/2$ and this will always be upperbounded by max of the seperate individuals since the x vector returns the smallest number among $\{x_1, \dots, x_n\}$.

overall meaning that $F(x)$ will decrease in the Gini index due to the marriage.

Problem 3

a) show that $f(x) = \max_{B \in S} x^T B x$ where

$$S = \{ B \in S^n \mid B_{ii} = A_{ii}, |B_{ij}| = A_{ij} \text{ for } i \neq j \}.$$

Also, $\bar{A}_{ij} = \begin{cases} A_{ii} & \text{if } i = j \\ -A_{ij} & \text{if } i \neq j \end{cases}$, $\bar{A} \succeq 0$ (psd).

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} |x_i||x_j| = |x|^T A |x|$$

We can write B as $B_{ij} = \begin{cases} \bar{A}_{ij} & \text{if } i = j \\ \pm A_{ij} & \text{if } i \neq j \end{cases}$

then,

$$x^T B x = \sum_{i=1}^{n} x_i^2 A_{ii} - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j |A_{ij}| \geq 0, \forall x$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 A_{ii} \geq \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j| A_{ij}.$$

QED

b) Show that every matrix in the set S is PSD.

Show $\sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} x_i x_j \geq 0$

$$B_{ij} = \pm A_{ij}$$

we know,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \bar{A}_{ij} x_i x_j \geq 0$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (-A_{ij}) x_i x_j \geq 0$$

define $y := c_i y_i = x_i$ where c_i is either -1 or 1.

Then, $$\sum_{i=1}^{n}\sum_{j=1}^{n} (-A_{ij}) c_i c_j y_i y_j \geq 0$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (-c_i c_j A_{ij}) y_i y_j \geq 0$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} y_i y_j \geq 0$$

hence every matrix in S is PSD.

c) If f is a convex function.

yes we showed $f(x) = \max_{B \in S} x^T B x$ and that $B \succeq 0$

so that means $x^T B x \succeq 0$.

Hence its the max of a convex function so $f(x)$ is convex.

Problem 4

Product of powers function : $\mathbb{R}^n_{++} \to \mathbb{R}$ given by

$$f(x) = x_1^{\theta_1}, \ldots, x_n^{\theta_n}$$

where $\theta \in \mathbb{R}^n$.

a) When $n=2$, $\theta \succeq 0$, and $1^T\theta = 1$, f is concave.

First $x_1^{\theta_1}$ is concave for all $0 \le \theta \le 1$. Applying perspective gives

$$t f(x/t) \to x_2 (x_1/x_2)^{\theta_1}$$

$$= x_1^{\theta_1} x_2^{1-\theta_1}$$

which is concave since perspective preserves convexity.

b) When $\theta \succeq 0$ and $1^T\theta = 1$, f is concave.

Proof by induction for general n. We know that the base case $n=1$ holds. For the induction step, we have $\theta \in \mathbb{R}^{n+1}_+$ and define $\bar{\theta} = (\theta_1, \ldots, \theta_n)$, $\bar{x} = (x_1, \ldots, x_n)$, and $1^T\theta = 1$. Then $\bar{x}^{\bar{\theta}/1^T\bar{\theta}}$ is concave by the induction assumption.

Then define the function $y^{1^T\bar{\theta}} z^{1-1^T\bar{\theta}}$ is concave by part (a) and is nondecreasing.

By composition we can write

$$\left(\bar{x}^{\bar{\theta}/1^T\bar{\theta}}\right)^{1^T\bar{\theta}} x_{n+1}^{1-1^T\bar{\theta}} = \bar{x}^{\bar{\theta}} x_{n+1}^{\bar{\theta}_{n+1}} = x^{\theta}$$

which is concave for $\theta \succeq 0$.

c) when $\theta \succeq 0$ and $1^T\theta \leq 1$, f is concave.

If $1^T\theta \leq 1$ then $x^{\theta/1^T\theta}$ is concave from part (b). Then define $y^{1^T\theta}$ which is concave and nondecreasing. Then by composition,

$$\left(x^{\theta/1^T\theta}\right)^{1^T\theta} = x^{\theta}$$

which is concave.

d) when $\theta \preceq 0$, f is convex.

If $\theta \preceq 0$ then $x^{\theta/1^T\theta}$ is concave by part (b). Then $z^{1^T\theta}$ is convex and nonincreasing since $1^T\theta < 0$. Then composition gives

$$\left(x^{\theta/1^T\theta}\right)^{1^T\theta} = x^{\theta}$$

which is convex.

Another approach is to see that this is the multiplication of log-convex functions which is convex. This is shown by $\theta_i \log x_i$ is convex function of x_i since $\theta_i \leq 0$ and therefore the sum $\sum_i \theta_i \log x_i$ is convex. By composition, the exp. of a convex function is convex

$$\exp\left(\sum_i \theta_i \log x_i\right) = x^{\theta}$$

is convex.

e) When $1^T\theta = 1$ and exactly <u>one</u> of the elements of θ is positive, f is convex.

If $1^T\theta = 1$ and $\theta \in \mathbb{R}^{n+1}$ let a single positive element is $\theta_{n+1} > 0$ so $\bar{\theta} = (\theta_1, \ldots, \theta_n) \leq 0$. Then $\bar{x} = (x_1, \ldots, x_n)$ makes $\bar{x}^{\bar{\theta}}$ convex by part (d). By applying the perspective transformation

$$x_{n+1}\left(\bar{x}/x_{n+1}\right)^{\bar{\theta}} = \bar{x}^{\bar{\theta}} x_{n+1}^{1-1^T\bar{\theta}} = \bar{x}^{\bar{\theta}} x_{n+1}^{\theta_{n+1}} = x^{\theta}$$

which is convex.

Problem 5

X is random vector in $\mathbb{R}^n$.

Scalar random variable $Y = \max_i X_i$ which has a CDF as $\phi(a) = \text{prob}(Y \leq a)$.

Define an indicator function as

$$\mathbb{1}(x,a) = \begin{cases} 1 & \text{if } \max_i X_i \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

then the CDF

$$\Phi(a) = \int_{\mathbb{R}^n} \underbrace{\mathbb{1}(x,a)}_{\text{log-concave}} \underbrace{p(x)}_{\text{log-concave}} \, dx^n$$

here $\log(\mathbb{1}(x,a)) \rightarrow \log(0) = -\infty$

or $\log(1) = 0$.

The indicator function is a convex set over which the points that its defined.

Hence, the CDF $\Phi(x,a)$ is log-concave.

QED

Problem 6

Formulate the following problems as linear programs.

a) minimize $\|Ax-b\|_1 + \|x\|_\infty$. $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

define slack variables $t, s \in \mathbb{R}^m$. Then we can formulate the LP as

$$\min \ \mathbf{1}^T t + s$$

$$\text{subject to } -t \preceq Ax - b \preceq t,$$

$$-s \leq x_i \leq s \quad i = 1, \dots, n.$$

b) Suppose $B \in \mathbb{R}^{m\times n}$ is random and takes finite set $\{B_1 \dots B_k\}$ with $\text{Prob}(B = B_i) = p_i$, $b \in \mathbb{R}^m$.

$$\text{minimize } \mathbb{E}\|Bx - b\|_\infty.$$

$$\Rightarrow \min_x \sum_{i=1}^{k} \|Bx - b\|_\infty \, p_i$$

$$= \min_x \sum_{i=1}^{k} \max |Bx_j - p_j| \text{ for } j = 1, \dots, m$$

then we can define a slack variable t and formulate the following as a linear program

LP: $$\min_{t} \; \mathbb{E}\, t$$

Subject to $-t \preceq Bx - b \preceq t$

OR $$\min_{t} \sum_{i}^{k} t_i \, p_i$$

Subject to $-t_i \leq B_i x - b \leq t_i$ for $i = 1, \dots, k$.

these LPs in part (a) and (b) are in epigraph form.