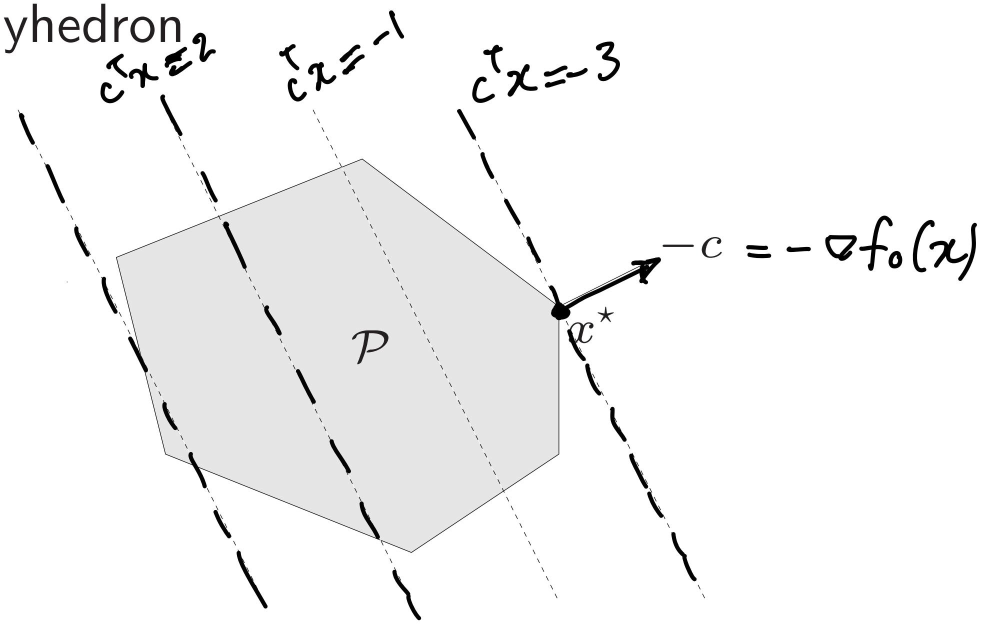
5. Linear and quadratic programming

- linear programming
- quadratic programming
- second-order cone programming

Linear program (LP)

$$
\begin{array}{ll}
\underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x + \cancel{d} \\
\text{subject to} & Gx \preceq h \\
 & Ax = b
\end{array}
$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

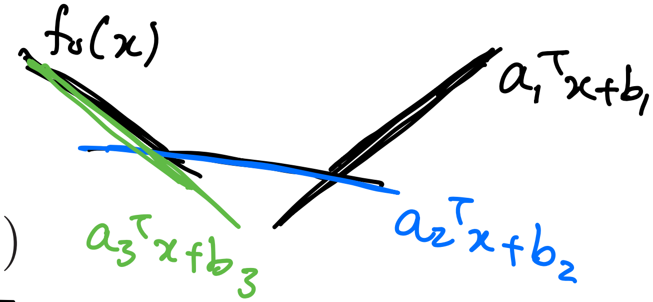
$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x = \sum c_j x_j \\ \text{subject to} & \underbrace{Ax \succeq b}_{a_i^T x \geq b_i}, \quad \underbrace{x \succeq 0} \end{array}$$

one of the earliest applications of LP during and after World War II: optimal diet for troops

(see G. Dantzig's interview for the whole history:

https://www.informs.org/Explore/History-of-O.R.-Excellence/Oral-Histories/George-Dantzig)

(convex) piecewise-linear minimization



$$\text{minimize} \quad \underline{\max_{i=1,\ldots,m}(a_i^T x + b_i)}$$

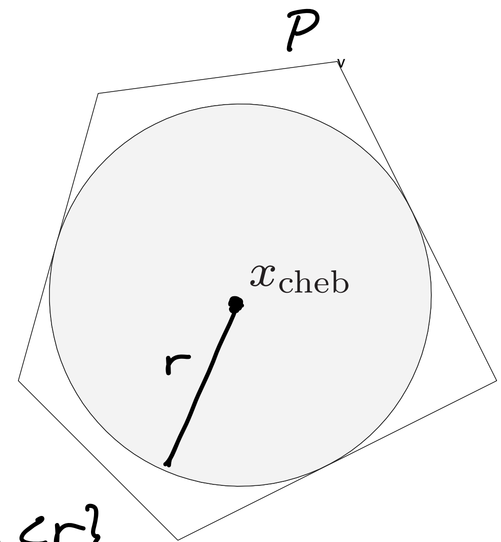
equivalent to an LP (epigraph form):

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to} & a_i^T x + b_i \le t, \quad i = 1, \ldots, m \end{array}$$

$$f_0(x) = \max_i \left(a_i^T x + b_i\right)$$

$$\begin{array}{ll} \displaystyle\min_{x \in \mathbb{R}^n,\, t \in \mathbb{R}} & t \\ \text{s.t.} & \displaystyle\max_{i=1,\ldots,m} \left(a_i^T x + b_i\right) \le t \iff a_i^T x + b_i \le t, \quad i = 1, \ldots, m \end{array}$$

Chebyshev center of a polyhedron



Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \dots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\} = \{x \mid \|x - x_c\|_2 \le r\}$$

- $a_i^T x \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \le r\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \underset{x_c,r}{\text{maximize}} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1, \dots, m \end{array}$$

$$a_i^T x \le b_i \quad \underbrace{\text{for all}} \quad x \in \{\underbrace{x_c + u} \mid \|u\|_2 \le r\}$$

$$\sup_{\|u\|_2 \le r} a_i^T (x_c + u) \le b_i$$

$$\sup_{\|u\|_2 \le r} (a_i^T x_c + a_i^T u) \le b_i$$

$$|a_i^T u| \le \|a_i\|_2 \|u\|_2$$

$\uparrow$ eq. if u in direction of a_i

$$a_i^T x_c + \underbrace{\sup_{\|u\|_2 \le r} \overbrace{a_i^T u}^{\|a_i\|_2 \|u\|_2}}_{r \|a_i\|_2} \le b_i \qquad i = 1, \dots, m$$

$$a_i^T x_c + \|a_i\|_2 r \le b_i \qquad i = 1, \dots, m$$

Announcements

- Check Natalia's announcements about her TA review & OH on Zoom (due to travel out of country) — back on Feb 14.
- HW5 due this Wed — note this HW is a bit harder than previous, ask Q's if you need help!
- Short HW6 will be assigned Wed & due next Monday (midnight)
- Midterm Fri Feb 16, in-class, closed book/notes, 2 letter-sized sheets double-sided

More on LP

LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

'standard' form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

converting to 'standard' form:

$$Ax + s = b$$

$$A(x^+ - x^-) + s = b$$

- inequality constraints: write $a_i^T x \leq b_i$ as

$$a_i^T x + s_i = b_i, \quad s_i \geq 0$$

$$\begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b$$

  s_i is *slack variable* associated with $a_i^T x \leq b_i$

$$\begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \succeq 0$$

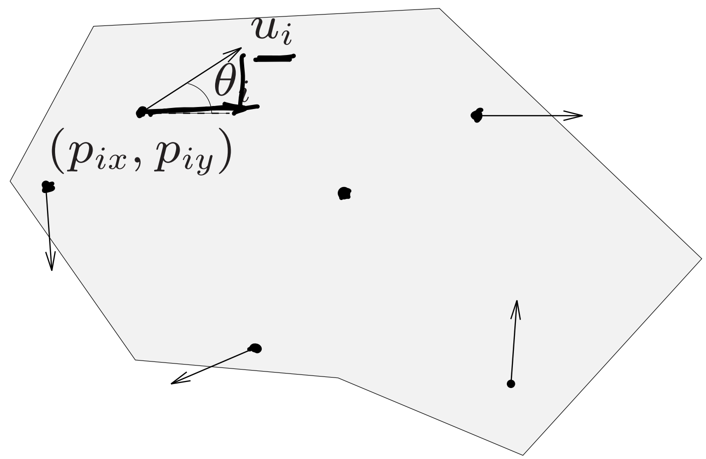
- unconstrained variables: write $x_i \in \mathbf{R}$ as

$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \geq 0$$

Example

force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbf{R}^2$
- n forces with magnitude u_i, acting at $p_i = (p_{ix}, p_{iy})$, in direction θ_i



- resulting horizontal force: $F_x = \sum_{i=1}^n u_i \cos\theta_i$
- resulting vertical force: $F_y = \sum_{i=1}^n u_i \sin\theta_i$
- resulting torque: $T = \sum_{i=1}^n (p_{iy} u_i \cos\theta_i - p_{ix} u_i \sin\theta_i)$

- force limits: $0 \le u_i \le 1$ (thrusters)
- fuel usage: $u_1 + \cdots + u_n$

problem: find thruster forces u_i that yield given desired forces and torques and minimize fuel usage
(if feasible)

can be expressed as LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \le u_i \le 1, \;\; i = 1, \ldots, n \end{array}$$

where

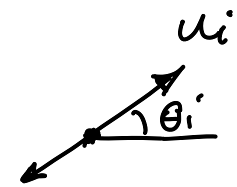
$$F = \left[\begin{array}{ccc} \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \cdots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{array} \right]_{3 \times n},$$

$$f^{\text{des}} = (F_x^{\text{des}},\; F_y^{\text{des}},\; T^{\text{des}}), \quad \mathbf{1} = (\,1,\; 1,\; \cdots\; 1\,)$$

thruster problem in 'standard' form

$$
\begin{array}{ll}
\text{minimize} & \left[\mathbf{1}^T \; 0\right] \begin{bmatrix} u \\ s \end{bmatrix} \\
\text{subject to} & \begin{bmatrix} u \\ s \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} F & 0 \\ I & I \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} f^{\mathrm{des}} \\ \mathbf{1} \end{bmatrix}
\end{array}
$$

some extensions (can express these as LP):



- opposing thruster pairs

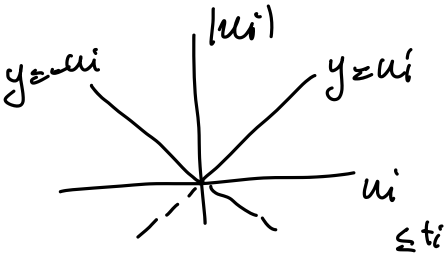
$$
\left\{
\begin{array}{ll}
\text{minimize} & \sum_i |u_i| \\
\text{subject to} & Fu = f^{\mathrm{des}} \\
& |u_i| \leq 1, \quad i = 1, \ldots, n
\end{array}
\right.
$$

$-1 \leq u_i \leq 1$

- given f^{des},

Chebychev error

$$
\begin{array}{ll}
\text{minimize} & \|Fu - f^{\mathrm{des}}\|_\infty \\
\text{subject to} & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n
\end{array}
$$



$|u_i| = \max\{u_i, -u_i\}$

$$\begin{cases} \min\limits_{u\in\mathbb{R}^n} & \sum\limits_{i=1}^{n} \overbrace{|u_i|}^{\le t_i} \\ \text{s.t.} & Fu = f_{des} \\ & |u_i| \le 1 \end{cases}$$

epigraph form:

epigraph variables t_i $\quad i=1,\dots,n$

$$\begin{cases} \min\limits_{\substack{u\in\mathbb{R}^n \\ t\in\mathbb{R}^n}} & \sum t_i \\ \text{s.t.} & Fu = f_{des} \\ & |u_i| \le t_i \rightarrow -t_i \le u_i \le t_i \\ & |u_i| \le 1 \rightarrow -1 \le u_i \le 1 \end{cases}$$

$$F = \begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}$$

- Chebychev error:

$$\begin{cases} \min\limits_{u} & \|Fu - f_{des}\|_\infty \rightarrow \max\limits_i |f_i^T u - f_{des,i}| \\ \text{s.t.} & |u_i| \le 1 \end{cases}$$

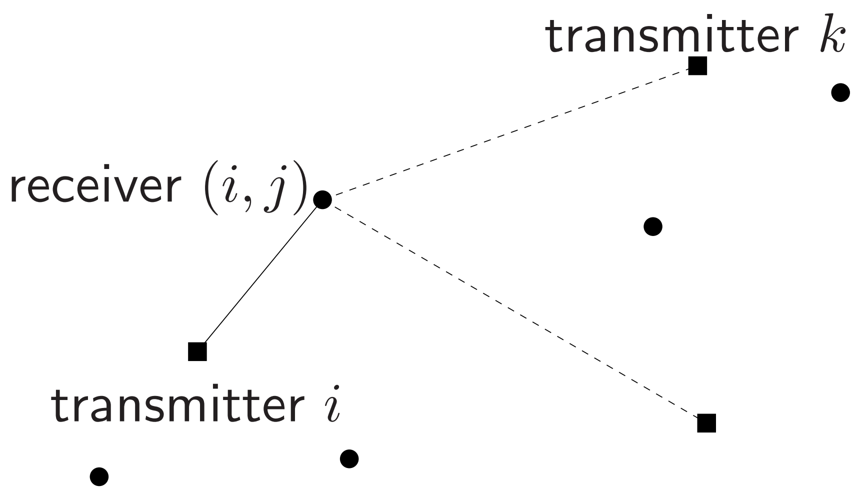
epigraph form:

$$\begin{cases} \min\limits_{\substack{u\in\mathbb{R}^n \\ t\in\mathbb{R}}} & t \\ & -t \le f_i^T u - f_{des,i} \le t \qquad i=1,2,3 \\ & -1 \le u_i \le 1 \end{cases}$$

Optimal transmitter power allocation

(generalized) linear fractional program (refer to section 4.2.3?)

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

at receiver (i, j):

- signal power: $S_{ij} = A_{iji}p_i$
- noise plus interference power: $I_{ij} = \sum_{k \neq i} A_{ijk}p_k + N_{ij}$
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle \min_{i,j} \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\mathsf{max}} \end{array}$$

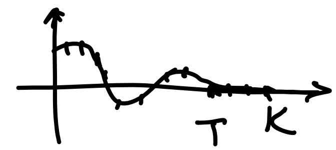
. . . a (generalized) linear fractional program

Minimum-time optimal control

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

$$x(t+1) = Ax(t) + Bu(t), \quad \text{with } x(0) = x_0 \quad \text{and} \quad \|u(t)\|_\infty \le 1, \; t = 0, 1, \ldots, K$$

variables: $u(0), \ldots, u(K)$



settling time $f(u(0), \ldots, u(K))$ is

$$\inf \{T \mid x(t) = 0 \text{ for } T \le t \le K+1\}$$

set of $K+1-T$ linear equation in $u(t)$'s

$x(1) = Ax(0) + Bu(0)$

$x(2) = Ax(1) + Bu(1) = A(Ax(0) + Bu(0)) + Bu(1)$

$= A^2 x(0) + AB\,u(0) + B\,u(1)$

$\vdots$

$x(t)$ depends linearly on all $u(t)$'s

Minimum-time optimal control

$$x(t+1) = Ax(t) + Bu(t), \quad \text{with } x(0) = x_0 \quad \text{and} \quad \|u(t)\|_\infty \le 1, \; t = 0, 1, \dots, K$$

variables: $u(0), \dots, u(K)$

settling time $f(u(0), \dots, u(K))$ is

$$\inf \{T \mid x(t) = 0 \text{ for } T \le t \le K+1\}$$

f is quasiconvex function of $(u(0), \dots, u(K))$: $S_T = \{u(0), \dots, u(K) \mid f(u(0), \dots, u(K)) \le T\}$

$f(u(0), u(1), \dots, u(K)) \le T$ if and only if for all $t = T, \dots, K+1$ intersection of

$$x(t) = A^t x_0 + A^{t-1} Bu(0) + \cdots + Bu(t-1) = 0$$

hyperplanes $\Rightarrow$ convex set

$\Rightarrow f$ is Q-cvx.

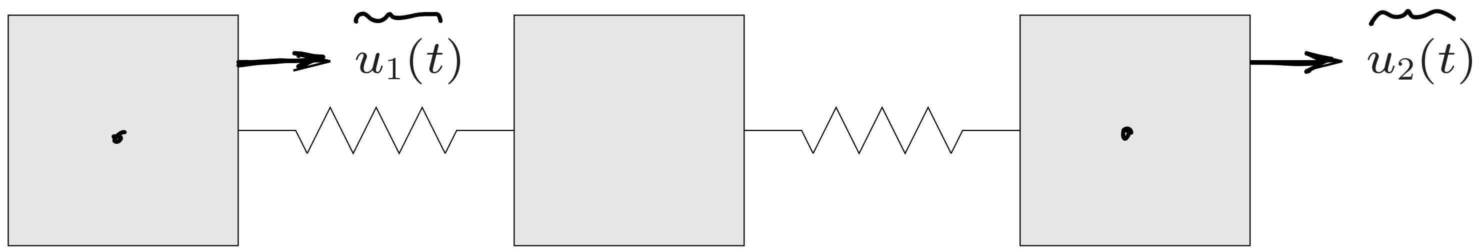
min-time optimal control problem:

$$\begin{array}{ll} \underset{u(0),\dots,u(K)}{\text{minimize}} & f(u(0), u(1), \dots, u(K)) \\ \text{subject to} & \|u(t)\|_\infty \le 1, \; t = 0, \dots, K \end{array}$$

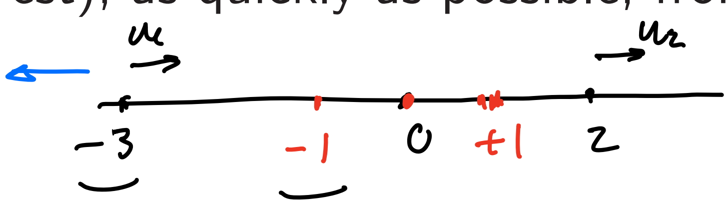
Min-time control example

three unit masses, connected by two unit springs with equilibrium length one

$u(t) \in \mathbf{R}^2$ is force on left and right mass over time interval $(0.15t, 0.15(t+1)]$

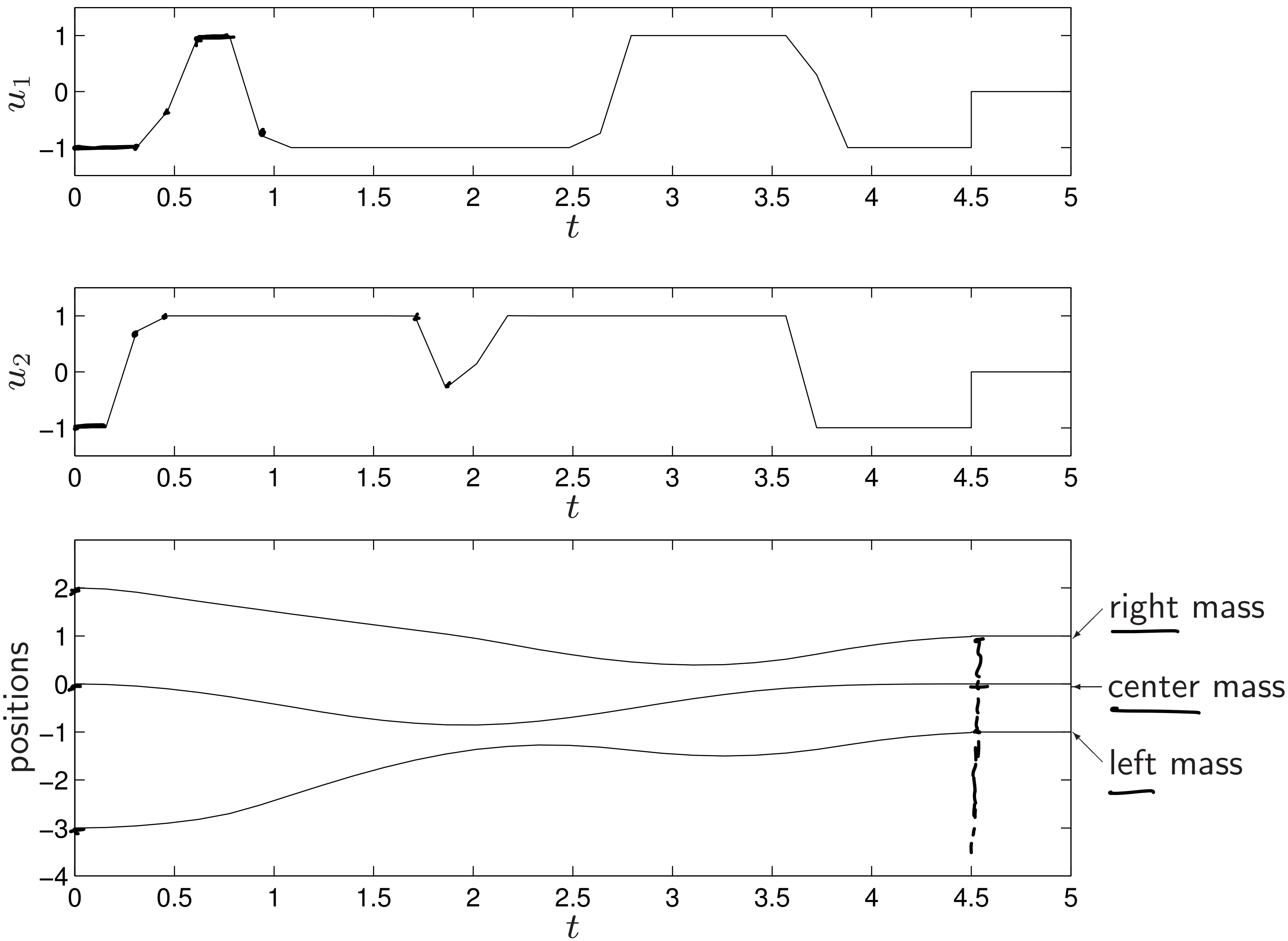


problem: pick $u(0), \ldots, u(K)$ to bring masses to positions $(-1, 0, 1)$ (at rest), as quickly as possible, from initial condition $(-3, 0, 2)$ (at rest)



bang-bang control

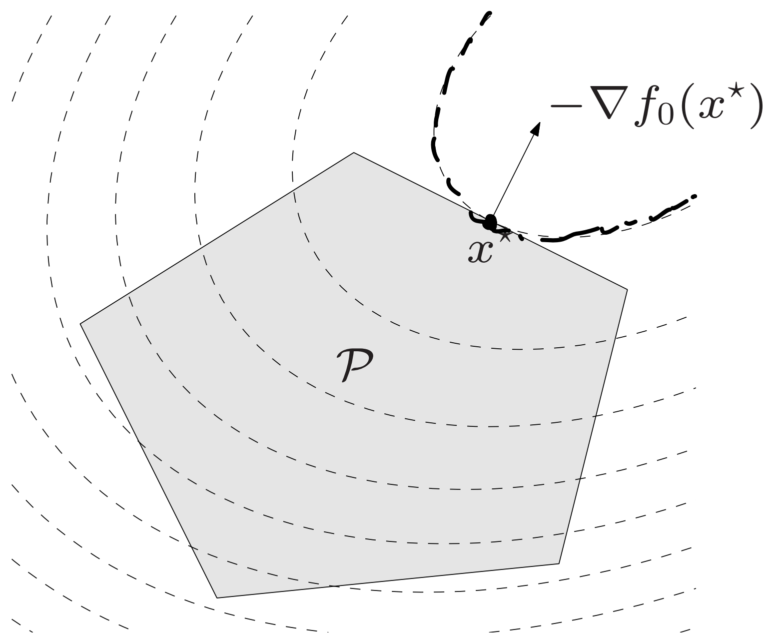
optimal solution:



Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \overbrace{\mathbf{var}(c^T x)}^{\text{risk}} \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

$$\begin{cases} \min_{x} & \underbrace{c^T x} \\ \text{s.t.} & Gx \preceq h \end{cases}$$

problem data: c, G, h

- in practice, uncertainty in data, e.g., uncertainty in $c \in \mathbb{R}^n$

$c \in \mathbb{R}^n$ is a random vector $c \sim D(\bar{c}, \underbrace{\Sigma})$ ⇝ given

$c^T x$ is also random variable:

- $\mathbb{E}_c(c^T x) = \bar{c}^T x$
- $\mathbb{E}_c\left[(c^T x - \bar{c}^T x)(c^T x - \bar{c}^T x)^T\right] = \mathbb{E}_c\left[x^T (c-\bar{c})(c-\bar{c})^T x\right] = x^T \underbrace{\mathbb{E}_c (c-\bar{c})(c-\bar{c})^T}_{\Sigma} x$

$$= x^T \Sigma x$$

$$\begin{cases} \min_{x} & \bar{c}^T x + \gamma \overbrace{x^T \Sigma x} \\ \text{s.t.} & Gx \preceq h \end{cases} \longrightarrow \text{a QP}$$

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Second-order cone programming

$x \in \mathbb{R}^n$

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

$\{(x, t) \mid \|x\|_2 \leq t,\ t > 0\}$

$(A_i \in \mathbf{R}^{n_i \times n},\ F \in \mathbf{R}^{p \times n})$

$x \rightarrow \begin{bmatrix} A_i x + b_i \\ c_i^T x + d_i \end{bmatrix} \updownarrow n_i$

- inequalities are called second-order cone (SOC) constraints:

$$(\underline{A_i x + b_i}, \underline{c_i^T x + d_i}) \in \text{second-order cone in } \mathbf{R}^{n_i + 1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP

- more general than QCQP and LP

LP

QP

QCQP

SOCP

SDP

Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

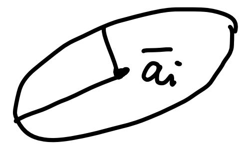
- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m \end{array}$$

deterministic approach via SOCP



- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\} \qquad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n \times n})$$

  center is $\bar{a}_i$, semi-axes determined by singular values/vectors of P_i

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

  is equivalent to the SOCP

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  (follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

$$a_i^T x \le b_i \quad \text{for all} \quad a_i \in \{ \underbrace{\bar{a}_i + P_i u} \mid \|u\|_2 \le 1 \}$$

$$\Updownarrow$$

$$\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x \le b_i$$

$$\bar{a}_i^T x + \underbrace{\sup_{\|u\|_2 \le 1} u^T (P_i^T x)} \le b_i$$

Cauchy-Schwartz

$|x^T y| \le \|x\|_2 \|y\|_2$

$$= \underbrace{\|u\|_2}_{\le 1} \|P_i^T x\|_2$$

$$= \|P_i^T x\|_2$$

$$\bar{a}_i^T x + \|P_i^T x\|_2 \le b_i \qquad i = 1, \dots, m$$

$$\|P_i^T x\|_2 \le b_i - \bar{a}_i^T x$$

stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence $\mathcal{N}(0,1)$

$$\mathbf{prob}(a_i^T x \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0,1)$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m, \end{array}$$

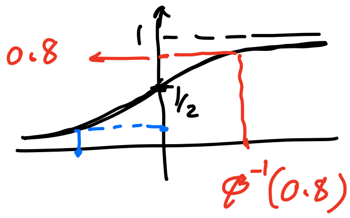
with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$$\Pr(a_i^T x \le b_i) = \Pr\left(\underbrace{\frac{a_i^T x - \bar{a}_i^T x}{\underbrace{\sqrt{x^T \Sigma_i x}}_{= \|\Sigma_i^{1/2} x\|_2}}}_{\sim N(0,1)} \le \frac{b_i - \bar{a}_i^T x}{\underbrace{\sqrt{x^T \Sigma_i x}}} \right)$$

$$= \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|} \right)$$

Φ : CDF of $N(0,1)$



$$\Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|} \right) \ge \eta$$

$$\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|} \ge \Phi^{-1}(\eta) \Rightarrow \qquad b_i \ge \underbrace{\Phi^{-1}(\eta)} \underbrace{\|\Sigma_i^{1/2} x\|_2}_{\text{SOC constraint}} + \bar{a}_i^T x$$