Assignment 5

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Problem 1

Let X be Katie's choice of path and Y be Pele's. Then

$$\begin{aligned}
E(T) &= \sum_X \sum_Y E(T|X=x, Y=y)P\{X=x\}P\{Y=y\} \\
&= \frac{1}{9}\Big(E(T|AB, BA) + E(T|AB, BC) + E(T|AB, BD) \\
&\quad + E(T|AC, BA) + E(T|AC, BC) + E(T|AC, BD) \\
&\quad + E(T|AD, BA) + E(T|AD, BC) + E(T|AD, BD)\Big) \\
&= \frac{1}{9}\Big(\frac{1}{2} + (1 + E(T)) + (1 + E(T)) \\
&\quad + (1 + E(T)) + 1 + (1 + E(T)) \\
&\quad + (1 + E(T)) + (1 + E(T)) + 1\Big) \\
&= \frac{1}{9}\left(2.5 + 6(1 + E(T))\right) \\
&= \frac{8.5}{9} + \frac{2}{3}E(T) \\
&= \frac{8.5}{3}
\end{aligned}$$

Problem 2

a)

Let X_T denote the number of umbrellas at the current location before trip T. So $S = \{0, 1, 2, 3, 4\}$. Let P be a transition matrix associated with X_T. So:

$$\text{P=}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & .8 & .2 \\ 0 & .8 & .2 & 0 \\ .8 & .2 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{aligned} \pi P &= \pi && (2) \\ \rightarrow \pi_0 &= .8\pi_3 && (3) \\ \pi_1 &= .8\pi_2 + .2\pi_3 && (4) \\ \pi_2 &= .8\pi_1 + \pi_2 && (5) \\ \pi_3 &= .8\pi_0 + \pi_1 && (6) \\ & && (7) \end{aligned}$$

From here we get:

$$\begin{aligned} \pi_3 &= \pi_1 && \text{(by (6))} \\ \pi_2 &= \pi_1 && \text{(from (5))} \\ \text{So: } \pi_3 &= \pi_2. \end{aligned}$$

So $\pi = .8\pi_3, \pi_3, \pi_3, \pi_3$

$$\text{We know that } \sum_i \pi_i = 1 = 3.8\pi_3$$

So $\pi_3 = \frac{5}{19}$ and our stationary distribution is:

$$\pi = \left[\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19}\right]$$

c)

```
matpow = function (M,n) {
# Finds matrix power M^n.
# Use this for n = integer greater than 1.
  ans = M
  for(i in 1:( n-1 )) {
    ans = ans %*% M
  }
  return (ans)
}

P = matrix(c(0,0,0,1,
             0,0,.8,.2,
             0,.8,.2,0,
             .8,.2,0,0), byrow=TRUE, ncol=4)

matpow(P,1000)
```

0.21052631578948	0.26315789473685	0.263157894736849	0.263157894736849
0.21052631578948	0.26315789473685	0.26315789473685	0.26315789473685
0.210526315789479	0.263157894736849	0.263157894736849	0.263157894736849
0.21052631578948	0.263157894736849	0.26315789473685	0.26315789473685

My calculated stationary dist:

```
matrix(c(4/19,5/19,5/19,5/19), byrow=TRUE,ncol=4)
```

0.210526315789474	0.263157894736842	0.263157894736842	0.263157894736842

d)

Fred only gets wet when he's in state zero (i.e. he's at a location with no umbrellas) and it starts to rain (with 0.2 probability) so:

$$\pi(0)P_{rain} = .21 * 0.2 = .0421$$

Problem 3

a)

We can show that this is not a Markov chain by proving that the Markov property does not hold. The Markov property states that

$$P(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_n = x_n \mid X_{n-1} = x_{n-1}).$$

From the problem its clear to see that $S = \{-1, 0, 1\}$
Let $X_0 = 1$ $X_0^* = -1$ be the starting states of two chains and $X_1 = 0$ be an intermediary state shared by both.
We want to see if

$$P\{X_2 = x_2 | X_1 = x_1, X_0 = x_0\} = P\{X_2 = x_2 | X_1 = x_1, X_0^* = x_0^*\}$$

.
The LHS sequence of Ys must be:

$$Y_{-1} = 1, Y_0 = 1, Y_1 = -1$$

And the RHS must be:

$$Y_{-1} = -1, Y_0 = -1, Y_1 = 1$$

Given that $X_2 = \frac{1}{2}(Y_1 + Y_2)$

$$P\{X_2 = 1 | X_1 = 0, X_0 = 1\} \neq P\{X_2 = 1 | X_1 = 0, X_0^* = -1\}$$

Because $X_2 \neq 1$ if $X_0 = 1$ but can if $X_0 = -1$. Thus the Markov property does not hold.

b)

Let $X_0 = 1$ $X_0^* = -1$ be the starting states of two chains and $X_1 ... X_{r-1} = 0$ be subsequent states shared by both chains. For the chain starting at $X_0 = 1$ all $Y_{odd} = -1, Y_{even} = 1$. For the chain starting at $X_0^* = 1$ all $Y_{odd} = 1, Y_{even} = -1$. This means that $Y_r^* \neq Y_r \forall r$ and thus that

$$P\{X_r = x_r | X_{r-1} = x_{r-1}, \ldots X_0 = x_0\} \neq P\{X_r = x_r | X_{r-1} = x_{r-1}, \ldots X_0^* = x_0^*\}$$

Problem 4

a)

P $= \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

$$\begin{aligned} \pi P &= \pi & (8) \\ \rightarrow \pi_1 &= (1-a)\pi_1 + \pi_2 b & (9) \\ \pi_2 &= \pi_1 a + \pi_2(1-b) & (10) \\ && (11) \end{aligned}$$

From (1) we get:

$$\begin{aligned} \pi_1 &= \pi_1 - \pi_1 a + \pi_2 b \\ &= \frac{\pi_2 b}{a} \end{aligned}$$

Plugging this into (2) we get:

$$\begin{aligned} \pi_2 &= \pi_2 b + \pi_2 - \pi_2 b \\ &= \pi_2 \end{aligned}$$

So: $\sum_i \pi = 1 = \frac{\pi_2 b}{a} + \pi_2$
Solving for π_2 we get $\pi_2 = \frac{a}{a+b}$
And thus: $\pi = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

b)

```
a <- 1/2
b <- 1/3

b/(a+b)
P = matrix(c(1-a, a,
             b, 1-b), byrow=TRUE, ncol=2)

p100  <- matpow(P,100)
sprintf("Prob that 100th question is true: %s", p100[1])
```

```
Prob that 100th question is true: 0.4
```

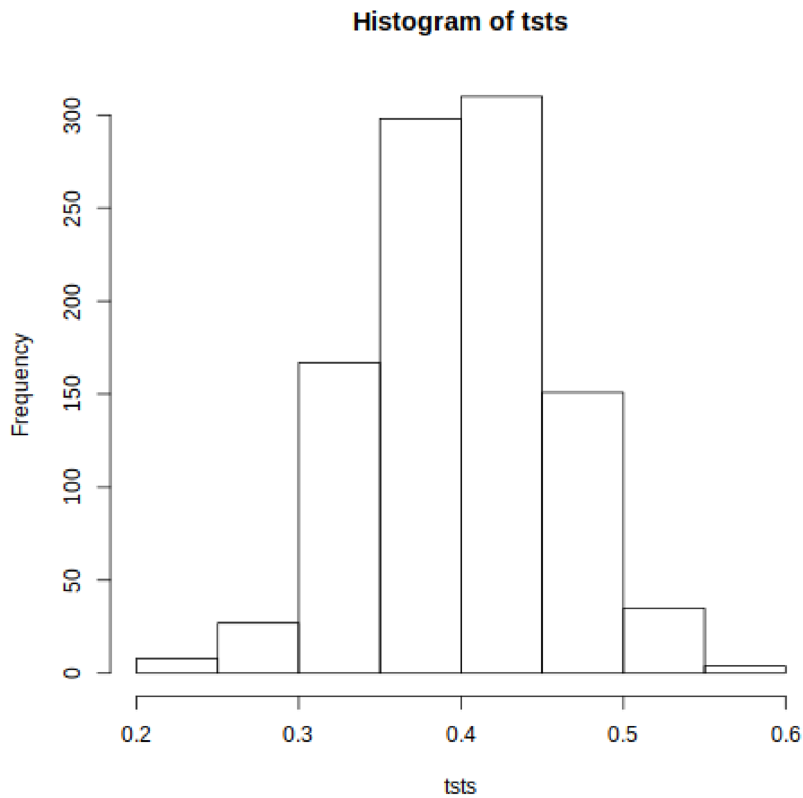
c)

```
simExam <- function(sz, P){
  # Here 1 == TRUE, 2 == FALSE
  start <- sample(1:2,size=1, prob = c(1/2,1/2))
  numTrue <- if(start == 1) 1 else 0
  prev <- start
  for(i in 2:sz)
  {
    prev <- sample(1:2, size = 1, prob = P[prev, ])
    numTrue <- if(prev == 1) numTrue + 1 else numTrue
  }
  return(numTrue / sz)
}

reps <- 1000
size <- 100
tsts <- rep(0, reps)
for(i in 1:reps){
  tsts[i] <- simExam(size, P)
}
sd(tsts)
mean(tsts)
sprintf("After %s exams: \n\t mean frac. of TRUEs: %s \n\t SD of TRUEs: %s ",
        reps, mean(tsts), sd(tsts))
```

After 1000 exams:
mean frac. of TRUEs: 0.40436
SD of TRUEs: 0.0575056283966822

```
hist(tsts)
```


Histogram of tsts
Frequency
0
50
100
150
200
250
300
0.2
0.3
0.4
0.5
0.6
tsts