Assignment 5

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Problem 1

a)

To prove: $E(W_n) = c_b E(W_{n-1})$

$$\begin{aligned} E(W_n) &= \sum_{W_{n-1}} E(W_n | W_{n-1} = w_{n-1}) && (1) \\ &= ((1-b)w_{n-1} + 2bw_{n-1})(0.5) && (2) \\ &\quad + ((1-b)w_{n-1} + 1.4bw_{n-1})(0.5) \\ &= (1 + .2b)w_{n-1} && (3) \\ &= c_b w_{n-1} && (4) \end{aligned}$$

Here c_b is maximized by when $b = 1$.
$E(W_n)$ is recursively defined so:

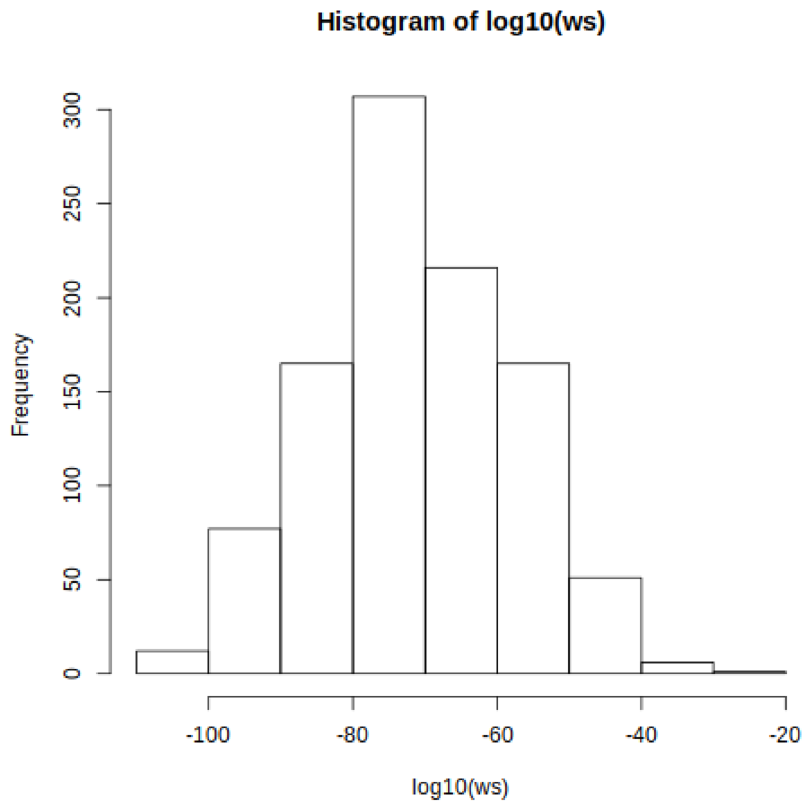
$$\begin{aligned} E(W_n) &= c_b E(w_{n-1}) && (5) \\ &= c_b c_b E(w_{n-2}) && (6) \\ &= c_b c_b ... c_b E(w_{n-n}) && (7) \\ &= c_b c_b ... c_b (1) && \text{(Given)} \\ &= c_b^n && (8) \end{aligned}$$

b)

```
worth <- function(b ){
  w <- 1
  for(j in 2:1460)
  {
```

```
    if(sample(1:2, 1) == 1)
    {
      w = (1-b)*w + 2*b*w
    }
    else
    {
      w = (1-b)*w +.4*b*w
    }
  }
  return(w)
}

n <- 1000
ws <- rep(0, n)
for(i in 1:n){
  ws[i] <- worth(1)
}
hist(log10(ws))
```



c)

$\mathrm{M}_i = \begin{cases} 2 & \text{prob: } .5 \\ .4 & \text{prob: } .5 \end{cases}$

So:

$\log M_i = \begin{cases} .30 & M_i = 2 \\ -.39 & M_i = .4 \end{cases}$

The LLN says as n $\to \infty, \overline{L_n} \to E(L_n)$

We are given: $L_n = X_1 + ... + X_n$ so:

$\overline{L_n} = E(X_1) + ... + E(X_n)$

Where: $E(X_n) = (.30)(.5) + (-.39)(.5)$

$= -0.04846$

Thus:

$$\overline{L_n} = n * -0.0486$$
$$= \text{-}\infty \quad \text{as n} \to \infty$$

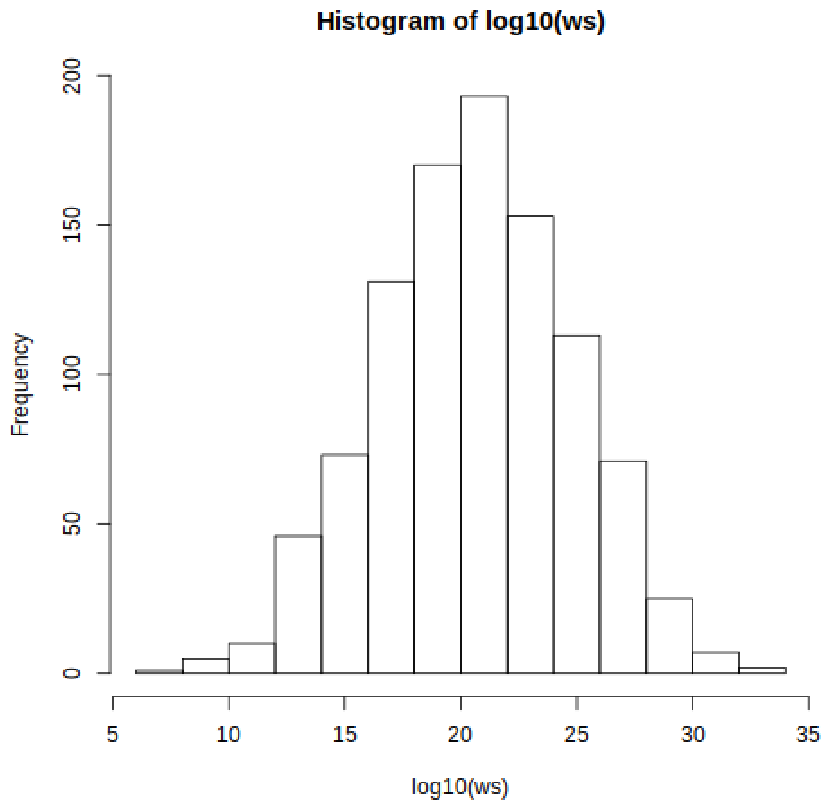
d)

$$\begin{aligned} E(X_n) &= E(\log M_i) \\ &= (0.5)(\log(1 - .6b)\log(1+b)) \end{aligned}$$

Now we take the derivative of this, set to zero, and solve for b which gives us:

$$b = \frac{1}{3}$$

```
ws <- rep(0, n)
for(i in 1:n){
  ws[i] <- worth(1/3)
}
hist(log10(ws))
```



2

a)

By the linearity of expectation we know that:

$$\begin{aligned} E(Z) &= aE(X) + bE(Y) \\ &= a\mu + b\mu && \text{(Given)} \\ &= (a+b)\mu \end{aligned}$$

So in order for Z to be unbiased estimator of μ

$$a + b = 1$$

b)

$$\begin{aligned} var(Z) &= a^2 var(X) + b^2 var(Y) \\ &= a^2 + 4b^2 && \text{(Given.)} \end{aligned} \tag{9}$$

3

a)

$$\begin{aligned} P\{|Y-\mu| \geq c\sigma\} &= P\{(Y-\mu)^2 \geq c^2\sigma^2\} \\ &\leq \frac{E((Y-\mu)^2)}{c^2\sigma^2} && \text{(By Markov's)} \\ &= \frac{Var(Y)}{c^2\sigma^2} \\ &= \frac{1}{c^2 n^2} \\ &\leq \frac{1}{c^2} \end{aligned} \tag{10}$$

b)

$E(X) = np = \frac{1}{6}6000 = 1000$
$var(X) = np(1-p) = 6000(\frac{1}{6})(\frac{5}{6}) = 833.33$

$c\sigma = 100$ so:

$$\begin{aligned} c &= \frac{100}{\sigma} \\ &= \frac{100}{\sqrt{833.3}} \\ &= 3.464 \end{aligned}$$

Thus: $P\{|X - E(X)| \geq 100\} \leq \dfrac{1}{3.454^2}$

c)

$$
\begin{aligned}
P\{|X - E(X)| \geq 100\} &= P\{|\frac{X - E(X)}{\sigma}| \geq \frac{100}{\sigma}\} && (11)\\
&= P\{|\frac{X - E(X)}{\sigma}| \geq c\} && (12)\\
&= P\{-c \leq Z \geq c\} && (13)
\end{aligned}
$$

$$P\{-c \leq Z \geq c\} = 0.00053$$

d)

```
pbinom(900, size = 6000, prob = 1/6) +
(1 -pbinom(1100, size = 6000, prob = 1/6))
```

4)

a)

$$\text{X} = \begin{cases} 0 & \text{prob: } .1 \\ 1 & \text{prob: } .4 \\ 2 & \text{prob: } .5 \end{cases}$$

Let Y be an r.v. representing senior parents. Y can take on a value between 0 and 6 with pmf:

$$\text{f}_y = \begin{cases} 0 & \text{prob: } .001 \\ 1 & \text{prob: } .012 \\ 2 & \text{prob: } .072 \\ 3 & \text{prob: } .184 \\ 4 & \text{prob: } .315 \\ 5 & \text{prob: } .3 \\ 6 & \text{prob: } .125 \end{cases}$$

b)

Let the r.v Z = $X_1 + \ldots + X_{1400}$ where X_i represents how many parents student is bringing, E(X) = .4 + .5(2) = 1.4

$$E(Z) = E(X_i) + ... + E(X_{1400})$$
$$= 1.4 * 1400$$
$$= 1960$$

Similarly:

$$var(Z) = var(X) * 1400$$
$$= 616 * 1400$$
$$= 8624000$$

So: $SD_Z = \sqrt{8}624000 = 24.8$

Now we can find $P\{Z \leq N\} \leq .95$ where N is the number of seats we have at our disposal.

Can now plug this into R to get the answer:

```
# qnorm returns the number whose cumulative distribution matches the probability
# given so all we do is plug in the appropriate vals and we have our answer!
  qnorm(p=.95, mean=1960, sd=24.8)
```

```
2000.7923699484
```