Proof without words - Exploration

Jake Bruner

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respond: Do you agree with the statement “to see” is “to understand”?

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quote nelson discuss the question

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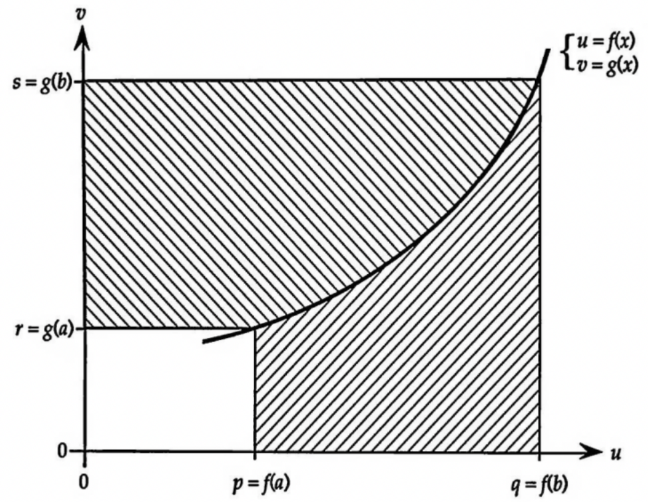
3 When Visual Proof Becomes Hectic

Consider Richard Courant's proof of integration by parts:



Area + Area $= qs - pr$

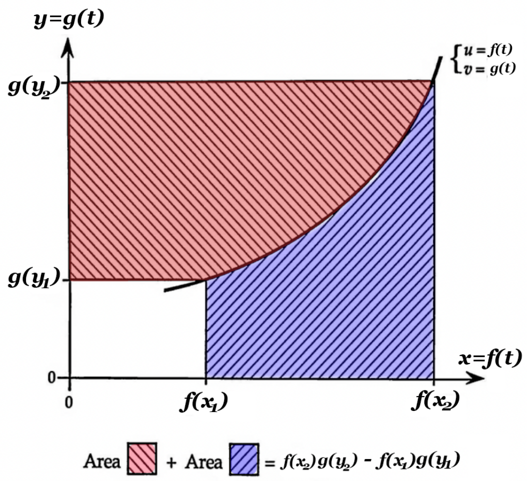
$$\int_r^s u\,dv + \int_p^q v\,du = uv\Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

—Richard Courant

This proof–although rigorous–is an extremely confusing jumble of variables upon first glance. Without text to aid, a few key assumptions are not immediately obvious. For instance, Courant's proof includes that the curve is bound by the parametric equation, $(f(x),\ g(x))$, made worse by the fact x is the parameterising variable. Furthermore, this 'visual' proof either requires the reader to figure out or have prior knowledge of how to integrate a parametric curve. Overall, the beauty of this proof cannot be realised without close examination, which detracts from its accessibility.

Attempting to depict this proof in a more straight-forward way, it might be useful to relabel the axis and variables to be more friendly to math conventions.



Breaking this down piece by piece, it's initially helpful to reiterate these parametric equations in terms of non-parametric functions so that integration and derivation are more explicit. Clearly here, y is a function of x and x is a function of y).

$$x(y) = x = f(t)$$
$$y(x) = y = g(t)$$

Given that these variables simply represent the x and y axis, it is possible to understand these integrals without knowing the process for parametric curves. Simply put, if one knows that an area under a curve can be represented by an integral with respect to the axis variable (usually x), it should logically be a similar process to see the area bounded by the vertical axis. For the x axis, the area in the blue is simply:

$$A_b = \int_{x_1}^{x_2} y \; dx$$

Now, the area bound by the curve and the y-axis takes the same form. Although, for many this seems like sacrilege, this idea is perfectly legal; one can imagine this as defining the horizontal axis as y and the vertical axis as x then finding the area under a curve. So hence, the area in red is simply:

$$A_r = \int_{y_1}^{y_2} x \; dy$$

Now, the sum of these two integrals would represent the total area of the rectangular section of the graph, minus the area in the white. Again, the total area $A_b + A_r$ is equal to the large rectangle $x_2 \times y_2$ minus the small, white rectangle $x_1 \times y_1$.

$$\begin{array}{cccc} A_b + & A_r = & \textit{large rectangle} - & \textit{small rectangle} \\ \displaystyle\int_{x_1}^{x_2} y\ dx + & \displaystyle\int_{y_1}^{y_2} x\ dy = & (x_2 \times y_2)\ - & (x_1 \times y_1) \end{array}$$

From here, its helpful to write the subtraction of the two areas as a product of the axis variables x and y evaluated at the same bounds as the integral. This can be thought as similar to the the reverse process of evaluating a definite integral, because:

$$xy\Big|_{x_1}^{x_2} = x_2 y_2 - x_1 y_1$$

and

$$yx\Big|_{y_1}^{y_2} = x_2 y_2 - x_1 y_1$$

(This makes even more sense when the functions are written explicitly.) So substituting this simplification into the original equation:

$$\int_{x_1}^{x_2} y\ dx + \int_{y_1}^{y_2} x\ dy = xy\Big|_{x_1}^{x_2} = yx\Big|_{y_1}^{y_2}$$

Rewritten in function notation:

$$\int_{x_1}^{x_2} y(x)\ dx + \int_{y_1}^{y_2} x(y)\ dy = x \cdot y(x)\Big|_{x_1}^{x_2} = y \cdot x(y)\Big|_{y_1}^{y_2}$$

The most conceptually foreign step is now converting all these bounds and function variables into the parametric variable t. Since the functions x and y are defined in terms of the functions $f(t)$ and $g(t)$, its possible to define x as a function of t instead of a function of y (and vice-versa) to eliminate the self-referential and self-inverse confusion. So hence, rewritten in terms of t:

$$\int_{t_1}^{t_2} y(t)\ dx(t) + \int_{t_1}^{t_2} x(t)\ dy(t) = x(t) \cdot y(t)\Big|_{t_1}^{t_2}$$

Rearranging we get the a form of integration by parts, given that:

$$d\big(y(t)\big) = y'(t)\ dt$$
$$d\big(x(t)\big) = x'(t)\ dt:$$

Hence:

$$\int_{t_1}^{t_2} x(t)\ d\big(y(t)\big) = x(t)y(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} y(t)\ d\big(x(t)\big)$$

Or sometimes written as:

$$\int_{t_1}^{t_2} x(t)y'(t)\ dt = x(t)y(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} y(t)x'(t)\ dt$$

Rewritten with indefinite integrals (if $t_1 \to -\infty$, and $t_2 \to \infty$), it is also:

$$\int x\ dy = xy - \int y\ dx$$

$\blacksquare$

4 Conclusion