# Game Theory from Mathematical Optimization and Economic Theory

Jacob Wyngaard

## 1 Introduction

Game Theory is a branch of economics (and applied mathematics) concerned with decision-making among multiple individuals whose objective functions depend on the decisions of other players. The field was popularized by John Nash's foundational work, and it has since been applied to areas such as nuclear strategy, sports modeling, negotiation, and more.

## 2 Basic Terminology

- **Players:** the decision-makers.
- **Payoff functions:** each player's objective function; it maps a profile of strategies (one strategy per player) to a payoff.
- **Move:** a point where a player must make a choice.
- **Play:** a particular set of moves and choices realized during the game.
- **Strategy:** a predetermined rule specifying a choice at every move where that player could be called upon to act.

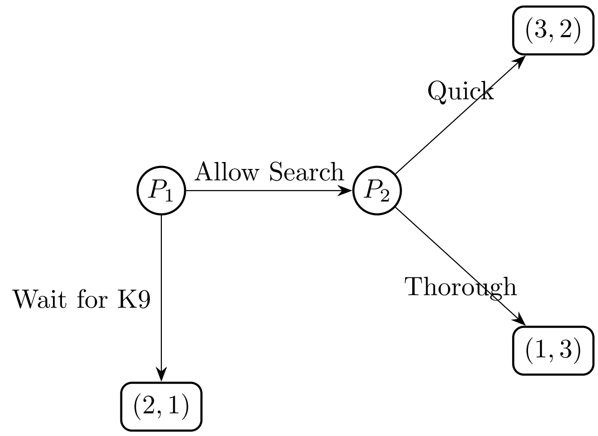
## 3 Forms of Games

Two common representations are *extensive form* (sequential decisions) and *normal form* (simultaneous or independent decisions).

### 3.1 Extensive Form (Decision Trees)

When decisions are made in sequence, a game is described in extensive form and represented by a decision tree. Decision trees are solved *backwards* via backward induction: at each decision node, the acting player selects an action that maximizes their payoff, anticipating rational play in future nodes.

**Illustrative structure.** Below is an example schematic):




### 3.2 Normal Form (Payoff Matrices)

When players choose independently and/or simultaneously, the game is described in normal form using a payoff matrix. We typically assume all players know the payoff matrix.

**Prisoner's Dilemma Payoff Matrix**

| | **Cooperate** | **Defect** |
|---|---|---|
| **Cooperate** | $\boxed{(3,3)}$ | $(0,5)$ |
| **Defect** | $(5,0)$ | $\boxed{(1,1)}$ |

*Each player has a dominant strategy to defect, leading to the Nash equilibrium* $(1,1)$*, even though mutual cooperation* $(3,3)$ *is Pareto superior.*

## 4 Two-Player Zero-Sum Games

In a two-player zero-sum game, one player's gain is exactly the other's loss. Thus, it is common to display only Player 1's payoffs; Player 2's payoffs are the negatives.

### 4.1 The Maximin / Minimax Idea

Each player chooses a strategy based on the *best worst-case* outcome:

- Player 1 computes the worst payoff achievable against each of their strategies, then selects the strategy with the largest such worst payoff (*maximin* or $\max_i \min_j \Pi_{ij}$).
- Player 2 similarly selects the strategy that minimizes Player 1's best possible outcome (*minimax* or $\min_j \max_i \Pi_{ij}$).

### 4.2 Saddle Points

A **saddle point** (in pure strategies) is an entry of the payoff matrix that is:

- the minimum in its row (so Player 2 cannot make Player 1 worse if Player 1 commits to that row), and

- the maximum in its column (so Player 1 cannot do better if Player 2 commits to that column).

If a saddle point exists, it yields a pure-strategy equilibrium and the game has a well-defined value in pure strategies.

### 4.3 Eliminating Strictly Dominated Strategies

A strategy is **strictly dominated** if there exists another strategy that yields a strictly better payoff against every possible opponent action. Such strategies can be eliminated to simplify the game without changing the equilibrium set of the reduced game.

### 4.4 Saddle Points Need Not Exist

Not every finite two-player zero-sum game has a saddle point in pure strategies. Some payoff matrices have no entry satisfying the saddle-point condition.

## 5 The Minimax Theorem and Mixed Strategies

Von Neumann's Minimax Theorem states that every finite two-player zero-sum game has a value when *mixed strategies* (randomization over pure strategies) are allowed. Equivalently, there exists an equilibrium pair of mixed strategies that satisfies:

$$\max_x \min_y V(x,y) \;=\; V(x^*,y^*) \;=\; \min_y \max_x V(x,y),$$

where $V(x,y)$ denotes Player 1's expected payoff under mixed strategies $x$ and $y$.

## 6 Two-Person Non-Zero-Sum Games

In non-zero-sum games, one player's win does not require the other's loss. Cooperation can sometimes yield outcomes that are better for both players.

### 6.1 Coordination / Cooperation Example

A stylized "push against the door" situation: if both players cooperate, both benefit; if one does not, outcomes may be poor for both.

### 6.2 Chicken and Mutually Assured Destruction

In some non-zero-sum games (e.g., "Chicken"), the presence of a disastrous outcome for mutual aggression leads to strategic bluffing and risk-taking.

## 7 Nash Equilibrium (Finite Games)

Nash proved that every finite game (any finite number of players and strategies) admits at least one equilibrium in mixed strategies. This generalizes the minimax perspective beyond the two-player zero-sum case.

# 8 Cooperative Games

In a **cooperative game**, players may communicate prior to play and can make binding agreements.

## 8.1 Nash Cooperative (Bargaining) Solution

For cooperative games *without side payments*, Nash's bargaining solution uses a **threat point** (disagreement payoff) and imposes axioms under which the solution is unique. Commonly listed assumptions include:

- **Symmetry:** relabeling the players does not change the solution.
- **Independence of linear transformations:** affine changes of utility scales do not change the solution appropriately.
- **Independence of irrelevant alternatives:** removing unchosen feasible payoffs does not change the selected outcome.
- **Pareto optimality:** no other feasible payoff makes one player better off without making the other worse off.

## 8.2 Cooperative Games with Side Payments

When side payments are allowed within coalitions, it is often sufficient to track only the total value that a coalition can generate ("the size of the pie").

## 8.3 Imputations and Rationality

In transferable-utility cooperative games, an **imputation** is a payoff vector that is:

- **Efficient:** the grand coalition's total value is fully allocated, and
- **Individually rational:** each player receives at least what they can guarantee alone.

## 8.4 Effective Coalitions and the Core

A coalition is **effective** at a payoff allocation if it can guarantee its members more by deviating and redistributing its coalition value. The **core** is the set of imputations that are not blocked by any coalition; equivalently, no coalition can deviate and make all its members strictly better off.

# 9 Games with Infinitely Many Players (High-Level Remarks)

Under suitable assumptions, as the number of players grows large, multiple solution concepts can converge. In many "large population" limits, equilibria exist and cooperative concepts (like the Shapley value and the core) may approach the same limiting prediction. This is one reason game theory is often described as being especially clean for two-player or infinite-player extremes.

# A Proof of Minimax Theorem

Proof reproduced from Rutgers lecture notes with additional commentary and explanation provided by the author.

**Theorem 1** (von Neumann Minimax Theorem)**.** *Let $A = (a_{i,j})$ be an $m \times n$ matrix representing the payoff matrix for a two-person, zero-sum game. Then the game has a value and there exists a pair of mixed strategies which are optimal for the two players.*

Let us recall the following definition where, for a mixed strategy pair $(x, y)$, we define

$$V(x, y) := \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \, a_{i,j} \, y_j.$$

**Definition 1.** *A pair of mixed strategies $(x^*, y^*)$ is said to be an* equilibrium point *for a two-person, zero-sum game provided*

$$V(x, y^*) \leq V(x^*, y^*) \quad \textit{for all } x \in X_m, \qquad V(x^*, y^*) \leq V(x^*, y) \quad \textit{for all } y \in Y_n.$$

Note that this is equivalent to the assertion that

$$\max_{x \in X_m} V(x, y^*) \; = \; V(x^*, y^*) \; = \; \min_{y \in Y_n} V(x^*, y).$$

**Theorem 2.** *Each of the following three conditions are equivalent:*

*(a)* *An equilibrium pair exists.*

*(b)*

$$v_A := \max_{x \in X_m} \min_{y \in Y_n} V(x, y) \; = \; \min_{y \in Y_n} \max_{x \in X_m} V(x, y) =: v_B.$$

*(c)* *There exists a $v \in \mathbb{R}$ and $x^{(0)} \in X_m$, $y^{(0)} \in Y_n$ such that*

$$\textit{(i)} \quad \sum_{i=1}^{m} a_{i,j} \, x_i^{(0)} \geq v, \qquad j = 1, 2, \ldots, n,$$

$$\textit{(ii)} \quad \sum_{j=1}^{n} a_{i,j} \, y_j^{(0)} \leq v, \qquad i = 1, 2, \ldots, m.$$

$\sum_{i=1}^{m} a_{i,j} \, x_i^{(0)}$ is Player 1's expected gain with the mixed strategy $x^{(0)}$ when Player 2 uses pure strategy $j$

$\sum_{j=1}^{n} a_{i,j} \, y_j^{(0)}$ is Player 2's expected loss with the mixed strategy $y^{(0)}$ when Player 1 uses pure strategy $i$

Thus, $(c)$ stipulates that there exists $v, x^{(0)}, y^{(0)}$ such that

- $x^{(0)}$ guarantees that Player 1 has at least an expected gain of $v$ when Player 2 uses a pure strategy
- $y^{(0)}$ guarantees that Player 2 has at most an expected loss of $v$ when Player 1 uses a pure strategy

*Proof.* To see that (a)⇒(b), consider an equilibrium pair $(x^*, y^*)$. Then

$$\begin{aligned} v_B := \min_{y\in Y_n} \max_{x\in X_m} V(x,y) &\le \max_{x\in X_m} V(x,y^*) = V(x^*,y^*) \\ &= \min_{y\in Y_n} V(x^*,y) \le \max_{x\in X_m} \min_{y\in Y_n} V(x,y) =: v_A. \end{aligned}$$

But since we always have $v_A \le v_B$, we must have equality throughout.

> Why do we have always $v_A \le v_B$?
>
> For any $x_0 \in X_m$ and $y_0 \in Y_n$, $\min_{y\in Y_n} V(x_0,y) \le V(x_0,y_0) \le \max_{x\in X_m} V(x,y_0)$
>
> Since this holds for any $x_0 \in X_m$, it must hold at $x_0 = \arg\max_{x\in X_m} \min_{y\in Y_n} V(x_0,y)$
>
> Since this holds for any $y_0 \in Y_n$, it must hold at $y_0 = \arg\min_{y\in Y_n} \max_{x\in X_m} V(x,y_0)$
>
> Thus, $\max_{x\in X_m} \min_{y\in Y_n} V(x,y) \le V(x_0,y_0) \le \min_{y\in Y_n} \max_{x\in X_m} V(x,y)$
>
> Therefore, $\max_{x\in X_m} \min_{y\in Y_n} V(x,y) \le \min_{y\in Y_n} \max_{x\in X_m} V(x,y)$

To see that (b)⇒(c), suppose that $v_A = v_B = v$. Let $x^{(0)}$ be a maximin strategy and $y^{(0)}$ be a minimax strategy. Then for all $j = 1, 2, \ldots, n$ and for all $i = 1, 2, \ldots, m$ we have

$$\begin{aligned} \sum_{i=1}^{m} a_{i,j}\, x_i^{(0)} = V\big(x^{(0)}, \beta_j\big) \;\ge\; \min_{y\in Y_n} V\big(x^{(0)}, y\big) \;=\; \max_{x\in X_m} \min_{y\in Y_n} V(x,y) \\ = v \;=\; \min_{y\in Y_n} \max_{x\in X_m} V(x,y) \;=\; \max_{x\in X_m} V\big(x, y^{(0)}\big) \\ \ge V\big(\alpha_i, y^{(0)}\big) = \sum_{j=1}^{n} a_{i,j}\, y_j^{(0)}. \end{aligned}$$

The left inequality gives (i) and the right inequality gives (ii).

> $\alpha_i$ represents Player 1 selecting strategy $i$
> $\beta_j$ represents Player 2 selecting strategy $j$

Finally, to see that (c)⇒(a), we see from (i) and (ii) that

$$V\big(x^{(0)}, y\big) \ge v \ge V\big(x, y^{(0)}\big) \quad \text{for all } x \in X_m,\ y \in Y_n.$$

Putting $x = x^{(0)}$ and $y = y^{(0)}$ gives $v = V\big(x^{(0)}, y^{(0)}\big)$, hence $(x^{(0)}, y^{(0)})$ is an equilibrium pair. □

Now we ask the question whether, given a game described by a payoff matrix $A$, any one (and hence all) of these conditions hold. That one does is the force of von Neumann's theorem.

We present Nash's proof of the theorem, which uses the Brouwer fixed point theorem. Recall:

**Theorem 3** (Brouwer Fixed Point Theorem). *Let $K \subset \mathbb{R}^p$ be a closed, bounded, and convex set. If $f : K \to K$ is continuous, then there is an $\hat{x} \in K$ such that $f(\hat{x}) = \hat{x}$.*

## Nash's proof of von Neumann's theorem

*Proof.* We know that the set $X_m \times Y_n$ of pairs of mixed strategies is a closed, bounded, and convex subset of $\mathbb{R}^{m+n}$. We define a transformation $T : X_m \times Y_n \to X_m \times Y_n$.

For $(x, y) \in X_m \times Y_n$, let

$$c_i(x,y) := \begin{cases} V(\alpha_i, y) - V(x,y), & \text{if this quantity is positive,} \\ 0, & \text{otherwise,} \end{cases}$$

$$d_j(x,y) := \begin{cases} V(x,y) - V(x,\beta_j), & \text{if this quantity is positive,} \\ 0, & \text{otherwise.} \end{cases}$$

> $c_i(x,y)$ represents how much Player 1 benefits by switching from strategy $x$ to strategy $i$
>
> $d_j(x,y)$ represents how much Player 2 benefits (lower losses) by switching from strategy $y$ to strategy $j$

Define $T(x,y) = (x', y')$ by

$$x'_i := \frac{x_i + c_i(x,y)}{1 + \sum_{k=1}^m c_k(x,y)}, \qquad y'_j := \frac{y_j + d_j(x,y)}{1 + \sum_{k=1}^n d_k(x,y)}.$$

Note that $x' \geq 0$ since $x \geq 0$, $c \geq 0$, and $1 + \sum_k c_k \geq 0$. Moreover,

$$\sum_{i=1}^m x'_i = \frac{\sum_{i=1}^m (x_i + c_i(x,y))}{1 + \sum_{k=1}^m c_k(x,y)} = \frac{1 + \sum_{i=1}^m c_i(x,y)}{1 + \sum_{k=1}^m c_k(x,y)} = 1.$$

Likewise $y' \geq 0$ and $\sum_{j=1}^n y'_j = 1$. Hence $T$ maps $X_m \times Y_n$ into itself.

> $T(x,y)$ shifts probabilistic "weight" to strategies that outperform the current $x$ and $y$
>
> Note that the outperforming strategies assume that the other player holds their strategy constant (aligning with the concept of the Nash equilibrium)

We first show that $(\hat{x}, \hat{y})$ is an equilibrium pair if and only if it is a fixed point of $T$. Note that $c_i(x,y)$ measures the amount that $\alpha_i$ is better than $x$ (if at all) as a response against $y$, and $d_j(x,y)$ measures the amount that $\beta_j$ is better than $y$ (if at all) as a response against $x$.

If $(\hat{x}, \hat{y})$ is an equilibrium pair, then $\hat{x}$ is good against $\hat{y}$, so $c_i(\hat{x}, \hat{y}) = 0$ for all $i$, hence $\hat{x}' = \hat{x}$. Similarly $d_j(\hat{x}, \hat{y}) = 0$ for all $j$, hence $\hat{y}' = \hat{y}$. Thus $T(\hat{x}, \hat{y}) = (\hat{x}, \hat{y})$.

> If $(x,y)$ is an equilibrium pair, then no strategies outperform $x$,$y$ and so no probabilistic weight is shifted

Conversely, suppose $(\hat{x}, \hat{y})$ is a fixed point of $T$. We show there must be at least one index $i_0$ for which both $\hat{x}_{i_0} > 0$ and $c_{i_0}(\hat{x}, \hat{y}) = 0$. To see this, recall that

$$V(\hat{x}, \hat{y}) = \sum_{i=1}^m \hat{x}_i \, V(\alpha_i, \hat{y}),$$

so we cannot have $V(\hat{x},\hat{y}) < V(\alpha_i,\hat{y})$ for every $i$ with $\hat{x}_i > 0$. Thus for some $i_0$ with $\hat{x}_{i_0} > 0$ we have $V(\alpha_{i_0},\hat{y}) - V(\hat{x},\hat{y}) \leq 0$, i.e. $c_{i_0}(\hat{x},\hat{y}) = 0$.

> We must have an $i_0$ such that $c_{i_0}(\hat{x},\hat{y}) = 0$ as not every pure strategy can be a better strategy than $x$. This follows from $x$ being a linear combination of pure strategies

Since $(\hat{x},\hat{y})$ is a fixed point,

$$\hat{x}_{i_0} = \hat{x}'_{i_0} = \frac{\hat{x}_{i_0} + c_{i_0}(\hat{x},\hat{y})}{1 + \sum_{k=1}^{m} c_k(\hat{x},\hat{y})} = \frac{\hat{x}_{i_0}}{1 + \sum_{k=1}^{m} c_k(\hat{x},\hat{y})}.$$

Because $\hat{x}_{i_0} > 0$, this forces $\sum_{k=1}^{m} c_k(\hat{x},\hat{y}) = 0$. Since each $c_k(\hat{x},\hat{y}) \geq 0$, we conclude $c_k(\hat{x},\hat{y}) = 0$ for all $k$.

Therefore $V(\alpha_k,\hat{y}) \leq V(\hat{x},\hat{y})$ for all $k$, so $\hat{x}$ is at least as good a response against $\hat{y}$ as any pure strategy $\alpha_k$; hence $\hat{x}$ is good against $\hat{y}$. A symmetric argument shows $\hat{y}$ is good against $\hat{x}$. Thus $(\hat{x},\hat{y})$ is an equilibrium pair.

Now that we know equilibrium pairs are precisely fixed points of $T$, it suffices to show $T$ has a fixed point. This follows from Brouwer, since $X_m \times Y_n$ is closed, bounded, convex, $T$ maps it into itself, and $T$ is continuous. $\square$