12 Calculus of Variations

The first approach to the control problem will be that of the calculus of variations.[1] The control problem treated in the classical calculus of variations is that of choosing a time path for a state variable connecting given initial and terminal points so as to maximize the value of the integral of a given function of the state variable, the time rate of change of the state variable, and time. Thus, the *classical calculus of variations problem* is:

$$\begin{aligned} \max_{\{x(t)\}} \quad J &= \int_{t_0}^{t_1} I(x(t), \dot{x}(t), t)\, dt \\ x(t_0) &= x_0 \\ x(t_1) &= x_1, \end{aligned} \qquad \textbf{(12.0.1)}$$

where $I(x, \dot{x}, t)$ is a given continuously differentiable function and t_0, t_1, x_0, and x_1 are given parameters. This problem can be considered the special case of the general control problem (11.1.21) in which there is no dependence on final considerations (the problem is one of Lagrange); there is only one state variable and one control variable; the control variable is simply the time rate of change of the state variable, the equation of motion being:

$$\dot{x} = u, \qquad (12.0.2)$$

so u is replaced by $\dot{x}$ in $I(\cdots)$; and the control variable can take any value:

$$\Omega = E. \tag{12.0.3}$$

Thus, the only restriction on the control trajectory is that it be a piecewise continuous function of time. Any trajectory $\{x(t)\}$ satisfying the boundary conditions in (12.0.1) and the continuity condition that $x(t)$ be continuous and $\dot{x}(t)$ be piecewise continuous functions of time is called *admissible*, and the classical calculus of variations problem is that of choosing an admissible trajectory which maximizes the integral objective functional. Some alternative admissible trajectories are shown in Fig. 12.1.

The classical calculus of variations problem can be considered the dynamic analogue of the classical programming problem. The replacement of u by $\dot{x}$ in the objective function is analogous to substitution in the objective function, using the equality constraints in classical programming. In addition, the consideration of inequality constraints, which led in the static case to the modern developments of linear and nonlinear programming, leads in the dynamic case to the modern developments of dynamic programming, the maximum principle, and modern treatments of the calculus of variations.

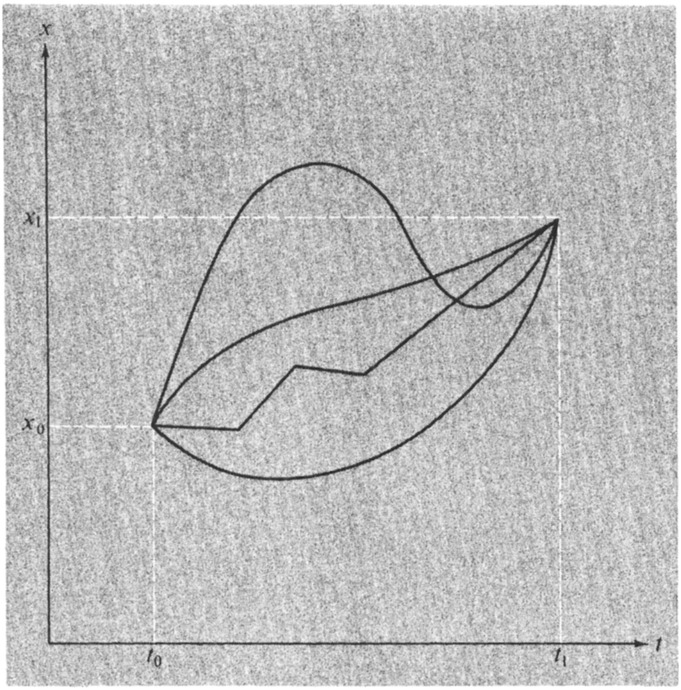

Fig. 12.1
Some Alternative Admissible Trajectories

12.1 Euler Equation

A solution to the calculus of variations problem (12.0.1) is an admissible trajectory $\{x(t)\}$ which maximizes the value of the integral objective functional. Assuming such a solution exists, it must satisfy certain necessary conditions which can be considered dynamic analogues of the necessary conditions for unconstrained classical programming problems. The necessary condition analogous to the first order condition that the derivative vanish is the *Euler equation* of the calculus of variations.

The necessary conditions in classical programming problems were obtained by considering small variations about the solution, where the solution was a point in Euclidean space. The necessary conditions for the classical calculus of variations problem can be obtained in an analogous way—by considering small variations about the solution trajectory. Assuming $\{x(t)\}$ is a solution trajectory, consider the variation about the solution trajectory $\{z(t)\}$ where:

$$z(t) = x(t) + \varepsilon\eta(t), \tag{12.1.1}$$

and $\eta(t)$ is any continuous function with piecewise continuous derivative for which:

$$\eta(t_0) = \eta(t_1) = 0. \tag{12.1.2}$$

The variation about the solution trajectory $\{z(t)\}$ satisfies both the boundary and the continuity conditions, and hence is an admissible trajectory. The parameter ε measures the "difference" between the solution trajectory $\{x(t)\}$ and the variation about the solution trajectory $\{z(t)\}$ where:

$$\lim_{\varepsilon \to 0} \{z(t)\} = \{x(t)\}. \tag{12.1.3}$$

The two trajectories are shown in Fig. 12.2.

The value of the objective functional for the variation about the solution trajectory $\{z(t)\}$ can be considered a function of ε:

$$J(\varepsilon) = \int_{t_0}^{t_1} I(x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}, t)\, dt, \tag{12.1.4}$$

and, since $\{x(t)\}$ is a solution, $J(\varepsilon)$ must be maximized at $\varepsilon = 0$, requiring that:

$$\frac{dJ}{d\varepsilon}(0) = 0 \tag{12.1.5}$$

for all $\eta(t)$ satisfying the appropriate continuity and boundary conditions.

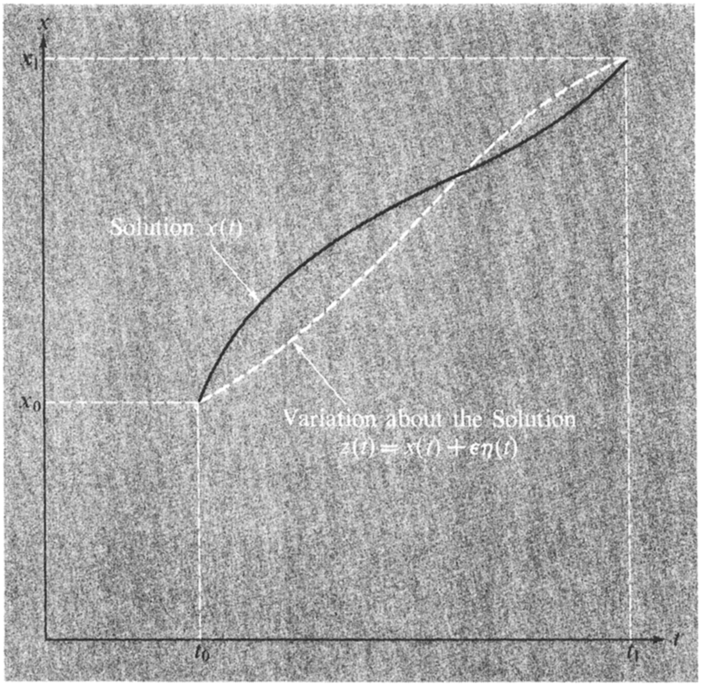

Fig. 12.2
Variation about the Solution Trajectory

But:

$$\frac{dJ}{d\varepsilon}(0) = \int_{t_0}^{t_1} \left(\frac{\partial I}{\partial x} \eta + \frac{\partial I}{\partial \dot{x}} \dot{\eta} \right) dt. \tag{12.1.6}$$

Integrating the second term by parts yields:

$$\frac{dJ}{d\varepsilon}(0) = \int_{t_0}^{t_1} \frac{\partial I}{\partial x} \eta \, dt + \left[\frac{\partial I}{\partial \dot{x}} \eta \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) \eta \, dt, \tag{12.1.7}$$

so, from the boundary conditions (12.1.2):

$$\frac{dJ}{d\varepsilon}(0) = \int_{t_0}^{t_1} \left[\frac{\partial I}{\partial x} - \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) \right] \eta \, dt = 0. \tag{12.1.8}$$

In order for the integral to vanish for all $\eta(t)$ satisfying the boundary and continuity conditions, it is necessary that the term in brackets vanish for all t between t_0 and t_1:

$$\frac{\partial I}{\partial x} - \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) = 0, \tag{12.1.9}$$

since otherwise $\eta(t)$ can be chosen to be nonzero at points where this term does not vanish, leading to a nonzero integral in (12.1.8), a result known as the *fundamental lemma of the calculus of variations*.

Equation (12.1.9) is the *Euler equation*.[2] It is a second order ordinary differential equation, as can be seen by taking the indicated total time derivative of $\partial I / \partial \dot{x}$, which is itself a function of x, $\dot{x}$, and t, and writing the Euler equation as:

$$\left(\frac{\partial^2 I}{\partial \dot{x}^2} \right) \frac{d^2 x}{dt^2} + \left(\frac{\partial^2 I}{\partial x \, \partial \dot{x}} \right) \frac{dx}{dt} + \left(\frac{\partial^2 I}{\partial t \, \partial \dot{x}} - \frac{\partial I}{\partial x} \right) = 0. \tag{12.1.10}$$

The associated boundary conditions are those given in the problem, the initial and terminal values:

$$\begin{aligned} x(t_0) &= x_0 \\ x(t_1) &= x_1. \end{aligned} \tag{12.1.11}$$

Any trajectory $\{x(t)\}$ satisfying the Euler equation (12.1.9) for all t, $t_0 \leq t \leq t_1$, and satisfying the boundary conditions (12.1.11) is called an *extremal*, and, if a solution exists to the classical calculus of variations problem, it is necessary that it be an extremal.

In the general case, the intermediate function (integrand) depends on three variables: $I(x, \dot{x}, t)$. If, however, the intermediate function does not depend explicitly on $\dot{x}$ then the Euler equation becomes:

$$\frac{\partial I}{\partial x} = 0, \tag{12.1.12}$$

as in the unconstrained classical programming problem. In this case the dynamic problem is in reality only a succession of static classical programming problems indexed by the time variable between t_0 and t_1. If the intermediate function does not depend explicitly on x, the Euler equation becomes:

$$\frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}}\right) = 0, \tag{12.1.13}$$

which can be integrated directly as:

$$\frac{\partial I}{\partial \dot{x}} = \text{constant.} \tag{12.1.14}$$

Finally, if the intermediate function does not depend explicitly on t then, since the Euler equation can always be written:

$$\frac{d}{dt}\left(I - \frac{\partial I}{\partial \dot{x}}\dot{x}\right) - \frac{\partial I}{\partial t} = 0, \tag{12.1.15}$$

the Euler equation implies in this case that

$$I - \frac{\partial I}{\partial \dot{x}}\dot{x} = \text{constant.} \tag{12.1.16}$$

An example of the special case in which the intermediate function does not depend explicitly on the state variable x is that of proving that the shortest distance between two points on a plane is a straight line. Letting t refer to distance rather than to time, the problem is that of finding a path $\{x(t)\}$ connecting $x(t_0) = x_0$ and $x(t_1) = x_1$ so as to minimize the distance traversed. But the distance traversed is:

$$\int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}\, dt, \tag{12.1.17}$$

since a differential element of arc length, ds, is $\sqrt{dt^2 + dx^2}$ or $\sqrt{1 + \dot{x}^2}\, dt$. Thus:

$$I(x, \dot{x}, t) = -\sqrt{1 + \dot{x}^2}, \tag{12.1.18}$$

which does not depend explicitly on x. The Euler equation, from (12.1.14), is:

$$\frac{\partial I}{\partial \dot{x}} = \frac{-\dot{x}}{\sqrt{1 + \dot{x}^2}} = \text{constant}, \tag{12.1.19}$$

which implies that $\dot{x}$ must be constant. Integrating, $x(t)$ must be linear:

$$x(t) = c_1 t + c_2, \tag{12.1.20}$$

where c_1 and c_2 are constants, determined from the boundary conditions as:

$$c_1 = \frac{x_1 - x_0}{t_1 - t_0} \qquad c_2 = \frac{x_0 t_1 - x_1 t_0}{t_1 - t_0}. \tag{12.1.21}$$

Thus it has been proved, using the Euler equation of the calculus of variations, that the shortest distance between two points on a plane is along the straight line connecting these points.

12.2 Necessary Conditions

The Euler equation is a necessary condition analogous to the first order condition that the derivative vanish in the static case. Some of the other necessary conditions that a solution to the classical calculus of variations problem must satisfy can be presented by analogy to the corresponding conditions in the static classical programming problem.

The condition analogous to the second order necessary condition in the static case is the *Legendre condition*, that the solution trajectory $\{x(t)\}$ must satisfy:

$$\frac{\partial^2 I}{\partial \dot{x}^2} \leq 0, \tag{12.2.1}$$

for all t between t_0 and t_1. This condition follows from the analysis of the variation about the solution trajectory, the second order necessary condition for $J(\varepsilon)$ in (12.1.4) to be maximized at $\varepsilon = 0$ being:

$$\frac{d^2 J}{d\varepsilon^2}(0) \leq 0 \tag{12.2.2}$$

for all $\eta(t)$ satisfying the appropriate continuity and boundary conditions.

The condition analogous to the one in the static case that the objective function be concave is the *Weierstrass condition*, that if $\{x(t)\}$ is the solution trajectory and $\{z(t)\}$ is any other admissible trajectory:

$$E(x, \dot{x}, t, \dot{z}) \leq 0, \tag{12.2.3}$$

where $E(\cdot \cdot \cdot)$ is the *Weierstrass excess function*, defined as:

$$E(x, \dot{x}, t, \dot{z}) = I(x, \dot{z}, t) - I(x, \dot{x}, t) - \frac{\partial I}{\partial \dot{x}}(x, \dot{x}, t)(\dot{z} - \dot{x}). \tag{12.2.4}$$

This condition is in fact always met if the intermediate function $I(x, \dot{x}, t)$ is a concave function when considered a function of the control variable $\dot{x}$.

The last of the necessary conditions to be presented here are the *Weierstrass-Erdmann corner conditions*, which have no direct analogue in static problems, since they depend in an essential way on time. While the trajectory $\{x(t)\}$ is continuous, the control trajectory $\{\dot{x}(t)\}$ need be only piecewise continuous and, hence, may actually consist of segments of curves joined at points called *corners* at which $\dot{x}(t)$ is discontinuous. Such a corner occurs at time τ in Fig. 12.3. The Weierstrass-Erdmann corner conditions require that $(\partial I/\partial \dot{x})$ and $(I - \partial I/\partial \dot{x}\dot{x})$ be continuous across the corner. Thus, if a corner occurs at time τ:

$$\begin{aligned} \left[\frac{\partial I}{\partial \dot{x}}\right]_{\tau-} &= \left[\frac{\partial I}{\partial \dot{x}}\right]_{\tau+} \\ \left[I - \frac{\partial I}{\partial \dot{x}}\dot{x}\right]_{\tau-} &= \left[I - \frac{\partial I}{\partial \dot{x}}\dot{x}\right]_{\tau+} \end{aligned} \tag{12.2.5}$$

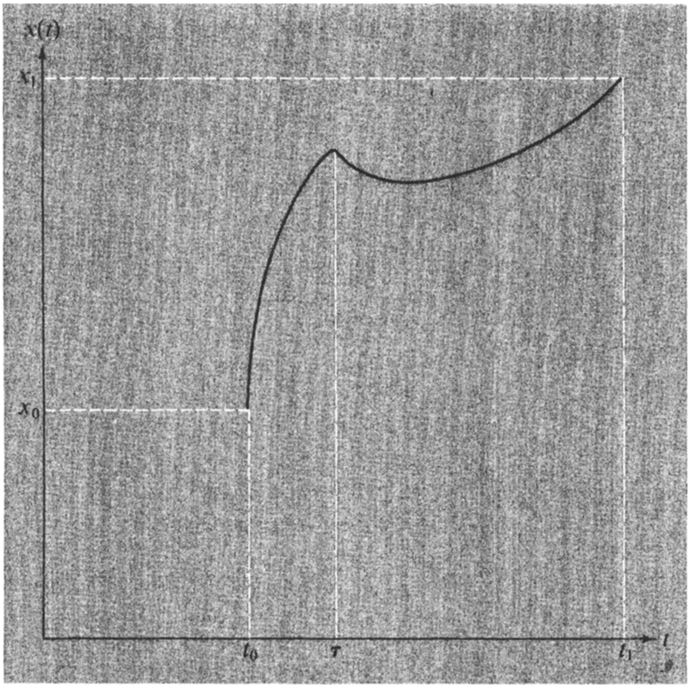

Fig. 12.3
A Corner Occurs at Time τ.

where $\tau-$ and $\tau+$ refer to the left and right hand limits respectively:

$$\left[\frac{\partial I}{\partial \dot{x}}\right]_{\tau-} = \lim_{\substack{t \to \tau \\ t < \tau}} \left[\frac{\partial I}{\partial \dot{x}}\right]$$
$$\left[\frac{\partial I}{\partial \dot{x}}\right]_{\tau+} = \lim_{\substack{t \to \tau \\ t > \tau}} \left[\frac{\partial I}{\partial \dot{x}}\right] \tag{12.2.6}$$

So far the problem under consideration is one with a single state variable. The classical calculus of variations problem with a vector of n state variables is:

$$\max_{\{\mathbf{x}(t)\}} J = \int_{t_0}^{t_1} I(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\, dt$$
$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{12.2.7}$$
$$\mathbf{x}(t_1) = \mathbf{x}_1$$

where $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ are the column vectors:

$$\begin{aligned} \mathbf{x}(t) &= (x_1(t), x_2(t), \ldots, x_n(t))' \\ \dot{\mathbf{x}}(t) &= (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t))'. \end{aligned} \tag{12.2.8}$$

The necessary conditions in this case are:

$$\begin{aligned} &\text{Euler equation: } \frac{\partial I}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\mathbf{x}}}\right) = \mathbf{0} \\ &\text{Boundary conditions: } \mathbf{x}(t_0) = \mathbf{x}_0, \qquad \mathbf{x}(t_1) = \mathbf{x}_1 \\ &\text{Legendre condition:} \\ &\quad \frac{\partial^2 I}{\partial \dot{\mathbf{x}}^2} \text{ negative definite or negative semidefinite} \\ &\text{Weierstrass condition: } E(\mathbf{x}, \dot{\mathbf{x}}, t, \dot{\mathbf{z}}) \le 0 \\ &\text{Weierstrass-Erdman corner conditions:} \\ &\quad \frac{\partial I}{\partial \dot{\mathbf{x}}} \text{ and } I - \frac{\partial I}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}} \text{ continuous across corners.} \end{aligned} \tag{12.2.9}$$

where:

$$\begin{aligned} \frac{\partial I}{\partial \mathbf{x}} &= \left(\frac{\partial I}{\partial x_1}, \frac{\partial I}{\partial x_2}, \ldots, \frac{\partial I}{\partial x_n}\right) \\ \frac{\partial I}{\partial \dot{\mathbf{x}}} &= \left(\frac{\partial I}{\partial \dot{x}_1}, \frac{\partial I}{\partial \dot{x}_2}, \ldots, \frac{\partial I}{\partial \dot{x}_n}\right) \\ \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\mathbf{x}}}\right) &= \left(\frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}_1}\right), \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}_2}\right), \ldots, \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}_n}\right)\right) \\ \frac{\partial^2 I}{\partial \dot{\mathbf{x}}^2} &= \begin{pmatrix} \frac{\partial^2 I}{\partial \dot{x}_1^2} & \frac{\partial I}{\partial \dot{x}_1\, \partial \dot{x}_2} & \cdots & \frac{\partial^2 I}{\partial \dot{x}_1\, \partial \dot{x}_n} \\ \frac{\partial^2 I}{\partial \dot{x}_2\, \partial \dot{x}_1} & \frac{\partial^2 I}{\partial \dot{x}_2^2} & \cdots & \frac{\partial^2 I}{\partial \dot{x}_2\, \partial \dot{x}_n} \\ \vdots & & & \\ \frac{\partial^2 I}{\partial \dot{x}_n\, \partial \dot{x}_1} & \frac{\partial^2 I}{\partial \dot{x}_n\, \partial \dot{x}_2} & \cdots & \frac{\partial^2 I}{\partial \dot{x}_n^2} \end{pmatrix} \\ E(\mathbf{x}, \dot{\mathbf{x}}, t, \dot{\mathbf{z}}) &= I(\mathbf{x}, \dot{\mathbf{z}}, t) - I(\mathbf{x}, \dot{\mathbf{x}}, t) - \frac{\partial I}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, t)(\dot{\mathbf{z}} - \dot{\mathbf{x}}). \end{aligned} \tag{12.2.10}$$

Thus, for example, there are n Euler equations:

$$\frac{\partial I}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}_j}\right) = 0, \qquad j = 1, 2, \ldots, n. \tag{12.2.11}$$

12.3 Transversality Condition

In the problem treated thus far, terminal time and terminal state are both given. In the case of a problem with a terminal surface, the condition

$$(\mathbf{x}(t), t)\varepsilon T \quad \text{at} \quad t = t_1 \tag{12.3.1}$$

defines the terminal time t_1 and terminal state $\mathbf{x}(t_1) = \mathbf{x}_1$. Suppose the terminal surface is given by the conditions:

$$\mathbf{T}(\mathbf{x}(t), t) = \mathbf{0} \quad \text{at} \quad t = t_1, \tag{12.3.2}$$

where $\mathbf{T}$ is a vector valued function of the state variables and time. The necessary conditions in this case can be derived using the variation about the solution approach. Suppose, in the single state variable problem, that $\{x(t)\}$ is the solution trajectory and $\{z(t)\}$ is the variation about the trajectory:

$$z(t) = x(t) + \varepsilon\eta(t). \tag{12.3.3}$$

The solution trajectory reaches the terminal surface at time t_1:

$$\mathbf{T}(x(t), t) = \mathbf{0} \quad \text{at} \quad t = t_1, \tag{12.3.4}$$

and the variation about the solution trajectory reaches the terminal surface at time $t_1(\varepsilon)$:

$$\mathbf{T}(z(t), t) = \mathbf{0} \quad \text{at} \quad t = t_1(\varepsilon) \tag{12.3.5}$$

where:

$$\lim_{\varepsilon \to 0} t_1(\varepsilon) = t_1, \tag{12.3.6}$$

as shown in Fig. 12.4. The objective functional evaluated for $\{z(t)\}$ is a function of ε:

$$J(\varepsilon) = \int_{t_0}^{t_1(\varepsilon)} I(x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}, t)\, dt, \tag{12.3.7}$$

and, since $J(\varepsilon)$ reaches a maximum at $\varepsilon = 0$, corresponding to the solution $\{x(t)\}$:

$$\frac{dJ}{d\varepsilon}(0) = I\Big|_{t_1(\varepsilon)} \frac{dt_1(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} + \int_{t_0}^{t_1}\left(\frac{\partial I}{\partial x}\eta + \frac{\partial I}{\partial \dot{x}}\dot{\eta}\right) dt = 0. \tag{12.3.8}$$

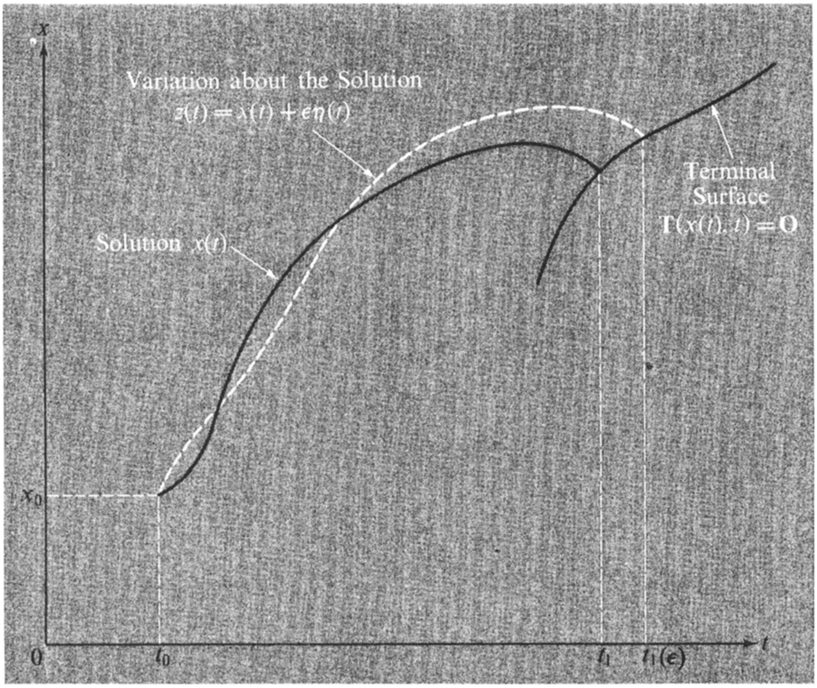

Fig. 12.4
Variation about the Solution Trajectory in the Case of a Terminal Surface

Integrating by parts, as before:

$$I\bigg|_{t_1(\varepsilon)} \frac{dt_1(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial I}{\partial \dot{x}}\bigg|_{t_1(\varepsilon)} \eta(t_1(\varepsilon)) + \int_{t_0}^{t_1}\left[\frac{\partial I}{\partial x} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}}\right)\right]\eta\, dt = 0 \quad (12.3.9)$$

Since the first terms do not depend on $\eta(\varepsilon)$, except for $t = t_1(\varepsilon)$, the Euler equation must hold as before:

$$\frac{\partial I}{\partial x} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}}\right) = 0. \qquad (12.3.10)$$

Thus:

$$I\bigg|_{t_1(\varepsilon)} \frac{dt_1(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial I}{\partial \dot{x}}\bigg|_{t_1(\varepsilon)} \eta(t_1(\varepsilon)) = 0. \qquad (12.3.11)$$

But the derivative $dt_1(\varepsilon)/d\varepsilon$ is obtained by differentiating:

$$\mathbf{T}(x(t_1(\varepsilon)) + \varepsilon\eta(t_1(\varepsilon)), t_1(\varepsilon)) = \mathbf{0} \qquad (12.3.12)$$

with respect to ε, yielding:

$$\frac{\partial \mathrm{T}}{\partial x}\left(\frac{dx}{dt_1(\varepsilon)}\frac{dt_1(\varepsilon)}{d\varepsilon} + \eta(t_1(\varepsilon)) + \varepsilon\frac{d\eta}{dt_1(\varepsilon)}\frac{dt_1(\varepsilon)}{d\varepsilon}\right) + \frac{\partial \mathrm{T}}{\partial t}\frac{dt_1(\varepsilon)}{d\varepsilon} = 0. \quad (12.3.13)$$

Taking the limit as $\varepsilon \to 0$:

$$\frac{\partial \mathrm{T}}{\partial x}\left(\frac{dx}{dt_1}\frac{dt_1}{d\varepsilon} + \eta(t_1)\right) + \frac{\partial \mathrm{T}}{\partial t}\frac{dt_1}{d\varepsilon} = 0, \quad (12.3.14)$$

and, combining with (12.3.11), yields the *transversality condition:*

$$\left[I - \frac{\partial I}{\partial \dot{x}}\dot{x}\right]_{t_1}\frac{\partial \mathrm{T}}{\partial x} - \left[\frac{\partial I}{\partial \dot{x}}\right]_{t_1}\frac{\partial \mathrm{T}}{\partial t} = 0. \quad (12.3.15)$$

Since:

$$\frac{\partial \mathrm{T}}{\partial x}\left(\frac{dx}{dt}\right)_{\mathrm{T}(\cdot\cdot)=0} = -\frac{\partial \mathrm{T}}{\partial t} \quad (12.3.16)$$

the condition can be written:

$$\left[I - \frac{\partial I}{\partial \dot{x}}\dot{x}\right]_{t_1} + \left[\frac{\partial I}{\partial \dot{x}}\right]_{t_1}\left(\frac{dx}{dt}\right)_{\mathrm{T}(\cdot\cdot)=0} = 0, \quad (12.3.17)$$

and, more generally, in the case of a vector of state variables the transversality condition is:

$$\left[I - \frac{\partial I}{\partial \mathbf{x}}\dot{\mathbf{x}}\right]_{t_1} + \left[\frac{\partial I}{\partial \dot{\mathbf{x}}}\right]_{t_1}\left(\frac{d\mathbf{x}}{dt}\right)_{\mathrm{T}(\cdot\cdot)=0} = 0, \quad \mathbf{(12.3.18)}$$

where $(d\mathbf{x}/dt)_{\mathrm{T}(\cdot\cdot)=0}$ is the gradient vector, a column vector normal to the terminal surface

12.4 Constraints

The calculus of variations approach can be used to characterize solutions of certain control problems with constraints.

One important type of constraint is the integral constraint, in which the integral of a given function is held constant. This problem, known as the *isoperimetric problem*, is of the form:

$$\max_{\{(\mathbf{x}t)\}} \quad J = \int_{t_0}^{t_1} I(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{x}(t_1) = \mathbf{x}_1 \tag{12.4.1}$$
$$K = \int_{t_0}^{t_1} G(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt = c,$$

where $G(\cdots)$ is a given continuously differentiable function and c is a given constant. The classic example of such a problem, for which the problem is named, is that of finding a curve of fixed length (constant perimeter) enclosing the largest area. The constraint is accounted for by introducing the Lagrange multiplier y and defining the functional:

$$J' = \int_{t_0}^{t_1} [I(\cdots) + yG(\cdots)]\, dt, \tag{12.4.2}$$

the necessary conditions being those for finding a maximum of J' with respect to the trajectory $\{x(t)\}$ and a minimum of J' with respect to the Lagrange multiplier, y. For example, the Euler equation is:

$$\frac{\partial}{\partial \mathbf{x}}(I(\cdots) + yG(\cdots)) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{x}}}(I(\cdots) + yG(\cdots))\right) = \mathbf{0} \tag{12.4.3}$$

which, together with the boundary conditions and constraint, characterizes the solution.

An important result for the Isoperimetric problem is the *Principle of Reciprocity*, which states that if $\mathbf{x}(t)$ maximizes J subject to the condition that K is constant, then normally $\mathbf{x}(t)$ minimizes K subject to the condition that J is constant. For example, the curve of fixed length that maximizes the enclosed area is also the curve that minimizes the length required to enclose a given area—the curve being a circle.

A second important type of constraint is a set of equality constraints connecting the state variables, their rate of change, and time. In this case the problem is:

$$\max_{\{\mathbf{x}(t)\}} \quad J = \int_{t_0}^{t_1} I(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$
$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{12.4.4}$$
$$\mathbf{x}(t_1) = \mathbf{x}_1$$
$$\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{b},$$

where $\mathbf{g}(\cdots)$ is a given column vector of r functions and $\mathbf{b}$ is a given column vector. It is assumed that $n > r$, where the difference $n - r$ is referred to as the *degrees of freedom* of the problem, and that the Jacobian matrix:

$$\frac{\partial \mathbf{g}}{\partial \dot{\mathbf{x}}} = \begin{pmatrix} \frac{\partial g_1}{\partial \dot{x}_1} & \frac{\partial g_1}{\partial \dot{x}_2} & \cdots & \frac{\partial g_1}{\partial \dot{x}_n} \\ \frac{\partial g_2}{\partial \dot{x}_1} & \frac{\partial g_2}{\partial \dot{x}_2} & \cdots & \frac{\partial g_2}{\partial \dot{x}_n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{\partial g_r}{\partial \dot{x}_1} & \frac{\partial g_r}{\partial \dot{x}_2} & \cdots & \frac{\partial g_r}{\partial \dot{x}_n} \end{pmatrix} \tag{12.4.5}$$

is of full row rank at all points on the solution trajectory—assumptions directly analogous to those employed in classical programming. The method of solution involves the introduction of r Lagrange multipliers:

$$\mathbf{y} = (y_1, y_2, \ldots, y_r). \tag{12.4.6}$$

Defining the Lagrangian function as:

$$L(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{y}) = I(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{y}[\mathbf{b} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t)], \tag{12.4.7}$$

the solution is obtained by choosing $\{\mathbf{x}(t)\}$ to maximize and $\mathbf{y}$ to minimize:

$$J' = \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t, \mathbf{y})\, dt, \tag{12.4.8}$$

leading to the Euler equation:

$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \mathbf{0}, \tag{12.4.9}$$

which, together with the boundary conditions and constraint, characterizes the solution.

A third important type of constraint is that of inequality constraints connecting the state variables, their rates of changes, and time. In this case the problem is:

$$\begin{aligned} \max_{\{x(t)\}} \ J &= \int_{t_0}^{t_1} I(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \mathbf{x}(t_1) &= \mathbf{x}_1 \\ \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t) &\leq \mathbf{b}, \end{aligned} \tag{12.4.10}$$

where $\mathbf{g}(\cdots)$ is again a column vector of r functions. Forming the Lagrangian as in (12.4.7), the solution must satisfy:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) &= \mathbf{0} \\ \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t) &\leq \mathbf{b} \\ \mathbf{y} &\geq \mathbf{0} \\ \mathbf{y}[\mathbf{b} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t)] &= 0, \end{aligned} \tag{12.4.11}$$

where the first n conditions are the Euler equations and the remaining conditions are the Kuhn-Tucker conditions as discussed in Chap. 4. The Kuhn-Tucker conditions imply the complementary slackness conditions that any Lagrange multiplier equals zero if the corresponding constraint is satisfied as a strict inequality and that any constraint is satisfied as an equality if the corresponding Lagrange multiplier is positive.

Thus the calculus of variations can be used to solve control problems involving certain types of constraints. The principal weakness of the classical calculus of variations, however, is that it cannot cope directly with problems in which the control variables are restricted to a given control set, a weakness overcome by the newer approaches of dynamic programming and the maximum principle.

PROBLEMS

12-A. Find the extremals of the problem with a single state variable, $x(t)$, and check the Legendre condition where:

1. $I = 4xt - \dot{x}^2$
2. $I = t\dot{x} - 2\dot{x}^2$
3. $I = \frac{1}{x}\sqrt{1 - \dot{x}^2}$
4. $I = x^2 - 6xt$
5. $I = \frac{-\dot{x}^2}{t^3}$

12-B. Find extremals of the problem with two state variables $(x_1(t), x_2(t))'$ and check Legendre conditions, where:

1. $I = \dot{x}_1^2 - \dot{x}_2^2 + 2x_1x_2 - 2x_2^2$
2. $I = \dot{x}_1^2 + x_2 + \dot{x}_1\dot{x}_2.$

12-C. Solve:

$$\min \int_0^{t_1} \frac{(1-\dot{x}^2)^{1/2}}{x}\,dt$$

$$x(0) = 0$$

$$x(t_1) = t_1 - 5.$$

12-D. Consider the problem:

$$\min \int_1^3 x^2(1-\dot{x})^2\,dt$$

$$x(1) = 0 \qquad x(3) = a.$$

1. Show that the solution is a line if $a = 0$ and if $a = 2$.
2. Show that if $0 < a < 2$, the solution entails a corner, and illustrate in a diagram several possible solutions if $a = 1$. Verify that these solutions satisfy the Euler equation and the Weierstrass-Erdmann corner conditions.
3. What happens if $a > 2$?

12-E. Obtain and exhibit geometrically several possible solutions to the problem:

$$\min \int_1^4 (1-\dot{x})^2(1+\dot{x})^2\,dt$$

$$x(1) = 0$$

$$x(4) = 1.$$

12-F. Show that the straight line solution to the problem of finding the shortest distance between two points satisfies the Legendre and the Weierstrass conditions.

12-G. Show that if the intermediate function $I(\cdots)$ is quadratic, then the optimal (closed loop) control is a linear function of the state variables.

12-H. A cable of length ℓ hangs between two level supports, and the shape of the hanging cable is given by the curve $x(t)$ for $t_0 \leq t \leq t_1$, where the supports are given as:

$$x(t_0) = x_0$$

$$x(t_1) = x_1.$$

The potential energy of the hanging cable,

$$V = \int mgx\,ds = mg\int_{t_0}^{t_1} x\sqrt{1+\dot{x}^2}\,dt,$$

is minimized when the cable hangs in equilibrium, subject to the condition that the length of the cable is fixed:

$$\ell = \int ds = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}\, dt.$$

Show that the curve of the hanging cable is the catenary:

$$x = c_1 \cosh\left(\frac{t + c_2}{c_1}\right) + c_3,$$

where c_1, c_2, and c_3 are constants determined from the parameters of the problem.

12-I. Using integration by parts, prove that in the problem with an explicit control variable:

$$\max_{\{u(t)\}} J = \int_{t_0}^{t_1} I(x, u, t)\, dt$$

$$\dot{x} = f(x, u, t)$$

$$x(t_0) = x_0$$

$$x(t_1) = x_1$$

the Euler equation is:[3]

$$\frac{\partial I}{\partial x} - \frac{\partial f/\partial x}{\partial f/\partial u}\frac{\partial I}{\partial u} - \frac{d}{dt}\left(\frac{\partial I/\partial u}{\partial f/\partial u}\right) = 0.$$

12-J. Show that for the case in which the intermediate function also depends on the vector of second derivatives, $\ddot{\mathbf{x}}$, in which the objective functional is:

$$J = \int_{t_0}^{t_1} I(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})\, dt$$

the Euler equation is:

$$\frac{\partial I}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\mathbf{x}}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial I}{\partial \ddot{\mathbf{x}}}\right) = \mathbf{0}.$$

Generalize to the case in which I depends on all time derivatives of $\mathbf{x}(t)$ up to and including the ℓ^{th}.

12-K. Prove that the transversality condition reduces to an orthogonality condition for functionals of the form:

$$J = \int_{t_0}^{t_1} A(x, t)\sqrt{1 + \dot{x}^2}\, dt.$$

In particular, show that the shortest line segment between a point and a given curve is perpendicular to the tangent to the curve at the point of contact.

12-L. Show that the Euler equation is automatically satisfied (and hence provides no way of solving the problem) if and only if the intermediate function is linear in $\dot{x}$:

$$I(x, \dot{x}, t) = A(x, t) + B(x, t)\dot{x}$$

where:

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial t}.$$

Why is this problem analogous to the problem of maximizing a function that is constant in value in the relevant region?

12-M. Show that the Euler equation for:

$$\int_{t_0}^{t_1} I(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$

is the same as the Euler equation for:

$$\int_{t_0}^{t_1} \{cI(\mathbf{x}, \dot{\mathbf{x}}, t) + I'(\mathbf{x}, \dot{\mathbf{x}}, t)\}\, dt$$

where c is a nonzero constant and:

$$I'(\mathbf{x}, \mathbf{x}, t) = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \mathbf{x}}\dot{\mathbf{x}}$$

where $\phi(\mathbf{x}, t)$ is any continuously differentiable function.

12-N. Verify the necessity of the Weierstrass condition by showing that the straight line $x = t$ satisfies both the Euler equation and Legendre condition for the problem,

$$\min J = \int_0^1 \dot{x}^3\, dt$$

$$x(0) = 0$$

$$x(1) = 1$$

but that the straight line does not satisfy the Weierstrass condition and does not in fact solve the problem.[4]

12-O. Verify that the Euler equation can always be written as in (12.1.15).

12-P. Derive the Legendre condition for the problem with a single state variable from the condition (12.2.2), where $J(\varepsilon)$ is the value of the objective functional for the variation about the solution trajectory:

$$z(t) = x(t) + \varepsilon\eta(t),$$

and where $\eta(t) = 0$ but $\dot{\eta}(t) \neq 0$, e.g., $\eta(t) = (\sin wt)/w$ for large w.

12-Q. One way of taking account of inequality restrictions on the control variables is by transforming variables. Thus, the restriction $\dot{x} \leq K$ can be taken into account by using the variable z, where $z^2 = K - \dot{x}$ and the restriction $|\dot{x}| \leq 1$ can be taken into account by using the variable θ, where $\dot{x} = \sin\theta$. In both cases develop the implied necessary conditions for the classical calculus of variations problem.[5]

FOOTNOTES

[1] The basic references for the calculus of variations are Bliss (1946), Gelfand and Fomin (1963), Dreyfus (1965), and Hestenes (1966).

[2] An alternative proof of the necessity of the Euler equation uses the discrete time approximation developed in Sec. 11.4. Dividing the time interval into N subintervals of equal length Δ:

$$J^N = \sum_{q=0}^{N-1} I(x^q, u^q, t^q)\Delta$$

$$t^q = t_0 + q\Delta$$

$$x^q = x(t^q)$$

$$u^q = \frac{x^q - x^{q-1}}{\Delta}$$

where:

$$\lim_{\substack{N\to\infty \\ \Delta\to 0 \\ N\Delta=(t_1-t_0)}} J^N = J.$$

In order to maximize J^N by choice of x^q it is necessary that:

$$\frac{\partial J^N}{\partial x^q} = 0,$$

but x^q appears in two terms of the sum:

$$\frac{\partial J^N}{\partial x^q} = \frac{\partial}{\partial x^q}\left[I\left(x^{q-1}, \frac{x^q - x^{q-1}}{\Delta}, t^{q-1}\right) + I\left(x^q, \frac{x^{q+1} - x^q}{\Delta}, t^q\right)\right]\Delta = 0$$

$$= \left[\frac{\partial I}{\partial u^{q-1}}\cdot\frac{1}{\Delta} + \frac{\partial I}{\partial x^q} - \frac{\partial I}{\partial u^q}\frac{1}{\Delta}\right]\Delta = 0$$

$$= \left[\frac{\partial I}{\partial x^q} - \left(\frac{\dfrac{\partial I}{\partial u^q} - \dfrac{\partial I}{\partial u^{q-1}}}{\Delta}\right)\right]\Delta = 0,$$

and taking the limit as $N \to \infty$, $\Delta \to 0$, $x^q \to x$, $u^q \to \dot{x}$ yields the Euler equation:

$$\frac{\partial I}{\partial x} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}}\right) = 0.$$

[3] See Bellman (1957, 1961).
[4] See Dreyfus (1965).
[5] See Valentine (1937) and Miele (1962).

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