15 Differential Games

A *differential game* is a situation of conflict or cooperation in which players choose strategies over time.[1] By contrast to the last four chapters, in a differential game there is more than one player, and the payoffs to each player depend on the control trajectories employed by all the players. On the other hand, by contrast to the games treated in Chapter 6, in a differential game the players make their moves over an interval of time, so the number of moves, and hence the number of strategies, are infinite.

Differential games can be classified in some of the same ways in which games were classified in Chapter 6. One classification is by the number of players—as a *two-person*, *three-person*, . . . , *n-person* differential game, where the control problem of Chapter 11 can be considered the special differential game in which there is only one player. Another classification is by the nature of the payoff functions, as *zero-sum* or *nonzero-sum*, depending on whether or not the sum of the payoffs to all players equals or does not equal zero (or, more generally, any constant). Yet another way of classifying differential games is as to whether the game is *stochastic*, containing random variables, or *deterministic*, otherwise.[2] One way of classifying differential games which does not appear in static games is by the nature of time. If time is measured in discrete units then the game is a *discrete differential game*, and if time is measured in continuous units then it is a *continuous differential game.*

15.1 Two-Person Deterministic Continuous Differential Games

The subject of this chapter will be two-person deterministic continuous differential games. The game is played over an interval of time:

$$t_0 \leq t \leq t_1 \tag{15.1.1}$$

where t_0, the initial time, is given, and t_1, the terminal time, is either given or determined by the game itself.

The game is played within a system described by a set of n state variables, summarized by the *state vector* $\mathbf{x}$, an $n \times 1$ column vector the entries of which can vary over time:

$$\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))' \tag{15.1.2}$$

starting from given initial values:

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{15.1.3}$$

and ending at terminal values:

$$\mathbf{x}(t_1) = \mathbf{x}_1. \tag{15.1.4}$$

The terminal time, t_1, is determined by the *terminal surface*, a surface in E^{n+1} described by the equations:

$$\mathbf{T}(\mathbf{x}_1, t_1) = \mathbf{0}. \tag{15.1.5}$$

It is assumed that the game is one of *perfect information* in that all players know the values of all current state variables. Each player chooses time paths for his vector of control variables, summarized by a *control trajectory*. Thus player 1 chooses the first control trajectory $\{\mathbf{u}^1(t)\}$:

$$\{\mathbf{u}^1(t)\} = \{(u_1^1(t), u_2^1(t), \ldots, u_{r_1}^1(t))' \mid t_0 \leq t \leq t_1\} \tag{15.1.6}$$

and player 2 chooses the second control trajectory $\{\mathbf{u}^2(t)\}$:

$$\{\mathbf{u}^2(t)\} = \{(u_1^2(t), u_2^2(t), \ldots, u_{r_2}^2(t))' \mid t_0 \leq t \leq t_1\}. \tag{15.1.7}$$

These control trajectories belong to given *control sets*:

$$\begin{aligned} &\{\mathbf{u}^1(t)\} \in U^1 \\ &\{\mathbf{u}^2(t)\} \in U^2, \end{aligned} \tag{15.1.8}$$

which require that the controls be piecewise continuous functions of time the values of which must at all times in the relevant interval belong to certain nonempty compact sets:

$$\begin{aligned} &\mathbf{u}^1(t) \in \Omega^1 \quad \text{for all} \quad t, \qquad t_0 \leq t < t_1, \qquad \Omega^1 \subset E^{r_1} \\ &\mathbf{u}^2(t) \in \Omega^2 \quad \text{for all} \quad t, \qquad t_0 \leq t \leq t_1, \qquad \Omega^2 \subset E^{r_2}. \end{aligned} \tag{15.1.9}$$

The *equations of motion* are the set of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t), \tag{15.1.10}$$

where $\mathbf{f}(\cdot \cdot \cdot)$ is assumed given and continuously differentiable. These equations of motion, together with the initial state (15.1.3) and the control trajectories chosen by the two players (15.1.6) and (15.1.7), determine the *state trajectory* $\{\mathbf{x}(t)\}$:

$$\{\mathbf{x}(t)\} = \{(x_1(t), x_2(t), \ldots, x_n(t))' \mid t_0 \leq t \leq t_1\}. \tag{15.1.11}$$

The *payoff* to each player depends on the control trajectories chosen by both players, where the payoff to player 1 is:

$$J^1 = J^1[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] = \int_{t_0}^{t_1} I^1(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + \mathbf{F}^1(\mathbf{x}_1, t_1), \tag{15.1.12}$$

and the payoff to player 2 is:

$$J^2 = J^2[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] = \int_{t_0}^{t_1} I^2(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + F^2(\mathbf{x}_1, t_1). \quad (15.1.13)$$

Each player seeks to maximize his own payoff by choice of his own control trajectory.

A *strategy* for a player is a rule for determining his control vector at any time as a function of the state variables at that time:

$$\left.\begin{aligned} \mathbf{u}^1(t) &= \mathbf{S}^1(\mathbf{x}(t)) \\ \mathbf{u}^2(t) &= \mathbf{S}^2(\mathbf{x}(t)) \end{aligned}\right\} \quad \text{for all} \quad t, \qquad t_0 \le t \le t_1, \quad (15.1.14)$$

where mixed strategies are *not* excluded. Since a strategy indicates the choices made by a player for any possible contemporaneous situation, as summarized by the state vector, the notion of strategy employed here conforms to that used in Chapter 6. It also represents, in terms of the control problem, a closed loop control, as discussed in Chapter 11. Since each player knows only his own strategy and gains information about the other player only by observing the evolution of the game, he must choose his control vector in response to current state variables. Thus, by its very nature, a differential game requires closed loop controls (strategies) rather than open loop controls.

Each player selects his strategy so as to maximize his own payoff, leading to optimal strategies $\mathbf{S}^{1*}(\mathbf{x})$, $\mathbf{S}^{2*}(\mathbf{x})$, and, given these strategies, the equations of motion become:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{S}^{1*}(\mathbf{x}), \mathbf{S}^{2*}(\mathbf{x}), t). \quad (15.1.15)$$

These equations can be integrated forward from the given initial state to determine the state trajectory $\{\mathbf{x}(t)\}$ and hence the payoffs to each player:

$$\begin{aligned} J^{1*} &= J^1[\{\mathbf{S}^{1*}(\mathbf{x})\}, \{\mathbf{S}^{2*}(\mathbf{x})\}] \\ J^{2*} &= J^2[\{\mathbf{S}^{1*}(\mathbf{x})\}, \{\mathbf{S}^{2*}(\mathbf{x})\}]. \end{aligned} \quad (15.1.16)$$

15.2 Two-Person Zero-Sum Differential Games

In a two-person zero-sum differential game the payoff to player 2 is the negative of the payoff to player 1. Letting J be the payoff to player 1:

$$J[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] = \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + F(\mathbf{x}_1, t_1), \quad (15.2.1)$$

player 1 seeks to maximize J by choice of $\{\mathbf{u}^1(t)\}$ and player 2 seeks to minimize J by choice of $\{\mathbf{u}^2(t)\}$. The problem is thus one of finding strategies:

$$\begin{aligned}\mathbf{u}^{1*}(t) &= \mathbf{S}^{1*}(\mathbf{x}(t))\\ \mathbf{u}^{2*}(t) &= \mathbf{S}^{2*}(\mathbf{x}(t))\end{aligned} \tag{15.2.2}$$

for which $\{\mathbf{u}^{1*}(t)\} \in U^1$ and $\{\mathbf{u}^{2*}(t)\} \in U^2$ form a saddle point of the payoff functional:

$$\begin{gathered}J[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^{2*}(t)\}] \leq J[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^{2*}(t)\}] \leq J[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^2(t)\}]\\ \text{for all} \quad \{\mathbf{u}^1(t)\} \in U^1, \qquad \{\mathbf{u}^2(t)\} \in U^2,\end{gathered} \tag{15.2.3}$$

where $J[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^{2*}(t)\}]$ is called the *value* of the differential game. The necessary conditions for controls satisfying this saddle point condition can be obtained by analogy to the conditions for optimal controls using the maximum principle.[3] Introduce a row vector of n costate variables:

$$\mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t)), \tag{15.2.4}$$

and define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}, t) = I(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t). \tag{15.2.5}$$

Necessary conditions for optimal strategies for the two players are that player 1 maximize the Hamiltonian by choice of his control vector and that player 2 minimize the Hamiltonian by choice of his control vector at all points of time in the relevant interval. Assuming the differential game satisfies certain regularity conditions and is *strictly determined*, in that a solution exists in pure rather than mixed strategies, a necessary condition for a solution is that the Hamiltonian be at a saddle point at all relevant points of time:[4]

$$\begin{gathered}H(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^{2*}, \mathbf{y}, t) \leq H(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, \mathbf{y}, t) \leq H(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^2, \mathbf{y}, t)\\ \text{for all} \quad \mathbf{u}^1 \in \Omega^1, \qquad \mathbf{u}^2 \in \Omega^2, \qquad \text{all} \quad t, \qquad t_0 \leq t \leq t_1,\end{gathered} \tag{15.2.6}$$

that is:

$$\begin{aligned}\max_{\mathbf{u}^1 \in \Omega^1} \min_{\mathbf{u}^2 \in \Omega^2} H(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}, t) &= \min_{\mathbf{u}^2 \in \Omega^2} \max_{\mathbf{u}^1 \in \Omega^1} H(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}, t)\\ &= H(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, \mathbf{y}, t).\end{aligned} \tag{15.2.7}$$

Thus, according to this result, which, by analogy to the maximum principle, can be called the *minimaximum principle*, a two-person zero-sum differential game that is strictly determined must satisfy at each point of time in the relevant interval the saddle point condition of a strictly determined (static) game. The remaining necessary conditions are the canonical equations and boundary conditions which are the same as those for the maximum principle:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad \mathbf{y}(t_1) = \frac{\partial F}{\partial \mathbf{x}_1} + \mathbf{v}\frac{\partial T}{\partial \mathbf{x}_1}, \tag{15.2.8}$$

where $\mathbf{v}$ is a row vector of Lagrange multipliers which can be eliminated to obtain the terminal transversality condition:

$$\left(H + \frac{\partial F}{\partial t_1}\right) + \left(\frac{\partial F}{\partial \mathbf{x}_1} - \mathbf{y}\right)\left(\frac{d\mathbf{x}}{dt}\right)_{\mathbf{T}(\cdots)=\mathbf{0}} = 0, \tag{15.2.9}$$

all variables and derivatives being evaluated at terminal time t_1.

If the problem is autonomous in that both $I(\cdots)$ and $\mathbf{f}(\cdots)$ are independent of any explicit dependence on time then the min-max value of the Hamiltonian is constant, which may be taken as zero. Thus in this case:[5]

$$\max_{\mathbf{u}^1 \in \Omega^1} \min_{\mathbf{u}^2 \in \Omega^2} I(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) = 0. \tag{15.2.10}$$

An example of a two-person zero-sum differential game which can be solved by the minimaximum principle is the quadratic objective functional-linear autonomous equations of motion game, which can be treated as the comparable control problem (the minimum effort servomechanism) was treated in Sec. 14.5. In this differential game the state vector can be decomposed into:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}, \tag{15.2.11}$$

where $\mathbf{x}^1$ summarizes the state variables relating to player 1 and $\mathbf{x}^2$ summarizes the state variables relating to player 2. The equations of motion are linear and autonomous:

$$\dot{\mathbf{x}}^1 = \mathbf{A}^1\mathbf{x}^1 + \mathbf{B}^1\mathbf{u}^1$$
$$\dot{\mathbf{x}}^2 = \mathbf{A}^2\mathbf{x}^2 + \mathbf{B}^2\mathbf{u}^2, \tag{15.2.12}$$

where $\mathbf{u}^1$ and $\mathbf{u}^2$ are the control vectors for player 1 and 2 respectively and are assumed to be unrestricted. Terminal time t_1 is assumed given, and the payoff to player 1 is:

$$J = \int_{t_0}^{t_1} [\mathbf{x}^{1\prime}\mathbf{C}^1\mathbf{x}^1 + \mathbf{x}^{2\prime}\mathbf{C}^2\mathbf{x}^2 + \mathbf{u}^{1\prime}\mathbf{D}^1\mathbf{u}^1 + \mathbf{u}^{2\prime}\mathbf{D}^2\mathbf{u}^2 + \mathbf{u}^{1\prime}\mathbf{D}^3\mathbf{u}^2]\, dt$$
$$+ [\mathbf{x}_1^{1\prime}\mathbf{E}^1\mathbf{x}_1^1 + \mathbf{x}_1^{2\prime}\mathbf{E}^2\mathbf{x}_1^2 + \mathbf{x}_1^{1\prime}\mathbf{E}^3\mathbf{x}_1^2], \tag{15.2.13}$$

where $\mathbf{D}^1$ is negative definite and $\mathbf{D}^2$ is positive definite. The Hamiltonian is:

$$\begin{aligned} H = [&\mathbf{x}^{1\prime}\mathbf{C}^1\mathbf{x}^1 + \mathbf{x}^{2\prime}\mathbf{C}^2\mathbf{x}^2 + \mathbf{u}^{1\prime}\mathbf{D}^1\mathbf{u}^1 + \mathbf{u}^{2\prime}\mathbf{D}^2\mathbf{u}^2 + \mathbf{u}^{1\prime}\mathbf{D}^3\mathbf{u}^2] \\ &+ \mathbf{y}^1[\mathbf{A}^1\mathbf{x}^1 + \mathbf{B}^1\mathbf{u}^1] + \mathbf{y}^2[\mathbf{A}^2\mathbf{x}^2 + \mathbf{B}^2\mathbf{u}^2], \end{aligned} \tag{15.2.14}$$

where the costate vector is:

$$\mathbf{y} = (\mathbf{y}^1\ \mathbf{y}^2). \tag{15.2.15}$$

By the minimaximum principle, necessary conditions for optimality are:

$$\begin{aligned} \frac{\partial H}{\partial \mathbf{u}^1} &= 2\mathbf{u}^{1\prime}\mathbf{D}^1 + \mathbf{u}^{2\prime}\mathbf{D}^3 + \mathbf{y}^1\mathbf{B}^1 = \mathbf{0} \\ \frac{\partial H}{\partial \mathbf{u}^2} &= 2\mathbf{u}^{2\prime}\mathbf{D}^2 + \mathbf{u}^{1\prime}\mathbf{D}^3 + \mathbf{y}^2\mathbf{B}^2 = \mathbf{0}, \end{aligned} \tag{15.2.16}$$

the second order conditions being satisfied by the assumptions that $\mathbf{D}^1$ is negative definite and $\mathbf{D}^2$ positive definite. The solutions for the optimal control vectors in terms of the costate variables are then:

$$\begin{aligned} \mathbf{u}^1 &= -\frac{1}{2}\,\mathbf{D}^{1^{-1}}\{\mathbf{D}^3\mathbf{u}^2 + \mathbf{B}^{1\prime}\mathbf{y}^{1\prime}\} \\ \mathbf{u}^2 &= -\frac{1}{2}\,\mathbf{D}^{2^{-1}}\{\mathbf{D}^3\mathbf{u}^1 + \mathbf{B}^{2\prime}\mathbf{y}^{2\prime}\}. \end{aligned} \tag{15.2.17}$$

But the differential equations for the costate variables are:

$$\begin{aligned} \dot{\mathbf{y}}^1 &= -\frac{\partial H}{\partial \mathbf{x}^1} = -2\mathbf{x}^{1\prime}\mathbf{C}^1 - \mathbf{y}^1\mathbf{A}^1 \\ \dot{\mathbf{y}}^2 &= -\frac{\partial H}{\partial \mathbf{x}^2} = -2\mathbf{x}^{2\prime}\mathbf{C}^2 - \mathbf{y}^2\mathbf{A}^2, \end{aligned} \tag{15.2.18}$$

so, assuming linear solutions of the form:

$$\begin{aligned} \mathbf{y}^1 &= \mathbf{x}^{1\prime}\mathbf{Q}^1(t) \\ \mathbf{y}^2 &= \mathbf{x}^{2\prime}\mathbf{Q}^2(t) \end{aligned} \tag{15.2.19}$$

leads to matrix Ricatti equations for $\mathbf{Q}^1(t)$ and $\mathbf{Q}^2(t)$, as in Sec. 14.5. The optimal closed loop controls are then:

$$\begin{aligned} \mathbf{u}^1 &= -\frac{1}{2}\,\mathbf{D}^{1^{-1}}\{\mathbf{D}^3\mathbf{u}^2 + \mathbf{B}^{1\prime}\mathbf{Q}^{1\prime}\mathbf{x}^1\} \\ \mathbf{u}^2 &= -\frac{1}{2}\,\mathbf{D}^{2^{-1}}\{\mathbf{D}^{3\prime}\mathbf{u}^1 + \mathbf{B}^{2\prime}\mathbf{Q}^{2\prime}\mathbf{x}^2\}, \end{aligned} \tag{15.2.20}$$

showing that the optimal controls for each player are linear functions of his own state variables and the control variables of the other player. An *equilibrium point* is reached when the choice of $\mathbf{u}^1$ by player 1 on the basis of the control of $\mathbf{u}^2$ by player 2 leads player 2 to optimally choose precisely the $\mathbf{u}^2$ that led player 1 to his original choice. These equilibrium points are obtained by solving the equations in (15.2.19) simultaneously for $\mathbf{u}^1$ and $\mathbf{u}^2$, as:

$$\begin{aligned}\mathbf{u}^1 &= [I - \tfrac{1}{4}\mathbf{D}^{1^{-1}}\mathbf{D}^3\mathbf{D}^{2^{-1}}\mathbf{D}^{3\prime}]^{-1}[-\tfrac{1}{2}\mathbf{D}^{1^{-1}}\mathbf{B}^{1\prime}\mathbf{Q}^{1\prime}(t)\mathbf{x}^1 \\ &\qquad + \tfrac{1}{4}\mathbf{D}^{1^{-1}}\mathbf{D}^3\mathbf{D}^{2^{-1}}\mathbf{B}^{2\prime}\mathbf{Q}^{2\prime}(t)\mathbf{x}^2] \\ \mathbf{u}^2 &= [I - \tfrac{1}{4}\mathbf{D}^{2^{-1}}\mathbf{D}^{3\prime}\mathbf{D}^{1^{-1}}\mathbf{D}^3]^{-1}[\tfrac{1}{4}\mathbf{D}^{2^{-1}}\mathbf{D}^{3\prime}\mathbf{D}^{1^{-1}}\mathbf{B}^{1\prime}\mathbf{Q}^{1\prime}(t)\mathbf{x}^1 \\ &\qquad - \tfrac{1}{2}\mathbf{D}^{2^{-1}}\mathbf{B}^{2\prime}\mathbf{Q}^{2\prime}(t)\mathbf{x}^2],\end{aligned} \tag{15.2.21}$$

where it has been assumed that the two inverse matrices exist. Under this assumption, the optimal control vector for each player is a linear function of the state vectors of both players, i.e., each player optimally uses a *linear decision rule*, linearly relating his control variables to the state variables.

15.3 Pursuit Games

The most important class of two-person zero-sum differential games from the viewpoint of either theory or applications is that of *pursuit games*, in which player 1 is the *pursuer*, and player 2 the *evader*.[6] The game ends when the pursuer is sufficiently close to the evader, at which point the pursuer is said to "capture" the evader, the "time to capture" being the duration of the game. The objective of the pursuer is to minimize the time to capture, and the objective of the evader is to maximize the time to capture. If the pursuer never comes sufficiently close to the evader to capture him, then the evader "escapes," and the time to capture is infinite. This description of the pursuit game is general enough to cover many instances of pursuit and evasion, including such diverse situations as the pursuit of the runner in a football game or the pursuit of a missile by an anti-missile.

The simplest pursuit game is that of *pursuit in the plane*, where the players are located at two points in the plane and move at fixed velocities, the velocity of the pursuer exceeding that of the evader. The control variables are the directions in which the players move. The definition of the state and control variables is indicated in Fig. 15.1. Line L is that of a reference direction, and line M passes through the coordinates of both players at any one time. The state variables are chosen to be those in the moving reference system:

$$\begin{aligned} x_1 &= \text{distance between player 1 and player 2;} \\ x_2 &= \text{angle between } L \text{ and } M. \end{aligned} \tag{15.3.1}$$

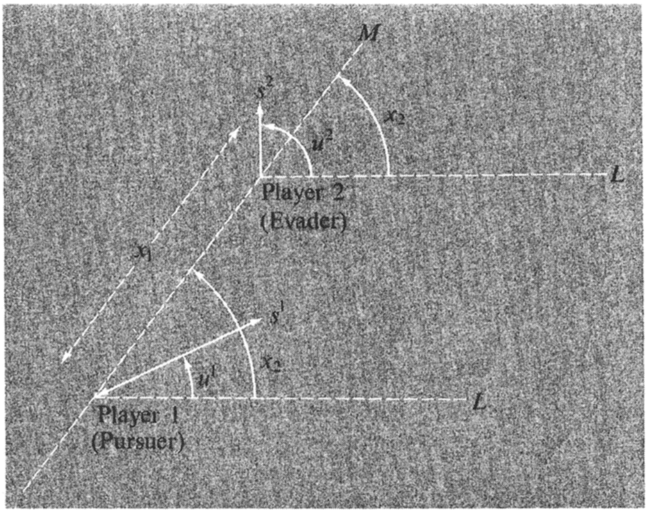

Fig. 15.1
Pursuit in the Plane

The control variables are the directions of movement:

$$\begin{aligned} u^1 &= \text{angle between velocity vector of player 1 and } L \\ u^2 &= \text{angle between velocity vector of player 2 and } L, \end{aligned} \tag{15.3.2}$$

where player 1 (pursuer) moves with speed s^1, player 2 (evader) moves with speed s^2 ($s^1 > s^2$), and:

$$\begin{aligned} 0 &\le u^1 < 2\pi \\ 0 &\le u^2 < 2\pi. \end{aligned} \tag{15.3.3}$$

The equations of motion are:

$$\begin{aligned} \dot{x}_1 &= -s^1 \cos(u^1 - x_2) + s^2 \cos(u^2 - x_2) \\ \dot{x}_2 &= \frac{-s^1 \sin(u^1 - x_2) + s^2 \sin(u^2 - x_2)}{x_1}. \end{aligned} \tag{15.3.4}$$

Note that if the pursuer moves directly toward the evader and the evader moves directly away from the pursuer, then:

$$\begin{aligned} u^1 &= x_2 \\ u^2 &= x_2, \end{aligned} \tag{15.3.5}$$

and the equations of motion become:

$$\begin{aligned}\dot{x}_1 &= s^2 - s^1\\ \dot{x}_2 &= 0,\end{aligned} \tag{15.3.6}$$

where the first equation states that the distance between the players is falling at a rate equal to the difference in their speeds.

Terminal time t_1 is determined as the time at which the distance between the players is reduced to a given distance ℓ:

$$x_1(t_1) = \ell, \tag{15.3.7}$$

at which time the pursuer "captures" the evader. The payoff to the pursuer (player 1) is:

$$J = -\int_{t_0}^{t_1} dt = -(t_1 - t_0).$$

The Hamiltonian is, therefore:

$$\begin{aligned}H = -1 &+ y_1(-s^1 \cos(u^1 - x_2) + s^2 \cos(u^2 - x_2))\\ &+ \frac{y_2}{x_1}(-s^1 \sin(u^1 - x_2) + s^2 \sin(u^2 - x_2)).\end{aligned} \tag{15.3.8}$$

By the minimaximum principle the Hamitonian should be maximized with respect to u^1 and minimized with respect to u^2. The first order conditions are:

$$\begin{aligned}\frac{\partial H}{\partial u^1} &= y_1 s^1 \sin(u^1 - x_2) - \frac{y_2}{x_1} s^1 \cos(u^1 - x_2) = 0\\ \frac{\partial H}{\partial u^2} &= -y_1 s^2 \sin(u^2 - x_2) + \frac{y_2}{x_1} s^2 \cos(u^2 - x_2) = 0,\end{aligned} \tag{15.3.9}$$

implying that:

$$\tan(u^1 - x_2) = \tan(u^2 - x_2) = \frac{y_2}{y_1 x_1}. \tag{15.3.10}$$

The differential equations for the costate variables are:

$$\begin{aligned}\dot{y}_1 &= -\frac{\partial H}{\partial x_1} = -\frac{y_2}{x_1^2}(-s^1 \sin(u^1 - x_2) + s^2 \sin(u^2 - x_2))\\ \dot{y}_2 &= -\frac{\partial H}{\partial x_2} = y_1(-s^1 \sin(u^1 - x_2) + s^2 \sin(u^2 - x_2))\\ &\quad + \frac{y_2}{x_1}(s^1 \cos(u^1 - x_2) - s^2 \cos(u^2 - x_2)).\end{aligned} \tag{15.3.11}$$

But from (15.3.10):

$$\sin(u^1 - x_2) = \frac{y_2}{y_1 x_1}\cos(u^1 - x_2)$$
$$\sin(u^2 - x_2) = \frac{y_2}{y_1 x_1}\cos(u^2 - x_2) \quad (15.3.12)$$

which imply that:

$$\dot{y}_2 = 0; \quad (15.3.13)$$

i.e., y_2 is constant through time. Also, since there is no constraint on the terminal value of x_2:

$$y_2(t_1) = 0, \quad (15.3.14)$$

so that y_2 must be zero everywhere:

$$y_2(t) = 0, \qquad t_0 \leq t \leq t_1. \quad (15.3.15)$$

Thus the value of the game is independent of the initial angle $x_2(t)$ since, by the sensitivity interpretation of the costate variable,

$$y_2(t_0) = \frac{\partial J^*}{\partial x_2(t_0)} = 0.$$

From (15.3.10), the solution is at u^1, u^2 where:

$$\sin(u^1 - x_2) = \sin(u^2 - x_2) = 0$$
$$\tan(u^1 - x_2) = \tan(u^2 - x_2) = 0 \quad (15.3.16)$$

so the optimal controls satisfy:

$$u^1 = x_2$$
$$u^2 = x_2 \quad (15.3.17)$$

which, as noted above, is the case in which the pursuer moves directly toward the evader, and the evader moves directly away from the pursuer. In this case the rate of change of the distance between the players is:

$$\dot{x}_1 = s^2 - s^1, \quad (15.3.18)$$

so:

$$x_1(t) = (s^1 - s^2)(t_0 - t) + x_1(t_0) \quad (15.3.19)$$

where $x_1(t_0)$ is the given initial distance between the players. By the definition of t_1:

$$x_1(t_1) = (s^1 - s^2)(t_0 - t) + x_1(t_0) = \ell \quad (15.3.20)$$

so the value of the game to player 1 (the pursuer) is:

$$J^* = -(t_1 - t_0) = -\left(\frac{x_1(t_0) - \ell}{s^1 - s^2}\right) \tag{15.3.21}$$

Optimal and non-optimal play of the pursuit in the plane game are shown in Fig. 15.2. The upper diagram shows optimal play, with the pursuer moving toward the evader, and the evader moving away from the pursuer along the line M connecting the two players. The lower diagram shows nonoptimal play, where the evader moves nonoptimally at a right angle to the line M. The pursuer, who optimally aims toward the evader at all times ($u^1 = x^2$), catches him in a shorter time.

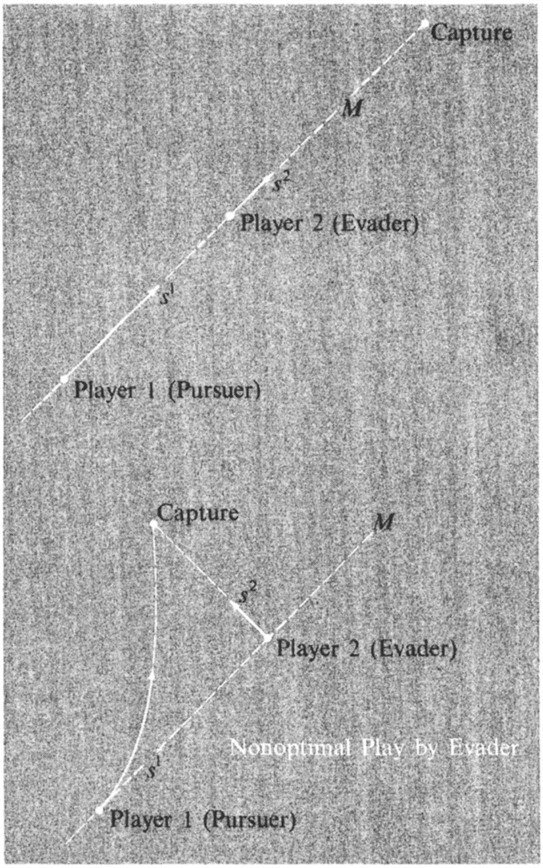

Fig. 15.2

Optimal and Nonoptimal Play of Pursuit in the Plane

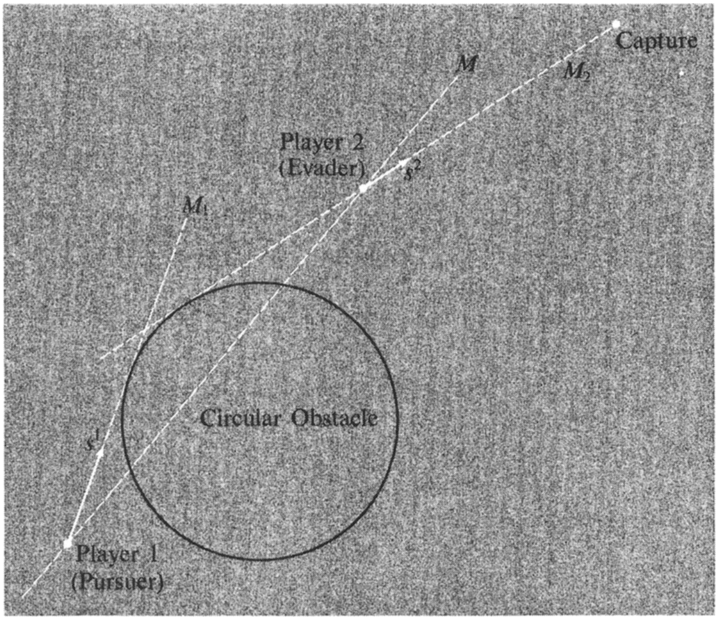

Fig. 15.3

Pursuit in the Plane, where a Circular Obstacle Lies Between Pursuer and Evader

Various extensions are possible for the game of pursuit in the plane. As one extension, consider the case in which there is an obstacle between the pursuer and evader, such as the circle shown in Fig. 15.3. The optimal policy of player 2, the evader, will then be to move along the line M_2 which is tangent to the circle and passes through his original position. The optimal policy of player 1, the pursuer, will be to move first along line M_1 which is tangent to the circle and passes through his original position, then to move along the circle, and finally to move along the line M_2, along which capture occurs. This optimal policy for each player is illustrated in Fig. 15.3. No other strategy of player 1 could shorten the time to capture, and no other strategy of player 2 could lengthen the time to capture, as compared to the strategies illustrated in Fig. 15.3.

If, in Fig. 15.3, the line M connecting the initial positions of the players passed through the center of the circle, then each player has two equally good tangents as possible paths. In this case the players might use mixed strategies, choosing a path with a random device such that either path can be chosen with probability 1/2. The set of all such symmetric positions is called a *dispersal surface*. This surface disappears the instant after the choices have been made, in which case one or both players may reverse their routes. If both reverse their routes, however, they may wind up on another dispersal surface.[7]

15.4 Coordination Differential Games

In a zero-sum game the players are in direct conflict, with the payoff to one player being the negative of the payoff to the other player. A *coordination game*, by contrast, is one in which the players are in complete accord, with the payoff to the players identical, both players seeking to maximize the payoff:

$$J[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] = \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + F(\mathbf{x}_1, t_1) \tag{15.4.1}$$

by choice of their control trajectories, $\{\mathbf{u}^1(t)\}$ and $\{\mathbf{u}^2(t)\}$ respectively. An illustration of such a game is the problem of collision avoidance among two moving craft (e.g., autos, boats, airplanes), where the payoff can be defined as zero or one, depending on whether the distance between the craft at the time they are closest together falls short of or exceeds some critical distance.

The solution to the two-person cooperative differential game can again be developed by analogy to the maximum principle solution to the control problem. In this case, assuming the differential game satisfies certain regularity conditions, the optimal controls necessarily satisfy the condition on the Hamiltonian function:

$$\max_{\mathbf{u}^1 \in \Omega^1} \max_{\mathbf{u}^2 \in \Omega^2} H(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}, t) = H(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, \mathbf{y}, t) \tag{15.4.2}$$

at all points of time in the relevant interval, a condition which can be called the *maximaximum principle*. The canonical equations, etc., are the same as those of the last section.

As an example of a two-person cooperative differential game, consider the case in which each player controls the acceleration in one direction of a unit mass with coordinates at $(x_1, x_2)'$. The differential equations are:

$$\begin{aligned} \ddot{x}_1 &= u^1 \\ \ddot{x}_2 &= u^2, \end{aligned} \tag{15.4.3}$$

where the constraints on the control variables are:

$$\begin{aligned} |u^1| &\leq 1 \\ |u^2| &\leq 1, \end{aligned} \tag{15.4.4}$$

stating that the maximum acceleration in either direction for each player is unity. The objective is to reach the origin in minimum time; i.e.:

$$J = -\int_{t_0}^{t_1} dt = -(t_1 - t_0), \tag{15.4.5}$$

where the initial position is given, and the mass is initially at rest:

$$\begin{aligned} x_1(t_0) &= x_{10} \\ x_2(t_0) &= x_{20} \\ \dot{x}_1(t_0) &= 0 \\ \dot{x}_2(t_0) &= 0. \end{aligned} \tag{15.4.6}$$

This coordination game is a differential game extension of the minimum time problem in which the control is the second derivative of the state variable as discussed in Sec. 14.5. Using the approach of that section, the differential equations (15.4.3) can be converted to first order by introducing new state variables x_3 and x_4 defined by:

$$\begin{aligned} \dot{x}_1 &= x_3, & x_1(t_0) &= x_{10} \\ \dot{x}_2 &= x_4, & x_2(t_0) &= x_{20} \\ \dot{x}_3 &= u^1, & x_3(t_0) &= 0 \\ \dot{x}_4 &= u^2, & x_4(t_0) &= 0. \end{aligned} \tag{15.4.7}$$

The Hamiltonian is:

$$H = -1 + y_1 x_3 + y_2 x_4 + y_3 u^1 + y_4 u^2, \tag{15.4.8}$$

and, by the maximaximum principle:

$$\begin{aligned} u^1 &= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \text{ if } y_3 \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \\ u^2 &= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \text{ if } y_4 \begin{Bmatrix} > \\ < \end{Bmatrix} 0. \end{aligned} \tag{15.4.9}$$

The canonical equations for the costate variables are:

$$\begin{aligned} \dot{y}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{y}_2 &= -\frac{\partial H}{\partial x_2} = 0 \\ \dot{y}_3 &= -\frac{\partial H}{\partial x_3} = -y_1 \\ \dot{y}_4 &= -\frac{\partial H}{\partial x_4} = -y_2 \end{aligned} \tag{15.4.10}$$

which have as solutions:

$$\begin{aligned} y_1 &= c_1 \\ y_2 &= c_2 \\ y_3 &= c_3 - c_1 t \\ y_4 &= c_4 - c_2 t, \end{aligned} \tag{15.4.11}$$

where c_1, c_2, c_3, and c_4 are constants. But since terminal velocities are free, it follows that:

$$\begin{aligned} y_3(t_1) &= 0 \\ y_4(t_1) &= 0. \end{aligned} \tag{15.4.12}$$

These terminal conditions and the above solutions for the costate variables imply that y_3 and y_4 cannot switch sign—they are either always positive, always negative, or zero.

Solutions to the problem are illustrated in Fig. 15.4. The solution starting

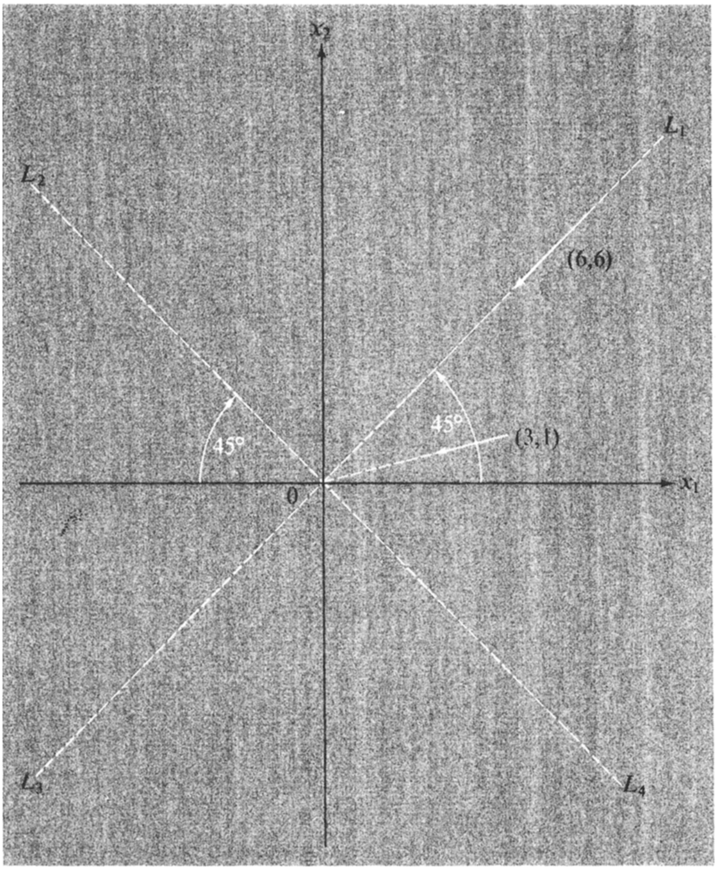

Fig. 15.4
A Cooperative Differential Game

from a point on the line OL_1, such as (6, 6), is obviously:

$$u^1 = -1, \qquad u^2 = -1, \tag{15.4.13}$$

where both y_3 and y_4 are negative. Since, by the sensitivity interpretation of the initial costate variables:

$$\begin{aligned} y_3(t_0) &= \frac{\partial J^*}{\partial x_{30}} \\ y_4(t_0) &= \frac{\partial J^*}{\partial x_{40}}, \end{aligned} \tag{15.4.14}$$

the negative values for the initial y_3 and y_4 in this case indicate that, other things being equal, an increase to positive initial velocities in either direction starting from points on the OL line would increase the time required to reach the origin.

Similarly, the solution starting from a point on the line OL_2 is:

$$u^1 = 1, \qquad u^2 = -1, \tag{15.4.15}$$

that starting from a point on the line OL_3 is:

$$u^1 = 1, \qquad u^2 = 1, \tag{15.4.16}$$

and that starting from a point on the line OL_4 is:

$$u^1 = -1, \qquad u^2 = 1. \tag{15.4.17}$$

What about points not on one of these lines, such as (3, 1)? A solution still lies along a line where, in this case:

$$u^1 = -1, \qquad u^2 = -\tfrac{1}{3}. \tag{15.4.18}$$

This solution is consistent with the above necessary conditions even though u^2 does not lie on the boundary. In this case:

$$\begin{aligned} y_2 &= 0 \\ y_4 &= 0, \end{aligned} \tag{15.4.19}$$

so, by the sensitivity interpretation of the costate variables, the value of the objective functional (the minimum time) is independent of the initial position and velocity in the vertical direction. But this is obviously so, since the only determinant of the time required in this case is the horizontal direction.

Starting from a higher position or a larger vertical velocity simply requires a different value for u^2, with no change in J^*. The optimal solution by this reasoning always lies along a line, and the optimal payoff, the minimum time, depends only on the larger of the initial coordinates.

15.5 Noncooperative Differential Games

A *noncooperative differential game* is a nonzero-sum differential game in which the players are not able to make binding commitments in advance of play on the strategies they will employ. In a two-person nonzero-sum differential game in which the payoffs to player 1 and player 2 are respectively:

$$\begin{aligned} J^1[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] &= \int_{t_0}^{t_1} I^1(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + F^1(\mathbf{x}_1, t_1) \\ J^2[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^2(t)\}] &= \int_{t_0}^{t_1} I^2(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)\, dt + F^2(\mathbf{x}_1, t_1), \end{aligned} \tag{15.5.1}$$

a noncooperative (Nash) equilibrium is a pair of strategies:

$$\begin{aligned} \mathbf{u}^{1*}(t) &= \mathbf{S}^{1*}(\mathbf{x}(t)) \\ \mathbf{u}^{2*}(t) &= \mathbf{S}^{2*}(\mathbf{x}(t)) \end{aligned} \tag{15.5.2}$$

having the property that neither player has an incentive to change his strategy, given the strategy of the other. Thus:

$$\begin{aligned} J^1[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^{2*}(t)\}] &\geq J[\{\mathbf{u}^1(t)\}, \{\mathbf{u}^{2*}(t)\}] \quad \text{for all} \quad \{\mathbf{u}^1(t)\} \in U^1 \\ J^2[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^{2*}(t)\}] &\geq J[\{\mathbf{u}^{1*}(t)\}, \{\mathbf{u}^2(t)\}] \quad \text{for all} \quad \{\mathbf{u}^2(t)\} \in U^2. \end{aligned} \tag{15.5.3}$$

Again proceeding by analogy to the maximum principle solution, the necessary conditions for a noncooperative (Nash) equilibrium under certain regularity assumptions can be developed in terms of the Hamiltonian concept.[8] The Hamiltonians for players 1 and 2 are, respectively:

$$\begin{aligned} H^1(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}^1, t) &= I^1(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) + \mathbf{y}^1\mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) \\ H^2(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, \mathbf{y}^2, t) &= I^2(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) + \mathbf{y}^2\mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t), \end{aligned} \tag{15.5.4}$$

where $\mathbf{y}^1$ is the row vector of costate variables for player 1 and $\mathbf{y}^2$ is the row vector of costate variables for player 2. Necessary conditions for a noncooperative (Nash) equilibrium are then the conditions that at each time in the relevant interval the control vectors represent a noncooperative (Nash) equilibrium for the nonzero sum (static) game in which the payoffs are $H^1(\cdot\cdot\cdot\cdot\cdot)$ and $H^2(\cdot\cdot\cdot\cdot\cdot)$:

$$\begin{aligned} H^1(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, \mathbf{y}^1, t) &\geq H^1(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^{2*}, \mathbf{y}^1, t) \quad \text{for all} \quad \mathbf{u}^1 \in \Omega^1 \\ H^2(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, \mathbf{y}^2, t) &\geq H^2(\mathbf{x}, \mathbf{u}^{1*}, \mathbf{u}^2, \mathbf{y}^2, t) \quad \text{for all} \quad \mathbf{u}^2 \in \Omega^2, \end{aligned} \tag{15.5.5}$$

i.e.:

$$\begin{aligned} \mathbf{u}^{1*}(t) = \mathbf{S}^{1*}(\mathbf{x}(t)) \quad &\text{maximizes} \quad H^1(\mathbf{x}, \mathbf{u}^1, \mathbf{S}^{2*}(\mathbf{x}), \mathbf{y}^1, t) \\ \mathbf{u}^{2*}(t) = \mathbf{S}^{2*}(\mathbf{x}(t)) \quad &\text{maximizes} \quad H^2(\mathbf{x}, \mathbf{S}^{1*}(\mathbf{x}), \mathbf{u}^2, \mathbf{y}^2, t). \end{aligned} \tag{15.5.6}$$

The canonical equations are:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) = \frac{\partial H^1}{\partial \mathbf{y}^1} = \frac{\partial H^2}{\partial \mathbf{y}^2} \\ \dot{\mathbf{y}}^1 &= -\frac{\partial H^1}{\partial \mathbf{x}} - \frac{\partial H^1}{\partial \mathbf{u}^2}\frac{\partial \mathbf{S}^{2*}}{\partial \mathbf{x}} \\ \dot{\mathbf{y}}^2 &= -\frac{\partial H^2}{\partial \mathbf{x}} - \frac{\partial H^1}{\partial \mathbf{u}^1}\frac{\partial \mathbf{S}^{1*}}{\partial \mathbf{x}}, \end{aligned} \tag{15.5.7}$$

where the last terms in the last two differential equations are "interaction terms," indicating the interaction of the strategy of one player on the Hamiltonian of the other.

PROBLEMS

15-A. Solve the two-person zero-sum game with payoff at terminal time for which the equations of motion are:

$$\dot{x}_1 = au^1 + b \sin u^2$$

$$\dot{x}_2 = -1 + b \cos u^2$$

and for which the scalar control variables satisfy:[9]

$$-1 \leq u^1 \leq 1$$

$$0 \leq u^2 < 2\pi.$$

15-B. In a certain two-person zero-sum differential game the equations of motion are:

$$\dot{x}_1 = u_1(1 + 2\sqrt{|x_1|}) + u_2$$
$$\dot{x}_2 = -1,$$

and the scalar control variables satisfy:

$$0 \leq u^1 \leq 1$$
$$0 \leq u^2 \leq 1.$$

The game starts at $x_2(t_0) > 0$ and terminates at $x_2(t_1) = 0$, at which the payoff to player 1 is:

$$J = \frac{1}{1 + [x_1(t_1)]^2}.$$

Show that the x_2 axis is a "singular surface" in that optimal trajectories are curves beginning on this axis.[10]

15-C. In a two-person zero-sum differential game the equations of motion are:

$$\dot{x}_1 = (u^1 - u^2)^2$$
$$\dot{x}_2 = -1,$$

and the scalar control variables satisfy:

$$|u^1| \leq 1$$
$$|u^2| \leq 1.$$

The game starts at $x_2(t_0) > 0$ and terminates at $x_2(t_1) = 0$, at which the payoff to player 1 is:

$$J = x_1(t_1).$$

Show that this game has no solution in pure strategies. Illustrate geometrically in the $(x_1, x_2)'$ plane.

15-D. Suppose in the pursuit problem the equations of motion are linear and separable:

$$\dot{\mathbf{x}}^1 = \mathbf{A}^1\mathbf{x}^1 + \mathbf{b}^1u^1$$
$$\dot{\mathbf{x}}^2 = \mathbf{A}^2\mathbf{x}^2 + \mathbf{b}^2u^2$$

where the scalar control variables satisfy

$$0 \leq |u^1| \leq 1$$

$$0 \leq |u^2| \leq 1.$$

The initial positions $x^1(t_0)$ and $x^2(t_0)$ are given, and the game terminates when:

$$x_1^1(t_1) = x_1^2(t_1).$$

Player 1 (2) seeks to minimize (maximize) the time to intercept, $t_1 - t_0$. Develop the solution.[11]

15-E. In the pursuit game in the plane the pursuer, player 1, exerts control on the coordinate x_1, and the evader, player 2, exerts control on the coordinate x_2, where:

$$\ddot{x}_1 + \alpha \dot{x}_1 = u^1, \qquad |u^1| \leq 1$$

$$\ddot{x}_2 + \beta \dot{x}_2 = u^2, \qquad |u^2| \leq 1.$$

Termination time occurs at time t_1 when:

$$x_1(t_1) = x_2(t_2).$$

Show that the payoff is finite (i.e., the game can be terminated) if $\alpha < \beta$.[12]

15-F. Derive the "main equation" of footnote 5 using the dynamic programming approach.

15-G. In the *goal-keeping differential game* player 1 is defending a scoring zone being approached by player 2, as in hockey, where player 1 is the goalie. The game is played on the $(x_1, x_2)'$ plane where the scoring zone lies on the x_1 axis and extends a distance L from each side of the x_2 axis. Player 1 starts from the scoring zone, moving away from this zone at a fixed veocity v^1 and controlling his lateral velocity:

$$\dot{x}_1^1 = u^1, \qquad x_1^1(t_0) \text{ given } \quad |u^1| \leq \bar{u}^1$$

$$\dot{x}_2^1 = v^1 \qquad x_2^1(t_0) = 0.$$

Player 2 starts from an upfield position moving toward the scoring zone at a fixed velocity v^2 and controlling his lateral velocity:

$$\dot{x}_1^2 = u^2, \qquad x_1^2(t_0) \text{ given}, \quad |u^2| \leq \bar{u}^2, \bar{u}^2 > \bar{u}^1$$

$$\dot{x}_2^2 = -v^2, \qquad x_2^2(t_0) > 0.$$

The game ends when the players pass:

$$x_2^1(t_1) = x_2^2(t_1),$$

at which point the payoff (loss) to player 1 (2) is:

$$J = \begin{Bmatrix} 1 \\ -(x_1^1(t_1) - x_1^2(t_1))^2 \end{Bmatrix} \text{ if } \begin{Bmatrix} |x_1^2(t_1)| > L + \dfrac{\bar{u}^2 x_2^2(t_1)}{v^2} \\ \text{otherwise} \end{Bmatrix}.$$

Interpret the payoff function and develop the solution as far as possible.[13]

15-H. A lion and a man are in a circular arena and have identical maximum velocities. Can the lion assure himself a meal?

15-I. An attacker and a defender lie at two points in the plane outside a certain target area. They move at the same speed and can control their own directions of movement. The defender captures the attacker when he comes sufficiently close to him, and he seeks to maximize the distance between the point of capture and the target area. The attacker seeks to come as close as possible to the target area. Assuming capture occurs outside the target area, show the optimal strategies geometrically.[14]

15-J. In a *dynamic model of a missile war* two countries, A and B, are engaged in a war between times t_0 and t_1. The state variables are the missiles remaining in each country, M_A and M_B, and the casualities in each country, C_A and C_B, the equations of motion being:

$$\dot{M}_A = -\alpha M_A - \beta M_B \beta' f_B$$
$$\dot{M}_B = -\beta M_B - \alpha M_A \alpha' f_A$$
$$\dot{C}_A = (1 - \beta')\beta M_B v_B$$
$$\dot{C}_B = (1 - \alpha')\alpha M_A v_A.$$

The control variables for A are α, the rate of fire, and α', the counterforce (targeting) proportion; the control variables for B are similarly β and β', where:

$$0 \le \alpha \le \bar{\alpha}, \quad \text{given } \bar{\alpha}$$
$$0 \le \alpha' \le 1$$
$$0 \le \beta \le \bar{\beta}, \quad \text{given } \bar{\beta}$$
$$0 \le \beta' \le 1.$$

In the equations of motion f_B is the effectiveness of B missiles against A missiles; i.e., the number of A missiles destroyed per B missile. Similarly, f_A

is the effectiveness of A missiles against B missiles, v_B is the effectiveness of B missiles against A cities, and v_A is the effectiveness of A missiles against B cities. Thus, the two terms in the equation for M_A show the loss of A missiles due to A firing decisions and due to destruction by B counterforce missiles, respectively. The boundary conditions are:

$$M_A(t_0) = M_{A_0}$$

$$M_B(t_0) = M_{B_0}$$

$$C_A(t_0) = 0$$

$$C_B(t_0) = 0.$$

Assuming t_1 is given, find the optimal rate of fire and targeting strategies for A and B, assuming the objective of A is to minimize $C_A(t_1) - C_B(t_1)$, and the objective of B is to minimize $C_B(t_1) - C_A(t_1)$.[15]

15-K. A *differential game of kind* (or *differential game of survival*) is a two-person zero-sum differential game in which one player wins and the other loses. The terminal surface can be divided into a surface on which player 1 wins, W, and one on which he loses, L. The space of state variables (some subset of E^n) can then be divided into a winning zone, WZ, consisting of all points from which player 1 can ensure termination in W, a losing zone, LZ, consisting of all points from which player 2 can ensure termination in L, and the remaining zone, N, in which neither player is assured of winning or losing.

1. Given the equations of motion, control set, and boundary conditions of Problem 15-C, suppose:

$$W = \{x_1(t_1) \mid x_1(t_1) > 0\}$$

$$L = \{x_1(t_1) \mid x_1(t_1) < 0\}.$$

   Show WZ, LZ, and N geometrically.

2. Again using the conditions of Problem 15-C), suppose:

$$W = \{x_1(t_1) \mid |x_1(t_1)| \le 1\}$$

$$L = \{x_1(t_1) \mid |x_1(t_1)| > 1\}.$$

   Show WZ, LZ, and N geometrically.

3. Assuming N is smooth, show that the normal vector to N, $\mathbf{V} = (V_1, \ldots, V_n)$, oriented to WZ, satisfies:[16]

$$\max_{\mathbf{u}^1 \in \Omega} \min_{\mathbf{u}^2 \in \Omega} [\mathbf{V} \cdot \mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t)] = 0.$$

FOOTNOTES

[1] The basic references for differential games are Isaacs (1965), Ho (1965), Simakova (1966), Berkovitz (1967b), and Owen (1968). The analysis of differential games uses many of the terms of game theory, such as "player," "strategy," and "payoff." These terms are discussed in Chapter 6.

[2] For discussion of stochastic differential games see Ho (1966).

[3] The proofs are similar to those presented in the last three chapters. For a proof using the calculus of variations approach see Berkovitz (1964); for proofs using the dynamic programming approach see Isaacs (1965) and Berkovitz (1967a); and for a proof using the maximum principle approach see Pontryagin et al. (1962). These proofs generally assume that optimal strategies exist for both players and that the differential game has a finite value. On the question of existence of solutions see Varaiya (1967).

[4] It might be recalled from Chapter 6 that games of perfect information are always strictly determined if they are finite games. Differential games, while games of perfect information, are infinite games and therefore might require mixed strategies, i.e., probability distributions over the alternative possible pure strategies in the control sets. For examples of differential games that are not strictly determined, requiring mixed strategy solutions, see Berkovitz (1967b) and Owen (1968). If, however, both the intermediate function $I(\cdots)$ and the equations of motion function $\mathbf{f}(\cdots)$ are *separable* in that the Hamiltonian can be separated into the sum of two functions, one of which depends only on $\mathbf{u}^1$ and the other only on $\mathbf{u}^2$ then the differential game is strictly determined and so has a solution in pure strategies. An example is the case in which both $I(\cdots)$ and $\mathbf{f}(\cdots)$ are linear, as discussed in Pontryagin et al. (1962).

[5] Isaacs (1965) replaces $\mathbf{y}$ by its sensitivity interpretation $\partial J^*/\partial \mathbf{x}$ as discussed in Sec. 14.4 and calls the equation:

$$\max_{\mathbf{u}^1 \in \Omega^1} \min_{\mathbf{u}^2 \in \Omega^2} \left[I(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}^1, \mathbf{u}^2, t) \right] = 0$$

the *main equation*. This equation is simply Bellman's equation for the problem. Also Isaacs (1965) writes the canonical equations as the *retrograde path equations*:

$$\overset{\circ}{\mathbf{x}} = -\frac{\partial H}{\partial \mathbf{y}}$$

$$\overset{\circ}{\mathbf{y}} = \frac{\partial H}{\partial \mathbf{x}},$$

where the superscript circle represents differentiation with respect to time but backward from terminal time, i.e.:

$$\overset{\circ}{\mathbf{z}} = \frac{dz}{d\tau}, \quad \text{where } \tau = t_1 - t = \text{time-to-go}.$$

[6] See Pontryagin et al. (1962), Ho and Baron (1965), Ho, Bryson, and Baron (1965), Isaacs (1965), and Simakova (1966).

[7] See Isaacs (1965).

[8] See Starr and Ho (1969).

[9] See Isaacs (1965).

[10] See Owen (1966).

[11] See Pontryagin et al. (1962) and Ho and Baron (1965).

[12] See Pshenichniy (1967).

[13] See Meschler (1967).

[14] See Isaacs (1965).

[15] See Intriligator (1967).

[16] See Isaacs (1965) and Owen (1968).

BIBLIOGRAPHY

Balakrishnan, A. V., and L. W. Neustadt, eds., *Mathematical Theory of Control.* New York: Academic Press, Inc., 1967.

Berkovitz, L. D., "A Variational Approach to Differential Games," in *Advances in Game Theory*, Annals of Mathematics Study No. 52, ed. M. Dresher, L. S. Shapley, and A. W. Tucker. Princeton, N.J.: Princeton University Press, 1964.

———, "Necessary Conditions for Optimal Strategies in a Class of Differential Games and Control Problems," *J. SIAM Control*, 5 (1967a):1–24.

———, "A Survey of Differential Games," in *Mathematical Theory of Control*, ed. A. V. Balakrishnan, and L. W. Neustadt. New York: Academic Press, Inc., 1967b.

Dresher, M., L. S. Shapley, and A. W. Tucker, eds., *Advances in Game Theory*, Annals of Mathematics Study No 52. Princeton, N.J.: Princeton University Press, 1964.

Ho, Y. C., "Differential Games and Optimal Control Theory," *Proc. Nat. Elect. Conf.*, 21 (1965):613–5.

———, "Optimal Terminal Maneuver and Evasion Strategy," *J. SIAM Control*, 4 (1966):421–8.

Ho, Y. C., and S. Baron, "Minimal Time Intercept Problems," *IEEE Trans. Autom. Control*, AC-10 (1965):200.

Ho, Y. C., A. E. Bryson, and S. Baron, "Differential Games and Optimal Pursuit-Evasion Strategies," *IEEE Trans. Autom. Control*, AC-10 (1965):385–9.

Intriligator, M. D., *Strategy in a Missile War: Targets and Rates of Fire.* Los Angeles, Calif.: UCLA Security Studies Project, 1967.

Isaacs, R., *Differential Games.* New York: John Wiley & Sons, Inc., 1965.

Meschler, P. A., "On a Goal-Keeping Differential Game," *IEEE Trans. Autom. Control*, AC-12 (1967):15–21.

Owen, G., *Game Theory.* Philadelphia, Pa.: W. B. Saunders Company, Inc., 1968.

Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, trans. by K. N. Trirogoff. New York: Interscience Publishers, John Wiley & Sons, Inc., 1962.

Pshenichniy, B. N., "Linear Differential Games" in *Mathematical Theory of Control*, ed. A. V. Balakrishnan and L. W. Neustadt. New York: Academic Press, Inc., 1967.

Simakova, E. N., "Differential Games," *Automat. i Telemekh.*, 27 (1966):161–78. Translated in *Automation and Remote Control*, (1966), 27:1980–98.

Starr, A. W., and Y. C. Ho, "Nonzero-Sum Differential Games," *Journal of Optimization Theory and Applications*, (1969), 3.

Varaiya, P. P , "On the Existence of Solutions to a Differential Game," *J. SIAM Control*, 5 (1967):153–62.