Control Theory Notes

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1 Big Picture: Static vs. Dynamic Economizing

- **Static (mathematical programming):** choose a finite-dimensional decision vector to maximize an objective over an opportunity set.
- **Dynamic (control problem):** choose a *time path* of decisions (a control trajectory) to maximize an *objective functional* subject to equations of motion.

2 Formal Statement of the Control Problem

2.1 Time Horizon

Time is continuous on an interval from initial time t_0 to terminal time t_1, where t_1 may be fixed or determined by achieving a goal state (such as intercepting an enemy missle):

$$t_0 \leq t \leq t_1.$$

2.2 State Variables and State Trajectory

The **state** at time t is an n-vector

$$x(t) = \big(x_1(t), \ldots, x_n(t)\big)^\top \in \mathbb{R}^n,$$

with $x_i(\cdot)$ continuous in time. The **state trajectory** is $\{x(t)\}_{t \in [t_0, t_1]}$ with initial condition $x(t_0) = x_0$. The terminal state $x(t_1)$ may be free or constrained.

2.3 Control Variables and Admissible Controls

The **control** at time t is an r-vector

$$u(t) = \big(u_1(t), \ldots, u_r(t)\big)^\top \in \mathbb{R}^r,$$

typically assumed *piecewise continuous*. Controls are constrained by a nonempty set

$$u(t) \in \Omega \subset \mathbb{R}^r \quad \text{for all } t \in [t_0, t_1],$$

often with Ω compact and convex.

Let $\mathcal{U}$ denote the set of all admissible control trajectories:

$$\mathcal{U} := \Big\{ u(\cdot) : [t_0, t_1] \to \Omega \;\Big|\; u(\cdot) \text{ piecewise continuous} \Big\}.$$

2.4 Equations of Motion (Dynamics)

The state evolves according to an ODE system

$$\dot{x}(t) = f\big(x(t), u(t), t\big),$$

where $f : \mathbb{R}^n \times \mathbb{R}^r \times [t_0, t_1] \to \mathbb{R}^n$ is given (typically C^1).

Example A common special case is

$$\dot{x}(t) = Ax(t) + Bu(t),$$

for given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$.

2.5 Terminal Conditions via a Terminal Surface

Terminal time/state constraints can be encoded by a **terminal surface**

$$(x(t), t) \in \mathcal{T} \subset \mathbb{R}^{n+1} \quad \text{at } t = t_1.$$

- **Terminal time given:** t_1 fixed, $x(t_1)$ free or constrained separately.
- **Terminal state given:** $x(t_1) = x_1$ fixed, t_1 free or fixed.

2.6 Objective Functional (Bolza Form)

A standard objective functional (in Bolza problem form) is

$$J[u(\cdot)] \;=\; \int_{t_0}^{t_1} I\big(x(t), u(t), t\big)\,\mathrm{d}t \;+\; F\big(x(t_1), t_1\big),$$

where I is the **intermediate function** and F is the **final function**.

2.7 The General Control Problem

$$\begin{aligned} \max_{u(\cdot)\in\mathcal{U}} \quad & J[u(\cdot)] \;=\; \int_{t_0}^{t_1} I(x(t), u(t), t)\,\mathrm{d}t + F(x(t_1), t_1) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \qquad x(t_0) = x_0, \\ & (x(t), t) \in \mathcal{T} \text{ at } t = t_1. \end{aligned}$$

3 Some Special Cases of the Objective

3.1 Time-Optimal Control

Move from initial to terminal conditions in minimum time. Equivalent to maximizing

$$J = -(t_1 - t_0).$$

(So minimizing t_1 when t_0 is fixed.)

3.2 Servomechanism / Tracking a Desired State

Given a desired trajectory $x^\circ(t)$, penalize deviation:

$$J = \int_{t_0}^{t_1} \phi\big(x^\circ(t) - x(t)\big)\, \mathrm{d}t.$$

Least-squares tracking uses a quadratic form with weights:

$$\phi\big(x^\circ(t) - x(t)\big) = \big(x^\circ(t) - x(t)\big)^\top D \big(x^\circ(t) - x(t)\big),$$

where D is (typically) negative definite (since the objective is maximization).

3.3 Minimum Effort

Penalize control magnitude:

$$J = \int_{t_0}^{t_1} u(t)^\top E\, u(t)\, \mathrm{d}t,$$

with E negative definite (again, for maximization).

3.4 Least Squares + Effort (Combined Quadratic Objective)

A common combined form is

$$J = \int_{t_0}^{t_1} \Big(x(t)^\top D\, x(t) + u(t)^\top E\, u(t)\Big)\, \mathrm{d}t,$$

(assuming the desired state is shifted to the origin).

4 Types of Control: Open Loop vs. Closed Loop

Open loop control: Compute the optimal control trajectory purely as a function of time:

$$u^*(t).$$

All decisions are planned at t_0 and then executed.

Closed loop (feedback) control: Compute the optimal control as a function of current state and time:

$$u^*(x(t), t).$$

This allows revision using new information embodied in the current state. The problem of finding such a feedback rule is called **synthesis**.

Feedback becomes crucial in **stochastic control** and **adaptive control** (parameter uncertainty).

5 Generalized Weierstrass Theorem (Existence of an Optimum)

Generalized Weierstrass Theorem A solution to the control problem exists if:

1. the objective functional $J[u(\cdot)]$ is **continuous** (as a functional of the control trajectory), and
2. the admissible control set $\mathcal{U}$ is **compact** in the relevant infinite-dimensional function space.

Proof Sketch: Let $J^* = \sup_{u(\cdot)\in\mathcal{U}} J[u(\cdot)]$.

Take a sequence $\{u_n(\cdot)\} \subset \mathcal{U}$ such that $J^* - J(u_n(\cdot)) < \frac{1}{n}$

Compactness gives a convergent subsequence $u_{n_k}(\cdot) \to u^*(\cdot) \in \mathcal{U}$.

Continuity gives $J[u_{n_k}(\cdot)] \to J[u^*(\cdot)]$, hence $J[u^*(\cdot)] = J^*$.