6 Game Theory

The problems treated thus far are those of a single decision-maker, for whom the economizing problem is summarized by the objective function, instruments, and constraints. This chapter introduces the possibility of more than one decision-maker, in which case the value of the objective function for any one decision-maker depends not only on his own choices but also on the choices of the others. *Game theory* is the study of such situations, situations in which conflict and cooperation play important roles.[1]

The study of situations involving more than one decision-maker is called "game-theory" because in mathematical form such situations are in many respects similar to those presented by common parlor games of strategy, such as matching pennies, tic-tac-toe, poker, bridge, and chess. Of course, implications of game theory range far beyond parlor games—to mathematics, economics, politics, and military strategy to name a few. Because of its foundations, however, much of the terminology of game theory is taken from parlor game situations.

Thus, the decision-makers are called *players* and the objective function is called a *payoff function*. The players may be individuals, groups of individuals (e.g., the partners in a bridge game), firms, nations, etc. The payoff function gives numerical *payoffs*, to each of the players. A *game* is then a collection of rules known to all players which determine what players may do and the outcomes and payoffs resulting from their choices.

A *move* is a point in the game at which players must make choices between alternatives, and any particular set of moves and choices is a *play* of the game. The essential feature of a game is that the payoff to any player typically depends not only on his own choices but also on the choices of the other players.[2]

Each player must take this joint dependence into account in selecting a *strategy*, a set of decisions formulated in advance of play specifying choices to be made in every possible contingency. The notion of a strategy is central to game theory, and the subject is in fact sometimes called "games of strategy."

6.1 Classification and Description of Games

There are several ways of classifying games: by the number of players, the number of strategies, the nature of the payoff function, and the nature of preplay negotiation.

Games can be classified by the *number of players*; for example, *two-person games, three-person games, . . . , n-person games*. The previous chapters can be considered studies of games in which there is only one player. Two players is the minimum number for conflict or cooperation to be present. Three or more players lead to the possibility of coalition formation, where a group of two or more players merge their interests and coordinate their strategies.

Another way of classifying games is by the *number of strategies*, as finite games or infinite games. This chapter will treat only finite games, in which the number of strategies available to each of the players is finite.

In most of the examples to be discussed the number of strategies is only two or three, but the same theory is applicable to games with a large, even astronomically large number of strategies.[3] By contrast, infinite games are those in which there are an infinite number of strategies available for one or more players.[4]

A third way of classifying games is by the *nature of payoff function*. One important type of payoff function is that of a *zero-sum game*, where the payoffs to the players sum to zero. In the two-person zero-sum game what one player gains the other player loses; i.e., the players are in direct conflict. At the opposite extreme is the two person *constant difference game*, in which the players gain or lose together, so they should rationally cooperate. In the general *nonzero-sum game* there are usually elements of both conflict and cooperation.

A final classification of games is by the *nature of preplay negotiation*. The game is a cooperative game if the players can form a coalition to discuss their strategies before the game is played and make binding agreements on strategy. If the players cannot coordinate their strategies in this way, then the game is a *noncooperative game*.

There are several ways in which a game can be described and analyzed. One way of describing a game is to summarize the rules of the game by indicating the moves, the information and choices available to the players, and the ultimate payoffs to all players at the end of play. A game described in this way is referred to as a *game in extensive form*, and the description usually takes the form of a *game tree*, such as that of Fig. 6.1 for a simplified two-person poker game. In this game both players ante \$5.00 and are dealt hands which are either "high" (H) or "low" (L). Player 1 has two alternatives: either he can "see" (S) or "raise" (R). If he chooses S, then the higher hand wins the pot or equal hands split the pot.

If he chooses R, then he adds \$5.00 to the pot, and then player 2 has two alternatives: either he can "fold" (F) or "call" (C). If he chooses F, then player 1 wins the pot regardless of the hands. If he chooses C, then he adds \$5.00 and the higher hand wins the pot or equal hands split the pot. The game tree of Fig. 6.1 indicates all possible events and their resulting payoffs.

A game in extensive form exhibits *perfect information* if no moves are made simultaneously and at each move all players know the choices made

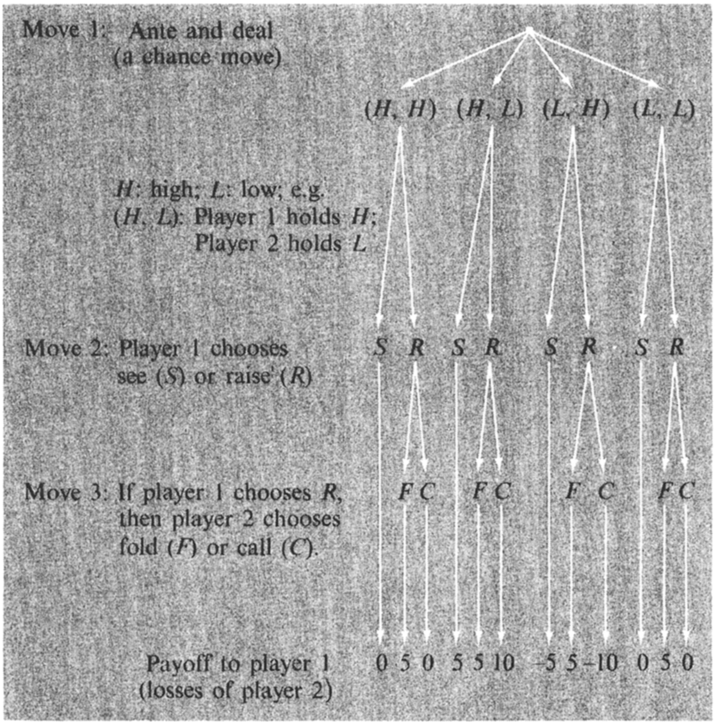

Fig. 6.1

Game Tree for a Game in Extensive Form—
Simplified Two-Person Poker

at every previous move, including chance moves. Tic-tac-toe and chess, for example, are games of perfect information. Poker, however, is a game of imperfect information since the players do not know certain choices made on chance moves, specifically the hands held by their opponent.

A second way of describing a game is to consider all possible strategies of each of the players and to indicate the payoffs to each of the players resulting from alternative combinations of strategies chosen by all players.

A game described in this way is referred to as a *game in normal form*, and the normal form can be derived from the extensive form. In the two-person game the normal form consists of two *payoff matrices*, showing the payoffs to each of the two players given alternative possible pairs of strategies. The two matrices are usually collapsed into a single matrix, each entry of which is a pair of numbers, the first being the payoff to player 1, and the second being the payoff to player 2, as shown in Fig. 6.2.

$$
\begin{array}{cc}
 & \text{Player 2 selects strategy} \\
 & \begin{array}{ccccc} S_1^2 & S_2^2 & \qquad & S_j^2 & \qquad S_n^2 \end{array} \\
\text{Player 1 selects strategy} \left\{ \begin{array}{c} S_1^1 \\ S_2^1 \\ \vdots \\ S_i^1 \\ \vdots \\ S_m^1 \end{array} \right. &
\begin{bmatrix}
(\Pi_{11}^1, \Pi_{11}^2) & (\Pi_{12}^1, \Pi_{12}^2) & \cdots & & (\Pi_{1n}^1, \Pi_{1n}^2) \\
(\Pi_{21}^1, \Pi_{21}^2) & (\Pi_{22}^1, \Pi_{22}^2) & & & (\Pi_{2n}^1, \Pi_{2n}^2) \\
\vdots & & & & \vdots \\
 & & & (\Pi_{ij}^1, \Pi_{ij}^2) & \\
\vdots & & & & \vdots \\
(\Pi_{m1}^1, \Pi_{m1}^2) & (\Pi_{m2}^1, \Pi_{m2}^2) & \cdots & & (\Pi_{mn}^1, \Pi_{mn}^2)
\end{bmatrix}
\end{array}
$$

Fig. 6.2

Payoff Matrices for a Two-Person Game

Player 1 selects a row of the matrix as his strategy, selecting one of the m strategies labelled $S_1^1, S_2^1, \ldots, S_m^1$. Similarly, player 2 selects a column of the matrix as his strategy, selecting one of the n strategies labelled $S_1^2, S_2^2, \ldots, S_n^2$. Once both players have selected their strategies, the payoff to each is indicated by the pair of entries appearing in the corresponding row and column of the matrix. For example, if player 1 chooses strategy S_1^1, and player 2 chooses strategy S_1^2, then the payoff to 1 is Π_{11}^1, and the payoff to 2 is Π_{11}^2. More generally, if player 1 chooses S_i^1, the player 2 chooses S_j^2, then the payoffs are Π_{ij}^1 and Π_{ij}^2 to players 1 and 2, respectively ($i = 1, \ldots, m$; $j = 1, \ldots, n$). The payoff matrices are of size $m \times n$ where m is the (finite) number of strategies available to player 1, and n is the (finite) number of strategies available to player 2. It is assumed that both players know all elements of the payoff matrices.

6.2 Two-person Zero-sum Games

The analysis of two-person zero-sum games is the most highly developed part of game theory. Using the normal form of the game, only the payoff matrix of the first player need be considered, since, by "zero-sum," it is meant that:

$$\Pi_{ij}^1 + \Pi_{ij}^2 = 0, \tag{6.2.1}$$

so the payoff to the second player is simply the negative of the payoff to the first player. The payoff matrix is shown in Fig. 6.3, where:

$$\Pi_{ij} = \Pi^1_{ij} = -\Pi^2_{ij}, \tag{6.2.2}$$

i.e., if player 1 chooses strategy S^1_i (the i^{th} row of the matrix), and player 2 chooses S^2_j (the j^{th} column of the matrix), then the payoff to player 1 is Π_{ij}, as shown, and the payoff to player 2 is understood to be $-\Pi_{ij}$. Such games are called *matrix games*, player 1 seeking to choose a row of the matrix so as to maximize the entry, and player 2 seeking to choose a column of the matrix so as to minimize the entry. Since the results of the analysis are not affected by adding a constant to each entry of the matrix, all *constant sum games*, characterized by:

$$\Pi^1_{ij} + \Pi^2_{ij} = a = \text{constant} \tag{6.2.3}$$

can be reduced to a matrix game as follows: the pairs of entries will be $(\Pi^1_{ij}, a - \Pi^1_{ij})$, and subtracting $a/2$ from each entry gives $(\hat{\Pi}^1_{ij}, \hat{\Pi}^2_{ij}) = (\Pi^1_{ij} - a/2, -\Pi^2_{ij} - a/2)$, in which case $\hat{\Pi}^1_{ij} = -\hat{\Pi}^2_{ij}$.

The basic assumption of two-person zero-sum game theory is that each player seeks to guarantee himself the maximum possible payoff regardless of what the opponent does. The largest possible guaranteed payoff, however, results from choosing a strategy that maximizes the payoff under the assumption that the player reveals his own strategy in advance and then allows the opponent to select his optimal strategy. If player 1 assumes that whatever row he picks, player 2 will choose the column maximizing the return to player 2 and thus minimizing the return to player 1, we can discard all the entries in the payoff matrix except the minimum payoff in each row. His optimal

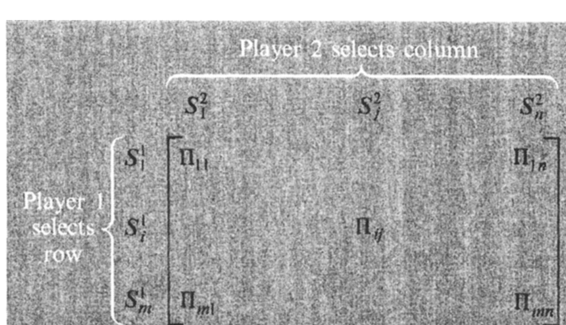

Fig. 6.3

Payoff Matrix for a Two-Person Zero-Sum Game strategy, which will ensure him the largest possible payoff regardless of the strategy chosen by the opponent is thus to select the row with the highest such minimum payoff. Player 1 therefore selects strategy i which solves the problem:

$$\max_i \min_j \Pi_{ij}, \tag{6.2.4}$$

maximizing over the set of row minima, a *maximin strategy*.

Player 2 similarly seeks to ensure the highest payoff to himself (the lowest payoff to his opponent) regardless of the strategy chosen by the opponent. Thus player 2 can discard all entries in the payoff matrix except for the maximum payoff in each column and then select as his optimal strategy the column with the smallest maximum payoff. Player 2 then selects strategy j, which solves the problem:

$$\min_j \max_i \Pi_{ij}, \tag{6.2.5}$$

minimizing over the set of column maxima, a *minimax strategy*.

If the first player selects the maximin strategy, then his payoff will be no less than the maximin value:

$$\Pi_{ij} \geq \max \min \Pi_{ij}, \tag{6.2.6}$$

and if the second player selects the minimax strategy, then his losses will be no greater than the minimax value:

$$\Pi_{ij} \leq \min \max \Pi_{ij}. \tag{6.2.7}$$

These strategies are consistent in that the two players end up with their guaranteed payoffs if:

$$\max \min \Pi_{ij} = \min \max \Pi_{ij} = \Pi_{ij}^*, \tag{6.2.8}$$

in which case the payoff matrix has a *saddle point* at Π_{ij}^*; the i, j element of the matrix being both the minimum in its row and the maximum in its column.

An example of such a game is shown in Fig. 6.4 for a game in which player 1 has a choice of two strategies and player 2 has a choice of three strategies. Player 1 figures that if he chooses row 1, then the opponent might choose column 2, resulting in a payoff of 1. Similarly, if he chooses row 2, then he figures the opponent might choose column 1, resulting in a payoff of -1. These are the row minima, shown in Fig. 6.4. Maximizing over these row minima, player 1 selects his first strategy, guaranteeing a payoff of 1 or more (more if player 2 selects column 1 or 3). Similarly, player 2 assumes the worst,

$$\begin{array}{rccc}
 & & & \text{Row Minima} \\
 & \left(\begin{matrix} 2 & 1 & 4 \\ -1 & 0 & 6 \end{matrix}\right) & & \begin{matrix} \boxed{1} \\ -1 \end{matrix} \\
\text{Column Maxima} & \begin{matrix} 2 & \boxed{1} & 6 \end{matrix} & &
\end{array}$$

Fig. 6.4

A Two-Person Zero-Sum Game with a Saddle Point (a Strictly Determined Game)

figures the opponent might select the first row if he chooses column 1 or 2 and the second row if he chooses column 3, leading to the column maxima 2, 1, and 6 as shown. Minimizing over these column maxima player 2 chooses his second strategy, guaranteeing a loss of not more than 1. Thus, in this game the choices are consistent:

$$\max \quad \min \quad \Pi_{ij} = \quad \min \quad \max \quad \Pi_{ij} = 1, \tag{6.2.9}$$

and the saddle point entry, 1, is the *value of the game* (to player 1). The saddle point represents an equilibrium in that if the opponent uses his saddle point strategy, then the optimal strategy is to play one's own saddle point strategy. Thus, in Fig. 6.4, given that player 1 uses his first strategy, it is optimal for player 2 to use his second strategy; and given that player 2 uses his second strategy, it is optimal for player 1 to use his first strategy. It is reasonable to expect, therefore, that in two-person zero-sum games with a saddle point, called *strictly determined games*, the players would in fact choose their saddle point strategies.

Not all two-person zero-sum games are strictly determined, however. In general:

$$\max \quad \min \quad \Pi_{ij} \leq \quad \min \quad \max \quad \Pi_{ij} \tag{6.2.10}$$

and games in which the strict inequality holds are *nonstrictly determined games* without a saddle point. An example of such a game is given in Fig. 6.5, for which:

$$\max \quad \min \quad \Pi_{ij} = -2 < 4 = \quad \min \quad \max \quad \Pi_{ij}. \tag{6.2.11}$$

If the players follow the rules developed thus far, player 1 selects strategy 1 and expects player 2 to select strategy 2 and a payoff of -2, while player 2 selects strategy 3 and expects player 1 to select strategy 2 and a payoff of 4. The outcome is $\Pi_{13} = 3$, which neither player expected! Furthermore, if

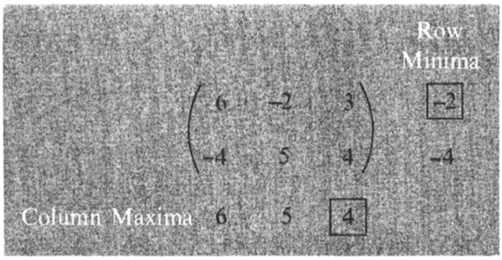

Fig. 6.5

A Two-Person Zero-Sum Game without a Saddle Point (a Nonstrictly Determined Game)

player 2 does select his third strategy, then player 1 would do better selecting his second, not his first, strategy. Similarly, if player 1 does select his first strategy, then player 2 would do better selecting his second, not his third, strategy. The solution concept as outlined so far seems to fail in such games.

The solution concept is still valid, however, if the concept of strategy is broadened to allow for *mixed strategies* (or *random strategies*), which are probability combinations of strategies discussed thus far, the *pure strategies*. For example, the pure strategies available to player 1 are simply the m rows of the payoff matrix. A mixed strategy for player 1 would be summarized by the probability (row) vector:

$$\mathbf{p}^1 = (p_1^1, p_2^1, \ldots, p_m^1) \tag{6.2.12}$$

where p_i^1 is the probability of selecting the i^{th} strategy, $i = 1, 2, \ldots, m$. For example $(1/3, 2/3, 0, \ldots, 0)$ represents the mixed strategy in which player 1 chooses row 1 with probability 1/3 and row 2 with probability 2/3. Of course, since $\mathbf{p}^1$ is a probability vector, it must satisfy the conditions that probabilities sum to unity and are nonnegative:

$$\sum_{i=1}^{m} p_i^1 = 1, \qquad p_i^1 \geq 0, \qquad i = 1, 2, \ldots, m. \tag{6.2.13}$$

Using vector notation, these restrictions are:

$$\mathbf{p}^1\mathbf{1}' = 1, \qquad \mathbf{p}^1 \geq \mathbf{0} \tag{6.2.14}$$

where $\mathbf{1}$ is a row vector all elements of which are unity:

$$\mathbf{1} = (1, 1, \ldots, 1).$$

Similarly, player 2 chooses a probability (column) vector:

$$\mathbf{p}^2 = (p_1^2, p_2^2, \ldots, p_n^2)' \tag{6.2.15}$$

where p_j^2 is the probability of selecting the j^{th} strategy, $j = 1, 2, \ldots, n$, and player 2 can choose any $\mathbf{p}^2$ provided it satisfies the restrictions:

$$\mathbf{1p}^2 = 1, \qquad \mathbf{p}^2 \geq \mathbf{0}. \tag{6.2.16}$$

Note that the pure strategies can be considered as special cases of these mixed strategies for which the probability vector is a unit vector. Thus, the row vector $(0, 1, 0, \ldots, 0)$ represents the choice by player 1 of the second row of the matrix, his second pure strategy, since it is chosen with probability one.

The fundamental theorem of two-person zero-sum games is the *minimax theorem*, which states that all finite games have a solution if mixed strategies are allowed.[5] Strictly-determined games have a solution, perhaps nonunique, in pure strategies, while nonstrictly determined games have a solution, perhaps nonunique, in which one or both of the players choose probability mixtures of their strategies.

Since probabilities of choosing strategies are employed, the payoff to player 1 (and loss to player 2) is no longer a single element of the payoff matrix, but is rather a weighted average of elements of the matrix, the weights being the probabilities. Specifically, the expected payoff to player 1, assuming he uses the probability vector $\mathbf{p}^1 = (p_1^1, p_2^1, \ldots, p_m^1)$ and player 2 chooses his j^{th} strategy is:

$$p_1^1\Pi_{1j} + p_2^1\Pi_{2j} + \cdots + p_m^1\Pi_{mj} = \mathbf{p}^1\mathbf{\Pi e}_j', \qquad j = 1, 2, \ldots, n \tag{6.2.17}$$

where $\mathbf{\Pi} = (\Pi_{ij})$ is the payoff matrix and $\mathbf{e}_j$ is the j^{th} unit vector (the j^{th} row of the identity matrix). Player 1, seeking the highest guaranteed expected payoff, chooses his probability vector so as to maximize the minimum expected payoff. Letting:

$$\Pi^1(\mathbf{p}^1) = \min_j \ \mathbf{p}^1\mathbf{\Pi e}_j', \tag{6.2.18}$$

player 1 acts so as to:

$$\max_{\mathbf{p}^1} \ \Pi^1(\mathbf{p}^1) = \max_{\mathbf{p}^1} \ \min_j \ \mathbf{p}^1\mathbf{\Pi e}_j'. \tag{6.2.19}$$

Similarly, the expected payoff to player 2, assuming he uses the probability vector $\mathbf{p}^2 = (p_1^2, p_2^2, \ldots, p_n^2)'$ and player 1 chooses his i^{th} strategy is:

$$\Pi_{i1}p_1^2 + \Pi_{i2}p_2^2 + \cdots + \Pi_{in}p_n^2 = \mathbf{e}_i\mathbf{\Pi p}^2, \qquad i = 1, 2, \ldots, m. \tag{6.2.20}$$

Player 2 seeks to minimize the maximum expected payoff:

$$\max_i \ \mathbf{e}_i\mathbf{\Pi p}^2. \tag{6.2.21}$$

Thus, player 2 chooses $\mathbf{p}^2$ so as to:

$$\min_{\mathbf{p}^2} \ \Pi^2(\mathbf{p}^2) = \min_{\mathbf{p}^2} \ \max_i \ \mathbf{e}_i \mathbf{\Pi} \mathbf{p}^2. \tag{6.2.22}$$

According to the minimax theorem there exist solutions to (6.2.19) and (6.2.22), $\mathbf{p}^{1*}$ and $\mathbf{p}^{2*}$ respectively, for which, letting:

$$V = \mathbf{p}^{1*} \mathbf{\Pi} \mathbf{p}^{2*} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i^{1*} \Pi_{ij} p_j^{2*}, \tag{6.2.23}$$

it follows that:

$$\mathbf{p}^1 \mathbf{\Pi} \mathbf{p}^{2*} \leq V \leq \mathbf{p}^{1*} \mathbf{\Pi} \mathbf{p}^2 \tag{6.2.24}$$

for all probability vectors $\mathbf{p}^1$ and $\mathbf{p}^2$, where:

$$\max_{\mathbf{p}^1} \ \mathbf{p}^1 \mathbf{\Pi} \mathbf{p}^{2*} = V = \min_{\mathbf{p}^2} \ \mathbf{p}^{1*} \mathbf{\Pi} \mathbf{p}^2. \tag{6.2.25}$$

Thus V, the *value of the game*, is simultaneously the maximized expected payoff to player 1 and the minimized expected loss to player 2. The minimax theorem therefore asserts the existence of at least one pair of mixed strategies $\mathbf{p}^1$, $\mathbf{p}^2$ such that max-min equals min-max for the expected payoff:

$$V = \max_{\mathbf{p}^1} \ \min_{\mathbf{p}^2} \ \mathbf{p}^1 \mathbf{\Pi} \mathbf{p}^2 = \min_{\mathbf{p}^2} \ \max_{\mathbf{p}^1} \ \mathbf{p}^1 \mathbf{\Pi} \mathbf{p}^2, \tag{6.2.26}$$

so that every finite game has a saddle point in probability space. The value of the game, V, is unique; however, the optimal mixed strategy probability vectors $\mathbf{p}^1$, $\mathbf{p}^2$, yielding V according to (6.2.23), need not be unique. If more than one pair of optimal mixed strategies exist, however, then these pairs form a closed convex polyhedral set and all of the pairs in this set yield the same value for the game.

One proof of the minimax theorem uses the duality theorem of linear programming. The fact that player 1 considers the minimum expected payoff, expressed in (6.2.18), can be stated as the linear inequalities:

$$\mathbf{p}^1 \mathbf{\Pi} \mathbf{e}_j' = \sum_{i=1}^{m} p_i^1 \Pi_{ij} \geq \Pi^1(\mathbf{p}^1), \qquad j = 1, 2, \ldots, n \tag{6.2.27}$$

or, equivalently, as:

$$\mathbf{p}^1 \mathbf{\Pi} - \Pi^1(\mathbf{p}^1) \mathbf{1} \geq \mathbf{0}, \tag{6.2.28}$$

where $\mathbf{1}$ is, as before, a row vector of ones. The problem for player 1 (6.2.19), can then be expressed as the linear programming problem:

$$\max_{\mathbf{p}^1} \ \Pi^1(\mathbf{p}^1)$$

subject to:

$$\mathbf{p}^1\mathbf{\Pi} - \Pi^1(\mathbf{p}^1)\mathbf{1} \geq \mathbf{0}$$
$$\mathbf{p}^1\mathbf{1}' = 1 \tag{6.2.29}$$
$$\mathbf{p}^1 \geq \mathbf{0}.$$

Similarly, the problem for player 2, who minimizes the maximum payoff is:

$$\min_{\mathbf{p}^2} \quad \Pi^2(\mathbf{p}^2)$$

subject to

$$\mathbf{\Pi}\mathbf{p}^2 - \mathbf{1}'\Pi^2(\mathbf{p}^2) \leq \mathbf{0}$$
$$\mathbf{1}\mathbf{p}^2 = 1 \tag{6.2.30}$$
$$\mathbf{p}^2 \geq \mathbf{0}.$$

The fact that these two problems are dual to one another is shown in the tableau of Fig. 6.6, which is similar to that of Fig. 4.3. This tableau summarizes the two problems, provided p_m^1 and p_n^2 are defined as:

$$p_m^1 = 1 - \sum_{i=1}^{m-1} p_i^1$$
$$p_n^2 = 1 - \sum_{j=1}^{n-1} p_j^2, \tag{6.2.31}$$

so that the probabilities sum to unity. Since feasible vectors exist for both opportunity sets (e.g., the unit vectors), by the existence theorem of linear

	p_1^2	p_2^2	$\cdots$	p_n^2	$-\Pi^2(\mathbf{p}^2)$	
p_1^1	Π_{11}	Π_{12}	$\cdots$	Π_{1n}	1	≤ 0
p_2^1	Π_{21}	Π_{22}	$\cdots$	Π_{2n}	1	≤ 0
$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$
p_m^1	Π_{m1}	Π_{m2}	$\cdots$	Π_{mn}	1	≤ 0
$-\Pi^1(\mathbf{p}^1)$	1	1	$\cdots$	1	1	$= 1-\Pi^2(\mathbf{p}^2)$, to max; i.e., min $\Pi^2(\mathbf{p}^2)$
	≥ 0	≥ 0	$\cdots$	≥ 0	$= 1-\Pi^1(\mathbf{p}^1)$, to min; i.e., max $\Pi^1(\mathbf{p}^1)$	

Fig. 6.6
Linear Programming Tableau for
the Dual Problems of Finding Optimal Mixed Strategies

programming, solutions $\mathbf{p}^{1*}$, $\mathbf{p}^{2*}$ exist for both problems. By the duality theorem, however:

$$\Pi^1(\mathbf{p}^{1*}) = \max_{\mathbf{p}^1} \Pi^1(\mathbf{p}^1) = V = \min_{\mathbf{p}^2} \Pi^2(\mathbf{p}^2) = \Pi^2(\mathbf{p}^{2*}) \quad (6.2.32)$$

where V is the value of the game. Thus the duality theorem of linear programming implies the minimax theorem of game theory. The complementary slackness theorem, furthermore, implies that:

$$\begin{aligned} &\text{either} \sum_{i=1}^{m} p_i^{1*}\Pi_{ij} = V \quad \text{or} \quad p_j^{2*} = 0, \qquad j = 1, 2, \ldots, n \\ &\text{either} \sum_{j=1}^{n} \Pi_{ij}p_j^{2*} = V \quad \text{or} \quad p_i^{1*} = 0, \qquad i = 1, 2, \ldots, m, \end{aligned} \quad (6.2.33)$$

results often referred to as the *strong minimax theorem*. They state, for example, that if the expected payoff to player 1 is larger than the value of the game for a particular pure strategy of player 2, then player 2 plays this strategy with probability zero.

In general it would be expected that the players of a two-person zero-sum game would use their optimal mixed strategies. In the special case of a strictly determined game, the optimal mixed strategies assign probability one to the pure strategies at the saddle point, i.e., the optimal mixed strategy vectors are unit vectors. In fact the number of nonzero elements in the optimal mixed strategy vectors need not exceed the minimum of the numbers of pure strategies available to the two players.

When the players use their mixed strategies, they do not reveal to their opponent the actual strategy to be employed in any one play of the game. The actual strategy is selected by a probability mechanism (e.g., toss of a coin. roll of dice, "wheel of fortune," table of random numbers), using the optimal probabilities. If the opponent knew the actual strategy to be used on a play of the game, he could then exploit this knowledge to his advantage. He cannot, however, gain any information from knowledge of the optimal probabilities employed.

The optimal mixed strategies can be obtained in the general case by solving the dual linear programming problems of Fig. 6.6. If, however, one player has only two (pure) strategies, then the solution for his optimal probabilities can be obtained graphically. An example is the nonstrictly determined game of Fig. 6.5, which is solved in Fig. 6.7. The horizontal axis measures p_2^1, the probability that player 1 chooses the second strategy (second row of the matrix). Since $p_2^1 = 1 - p_1^1$, the points 0 and 1 represent the two pure strategies of choosing row 1 and row 2, respectively. The vertical axis measures the payoff to player 1, and each of the lines is obtained by assuming the opponent,

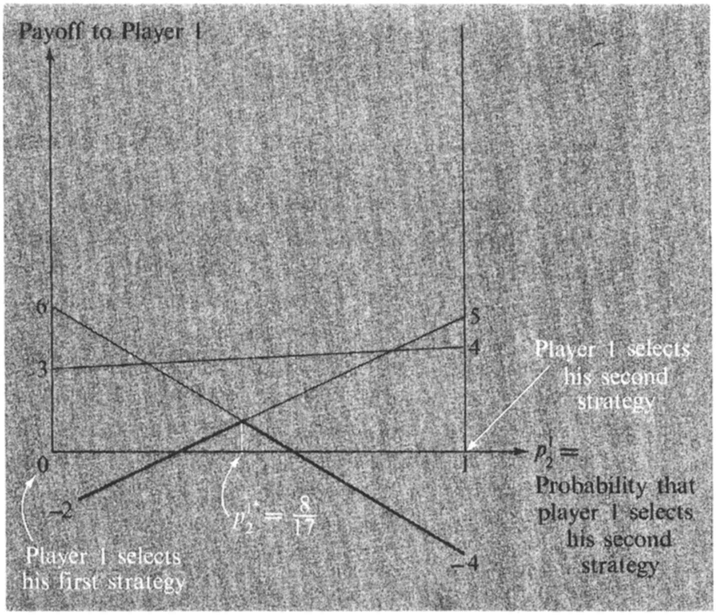

Fig. 6.7

Graphical Solution to the Game of Fig. 6.5 for Player 1

player 2, will select one of his pure strategies. For example, if player 2 chooses the first column, the payoff to player 1 is 6 if he chooses the first row ($p_2^1 = 0$) and -4 if he chooses the second row ($p_2^1 = 1$), shown as the 6 intercept on the left side of the diagram and the -4 intercept on the right side of the diagram. The line connecting the two intercepts summarizes the payoff implication of all mixed strategies. Since player 1 assumes the worst, the only relevant locus to player 1 is the heavy locus in the shape of an upside-down "V." The points on this locus show the smallest expected payoff to player 1 as his probability of choosing row 2 varies. Maximizing the expected payoff calls for $p_2^{1*} = 8/17$, which could be obtained either geometrically or algebraically from:

$$-2(1 - p_2^1) + 5p_2^1 = 6(1 - p_2^1) - 4p_2^1. \tag{6.2.34}$$

Thus, player 1 should choose his first strategy with probability 9/17 and his second strategy with probability 8/17. The value of the game is then:

$$V = -2\left(\frac{9}{17}\right) + 5\left(\frac{8}{17}\right) = 6\left(\frac{9}{17}\right) - 4\left(\frac{8}{17}\right) = \frac{22}{17}. \tag{6.2.35}$$

6.3 Two-person Nonzero-sum Games

In nonzero-sum games it is generally not true that what one player wins, the other loses—there is the possibility of mutual gain or loss. Because the players are not in complete conflict, there is scope for threats, bluffs, communication of intent, learning, and teaching phenomena. For example, while it was obviously undesirable to reveal one's strategy in advance in zero-sum games, in nonzero-sum games it is sometimes desirable to reveal a strategy to be able to coordinate with the other player or influence the other player in reaching a desirable outcome.

The desirability of communication and strategy coordination is obvious in *coordination games*, in which the payoffs are the same for both players, or, more generally, the payoffs differ by a constant amount so the players gain or lose utility together. As an example, suppose two men are caught in a house on fire. The door is jammed but could be opened if both push against it. The payoffs are shown in Fig. 6.8, which employs the format of Fig. 6.2. If they both push against the door, they will escape injury and receive a payoff of 100. Otherwise, they will both be injured, receiving a payoff of zero. It is obvious that their best strategies are to cooperate.

Communication is sometimes desirable even when the payoffs are not constant difference; i.e., even when there are elements of conflict. This point is illustrated by the "Prisoners' Dilemma" game. In this game there are two prisoners, each of whom can either confess or not confess to a particular crime. If neither confesses, they both will be set free; and if both confess, they

		Man 2: Push against door	Man 2: Don't push against door
Man 1	Push against door	(100, 100)	(0, 0)
	Don't push against door	(0, 0)	(0, 0)

Fig. 6.8

A Coordination Game:
Two Men Caught in a House on Fire

$$
\begin{array}{ll|cc}
 & & \multicolumn{2}{c}{\text{Prisoner 2}} \\
 & & \text{Don't confess} & \text{Confess} \\
\hline
\text{Prisoner 1} & \text{Don't confess} & (4, 4) & (0, 5) \\
 & \text{Confess} & (5, 0) & (2, 2)
\end{array}
$$

Fig. 6.9

The Prisoners' Dilemma

face moderate jail sentences. If, however, one confesses, and the other doesn't, the one who confesses will be rewarded and set free, while the one who doesn't confess will face the maximum jail sentence. The payoffs are shown in Fig. 6.9, where the utility of being set free is 4, the utility of the moderate jail sentence is 2, the utility of being rewarded and set free is 5, and the utility of the maximum jail sentence is 0.

Taking the viewpoint of player 1, it is apparent that the strategy of confessing dominates that of not confessing in that, whatever player 2 does, player 1 is better off confessing. If player 2 does not confess, player 1 can increase his payoff from 4 to 5 by confessing, while if player 2 confesses, player 1 can increase his payoff from 0 to 2 by confessing. By this reasoning, player 1 confesses. The same considerations apply to player 2, who also confesses, so they both confess and receive a payoff of 2 each. But from the payoff matrix it is clear that they would *both* be better off if they don't confess, since they would then obtain a payoff of 4 each. The prisoners, therefore, face a dilemma. If they could rely on each other or somehow convince each other that they would not confess, they would both be better off. But each realizes that the other would then be tempted to renege on the agreement and confess. The need for coordination and communication is apparent since in this example individually rational behavior can lead to inferior outcomes for all individuals.

The example is not an isolated one; there are many important social, economic, and political situations in which such a paradox appears. An economic example is the choice between free trade and protectionism: All countries are better off with free trade; however, a single country can, in the free-trade situation, improve its own position by a tariff.

Another example of a nonzero-sum game is the *game of chicken*. In this game two teenagers in automobiles drive toward each other at high speed, and the first one to swerve "loses." The payoffs are shown in Fig. 6.10: if one swerves and the other doesn't, the "winning" player receives 5, and the "losing" (swerving) player receivers −5. If both swerve, the contest is a

		Driver 2	
		Swerve	Don't swerve
Driver 1	Swerve	(0, 0)	(−5, 5)
	Don't swerve	(5, −5)	(−100, −100)

Fig. 6.10
The Game of Chicken

draw; both receive zero. If neither swerves, they crash, each receiving -100. The game is like the Prisoners' Dilemma except here neither player has a dominating strategy that is best under all assumptions concerning the other player. The dilemma remains, however, since if each convinced the other he were going to swerve, they could both attain the draw; but there are strong temptations to renege on any such agreement and thereby win. If both renege, the outcome is disaster.

Nonzero-sum games can be either cooperative or noncooperative games. In *noncooperative games* the players make their decisions independently either because coordination is forbidden, or enforceable agreements are not possible. An example of the former is the antitrust laws which deem certain types of collusion illegal; an example of the latter is international trade agreements which are difficult or impossible to enforce.

One approach to noncooperative games is that of identifying the *equilibrium point*(*s*) of the game, that is, the point(s) at which neither player has an incentive to change his strategy if acting unilaterally.[6] For example, in the Prisoners' Dilemma game of Fig. 6.9 the (2, 2) point at which both confess is an equilibrium point since each would be worse off if he changed his strategy while the other player held his strategy fixed. None of the other points are equilibrium points. For example, at (0, 5) player 1 could unilaterally increase his payoff by switching to confess. In this game there is only one equilibrium point. In the game of chicken, however, there are two equilibrium points: at $(5, -5)$ and $(-5, 5)$, where one swerves, and the other doesn't.

To specify the notion of an equilibrium point in terms of mixed strategies, assume as in Fig. 6.2 that if player 1 selects strategy S_i^1, and player 2 selects strategy S_j^2, then the payoff to player 1 is Π_{ij}^1, and the payoff to player 2 is Π_{ij}^2. Assuming p_i^1 is the probability of player 1 selecting the i^{th} pure strategy, S_i^1, $i = 1, 2, \ldots, m$, the mixed strategy for player 1 is summarized by the vector:

$$\mathbf{p}^1 = (p_1^1, p_2^1, \ldots, p_m^1), \qquad \text{where} \quad \mathbf{p}^1\mathbf{1}' = 1, \qquad \mathbf{p}^1 \geq \mathbf{0}. \qquad (6.3.1)$$

Similarly, if p_j^2 is the probability of player 2 selecting the j^{th} pure strategy, $S_j^2, j = 1, 2, \ldots, n$, the mixed strategy for player 2 is summarized by the vector:

$$\mathbf{p}^2 = (p_1^2, p_2^2, \ldots, p_n^2)', \qquad \text{where} \quad \mathbf{1p}^2 = 1, \qquad \mathbf{p}^2 \geq 0. \tag{6.3.2}$$

An equilibrium point in mixed strategies is then the pair of vectors $\mathbf{p}^{1*}$ and $\mathbf{p}^{2*}$, each of which is an optimal mixed strategy, in the sense of maximizing expected payoff, assuming the other player uses his (optimal) mixed strategy. Thus:

$$\begin{aligned} \mathbf{p}^1\mathbf{\Pi}^1\mathbf{p}^{2*} &\leq \mathbf{p}^{1*}\mathbf{\Pi}^1\mathbf{p}^{2*} \quad \text{for all} \quad \mathbf{p}^1 \\ \mathbf{p}^{1*}\mathbf{\Pi}^2\mathbf{p}^2 &\leq \mathbf{p}^{1*}\mathbf{\Pi}^2\mathbf{p}^{2*} \quad \text{for all} \quad \mathbf{p}^2. \end{aligned} \tag{6.3.3}$$

Such an equilibrium pair of mixed strategy vectors exists for all two-person finite games, but need not be unique, nor even yield unique (expected) payoffs. More generally, a mixed strategy equilibrium exists for every n-person game with a finite number of strategies. The equilibrium is a set of mixed strategies for the players such that none of the players could improve his position by a unilateral change in his mixed strategies.

6.4 Cooperative Games

A *cooperative game* is a nonconstant sum game in which the players can discuss their strategies before play and make binding agreements on strategies they will employ; i.e., the players can form coalitions. The basic problem of a cooperative game is then that of dividing the coalition payoff among the members of the coalition. An important distinction to be drawn in cooperative games is that between those with side payments, in which payoffs are transferable, and those without side payments, in which payoffs are not transferable.

The *Nash cooperative solution* is an approach to the cooperative game without side payments in the case of two players.[7] The players reach an agreement on coordinating their strategies where failure to reach such an agreement would give each player a certain fixed payoff known as the *threat payoff*. For example, the threat point might be the max-min payoffs in the corresponding noncooperative game.

Nash specified some reasonable assumptions under which the solution to this bargaining game is unique. His first assumption is that of *symmetry*, that the solution does not depend on the numbering of the players. His second assumption is that of *independence of linear transformations*, that the solution is invariant under monotonic linear transformations of the payoffs. His third assumption is that of *independence of irrelevant alternatives*, that the solution is invariant if any of the potential choices not employed in the solution are deleted. His fourth assumption is that of *Pareto optimality*, that the solution cannot occur at a set of payoffs for which there exists an alternative feasible set of payoffs for which one or both players are better off. Under these assumptions the unique solution is obtained at payoffs (Π^{1*}, Π^{2*}), which maximize the product of the excess of the payoffs over the threat payoffs:

$$\max_{\Pi^1,\Pi^2} (\Pi^1 - T^1)(\Pi^2 - T^2) \tag{6.4.1}$$

where Π^1, Π^2 are the payoffs to the two players, and T^1, T^2 are the payoffs to the two players at the threat point. Geometrically, the solution is shown in Fig. 6.11. The set of all possible payoffs is the shaded set, which is convex since the players can use mixed strategies. The heavy curve on the boundary shows the *payoff frontier*, the set of all payoff pairs which satisfies the Pareto optimality assumption. The threat point is at T, and the Nash solution is at S, where the payoff frontier reaches the highest contour, the contours being rectangular hyperbolas with origin at T. The solution is unique and is in the *negotiation set*, the set of all points on the payoff frontier which gives both players a higher payoff than their threat payoffs.

Cooperative games with side payments are games in which binding agreements can be made about strategy, and payoffs can be transferred between players. Because there are side payments, only the total payoff to each possible coalition need be considered. Such games can be analyzed using the *characteristic function description* of a game, which characterizes all possible coalitions by indicating the maximum total payoff which each coalition can guarantee itself. Given the set of players in an n-person game:

$$N = \{1, 2, \dots, n\} \tag{6.4.2}$$

a coalition is any subset S of N, and the characteristic function indicates the payoff S can guarantee itself. The characteristic function is thus a real-valued function, the domain of which is the 2^n possible subsets of N.[8] It can be written:

$$v(S), \qquad \text{where} \quad S \subset N. \tag{6.4.3}$$

An example of a characteristic function for a three-person game is given in Fig. 6.12, where the four lines give values of the characteristic function for coalitions of 0, 1, 2, and 3 players, respectively. The first line states the convention that the maximum payoff to the empty set is zero. The second line states that the payoff to any player acting alone is zero. The third line gives the payoffs to the three possible coalitions each of which consists of two players. This line indicates that, while 1 and 2 acting together can guarantee

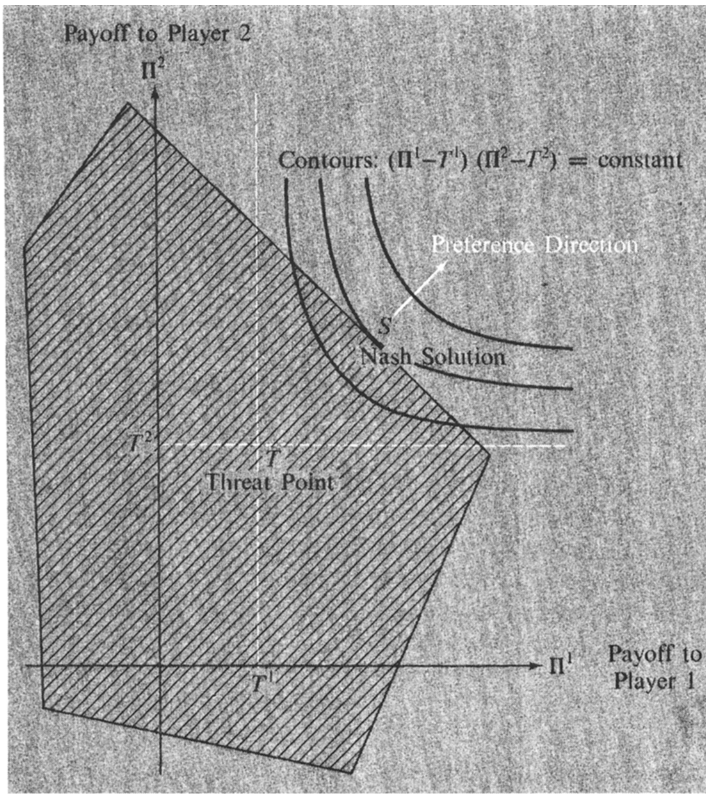

Fig. 6.11

The Nash Solution to the Bargaining Problem

$$v(\phi) = 0$$
$$v(1) = 0, \quad v(2) = 0, \quad v(3) = 0$$
$$v(1, 2) = 0.1, \quad v(1, 3) = 0.2, \quad v(2, 3) = 0.2$$
$$v(1, 2, 3) = v(N) = 1$$

Fig. 6.12

A Three-Person Game in Characteristic Function Form themselves .1, either 1 and 3 or 2 and 3 acting as a coalition can guarantee themselves .2. Finally, the last line indicates that if all players join in a "grand coalition," their payoff would be unity. This game is in $0 - 1$ *normalized form* in that the payoff to individual players is zero, while the payoff to the grand coalition of all players is unity:

$$\begin{aligned} v(i) &= 0 \quad \text{all} \quad i \in N \\ v(N) &= 1 \quad \text{where} \quad N = \{1, \ldots, n.\} \end{aligned} \tag{6.4.4}$$

The characteristic function exhibits superadditivity:

$$v(A \cup B) \geq v(A) + v(B) \quad \text{for all disjoint subsets} \quad A, B, \tag{6.4.5}$$

that is, if coalitions A and B have no player in common and are merged into a single coalition, then the payoff to the merged coalition is larger than or equal to the sum of the payoffs to the separate coalitions. Superadditivity is reasonable because it would be irrational for coalitions to form if they reduce the payoff as compared to smaller coalitions acting alone.

An *imputation* is a vector in Euclidean n-space summarizing the payoffs to each of the players in the game:

$$\mathbf{\Pi} = (\Pi^1, \Pi^2, \ldots, \Pi^n), \tag{6.4.6}$$

where Π^i is the payoff to the i^{th} player, $i = 1, 2, \ldots, n$. An example of an imputation for the game of Fig. 6.12 is (.3, .2, .5), where player 1 receives .3, player 2 receives .2 and player 3 receives .5 . Assuming all players and payoffs are accounted for, the total of the payoffs to all players equals the payoff to the grand coalition of all players:

$$v(N) = \sum_{i \in N} \Pi^i = \sum_{i=1}^{n} \Pi^i \tag{6.4.7}$$

which is the assumption of *group rationality*. It is also reasonable to assume that each player receives at least as much as he would obtain by independent action:

$$\Pi^i \geq v(\{i\}), \qquad \text{all} \quad i \in N \tag{6.4.8}$$

which is the assumption of *individual rationality*. These assumptions limit the number of possible imputations. For example, in normalized games the only acceptable imputations are vectors with nonnegative components which sum to unity. The remaining imputations still form an extremely large set, so the next step is to suggest criteria of admissibility or dominance among imputations to limit the number of imputations under consideration.

A weak criterion of dominance among imputations is the "von Neumann-Morgenstern solution." A set of players is *effective* for an imputation if they can, by forming a coalition, obtain at least as much for themselves as they jointly receive in the imputation. Thus, coalition S is effective for imputation $\mathbf{\Pi} = (\Pi^1, \ldots, \Pi^n)$ if:

$$v(S) \geq \sum_{i \in S} \Pi^i. \tag{6.4.9}$$

For example, for the game described in Fig. 6.12 the set of players 2, 3 is effective for the imputation (.95, 0, .05) since if they formed their own coalition, they would jointly receive .2, which is more than they receive in the imputation. Imputation $\mathbf{\Pi}_1 = (\Pi_1^1, \Pi_1^2, \ldots, \Pi_1^n)$ *dominates* imputation $\mathbf{\Pi}_2 = (\Pi_2^1, \Pi_2^2, \ldots, \Pi_2^n)$ if there is a coalition of players effective for $\mathbf{\Pi}_1$ such that every player in the coalition receives more in $\mathbf{\Pi}_1$ than in $\mathbf{\Pi}_2$; that is, if there is a coalition of players S which is effective for $\mathbf{\Pi}_1$:

$$v(S) \geq \sum_{\text{all } i \in S} \Pi_1^i \tag{6.4.10}$$

every member of which receives more in $\mathbf{\Pi}_1$ than in $\mathbf{\Pi}_2$:

$$\Pi_1^i > \Pi_2^i \quad \text{for all} \quad i \in S. \tag{6.4.11}$$

For example, for the game described in Fig. 6.12 the imputation $\mathbf{\Pi}_1 = (.1, .8, .1)$ dominates $\mathbf{\Pi}_2 = (.05, .9, .05)$ since the coalition $\{1, 3\}$ is effective for $\mathbf{\Pi}_1$, where both players 1 and 3 receive more in $\mathbf{\Pi}_1$ than in $\mathbf{\Pi}_2$. By threatening independent action, the coalition $\{1, 3\}$ can ensure that the imputation (.05, .9, .05) will never be used. A set of imputations is a *von Neumann-Morgenstern solution* if no imputation in the set dominates any other imputation in the set, and any imputation not in the set is dominated by some imputation in the set. This weak notion of dominance generally narrows down the choice of imputations but typically does not yield a unique imputation. Indeed, a von Neumann-Morgenstern solution often contains an infinite number of imputations, and, in the case of more than two players, the number of von Neumann-Morgenstern solutions (i.e., the number of sets of imputations, each of which is a von Neumann-Morgenstern solution) may itself be large or even infinite. Furthermore, despite the large number of remaining imputations in von Neumann-Morgenstern solutions in most games, there are several examples of games with no von Neumann-Morgenstern solution.[9]

A stronger criterion of dominance among imputations is the "core," which is a subset of every von Neumann-Morgenstern solution, if such a solution exists. The number of imputations to be considered is narrowed down in the core by requiring that every coalition exercise the same degree

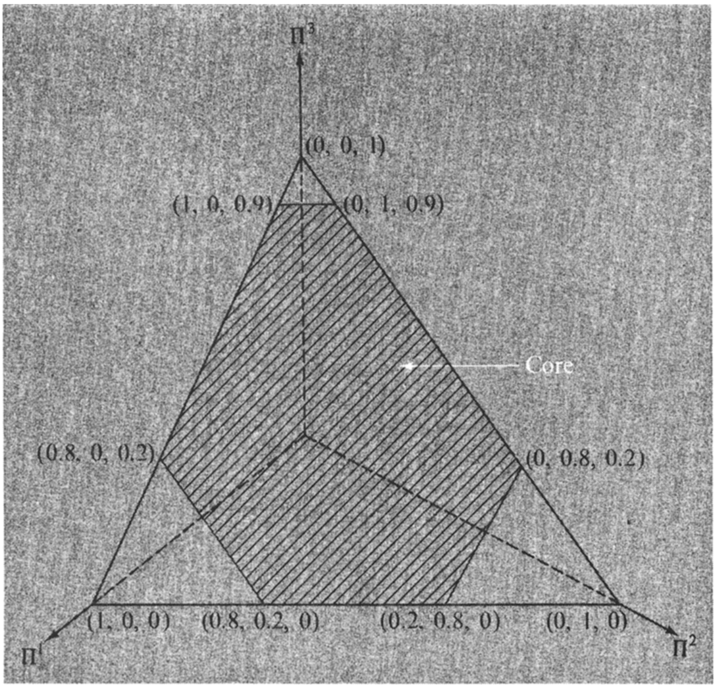

Fig. 6.13
The Core for the Game of Fig. 6.12

of rationality as an individual player, so that the imputation allocates to each coalition at least as much as it could obtain by independent action. The *core* is the set of all such undominated imputations, namely those imputations $\mathbf{\Pi} = (\Pi^1, \ldots, \Pi^n)$ satisfying:

$$\sum_{i \in S} \Pi^i \geq v(S) \quad \text{for every subset } S \text{ of } N. \tag{6.4.12}$$

The core is then the set of imputations satisfying "coalition rationality," including "individual rationality," where the subsets consist of individual players; "group rationality," where the subset is the grand coalition of all players; and the rationality of all intermediate size coalitions. The core of the three-person game described in Fig. 6.12 is shown geometrically in Fig. 6.13. The equilateral triangle represents the boundary of the simplex in E^3, the set of imputations (Π^1, Π^2, Π^3) such that:

$$\begin{aligned} \Pi^i &\geq 0, \qquad i = 1, 2, 3, \\ \Pi^1 + \Pi^2 + \Pi^3 &= 1, \end{aligned} \tag{6.4.13}$$

where the vertices represent imputations for which one player takes all. The shaded area is the core. It would seem reasonable to assume that if the core exists, then the imputation chosen should be in the core, since then all coalitions are accounted for. Unfortunately, however, in many games the core is empty; i.e., no imputation satisfies the conditions of coalition rationality for all coalitions. For example, if, in the three-person game of Fig. 6.12 all coalitions of two players receive .8 then the core would be empty.

The number of imputations in the core is generally either zero (i.e., the core is empty) or many (e.g., Fig. 6.13). Only infrequently does the core consist of a unique imputation. A unique imputation is, however, always obtained via the *Shapley value*, an imputation based on the "power" of each of the players as reflected in the additional payoff resulting from the addition of this player to the coalitions not including him.[10] Thus, for the game described in Fig. 6.12 the third player has more power than the other players and should obtain more than they, since the two two-player coalitions with player 3 obtain .2, while the one without him obtains .1 . Assuming each player receives the average of his contribution to all coalitions of which he is a potential member, the payoff to the i^{th} player is the expected value of $v(S \cup \{i\}) - v(S)$, where S is any subset of players excluding player i, and $S \cup \{i\}$ is the same subset including player i. The expected value is the payoff:

$$\Pi^i = \sum_{\text{all } S \subset N} \gamma_n(S)[v(S \cup \{i\}) - v(S)] \tag{6.4.14}$$

where $\gamma_n(S)$ is the weighting factor:

$$\gamma_n(S) = \frac{s!\,(n-s-1)!}{n!}, \tag{6.4.15}$$

s being the number of players in S. This weighting factor is based on the facts that the n-person coalition can be formed in $n!$ different ways; the s players in coalition S before player i joins it can be arranged in $s!$ different ways; and the $n - s - 1$ players not in the enlarged coalition can be arranged in $(n - s - 1)!$ different ways. Thus $\gamma_n(S)$ is simply the probability that a player joins coalition S, assuming the n ways of forming an n-player coalition are all equally probable. In the game described in Fig. 6.12 there are four cases to consider for each player. For player 1 the cases are:

$$\begin{aligned} v(\{1\}) - v(\phi) &= 0 \\ v(\{1, 2\}) - v(\{2\}) &= .1 \\ v(\{1, 3\}) - v(\{3\}) &= .2 \\ v(\{1, 2, 3\}) - v(\{2, 3\}) &= .8, \end{aligned} \tag{6.4.16}$$

and the weights applied to these four cases are $\frac{2}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, and $\frac{2}{6}$, respectively. The payoff to player 3 should, therefore, be:

$$\Pi^3 = (\tfrac{2}{6})0 + \tfrac{1}{6}(.1) + \tfrac{1}{6}(.2) + \tfrac{2}{6}(.8) = \tfrac{19}{60}. \qquad (6.4.17)$$

Similarly, the payoff to player 2 is 19/60, and the payoff to player 3 is 22/60. Thus, the Shapley value imputation for this game is $(\frac{19}{60}, \frac{19}{60}, \frac{22}{60})$.

6.5 Games With Infinitely Many Players

An interesting and important problem in n-person games is that of determining what happens when the number of players increases without limit.[11] The remarkable outcome is that under certain assumptions regarding the game and the manner in which the number of players increases many of the different solution concepts developed in the last sections all converge to the same solution. With infinitely many players there always exists an equilibrium point; and the equilibrium points, the core, and the Shapley value all converge as n increases without limit to this equilibrium point. This result is truly remarkable since these solution concepts are all based on different approaches. For example, equilibrium point(s) are generally not Pareto optimal (e.g., (2, 2) in Fig. 6.9), but, as the number of players increases without limit, such point(s) move onto the surface of Pareto optimal points. The core, on the other hand, can be considered an area on the Pareto optimal surface which, as the number of players increases without limit, shrinks to a single point or set of points. Finally, the Shapley value is not necessarily in the core, but it converges to the same limit as the core. Thus, while there are many approaches and correspondingly many solution concepts for games with a finite number of players (excluding the simplest case of two-person zero-sum games where the minimax solution is compelling), there is a single solution, not necessarily unique, to games with an infinity of players. Game theory, therefore, provides a satisfactory analysis of games with one or two players and of games with an infinity of players, but not a unique satisfactory analysis for games with a finite number of three or more players. In this respect game theory resembles mechanics, which provides solutions to one-body or two-body problems and, via statistical mechanics, provides solutions to problems when the number of bodies is of the order of 10^{23} or more, but provides no satisfactory analysis to date when the number of bodies falls in the intermediate range (e.g., the famous three body problem).

PROBLEMS

6-A. Solve the following two-person zero-sum games:

1. $\begin{pmatrix} 4 & 0 \\ 6 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 15 & 0 & -2 \\ 0 & -15 & -1 \\ 1 & 2 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 4 & -3 \\ 0 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 5 & 3 & 2 \\ 3 & 4 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \end{pmatrix}$

6. $\begin{pmatrix} -3 & 6 \\ 8 & -2 \\ 6 & 3 \end{pmatrix}$

7. $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad a > b > c$

6-B. Solve the following zero-sum game: players 1 and 2 independently choose a number 1, 2, or 3. If the numbers are equal, player 1 pays player 2 that amount. If they are not equal, then player 2 pays player 1 an amount equal to the number that player 1 has chosen.

6-C. A simplified two-person poker game is presented in extensive form in Fig. 6.1. Obtain the normal form of the game and solve the game. Is the solution dependent on the amount of the ante? The amount of the raise?

6-D. A two-person zero-sum game is *fair* if the value of the game is zero.

1. Show that symmetric games, for which the payoff matrix is skew-symmetric ($\mathbf{\Pi} = -\mathbf{\Pi}'$) are fair and that in this case the optimal probability vectors are simply transposes of one another.

2. Construct an example of a nonsymmetric strictly determined game which is fair.
3. Construct an example of a nonsymmetric nonstrictly determined game which is fair.

6-E. Prove that for two-person zero-sum games:

1. The saddle point need not be unique, but the value of the game is unique.
2. The value of the game is a nondecreasing continuous function of the components of the payoff matrix.

6-F. Show that for nonstrictly determined two-person zero-sum games:

1. If the opponent uses his optimal mixed strategy then any pure strategy cannot yield a higher expected payoff than the optimal mixed strategy.
2. If the opponent uses an optimal mixed strategy then playing any pure strategy used with nonzero probability in some optimal mixed strategy yields the value of the game, while playing any pure strategy with zero probability in every optimal mixed strategy yields less than the value of the game.
3. Any dominated pure strategy is used with zero probability in an optimal mixed strategy.
4. If there are two optimal mixed strategies, then any convex linear combination of these strategies is also an optimal mixed strategy.

6-G. In certain two-person zero-sum games the payoffs can be transformed by any monotonic transformation, changing payoff Π_{ij} to Π_{ij}' where:

$$\Pi_{ij}' = \phi(\Pi_{ij}), \qquad \phi' > 0.$$

In certain two-person zero-sum games the payoffs can be transformed by any monotonic linear transformation, changing payoff to Π_{ij}' where:

$$\Pi_{ij}' = a\Pi_{ij} + b, \qquad a > 0.$$

1. Show that in a strictly determined game any monotonic transformation does not change the optimal (pure) strategies and changes the value of the game by the monotonic transformation.
2. Show that in a nonstrictly determined game any monotonic linear transformation does not change the optimal (mixed) strategies and changes the value of the game by the monotonic linear transformation.
3. Show by example that monotonic nonlinear transformations, which do not change the optimal pure strategies in strictly determined games, can change the optimal mixed strategies in nonstrictly determined games.

6-H. Show that the optimal mixed strategies $\mathbf{p}^{1*}$ and $\mathbf{p}^{2*}$ for the 2×2 nonstrictly determined matrix game summarized by the matrix $\mathbf{\Pi}$, assumed nonsingular, are:

$$\mathbf{p}^{1*} = \frac{1}{V}\,\mathbf{1}\mathbf{\Pi}^{-1}$$

$$\mathbf{p}^{2*} = \frac{1}{V}\,\mathbf{\Pi}^{-1}\mathbf{1}'$$

where $\mathbf{1} = (1, 1)$, and V, the value of the game, is:

$$V = \frac{1}{\mathbf{1}\mathbf{\Pi}^{-1}\mathbf{1}'}.$$

Extend the results to case in which $\mathbf{\Pi}$ is a singular 2×2 matrix. Apply these results to find optimal mixed strategies for the matrix games:

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix},$$

verifying the results, using the graphical method.

6-I. In Sec. 6.1 it was shown that the duality theorem of linear programming implies the minimax theorem of game theory. Prove the converse, that the minimax theorem implies the duality theorem. Hint: represent the dual problems of linear programming, as developed in Chapter 5, as the two-person zero-sum game for which the payoff matrix is the skew-symmetric matrix:

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{0} & \mathbf{A} & -\mathbf{b} \\ -\mathbf{A}' & \mathbf{0} & \mathbf{c}' \\ \mathbf{b}' & -\mathbf{c} & 0 \end{pmatrix}.$$

6-J. In a certain infinite two-person zero-sum game player 1 picks an integer i, player 2 picks an integer j, and the payoff to player 1 is $i - j$. Player 1 uses the mixed strategy:

$$p_i = \begin{Bmatrix} 1/2i & \text{if} \quad i = 2^k, & \text{where} \quad k \quad \text{is an integer} \\ 0 & & \text{otherwise} \end{Bmatrix}.$$

Show that this mixed strategy gives an infinite expected payoff for player 1 against any pure strategy of player 2.

6-K. In a *game against nature* there is only one player, called a "decision-maker," who must make a "decision," where the outcome depends not only on his decision but also on the "state of nature."[12] The payoff matrix describing such a game is similar to that of Fig. 6.3, where player 1 is the decision-maker, with m alternative possible decisions and player 2 is "nature," with n alternative possible states of nature. Some of the alternative criteria used to pick a single decision are:

1. The *Laplace criterion* ("Principle of Insufficient Reason") which assumes all states are equally likely and so leads to the choice of the row of the matrix which maximizes the row average.
2. The *minimax criterion* which assumes nature is a malevolent opponent and so leads to the choice of the row of the matrix which contains the maximin element that maximizes over the set of column minima.
3. The *maximax criterion* which assumes nature is a benevolent partner and so leads to the choice of the row of the matrix which contains the maximax element that maximizes over the set of column maxima.
4. The *minimax regret criterion* which assumes that any decision is compared to the decision which would have been made had the state of nature been known. This criterion leads to the choice of the row of the matrix which contains the minimax regret element that minimizes the maximum regret, where regret is the absolute value of the difference between any particular payoff and the payoff that would have been obtained had the state of nature been known. Develop specific numerical matrices representing games against nature for which all four criteria give the same result and for which all four criteria give different results.

6-L. There are three general classes of outcomes in zero-sum games in which each player has two possible strategies:

1. Both players have a dominating strategy,
2. Only one player has a dominating strategy,
3. Neither player has a dominating strategy,

where one strategy *dominates* another if it yields no lower payoff than the other for all strategies of the opponent and higher payoff for some strategies of the opponent. For each of these classes give several numerical examples of different types of games.

6-M. Compare the Prisoners' Dilemma to the Altruists' Dilemma:

$$\begin{bmatrix} (4,4) & (0,5) \\ (5,0) & (2,2) \end{bmatrix} \qquad \begin{bmatrix} (4,4) & (5,0) \\ (0,5) & (2,2) \end{bmatrix}$$

Prisoners' Dilemma Altruists' Dilemma

in terms of the relationship between individual rationality and social efficiency. Why the name "Altruists' Dilemma"?

6-N. In the game of "Battle of the Sexes" the players are a man and woman, each of whom decides whether to go to a prize fight (the first strategy) or a fashion show (the second strategy). The man prefers the prize fight, and the woman the fashion show; but, in any case, they prefer to go together. The payoff matrix is:

$$\begin{bmatrix} (4, 1) & (0, 0) \\ (0, 0) & (1, 4) \end{bmatrix}.$$

1. Solve the bargaining problem, using the Nash solution to the cooperative game, assuming the threat point is (0, 0).
2. Solve the bargaining problem, using the Nash solution if the threat point is that of max-min for each player. Show that this solution is the same as the Shapley value of the cooperative game where:

$$v(\phi) = 0, \qquad v(N) = 1,$$
$$\text{and } v(\{\text{man}\}) \text{ and}$$
$$v(\{\text{woman}\}) \text{ are the max-min values.}$$

6-O. Contrast the following solution concepts for the Prisoners' Dilemma, Altruists' Dilemma, Game of Chicken, and Battle of the Sexes:

1. Equilibrium point(s);
2. Max-min; i.e., each player maximizes his own minimum payoff;
3. Min-max; i.e., each player minimizes the opponent's maximum payoff;
4. Max-sum; i.e., maximize the sum of the payoffs;
5. Max-diff; i.e., maximize difference between one's own payoff and the opponent's payoff.

6-P. In a certain bargaining problem two men are offered \$100 if they can agree on a division of the money. Man 1 has $\$W_1$ of wealth, and his utility function is logarithmic, so that his payoff, assuming he receives $\$X$ of the \$100, is:

$$\Pi^1 = \ln (W_1 + X), \qquad 0 \leq X \leq 100.$$

Similarly, man 2 has $\$W_2$ of wealth, and his payoff after receiving the remaining $\$100 - X$ is:

$$\Pi^2 = \ln (W_2 + 100 - X).$$

1. Suppose both men are wealthy compared to the amount to be decided: $W_i \gg 100, \quad i = 1, 2$. According to the Nash solution, how is the money divided? (Hint: $\ln (1 + z) \approx z$ if z is small.)

2. Suppose man 2 is wealthy, but the wealth of man 1 totals only \$100 before the division of the money: $W_1 = 100$, $W_2 \gg 100$. How is the money divided? Is this division "fair"?

6-Q. The game of "odd man out" is a three-person game in which each of the players independently chooses "heads" or "tails." If all players choose the same, the house pays each player \$1; otherwise, the odd man pays each of the others a dollar. Find the characteristic function.

6-R. For the notion of dominance for imputations introduced for the von Neumann-Morgenstern solution, give examples to illustrate the possibility:

1. $\boldsymbol{\Pi}_1$ dominates $\boldsymbol{\Pi}_2$ and $\boldsymbol{\Pi}_2$ doesn't dominate $\boldsymbol{\Pi}_1$
2. $\boldsymbol{\Pi}_1$ dominates $\boldsymbol{\Pi}_2$ and $\boldsymbol{\Pi}_2$ dominates $\boldsymbol{\Pi}_1$
3. Neither $\boldsymbol{\Pi}_1$ nor $\boldsymbol{\Pi}_2$ dominate the other.

6-S. For a game in characteristic function form prove that the core is a subset of the von Neumann-Morgenstern solution.

6-T. A game is *constant sum in characteristic function form* if:

$$v(S) + v(N \sim S) = v(N) \quad \text{for all subsets} \quad S \quad \text{of} \quad N.$$

1. Show by example that a game can be constant sum in characteristic function form but not constant sum in normal form;
2. Prove that all finite games which are constant sum in normal form are also constant sum in characteristic function form.

6-U. A three-person game has a normalized characteristic function for which the maximum guaranteed payoff to all two-person coalitions is the parameter p. Find the core and the von Neumann-Morgenstern solution(s) if $p = 0, p = 1/3, p = 2/3, p = 1$.

6-V. In a corporation a simple majority of shares is required for control, but dividends are paid equally on all shares, regardless of whether or not the shares are owned by the controlling interests. If the i^{th} shareholder holds S_i shares, $i = 1, \ldots, n$, and m is the total number of shares outstanding:

$$m = \sum_{i=1}^{n} S_i$$

then the characteristic function is:

$$v(S) = \begin{cases} 0 & \text{for} \quad n_s \leq \dfrac{m}{2} \\[2ex] \dfrac{n_s}{m} & \text{for} \quad n_s > \dfrac{m}{2} \end{cases}$$

where n_s, the number of shares controlled by coalition S, is:

$$n_s = \sum_{\text{all } i \in S} S_i.$$

Show that, assuming the S_i are not all equal, the core consists of the single imputation $(S_1/m, S_2/m, \ldots . S_n/m)$. Interpret this result.

FOOTNOTES

[1] The basic references in game theory are von Neumann and Morgenstern (1944), Luce and Raiffa (1957), Shubik, ed. (1964), and Owen (1968).

[2] Payoffs are measured in terms of utility, as discussed in Chapter 7. If the payoffs depend on the outcome of random events with known probabilities ("chance moves") then the payoffs are expected utilities, i.e., utilities weighted by probabilities.

3. In parlor games the number of strategies is typically astronomical but finite. Consider, for example, chess, where a strategy would be a set of rules as to choices to make given all possible choices of the opponent. In particular consider the first move in which first White and then Black choose from one of the 20 available alternatives (moving one or two spaces for each pawn and two jump moves for each Knight). White chooses first, and for each of his 20 possible choices Black must select one of 20 possible choices. Thus the number of strategies available for Black assuming the game ended after the first move is 20^{20} or about 10^{26}—a truly astronomical number.

[4] For discussions of infinite games see Dresher, Tucker, and Wolfe, eds. (1957); Karlin (1959); and Dresher, Shapley, and Tucker, eds (1964). An example of an infinite game is a game on the unit square, where each of the strategies available to the two players is a real number lying between zero and unity. Another example is a game in which each player chooses as a strategy a time path from a set of alternative possible time paths. The latter game, called a *differential game* and discussed in Chapter 15, involves an infinite number of moves and hence an infinite number of strategies.

[5] The minimax theorem was proved by von Neumann (1928). See Gale, Kuhn, and Tucker (1951) and Nash (1951). The theorem can be proved in several ways, including the duality theorem of linear programming, fixed point theorems, and separation theorems for convex sets. The theorem is not valid for infinite games. An example is "choose a number," in which the two players each write down a number and the one with the larger number is paid a certain sum by the one with the smaller number. Such games have no solution in pure or mixed strategies.

[6] Equilibrium points are also called "Nash equilibrium" points after Nash (1950). Equilibrium points are by no means the only possible approach to noncooperative games. Other approaches are max-min (maximize one's own minimum payoff, as in Sec. 6.1); max-max (maximize one's own maximum payoff); min-max (minimize the opponent's maximum payoff); max-sum (maximize the sum of payoffs); and max-diff (maximize the difference in payoffs).

[7] See Nash (1950b, 1953) and Harsanyi (1956). For extensions to more than two players see Harsanyi (1959, 1963).

[8] Cooperative games *without* side payments can be analyzed using a vector-valued characteristic function, which characterizes all possible coalitions by indicating the maximum payoffs each member of the coalition can guarantee himself. See Aumann (1967).

[9] See Lucas (1967).

[10] See Shapley (1953) and Selten (1964).

[11] See Shubik (1959b), Debreu and Scarf (1963), Aumann (1964), and Shapley and Shubik (1967). See also Sec. 10.2.

[12] See Milnor (1954).

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