11 The Control Problem

The static economizing problem was that of allocating resources among competing ends at a given point in time. In mathematical terms, the problem was that of choosing values for certain variables, called *instruments*, from a given set, called the *opportunity set*, so as to maximize a given function, called the *objective function*. When expressed in this form the problem was referred to as the *mathematical programming problem*.

The dynamic economizing problem is that of allocating scarce resources among competing ends over an interval of time from *initial time* to *terminal time*. In mathematical terms the problem is that of choosing time paths for certain variables, called *control variables*, from a given class of time paths, called the *control set*. The choice of time paths for the control variables implies, via a set of differential equations, called the *equations of motion*, time paths for certain variables describing the system, called the *state variables*, and the time paths of the control variables are chosen so as to maximize a given functional depending on the time paths for the control and the state variables, called the *objective functional*. When presented in this form the problem is referred to as the *control problem*.

A classic example of the control problem is that of determining optimal missile trajectories. In this problem the control variables are the timing, magnitude, and direction of various thrusts that can be exerted on the missile. These thrusts are chosen subject to certain constraints; for example, the total amount of propellant available. The state variables, which describe the missile trajectory, are the mass of the missile and the position and velocity of the missile relative to a given coordinate system. The influence of the thrusts on the state variables is summarized by a set of differential equations obtained from the laws of physics. The mission to be accomplished is then represented as the maximization of an objective functional. For example, in the *Apollo Mission Problem* the objective is that of maximizing terminal payload given a terminal position on the surface of the moon and given terminal velocity sufficiently small so that the men and equipment aboard will survive the lunar impact.

11.1 Formal Statement of the Problem

A formal statement of the control problem is comprised of *time*, the *state variables*, the *control variables*, the *equations of motion*, the *determination of terminal time*, and the *objective functional*.[1]

Time, t, is measured in continuous units and is defined over the *relevant interval* from *initial time* t_0, which is typically given, to *terminal time* t_1, which must often be determined. Thus the relevant interval is:[2]

$$t_0 \le t \le t_1. \tag{11.1.1}$$

At any time t in the relevant interval the state of the system is characterized by n real numbers, $x_1(t), x_2(t), \ldots, x_n(t)$, called *state variables*, and summarized by the *state vector*:

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))', \tag{11.1.2}$$

an n dimensional column vector which can be interpreted geometrically as a point in Euclidean n-space, E^n. Each state variable is assumed to be a continuous function of time, so the *state trajectory:*

$$\{\mathbf{x}(t)\} = \{\mathbf{x}(t) \in E^n \mid t_0 \le t \le t_1\} \tag{11.1.3}$$

is a continuous vector valued function of time, the value of which at any time t in the relevant interval is the state vector (11.1.2). Geometrically, the state trajectory is a path of points in E^n, starting at the *initial state:*

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{11.1.4}$$

which is assumed given, and ending at the *terminal state:*

$$\mathbf{x}(t_1) = \mathbf{x}_1, \tag{11.1.5}$$

which must often be determined.

At any time t in the relevant interval the choices (decisions) to be made are characterized by r real numbers, $u_1(t), u_2(t), \ldots, u_r(t)$, called *control variables* and summarized by the *control vector*:

$$\mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_r(t))', \tag{11.1.6}$$

an r dimensional column vector which can be interpreted geometrically as a point in E^r. Each control variable is required to be a piecewise continuous function of time, so the *control trajectory:*

$$\{\mathbf{u}(t)\} = \{\mathbf{u}(t) \in E^r \mid t_0 \le t \le t_1\} \tag{11.1.7}$$

is a piecewise continuous-vector-valued function of time, the value of which, at any time t in the relevant interval, is the control vector (11.1.6). Geometrically, the control trajectory is a path of points in E^r that is continuous, except possibly for a finite number of discrete jumps.

The control variables are chosen subject to certain constraints on their possible values, summarized by the restriction that the control vector at all times in the relevant interval must belong to a given nonempty subset of Euclidean r-space Ω:

$$\mathbf{u}(t) \in \Omega, \qquad t_0 \leq t \leq t_1, \tag{11.1.8}$$

where Ω is usually assumed compact (closed and bounded), convex, and time invariant. The control trajectory (11.1.7) is *admissible* if it is a piecewise continuous vector valued function of time the value of which at any point of time in the relevant interval belongs to Ω. The contıol set, U, is the set of all admissible control trajectories, i.e., control trajectories which are piecewise continuous functions of time over the relevant time interval the values of which at all times in this interval belong to Ω. The control trajectory must belong to this control set:

$$\{\mathbf{u}(t)\} \in U. \tag{11.1.9}$$

The state trajectory $\{\mathbf{x}(t)\}$ is characterized by *equations of motion*, a set of n differential equations giving the time rate of change of each state variable as a function of the state variables, the control variables, and time:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{11.1.10}$$

or, written out in full:

$$\frac{dx_j}{dt}(t) = \dot{x}_j(t) = f_j(x_1(t), x_2(t), \ldots, x_n(t); u_1(t), u_2(t), \ldots, u_r(t); t),$$
$$j = 1, 2, \ldots, n, \tag{11.1.11}$$

where each of the n functions $f_1(\cdots), f_2(\cdots), \ldots, f_n(\cdots)$ is assumed given and continuously differentiable. If the differential equations do not depend explicitly on time then the equations of motion are *autonomous*. An important example is the linear autonomous equations of motion:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{11.1.12}$$

where $\mathbf{A}$ is a given $n \times n$ matrix and B is a given $n \times r$ matrix.

The boundary conditions on the equations of motion are the given initial values of the state variables (11.1.4). Given these initial values and given a control trajectory $\{\mathbf{u}(t)\}$, there exists a unique state trajectory $\{\mathbf{x}(t)\}$ satisfying the equations of motion and boundary conditions, which can be obtained by integrating the differential equations forward from $\mathbf{x}_0$. A state trajectory obtained from the equations of motion and initial state using an admissible control is called *feasible*, and any state vector reached on a feasible trajectory in finite time is called *reachable*.

Terminal time, t_1, is defined by:

$$(\mathbf{x}(t), t) \in T \quad \text{at} \quad t = t_1, \tag{11.1.13}$$

where T is a given subset of E^{n+1}, called the *terminal surface.* Important special cases are the *terminal time problem*, in which t_1 is given explicitly as a parameter of the problem, and the *terminal state problem*, in which $\mathbf{x}(t_1)$ is given explicitly as a vector of parameters of the problem.

The *objective functional* is a mapping from control trajectories to points on the real line, the value of which is to be maximized. It will generally be assumed to be of the form:[3]

$$J = J\{\mathbf{u}(t)\} = \int_{t_0}^{t_1} I(\mathbf{x}(t), \mathbf{u}(t), t)\, dt + F(\mathbf{x}_1, t_1), \tag{11.1.14}$$

where the integrand in the first term, $I(\cdot\cdot\cdot)$, called the *intermediate function*, shows the dependence of the functional on the time paths of the state variables, control variables, and time within the relevant time interval:

$$I(\mathbf{x}, \mathbf{u}, t) = I(x_1(t), x_2(t), \ldots, x_n(t); u_1(t), u_2(t), \ldots, u_r(t); t)$$

where:

$$t_0 \le t \le t_1. \tag{11.1.15}$$

The second term $F(\cdot\,\cdot)$, called the *final function*, shows the dependence of the functional on the terminal state and terminal time:

$$F(\mathbf{x}_1, t_1) = F(x_1(t_1), x_2(t_1), \ldots, x_n(t_1); t_1). \tag{11.1.16}$$

Both $I(\cdot\cdot\cdot)$ and $F(\cdot\,\cdot)$ are assumed given and continuously differentiable. The objective functional is written in (11.1.14) as a functional in the control trajectory since, given $\mathbf{f}(\cdot\cdot\cdot)$ and $\mathbf{x}_0$, the trajectory $\{\mathbf{u}(t)\}$ determines the trajectory $\{\mathbf{x}(t)\}$.

With the objective functional as given in (11.1.14) the problem is usually referred to as a *Problem of Bolza*. If the final function is identically zero, so:

$$J = \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}, t)\, dt, \tag{11.1.17}$$

then the problem is usually referred to as a *Problem of Lagrange*, while if the intermediate function is identically zero, so:

$$J = F(\mathbf{x}_1, t_1), \tag{11.1.18}$$

then the problem is usually referred to as a *Problem of Mayer*. It might appear that the Problem of Bolza is more general than either the Problem of Lagrange or the Problem of Mayer, but, by suitable definitions of variables, all three problems are equivalent. For example, the Problem of Bolza can be converted to a Problem of Mayer by defining the added state variable x_{n+1} as:

$$\dot{x}_{n+1}(t) = I(\mathbf{x}, \mathbf{u}, t) \qquad (11.1.19)$$
$$x_{n+1}(t_0) = 0,$$

in which case (11.1.14) becomes:

$$J = x_{n+1}(t_1) + F(\mathbf{x}_1, t_1), \qquad (11.1.20)$$

which is the objective functional for a Problem of Mayer.

To summarize, the general control problem is:

$$\begin{aligned} &\max_{\{\mathbf{u}(t)\}} \quad J = \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}, t)\, dt + F(\mathbf{x}_1, t_1) \\ &\text{subject to:} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ &t_0 \ \text{ and } \ \mathbf{x}(t_0) = \mathbf{x}_0 \ \text{ given} \\ &(\mathbf{x}(t), t) \in T \ \text{ at } \ t = t_1 \\ &\{\mathbf{u}(t)\} \in U. \end{aligned} \qquad \textbf{(11.1.21)}$$

The geometry of this problem is shown in Fig. 11.1 for the case of one state variable. Starting at the given initial state x_0 at initial time t_0, the state

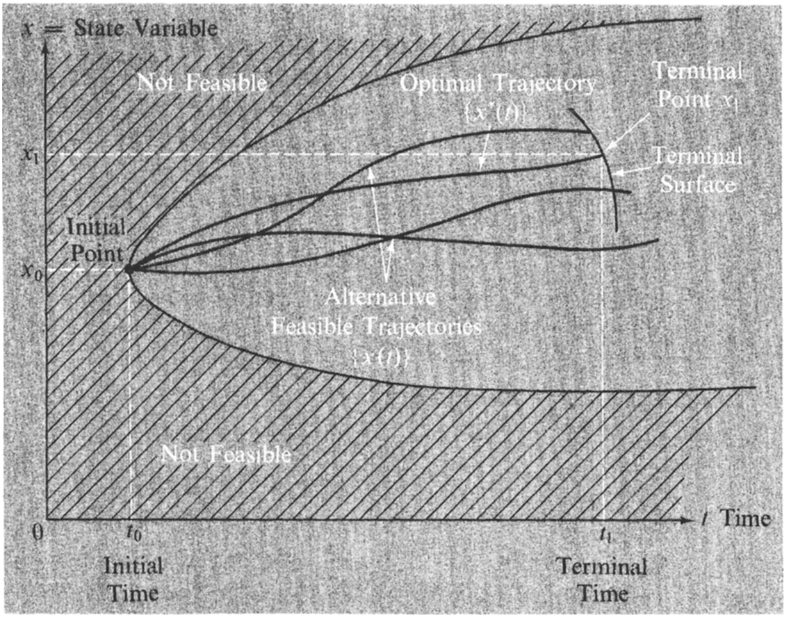

Fig. 11.1
The Geometry of the Control Problem in the Case of One State Variable

trajectory $\{x(t)\}$ must be chosen from the set of feasible trajectories, each of which results from using an admissible control trajectory $\{\mathbf{u}(t)\}$. The particular feasible state trajectory that is optimal $\{x^*(t)\}$, must hit the terminal surface and must maximize the objective functional among the set of all such trajectories.

11.2 Some Special Cases

The objective functional (11.1.14), or, equivalently, (11.1.17) or (11.1.18), is a very important one, in that it subsumes several important special cases. The first special case is the *time optimal control problem*, in which the objective is to move the state variables from given initial values to given terminal values in minimum time. In this case the objective function is:

$$J = -(t_1 - t_0), \tag{11.2.1}$$

which results from the Problem of Lagrange for which $I(\cdots) = -1$. Since t_0 is given, an equivalent problem is the Problem of Mayer for which $F(\cdot\,\cdot) = -t_1$. The classic example of a minimum time problem, dating back to the seventeenth century, is the *Brachistochrone problem* of designing a curve such that a particle sliding frictionlessly along the curve under the influence of gravity moves from a given upper point to a given lower point in minimum time. Another example is that of steering a ship so as to reach some given destination in minimum time.

A second special case is that of a *servomechanism*, in which a desired state $\mathbf{x}^0(t)$ is specified for each time in the relevant interval, and the objective is that of ensuring that the actual state vector is sufficiently close to the desired state at any time in the interval. For example, in heating a home the state variable is the room temperature, and one wants to keep the actual room temperature reasonably close to a desired temperature. In this case the objective functional takes the form:

$$J = \int_{t_0}^{t_1} \phi\,(\mathbf{x}^0(t) - \mathbf{x}(t))\,dt, \tag{11.2.2}$$

where $\varphi(\cdot)$ is a function measuring the negative of the cost of the discrepancy between desired and actual states. For example, using the *least squares criterion* $\varphi(\cdot)$ is the quadratic form:

$$\varphi(\mathbf{x}^0(t) - \mathbf{x}(t)) = (\mathbf{x}^0(t) - \mathbf{x}(t))'\mathbf{D}(\mathbf{x}^0(t) - \mathbf{x}(t)), \tag{11.2.3}$$

where $\mathbf{D}$ is a given negative definite matrix of weights. Expanding the product and dropping the constant term, which is irrelevant as far as the maximization is concerned, in this case the intermediate function is the sum of a linear and a quadratic term, so:

$$J = \int_{t_0}^{t_1} (\mathbf{cx} + \mathbf{x}'\mathbf{Dx})\, dt, \tag{11.2.4}$$

where $\mathbf{c}$ is the row vector $-2\mathbf{x}^0(t)'\mathbf{D}$.

The third special case is that of *minimum effort*, in which case the objective functional depends only on the control trajectory. In the quadratic case:

$$J = \int_{t_0}^{t_1} \mathbf{u}(t)'\mathbf{Eu}(t)\, dt, \tag{11.2.5}$$

where E is a given negative definite matrix of weights. This case and the last case can be combined to form the objective functional:

$$J = \int_{t_0}^{t_1} (\mathbf{cx} + \mathbf{x}'\mathbf{Dx} + \mathbf{u}'\mathbf{Eu})\, dt, \tag{11.2.6}$$

where $\mathbf{c}$ is a given row vector and $\mathbf{D}$ and $\mathbf{E}$ are given negative definite matrices. There is no loss in generality in assuming that the desired state is the origin $\mathbf{x}^0(t) = \mathbf{0}$, the actual state being measured from the desired state, in which case $\mathbf{c} = \mathbf{0}$ and:

$$J = \int_{t_0}^{t_1} (\mathbf{x}'\mathbf{Dx} + \mathbf{u}'\mathbf{Eu})\, dt, \tag{11.2.7}$$

which is the objective functional of the *least squares minimum effort servo-mechanism*.

11.3 Types of Control

There are two types of control which can be envisaged for the control problem. One is *open loop control*, in which the optimal control trajectory, solving (11.1.21), is determined as a function of time

$$\{\mathbf{u}^*(t)\}. \tag{11.3.1}$$

This open loop control is completely specified at the initial time t_0, and the state trajectory $\{\mathbf{x}(t)\}$ is determined by integrating the equations of motion forward from their prescribed initial values, using the open loop control.

The other type of control is *closed loop control*, in which the optimal control trajectory is determined as a function of the current state variables and time:

$$\{\mathbf{u}^*(\mathbf{x}(t), t)\}. \tag{11.3.2}$$

By contrast to open loop control, in which all decisions are made in advance, in closed loop control the decisions may be revised in the light of new information embodied in the current state variables. The problem of obtaining the optimal closed loop control is called that of *synthesis*.

Familiar examples of the distinction between open loop and closed loop control are clothes dryers and home heating systems. Most clothes dryers are regulated by open loop control, by a timer which must be set in advance. A home heating system, by contrast, is typically regulated by a thermostat which turns the furnace on if the room temperature is too low and turns it off if the room temperature is too high. Thus the control of the furnace depends on the current state variable, the room temperature.

Examples of open loop and closed loop also exist in the economy. Automatic stabilizers, such as unemployment insurance and the progressive income tax are closed loop systems, where added unemployment results in more government payments via unemployment insurance, thereby counteracting the added unemployment. Similarly, added inflation results in proportionately larger taxes via the progressive income tax, thereby counteracting the added inflation. In both cases the control variables (benefit payments in the first case; tax receipts in the second case) respond to the current state of the economy. Another example of a closed loop system in the economy is monetary policy as carried out by the Federal Reserve System, which responds to current economic variables in its control of money and credit. There have been proposals, however, to convert this closed loop system into an open loop one in which some rate of expansion of the money supply, such as five percent per year, is decided in advance and carried out without regard to current economic conditions.

The two types of control and other aspects of the control problem are shown schematically for the terminal time problem in Fig. 11.2. The givens appear in circles: initial time and state, the equations of motion, the control set, and the objective functional. The trajectories to be determined—the control trajectory and state trajectory—appear in boxes. The two types of control appear in diamonds: open loop control and closed loop control. The arrows show the interrelations between various parts of the problem. For example, the equations of motion use the current state, control, and time to determine the time rate of change of the state variables, thereby influencing the state trajectory.

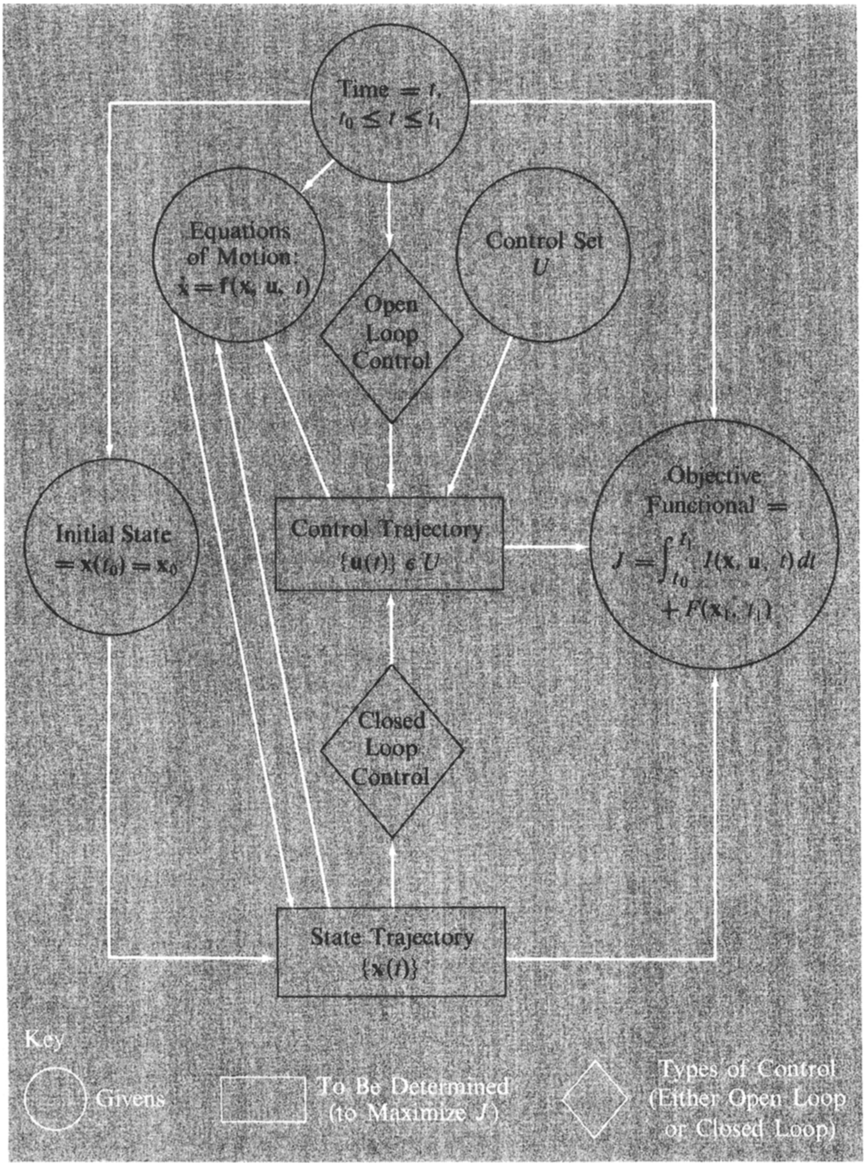

Fig. 11.2

The Control Problem, for which Terminal Time is Given

It will generally be assumed that the control problem contains no random variables and that all relevant parameters, functions, and sets in (11.2.1) are completely specified. In such a case open loop and closed loop control yield identical results, so the emphasis will be on open loop control, which is typically more easily determined than closed loop control. By contrast, closed loop control is generally superior to open loop control in yielding a higher maximum for the objective functional in the case of *stochastic control*, in which random variables with given distributions appear in the problem, and in the case of *adaptive control*, in which initial uncertainties about the parameters, functions, or sets of the problem are reduced or eliminated as the process unfolds. These two cases will not be discussed here.[4]

11.4 The Control Problem as One of Programming in an Infinite Dimensional Space; the Generalized Weierstrass Theorem

The control problem can be considered one of mathematical programming in an infinite dimensional space. Consider the following control problem:

$$\begin{gathered} \max_{\{u(t)\}} \quad J = \int_{t_0}^{t_1} I(x, u)\, dt \\ \dot{x} = f(x, u) \\ t_0 \quad \text{and} \quad x(t_0) = x_0 \quad \text{given} \\ t_1 \quad \text{given} \\ \{u(t)\} \in U. \end{gathered} \tag{11.4.1}$$

By contrast to (11.1.21), this problem is: autonomous, the equations of motion and objective functional showing no explicit dependence on time; one of Lagrange, there being no dependence of the objective functional on terminal state or time; one of terminal time, since t_1 is given and $x(t_1)$ is free; and one involving only a single control and a single state variable.

Since the relevant time interval is given, it can be divided into N subintervals of equal length Δ:

$$\Delta = \frac{t_1 - t_0}{N}. \tag{11.4.2}$$

Time is measured in discrete units, where:

$$t = t_0 + q\Delta, \tag{11.4.3}$$

q being an index ranging from 0 (corresponding to $t = t_0$) to N (corresponding to $t = t_1$). The state and control are measured at the discrete time points:

$$\begin{aligned} x^q &= x(t_0 + q\Delta) \\ u^q &= u(t_0 + q\Delta). \end{aligned} \tag{11.4.4}$$

Now consider the mathematical programming problem in the $N + 1$ variables $u^0, u^1, \ldots, u^N$:

$$\begin{aligned} &\max_{u^0, u^1, \ldots, u^N} J^N = \sum_{q=0}^{N} I(x^q, u^q)\,\Delta \\ &x^{q+1} - x^q = f(x^q, u^q)\,\Delta, \qquad q = 0, 1, \ldots, N-1 \\ &x^0 = x_0, \quad \text{given} \quad u^q \in \Omega, \end{aligned} \tag{11.4.5}$$

where Δ is a given positive parameter. The objective function of this problem approaches the objective functional of (11.4.1) as N increases without limit and Δ decreases to zero, where $N\Delta$ is fixed as $(t_1 - t_0)$:

$$\lim_{\substack{N\to\infty \\ \Delta\to 0 \\ N\Delta=(t_1-t_0)}} J^N = J \tag{11.4.6}$$

By the same limiting process, the difference equation of (11.4.5) approaches the differential equation of (11.4.1). Thus, the control problem can be considered a mathematical programming problem in infinite dimensional space, the space being that of all piecewise continuous real valued functions $u(t)$ defined over the interval $t_0 \le t \le t_1$.

A fundamental theorem of mathematical programming, the Weierstrass theorem, discussed in Sec. 2.3, gave conditions sufficient for the existence of a maximum, namely the conditions that the objective function be continuous and the opportunity set be compact. This theorem can be generalized to infinite dimensional space to obtain the fundamental existence theorem for control problems, the *generalized Weierstrass theorem.* According to this theorem, there exists a solution to the general control problem (11.1.21) if the objective functional $J\{\mathbf{u}(t)\}$ is a continuous functional in the control trajectories and the subset of the infinite dimensional space to which the control trajectory is confined, U, is compact.[5] An important special case for which solutions exist is that in which the functions $I(\cdots)$ and $\mathbf{f}(\cdots)$ are linear in $\mathbf{u}$.

FOOTNOTES

[1] The basic references for the control problem are Pontryagin et al. (1962), Zadeh and Desoer (1963), Feldbaum (1965), Athans and Falb (1966), Hestenes (1966), and Lee and Markus (1967). For historically important papers dealing with the control problem see Bellman and Kalaba, eds. (1964) and Oldenburger, ed. (1966).

[2] For control problems in which time is measured in discrete units $t = 0, 1, 2, \ldots$, see Chang (1961), Aris (1964), Fan and Wang (1964), and Wilde and Beightler (1967). See also Secs. 11.4 and 13.4.

[3] Note that the standard notation of the control problem differs from that for the programming problem. The dynamic analogue of the instrument vector $\mathbf{x}$ of mathematical programming is the control trajectory $\{\mathbf{u}(t)\}$, *not* the state trajectory $\{\mathbf{x}(t)\}$.

[4] For discussions of stochastic control see Aoki (1967) and Kushner (1967). For discussions of adaptive control see Bellman (1961), Mishkin and Braun (1961), and Murphy (1965).

[5] To prove the generalized Weierstrass theorem, let J^* be the supremum of $J\{\mathbf{u}(t)\}$ over all $\{\mathbf{u}(t)\} \in U$, that is:

$$J\{\mathbf{u}(t)\} \leq J^* \quad \text{for all} \quad \{\mathbf{u}(t)\} \in U.$$

Choose a sequence of control trajectories $\{\mathbf{u}^p\}$ such that:

$$J^* - \frac{1}{p} < J\{\mathbf{u}^p\} \leq J^*.$$

Since U is compact the sequence contains a subsequence $\{\mathbf{u}^{p_k}\}$ converging to some control trajectory $\{\mathbf{u}^*\} \in U$. Then:

$$J^* - \frac{1}{p_k} < J\{\mathbf{u}^{p_k}\} \leq J^*$$

and so:

$$\lim_{p_k \to \infty} J\{\mathbf{u}^{p_k}\} = J^*.$$

But, since J is continuous:

$$\lim_{p_k \to \infty} J\{\mathbf{u}^{p_k}\} = J\{\mathbf{u}^*\},$$

so the optimal control trajectory is $\{\mathbf{u}^*\} \in U$, for which $J\{\mathbf{u}^*\} = J^*$.

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