14 Maximum Principle

The maximum principle is the third approach to the control problem, an approach which is often the most useful since, by contrast to the classical calculus of variations, it can cope directly with general constraints on the control variables and, by contrast to dynamic programming, it usually suggests the nature of the solution.[1] The maximum principle therefore has been the basic approach to computing optimal controls in many important problems in mathematics, engineering, and economics.

The maximum principle problem is the general control problem:

$$
\begin{aligned}
\max_{\{\mathbf{u}(t)\}} J &= \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}, t)\, dt + F(\mathbf{x}_1, t_1) \\
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\
\mathbf{x}(t_0) &= \mathbf{x}_0 \\
\mathbf{x}(t_1) &= \mathbf{x}_1 \\
\{\mathbf{u}(t)\} &\in U,
\end{aligned}
\tag{14.0.1}
$$

where $I(\cdots)$, $F(\cdot\cdot)$, and $\mathbf{f}(\cdots)$ are given continuously differentiable functions; t_0, $\mathbf{x}_0$ are given parameters; t_1 or $\mathbf{x}_1$ are given parameters (or $\mathbf{T}(\mathbf{x}, t) = \mathbf{0}$

defines the terminal surface); and $\{\mathbf{u}(t)\}$, the control trajectory, must belong to the given control set U, requiring that $\mathbf{u}(t)$ be a piecewise continuous function of time the values of which must belong to the set Ω, a given nonempty compact subset of E^r.

14.1 Costate Variables, the Hamiltonian, and the Maximum Principle

In earlier chapters the method of Lagrange multipliers was applied to various problems of static optimization. The method was that of introducing new variables, Lagrange multipliers, one for each constraint; defining a Lagrangian expression; and finding a saddle point of this expression, maximizing with respect to the choice variables and minimizing with respect to the Lagrange multipliers. The maximum principle can be considered the extension of the method of Lagrange multipliers to dynamic optimization (control) problems. Consider the control problem, (14.0.1), in the special case in which terminal time is given and the control variables are unconstrained. This problem is one of maximization subject to constraints, where the expression to be maximized is the objective functional:

$$J = \int_{t_0}^{t_1} I(\mathbf{x}, \mathbf{u}, t)\, dt + F(\mathbf{x}_1, t_1), \tag{14.1.1}$$

and the constraints are the n differential equations, which can be written:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}}(t) = \mathbf{0}, \qquad t_0 \le t \le t_1. \tag{14.1.2}$$

Proceeding in a way analogous to that in static problems, add to the problem a (row) vector of new variables, one for each of the n constraints:

$$\mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t)). \tag{14.1.3}$$

These new variables are called *costate variables*, and they are the dynamic equivalents of the Lagrange multipliers of static problems of maximization subject to constraints.[2] Since each of the costate variables corresponds to one of the differential equations of motion, which is itself defined over the entire time interval from t_0 to t_1, the costate variables in general vary over time, as indicated in (14.1.3), and are assumed to be nonzero continuous functions of time.

Again proceeding by analogy to the static case, the next step is to define a Lagrangian function which equals the expression to be maximized plus the inner product of the Lagrange multiplier vector and the constraints. Since the constraints and costate variables are defined over the entire time interval, however, the inner product is properly treated under the integral sign, the Lagrangian expression being:

$$\begin{aligned} L &= J + \int_{t_0}^{t_1} \mathbf{y}[\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}}]\, dt \\ &= \int_{t_0}^{t_1} \{I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}[\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}}]\}\, dt + F(\mathbf{x}_1, t_1). \end{aligned} \tag{14.1.4}$$

Yet again, by analogy to the static case, a saddle point of the Lagrangian would yield the solution. Here, however, the saddle point is in the space of functions, where $(\{\mathbf{u}^*(t)\}, \{\mathbf{y}^*(t)\})$ represent a saddle point if:

$$L(\{\mathbf{u}(t)\}, \{\mathbf{y}^*(t)\}) \le L(\{\mathbf{u}^*(t)\}, \{\mathbf{y}^*(t)\}) \le L(\{\mathbf{u}^*(t)\}, \{\mathbf{y}(t)\}). \tag{14.1.5}$$

The control trajectory $\{\mathbf{u}^*(t)\}$ then solves the control problem. By the second inequality:

$$\int_{t_0}^{t_1} \{(\mathbf{y}^* - \mathbf{y})[\mathbf{f}(\mathbf{x}^*, \mathbf{u}^*, t) - \dot{\mathbf{x}}^*]\}\, dt \le 0, \tag{14.1.6}$$

which, holds for all continuous $\{\mathbf{y}(t)\}$ only if:

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*, t), \tag{14.1.7}$$

since otherwise $\{\mathbf{y}(t)\}$ can be chosen at points where this equality is not satisfied in such a way that the integral in (14.1.6) is positive. Thus the equations of motion are satisfied along the optimal trajectory. But, from the first inequality in (14.1.5):

$$J\{\mathbf{u}^*(t)\} \geq J\{\mathbf{u}(t)\} + \int_{t_0}^{t_1} \{\mathbf{y}^*[\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}}]\}\, dt, \tag{14.1.8}$$

so, for all control trajectories $\{\mathbf{u}(t)\}$ satisfying the equations of motion:

$$J\{\mathbf{u}^*(t)\} \geq J\{\mathbf{u}(t)\}, \tag{14.1.9}$$

and therefore $\{\mathbf{u}^*(t)\}$ is the optimal trajectory. The optimal value of the objective functional is then the value of the Lagrangian at the saddle point.

Now consider the necessary conditions for such a saddle point. From (14.1.4) a change in the costate variable trajectory to $\{\mathbf{y}(t) + \Delta\mathbf{y}(t)\}$ where $\Delta\mathbf{y}(t)$ is any continuous function of time would change the Lagrangian by:

$$\Delta L = \int_{t_0}^{t_1} \Delta\mathbf{y}[\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}}]\, dt. \tag{14.1.10}$$

Setting the change in the Lagrangian equal to zero, the first order necessary condition for minimizing L by choice of $\{\mathbf{y}(t)\}$, requires, from the fundamental lemma of the calculus of variations, that the equations of motion be satisfied:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t). \tag{14.1.11}$$

So, obtaining the equations of motion here as necessary conditions is completely analogous to obtaining the constraints as necessary conditions in static problems.

To develop the remaining necessary conditions, note that the term $-\mathbf{y}(t)\dot{\mathbf{x}}(t)$ in (14.1.4) can be integrated by parts to yield:

$$\begin{aligned} L = &\int_{t_0}^{t_1} \{I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \dot{\mathbf{y}}\mathbf{x}\}\, dt \\ &+ F(\mathbf{x}_1, t_1) - [\mathbf{y}(t_1)\mathbf{x}(t_1) - \mathbf{y}(t_0)\mathbf{x}(t_0)]. \end{aligned} \tag{14.1.12}$$

The first two expressions under the integral sign are defined to be the *Hamiltonian function:*

$$H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) \equiv I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{14.1.13}$$

that is, the Hamiltonian function is defined as the sum of the intermediate function (integrand) of the objective functional plus the inner product of the vector of costate variables and the vector of functions defining the rate of change of the state variables. Thus:

$$L = \int_{t_0}^{t_1} \{H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) + \dot{\mathbf{y}}\mathbf{x}\}\, dt + F(\mathbf{x}_1, t_1) - [\mathbf{y}(t_1)\mathbf{x}(t_1) - \mathbf{y}(t_0)\mathbf{x}(t_0)]. \tag{14.1.14}$$

Consider the effect of a change in the control trajectory from $\{\mathbf{u}(t)\}$ to $\{\mathbf{u}(t) + \Delta\mathbf{u}(t)\}$ with a corresponding change in the state trajectory from $\{\mathbf{x}(t)\}$ to $\{\mathbf{x}(t) + \Delta\mathbf{x}(t)\}$. The change in the Lagrangian is:

$$\Delta L = \int_{t_0}^{t_1} \left\{\frac{\partial H}{\partial \mathbf{u}}\Delta\mathbf{u} + \left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\mathbf{y}}\right)\Delta\mathbf{x}\right\} dt + \left[\frac{\partial F}{\partial \mathbf{x}_1} - \mathbf{y}(t_1)\right]\Delta\mathbf{x}_1, \tag{14.1.15}$$

where:

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{u}} &= \left(\frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2}, \ldots, \frac{\partial H}{\partial u_r}\right)\\ \frac{\partial H}{\partial \mathbf{x}} &= \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_n}\right).\end{aligned} \tag{14.1.16}$$

For a maximum it is necessary that the change in the Lagrangian vanish, implying, since (14.1.15) must hold for any $\{\Delta\mathbf{u}(t)\}$, that:

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0}, \qquad t_0 \le t \le t_1 \tag{14.1.17}$$

$$\dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad t_0 \le t \le t_1 \tag{14.1.18}$$

$$\mathbf{y}(t_1) = \frac{\partial F}{\partial \mathbf{x}_1}. \tag{14.1.19}$$

Necessary conditions (14.1.17) state that the Hamiltonian function is maximized by choice of the control variables at each point along the optimal trajectory, the r conditions in (14.1.17) being those for an interior maximum since in the problem under consideration there are no constraints on the values taken by the control variables. More generally, if there are restrictions on the values taken by the control variables, condition (14.1.17) becomes:

$$\max_{\{\mathbf{u}\in\Omega\}} H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) \quad \text{for all} \quad t, \qquad t_0 \le t \le t_1, \tag{14.1.20}$$

i.e., the Hamiltonian function is maximized at each point of time along the optimal trajectory by choice of the control variables.[3] Thus, at any time t in the relevant interval there is either an interior solution at which:

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0}, \tag{14.1.21}$$

as in classical programming, or a boundary solution at which

$$\frac{\partial H}{\partial \mathbf{n}} \geq \mathbf{0}, \tag{14.1.22}$$

where $\mathbf{n}$ is an outward pointing normal on the boundary of Ω, as in non-linear programming. These possibilities are illustrated in the scalar case ($r = 1$) in Fig. 14.1.

Necessary conditions (14.1.18) and (14.1.19) are differential equations and boundary conditions respectively for the costate variables. The differential equations require that the time rate of change of each costate variable is

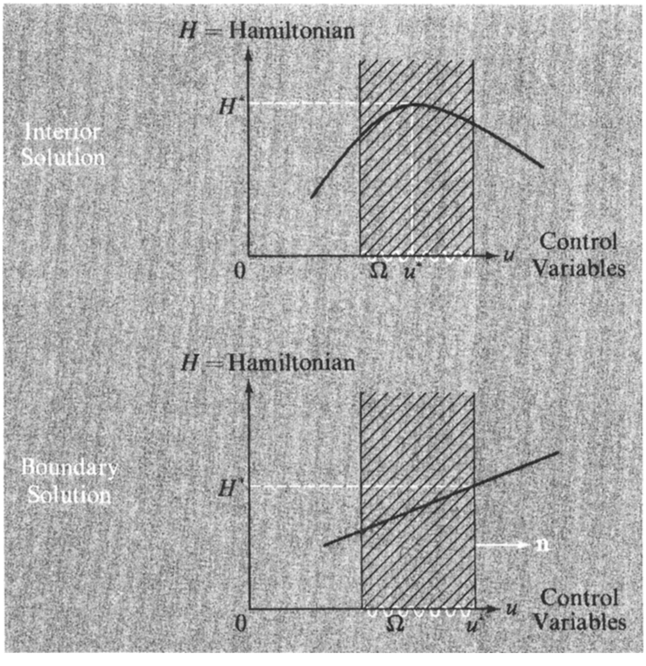

Fig. 14.1

The Maximum Principle in the Scalar Case ($r = 1$) at a Given Time $t(t_0 \leq t \leq t_1)$

the negative of the partial derivative of the Hamiltonian function with respect to the corresponding state variable, and the boundary conditions state that the terminal value of each costate variable is the partial derivative of the final function with respect to the corresponding state variable.

The differential equations for the state variables, i.e., the equations of motion, can be expressed, in terms of the Hamiltonian, as:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}. \tag{14.1.23}$$

These differential equations for the state variables and the differential equations for the costate variables plus all boundary conditions are called the *canonical equations:*

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial H}{\partial \mathbf{y}}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \dot{\mathbf{y}} &= -\frac{\partial H}{\partial \mathbf{x}}, \qquad \mathbf{y}(t_1) = \frac{\partial F}{\partial \mathbf{x}_1}, \end{aligned} \tag{14.1.24}$$

a set of $2n$ differential equations of which half have boundary conditions at initial time and n have boundary conditions at terminal time.

Consider now the change in the Hamiltonian over time. Since $H = H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t)$:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial H}{\partial \mathbf{u}}\dot{\mathbf{u}} + \dot{\mathbf{y}}\frac{\partial H}{\partial \mathbf{y}} + \frac{\partial H}{\partial t}, \tag{14.1.25}$$

using the equations of motion and collecting terms:

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\mathbf{y}}\right)\mathbf{f}(\mathbf{x}, \mathbf{u}, t) + \left(\frac{\partial H}{\partial \mathbf{u}}\right)\dot{\mathbf{u}} + \frac{\partial H}{\partial t}. \tag{14.1.26}$$

Along the optimal trajectory the first term vanishes because of the differential equation for the costate variable. The second term vanishes because either the partial derivative vanishes for an interior solution or $\dot{\mathbf{u}}$ vanishes for a boundary solution. Thus, along the optimal trajectory:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{14.1.27}$$

In particular, if the problem is autonomous in that both $I(\cdots)$ and $\mathbf{f}(\cdots)$ show no explicit dependence on time then the Hamiltonian shows no explicit dependence on time and, since $dH/dt = 0$, along the optimal trajectory the value of the Hamiltonian is constant over time.

To summarize, the maximum principle technique involves adding to the problem n costate variables $\mathbf{y}(t)$, defining the Hamiltonian function as:

$$H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) = I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{14.1.28}$$

and solving for trajectories $\{\mathbf{u}(t)\}$, $\{\mathbf{y}(t)\}$, and $\{\mathbf{x}(t)\}$ satisfying.[4]

$$\begin{gathered} \max_{\{\mathbf{u}\in\Omega\}} H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) \quad \text{for all} \quad t, \quad t_0 \le t \le t_1 \\ \dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}}, \qquad \mathbf{y}(t_1) = \frac{\partial F}{\partial \mathbf{x}_1}. \end{gathered} \tag{14.1.29}$$

These conditions are necessary for a local maximum.[5] The form of the solution for the optimal control often follows readily from the maximization of the Hamiltonian, which usually gives the optimal control variables not as functions of time but rather as functions of the costate variables. To then obtain the control variables as functions of time then requires the time paths of the costate variables, which entails solving a two point boundary value problem—the canonical equations—$2n$ differential equations of which n have *initial* boundary conditions (those for the state variables) and n have *terminal* boundary conditions (those for the costate variables).

14.2 The Interpretation of the Costate Variables

The maximum principle, as already seen, can be considered a dynamic generalization of the method of Lagrange multipliers and, just as the Lagrange multipliers of static problems yield information on the sensitivity of the solution, the costate variables of the maximum principle yield information on the sensitivity of the solution to variations in parameters.

The Lagrangian defined above in (14.1.4) equals the optimal value of the objective function, when evaluated at the solution $\{\mathbf{u}^*(t)\}$, $\{\mathbf{y}^*(t)\}$, and $\{\mathbf{x}^*(t)\}$. Thus, from (14.1.14):

$$\begin{aligned} J^* = \int_{t_0}^{t_1} \{H(\mathbf{x}^*, \mathbf{u}^*, \mathbf{y}^*, t) + \dot{\mathbf{y}}^*\mathbf{x}^*\}\, dt \\ + F(\mathbf{x}_1^*, t_1) - [\mathbf{y}^*(t_1)\mathbf{x}^*(t_1) - \mathbf{y}^*(t_0)\mathbf{x}(t_0)]. \end{aligned} \tag{14.2.1}$$

The sensitivities of the solution to changes in parameters, namely the parameters t_0, t_1, and $\mathbf{x}(t_0)$, are indicated by the partial derivatives of J^* with respect to these variables.

The sensitivity of the optimal value of the objective functional to a change in the initial time t_0 is given by:

$$\begin{aligned}\frac{\partial J^*}{\partial t_0} &= -[H^* + \dot{\mathbf{y}}^*\mathbf{x}^*]_{t_0} + [\mathbf{y}^*\dot{\mathbf{x}}^* + \dot{\mathbf{y}}^*\mathbf{x}^*]_{t_0} \\ &= -[H^* - \mathbf{y}^*\dot{\mathbf{x}}^*]_{t_0} \\ &= -[I(\mathbf{x}^*, \mathbf{u}^*, t)]_{t_0},\end{aligned} \tag{14.2.2}$$

that is, by the negative of the initial value of the intermediate function. Shifting the initial time, therefore, reduces J^* by the portion of the intermediate function lost due to the change in initial time.

The sensitivity of J^* to changes in the terminal time, t_1, is given by:

$$\begin{aligned}\frac{\partial J^*}{\partial t_1} &= [H^* + \dot{\mathbf{y}}^*\mathbf{x}^*]_{t_1} + \frac{\partial F}{\partial \mathbf{x}(t_1)}\frac{d\mathbf{x}^*(t_1)}{dt_1} + \frac{\partial F}{\partial t_1} - [\dot{\mathbf{y}}^*\mathbf{x}^* + \mathbf{y}^*\dot{\mathbf{x}}^*]_{t_1} \\ &= [I(\mathbf{x}^*, \mathbf{u}^*, t)]_{t_1} + \frac{\partial F}{\partial \mathbf{x}(t_1)}\frac{d\mathbf{x}^*(t_1)}{dt_1} + \frac{\partial F}{\partial t_1}(\mathbf{x}^*(t_1),t_1)\end{aligned}$$

that is, by the terminal value of the intermediate function plus the increase in the final function.

The sensitivities of the optimal value of the objective functional to changes in the initial state $\mathbf{x}(t_0)$ are given by:

$$\frac{\partial J^*}{\partial \mathbf{x}(t_0)} = \mathbf{y}^*(t_0), \tag{14.2.4}$$

that is, by the initial value of the corresponding optimal costate variable. If, in particular, one of the initial costate variables vanishes then the solution is insensitive to small changes in the initial value of the corresponding state variable. This result indicates the interpretation of the initial costate variables as the changes in the optimal value of the objective functional due to changes in the corresponding initial state variables. To the extent that the objective functional has the dimension of an economic value, i.e., price times quantity, such as revenue, cost, or profit, and the state variable has the dimension of an economic quantity, then the costate variable has the dimension of a price—*a shadow price*. Thus, to any dynamic economizing problem of allocation over time there corresponds a dual problem of valuation over time, namely, the problem of determining time paths for the costate variables. This interpretation of the costate variables is obviously the dynamic analogue to the interpretation of the Lagrange multipliers of static economizing problems.

14.3 The Maximum Principle and the Calculus of Variations

The necessary conditions of the classical calculus of variations can be derived from the maximum principle.[6] In the classical calculus of variations problem the control variables are the rates of change of the state variables and the control variables are unrestricted in value:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{u} \\ \Omega &= E^r. \end{aligned} \tag{14.3.1}$$

The Hamiltonian is

$$H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t) = I(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{y}\dot{\mathbf{x}}, \tag{14.3.2}$$

and maximizing the Hamiltonian by choice of $\dot{\mathbf{x}}$ requires, as a first order necessary condition, that:

$$\frac{\partial H}{\partial \dot{\mathbf{x}}} = \frac{\partial I}{\partial \dot{\mathbf{x}}} + \mathbf{y} = \mathbf{0}, \tag{14.3.3}$$

so that:

$$\mathbf{y} = -\frac{\partial I}{\partial \dot{\mathbf{x}}}. \tag{14.3.4}$$

Differentiating with respect to time:

$$\dot{\mathbf{y}} = -\frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\mathbf{x}}}\right), \tag{14.3.5}$$

but, by the canonical equation for the costate variables:

$$\dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial I}{\partial \mathbf{x}}. \tag{14.3.6}$$

Combining (14.3.5) and (14.3.6) yields the Euler equation of the calculus of variations:

$$\frac{\partial I}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\mathbf{x}}}\right) = \mathbf{0}. \tag{14.3.7}$$

The second order necessary condition for the maximization of the Hamiltonian is the condition on the Hessian matrix of second order partial derivatives of the Hamiltonian function:

$$\left(\frac{\partial^2 H}{\partial \dot{\mathbf{x}}^2}\right) \text{ negative definite or negative semidefinite,} \tag{14.3.8}$$

which yields the Legendre condition:

$$\left(\frac{\partial^2 I}{\partial \dot{\mathbf{x}}^2}\right) \text{ negative definite or negative semidefinite.} \tag{14.3.9}$$

Again by the maximum principle, if $\mathbf{u} = \dot{\mathbf{x}}$ is the optimal control then for any other control $\dot{\mathbf{z}}$:

$$H(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{y}, t) \geq H(\mathbf{x}, \dot{\mathbf{z}}, \mathbf{y}, t), \tag{14.3.10}$$

so that, by (14.3.2):

$$I(\mathbf{x}, \dot{\mathbf{x}}, t) + \mathbf{y}\dot{\mathbf{x}} \geq I(\mathbf{x}, \dot{\mathbf{z}}, t) + \mathbf{y}\dot{\mathbf{z}}. \tag{14.3.11}$$

Using (14.3.4) and rearranging yields the Weierstrass condition:

$$E(\mathbf{x}, \dot{\mathbf{x}}, t, \dot{\mathbf{z}}) = I(\mathbf{x}, \dot{\mathbf{z}}, t) - I(\mathbf{x}, \dot{\mathbf{x}}, t) - \frac{\partial I}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, t)(\dot{\mathbf{z}} - \dot{\mathbf{x}}) \leq 0. \tag{14.3.12}$$

Finally, according to the maximum principle both $\mathbf{y}$ and H are continuous functions of time. But:

$$\begin{aligned} \mathbf{y} &= -\frac{\partial I}{\partial \dot{\mathbf{x}}} \\ H &= I - \frac{\partial I}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}}, \end{aligned} \tag{14.3.13}$$

so that $\partial I/\partial \dot{\mathbf{x}}$ and $I - (\partial I/\partial \dot{\mathbf{x}})\dot{\mathbf{x}}$ are continuous functions of time, yielding the Weierstrass-Erdmann corner conditions:

$$\frac{\partial I}{\partial \dot{\mathbf{x}}} \text{ and } I - \frac{\partial I}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}} \text{ continuous across corners.} \tag{14.3.14}$$

Thus the necessary conditions of the calculus of variations have been derived from the maximum principle. Special cases of the calculus of variations can also be readily treated using the maximum principle. For example, if the intermediate function $I(\cdots)$ does not depend explicitly on time the problem is autonomous, in which case, by (14.1.27) the Hamiltonian is constant along the optimal path, so:

$$H = I - \frac{\partial I}{\partial \dot{\mathbf{x}}} \dot{\mathbf{x}} = \text{constant}, \tag{14.3.15}$$

which is the condition obtained in Chapter 12 (12.1.16) for this case.

14.4 The Maximum Principle and Dynamic Programming

The maximum principle and dynamic programming approaches both apply to the same type of general control problem, so there are close relationships between the two approaches.[7]

In dynamic programming the optimal performance function $J(\mathbf{x}, t)$ is defined as the optimal value of the objective functional for the problem beginning at initial state $\mathbf{x}$ and initial time t, and the approach requires the solution to the fundamental partial differential equation—Bellman's equation:

$$-\frac{\partial J^*}{\partial t} = \max_{\{\mathbf{u}\}} \left[I(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right]. \tag{14.4.1}$$

The relationship between this approach and that of the maximum principle is based on equation (14.2.4), which states that the change in the optimal value of the objective functional with respect to the initial state is the initial value of the costate variable. In terms of the optimal performance function:

$$\frac{\partial J^*}{\partial \mathbf{x}} = \mathbf{y}. \tag{14.4.2}$$

The expression in square brackets in Bellman's equation is therefore the Hamiltonian function:

$$I(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}, t) = H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t), \tag{14.4.3}$$

and (14.4.1) can be written:

$$-\frac{\partial J^*}{\partial t} = \max_{\{\mathbf{u}\}} [H(\mathbf{x}, \mathbf{u}, \mathbf{y}, t)]. \tag{14.4.4}$$

The maximization called for in this equation is that of maximizing the Hamiltonian by choice of control variables within the control set, which is, of course, the maximum principle itself, Assuming $\mathbf{u}$ is the control maximizing the Hamiltonian:

$$-\frac{\partial J^*}{\partial t} = H\left(\mathbf{x}, \mathbf{u}, \frac{\partial J^*}{\partial \mathbf{x}}, t\right), \tag{14.4.5}$$

an equation called the *Hamiltonian-Jacobi equation*. Taking a derivative with respect to $\mathbf{x}$:

$$-\frac{\partial^2 J^*}{\partial \mathbf{x}\,\partial t} = \frac{\partial H}{\partial \mathbf{x}} + \left(\frac{\partial H}{\partial \mathbf{y}}\right)' \frac{\partial^2 J^*}{\partial \mathbf{x}^2}. \tag{14.4.6}$$

Differentiating (14.4.2), however:

$$\dot{\mathbf{y}} = (\dot{\mathbf{x}})' \frac{\partial^2 J^*}{\partial \mathbf{x}^2} + \frac{\partial^2 J^*}{\partial t\,\partial \mathbf{x}}. \tag{14.4.7}$$

Combining the last two equations and using the equality of the second order mixed partial derivatives (since $J^*(\mathbf{x}, t)$ is assumed continuously differentiable in dynamic programming) yields the canonical equations of the maximum principle:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}} \qquad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{14.4.8}$$

Finally, the terminal boundary condition on Bellman's equation implies the terminal boundary condition on the costate variables, since:

$$J^*(\mathbf{x}_1, t_1) = F(\mathbf{x}_1, t_1) \tag{14.4.9}$$

implies that:

$$\mathbf{y}(t_1) = \frac{\partial J^*}{\partial \mathbf{x}}(\mathbf{x}_1, t_1) = \frac{\partial F}{\partial \mathbf{x}_1}. \tag{14.4.10}$$

The dynamic programming conditions, namely Bellman's equation and its boundary condition, therefore imply the maximum principle conditions. The maximum principle does not, however, imply Bellman's equation since the maximum principle does not require the assumption basic to dynamic programming that the optimal performance function be continuously differentiable. In addition, as far as computing optimal controls the two methods represent two very different approaches to the dynamic economizing problem: dynamic programming leads to a nonlinear partial differential equation, while the maximum principle leads to two sets of ordinary differential equations. The maximum principle is often a more fruitful method of approach because it, in essence, breaks up the solution of Bellman's equation into two steps, the first step being that of solving for the optimal controls as functions of the costate variables and the second step being that of solving for the time paths of the costate variables. The first step can generally be easily taken, and it often yields insight into the nature of the solution, allowing for solution by other means. The second step is more difficult, involving the solution to a two-point boundary value problem. On the other hand, dynamic programming requires that both steps be taken together—via solving Bellman's equation. For an analytic solution, therefore, the maximum principle approach is generally more useful than the dynamic programming approach. For numerical solutions, however, both methods lead to similar computer programs and similar problems on storage capacity ("curse of dimensionality"), dynamic programming requiring an approximate solution to a nonlinear partial differential equation and the maximum principle requiring an approximate solution to a two-point boundary value problem.[8]

14.5 Examples

Some examples will now be given to illustrate the maximum principle approach to control problems. As a first example, consider the linear time optimal problem of transferring state variables from given initial values to specified terminal values in minimum time, where the equations of motion are linear and autonomous. For simplicity, only a single control variable ($r = 1$) is treated, and this control variable is constrained to take values between -1 and $+1$. The problem is then:

$$\max_{\{u(t)\}} J = -\int_{t_0}^{t_1} dt = -(t_1 - t_0)$$

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ &t_0 \quad \text{and} \quad \mathbf{x}(t_0) \quad \text{given} \\ &\mathbf{x}(t_1) \quad \text{given} \\ &-1 \le u(t) \le 1 \quad \text{and} \quad u(t) \quad \text{piecewise continuous.} \end{aligned} \tag{14.5.1}$$

The Hamiltonian is:

$$H = -1 + \mathbf{y}(\mathbf{A}\mathbf{x} + \mathbf{b}u), \tag{14.5.2}$$

which is linear in the control variable. By the maximum principle, the optimal control is:

$$u^* = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad \text{if} \quad \mathbf{yb} \begin{Bmatrix} > \\ < \end{Bmatrix} 0, \tag{14.5.3}$$

or, in terms of the signum function defined as:

$$\operatorname{sgn} z = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad \text{if} \quad z \begin{Bmatrix} > \\ < \end{Bmatrix} 0, \tag{14.5.4}$$

the optimal control is:[9]

$$u^* = \operatorname{sgn}(\mathbf{yb}). \tag{14.5.5}$$

The optimal control therefore always lies at any one time on a boundary of the control set but, over time, can switch from one boundary point to the other. Such a solution is known as *bang-bang control*, and the fact that the solution to problem (14.5.1) is the same as the solution to the problem in which the control variable is restricted to only the two values $+1$ and -1 is called the *bang-bang principle*.[10] The function **yb** is known as the *switching function* since the optimal control switches between the two values $+1$ and -1 when **yb** changes sign. The time path of the costate variable, which gives the time path of the switching function, is characterized by the differential equations:

$$\dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{yA}. \tag{14.5.6}$$

If the characteristic roots of the $n \times n$ matrix **A** are real distinct and negative then an optimal control exists for which at most $n - 1$ switches in sign are needed, i.e., the time interval $t_0 \leq t \leq t_1$ can be divided into n subintervals in each of which the optimal control takes either the maximum value ($u^* = 1$) or the minimum value ($u^* = -1$).[11]

As a special case of the first example, consider the problem of minimum time in which the control variable is the second derivative of the (single) state variable:

$$u = \ddot{x}_1 = \frac{d^2 x_1}{dt^2}. \tag{14.5.7}$$

To give a physical example of this special case, u can be considered the force applied to a unit mass where x_1 is a measure of the distance of the mass from a given point, equation (14.5.7) stating that force (u) equals mass (1) times acceleration ($\ddot{x}_1$). Since the formulation of the general control problem entails only first derivatives it is convenient to represent (14.5.7) by the two equations of motion:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned} \tag{14.5.8}$$

or, in terms of the general linear equations of motion in (14.5.1):

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{14.5.9}$$

The problem will be assumed to be that of driving the state variables from given initial values $(x_1(t_0), x_2(t_0))'$ to the origin $(0, 0)'$ in minimum time. The Hamiltonian is:

$$H = -1 + y_1 x_2 + y_2 u, \tag{14.5.10}$$

so, by the maximum principle:

$$u^* = \operatorname{sgn}(y_2). \tag{14.5.11}$$

The differential equations for the costate variables are:

$$\begin{aligned} \dot{y}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{y}_2 &= -\frac{\partial H}{\partial x_2} = -y_1, \end{aligned} \tag{14.5.12}$$

implying that:

$$\begin{aligned} y_1 &= c_1 \\ y_2 &= -c_1 t + c_2, \end{aligned} \tag{14.5.13}$$

where c_1 and c_2 are constants, determined from the initial conditions. Since y_2 can change sign at most once, the optimal solution requires at most one switch in the control variable, a result consistent with the above general principle on the maximum number of switches necessary.

An elegant way of illustrating the solution to this problem is via the phase plane for the variables x_1 and $x_2 = \dot{x}_1$. By the bang-bang principle only $u = 1$ and $u = -1$ need be considered. If $u = 1$ then the equations of motion imply:

$$x_1 = \frac{1}{2}x_2^2 + c, \qquad c = \text{constant}, \tag{14.5.14}$$

and if $u = -1$ then they imply:

$$x_1 = -\frac{1}{2}x_2^2 + c, \qquad c = \text{constant}. \tag{14.5.15}$$

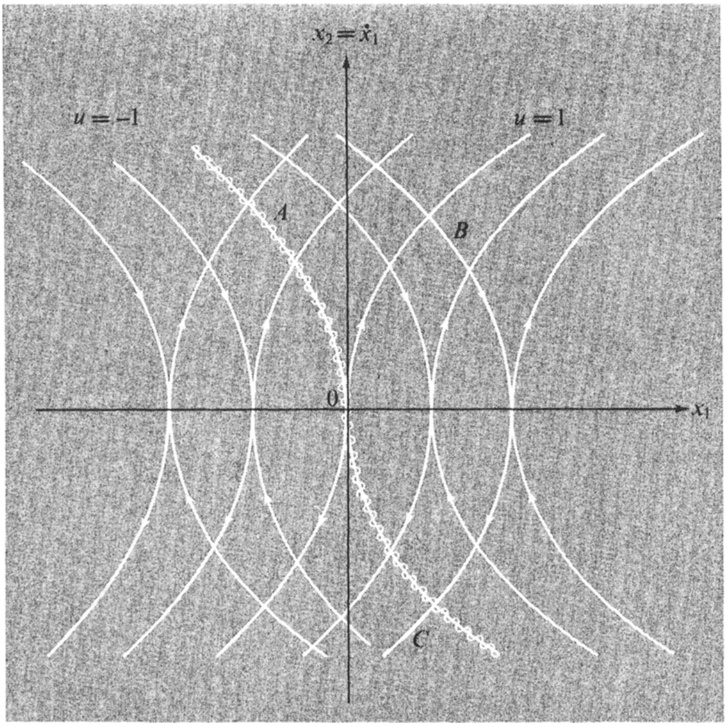

Fig. 14.2

Phase Plane Solution to the Problem of Minimum Time where the Control Is the Second Derivative of the State Variable

A few of these curves are shown in Fig. 14.2, those with arrows pointing up for $u = 1$ (in which case $x_2 = \dot{x}_1$ increases), and those with arrows pointing down for $u = 1$ (in which case $x_2 = \dot{x}_1$ decreases). The optimal trajectory for moving the state variables from any point in the plane to any other point in the plane, in particular the origin, involves moving along one or two of these curves. All initial points on the heavy shaded curve require no switch; and all those elsewhere require one switch in the optimal control. For example, moving from point A to the origin requires no switch ($u = -1$) while moving from point B to the origin requires one switch—at C—from $u = -1$ to $u = +1$.

As a second example of the maximum principle, consider the minimum effort servomechanism where the equations of motion are linear and autonomous:

$$\max_{\{\mathbf{u}(t)\}} J = \frac{1}{2}\int_{t_0}^{t_1} (\mathbf{x}'\mathbf{D}\mathbf{x} + \mathbf{u}'\mathbf{E}\mathbf{u})\,dt + \tfrac{1}{2}\mathbf{x}_1'\mathbf{F}\mathbf{x}_1$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{14.5.16}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$$\mathbf{x}(t_1) = \mathbf{x}_1,$$

where $\mathbf{D}$ and $\mathbf{F}$ are given negative definite matrices of order n, the matrix $\mathbf{E}$ is a given negative definite of order r, and $\mathbf{A}$ and $\mathbf{B}$ are given matrices of size $n \times n$ and $n \times r$ respectively. It will be assumed here that $\mathbf{u}$ can take any values, i.e., $\Omega = E^r$.

The Hamiltonian is:

$$H = \tfrac{1}{2}(\mathbf{x}'\mathbf{D}\mathbf{x} + \mathbf{u}'\mathbf{E}\mathbf{u}) + \mathbf{y}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}), \tag{14.5.17}$$

and, by the maximum principle:

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{u}'\mathbf{E} + \mathbf{y}\mathbf{B} = \mathbf{0}, \tag{14.5.18}$$

so that the solution for the optimal control is

$$\mathbf{u}^* = -\mathbf{E}^{-1}\mathbf{B}'\mathbf{y}', \tag{14.5.19}$$

a linear function of the costate variables. The canonical equations are:

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial H}{\partial \mathbf{y}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{E}^{-1}\mathbf{B}'\mathbf{y}', \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \dot{\mathbf{y}} &= -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{x}'\mathbf{D} - \mathbf{y}\mathbf{A}, \qquad \mathbf{y}(t_1) = \mathbf{x}_1'\mathbf{F}. \end{aligned} \tag{14.5.20}$$

Assuming a linear solution of the form:

$$\mathbf{y} = \mathbf{x}'\mathbf{Q}(t), \tag{14.5.21}$$

where $\mathbf{Q}$ is an $n \times n$ matrix with elements varying over time, leads to the matrix Ricatti equation for $\mathbf{Q}(t)$:

$$\dot{\mathbf{Q}} - \mathbf{Q}\mathbf{B}\mathbf{E}^{-1}\mathbf{B}'\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{A}'\mathbf{Q} + \mathbf{D} = \mathbf{0} \tag{14.5.22}$$

with the boundary condition:

$$\mathbf{Q}(t_1) = \mathbf{F}. \tag{14.5.23}$$

The optimal closed loop control is then:

$$\mathbf{u}^*(t) = -\mathbf{E}^{-1}\mathbf{B}'\mathbf{Q}'(t)\mathbf{x}(t). \tag{14.5.24}$$

Thus, for a minimum effort servomechanism with linear autonomous equations of motion the optimal controls are linear functions of the state variables. This result is a dynamic extension of the linear decision rule for programming problems with quadratic objective functions and linear constraints.

PROBLEMS

14-A. Using the approach of Sec. 14.1, prove that in problems with a terminal surface:

$$\mathbf{T}(\mathbf{x}(t), t) = \mathbf{0} \quad \text{at} \quad t = t_1,$$

the maximum principle transversality condition is:

$$\left(\mathbf{H}\Big|_{t_1} + \frac{\partial F}{\partial t_1}\right) + \left(\frac{\partial F}{\partial \mathbf{x}_1} - \mathbf{y}\right)\left(\frac{d\mathbf{x}}{dt}\right)\Big|_{T(\cdot\cdot)=0} = 0.$$

14-B. Using the maximum principle show that the Euler equation for the calculus of variations problem with an explicit control variable in the case of one state variable is:

$$\left(\frac{\partial I}{\partial x} - \frac{\partial I/\partial u}{\partial f/\partial u}\frac{\partial f}{\partial x}\right) - \frac{d}{dt}\left(\frac{\partial I/\partial u}{\partial f/\partial u}\right) = 0.$$

14-C. Show that the optimal controls as obtained from the maximum principle satisfy the Principle of Optimality: if $\{\mathbf{u}^*(t)\}$ is an optimal control and $\{\mathbf{x}^*(t)\}$ is the corresponding optimal trajectory for $t_0 \le t \le t_1$ where $\mathbf{x}(t_0) = \mathbf{x}_0$ then $\{\mathbf{u}^*(t)\}$ for $\tau \le t \le t_1$ is an optimal control for the problem beginning at time τ and state $\mathbf{x}^*(\tau)$.

14-D. In the following control problem x is a single state variable and u is a single control variable:

$$\max J = \int_0^2 (2x - 3u - \alpha u^2)\, dt$$

$$\dot{x} = x + u$$

$$x(0) = 5$$

$$0 \le u \le 2.$$

Using the maximum principle, solve for the optimal control if $\alpha = 0$ and also if $\alpha = 1$.

14-E. Using the maximum principle, solve the following control problem:

$$\max J = \int_0^1 (x_1^2 - u^2)\, dt$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$x_1(0) = x_2(0) = 1$$

$$x_1(1) = x_2(1) = 0.$$

14-F. Solve the following problem of Mayer using the maximum principle:

$$\max 8x_1(18) + 4x_2(18)$$

$$\dot{x}_1 = 2x_1 + x_2 + u$$

$$\dot{x}_2 = 4x_1 - 2u$$

$$x_1(0) = x_{10}$$

$$x_2(0) = x_{20}$$

$$0 \le u \le 1.$$

14-G. In the minimum time problem for which the control is the second derivative of the state variable, show that the time required to move from $(x_1, x_2)'$ to the origin is:

$$\left\{ \begin{matrix} x_2 + \sqrt{4x_1 + 2x_2^2} \\ -x_2 + \sqrt{-4x_1 + 2x_2^2} \\ |x_2| \end{matrix} \right\} \quad \text{if} \quad x_1 \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} - \tfrac{1}{2} x_2 \, |x_2|.$$

14-H. Solve the time optimal control problems of reaching the origin in minimum time in which the equation(s) of motion and control set are:

1. $\ddot{x} + 2b\dot{x} + x = u$

$$|u| \le 1$$

2.
$$\dot{x}_1 = u_1 x_2$$

$$\dot{x}_2 = u_2$$

$$|u_1| \le 1, \qquad |u_2| \le 1$$

3. $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{u}$$

$$\sum_{j=1}^{r} u_j^2 \leq 1$$

14-I. Solve:

$$\max_{\{u(t)\}} J = -\int_{t_0}^{t_1} |u| \, dt$$

$$\ddot{x} = u$$

$$x(t_0) = x_0$$

$$x(t_1) = x_1$$

$$|u| \leq 1.$$

(The solution is known as "bang-bang with coasting.")

14-J. Solve:

$$\max J = -\int_0^1 u^2 \, dt$$

$$\dot{x} = x + u$$

$$x(0) = 1$$

$$x(1) = 0,$$

and show that the optimal control varies exponentially over time.

14-K. Using the maximum principle, prove that the shortest distance from a given point to a given line is along a straight line perpendicular to the given line.

14-L. The Speedrail Company is building an ultra high speed train to convey passengers between Boston and Washington, a flat distance of 400 miles.

1. What is the shortest possible duration of the trip if the only constraint is that the maximum acceptable level of acceleration is $2g$, where g, the acceleration due to gravity, is 32 feet/sec^2?
2. What is the shortest possible duration of the trip if, in addition to the acceleration constraint, there is also the constraint that velocity cannot exceed 360 miles/hour (= 528 feet/sec)?

14-M. Find the path which minimizes the time required to climb a rotationally symmetric mountain of height h using a car with velocity v dependent on the angle of inclination α, where $v(0) = v_0$; $v(\pi/2) = 0$; and $v(\alpha)$ and $dv/d\alpha$ are monotonically decreasing functions.[12]

14-N. A boat moves with constant unit velocity in a stream moving at constant velocity s. The problem is that of finding the optimal steering angle which minimizes the time required to move between two given points. If x_1 and x_2 are the positions of the boat parallel to and perpendicular to the stream velocity, respectively, and θ is the steering angle, the equations of motion are:

$$\dot{x}_1 = s + \cos\theta$$
$$\dot{x}_2 = \sin\theta.$$

Find the optimal steering program.[13]

14-O. Suppose that in a country at time t there are $S(t)$ scientists engaged in either teaching or research. The number of teaching scientists (educators) is $E(t)$, and the number of research scientists (researchers) is $R(t)$, where:

$$S(t) = E(t) + R(t).$$

New scientists are produced by educators where it takes $1/\gamma$ educators to produce a new scientist in one year. Scientists leave the field of science due to death, retirement, and transfer at the rate δ per year. Thus:

$$\dot{S}(t) = \gamma E(t) - \delta S(t).$$

(For the U.S. currently the parameters have been estimated as: $\gamma = .14$, $\delta = .02$). By means of various incentives a science policy maker can influence the proportion of new scientists entering teaching, $\alpha(t)$, where:

$$\dot{E}(t) = \alpha(t)\,\gamma E(t) - \delta E(t)$$
$$\dot{R}(t) = (1 - \alpha(t))\gamma E(t) - \delta R(t)$$
$$0 < \bar{\alpha} \le \alpha(t) \le \bar{\bar{\alpha}} < 1.$$

Find the optimal allocation policy if the objective is to minimize the time required to attain given numbers of teaching and research scientists.[14]

14-P. Find the advertising policy which maximizes sales over a period of time where the rate of change of sales decreases at a rate proportional to sales but increases at a rate proportional to the rate of advertising as applied to the share of the market not already purchasing the product. The problem is:

$$\max_{\{A(t)\}} \int_{t_0}^{t_1} S(t)\,dt$$
$$\dot{S} = -aS + bA\left[1 - \frac{S}{M}\right]$$
$$S(t_0) = S_0$$
$$0 \le A(t) \le \bar{A}$$

where S is sales; A is advertising; M is the extent of the market; and t_0, t_1, a, b, S_0, and $\bar{A}$ are given positive parameters.[15]

14-Q. In the last problem suppose the effect of advertising on sales cumulates over time, so:

$$\dot{S} = -aS + b\int_0^\infty A(t-\tau)e^{-\tau}\,d\tau.$$

Show that this equation can be written as a second order differential equation using the change of variable $X = t - \tau$. Solve the problem by rewriting the second order equation as two first order equations and using the maximum principle.[16]

FOOTNOTES

[1] The basic references for the maximum principle are Pontryagin et al. (1962), Athans and Falb (1966), Hestenes (1966), Leitmann (1966) and Lee and Markus (1967).

[2] There is no standard name or notation for the costate variables. Other names are "multipliers," "auxiliary variables," "adjoint variables," and "dual variables." Other notation is $\mathbf{\Psi}$, $\mathbf{z}$, $\boldsymbol{\lambda}$, and $\mathbf{p}$. The notation here, $\mathbf{y}$, is chosen to conform to that used in the static theory developed in Chapters. 2–6.

[3] It is assumed that the $r \times r$ Hessian matrix $\partial^2 H/\partial \mathbf{u}^2$ is negative definite or negative semidefinite at each time in the relevant interval.

[4] This statement of the maximum principle is based on certain regularity assumptions that are analogous to the contraint qualification assumptions of nonlinear programming problems (see p. 57). Without these assumptions one must assign a nonnegative costate variable y_0 to the intermediate function, so that the Hamiltonian is:

$$H' = y_0 I(\mathbf{x}, \mathbf{u}, t) + \mathbf{y}\mathbf{f}(\mathbf{x}, \mathbf{u}, t).$$

Under the regularity assumptions y_0 is necessarily positive at the solution, so the set of all $n + 1$ costate variables can be normalized by setting y_0 equal to unity, in which case H' reduces to H. Without the further assumptions y_0 can vanish at the solution, a case analogous to a solution at a cusp in nonlinear programming problems not satisfying the constraint qualification condition.

[5] The maximum principle conditions are, in general, not sufficient, nor do they necessarily yield a unique solution or a global maximum. The conditions are, however, necessary and sufficient if the Hamiltonian is linear in the control variables [Rozonoer (1959)] or if the maximized Hamiltonian is a concave function of the state variables [Mangasarian (1966)].

[6] See Berkovitz (1961), Kalman (1963) and Hestenes (1966).

[7] See Desoer (1961) and Feldbaum (1965).

[8] On the numerical solution to two point boundary value problems see Balakrishnan and Neustadt, eds. (1964).

[9] Note that u^* is not defined at points where $\mathbf{yb} = 0$, and the problem is *singular* if this condition persists over a finite interval of time. See Athans and Falb (1966) and Kelley, Kopp, and Moyer (1967).

[10] See Bellman, Glicksberg, and Gross (1956, 1958); LaSalle (1961); and Halkin (1965). The bang-bang principle is important in engineering applications where it is typically less expensive to provide the capability of obtaining the extremes than to provide the capability of obtaining the extremes plus all intermediate values. The home thermostat is an example, where a device turning the furnace on or off is less expensive than a device regulating the intensity of the furnace.

[11] See Bellman, Glicksberg, and Gross (1956, 1958), Bushaw (1958); LaSalle (1959, 1960, 1961); and Feldbaum (1965). Note that if the characteristic roots of A are real and negative then the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

is stable but the system:

$$\dot{\mathbf{y}} = -\mathbf{A}\mathbf{y}$$

is then *unstable* since the characteristic roots of $-\mathbf{A}$ are real and positive. This result, known as *dual instability* greatly increases the difficulty in solving the two point boundary value problem since small errors in y tend to be magnified if the costate differential equations are integrated forward from initial time while small errors in x tend to be magnified if the state differential equations (equations of motion) are integrated backward from terminal time. For a discussion of dual instability in relation to dynamic input-output systems in economics, where either the system for determining the outputs or that for determining the prices is unstable, see Solow (1959) and Jorgenson (1960).

[12] See Courant (1962).

[13] See Leitmann (1966).

[14] See Intriligator and Smith (1966).

[15] See Connors and Teichroew (1967).

[16] *Ibid.*

BIBLIOGRAPHY

Athans, M., and P. L. Falb, *Optimal Control.* New York: McGraw-Hill Book Company, Inc., 1966.

Balakrishnan, A. V., and L. W. Neustadt, eds., *Computing Methods in Optimization Problems.* New York: Academic Press Inc., 1964.

Bellman, R., ed., *Mathematical Optimization Techniques.* Berkeley and Los Angeles, Calif.: University of California Press, 1963.

Bellman, R., I. Glicksberg, and O. Gross, "On the 'Bang-Bang' Control Problem," *Quarterly of Applied Mathematics,* 14 (1956):11–18. Reprinted in *Optimal and Self-Optimizing Control,* ed. R. Oldenburger. Cambridge, Mass.: M.I.T. Press, 1966.

———, *Some Aspects of the Mathematical Theory of Control Processes,* R-313. Santa Monica, Calif.: Rand Corp., 1958.

Berkovitz, L., "Variational Methods in Problems of Control and Programming," *J. Math. Anal. and Appl.,* 3 (1961):145–69. Reprinted in *Optimal and Self-Optimizing Control,* ed. R. Oldenburger. Cambridge, Mass.: M.I.T. Press, 1966.

Bushaw, D. W., "Optimal Discontinuous Forcing Terms," in *Contributions to*

Nonlinear Oscillations, vol. 4, Annals of Mathematics Study No. 24, ed. S. Lefschetz. Princeton, N.J.: Princeton University Press, 1958.

Connors, M. M., and D. Teichroew, *Optimal Control of Dynamic Operations Research Models*. Scranton, Pa.: International Textbook Co., 1967.

Courant, R., *Calculus of Variations, with Supplementary Notes and Exercises*, revised and amended by J. Moser. New York: Courant Institute of Mathematical Sciences, New York University Press, 1962.

Desoer, C. A., "Pontryagin's Maximum Principle and the Principle of Optimality," *J. Franklin Institute*, 271 (1961):361–7.

Feldbaum, A. A., *Optimal Control Systems*. New York: Academic Press Inc., 1965.

Halkin, H., "A Generalization of LaSalle's 'Bang-Bang' Principle," *J. SIAM Control*, 2 (1965):199–203.

Hestenes, M. R., *Calculus of Variations and Optimal Control Theory*. New York: John Wiley & Sons, Inc., 1966.

Intriligator, M. D., and B. L. R. Smith, "Some Aspects of the Allocation of Scientific Effort Between Teaching and Research," *American Economic Review*, 61 (1966):494–507.

Jorgenson, D., "A Dual Stability Theorem," *Econometrica*, 28 (1960):892–9.

Kalman, R. E., "The Theory of Optimal Control and the Calculus of Variations," in *Mathematical Optimization Techniques*, ed. R. Bellman. Berkeley and Los Angeles, Calif.: University of California Press, 1963.

Kelley, H. J., R. E. Kopp, and H. G. Moyer, "Singular Extremals," in *Topics in Optimization*, ed. G. Leitmann. New York: Academic Press Inc., 1967.

LaSalle, J. P., "Time Optimal Control Systems," *Proc. Nat. Acad. Sci., USA*, 45 (1959):573–7.

———, "The Time Optimal Control Problem," in *Contributions to the Theory of Nonlinear Oscillations*, vol. V, Annals of Mathematics Study No. 45, eds. L. Cesari, J. LaSalle, and S. Lefschetz. Princeton, N.J.: Princeton University Press, 1960.

———, "The 'Bang-Bang' Principle," *Proc. of the IFAC, Moscow, 1960*. London: Butterworths, 1961. Reprinted in *Optimal and Self-Optimizing Control*, ed. R. Oldenburger. Cambridge, Mass.: M.I.T. Press, 1966.

Lee, E. B., and L. Markus, *Foundations of Optimal Control Theory*. New York: John Wiley & Sons, Inc., 1967.

Leitmann, G., *An Introduction to Optimal Control*. New York: McGraw-Hill Book Company, Inc., 1966.

Leitmann, G., ed., *Topics in Optimization*. New York: Academic Press Inc., 1967.

Mangasarian, O. L., "Sufficient Conditions for the Optimal Control of Nonlinear Systems," *J. SIAM Control*, 4 (1966):139–52.

Oldenburger, R., ed., *Optimal and Self-Optimizing Control*. Cambridge, Mass.: M.I.T. Press, 1966.

Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, *The Mathematical Theory of Optimal Processes*, trans. by K. N. Trirogoff. New York: Interscience Publishers, John Wiley & Sons, Inc., 1962.

Rozonoer, L. I., "L. S. Pontryagin's Maximum Principle in Optimal Control

Theory," *Automat. i. Telemekh.*, 20 (1959):1320–34, 1441–58, 1561–78. Translated in *Automation and Remote Control*, 20 (1960):1288–302, 1405–21, 1517–32. Reprinted in *Optimal and Self-Optimizing Control*, ed. R. Oldenburger. Cambridge, Mass.: M.I.T. Press, 1966.

Solow, R. M., "Competitive Valuation in a Dynamic Input-Output System," *Econometrica*, 27 (1959):30–53.