Axioms of Probability

Sample space (Ω): This is the set of all possible outcomes of some random experiment.

Event: A subset of Ω.

Ex: A coin is flipped twice

$$\Omega = \{HH, HT, TH, TT\}$$

$$A = \{\text{observing exactly one } H\}$$

$$= \{HT, TH\}.$$

We think of a collection of events as a collection of sets (denoted by $\mathcal{F}$).

Definition: A collection of sets $\mathcal{F}$ is called a σ-algebra if the following are satisfied:

i) $\phi \in \mathcal{F}$

ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

iii) If $A_1, A_2, \ldots \in \mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Examples: ① $\mathcal{F} = \{\phi, \Omega\}$

② $\mathcal{F} = \{\phi, A, A^c, \Omega\}$

De Morgan's laws

Consider a sequence of set $A_1, A_2, \ldots, A_n, \ldots \subset \Omega$.
Then

① $$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

② $$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$$

Proof : ① Let $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$

$$\Rightarrow x \notin \bigcup_{i=1}^{\infty} A_i$$

$$\Rightarrow x \notin A_i, \quad \forall i$$

$$\Rightarrow x \in A_i^c, \quad \forall i$$

$$\Rightarrow x \in \bigcap_{i=1}^{\infty} A_i^c$$

The converse also clearly holds $\square$

② Left as an exercise.

Problem : Let $\mathcal{F}$ be a σ- algebra. If $A, B \in \mathcal{F}$, then prove that $A \cap B \in \mathcal{F}$.

Proof : Suppose $A, B \in \mathcal{F}$.

Then $A^c, B^c \in \mathcal{F}$. Since $\mathcal{F}$ is closed under unions,

$$A^c \cup B^c \in \mathcal{F}$$

$$\Rightarrow (A^c \cup B^c)^c \in \mathcal{F}$$

$$\overset{\text{De Morgan}}{\Longrightarrow} A \cap B \in \mathcal{F}$$

$\square$

Axioms of probability.

Let $(\Omega, \mathcal{F})$ be a measurable space. A probability measure $P : \mathcal{F} \to [0,1]$ is a set function satisfying the following :

① $P(\Omega) = 1$

② If $A_1, A_2, \ldots \in \mathcal{F}$ is a sequence of mutually exclusive events $(A_i \cap A_j = \emptyset, \forall i \neq j)$, the

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$