① Let $P(H) = p$
$P(T) = q$

Define $R = \{$A run of 5 heads before a run of 5 tails$\}$

Let $H_1 = \{$first flip is a head$\}$.

Then,

$$P(R) = P(R|H_1)P(H_1) + P(R|T_1)P(T_1)$$

$$P(R) = p\,P(R|H_1) + q\,P(R|T_1) \quad \text{——— (1)}$$

Given H_1, we can obtain a run of 5 heads if the next 4 flips (flips 2-5) are all heads.

$$A = \{\text{flips } 2-5 \text{ are } Hs\}$$

Therefore,

$$P(R|H_1) = \underbrace{P(R|H_1 \cap A)}_{\substack{=1 \\ \text{(Sequence of} \\ \text{5H is completed)}}} P(A|H_1) + P(R|H_1 \cap A^c)P(A^c|H_1) \quad \text{——— (2)}$$

Also, $P(R|H_1 \cap A^c) = P(R|T_1)$ because if $H_1 \cap A^c$ occurs, a tail occurs at some point in the next 4 trials & we have to start over again. The situation would be exactly as if we started out with a T.

Also, due to independence,

$$P(A \mid H_1) = P(A) = p^4$$

(2) $\Rightarrow$ $P(R \mid H_1) = p^4 + P(R \mid T_1)(1-p^4)$ —— (3)

Now, to obtain a similar expression for $P(R \mid T_1)$, define

$$B = \{\text{2-5 are tails}\}$$

$$P(R \mid T_1) = \underbrace{P(R \mid T_1 \cap B)}_{=0}\underbrace{P(B \mid T_1)}_{q^4} + \underbrace{P(R \mid T_1 \cap B^c)}_{P(R \mid H_1)}\underbrace{P(B^c \mid T_1)}_{1-q^4}$$

$P(R \mid T_1) = (1-q^4)\, P(R \mid H_1)$ —— (4)

Now, solve (3), (4):

$$P(R \mid H_1) = p^4 + (1-q^4)\, P(R \mid H_1)(1-p^4)$$

$$\Rightarrow P(R \mid H_1) = \frac{p^4}{1-(1-p^4)(1-q^4)}$$

$$P(R \mid T_1) = \frac{p^4(1-q^4)}{1-(1-p^4)(1-q^4)}$$

(1)

$$\Rightarrow \boxed{P(R) = \frac{p^4 + p^4 q (1-q^4)}{1-(1-p^4)(1-q^4)}}$$

(2) We proved this in class.

See lecture 8 page 3

(3) Let $\{X_n\}$ be i.i.d such that $X_n \sim \text{Poisson}(1)$.

Then $E(X_n) = 1$, $V(X_n) = 1$,

Therefore,

$$E\left(\sum_{i=1}^{n} X_i\right) = n$$

$$V\left(\sum_{i=1}^{n} X_i\right) = n$$

& $\sum_{i=1}^{n} X_i \sim \text{Poisson}(n)$ (because of indepence)

By CLT,

$$\frac{\dfrac{\sum_{i=1}^{n} X_i}{n} - 1}{\dfrac{1}{\sqrt{n}}} \longrightarrow N(0,1)$$

or,

$$\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} \longrightarrow N(0,1).$$

Note that

$$P\left(\sum_{i=1}^{n} X_i \leq n\right) = \sum_{k=0}^{n} \frac{e^{-n} \cdot n^k}{k!}$$

By CLT,

$$P\left[\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} \leq \frac{n-n}{\sqrt{n}}\right] = P\left[\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} \leq 0\right]$$

$$\longrightarrow \Phi(0) = \frac{1}{2}$$

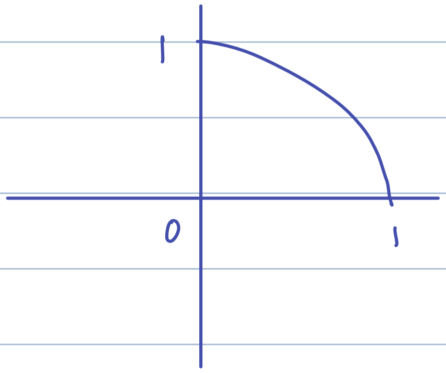
Therefore, $$\sum_{k=0}^{n} \frac{e^{-n} \cdot n^k}{k!} \longrightarrow \frac{1}{2}$$

(4) See lecture 25 page 2

(5)

$$f(x,y) = \begin{cases} c x^3 y \ ; & x, y \geq 0, \quad x^2+y^2 \leq 1 \\ 0 \ ; & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx$$



$$= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} c x^3 y \, dy \, dx$$

$$= \int_{x=0}^{x=1} c\,x^3(1-x^2)\,dx$$

$$= c\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = c\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{c}{12} \Rightarrow C = 12$$

$$f_Y(y) = \int_{x=0}^{x=\sqrt{1-y^2}} 12x^3 y\,dx = 3y \cdot x^4 \Big|_0^{\sqrt{1-y^2}}$$

$$= 3y(1-y^2)^2 \quad ; \quad 0 < y < 1$$

$$f_{X|Y}(x|y) = \frac{12x^3 y}{3y(1-y^2)^2} = \frac{4x^3}{(1-y^2)^2} \quad ; \quad \begin{array}{l} x, y \geq 0 \\ x^2 + y^2 \leq 1 \end{array}$$

$$E[X|Y=y] = \int_{x=0}^{x=\sqrt{1-y^2}} x \cdot \frac{4x^3}{(1-y^2)^2}\,dx$$

$$= \frac{4}{(1-y^2)^2} \cdot \frac{x^5}{5}\Big|_0^{\sqrt{1-y^2}}$$

$$\boxed{E[X|Y=y] = \frac{4}{5}\sqrt{1-y^2}} \quad ; \quad 0 < y < 1.$$

⑥ $X \mid Y = y \sim U(-y, y)$, $\quad Y \sim U(0,1)$

Recall: $\quad W \sim U(a,b) \Rightarrow \quad E(W) = \dfrac{a+b}{2}$

$$V(W) = \frac{(b-a)^2}{12}$$

$$E[X \mid Y=y] = \frac{-y+y}{2} = 0$$

$$E[X^2 \mid Y=y] = \frac{(y-(-y))^2}{12} = \frac{y^2}{3}$$

$$E[X^2] = E[E[X^2 \mid Y]] = E\left[\frac{Y^2}{3}\right] = \frac{1}{3} E[Y^2]$$

$$= \frac{1}{3} \int_0^1 y^2 \, dy = \frac{1}{9}$$

$$V(X) = E(X^2) - (E(X))^2 = \boxed{\frac{1}{9}}$$

⑦ Let $A_i = \{$ ith coin is selected $\}$

$B_n = \{$ first n flips are heads $\}$

$H_{n+1} = \{$ $(n+1)$ flip is a head $\}$

$$P(H_{n+1} \mid F_n) = \sum_{i=0}^{99} P(H_{n+1} \mid B_n \cap A_i) \, P(A_i \mid B_n)$$

Due to indepence,

$$P(H_{n+1} \mid A_i \cap B_n) = P(H_{n+1} \mid A_i) = \frac{i}{99}$$

By Bayes' rule

$$P(A_i \mid B_n) = \frac{P(B_n \mid A_i) P(A_i)}{\sum_{j=0}^{99} P(B_n \mid A_j) P(A_j)}$$

$$= \frac{\left(\frac{i}{99}\right)^n \cdot \cancel{\frac{1}{100}}}{\sum_{j=0}^{99} \left(\frac{j}{99}\right)^n \cdot \cancel{\frac{1}{100}}}$$

$$= \boxed{\frac{\left(\frac{i}{99}\right)^n}{\sum_{j=0}^{99} \left(\frac{j}{99}\right)^n}}$$

⑧ Let $X, Y \sim U(0,1)$

Notice that

$$E\left(\left|X-\tfrac{1}{2}\right|\right) = \int_0^{\frac{1}{2}} -(x-\tfrac{1}{2})\,dx + \int_{\frac{1}{2}}^{1} (x-\tfrac{1}{2})\,dx = \frac{1}{4}$$

Similarly,

$$E\left|Y-\tfrac{1}{2}\right| = \frac{1}{4}$$

Since $|X-Y| \le |X-\tfrac{1}{2}| + |Y-\tfrac{1}{2}|$

$$\Rightarrow E|X-Y| \le \frac{1}{2} \qquad \square$$

⑨ See lecture 24, pages 7-9

⑩ $$(x \;\; y \;\; z)\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 1 & 0 \end{pmatrix} = (x \;\; y \;\; z)$$

$$0.5x + 0.5y = x \Rightarrow x = y$$

$$0.1y + z = y \Rightarrow z = 0.9y$$

$$x + y + z = 1 \Rightarrow 2.9y = 1 \Rightarrow y = \frac{1}{2.9} = x, \quad z = \frac{0.9}{2.9}$$

$$\pi = \left(\frac{1}{2.9} \quad \frac{1}{2.9} \quad \frac{0.9}{2.9} \right)$$

(11) See lecture 21, page 4

(12) See lecture 23, page 1

(13) See lecture 24, page 2-3

(14) Leture 21, page 7

(15) $P(X_n = 1) = a_n$

$P(X_n = 0) = 1 - a_n$

Suppose $\lim_{n \to \infty} a_n = 0$

Let $\varepsilon > 0$.

$$P(|X_n| \geq \varepsilon) \leq \frac{E|X_n|}{\varepsilon} = \frac{a_n}{\varepsilon} \to 0 \quad \text{as} \quad n \to \infty.$$

$$\Rightarrow X_n \xrightarrow{P} 0 \quad \square$$

(16) (a) $P =$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix} \end{array}$$

(b) If the chain begins in 1 or 2, then it never enters 3, 4. Therefore, if $X_0 = 1$ or $X_0 = 2$,

$$(a \quad b)\begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} = (a \quad b)$$

$$0.5a + 0.25b = a \Rightarrow b = 2a$$

$$a + b = 1 \Rightarrow a = \frac{1}{3}, \quad b = \frac{2}{3}$$

$$\Rightarrow \pi = \left(\frac{1}{3} \quad \frac{2}{3} \quad 0 \quad 0\right).$$

Similarly, if $X_0 = 3$ or $X_4 = 4$,

$$(c \quad d)\begin{pmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{pmatrix} = (c \quad d)$$

$$0.25c + 0.75d = c \Rightarrow c = d$$

$$c + d = 1 \Rightarrow c = d = \frac{1}{2}$$

Therefore, $\pi = \left(0 \quad 0 \quad \frac{1}{2} \quad \frac{1}{2}\right)$.

(8) Let $S = \{1, 2\}$

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} \end{array}$$

We are given that $P(X_0 = 1) = P(X_0 = 2) = \frac{1}{2}$

$$P^3 \approx \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} P^3 \approx \begin{pmatrix} 0.67 & 0.33 \end{pmatrix}$$

$$\Rightarrow \boxed{P(X_3 = 1) \approx 0.67}$$

$$P(X_4 = 1 \mid X_0 = 1) = p_{11}^{(4)}$$

Since $P^4 \approx \begin{pmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{pmatrix}$, $p_{11}^{(4)} \approx 0.67$

(19) See lecture 18, page 5

(20) Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t

$$|x - k| < \delta \implies |g(x) - g(k)| < \varepsilon.$$

or $$|g(x) - g(k)| \geq \varepsilon \implies |x - k| \geq \delta$$

$$\Rightarrow P\left(|g(X_n) - g(k)| \geq \varepsilon\right) \leq P\left(|X_n - k| \geq \varepsilon\right) \to 0$$

$$\Rightarrow g(X_n) \xrightarrow{P} g(k) \quad \square$$

(21) (a) $E[X \mid \lambda] = \lambda$

(b) $P\left(E[X \mid \lambda] > 1.5\right)$

$$= P(\lambda > 1.5) = \int_{1.5}^{2} \frac{1}{2-1} = \boxed{0.5}$$

22

$$P(A \mid B \cap C)\, P(C \mid B) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(B)}$$

$$= \frac{P(A \cap B \cap C)}{P(B)} \quad \text{——} \quad (1)$$

$$P(A \mid B \cap C^c)\, P(C^c \mid B) = \frac{P(A \cap B \cap C^c)}{P(B \cap C^c)} \cdot \frac{P(B \cap C^c)}{P(B)}$$

$$= \frac{P(A \cap B \cap C^c)}{P(B)} \quad \text{——} \quad (2)$$

$(1) + (2) \Rightarrow$

$$P(A \mid B \cap C)\, P(C \mid B) + P(A \mid B \cap C^c)\, P(C^c \mid B)$$

$$= \frac{P(A \cap B \cap C) + P(A \cap B \cap C^c)}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)} = P(A \mid B)$$

(23)

$$E\left[\max\{Z-c, 0\}\right] = \int_c^{\infty} (z-c)\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \underbrace{\int_c^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz}_{\text{Let } u = \frac{z^2}{2}} - c \cdot \underbrace{\int_c^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz}_{1-\Phi(c)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{c^2}{2}}^{\infty} e^{-u}\, du + c\left(1-\Phi(c)\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} + c\left(1-\Phi(c)\right)$$

(24) When $X \geq 0$,

$$E[X] = \int_0^{\infty} x \cdot f(x)\, dx$$

Note that

$$\int_0^{\infty} P(X > t)\, dt = \int_{t=0}^{\infty} \int_{y=t}^{\infty} f(y)\, dy\, dt$$

$$= \int_{y=0}^{y=\infty} \int_{t=0}^{t=y} f(y)\, dt\, dy = \int_{y=0}^{y=\infty} y f(y)\, dy = E(X)$$

$\square$

25 See lecture 11, page 3

26 – 26 } see lecture 19