Math 600 Lecture 7

Prove or give a counterexample:

- If $E \subset \mathbb{R}^2$ is open, then $E = E'$.
- If $E \subset \mathbb{R}^2$ is closed, then $E = E'$.

Compact sets

Compactness is highly valued, because of theorems like the following:

- If $E \subset X$ is compact and $\{x_n\}$ is a sequence in E, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $x \in E$ such that $x_{n_k} \to x$.
- If $f: X \to \mathbb{R}$ is continuous and $E \subset X$ is compact, then there exists $x \in E$ such that

$$f(x) = \max\{f(t) \mid t \in E\}.$$

However, as we will see, compactness is an abstract and nonintuitive concept.

Definition: Let (X, d) be a metric space and let $E \subset X$. We say that E is compact iff the following condition is satisfied: For every collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $E \subset \bigcup_{\alpha \in A} G_\alpha$, there exists a finite subcollection $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\} \subset \{G_\alpha \mid \alpha \in A\}$ such that $E \subset \bigcup_{j=1}^{n} G_{\alpha_j}$.

(Informally, we call $\{G_\alpha \mid \alpha \in A\}$ an open cover of E iff each $G_\alpha \subset X$ is open and $E \subset \bigcup_{\alpha \in A} G_\alpha$. We refer to $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ as a finite subcover.)

Examples

- $(0,1) \subset \mathbb{R}$ is not compact. For instance, if $G_n = (\frac{1}{n}, 1)$ $\forall n \in \mathbb{Z}^+$, then $\{G_n \mid n \in \mathbb{Z}^+\}$ is an open cover of $(0,1)$. But any finite subcover $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$ satisfies $G_{n_j} \subset (\frac{1}{\ell}, 1)$ $\forall j = 1, \dots, k$, where $\ell = \max\{n_1, \dots, n_k\}$. Thus $(0,1) \not\subset \bigcup_{j=1}^{k} G_{n_j}$.
- $[0,1] \subset \mathbb{R}$ is compact, though this is not easy to prove directly. We will see later that every closed and bounded subset of $\mathbb{R}$ (or even $\mathbb{R}^k$) is compact

<u>Definition</u>: Let (X,d) be a metric space. We say that $E \subset X$ is <u>bounded</u> iff there exist $x \in X$ and $R > 0$ such that

$$\forall y \in E, \ d(y,x) \leq R.$$

<u>Theorem</u>: Let (X,d) be a metric space and let $E \subset X$ be compact. Then E is closed. Also, if F is a closed subset of X and $F \subset E$, then F is also compact.

<u>Proof:</u> Suppose $E \subset X$ is compact. We will prove that E is closed by showing that E^c is open. Suppose $x \in E^c$ and, for each $y \in E$, define

$$r_y = \tfrac{1}{2} d(y,x) \quad (r_y > 0 \text{ because } y \neq x).$$

Then $\{B_{r_y}(y) \mid y \in E\}$ is an open cover of E and hence there exist $y_1, \dots, y_n \in E$ such that

$$E \subset \bigcup_{j=1}^{n} B_{r_{y_j}}(y_j).$$

Since

$$B_{r_y}(y) \cap B_{r_y}(x) = \emptyset \quad \forall y \in Y,$$

if we define

$$r = \min\{r_{y_1}, \dots, r_{y_n}\},$$

then

$$B_{r_{y_j}}(y) \cap B_r(x) = \emptyset \quad \forall j = 1, \dots, n$$

$$\Rightarrow E \cap B_r(x) = \emptyset.$$

Thus $B_r(x) \subset E^C$; since x was chosen arbitrarily, this proves that E^C is open.

Now suppose $E \subset X$ is compact, F is a closed subset of X, and $F \subset E$. Suppose $\{G_\alpha \mid \alpha \in A\}$ is an open cover of F. Since F is closed, F^C is open, and clearly

$$\{G_\alpha \mid \alpha \in A\} \cup \{F^C\}$$

is an open cover of E (since $E = (E \cap F) \cup (E \cap F^C)$). But then, since E is compact, there is a finite subcover, either of the form

$$\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$$

or

$$\{G_{\alpha_1}, \dots, G_{\alpha_n}, F^C\}.$$

Since $F \subset E$, we must have

$$(*) \qquad F \subset \bigcup_{j=1}^{n} G_{\alpha_j}$$

(regardless of whether F^C belongs to the subcover or not, we still have $(*)$, since $F \cap F^C = \emptyset$). This shows that F is compact. //

<u>Corollary</u>: If (X,d) is a metric space, $E \subset X$ is compact, and $F \subset X$ is closed, then $F \cap E$ is compact.

<u>Proof:</u> Since E is compact, it is closed; hence $F \cap E$ is the intersection of two closed sets and hence is closed. But then, since $F \cap E \subset E$, $F \cap E$ is compact by the previous theorem. //

Given $Y \subset X$, we have previously defined $E \subset Y \subset X$ is open relative to Y. We saw that E can be open relative to Y even if E is not open relative to X. Similarly, we can say that $E \subset Y \subset X$ is <u>compact relative to Y</u> iff E is compact as a subset of the metric space Y (using the metric inherited from X). It turns out, though, that E is compact relative to Y iff E is compact relative to X.

<u>Theorem</u>: Let (X,d) be a metric space and suppose $Y \subset X$. Then $E \subset Y$ is compact relative to Y iff E is compact relative to X.

<u>Proof:</u> Suppose first that E is compact relative to X. We wish to show that E is compact relative to Y, so assume that

$$E \subset \bigcup_{\alpha \in A} U_\alpha,$$

where $U_\alpha \subset Y$ is open relative to Y for all $\alpha \in A$. By an earlier theorem, for each $\alpha \in A$, there exists an open subset G_α of X such that

$$U_\alpha = Y \cap G_\alpha.$$

But then $U_\alpha \subset G_\alpha$ $\forall \alpha \in A$ and hence

$$E \subset \bigcup_{\alpha \in A} G_\alpha.$$

Since E is compact relative to X, it follows that there exist $\alpha_1, \ldots, \alpha_n \in A$ such that

$$E \subset \bigcup_{j=1}^{n} G_{\alpha_j}.$$

Since $E \subset Y$, it follows that

$$E \subset Y \cap \left(\bigcup_{j=1}^{n} G_{\alpha_j}\right) = \bigcup_{j=1}^{n} (Y \cap G_{\alpha_j}) = \bigcup_{j=1}^{n} U_{\alpha_j}.$$

Thus $\{U_\alpha \mid \alpha \in A\}$ contains a finite subcover of E, which shows that E is compact relative to Y.

Conversely, suppose E is compact relative to Y. We wish to show that E is compact relative to X, so assume that $\{G_\alpha \mid \alpha \in A\}$ is an open cover of E, where each G_α is open in X. The

$$E \subset \bigcup_{\alpha \in A} G_\alpha \text{ and } E \subset Y$$

$$\Rightarrow E \subset Y \cap \left(\bigcup_{\alpha \in A} G_\alpha\right) = \bigcup_{\alpha \in A} (Y \cap G_\alpha).$$

Since each $Y \cap G_\alpha$ is open relative to Y and E is compact relative to Y, there exist $\alpha_1, \ldots, \alpha_n \in A$ such that

$$E \subset \bigcup_{j=1}^{n} (Y \cap G_{\alpha_j}) \subset \bigcup_{j=1}^{n} G_{\alpha_j}.$$

Thus $\{G_\alpha \mid \alpha \in A\}$ contains a finite subcover of E, and hence E is compact relative to X. //

<u>Note</u>: Let (X,d) be a metric space.

- Every metric space is open relative to itself (why?). Thus, if $Y \subset X$ is <u>any</u> subset, then Y is open relative to Y.

- Ditto for closed sets: Every metric space is closed relative to itself.

- However, compactness is a more intrinsic property of a set. Whether or not a set is compact does not depend on whether it is regarded as a subset of another set or not.