Math 672 Lecture 8

Recall: If $T: V \to W$, then

$\mathcal{N}(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V,

$\mathcal{R}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.

We learned one simple but important fact from the fact that $\mathcal{R}(T)$ is a subspace of W: If $\mathcal{R}(T)$ is a proper subspace of W, then the equation $T(v) = w$ fails to have a solution v for most $w \in W$.

<u>Definition</u>: Let X and Y be sets. Then $f: X \to Y$ is called <u>surjective</u> (<u>onto</u>) iff $\mathcal{R}(f) = Y$, that is, iff for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The above applies to linear maps $T: V \to W$; T is surjective iff $\mathcal{R}(T) = W$.

<u>Definition</u>: Let X and Y be sets. Then $f: X \to Y$ is called <u>injective</u> (one-to-one) iff $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Note that f is a well-defined function iff every input (element of X) corresponds to a unique output (element of $\mathcal{R}(f)$). The function f is injective iff every output (element of $\mathcal{R}(f)$) corresponds to a unique input (element of X).

<u>Theorem</u>: Let $T: V \to W$ be linear. Then T is injective iff $\mathcal{N}(T) = \{0\}$.

<u>Proof</u>: Suppose first that $\mathcal{N}(T) = \{0\}$. Let $u, v \in V$ satisfy $T(u) = T(v)$. We must show that $u = v$. But

$$
\begin{aligned}
T(u) = T(v) &\Rightarrow T(u) - T(v) = 0 \\
&\Rightarrow T(u - v) = 0 \quad (\text{by linearity of } T) \\
&\Rightarrow u - v \in \mathcal{N}(T) \quad (\text{by definition of } \mathcal{N}(T)) \\
&\Rightarrow u - v = 0 \quad (\text{since } \mathcal{N}(T) = \{0\} \text{ by assumption}) \\
&\Rightarrow u = v.
\end{aligned}
$$

This proves that T is injective.

Conversely, suppose $\mathcal{N}(T) \neq \{0\}$. Then there exists $z \neq 0$ such that $z \in \mathcal{N}(T)$. But then

$$z \neq 0 \text{ and } T(z) = T(0),$$

which shows that T is not injective. //

The fundamental theorem of linear algebra Let $T: V \to W$ be linear, where V is finite-dimensional. Then

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V).$$

Proof: Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $\mathcal{N}(T)$ (where $k=0$ if $\mathcal{N}(T) = \{0\}$), and extend $\{v_1, v_2, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Then $\dim \mathcal{N}(T) = k$, $\dim(V) = n$, and it suffices to prove that $\dim \mathcal{R}(T) = n-k$. We can do this by proving that

$$\{T(v_{k+1}), \ldots, T(v_n)\}$$

is a basis for $\mathcal{R}(T)$.

First we prove linear independence. Suppose $\alpha_{k+1}, \ldots, \alpha_n \in F$ and

$$\alpha_{k+1} T(v_{k+1}) + \cdots + \alpha_n T(v_n) = 0.$$

By linearity, this implies that

$$T(\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n) = 0$$

$$\Rightarrow \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n \in \mathcal{N}(T)$$

$$\Rightarrow \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = \alpha_1 v_1 + \cdots + \alpha_k v_k \text{ for some } \alpha_1, \ldots, \alpha_k \in F$$

(since $\{v_1, \ldots, v_k\}$ is a basis for $\mathcal{N}(T)$)

$$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_k v_k - \alpha_{k+1} v_{k+1} - \cdots - \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_k = \alpha_{k+1} = \cdots = \alpha_n = 0 \quad (\text{since } \{v_1, \dots, v_n\} \text{ is linearly independent})$$

$$\Rightarrow \alpha_{k+1} = \cdots = \alpha_n = 0.$$

Thus we have proven that $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

Now we show that $\{T(v_{k+1}), \dots, T(v_n)\}$ spans $R(T)$. Let $w \in R(T)$; Then there exists $v \in V$ such that $T(v) = w$. Since $\{v_1, \dots, v_n\}$ is a basis for V, there exist $\alpha_1, \dots, \alpha_n \in F$ such that

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

But then

$$T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = w$$

$$\Rightarrow \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = w$$

$$\Rightarrow \alpha_{k+1} T(v_{k+1}) + \cdots + \alpha_n T(v_n) = w \quad (\text{since } T(v_1) = \cdots = T(v_k) = 0 \text{ because } v_1, \dots, v_k \in N(T))$$

$$\Rightarrow w \in \operatorname{span}(T(v_{k+1}), \dots, T(v_n)).$$

Thus $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$, and the proof is complete. //

As you might guess from the name, the fundamental theorem has some important consequences.

Recall that $f: X \to Y$ is called <u>bijective</u> iff it is both injective and surjective.

<u>Theorem</u>: Let V and W be finite-dimensional vector spaces and let $T: V \to W$ be linear.

- If T is injective, then $\dim(W) \geq \dim(V)$.
- If T is surjective, then $\dim(V) \geq \dim(W)$.
- If T is bijective, then $\dim(V) = \dim(W)$.

<u>Proof</u>: Recall that

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V)$$

and obviously

$$\dim(W) \geq \dim(\mathcal{R}(T)).$$

If T is injective, then

$$\dim(\mathcal{N}(T)) = 0$$

$$\Rightarrow \dim(\mathcal{R}(T)) = \dim(V)$$

$$\Rightarrow \dim(W) \geq \dim(V).$$

If T is surjective, then $W = \mathcal{R}(T)$ and

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V)$$

$$\Rightarrow \dim(W) = \dim(\mathcal{R}(T)) \leq \dim(V). \;//$$

<u>Theorem</u>: Let V and W be finite-dimensional vector spaces satisfying $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Then T is injective iff T is surjective (and hence T is bijective iff (T is injective or T is surjective)).

<u>Proof</u>: We have

$$
\begin{aligned}
T \text{ is injective} &\Leftrightarrow \dim(\mathcal{N}(T)) = 0 \quad (\text{since } T \text{ is injective iff } \mathcal{N}(T) = \{0\}) \\
&\Leftrightarrow \dim(\mathcal{R}(T)) = \dim(V) \quad (\text{by the fundamental theorem}) \\
&\Leftrightarrow \dim(\mathcal{R}(T)) = \dim(W) \quad (\text{since } \dim(V) = \dim(W) \text{ by assumption}) \\
&\Leftrightarrow \mathcal{R}(T) = W \quad (\text{since } \mathcal{R}(T) \subseteq W) \\
&\Leftrightarrow T \text{ is surjective. } //
\end{aligned}
$$

Note that if $\dim(V) > \dim(W)$ and $T : V \to W$ is linear, then $\mathcal{N}(T)$ is nontrivial, meaning that the equation $T(v) = 0$ must have nontrivial solutions.

As an application of this fact, consider the following <u>homogeneous</u> system of linear equations:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = 0 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = 0 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = 0 \end{cases}$$

$\iff$ $Ax = 0$, where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $0 \in \mathbb{R}^m$

$\iff$ $T(x) = 0$, where $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(x) = Ax$ $\forall x \in \mathbb{R}^n$.

If $n > m$, then $\dim(\mathcal{N}(T)) > 0$ and $T(x) = 0$ must have nontrivial solutions. <u>Thus a system of m homogeneous linear equations in n unknowns in which $n > m$ (more unknowns than equations) must have nontrivial solutions.</u>

On the other hand, consider an <u>inhomogeneous</u> system:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m \end{cases}$$

$\iff Ax = b$, where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

$\iff T(x) = b$, where $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(x) = Ax$ $\forall x \in \mathbb{R}^n$.

If $m > n$, then T cannot be surjective $(\dim(R(T)) \le \dim(V) < \dim(W))$ and $T(x) = b$ fails to have a solution for some (most) $b \in \mathbb{R}^m$.

Thus an inhomogeneous system of m linear equations in n unknowns in which $m > n$ (more equations than unknowns) fails to have a solution for some (most) values of the right-hand side.