Math 600 Lecture 19

Theorem: Let $\{x_n\}$, $\{y_n\}$ be sequences of real numbers. If at least one of $\sum_{n=0}^{\infty} x_n$, $\sum_{n=0}^{\infty} y_n$ converges absolutely, then the product converges and

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x_k y_{n-k}\right) = \left(\sum_{m=0}^{\infty} x_m\right)\left(\sum_{n=0}^{\infty} y_n\right).$$

Proof: Wlog, assume $\sum_{n=0}^{\infty} x_n$ converges absolutely and $\sum_{n=0}^{\infty} y_n$ converges. Define

$$s = \sum_{n=0}^{\infty} x_n, \quad t = \sum_{n=0}^{\infty} y_n,$$

and, for all $n \in \mathbb{Z}^+$,

$$s_n = \sum_{k=0}^{n} x_k, \quad t_n = \sum_{k=0}^{n} y_k, \quad u_n = \sum_{k=0}^{n} z_k,$$

where

$$z_k = \sum_{j=0}^{k} x_j y_{k-j}.$$

Also, write

$$d_n = t_n - t \quad \forall n \in \mathbb{Z}^+.$$

Note that

$$u_n = x_0y_0 + (x_0y_1 + x_1y_0) + (x_0y_2 + x_1y_1 + x_2y_0) + \cdots + (x_0y_n + x_1y_{n-1} + \cdots + x_ny_0)$$

$$= x_0(y_0 + y_1 + \cdots + y_n) + x_1(y_0 + y_1 + \cdots + y_{n-1}) + x_2(y_0 + y_1 + \cdots + y_{n-2})$$

$$+\cdots+x_n y_0$$

$$= x_0 t_n + x_1 t_{n-1} + x_2 t_{n-2} + \cdots + x_n t_0$$

$$= x_0(t+d_n) + x_1(t+d_{n-1}) + x_2(t+d_{n-2}) + \cdots + x_n(t+d_0)$$

$$= (x_0+x_1+\cdots+x_n)t + x_0 d_n + x_1 d_{n-1} + \cdots + x_n d_0$$

$$= s_n t + \sum_{k=0}^{n} d_k x_{n-k}$$

We know that $s_n t \to st$, and we want to prove that $u_n \to st$.

Thus it suffices to prove that

$$\sum_{k=0}^{n} d_k x_{n-k} \to 0.$$

Let $\varepsilon > 0$ be given. Since $t = \sum_{n=0}^{\infty} y_n$,

$$d_n = t_n - t \to 0.$$

Hence there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow |d_n| < \frac{\varepsilon}{2S'},$$

where

$$S' = \sum_{n=0}^{\infty} |x_n|.$$

But then

$$n \geq N \Rightarrow \left| \sum_{k=0}^{n} d_k x_{n-k} \right| \leq \sum_{k=0}^{n} |d_k||x_{n-k}|$$

$$= \sum_{k=0}^{N-1} |d_k|\,|x_{n-k}| + \sum_{k=N}^{n} |d_k||x_{n-k}|$$

$$< \sum_{k=0}^{N-1} |d_k||x_{n-k}| + \frac{\varepsilon}{2s'} \sum_{k=N}^{n} |x_{n-k}|.$$

Now,

$$\sum_{k=N}^{n} |x_{n-k}| \leq \sum_{k=0}^{\infty} |x_k| = s'.$$

Also, if

$$M = \sum_{k=0}^{N-1} |d_k|,$$

then, there exists $N' \in \mathbb{Z}^+$ such that

$$n \geq N' \Rightarrow |x_n| < \frac{\varepsilon}{2M}.$$

But then

$$n \geq N+N' \Rightarrow n-(N-1) \geq N' \Rightarrow |x_{n-k}| < \frac{\varepsilon}{2M} \quad \forall k=0,1,\ldots,N-1$$

$$\Rightarrow \sum_{k=0}^{N-1} |d_k||x_{n-k}| < \frac{\varepsilon}{2M} \sum_{k=0}^{N-1} |d_k| = \frac{\varepsilon}{2M} \cdot M = \frac{\varepsilon}{2}.$$

Thus we obtain

$$n \geq N+N' \Rightarrow \left| \sum_{k=0}^{n} d_k x_{n-k} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\sum_{k=0}^{n} d_k x_{n-k} \to 0,$$

as desired. //

Power series

Given a sequence $\{c_n\}$ of real numbers and $a \in \mathbb{R}$, we can define a a real-valued function f by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the domain is the set of all x for which the series converges.

Such a series is called a <u>power series</u>.

We can (almost) determine the domain of f by the root test.

We compute

$$\limsup_{n\to\infty} \left| c_n (x-a)^n \right|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n} |x-a|$$

$$= \left(\limsup_{n\to\infty} |c_n|^{1/n} \right) |x-a| \quad \text{(since } |x-a| \text{ is constant w.r.t. } n\text{)}$$

We see that the series converges iff

$$|x-a| < R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$$

and it diverges iff

$$|x-a| > R.$$

Note two special cases:

- $\limsup_{n\to\infty} |c_n|^{1/n} = 0$. Then the series converges for all $x \in \mathbb{R}$.

We write $R=\infty$ and the domain of f is $\mathbb{R}=(-\infty,\infty)$.

- $\limsup_{n\to\infty} |c_n|^{1/n}=\infty$. Then the series converges only for $x=a$.

  We write $R=0$ and the domain of f is the degenerate interval $[a,a]$.

If

$$0<\limsup_{n\to\infty}|c_n|^{1/n}<\infty,$$

then $0<R<\infty$ and the domain of f contains

$$(a-R,a+R).$$

It may or may not contain $a-R$ and $a+R$. Thus the domain, in this case, is one of the following intervals:

$$(a-R,a+R),\ [a-R,a+R),\ (a-R,a+R],\ [a-R,a+R].$$

We call R the <u>radius of convergence</u> of the power series, and the <u>interval of convergence</u> is the domain of f (which is always an interval, if we are willing to call $[a,a]$ an interval).

<u>Note:</u> If we start with $a\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ (or $f:I\to\mathbb{R}$, where I is an open interval containing a), then we can define the Taylor series of f at a (or in powers of $x-a$) as follows

- $p_0(x)=f(a)$ is the unique constant polynomial that agrees with f at $x=a$.

- $p_1(x) = f(a) + f'(a)(x-a)$ is the unique linear polynomial that agrees with f and f' at $x=a$ (that is, $p(a) = f(a)$ and $p'(a) = f'(a)$).

- $p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ is the unique quadratic polynomial that agrees with f, f', and f'' at $x=a$.

$$\vdots$$

- $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the unique polynomial of degree n that agrees with $f, f', \ldots, f^{(n)}$ at $x=a$:

$$p_n(a) = f(a),$$
$$p_n'(a) = f'(a),$$
$$p_n''(a) = f''(a),$$
$$\vdots$$
$$p_n^{(n)}(a) = f^{(n)}(a).$$

Thus, it is natural to define the Taylor series of f at $x=a$ (or in powers of $x-a$) by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(assuming f is infinitely differentiable). We can then ask:

- does the series converge?

- does the series converge to $f(x)$?