Ex: Let P denote the transition matrix of $X = \{X_n\}$ & Q denote the transition matrix of $Y = \{Y_n\}$.

If X, Y are independent & regular MCs, prove that $Z_n = (X_n, Y_n)$ is a regular MC.

$$P(Z_n = (k, l) \mid Z_{n-1} = (i, j), Z_{n-2}, \ldots, Z_0)$$

$$= P(X_n = k \mid X_{n-1} = i, \ldots, X_0) \times$$
$$P(Y_n = l \mid Y_{n-1} = j, \ldots, Y_0)$$

$$= p_{ik} \cdot q_{jl}.$$

Since P, Q are regular, $\exists\, n_0, m_0 < \infty$ such that $P^{n_0} > 0$ & $Q^{m_0} > 0$

Therefore, $\left(P^{n_0}\right)^{m_0} = P^{n_0 m_0} > 0$

$$\left(Q^{m_0}\right)^{n_0} = Q^{n_0 m_0} > 0$$

Therefore, $p_{ik}^{(m_0 n_0)} \, q_{jl}^{(m_0 n_0)} > 0 \quad \forall i, j, k, l.$

MATH 630

[redacted]

Name:

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(1) Suppose the numbers of families that check into a hotel on successive days are independent Poisson random variables with mean λ. Also suppose that the number of days that a family stays in the hotel is a geometric random variable with parameter p, $0 < p < 1$. (Thus, a family who spent the previous night in the hotel will, independently of how long they have already spent in the hotel, check out the next day with probability p.) Also suppose that all families act independently of each other. Under these conditions it is easy to see that if X_n denotes the number of families that are checked in the hotel at the beginning of day n then X_n is a Markov chain.

(a) Find the transition matrix.

(b) Find $E(X_n|X_0)$.

Let $X_n = i$ be the number of families checked in at the beginning of day n.

Let R_i be the number of families that remain another day. Then, $R_i \sim B(i, q)$ $(q := 1-p)$.

Let $N \sim P(\lambda)$ be the number of new families that check on day n. Then

$$P(X_{n+1} = j \mid X_n = i) = P(R_i + N = j)$$

$$= \sum_k P(N = j-k) \cdot P(R_i = k)$$

$$= \sum_{k}^{\min\{i,j\}} \frac{e^{-\lambda} \lambda^{j-k}}{(j-k)} \binom{i}{k} q^k (1-p)^{i-k} = P_{ij}$$

(b) $E[X_n \mid X_{n-1} = i] = E[R_i + N]$

$$= iq + \lambda$$

Therefore, $E[X_n \mid X_{n-1}] = q X_{n-1} + \lambda$

Taking the expectation of both sides yields

$$E[X_n] = q E[X_{n-1}] + \lambda$$

$$= q \left[q E[X_{n-2}] + \lambda \right] + \lambda$$

$$\vdots$$

$$= q^n E[X_0] + q^{n-1} \lambda + \cdots + \lambda$$

$$= q^n E[X_0] + \lambda \frac{(1-q^n)}{1-q}$$

Therefore,

$$E[X_n] = E[E[X_n \mid X_0]] = E\left[q^n E[X_0] + \lambda \frac{(1-q^n)}{1-q} \right]$$

$$\Rightarrow \boxed{E[X_n \mid X_0] = q^n E[X_0] + \lambda \frac{(1-q^n)}{1-q}}$$

Classification of States

Definition: State j is said to accessible frome state i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

Definition: i & j are accessible from each other are said to communicate. This is denoted by $i \leftrightarrow j$.

By definition, $i \leftrightarrow i$ $\left(P_{ii}^{(0)} = 1 \right)$.

Remark: X is irreducible if all states communicate with each other.

Definition: Let f_i denote the probability, starting in state i, the process will reenter state i.

- State i is said to be recurrent if $f_i = 1$.
- State i is said to be transient if $f_i < 1$

Let

$$I_n = \begin{cases} 1 \; ; & X_n = i \\ 0 \; ; & X_n \neq i \end{cases}$$

$\sum_{n=0}^{\infty} I_n$ represents the number of times the process visits state i. Then

$$E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P(X_n = i \mid X_0 = i)$$

$$= \sum_{n=0}^{\infty} P_{ij}^{(n)}$$

Theorem :

i) i is recurrent iff $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$

ii) i is transient iff $\sum_{n=0}^{\infty} P_{ii}^{(n)} < \infty$.