Math 600 Lecture 22

Definition: Let $(X,d_X),(Y,d_Y)$ be metric spaces and let $f:X\to Y$ be a function. We say that f is <u>uniformly continuous</u> on X iff

$$\forall \varepsilon>0\ \exists \delta>0\ (u,x\in X \text{ and } d_X(u,x)<\delta) \Rightarrow d_Y(f(u),f(x))<\varepsilon.$$

Note the difference with the definition of "f is continuous at $x\in X$":

$$\forall \varepsilon>0\ \exists \delta>0\ (u\in X \text{ and } d_X(u,x)<\delta) \Rightarrow d_Y(f(u),f(x))<\varepsilon.$$

If we say that f is continuous at x, δ depends on both x and ε; if we say that f is uniformly continuous on X, δ depends only on ε (i.e. the same δ works for all $x\in X$).

Examples:

1. $f:[0,1]\to\mathbb{R}$, $f(x)=x^2$. Let $\varepsilon>0$ be given and define $\delta=\frac{\varepsilon}{2}$. Then

$$u,x\in[0,1] \text{ and } |u-x|<\delta$$

$$\Rightarrow |f(u)-f(x)| = |u^2-x^2| = |u-x||u+x| < \delta\cdot 2 \qquad (\text{since } |u+x| = u+x \le 1+1=2)$$

$$= \frac{\varepsilon}{2}\cdot 2 = \varepsilon.$$

Thus f is uniformly continuous on $[0,1]$

2. $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$. Let $\varepsilon>0$ be given and consider

$$|f(u)-f(x)| = \left|\frac{1}{u}-\frac{1}{x}\right| = \frac{|x-u|}{ux}.$$

Note that

$$|x-u|<\delta \Rightarrow |f(u)-f(x)| < \frac{\delta}{ux}.$$

Since $\frac{1}{ux} \to \infty$ as $x \to 0$, there is no way to choose δ such that

$$\frac{\delta}{ux} < \varepsilon \quad \forall x \in (0,1).$$

Thus f is not uniformly continuous on $(0,1)$.

<u>Theorem</u>: Let (X,d_X), (Y,d_Y) be metric spaces, let X be compact, and let $f: X \to Y$ be continuous on X. Then f is uniformly continuous on X.

<u>Proof:</u> Let $\varepsilon > 0$ be given. Since f is continuous on X, for all $x \in X$, there exists $\delta_x > 0$ such that

$$u \in B_{\delta_x}(x) \Rightarrow d_Y(f(u), f(x)) < \frac{\varepsilon}{2}.$$

For each $x \in X$, define $U_x = B_{\frac{1}{2}\delta_x}(x)$. Then $\{U_x\}$ is an open cover of X. Since X is compact, there exist $x_1, \ldots, x_n \in X$ such that

$$(*) \qquad X \subset \bigcup_{j=1}^{n} U_{x_j}.$$

Define

$$\delta = \frac{1}{2} \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}.$$

Now suppose $u, x \in X$ and $d_X(u,x) < \delta$. By $(*)$, there exists $j \in \{1, \ldots, n\}$ such that

$$x \in U_{x_j} \Rightarrow d_X(x, x_j) < \frac{1}{2}\delta_{x_j}$$

It follows that

$$
\begin{aligned}
d_X(u,x_i) &\le d_X(u,x) + d_X(x,x_i) \\
&< \tfrac{1}{2}\delta + \tfrac{1}{2}\delta_{x_i} \le \tfrac{1}{2}\delta_{x_j} + \tfrac{1}{2}\delta_{x_j} = \delta_{x_j}
\end{aligned}
$$

Thus u and x both lie in $B_{\delta_{x_j}}(x_j)$. But then

$$
\begin{aligned}
d_Y(f(u),f(x)) &\le d_Y(f(u),f(x_i)) + d_Y(f(x_i),f(x)) \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

This shows that f is uniformly continuous on X. //

Recall that $E \subset X$ is <u>connected</u> iff it is not possible to write $E = A \cup B$, where A and B are nonempty and $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

<u>Theorem</u>: Let (X,d_X), (Y,d_Y) be metric spaces and let $f: X \to Y$ be continuous. If $E \subset X$ is connected, then $f(E) \subset Y$ is connected.

<u>Proof:</u> We prove the contrapositive. Suppose $E = C \cup D$, where C and D are nonempty and $\bar{C} \cap D = C \cap \bar{D} = \emptyset$. Then

$$E \subset f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$$

$$\Rightarrow E = (E \cap f^{-1}(C)) \cup (E \cap f^{-1}(D))$$

$$\Rightarrow E = A \cup B,\ A = E \cap f^{-1}(C),\ B = E \cap f^{-1}(D).$$

Since C is nonempty and $C \subset f(E)$, there exists $x \in E$ such that $f(x) \in C$; that is, $x \in E \cap f^{-1}(C)$. Thus A is nonempty. Similarly, B is nonempty.

Note that

$$A \subset f^{-1}(C) \Rightarrow A \subset f^{-1}(\bar{C})$$

$$\Rightarrow \bar{A} \subset f^{-1}(\bar{C}) \text{ (since } f^{-1}(\bar{C}) \text{ is closed)}$$

$$\Rightarrow f(\bar{A}) \subset \bar{C}$$

and

$$B \subset f^{-1}(D) \Rightarrow f(B) \subset D$$

Thus

$$x \in \bar{A} \cap B \Rightarrow f(x) \in f(\bar{A} \cap B) \subset f(\bar{A}) \cap f(B) \subset \bar{C} \cap D = \emptyset.$$

Therefore, $\bar{A} \cap B = \emptyset$. By similar reasoning, $A \cap \bar{B} = \emptyset$.

Hence we have proven that if $f(E)$ is disconnected, then so is E. //

<u>Corollary</u> (the intermediate value theorem): Let $f: I \to \mathbb{R}$ be continuous, where $I \subset \mathbb{R}$ is an interval. If $a, b \in I$ with $a < b$, $f(a) \neq f(b)$, and v lies between $f(a)$ and $f(b)$ (i.e. $f(a) < v < f(b)$ or $f(b) < v < f(a)$), then there exists $c \in (a,b)$ such that $f(c) = v$.

<u>Proof:</u> Since $[a,b]$ is connected and f is continuous, $f([a,b])$ is continuous.

Hence, by an earlier theorem, $f([a,b])$ is an interval. The result follows. //

<u>Definition</u>: Let $f: D \to \mathbb{R}$, where $D \subset \mathbb{R}$. If $(a,b) \subset D$ for some $a, b \in \mathbb{R}$, $a < b$, then we say that

$$\lim_{x \to a^+} f(x) = L$$

iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \left(x \in (a, a+\delta) \Rightarrow |f(x) - L| < \varepsilon \right).$$

Similarly, if $(c,a) \subset D$ for some $a, c \in \mathbb{R}$, $c < a$, then we say that

$$\lim_{x \to a^-} f(x) = L$$

iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \left(x \in (a-\delta, a) \Rightarrow |f(x) - L| < \varepsilon \right).$$

Types of discontinuities

1. Removable $\lim_{x \to a} f(x)$ exists but $f(a) \neq \lim_{x \to a} f(x)$ or $f(a)$ is undefined.

This is called removable because we can just redefine f at $x = a$ and make the redefined function continuous at $x = a$. (Note that this concept applies to a general $f: X \to Y$, where X and Y are any metric spaces.)

Trivial example: $f: (-\infty, 2) \cup (2, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x^2 - 4}{x - 2}$. Since $f(x) = x + 2$ for all x in the domain of f, we should define $f(2) = \lim_{x \to 2} (x+2) = 4$.

The redefined f is continuous:

$$f: \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases} \quad \text{(or simply } f(x) = x + 2\text{)}.$$

Important example: $f: (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\sin(x)}{x}$.

It can be shown that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and hence we can redefine f to make it continuous at $x=0$:

$$f : \mathbb{R} \to \mathbb{R},$$

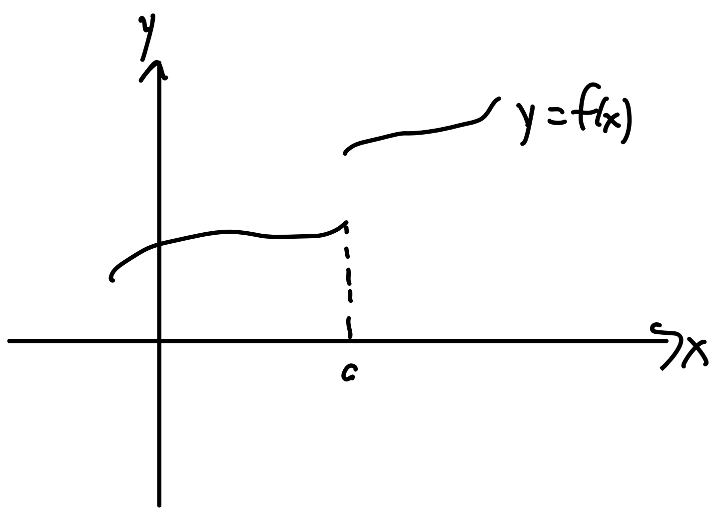
$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

2. <u>Jump</u> If $f : I \to \mathbb{R}$, where I is an interval, and a lies in the interior of I, we say that f has a <u>jump discontinuity</u> at $x=a$ iff

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to c^+} f(x)$$

exist, and

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to c^+} f(x).$$



3. <u>Infinite</u> If $f: I \to \mathbb{R}$, where I is an interval, and a lies in the interior of I (or $f: (I \setminus \{a\}) \to \mathbb{R}$), we say that f has an <u>infinite discontinuity</u> at $x=a$ iff

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

The above list is not exhaustive. A famous example is

$$f: \mathbb{R} \to \mathbb{R},$$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$