Math 600 Lecture 6

<u>Theorem</u>: Let (X,d) be a metric space and $E \subset X$. Then E is open iff E^c is closed. (Equivalently, E is closed iff E^c is open.)

<u>Proof</u>: Suppose first that E is open. We must show that E^c contains all of its limit points. Equivalently, we must show that if $x \in E$, then x is not a limit point of E^c (this is equivalent since $X = E \cup E^c$ and $E \cap E^c = \emptyset$). So assume that $x \in E$. Then there exists $r > 0$ such that $B_r(x) \subset E$, which implies that $B_r(x) \cap E^c = \emptyset$. But this implies that x is not a limit point of E^c. Hence E^c must contain all of its limit points, and hence E^c is closed.

Conversely, suppose E^c is closed. Then E^c contains all of its limit points. Let $x \in E$. Then, since $x \notin E^c$, x is not a limit point of E^c, and hence there exists $r > 0$ such that $B_r(x) \cap E^c = \emptyset$. But this implies that $B_r(x) \subset E$. Since $x \in E$ was arbitrary, this proves that E is open. //

<u>Theorem</u>: Let X be a set, let A be another set (not necessarily a subset of X), and, for each $\alpha \in A$, let E_α be a subset of X. Then

$$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c,$$

$$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

<u>Proof:</u> We have

$$x \in \left(\bigcup_{\alpha\in A} E_\alpha\right)^c \iff x \notin \bigcup_{\alpha\in A} E_\alpha$$

$$\iff \forall \alpha \in A,\ x \notin E_\alpha$$

$$\iff \forall \alpha \in A,\ x \in E_\alpha^c$$

$$\iff x \in \bigcap_{\alpha\in A} E_\alpha^c.$$

Thus

$$\left(\bigcup_{\alpha\in A} E_\alpha\right)^c = \bigcap_{\alpha\in A} E_\alpha^c.$$

Similarly,

$$x \in \left(\bigcap_{\alpha\in A} E_\alpha\right)^c \iff x \notin \bigcap_{\alpha\in A} E_\alpha$$

$$\iff \exists \alpha \in A,\ x \notin E_\alpha$$

$$\iff \exists \alpha \in A,\ x \in E_\alpha^c$$

$$\iff x \in \bigcup_{\alpha\in A} E_\alpha^c,$$

and thus

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c. \quad //$$

Theorem: Let (X,d) be a metric space.

- Suppose A is a set and, for all $\alpha\in A$, $E_\alpha \subset X$ is open. Then

$$\bigcup_{\alpha\in A} E_\alpha$$

is open.

- Suppose $E_1,\dots,E_n$ are open subsets of X. Then

$$\bigcap_{k=1}^{n} E_k$$

is open.

Proof: We have

$$x\in \bigcup_{\alpha\in A} E_\alpha \Rightarrow \exists \alpha'\in A,\ x\in E_{\alpha'}$$

$$\Rightarrow \exists r>0,\ B_r(x)\subset E_{\alpha'} \text{ (since } E_{\alpha'} \text{ is open)}$$

$$\Rightarrow B_r(x) \subset \bigcup_{\alpha\in A} E_\alpha \text{ (since } E_{\alpha'} \subset \bigcup_{\alpha\in A} E_\alpha\text{)}.$$

Thus every $x\in \bigcup_{\alpha\in A} E_\alpha$ is an interior point and hence $\bigcup_{\alpha\in A} E_\alpha$ is open.

Now consider open sets $E_1,\dots,E_n$. If $x\in \bigcap_{k=1}^{n} E_k$, then

$$\forall k=1,\dots,n,\ x\in E_k$$

$$\Rightarrow \forall k=1,\dots,n\ \exists r_k>0,\ B_{r_k}(x)\subset E_k.$$

If we define $r=\min\{r_1,\dots,r_n\}$, then

$$\forall k=1,\dots,n,\ r\le r_k \Rightarrow \forall k=1,\dots,n,\ B_r(x)\subset B_{r_k}(x)\subset E_k$$

$$\Rightarrow B_r(x) \subset \bigcap_{k=1}^{n} E_k.$$

Thus we have shown that every $x \in \bigcap_{k=1}^{n} E_k$ is an interior point, and hence $\bigcap_{k=1}^{n} E_k$ is open. //

<u>Corollary</u>: Let (X,d) be a metric space.

- If A is a set and, for all $\alpha \in A$, E_α is a closed subset of X, then

$$\bigcap_{\alpha \in A} E_\alpha$$

is closed.

- If $n \in \mathbb{Z}^+$ and $E_1, \dots, E_n$ are closed subsets of X, then

$$\bigcup_{k=1}^{n} E_k$$

is closed.

<u>Proof:</u> We have that

$$\left(\bigcap_{\alpha A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

and

$$E_\alpha \text{ closed } \forall \alpha \in A \Rightarrow E_\alpha^c \text{ is open } \forall \alpha \in A$$

$$\Rightarrow \bigcup_{\alpha \in A} E_\alpha^c \text{ is open by the previous result}$$

But then $\bigcap_{\alpha A} E_\alpha$ is closed (because its complement is open).

Similarly,

$$E_\alpha \text{ closed } \forall k=1,\dots,n \Rightarrow E_\alpha^c \text{ is open } \forall \alpha \in A$$

$$\Rightarrow \bigcap_{k=1}^{n} E_\alpha^c \text{ is open by the previous result}$$

Thus $\bigcup_{k=1}^{n} E_\alpha$ is closed because

$$\left(\bigcup_{k=1}^{n} E_\alpha\right)^c = \bigcap_{k=1}^{n} E_\alpha^c \text{ is open.} //$$

<u>Theorem</u>: Let (X,d) be a metric space and assume $E \subset X$. The

1. $\bar{E}$ is closed
2. $E = \bar{E}$ iff E is closed
3. If $F \subset X$ is closed and $E \subset F$, then $\bar{E} \subset F$ (i.e. $\bar{E}$ is the smallest open set containing E).

<u>Proof</u>: Exercise.

<u>Definition</u>: Let (X,d) be a metric space and let $Y \subset X$.

Note that Y is a metric space under the same metric. (Note: Technically, it's not the same metric because the domain is different. The metric on Y is $d_Y : Y \times Y \to \mathbb{R}$ defined by $d_Y(y_1, y_2) = d(y_1, y_2)$ $\forall y_1, y_2 \in Y$.) Given $E \subset Y$, we say that E is <u>open relative to Y</u> iff for all $x \in E$, there exists $r>0$ such that $y \in Y$ and $d(y,x) < r$ implies that $y \in E$ (that is, iff for all $x \in E$, there exists $r>0$ such that $B_r(x)\cap Y\subset E$).

Note: $B_r(x)\cap Y=\{y\in Y \mid d(y,x)<r\}$ is simply the ball of radius r centered at x, assuming that Y is the metric space. Thus, if we are discussing both X and $Y\subset X$, we will write $B_r(x)$ for the ball in X and $Y\cap B_r(x)$ for the ball in Y.

Example: Let $S=\{r\in\mathbb{Q} \mid 0<r<1\}$. If regarded as a subset of $\mathbb{R}$, S is not open (why?). But S is open relative to $\mathbb{Q}$.

Theorem: Let (X,d) be a metric space and let $Y\subset X$. Then $S\subset Y$ is open relative to Y iff there exists an open set E in X such that $S=Y\cap E$.

Proof: Suppose first that $S=Y\cap E$, where $E\subset X$ is open. If $y\in S$, then $y\in E$ and hence, since E is open, there exists $r>0$ such that $B_r(y)\subset E$. But then $Y\cap B_r(y)\subset Y\cap E=S$, which shows that y is an interior point of S (relative to Y). Since y was chosen arbitrarily, this shows that S is open relative to Y.

Conversely, suppose that $S\subset Y$ is open relative to Y. Then, for each $y\in S$, there exists $r_y>0$ such that $Y\cap B_{r_y}(y)\subset S$. Define

$$E=\bigcup_{y\in S} B_{r_y}(y).$$

Then E is open in X (because an arbitrary union of open sets is open) and

$$Y \cap E = Y \cap \left(\bigcup_{y \in S} B_{r_y}(y) \right)$$

$$= \bigcup_{y \in S} \left(Y \cap B_{r_y}(y) \right),$$

which shows that $Y \cap E \subset S$ (since $Y \cap B_{r_y}(y) \subset S \ \forall y \in S$). Since $S \subset Y \cap E$ obviously holds (by definition, $y \in B_{r_y}(y) \subset E \ \forall y \in S$), we see that $S = Y \cap E$, as desired. //