Joint and conditional Distributions

Definition : $F : \mathbb{R}^2 \to [0,1]$ is a joint distribution function of X & Y if

$$F(x,y) = P(X \le x, Y \le y)$$

If X, Y are discrete, the joint mass function $p : \mathbb{R}^2 \to [0,1]$ is given by

$$p(x,y) = P(X = x, Y = y).$$

$$A_x = \{ \omega : X(\omega) = x \}$$
$$B_x = \{ \omega : Y(\omega) = y \}$$

$$p(x,y) = P(A_x \cap B_y)$$

$$P(X = x) = P\left(\bigcup_y \left(\{X = x\} \cap \{Y = y\} \right) \right)$$
$$= \sum_y P(X = x, Y = y)$$
$$= \sum_y p(x,y)$$

The marginal mass function of X :

$$p_X(x) = \sum_y p(x,y)$$

Similarly,

$$p_Y(y) = \sum_x p(x,y)$$

Definition : $E[XY] = \sum_x \sum_y xy\, p(x,y)$

Theorem : $E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p(x,y)$.

Theorem : If X, Y are independent,
then $E(XY) = E(X) \cdot E(Y)$.

Proof : $A_x = \{X = x\}$
$B_y = \{Y = y\}$

$$XY = \sum_x \sum_y xy\, I_{A_x \cap B_y}$$

$$E(XY) = \sum_x \sum_y xy\, E(I_{A_x \cap B_y})$$

$$= \sum_x \sum_y xy\, P(A_x \cap B_y)$$

$$= \sum_x \sum_y xy\, P(A_x) \cdot P(B_y)$$

$$= \sum_x \sum xy \cdot p(x,y)$$

$\square$

Remark : ① If g, h are functions & X, Y are independent, then

$$E[g(X)\, h(Y)] = \sum_x \sum_y g(x)\, h(y)\, p(x,y)$$

② If $E(XY) = E(X) \cdot E(Y)$, then X & Y are called uncorrelated.

Ex: Let $X \sim Geo(\alpha)$, $Y \sim Geo(\beta)$ be independent. Find the pmf of

$$Z = \min\{X, Y\}.$$

Solution:

$$P(Z > k) = P(X > k, Y > k)$$
$$= P(X > k)\, P(Y > k)$$
$$= (1-\alpha)^k \cdot (1-\beta)^k = ((1-\alpha)(1-\beta))^k$$

Let $k \in \mathbb{N}$

$$P(Z = k) = P(Z > k-1) - P(Z > k)$$
$$= ((1-\alpha)(1-\beta))^{k-1} - ((1-\alpha)(1-\beta))^k$$
$$= ((1-\alpha)(1-\beta))^{k-1} (1 - (1-\alpha)(1-\beta))$$

$$\Longrightarrow \quad Z \sim Geo((1-\alpha)(1-\beta))$$

Definition: The covariance of X & Y is

$$\operatorname{cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

Theorem: $V(X+Y) = V(X) + V(Y) + 2\operatorname{cov}(X, Y)$

$$E[(X+Y)^2] - (E(X+Y))^2$$

$$= E[X^2 + 2XY + Y^2] - (E(X))^2 - 2E(X)E(Y) - (E(Y))^2$$
$$= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 + 2[E(XY) - E(X)E(Y)]$$
$$= V(X) + V(Y) + 2\operatorname{cov}(X, Y)$$

Theorem: If X & Y are independent, then $\mathrm{cov}(X,Y) = 0$.

Remark: If X & Y are independent, then $V(X+Y) = V(X) + V(Y)$.

Theorem: Cauchy - Schwartz inequality.

If $E(X^2), E(Y^2) < \infty$, then

$$(E(XY))^2 \leq E(X^2)\, E(Y^2)$$

Theorem: $E(aX+bY)^2 \geq 0$, for any $a, b \in \mathbb{R}$.

$0 \leq E(aX+bY)^2 = a^2 E(X^2) + 2ab\, E(XY) + b^2 E(Y)^2$ is a quadratic function of a.

Since $E[(aX+bY)^2] \geq 0$, this can have at most one real solution. Therefore,

$$\text{Discriminant} = (2b\, E(XY))^2 - 4\,(E(X^2)\, b^2 E(Y)^2) \leq 0$$

$$\Rightarrow E(XY)^2 \leq E(X^2)\, E(Y^2)$$

$\square$

Sums of random variables

Theorem: If X & Y are independent discrete rvs, then

$$P(X+Y=z) = \sum_x P_{X,Y}(x, z-x)$$

Proof: $\{X+Y=z\} = \bigcup \left\{ \{X=x\} \cap \{Y=z-x\} \right\}$

$$\Rightarrow P(X+Y=z) = \sum_x P(X=x, Y=z-x)$$

Remark: If X & Y are independent, then

$$P(X+Y=z) = P_{X+Y}(z) = \sum_x P_X(x) P_Y(z-x)$$

$$\text{or} = \sum_y P_X(z-y) P_Y(y)$$

This sum is called the convolution of X & Y

Notation: $P_{X+Y} = P_X * P_Y$

Ex: Sums of independent binomial.

$X_1 \sim B(n_1, p), \quad X_2 \sim B(n_2, p)$

$$P(X_1 + X_2 = k) = \sum_{m=0}^{k} P(X_1 = m) \cdot P(X_2 = k-m)$$

$$= \sum_{m=0}^{k} \binom{n_1}{m} p^m q^{n_1 - m} \cdot \binom{n_2}{k-m} p^{k-m} q^{n_2 - (k-m)}$$

$$= p^k q^{(n_1+m_2)-k} \cdot \underbrace{\sum_{m=0}^{k} \binom{n_1}{m}\binom{n_2}{k-m}}_{\binom{n_1+n_2}{k}} = \binom{n_1+n_2}{k} p^k q^{(n_1+n_2)-k}$$

$\Rightarrow X_1 + X_2 \sim B(n_1+n_2, p)$