Math 600 Lecture 16

Definition: Let $\{x_n\}$ be a sequence of real numbers. We call

$$(*) \qquad \sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots$$

the infinite series with terms $x_1, x_2, x_3, \ldots$ For each $N \in \mathbb{Z}^+$,

$$\sum_{n=1}^{N} x_n = x_1 + x_2 + \cdots + x_N$$

is called a partial sum of the infinite series. We say that the series $(*)$ converges and write

$$\sum_{n=1}^{\infty} x_n = s$$

iff the sequence of partial sums converges to s:

$$\lim_{N \to \infty} \sum_{n=1}^{N} x_n = s.$$

The series $(*)$ diverges iff the sequence of partial sums diverges.

Note that we sometimes deal with series of the form

$$\sum_{n=0}^{\infty} x_n$$

or

$$\sum_{n=n_0}^{\infty} x_n \qquad (n_0 \in \mathbb{Z}).$$

Example: Note the following algebraic identity:

$$(1-r)(1+r+r^2+\cdots+r^N) = 1-r^{N+1}.$$

Proof: $(1-r)(1+r+r^2+\cdots+r^N) = 1+\cancel{r}+\cancel{r^2}+\cdots+\cancel{r^N}$
$-(\cancel{r}+\cancel{r^2}+\cancel{r^3}+\cdots+r^{N+1}) = 1-r^{N+1}$ //

Thus

$$\sum_{n=0}^{N} r^n = 1+r+\cdots+r^N = \frac{1-r^{N+1}}{1-r}$$

$$\Rightarrow \lim_{N\to\infty} \sum_{n=0}^{N} r^n = \frac{1}{1-r}, \text{ provided } |r|<1.$$

$$\Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ if } |r|<1.$$

If $|r|\geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.

We call such a series a <u>geometric series</u>.

Note that, if $M>N$, then

$$\sum_{n=1}^{M} x_n - \sum_{n=1}^{N} x_n = \sum_{n=N+1}^{M} x_n.$$

The following theorem is the Cauchy criterion for convergence.

Theorem: Let $\{x_n\}$ be a sequence of real numbers. Then

$$\sum_{k=1}^{\infty} x_k$$

converges iff, for all $\varepsilon>0$, there exists $N\in\mathbb{Z}^+$ such that

$$m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} x_k \right| < \varepsilon.$$

Proof: Just apply the Cauchy criterion for sequences (a sequence converges iff it is Cauchy) to the sequence of partial sums. //

Theorem: If $\{x_n\}$ is a sequence of real numbers and

$$\sum_{n=1}^{\infty} x_n$$

converges, then $x_n \to 0$.

Proof: Let $\varepsilon > 0$. Since the series converges, the previous theorem implies that there exists $N \in \mathbb{Z}^+$ such that

$$m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} x_k \right| < \varepsilon.$$

In particular,

$$n \geq N \Rightarrow \left| \sum_{k=n}^{n} x_k \right| < \varepsilon \Rightarrow |x_n| < \varepsilon.$$

Thus $x_n \to 0$. //

The converse of the previous theorem is not true.

Example: Consider the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Write

$$S_N = \sum_{n=1}^{N} \frac{1}{n}$$

and note that

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = S_2 + \frac{1}{3} + \frac{1}{4} > S_2 + \frac{2}{4} = S_2 + \frac{1}{2} \geq 1 + 2 \cdot \frac{1}{2}$$

$$S_8 = S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > S_4 + \frac{4}{8} = S_4 + \frac{1}{2} \geq 1 + 3 \cdot \frac{1}{2}$$

$$S_{16} = S_8 + \frac{1}{9} + \cdots + \frac{1}{16} > S_8 + \frac{8}{16} = S_8 + \frac{1}{2} \geq 1 + 4 \cdot \frac{1}{2}$$

$$\vdots \qquad \vdots$$

Continuing in this fashion, it is easy to show that

$$S_{2^n} > 1 + \frac{n}{2}$$

Since $\{s_n\}$ is increasing, it follows that $s_n \to \infty$.

<u>Theorem</u>: Let $\{x_n\}$ be a sequence of nonnegative real numbers. Then $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums is bounded.

<u>Proof</u>: This follows immediately from the fact that

$$x_n \geq 0 \ \forall n \Rightarrow \left\{ \sum_{n=1}^{N} x_n \right\} \text{ is increasing.} //$$

Theorem (The Cauchy condensation test): Let $\{x_n\}$ be a decreasing sequence of nonnegative real numbers. Then

$$\sum_{n=1}^{\infty} x_n$$

converges iff

$$\sum_{k=0}^{\infty} 2^k x_{2^k}$$

converges.

Proof: Since the partial sums of both sequences are increasing, it suffices to prove that

$$(*) \qquad \sum_{k=1}^{2^n-1} x_k \leq \sum_{k=0}^{n-1} 2^k x_{2^k} \leq 2 \sum_{k=1}^{2^{n-1}} x_k .$$

(Why?) We have

$$\sum_{k=1}^{2^1-1} x_k = x_1$$

$$\sum_{k=0}^{0} 2^k x_{2^k} = x_1$$

$$2 \sum_{k=1}^{2^{n-1}} x_k = 2x_1$$

So $(*)$ holds for $n=1$.

Suppose, by way of induction, that $(*)$ holds for some $n \geq 1$.

Then

$$\sum_{k=1}^{2^{n+1}-1} x_k = \sum_{k=1}^{2^n-1} x_k + \overbrace{x_{2^n} + \cdots + x_{2^{n+1}-1}}^{2^n \text{ terms}}$$

$$\le \sum_{k=0}^{n-1} 2^k x_{2^k} + 2^n x_{2^n} \quad (\text{since } x_{2^n+j} \le x_{2^n}\ \forall j \ge 0)$$

$$= \sum_{k=0}^{n} 2^k x_{2^k}$$

$$\sum_{k=0}^{n} 2^k x_{2^k} = \sum_{k=0}^{n-1} 2^k x_{2^k} + 2^n x_{2^n}$$

$$\le 2 \sum_{k=1}^{2^{n-1}} x_k + 2\left(2^{n-1} x_{2^n}\right)$$

$$\le 2 \sum_{k=1}^{2^{n-1}} x_k + 2\left(x_{2^{n-1}+1} + \cdots + x_{2^n}\right)$$

$$= 2 \sum_{k=1}^{2^n} x_k .$$

Thus,

$$\sum_{k=1}^{2^{n+1}-1} x_k \le \sum_{k=0}^{n} 2^k x_{2^k} \le 2 \sum_{k=1}^{2^n} x_k$$

and hence, by induction, it follows that

$$\sum_{k=1}^{2^n-1} x_k \le \sum_{k=0}^{n-1} 2^k x_{2^k} \le 2 \sum_{k=1}^{2^{n-1}} x_k$$

holds for all $n \in \mathbb{Z}^+$. //