MATH 630

Convergence of Random Variables and Law of Large Numbers - Problems

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(1) Let $\{X_n\}$ be a sequence of independent random variables and $X_n \sim N(c^n, \frac{\sqrt{n}}{2})$ where $0 < c < 1$. Does $\{X_n\}$ obey WLLN? SLLN?

$$\frac{1}{n^2} V(X_1 + \cdots + X_n) = \frac{1}{n^2}\left[\frac{\sqrt{1}}{2} + \frac{\sqrt{2}}{2} + \cdots + \frac{\sqrt{n}}{2}\right]$$

$$\leq \frac{1}{n^2}\left[\frac{\sqrt{n}}{2} + \cdots + \frac{\sqrt{n}}{2}\right]$$

$$= \frac{n}{n^2} \cdot \frac{\sqrt{n}}{2} = \frac{1}{2\sqrt{n}} \longrightarrow 0$$

Therefore, by Markov's theorem, $\{X_n\}$ obeys WLLN.

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} = \sum_{n=1}^{\infty} \frac{\frac{\sqrt{n}}{2}}{n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$$

$\Rightarrow$ By Kolmogorov's theorem, $\{X_n\}$ obeys SLLN.

(2) Let $\{X_n : n \geq 2\}$ be a sequence of independent random variables such that

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n\ln(n)}$$

$$P(X_n = 0) = 1 - \frac{1}{n\ln(n)}$$

Determine whether $\{X_n : n \geq 2\}$ obeys WLLN.

$E(X_n) = 0$ & $E(X_n^2) = n^2 \cdot \dfrac{1}{2n\ln(n)} + n^2 \cdot \dfrac{1}{2n\ln(n)}$

$$E(X_n^2) = \frac{n}{\ln(n)} = V(X_n)$$

Then,

$$V(X_1 + \dots + X_n) = \sum_{i=1}^{n} V(X_i) = \left[\frac{2}{\ln(2)} + \frac{3}{\ln(3)} + \dots + \frac{n}{\ln(n)}\right]$$

$$\leq \frac{n}{\ln(n)} + \dots + \frac{n}{\ln(n)}$$

$$= \frac{n^2}{\ln(n)}$$

Let $S_n = X_1 + \dots + X_2$. Then,

$E(S_n^2) = E(X_1^2 + \dots + X_n^2)$ (Because $X_1, \dots X_n$ are independent & $E(X_i) = 0$ $\forall i$.)

$$\leq \frac{n^2}{\ln(n)}.$$

Also, $\dfrac{E(S_n)}{n} = 0$.

Therefore,

$$E\left[\left(\frac{S_n}{n}-0\right)^2\right] = E\left(\frac{S_n^2}{n^2}\right) \leq \frac{1}{\ln(n)} \longrightarrow 0$$

$$\Rightarrow \quad \frac{S_n}{n} \xrightarrow{r=2} 0 .$$

Let $\varepsilon > 0$. Then, by Chebyshev's theorem,

$$P\left[\left|\frac{S_n}{n}\right| \geq \varepsilon\right] \leq \frac{E\left[\left(\frac{S_n}{n}\right)^2\right]}{\varepsilon^2} \longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

Hence, $\frac{S_n}{n} \xrightarrow{P} 0 \Rightarrow \{X_n\}$ obeys WLLN $\square$

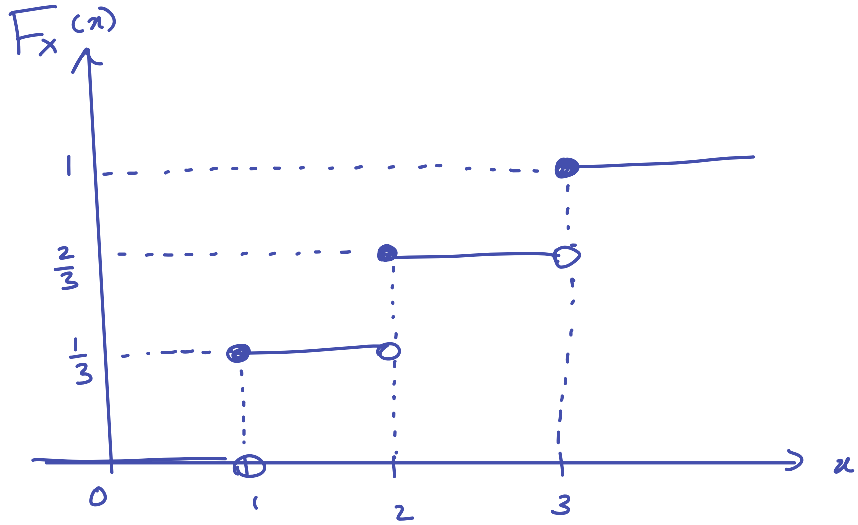
(3) Let $\{X_n\}$ be a sequence of random variables such that

$$P(X_n = k) = \frac{n+k}{3n+10}$$

and

$$P(X = k) = \frac{1}{3}$$

for $k = 1, 2, 3$. Prove that $X_n \to X$ in distribution.



- $x < 1$: Then $F_X(x) = 0$ &

$$F_n(x) = P(X_n \le x) = 0$$

$$\Rightarrow \lim_{n \to \infty} F_n(x) = F(x).$$

- $1 < x < 2$: $F_X(x) = \frac{1}{3}$

$$F_n(x) = \frac{n+1}{3n+10} \longrightarrow \frac{1}{3} \Rightarrow \lim_{n \to \infty} F_n(x) = F(x) = \frac{1}{3}$$

- $2 < x < 3$: $F_X(x) = \frac{2}{3}$

$$F_n(x) = P(X_n = 1) + P(X_n = 2) = \frac{n+1}{3n+10} + \frac{n+2}{3n+10} \longrightarrow \frac{2}{3} = F(x)$$

- $x > 3$: Similar to previous cases.

(4) If $\lim_{n\to\infty} E\left(\frac{|X_n|}{1+|X_n|}\right) = 0$, then prove that $X_n \to 0$ in probability.

Let $g(x) = \frac{x}{x+1}$. $g'(x) > 0 \quad \forall x \geq 0$.

Let $\varepsilon > 0$. By Markov's inequality,

$$P\left[\,|X_n| \geq \varepsilon\right] = P\left[g|X_n| \geq g(\varepsilon)\right]$$

$$\leq \frac{E\left[g|X_n|\right]}{g(\varepsilon)}$$

$$= \frac{E\left[\frac{|X_n|}{|X_n|+1}\right]}{\frac{\varepsilon}{\varepsilon+1}} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow X_n \xrightarrow{P} 0.$$

$\square$

(5) You are given the following information about $\{X_n\}$.

- For some $M > 0$, $V(X_n) \leq M$ for all $n = 1, 2, ...$
- $cov(X_i, X_j) < 0$ for all $i \neq j$.

Prove that $\{X_n\}$ obeys WLLN.

$$\frac{1}{n^2} V(\sum X_i) = \frac{1}{n^2} \left[\sum_{i=1}^{n} V(X_i) + 2\sum_{i<j} \underbrace{cov(X_i, X_j)}_{<0} \right]$$

$$\leq \frac{1}{n^2} \cdot \sum_{i=1}^{n} \underbrace{V(X_i)}_{\leq M}$$

$$\leq \frac{1}{n^2} Mn = \frac{M}{n} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $\{X_n\}$ obeys WLLN.

(6) Let $\{X_n\}$ be a sequence of random variables with finite mean and $V(X_k) = \sigma^2 < \infty$ for all k. If $cov(X_i, X_j) \leq 0$ for all $i \neq j$, prove that

$$Y_n/n \xrightarrow{2} 0$$

where $Y_n = \sum_{i=1}^{n}(X_i - E(X_i))$.

$$0 \leq E\left[\left(\frac{1}{n} Y_n\right)^2\right] = \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n}(X_i - E(X_i))\right)^2\right]$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} \underbrace{E(X_i - E(X_i))^2}_{V(X_i) = \sigma^2} + \sum_{i \neq j} \underbrace{E\left[(X_i - E(X_i))(X_j - E(X_j))\right]}_{\leq 0}\right]$$

$$\leq \frac{1}{n^2} \cdot \sigma^2 n = \frac{\sigma^2}{n} \to 0 .$$

$$\Rightarrow \quad \frac{Y_n}{n} \xrightarrow{r=2} 0 .$$

Notation: $\not\to$ Does not converge

(7) If $X_n \to X$ a.s. and $Y_n \to Y$ a.s., then prove that $X_n + Y_n \to X + Y$ a.s.

$X_n + Y_n$ does not convergerge to Y if $X_n \not\to X$ or $Y_n \not\to Y$. Therefore,

$$\{X_n + Y_n \not\to X+Y\} \subset \{X_n \not\to X\} \cup \{Y_n \not\to Y\}$$

$$P\{X_n + Y_n \not\to X+Y\} \leq \underbrace{P(X_n \not\to X)}_{=0} + \underbrace{P(Y_n \not\to Y)}_{=0}$$

$$\Rightarrow P(X_n + Y_n \not\to X+Y) = 0 \quad \square$$

(8) Let $X_n \sim \exp(n)$. Prove that $X_n \to 0$ in probability.

$E(X_n) = \frac{1}{n}$. Let $\varepsilon > 0$.

$$P\left[|X_n - 0| \geq \varepsilon\right] \leq \frac{E|X_n|}{\varepsilon} = \frac{E(X_n)}{\varepsilon} = \frac{1}{n\varepsilon} \to 0$$

$\square$

$X_n \xrightarrow{P} 0$

(9) Prove or give a counter example. If $\cancel{Y_n \xrightarrow{P} 0}$., then $\lim_{n\to\infty} E(X_n) = 0$.

This is not true. Consider

$$P(X_n = n) = \frac{1}{n}, \quad P(X_n = 0) = 1 - \frac{1}{n}.$$

Let $\varepsilon > 0$. Then,

$$P(X_n \geq \varepsilon) = \begin{cases} \frac{1}{n} ; & n \geq \varepsilon \\ 0 ; & n < \varepsilon \end{cases}$$

Therefore, $P(|X_n| \geq \varepsilon) \longrightarrow 0 \implies X_n \xrightarrow{P} 0$.

However, $E(X_n) = 1 \quad \forall n \in \mathbb{N}$.