Math 672 Lecture 27

Theorem (the spectral theorem for complex normal operators): Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$ be normal. Then there exists an orthonormal basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal.

Proof: By Schur's theorem, there exists an orthonormal basis $\mathcal{B}$ of V such that $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular. We will show that A must be diagonal. Recall that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T^*) = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)^* = A^*$. Write $\mathcal{B} = \{v_1, \ldots, v_n\}$. We have

$$T(v_j) = \sum_{i=1}^{j} A_{ij} v_i, \quad T^*(v_j) = \sum_{i=1}^{n} (A^*)_{ij} v_i = \sum_{i=1}^{n} \bar{A}_{ji} v_i$$

(using the fact that A is upper triangular and hence A^* is lower triangular). In particular,

$$T(v_1) = A_{11} v_1, \quad T^*(v_1) = \sum_{i=1}^{n} \bar{A}_{1i} v_i$$

and

$$\|T(v_1)\|^2 = \|T^*(v_1)\|^2 \Rightarrow |A_{11}|^2 = \sum_{i=1}^{n} |A_{1i}|^2$$
$$\Rightarrow \sum_{i=2}^{n} |A_{1i}|^2 = 0$$

$$\Rightarrow A_{12} = A_{13} = \cdots = A_{1n} = 0.$$

It follows that

$$T(v_2) = A_{12}v_1 + A_{22}v_2 = A_{22}v_2 \quad (\text{since } A_{12} = 0),$$

$$T^*(v_2) = \sum_{i=2}^{n} \overline{A}_{2i} v_i,$$

so

$$\|T(v_2)\|^2 = \|T^*(v_2)\|^2 \Rightarrow |A_{22}|^2 = \sum_{i=2}^{n} |A_{2i}|^2$$

$$\Rightarrow \sum_{i=3}^{n} |A_{2i}|^2 = 0$$

$$\Rightarrow A_{23} = A_{24} = \cdots = A_{2n} = 0.$$

Continuing in this fashion, we can show that all entries of A above the diagonal are zero, and hence A is diagonal. //

<u>Alternate proof:</u> We argue by induction on the dimension of V. If $\dim(V) = 1$, the result holds because every 1×1 matrix is diagonal. Suppose the result holds for vector spaces of dimension $n-1$, where $n \geq 2$, let $\dim(V) = n$, and let $T \in \mathcal{L}(V)$ be normal. Since V is a complex vector space, T has an eigenpair λ_1, v_1. Wlog we can assume that $\|v_1\| = 1$. Define $U = \operatorname{span}(v_1)$; then $V = U \oplus U^\perp$

Now, since T is normal,

$$T(v_1) = \lambda_1 v_1 \Rightarrow T^*(v_1) = \overline{\lambda_1} v_1$$
$$\Rightarrow U \text{ is invariant under } T^*$$
$$\Rightarrow U^\perp \text{ is invariant under } T \text{ (by an earlier result).}$$

Of course, U is also invariant under T and hence $U^\perp$ is invariant under T^*.

Define $S: U^\perp \to U^\perp$ by $S(u) = T(u)\ \forall u \in U^\perp$. We claim that S is normal. For $u, w \in U^\perp$,

$$\langle S(u), w\rangle = \langle T(u), w\rangle = \langle u, T^*(w)\rangle.$$

Since $T^*(w) \in U^\perp$ for all $w \in U^\perp$, this shows that $S^*: U^\perp \to U^\perp$ is defined by $S^*(w) = T^*(w)\ \forall w \in U^\perp$. It now follows that

$$\forall u \in U^\perp,\ (S^*S)(u) = S^*(S(u)) = S^*(T(u)) = T^*(T(u))$$
$$= T(T^*(u))$$
$$= T(S^*(u))$$
$$= S(S^*(u)).$$

Thus S is normal. Applying the induction hypothesis, there is an orthonormal basis $\{v_2, \ldots, v_n\}$ for $U^\perp$ such that

$$S(v_j) = \lambda_j v_j \text{ for } j = 2, 3, \ldots, n$$

$\Rightarrow T(v_j) = \lambda_j v_j$ for $j = 2, 3, \dots, n$

$\Rightarrow T(v_j) = \lambda_j v_j$ for $j = 1, 2, \dots, n$.

Since $\{v_1, v_2, \dots, v_n\}$ is orthonormal, this shows that T is diagonalizable by an orthonormal basis, and the proof by induction is complete. //

Next, we wish to prove the spectral theorem for real self-adjoint operators. (Note that the preceding theorem applies to a self-adjoint operator on a complex space.) The proof is similar to the alternate proof above, given the following lemma.

Lemma: Let V be a real inner product space and let $T \in \mathcal{L}(V)$ be self-adjoint. Then T has an eigenvalue.

We will give the proof of the lemma after the theorem.

Theorem (The spectral theorem for real self-adjoint operators): Let V be a real inner product space and let $T \in \mathcal{L}(V)$ be self-adjoint. Then there exists an orthonormal basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal.

Proof: We argue by induction on $\dim(V)$. If $\dim(V)=1$, the result is obviously true, so suppose it holds in every real inner product space of dimension $n-1$, where $n\geq 2$. Let V be a real inner product space of dimension n and let $T\in\mathcal{L}(V)$ be self-adjoint. By the preceding lemma, T has an eigenpair λ_1, v_1 ($\lambda_1\in\mathbb{R}$). Assume, wlog, that $\|v_1\|=1$, and define $U=\text{span}(v_1)$. Then $V=U\oplus U^\perp$.

We know that $U^\perp$ is invariant under $T^*=T$. It is straightforward to show that $S=T|_{U^\perp}$ is a self-adjoint element of $\mathcal{L}(U^\perp)$. Hence, by the induction hypothesis, there exists an orthonormal basis $\{v_2,v_3,\ldots,v_n\}$ of $U^\perp$ and scalars $\lambda_2,\lambda_3,\ldots,\lambda_n\in\mathbb{R}$ such that

$$S(v_j)=\lambda_j v_j \text{ for } j=2,3,\ldots,n.$$

But then $T(v_j)=S(v_j)=\lambda_j v_j$ for $j=2,3,\ldots,n$ and hence $\{v_1,v_2,\ldots,v_n\}$ is an orthonormal basis for V with

$$T(v_j)=\lambda_j v_j,\ j=1,2,\ldots,n.$$

This completes the proof by induction. //

It remains to prove that every real, self-adjoint operator has a (real) eigenvalue.

- <u>Proof 1</u>: Complexify V and T to get $V_{\mathbb{C}}$ and $T_{\mathbb{C}}$. Prove that $T_{\mathbb{C}}$ is a self-adjoint element of $\mathcal{L}(V_{\mathbb{C}})$. Then $T_{\mathbb{C}}$ has an eigenvalue λ (since $V_{\mathbb{C}}$ is a complex space) that must be real since $T_{\mathbb{C}}$ is self-adjoint. Prove that λ is also an eigenvalue of T.

  This is straightforward but a bit tedious.

- <u>Proof 2</u>:

  <u>Lemma</u>: Let V be a finite-dimensional real inner product space, let $T\in\mathcal{L}(V)$ be self-adjoint, and let $b,c\in\mathbb{R}$ satisfy $b^2-4c<0$. Then

$$T^2+bT+cI$$

  is invertible.

  <u>Proof</u>: For any $v\in V$, $v\neq 0$, we have

$$\begin{aligned}\langle (T^2+bT+cI)(v),v\rangle &= \langle T^2(v),v\rangle + b\langle T(v),v\rangle + c\langle v,v\rangle\\ &= \langle T(v),T(v)\rangle + b\langle T(v),v\rangle + c\langle v,v\rangle\\ &\geq \|T(v)\|^2 - b\|T(v)\|\,\|v\| + c\|v\|^2\\ &= \|T(v)\|^2 - b\|T(v)\|\,\|v\| + \frac{b^2}{4}\|v\|^2\\ &\quad + \left(c-\frac{b^2}{4}\right)\|v\|^2\\ &= \left(\|T(v)\|-\frac{b}{2}\|v\|\right)^2 + \frac{4c-b^2}{4}\|v\|^2\\ &\geq \frac{4c-b^2}{4}\|v\|^2 > 0 \quad (\text{since } 4c-b^2>0).\end{aligned}$$

Note how we used the Cauchy-Schwarz inequality:

$$|\langle T(v),v\rangle| \leq \|T(v)\|\,\|v\|$$

$$\Rightarrow \langle T(v),v\rangle \geq -\|T(v)\|\,\|v\|.$$

It follows that $(T^2+bT+cI)(v)\neq 0$ for all $v\neq 0$. Therefore $T^2+bT+cI$ is nonsingular and hence invertible. //

Now let $v\in V$ be nonzero and consider $\{v, T(v), \ldots, T^n(v)\}$, where $n=\dim(V)$. Since this set is linearly dependent, there exist $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, not all 0, such that

$$\alpha_0 v+\alpha_1 T(v)+\cdots+\alpha_n T^n(v)=0.$$

The polynomial

$$\alpha_0+\alpha_1 x+\cdots+\alpha_n x^n$$

can be factored (over $\mathbb{R}$) into a product of linear and irreducible quadratic factors:

(In principle, k or ℓ could be 0. But we will see below that ℓ must be positive.)

$$\alpha_0+\alpha_1 x+\cdots+\alpha_n x^n = c\,(x^2+b_1x+c_1)\cdots(x^2+b_kx+c_k)(x-\lambda_1)\cdots(x-\lambda_\ell)$$

$(b_1,\ldots,b_k, c_1,\ldots,c_k, \lambda_1,\ldots,\lambda_\ell\in\mathbb{R},\ b_j^2-4c_j<0\ \forall j=1,2,\ldots,k)$. It follows that

$$\alpha_0 I+\alpha_1 T+\cdots+\alpha_n T^n = c\,(T^2+b_1T+c_1I)\cdots(T^2+b_kT+c_kI)\cdot(T-\lambda_1 I)\cdots(T-\lambda_\ell I).$$

Now, $\alpha_0 I+\alpha_1 T+\cdots+\alpha_n T^n$ is singular and each

$$T^2+b_jT+c_jI$$

is nonsingular (by the preceding lemma). It follows that there must be at least one factor of the form $T-\lambda_j I$ that is singular. Thus T has at least one eigenvalue $\lambda_j \in \mathbb{R}$. //