Math 600 Lecture 33

Up till now, we have only defined $\int_a^b f$ for $b>a$. We have the convention that $\int_a^a f = 0$. We now define, for $b>a$,

$$\int_b^a f = -\int_a^b .$$

We know that

$$\int_a^b f = \int_a^c f + \int_c^b$$

holds if $a<c<b$. Now suppose $a<b<c$. Then

$$\int_a^c f = \int_a^b f + \int_b^c$$

$$\Rightarrow \int_a^b f = \int_a^c f - \int_b^c f = \int_a^c f + \int_c^b f \quad \left(\text{using } \int_c^b f = -\int_b^c\right).$$

Thus

$$\int_a^b f = \int_a^c f + \int_c^b f$$

holds for all a,b,c, regardless of the order of a,b,c (as long as f is Riemann integrable on $[\min\{a,b,c\}, \max\{a,b,c\}]$). You can check the other orders of a,b,c.

Now suppose that f is continuous on $[a,b]$, $x_0 \in [a,b]$, and we define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_{x_0}^{x} f .$$

Then the fundamental theorem of calculus applies:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[\int_{x_0}^{x+h} f - \int_{x_0}^{x} f \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f = f(x).$$

(The point is that $\int_{x_0}^{x+h} f = \int_{x_0}^{x} f + \int_{x}^{x+h} f$ regardless of the order of the points $x_0, x, x+h$.)

Similarly, if $b > a$, then

$$\int_b^a f' = -\int_a^b f' = -(f(b) - f(a)) = f(a) - f(b).$$

Thus

$$\int_a^b f' = f(b) - f(a)$$

regardless of whether $a < b$ or $a > b$.

Uniform convergence and differentiation

Theorem: Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable on $[a,b]$ (that is, f_n' exists and is continuous on $[a,b]$) for all $n \in \mathbb{Z}^+$, assume that $f_n' \to g$ uniformly, where $g : [a,b] \to \mathbb{R}$, and assume that there exists $x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ is convergent. Then there exists $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ uniformly on $[a,b]$ and $g = f'$.

Proof: Suppose $f_n(x_0) \to c$. For all $x \in [a,b]$, we have

$$\int_{x_0}^{x} f_n' = f_n(x) - f_n(x_0)$$

$$\Rightarrow f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'$$

$$\Rightarrow f_n(x) \to c + \int_{x_0}^{x} g$$

(recall that $f_n' \to g$ uniformly on $[x_0, x]$ or $[x, x_0]$ implies that $\int_{x_1}^{x} f_n' \to \int_{x_0}^{x} g$).

Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = c + \int_{x_0}^{x} g.$$

Since each f_n' is continuous and $f_n' \to g$ uniformly on $[a,b]$, g is continuous on $[a,b]$ and hence, by the fundamental theorem of calculus, f is differentiable and $f' = g$. So far, we have only shown that $f_n \to f$ pointwise.

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow \left(|f_n(x_0) - c| < \frac{\varepsilon}{2} \text{ and } |f_n'(x) - g(x)| < \frac{\varepsilon}{2(b-a)} \ \forall x \in [a,b] \right).$$

Then

$$n \geq N \Rightarrow |f_n(x) - f(x)| = \left| f_n(x_0) + \int_{x_0}^{x} f_n' - c - \int_{x_0}^{x} g \right|$$

$$\leq |f_n(x_0) - c| + \left| \int_{x_0}^{x} (f_n' - g) \right|$$

$$\leq |f_n(x_0) - c| + \left| \int_{x_0}^{x} |f_n' - g| \right|$$

$$< \frac{\varepsilon}{2} + |x - x_0| \frac{\varepsilon}{2(b-a)}$$

$$< \frac{\varepsilon}{2} + (b-a)\frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f_n \to f$ uniformly on $[a,b]$. //

Applications to power series

Let $\{c_n\}$ be a sequence of real numbers, let $x_0 \in \mathbb{R}$, and define $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \forall x \in (x_0 - R, x_0 + R),$$

where $R > 0$ (possibly $R = \infty$) is the radius of convergence of the power series.

Recall that

$$R = \frac{1}{\limsup\limits_{n\to\infty} |c_n|^{1/n}} \quad \left(R = \infty \text{ if } \limsup_{n\to\infty} |c_n|^{1/n} = 0 \right).$$

(Note: We are assuming that $R > 0$; that is, we do not allow $\{c_n\}$ to be such that $R = 0$. Also, we don't care if the series converges or diverges at the endpoints.)

Let us define $f_n : (x_0 - R, x_0 + R) \to \mathbb{R}$ by

$$f_n(x) = \sum_{k=0}^{n} c_k (x - x_0)^k \quad \text{(the nth partial sum).}$$

Then we know that $f_n \to f$ pointwise on $(x_0 - R, x_0 + R)$.

Theorem 1: If $0 < r < R$, then the series converges uniformly to f on $[x_0 - r, x_0 + r]$ (i.e. $f_n \to f$ uniformly on $[-r, r]$). Thus f is continuous on $(x_0 - R, x_0 + R)$.

Proof: Note that

$$|c_n (x - x_0)^n| = |c_n| |x - x_0|^n \leq |c_n| r^n \quad \forall x \in [x_0 - r, x_0 + r]$$

and

$$\sum_{n=0}^{\infty} |c_n| r^n$$

converges (since

$$\limsup_{n \to \infty} \left| |c_n| r^n \right|^{1/n} = r \limsup_{n \to \infty} |c_n|^{1/n} = \frac{r}{R} < 1),$$

Thus, by the Weierstrass M-test, $f_n \to f$ uniformly on $[x_0 - r, x_0 + r]$, and hence f is continuous on $[x_0 - r, x_0 + r]$. Since every $x \in (x_0 - R, x_0 + R)$ lies in $[x_0 - r, x_0 + r]$ for some $r \in (0, R)$, it follows that f is continuous at every $x \in (x_0 - R, x_0 + R)$. //

Theorem 2: f is differentiable on $(x_0 - R, x_0 + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$$

(i.e. "term-by-term" differentiation is valid).

Proof: Note that

$$\limsup_{n\to\infty} |nc_n|^{1/n} = \limsup_{n\to\infty} n^{1/n}|c_n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n}$$

Since $n^{1/n} \to 1$ as $n\to\infty$. Moreover,

$$|(x-x_0)^{n-1}|^{1/n} = |x-x_0|^{\frac{n-1}{n}} = |x-x_0|^{1-\frac{1}{n}} = \frac{|x-x_0|}{|x-x_0|^{1/n}} \to |x-x_0|$$

Since $|x-x_0|^{1/n} \to 1$ as $n\to\infty$ (for $x\neq x_0$). It follows that

$$\sum_{n=1}^{\infty} nc_n(x-x_0)^{n-1}$$

has the same radius of convergence as does $\sum_{n=0}^{\infty} c_n(x-x_0)^n$. Define $g:(x_0-R, x_0+R)\to\mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} nc_n(x-x_0)^{n-1} = \lim_{n\to\infty} f_n'(x) \quad \forall x\in(x_0-R, x_0+R).$$

By the previous theorem, $f_n'\to g$ on every interval $[x_0-r, x_0+r]$ when $r\in(0,R)$, and $g=f'$ by the first theorem above. //