Math 600 Lecture 17

Convergence tests

Theorem (the comparison test): Let $\{x_n\}, \{y_n\}$ be sequences of real numbers and suppose there exists $n_0 \in \mathbb{Z}^+$ such that

$$|x_n| \le y_n \quad \forall n \ge n_0.$$

Then, if

$$\sum_{n=1}^{\infty} y_n$$

converges, then so does

$$\sum_{n=1}^{\infty} x_n.$$

Proof: Assume that $\sum_{n=1}^{\infty} y_n$ converges and let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that

$$m \ge n \ge N \Rightarrow \left| \sum_{k=n}^{m} y_k \right| < \varepsilon$$

$$\Rightarrow \sum_{k=n}^{m} y_k < \varepsilon \quad (\text{since } y_k \ge 0 \ \forall k).$$

Define $N' = \max\{N, n_0\}$. Then

$$m \ge n \ge N' \Rightarrow \left| \sum_{k=n}^{m} x_k \right| \le \sum_{k=n}^{m} |x_k| \quad (\text{by the triangle inequality})$$

$$\le \sum_{k=n}^{m} y_k \quad (\text{by hypothesis})$$

$$< \varepsilon.$$

Thus, by the Cauchy criterion, $\sum_{n=1}^{\infty} x_n$ converges. //

Corollary: If $\{x_n\}$, $\{y_n\}$ are sequences of nonnegative real numbers, there exists $n_0 \in \mathbb{Z}^+$ such that $y_n \geq x_n$ for all $n \geq n_0$, and

$$\sum_{n=1}^{\infty} x_n$$

diverges, then

$$\sum_{n=1}^{\infty} y_n$$

diverges.

Examples:

1. Does $\sum_{n=1}^{\infty} \frac{1}{n+100}$ converge or diverge?

Solution: Note that

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

diverges to ∞ and

$$\frac{1}{n+100} \geq \frac{1}{2n} \quad \forall n \geq 100.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n+100}$$

diverges.

2. Does $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converge or diverge?

<u>Solution</u>: It converges, since

$$\frac{1}{n2^n} \le \frac{1}{2^n} \quad \forall n \ge 1$$

and

$$\sum_{n=1}^{\infty} 2^{-n}$$

is a convergent geometric series

<u>Theorem</u> (the root test) Let $\{x_n\}$ be a sequence of real numbers and define

$$L = \limsup |x_n|^{1/n}.$$

1. If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges

<u>Proof:</u> 1. Suppose $L < 1$ and choose any $r \in (L, 1)$. Then there exists $N' \in \mathbb{Z}^+$ such that

$$N \ge N' \Rightarrow \sup\{|x_n|^{1/n} \mid n \ge N\} < r$$

and thus

$$n \ge N' \Rightarrow |x_n|^{1/n} < r \Rightarrow |x_n| < r^n.$$

Hence, by comparison with the convergent geometric series $\sum_{n=1}^{\infty} r^n$, we see that $\sum_{n=1}^{\infty} x_n$ converges.

2. Suppose $L>1$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k}|^{1/n_k} \to L>1$, and there exists $N \in \mathbb{Z}^+$ such that

$$k \geq N \Rightarrow |x_{n_k}|^{1/n_k} > 1 \Rightarrow |x_{n_k}| > 1.$$

Thus $x_n \not\to 0$ and hence $\sum_{n=1}^{\infty} x_n$ is divergent. //

Note: If $L=1$, then no conclusion can be drawn. For instance:

$$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow L = \limsup_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1 \quad (\text{since } n^{1/n} \to 1 \text{ as } n\to\infty)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow L = \limsup_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2$$

$$= \left(\lim_{n\to\infty} \frac{1}{n^{1/n}}\right)^2 = 1.$$

But the first series diverges, while the second converges.

<u>Theorem</u> (the ratio test) Let $\{x_n\}$ be a sequence of nonzero real numbers (or, at least, $x_n \neq 0$ $\forall n$ sufficiently large).

1. If $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If there exist $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \geq 1,$$

then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof: 1. Suppose $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$. Then there exists $N \in \mathbb{Z}^+$ and $\beta \in (0,1)$ such that

$$n \geq N \Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < \beta \Rightarrow |x_{n+1}| \leq \beta |x_n|.$$

But then

$$|x_{N+1}| \leq \beta |x_N|,$$

$$|x_{N+2}| \leq \beta |x_{N+1}| \leq \beta^2 |x_N|,$$

$$|x_{N+3}| \leq \beta |x_{N+2}| \leq \beta^3 |x_N|,$$

$$\vdots$$

and hence

$$|x_{N+k}| \leq \beta^k |x_N|.$$

Since $0 < \beta < 1$,

$$\sum_{k=0}^{\infty} \beta^k$$

is convergent and hence

$$\sum_{k=0}^{\infty} x_{N+k} \text{ converges}$$

$$\Rightarrow \sum_{n=0}^{\infty} x_n \text{ converges.}$$

2. If

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 \quad \forall n \geq N,$$

then

$$|x_{N+1}| \geq |x_N|,$$

$$|x_{N+2}| \geq |x_{N+1}| \geq |x_N|$$

$$\vdots$$

Thus

$$n \geq N \Rightarrow |x_n| \geq |x_N| > 0$$

and hence $x_n \not\to 0$. Therefore, $\sum_{n=1}^{\infty} x_n$ diverges. //

Note that the second condition of the root test is <u>not</u> equivalent to

$$\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| \geq 1.$$

In fact, for $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

$$\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \liminf_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = \frac{1}{1} = 1.$$

Yet the series converges.