Math 672 Lecture 16

We mentioned that the main significance of invariant subspaces is that they allow us to understand the structure of the operator as reflected in its matrix representation.

In particular, if $T \in \mathcal{L}(V)$, $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V, and $U = \operatorname{span}(u_1, \ldots, u_k)$, $W = \operatorname{span}(u_{k+1}, \ldots, u_n)$ are invariant under T, then $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is block diagonal:

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T) = \left[\begin{array}{c|c} A^{(1,1)} & 0 \\ \hline 0 & A^{(2,2)} \end{array}\right].$$

Suppose we are fortunate enough to have a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V such that each v_j is an eigenvector of T:

$$T(v_j) = \lambda_j v_j, \quad j = 1, 2, \ldots, n.$$

Then

$$T\left(\sum_{j=1}^{n} x_j v_j\right) = \sum_{j=1}^{n} x_j T(v_j) = \sum_{j=1}^{n} \lambda_j x_j v_j,$$

that is,

$$\mathcal{M}_{\mathcal{B}}(v)=x \Rightarrow \mathcal{M}_{\mathcal{B}}(T(v))=\begin{bmatrix}\lambda_1 x_1\\ \lambda_2 x_2\\ \vdots\\ \lambda_n x_n\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

$$=D\,\mathcal{M}_{\mathcal{B}}(v)$$

$(D=\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n))$. Thus

$D\in F^{n\times n}$ is <u>diagonal</u> iff $D_{ij}=0\ \forall i,j=1,2,\dots,n,\ i\neq j$.

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)=D$$

is diagonal in this case. This is the best possible case.

In this case, we say that T is <u>diagonalizable</u>.

As we will see, not every $T\in\mathcal{L}(V)$ is diagonalizable, even when V is a complex vector space (i.e. a vector space over $\mathbb{C}$).

However, it is always possible to choose a basis $\mathcal{B}=\{v_1,v_2,\dots,v_n\}$ so that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular, <u>provided V is a complex vector space</u>.

Note that $A\in F^{n\times n}$ is called <u>upper triangular</u> iff

$$A_{ij}=0\quad\forall i,j=1,2,\dots,n,\ i>j.$$

Examples

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ is diagonal}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \text{ is upper triangular.}$$

Lemma: Given $T \in \mathcal{L}(V)$ and a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ for V, $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular iff

$$T(v_j) \in \operatorname{span}(v_1, v_2, \ldots, v_j) \quad \forall j = 1, 2, \ldots, n.$$

Proof: Recall that, if $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$, then

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i.$$

Thus

$$T(v_j) \in \operatorname{span}(v_1, \ldots, v_j) \iff A_{ij} = 0 \ \forall i > j.$$

The result follows. //

Note that

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T) \text{ is upper triangular} \Rightarrow T(v_1) \in \operatorname{span}(v_1)$$

$$\Rightarrow v_1 \text{ is an eigenvector of } T.$$

This suggests a proof of the following theorem.

Theorem: Let V be an n-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Then there exists a basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular.

Proof: We argue by induction on n, the dimension of V.

First, every matrix $A \in F^{1\times 1}$ is upper triangular, so the result holds for $n=1$.

Suppose the result holds for all complex vector spaces of dimension $n-1$. Let V be a complex vector space of dimension n and let $\lambda_1 \in \mathbb{C}$, $v_1 \in V$ be an eigenvalue/eigenvector pair of T. Extend $\{v_1\}$ to a basis $\{v_1, w_2, \ldots, w_n\}$ of V and define

$$U = \text{span}(v_1), \quad W = \text{span}(w_2, \ldots, w_n).$$

We know that $V = U \oplus W$. Define $P \in \mathcal{L}(V, W)$ by

$$P(u+w) = w \quad \forall\, u+w \in U \oplus W = V$$

and note that

$$(I-P)(u+w) = u \quad \forall\, u+w \in U \oplus W.$$

Define $S \in \mathcal{L}(W, W)$ by

$$S(w) = P(T(w)) \quad \forall\, w \in W.$$

Since $\dim(W) = n-1$, there exists a basis $\mathcal{B}' = \{v_2, \dots, v_n\}$ of W such that $\mathcal{M}_{\mathcal{B}',\mathcal{B}'}(S)$ is upper triangular.

Define

$$A' = \mathcal{M}_{\mathcal{B}',\mathcal{B}'}(S) \in F^{(n-1)\times(n-1)}$$

and write the entries of A' as

$$A'_{ij}, \quad 2 \leq i,j \leq n.$$

Thus

$$Sv_j = \sum_{i=2}^{j} A'_{ij} v_i, \quad j = 2, 3, \dots, n.$$

For each v_j, $j = 2, \dots, n$, we have

$$(I-P)(T(v_j)) \in U = \operatorname{span}(v_1)$$

$$\Rightarrow (I-P)(T(v_j)) = c_j v_1 \text{ for some } c_j \in \mathbb{C}.$$

Thus

$$T(v_1) = \lambda_1 v_1 \in \operatorname{span}(v_1),$$

$$T(v_j) = P(T(v_j)) + (I-P)(T(v_j))$$

$$= (PT)(v_j) + c_j v_1$$

$$= S(v_j) + c_j v_1$$

$$= \sum_{i=2}^{n} A'_{ij} v_i + c_j v_1$$

$$= \sum_{i=1}^{n} A_{ij} v_i \qquad \text{for } j = 2, 3, \ldots, n,$$

where $A \in F^{n \times n}$ is defined by

$$A_{ij} = \begin{cases} \lambda_1 & \text{if } i = j = 1 \\ 0 & \text{if } j = 1, i = 2, \ldots, n \\ c_j & \text{if } i = 1, j = 2, \ldots, n \\ A'_{ij} & \text{if } i = 2, 3, \ldots, n \end{cases}$$

We have

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i \quad \forall j = 1, 2, \ldots, n$$

$$\Rightarrow \quad A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T), \quad \mathcal{B} = \{v_1, v_2, \ldots, v_n\},$$

and A is upper triangular. This completes the proof by induction. //

<u>Theorem</u>: Let V be a finite-dimensional vector space over F, let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, let $T \in \mathcal{L}(V)$, and suppose $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular. Then T is singular iff at least one diagonal entry of A is zero.

<u>Proof:</u> Suppose first that $A_{kk}=0$. We have

$$T(v_j) \in \operatorname{span}(v_1,\dots,v_j) \subseteq \operatorname{span}(v_1,\dots,v_{k-1}) \ \forall j=1,2,\dots,k-1$$

(since A is upper triangular) and

$$T(v_k) = \sum_{i=1}^{k} A_{ik} v_i = \sum_{i=1}^{k-1} A_{ik} v_i \quad (\text{since } A_{kk}=0)$$

$$\Rightarrow T(v_k) \in \operatorname{span}(v_1,\dots,v_{k-1}).$$

Thus

$$\{T(v_1),\dots,T(v_k)\} \subseteq \operatorname{span}(v_1,\dots,v_{k-1})$$

$\Rightarrow \{T(v_1),\dots,T(v_k)\}$ is linearly dependent

$\Rightarrow T$ is singular (Why?).

Conversely, suppose all of the diagonal entries of A are nonzero.

Then

$$T(v_1) = A_{11} v_1 \neq 0 \quad (\text{since } v_1 \neq 0,\ A_{11} \neq 0)$$

$\Rightarrow \{T(v_1)\}$ is linearly independent

Now assume, by way of induction, that $\{T(v_1),\dots,T(v_{k-1})\}$ is linearly independent. We have

$$\{T(v_1),\dots,T(v_{k-1})\} \subseteq \operatorname{span}(v_1,\dots,v_{k-1})$$

and

$$T(v_k) = \sum_{i=1}^{k} A_{ik} v_i = \sum_{i=1}^{k-1} A_{ik} v_i + A_{kk} v_k \notin \text{span}(v_1, \ldots, v_{k-1})$$

since $A_{kk} \neq 0$. Thus $\{T(v_1), \ldots, T(v_k)\}$ is linearly independent.

By induction, it follows that

$$\{T(v_1), \ldots, T(v_n)\} \text{ is linearly independent}$$

$\Rightarrow$ $\dim(\mathcal{R}(T)) \geq n$

$\Rightarrow$ $\mathcal{R}(T) = V$

$\Rightarrow$ T is surjective

$\Rightarrow$ T is injective (since $T \in \mathcal{L}(V)$)

$\Rightarrow$ T is nonsingular.

This completes the proof. //

<u>Corollary</u>: Let V be a finite-dimensional vector space over F, let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, let $T \in \mathcal{L}(V)$, and suppose $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular. Then the eigenvalues of T are precisely the diagonal entries of A.

<u>Proof:</u> For any $\lambda \in I$, $\mathcal{M}_{\mathcal{B},\mathcal{B}}(\lambda I) = \lambda I$ (the first "I" is the

identity operator on V and the second is the identity matrix in $F^{n\times n}$). (The proof is straightforward.) Thus

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T-\lambda I) = A-\lambda I$$

and

$$\lambda\in F \text{ is an eigenvalue of } T$$

$\iff$ $T-\lambda I$ is singular

$\iff$ At least one diagonal entry of $A-\lambda I$ is zero

$\iff$ $A_{kk}-\lambda=0$ for some k, $1\le k\le n$

$\iff$ λ equals one of the diagonal entries of A. //