Proposition: Let $A, B \in \mathcal{F}$.

(i) $P(\phi) = 0$;

(ii) $A \subset B \implies P(A) \leq P(B)$

(iii) $P(A^c) = 1 - P(A)$

(iv) $P(A - B) = P(A) - P(A \cap B)$

(v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: (i) $P(\Omega) = 1$

$$1 = P(\Omega) = P(\Omega \cup \phi)$$

$$= \underbrace{P(\Omega)}_{=1} + P(\phi) \quad (\text{because } \Omega \cap \phi = \phi)$$

$$=$$

$$\implies P(\phi) = 0$$

(iii) $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c) \quad (A \cap A^c = \phi)$

$$\implies P(A^c) = 1 - P(A)$$

iv) $P(A \cup B) = P((A \cup B) \cap \Omega)$

$$= P[(A \cup B) \cap (A \cup A^c)]$$

$$= P[((A \cup B) \cap A) \cup ((A \cup B) \cap A^c)]$$

$$= P[A \cup (B \cap A^c)]$$

$$= P[A] + P(B \cap A^c)$$

$$= P(A) + P(B) - P(A \cap B) \quad \square$$

In general,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} P(A_1 \cap \dots \cap A_n)$$

This is called the inclusion - exclusion principal.

Ex: $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.1$

Find the probability that exactly one of the events A or B occurs.

Symmetric difference:

$$A \triangle B := (A \cap B^c) \cup (B \cap A^c)$$

$$\begin{aligned} P(A \triangle B) &= P(A \cap B^c) + P(B \cap A^c) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= 0.5 + 0.4 - 2(0.1) \\ &= \boxed{0.7} \end{aligned}$$

Theorem: Suppose Ω is a finite set. If all outcomes are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|} \quad \text{for any event } A.$$

Proof: Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$

$$1 = P(\Omega) = \sum_{i=1}^{n} P(\{\omega_i\})$$

$$= n\, P(\{\omega_i\})$$

$$\Rightarrow P(\{\omega_i\}) = \frac{1}{n}$$

$$P(A) = \sum_{i:\omega_i \in A} \underbrace{P(\{\omega_i\})}_{=\frac{1}{n}}$$

$$= \frac{|A|}{|\Omega|} \qquad (|\Omega| = n)$$

$\square$

Combinatorial Analysis

The basic principal of counting : Consider k experiments, each with $n_1, n_2, \ldots, n_k$ outcomes. Then the total outcomes of k experiments is $n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$.

Ex : (a) If you roll a die 10 times, 6^{10} outcomes are possible

(b) How many 5- character passwords can be created using only letters (A-Z, a-z) or numbers (0-9) if the password must contain at least one uppercase letter ? Assume characters can repeat.

Total # of passwords $= 62^5$
Passwords without uppercase letters $= 36^5$

$$\boxed{\text{Answer} = 62^5 - 36^5}$$

The number of ways k distinct objects chosen from a set of n distinct objects can be permuted $= \dfrac{n!}{(n-k)!}$

The of ways k distinct objects can be chosen from a set of n distinct objects $= \dfrac{n!}{k!(n-k)!} = \binom{n}{k}$

If n objects contain n_1 objects that are alike, ..., n_k objects that are alike ($\sum n_i = n$), then the number of permutations of the n objects $= \dfrac{n!}{n_1!\, n_2! \cdots n_k!}$

Ex: (1) A fair die is rolled 4 times. What is the probability that no number appears twice?

$|\Omega| = 6^4$

$|A| = \dfrac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3$

$P(A) = \dfrac{|A|}{|\Omega|} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3}{6^4}$

(2) A teacher distributes 10 identical pieces of candy between 3 students - A, B, C. What is the probability that A gets at least two pieces of candy?

$$|\Omega| = \frac{12!}{2!\,10!} = \binom{12}{2}$$

$$|A| = \frac{10!}{2!\,8!} = \binom{8}{2}$$

$$P(A) = \frac{\binom{8}{2}}{\binom{12}{2}}$$