Math 600
Detailed course outline

Mark S. Gockenbach

All references are to the course textbook, *Fundamentals of Mathematical Analysis* (third edition), by Walter Rudin.

Proposed weekly schedule

Week	Dates	Chapter(s)	Exams
1	Aug. 28, 30 (no Monday class)	1	
2	Sep. 4, 6 (no Monday class)	1, 2	
3	Sep. 9, 11, 13	2	
4	Sep. 16, 18, 20	2	
5	Sep. 23, 25, 27	3	
6	Sep. 30, Oct. 2, 4	3	Exam 1 (Sep. 30)
7	Oct. 7, 9, 11	3	
8	Oct. 14, 16, 18	3, 4	
9	Oct. 21, 23, 25	4	
10	Oct. 28, 30, Nov. 1	5	
11	Nov. 4, 6, 8	6	Exam 2 (Nov. 4)
12	Nov. 11, 13, 15	6	
13	Nov. 18, 20, 22	7	
14	Dec. 2, 4 6	7,8	
15	Dec. 9 (no Wednesday, Friday classes)	8	

Chapter 1: The real and complex number systems The natural numbers can be constructed, using only set theory, to satisfy the Peano axioms. One can then construct the integers from the natural numbers, the rational numbers from the integers (as the *field of quotients* of the integers, a construction that represents a rational number as a certain equivalence class of pairs of integers), the real numbers from the rational numbers (using Dedekind cuts, as described in the Appendix to Chapter 1, pp. 17ff), and finally the complex numbers from the real numbers. However, all of this takes a lot of time and background, and we simply assume that $\mathbb{R}$ (the set of real numbers), together with the usual operations of addition and multiplication, is an *ordered field* that satisfies the *least upper bound property.* This characterization of $\mathbb{R}$ will give us all of the properties that we need to do analysis.

This chapter contains a lot of background material that should be somewhat familiar: elementary set theory, functions and related concepts, a precise definition of order, and the definition of a field. We introduce upper bounds, least upper bound, and the *least upper bound property*, that every set with an upper bound has a least (smallest) upper bound.

As noted above, we take as our assumption that $\mathbb{R}$ is an ordered field that satisfies the least upper bound property. (In fact, $\mathbb{R}$ is the only such field, up to isomorphism, but proving this, like constructing $\mathbb{R}$, would take us too far afield.)

Finally, as background, we discuss the cardinality of sets, specifically, the question of when two sets have the same cardinality (size). A fundamental fact is that $\mathbb{Q}$ (the set of rational numbers) is "smaller" (that is, has a smaller cardinality) than $\mathbb{R}$. (**Note: This material on cardinality appears in Chapter 2, but I think it naturally fits with the background material in Chapter 1.**)

Chapter 2: Basic topology "Basic topology" is the author's title for Chapter 2, but the chapter would be better titled "Metric spaces" or "Basic topology in metric spaces". In a *topological space*, the fundamental concept is open set; in fact, a topological space is simple a set together with a collection of subsets, which are called open, that satisfies certain properties. A *neighborhood* of a point is simply an open set containing that point. In a toppological space X, a sequence $\{x_n\}$ converges to $x \in X$ if and only if, given every neighborhood U of x, there exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implies that $x_n \in U$. Similarly, if X and Y are topological spaces, then $f : X \to Y$ is continuous at $x \in X$ if and only if, for all neighborhoods V of $f(y)$, there exists a neighborhood U of x such that $t \in U$ implies that $f(t) \in V$.

A *metric space* is a set X together a *metric* $d : X \times X \to [0, \infty)$ that defines a notion of distance between points of X. In a metric space, a set U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that

$$B_\epsilon(x) = \{t \in X \,:\, d(t,x) < \epsilon\}$$

is contained in U. It can be shown that the collection of open sets in X, defined in this fashion, forms a topological space. However, we tend to use the metric to define most concepts, rather than rely on the more general definitions based on topology. For instance, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$, then we say that f is continuous at $x \in X$ if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$t \in X \text{ and } d_X(t,x) < \delta \implies d_Y(f(t), f(x)) < \epsilon.$$

Concepts studied in this chapter include open and closed sets, limit points, interior and closure of a set, dense subsets, compact sets, connected sets, and so forth. In future chapters, these concepts will be used to define convergence of sequences and series, continuity of functions, uniform convergence of sequences of functions, etc.

The topic of compactness is especially important. If a subset of a metric is compact, then every subsequence contained in that set has a convergent subsequence, and this property can be used to prove many results. For instance, a continuous function defined on a compact set attains its maximum and minimum on that set.

In addition to the discussion of abstract metric spaces, the topology of $\mathbb{R}$ and $\mathbb{R}^k$ are studied. We learn that, in an Euclidean space $\mathbb{R}^k$ (including the case $k = 1$, which is essentially the case of $\mathbb{R}$, as $\mathbb{R}^1$ is isomorphic to $\mathbb{R}$), a set is compact if and only if it is closed and bounded. Also, we learn that the only connected subsets of $\mathbb{R}$ are intervals (a result that seems intuitively obvious).

The standard metric on $\mathbb{R}^k$ is defined by the Euclidean norm:

$$d(x,y) = \|x - y\|_2, \text{ where } \|x\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2}.$$

However, an interesting point is that there is only one topology on $\mathbb{R}^k$ under which (vector) addition and scalar multiplication are continuous. This is the topology generated by the above metric, but the fact that there is only one topology means that any other norm defined on $\mathbb{R}^k$ defines a metric space with same open sets (even though distance is measured differently). When you get to functional analysis (for example, in Math 806), you will learn that, in infinite-dimensional spaces, different topologies are possible. Compactness is rare in the norm topology (for instance, no open set in an infinite-dimensional normed space is compact) and "weaker" topologies are defined under which more sets are compact.

Chapter 3: Numerical sequences and series Once again, the author's chapter title is a bit misleading, now because sequences are initially treated in an abstract metric space (so they do not have to be "numerical" sequences). After defining convergence of a sequence in a metric space, it is proven (as mentioned above) that every sequence in a compact set has a convergent subsequence. A *Cauchy* sequence is a sequence that "ought" to converge, and a *complete* metric space is one in which every Cauchy sequence does converge. A compact metric space is always complete.

Properties of sequences and convergence in $\mathbb{R}$ (or $\mathbb{C}$) are studied, such as the fact that every bounded monotonic sequence converges, and that

$$s_n \to s \text{ and } t_n \to t \implies s_n \pm t_n \to s \pm t,\ s_n t_n \to st, \text{ and } \frac{s_n}{t_n} \to \frac{s}{t} \text{ (if } t \neq 0).$$

Similarly, in $\mathbb{R}^k$,

$$s_n \to s \text{ and } t_n \to t \implies s_n \pm t_n \to s \pm t \text{ and } s_n \cdot t_n \to s \cdot t$$

(where $x \cdot y$ denotes the dot product of $x, y \in \mathbb{R}^k$). In $\mathbb{R}$, a bounded monotone sequence always has a limit; more specifically, if $\{x_k\}$ is bounded above and increasing, then it has a limit in $\mathbb{R}$, and similarly if $\{x_k\}$ is bounded below and decreasing.

If a sequence $\{x_k\} \subset \mathbb{R}$ fails to converge, it is frequently useful to consider its *limit inferior* (the smallest limit of any convergent subsequence) and *limit superior* (the largest limit of any convergent subsequence).

An *infinite series* of real or complex numbers is defined in terms of its sequence of partial sums:

$$\sum_{n=1}^{\infty} a_n \text{ converges to } a \iff \lim_{N \to \infty} \left(\sum_{n=1}^{N} a_n \right) = a.$$

There are a number of useful tests for convergence of infinite series: comparision test, alternating series test, ratio test, root test, integral test; these are proved and illustrated. The most general test for convergence is the *Cauchy criterion* (that a series converges if and only if its sequence of partial sums is Cauchy).

A *power series* is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

(the sequence of *coefficients* $\{a_n\}$ and the number c are given). Such a series defines a function whose domain is the set of all x for which the series converges:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n.$$

It turns out that the domain of f is always an interval that is symmetric (or at least nearly symmetric) about the point c. Specifically, the domain can have any of the following forms:

$$(c-R, c+R),\ (c-R, c+R],\ [c-R, c+R),\ [c-R, c+R],\ (-\infty, \infty),\ \{c\}.$$

In the first four cases, R is a positive number called the *radius of convergence*, while we say that $R = \infty$ if the interval is $(-\infty, \infty)$ and $R = 0$ is the domain is the degenerate interval $\{c\}$. The radius of convergence of a power series can always be determined (at least in principle) using the root test.

A convergent infinite series $\sum_{n=0}^{\infty} a_n$ is said to be *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |a_n|$ is also convergent; otherwise, $\sum_{n=0}^{\infty} a_n$ is said to be *conditionally convergent*. Absolutely convergent series are well behaved; for instance, the product of two convergent series is also convergent, provided at least one of them is absolutely convergent. Also, if $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then every rearrangement of the terms yields a convergent series (with the same sum).

Chapter 4: Continuity If (X, d_X) and (Y, d_Y) are metric spaces, then $f : X \to Y$ is said to be *continuous* at $t \in X$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d_X(x, t) < \delta$ imply that $d_Y(f(x), f(t)) < \epsilon$. Two equivalent conditions are

$$\lim_{x \to t} f(x) = f(t) \tag{1}$$

and

$$\{x_n\} \subset X \text{ and } x_n \to t \implies f(x_n) \to f(t).$$

(Note that we have to define what is meant by the limit in (1).)

We have already mentioned another equivalent condition for continuity at $t \in X$: f is continuous at $t \in X$ if and only if for every neighborhood V of $f(t)$, there exists a neighborhood U of t such that $f(U) \subset V$ (that is, $f(x) \in V$ for all $x \in U$).

We say that $f : X \to Y$ is *continuous* if it is continuous at every $t \in X$.

The composition of continuous functions is always continuous. Also, for real-valued functions, the sum, difference, product, and quotient of continuous functions are all continuous (as long as, in a quotient, we avoid division by zero).

Continuity and compactness have a rich interplay. The image of a compact set under a continuous function is compact. A continuous real-valued (or $\mathbb{R}^k$-valued) function on a compact set must be bounded; moreover, a real-valued function on a compact set must attain its maximum and minimum values.

A function $f : X \to Y$ is *uniformly continuous* if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$. The point is that δ depends only on ϵ and not on the point in X (as in the definition of "f is continuous at $t \in X$"). An important theorem states that a continuous function on a compact set is always uniformly continuous.

Just as the continuous image of a compact set is compact, so the continuous image of a connected set is connected. This yields the intermediate value theorem of calculus.

This chapter concludes with a discussion of one-sided limits of real-valued functions of a real variable, continuity from the right and from the left, and related results. Discontinuities can be classified according to type (*removable*, *jump*, etc.).

Chapter 5: Differentiation This chapter defines the derivative of a real-valued function of a real variable and proves the basic results of calculus: differentiability implies continuity, linearity of the derivative operator, the product, quotient, and chain rules for differentiation, and Fermat's theorem (local maxima and local minima are stationary points).

The mean value theorem (MVT) and generalized MVT are used to prove many results (such as the relationship between the signs of the derivative and monotonicity of the function).

Two important results related to differentiability are L'Hôpital's rule and (especially) Taylor's theorem. The former is probably familiar from calculus class (though, of course, we emphasize the proof), but few undergraduate students see a thorough treatment of the latter.

The chapter ends with a mean value inequality for vector-values functions of a real variable.

Chapter 6: The Riemann-Stieltjes integral Because of a lack of time, we will treat only the Riemann integral (the integral taught in undergraduate calculus courses). The Riemann integral is most conveniently defined in terms of upper and lower Darboux sums, although we will also discuss the definition given in most calculus books (the Riemann integral as the limit of Riemann sums).

We prove a number of existence theorems. In particular, $f : [a, b] \to \mathbb{R}$ is Riemann integrable if f is continuous or if f is monotone or if f is bounded and has only finitely many discontinuities. We also prove the elementary properties of the Riemann integral (such as linearity, the triangle inequality for integrals, etc.).

We prove the standard results of calculus: both versions of the fundamental theorem, u-substitution, and integration by parts. We end with a brief study of integration of vector-values functions of a real variable.

Chapter 7: Sequences and series of functions A sequence of functions (or a series of function, which is understood in terms of the sequence of partial sums) can converge pointwise or uniformly. Uniform convergence is sufficient for many desirable results. For instance, the uniform limit of continuous functions is continuous (but the pointwise limit of continuous function might be discontinuous), the limit of a sequence of integrals is the integral of the limit of the integrands, *if* the integrands converge uniformly, etc. We will learn the Cauchy criterion for the uniform convergence of a sequence of functions and the Weierstrass M-test for the uniform convergence of a sequence of functions.

We will briefly discuss $C(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$, where X is a given metric space. Under the so-called sup-norm, $C(X)$ is a complete metric space.

If time permits, we will discuss equicontinuity, the Arzela-Ascoli theorem (or, at least, a version of it), and the Stone-Weierstrass theorem.

Chapter 9: Functions of several variables If $f : \mathbb{R} \to \mathbb{R}$, then the derivative $f'(a)$ of f at the point a is defined (at least in calculus classes) to be a number. If $f : \mathbb{R}^n \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{R}^m$ or $f : \mathbb{R}^n \to \mathbb{R}^m$, how should we define "the" derivative of f? What is the role of partial derivatives?

The "grown-up" definition of the derivative is the following: $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a if and only if there exists a linear map $\ell : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - \ell(h)}{h} = 0. \tag{2}$$

Every linear map $\ell : \mathbb{R} \to \mathbb{R}$ has the form $\ell(x) = mx$ for some constant m; when (2) holds, we denote the constant m that defines ℓ by $f'(a)$. Indeed,

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - mh}{h} = 0 \iff \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = m,$$

so the relationship with the usual definition is clear.

By analogy, we say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if and only if there exists a linear map $\ell : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \ell(h)\|}{\|h\|} = 0. \tag{3}$$

Since every linear map $\ell : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $\ell(x) = Ax$ for some matrix $A \in \mathbb{R}^{m\times n}$, we could say that the derivative of f at a *is* the matrix $J \in \mathbb{R}^{m\times n}$ that satisfies

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - Jh\|}{\|h\|} = 0;$$

this matrix turns out to be

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix},$$

which is called the *Jacobian matrix* (and which I will denote by $f'(a)$). The right way to think about this is that the derivative of f at a is the linear map ℓ and $f'(a)$ denotes the representative of this linear map (a scalar if $\ell : \mathbb{R} \to \mathbb{R}$ and an $m \times n$ matrix if $\ell : \mathbb{R}^n \to \mathbb{R}^m$). We can discuss the derivatives of $f : \mathbb{R} \to \mathbb{R}^m$ or $f : \mathbb{R}^m \to \mathbb{R}$ similarly.

If time permits, we will discuss the chain rule for functions of several variables, the contractive mapping theorem, and two related theorems, the inverse function theorem and the implicit function theorem. If we have even more time, we will discuss Taylor's theorem for functions of several variables.