Math 672 Lecture 19

<u>Definition</u>: Let V be an inner product space over $\mathbb{R}$. We say that $u, v \in V$ are <u>orthogonal</u> iff $\langle u, v \rangle = 0$. We say that $\{u_1, u_2, \ldots, u_n\} \subseteq V$ is orthogonal iff $\langle u_i, u_j \rangle = 0$ for all $i, j = 1, \ldots, n$, $i \neq j$.

<u>Note</u>: "Orthogonal" is a generalization of "perpendicular" for Euclidean vectors. For $x, y \in \mathbb{R}^2$, the law of cosines shows that $x \cdot y = \|x\| \|y\| \cos\theta$, where θ is the angle between x and y. Thus $x \cdot y = 0$ iff $\theta = \frac{\pi}{2} = 90°$.

<u>Lemma</u>: Let V be an inner product space over $\mathbb{R}$. Then 0 is orthogonal to every $v \in V$, and if $u \in V$ is orthogonal to every $v \in V$, then $u = 0$.

<u>Proof</u>: We have

$$\langle v, 0 \rangle = 0 \quad \forall v \in V$$

and

$$\langle v, u \rangle = 0 \ \forall v \in V \Rightarrow \langle u, u \rangle = 0$$
$$\Rightarrow u = 0 \text{ (by definition of inner product).} //$$

<u>Theorem</u>: Let V be an inner product space over $\mathbb{R}$ and let u, v be vectors in V. Then u and v are orthogonal iff

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

(This is the Pythagorean theorem for inner product spaces.)

<u>Proof:</u> Note that

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2,$$

so

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \iff \langle u,v\rangle = 0. //$$

The above result explains why we can regard orthogonality as a generalization of perpendicularity.

<u>Example:</u> Consider $V = C[0,1]$ under the L^2 inner product:

$$\langle f,g\rangle_{L^2} = \int_0^1 f(t)g(t)\,dt, \quad \|f\|_{L^2} = \sqrt{\int_0^1 |f(t)|^2\,dt}\ .$$

Define $f_1, f_2 \in C[0,1]$ by

$$f_1(t) = \sin(\pi t), \quad f_2(t) = \sin(2\pi t).$$

Then

$$\begin{aligned}\langle f_1, f_2\rangle_{L^2} &= \int_0^1 \sin(\pi t)\sin(2\pi t)\,dt \\ &= \int_0^1 2\sin^2(\pi t)\cos(\pi t)\,dt \\ &= \frac{1}{\pi}\int_0^0 2u^2\,du \\ &= 0.\end{aligned}$$

$u = \sin(\pi t)$

$du = \pi\cos(\pi t)\,dt$

$t = 0 \leftrightarrow u = 0$

$t = 1 \leftrightarrow u = 0$

Thus f_1 and f_2 are orthogonal, and hence

$$\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 = \|f_1+f_2\|_{L^2}^2$$

$$\iff \int_0^1 \sin^2(\pi t)\,dt + \int_0^1 \sin^2(2\pi t)\,dt = \int_0^1 (\sin(\pi t)+\sin(2\pi t))^2\,dt.$$

<u>Norms and inner products on complex vector spaces</u>

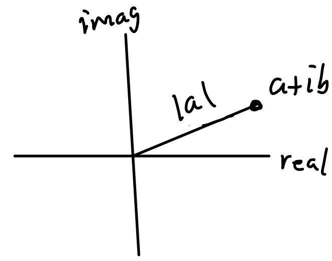
Let V be a vector space over $\mathbb{C}$. The definition of "norm" is unchanged; in particular, a norm on V is still a real-valued function. However, the definition of inner product must change.

<u>Definition</u>: Let V be a complex vector space. An <u>inner product</u> on V is a function $(u,v) \longmapsto \langle u,v\rangle$ mapping $U\times V$ into $\mathbb{C}$ satisfying the following properties:

- $\langle u,u\rangle \geq 0 \ \forall u\in V$ and $\langle u,u\rangle = 0$ iff $u=0$. (Note: $\langle u,u\rangle \in \mathbb{C}$ by assumption, but $\langle u,u\rangle \geq 0$ should be interpreted as "$\langle u,u\rangle$ is real and nonnegative.")
- $\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle \ \forall u,v,w\in V \ \forall \alpha,\beta\in\mathbb{C}$.
- $\langle v,u\rangle = \overline{\langle u,v\rangle} \ \forall u,v\in V$ (where $\overline{a}$ is the complex conjugate of $a\in\mathbb{C}$).

Recall that if $a=\alpha+i\beta\in\mathbb{C}$, then $\overline{a}=\alpha-i\beta$.

We have



- $a\bar{a} = \alpha^2+\beta^2 = |a|^2 \quad \forall\, a=\alpha+i\beta \in \mathbb{C}$
- $\overline{ab} = \bar{a}\bar{b} \quad \forall\, a,b\in\mathbb{C}$
- $\overline{a+b} = \bar{a}+\bar{b} \quad \forall\, a,b\in\mathbb{C}$

Note that the second and third properties of a complex inner product implies that

$$\langle w, \alpha u+\beta v\rangle = \overline{\langle \alpha u+\beta v, w\rangle} = \overline{\alpha\langle u,w\rangle+\beta\langle v,w\rangle}$$
$$= \bar{\alpha}\,\overline{\langle u,w\rangle} + \bar{\beta}\,\overline{\langle v,w\rangle}$$
$$= \bar{\alpha}\langle w,u\rangle + \bar{\beta}\langle w,v\rangle.$$

Thus $\langle \cdot,\cdot\rangle$ is linear in the first argument and <u>conjugate linear</u> in the second. For this reason, a complex inner product is sometimes called <u>sesquilinear</u> ("sesqui" means "one-and-a-half").

The Cauchy-Schwarz inequality still holds for a complex inner product.

<u>Theorem</u>: Let V be a complex inner product space. Then

$$|\langle u,v\rangle| \le \langle u,u\rangle^{1/2}\langle v,v\rangle^{1/2} \quad \forall\, u,v\in V$$

and equality holds iff one of u,v is a multiple of the other.

Proof: If $v=0$, then the inequality holds, as an equation, because both sides are zero. Also, $v=0\cdot u$ in this case.

Suppose $v\neq 0$ and define

$$\lambda = \frac{\langle u,v\rangle}{\langle v,v\rangle}.$$

Then

$$\langle u-\lambda v, u-\lambda v\rangle \geq 0$$

and

$$\langle u-\lambda v, u-\lambda v\rangle = \langle u, u-\lambda v\rangle - \lambda\langle v, u-\lambda v\rangle \quad \text{(linearity in the first argument)}$$

$$= \langle u,u\rangle - \bar{\lambda}\langle u,v\rangle - \lambda\langle v,u\rangle + \lambda\bar{\lambda}\langle v,v\rangle \quad \text{(conjugate linearity in the second)}$$

$$= \langle u,u\rangle - \frac{\overline{\langle u,v\rangle}}{\langle v,v\rangle}\langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle}\overline{\langle u,v\rangle} + \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle^2}\langle v,v\rangle$$

$$= \langle u,u\rangle - 2\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} + \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}$$

$$= \langle u,u\rangle - \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}.$$

Thus

$$\langle u,u\rangle - \frac{|\langle u,v\rangle|^2}{\langle v,v\rangle} \geq 0$$

$$\Rightarrow |\langle u,v\rangle|^2 \le \langle u,u\rangle\langle v,v\rangle$$

$$\Rightarrow |\langle u,v\rangle| \le \langle u,u\rangle^{1/2}\langle v,v\rangle^{1/2}.$$

Moreover, equality holds iff $u-\lambda v=0$, that is, iff $u=\lambda v$. //

It then follows, just as in the real case, that

$$\|v\| = \sqrt{\langle v,v\rangle} \quad \forall v\in V$$

defines a norm on V.

The Pythagorean theorem still holds, but only in one direction.

<u>Theorem</u>: Let V be a complex inner product space. If $u,v\in V$ are orthogonal, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

<u>Proof</u>: Note that $\langle u,v\rangle=0$ iff $\langle v,u\rangle=0$, so

$$\begin{aligned}\|u+v\|^2 = \langle u+v,u+v\rangle &= \langle u,u\rangle + \cancelto{0}{\langle u,v\rangle} + \cancelto{0}{\langle v,u\rangle} + \langle v,v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2. //\end{aligned}$$

If $u, v \in V$ and $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, then we can only say that

$$\langle u, v\rangle + \langle v, u\rangle = 0$$

$$\Leftrightarrow \langle u, v\rangle + \overline{\langle u, v\rangle} = 0$$

$$\Longleftrightarrow \operatorname{Re}(\langle u, v\rangle) = 0$$

Note that

$$\begin{aligned}
\langle u, iv\rangle + \langle iv, u\rangle &= -i\langle u, v\rangle + i\langle v, u\rangle \\
&= -i\langle u, v\rangle + i\overline{\langle u, v\rangle} \\
&= -i\left(\langle u, v\rangle - \overline{\langle u, v\rangle}\right) \\
&= -i \cdot 2i \operatorname{Im}(\langle u, v\rangle) \\
&= 2 \operatorname{Im}(\langle u, v\rangle)
\end{aligned}$$

(If $a = \alpha + i\beta$, then the <u>real part</u> of a is $\operatorname{Re}(a) = \alpha$ and the <u>imaginary part</u> of a is $\operatorname{Im}(a) = \beta$. Note that

$$a + \overline{a} = \alpha + i\beta + \alpha - i\beta = 2\alpha = 2\operatorname{Re}(a),$$
$$a - \overline{a} = \alpha + i\beta - (\alpha - i\beta) = 2i\beta = 2i\operatorname{Im}(a).)$$

Thus, if we have both

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \text{ and } \|u+iv\|^2 = \|u\|^2 + \|iv\|^2 = \|u\|^2 + \|v\|^2,$$

then we obtain

$$\operatorname{Re}(\langle u, v\rangle) = 0 \text{ and } \operatorname{Im}(\operatorname{Re}(\langle u, v\rangle)) = 0$$

$\Rightarrow \langle u,v\rangle = 0$

The following theorem is unchanged in the complex case.

<u>Theorem</u>: Let V be a complex normed vector space. Then the norm $\|\cdot\|$ of V is defined by an inner product iff the parallelogram law holds:

$$\|u+v\|^2+\|u-v\|^2 = 2(\|u\|^2+\|v\|^2) \quad \forall u,v\in V.$$

<u>Proof:</u> If $\|\cdot\|$ is defined by an inner product $\langle\cdot,\cdot\rangle$, then, for $u,v\in V$,

$$\begin{aligned}\|u+v\|^2+\|u-v\|^2 &= \langle u+v,u+v\rangle + \langle u-v,u-v\rangle \\ &= \langle u,u\rangle + \cancel{\langle u,v\rangle} + \cancel{\langle v,u\rangle} + \langle v,v\rangle \\ &\quad + \langle u,u\rangle - \cancel{\langle u,v\rangle} - \cancel{\langle v,u\rangle} + \langle v,v\rangle \\ &= 2\|u\|^2 + \|v\|^2.\end{aligned}$$

The proof of the converse is even trickier than in the real case, and will be omitted. //