Chapter 15: Radiation and scattering (08 Apr 2021)

A. Background and scope.

Chapters 12 – 14 covered plane waves of the general form $e^{i\vec{k}\cdot\vec{r}-i\omega t}$, specifically how they interact with materials in propagation and reflection; how they can be combined coherently to form polarization states, short pulses, and interfere in thin films; and how they propagate when confined laterally in waveguides. These treatments are based entirely on fields, not potentials. This is done for several reasons. First, the propagation (wave) equation for fields can be solved analytically even if the dielectric function or the permeability (but not both) is anisotropic. This enables the treatment of crystal optics, and more generally, to much of the current work in metamaterials. Second, the configurations that we have investigated so far are largely planar, and thus consistent with wavefronts that are also planar. Third, boundary conditions are conveniently written in terms of fields. Fourth, solutions of the homogeneous wave equation include elementary excitations (plasmons) that are associated with planar interfaces, another area of current interest. These are also conveniently expressed in terms of fields.

In the above applications, except for crystal optics and metamaterials, the "extra" term in the wave equation for fields causes no difficulties. However, there are limitations. The Green function of the wave equation for fields is so messy no one ever writes it down. Hence the practical solutions are those involving the homogeneous equation. As seen in Ch. 14, the one exception where sources are involved is the waveguide, although the inhomogeneous equation is bypassed through the use of energy conservation. These limitations exclude wide classes of important phenomena, for example radiation, scattering, and diffraction, all of which can be described as superpositions of waves emitted by point sources. The associated waves are not planar but spherical (think Huygens' Principle.)

For isotropic materials, potentials exhibit no such complications. As found in Ch. 2, their wave equations

$$\left(\nabla^2 - \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho\,; \tag{15.1a}$$

$$\left(\nabla^2 - \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{J}\,; \tag{15.1b}$$

are naturally inhomogeneous, although this does require μ and ε to be scalars. In practice this is not a serious limitation, since for most of our applications $\mu = \varepsilon = 1$.

The major advantage of Eqs. (15.1) is the existence of a relatively simple – and analytic – Green function. This significantly enhances our understanding of the physics of radiation, scattering, and diffraction. Although the solutions are of the spherical-wave form $e^{ikr-i\omega t}$, in the far-field (radiation) limit the locally flat approximation $e^{ikr} \cong e^{i\vec{k}\cdot\vec{r}}$ collapses these solutions to calculations familiar with plane waves, i.e., where the gradient operator ∇ is replaced by $i\vec{k}$.

Equations (15.1a) and (15.1b) are conveniently written in 4-vector form

$$\left(\nabla^2 - \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)(\phi, \vec{A}) = -\frac{4\pi}{c}(c\rho, \vec{J}), \tag{15.2}$$

where the scalar 4-current is $c\rho$. The 4-vector notation $(c\rho, \vec{J})$ and $(\phi, \vec{A})$ emphasizes the fact that Maxwell's Equations treat ϕ and $\vec{A}$ equivalently. When $\vec{E}$ and $\vec{H}$ are required, they are calculated from ϕ and $\vec{A}$. The 4-vector form also appears in the definition of the Lorentz gauge,

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \tag{15.3}$$

which is used to obtain Eqs. (15.1) and (15.2), and in charge conservation,

$$\nabla\cdot\vec{J} + \frac{1}{c}\frac{\partial(c\rho)}{\partial t} = 0, \tag{15.4}$$

although this is usually written without the c's. The conservation laws Eqs. (15.3) and (15.4) are manifestations of the 4-divergence, which is usually written

$$\frac{\partial}{\partial X^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right). \tag{15.5}$$

The phenomena described in Ch. 15 are our first serious encounter with the speed of light, so it is worth making some qualitative remarks to establish perspective and eliminate misconceptions. Although electrodynamics is relativistic in the sense that Maxwell's Equations follow from Coulomb's Law of electrostatics and special relativity, the classical treatment of radiation, scattering, and diffraction depends on c only to order c^{-1}. Time dilation and foreshortening of the dimension in the direction of motion, two of the best-known features of special relativity, are of order c^{-2}. Hence the phenomena discussed in Ch. 15 have no direct connection to special relativity. Time is treated as in

Newtonian physics: the time t' describing the configuration emitting radiation is measured with the same clock used by the observer.

This c^{-1} approximation is more accurately termed retardation physics, the objective of which is to describe what is seen by the observer at $(\vec{r},t)$ as a result of an action, for example emission of a ray at $(\vec{r}\,',t')$. Although taken only to c^{-1}, this difference can be substantial, and in some cases, unexpected. For example, it is the origin of the contribution of ϕ to far-field radiation. Differences between reality and perception are well known in astronomy: we perceive the sun, Alpha Centauri, and Andromeda not as they are now, but as they were 8 minutes, 4.2 years, and 6.2×10^6 years ago, respectively. In this case "now" is used in the Newtonian sense that the same clock applies everywhere, even though we may not know exactly where the sun, Alpha Centauri, and Andromeda are currently located. This discussion continues in Sec. D.

Restricting the treatment of radiation to order c^{-1} does limit our scope. For example we cannot treat synchrotron radiation, which results from charges moving at relativistic speeds. For this reason Jackson defers the treatment of radiation from moving charges until he covers special relativity. However, c^{-1} effects are sufficient to describe nearly all everyday phenomena, and working with the approximation we obtain physical insights that are lacking in the more exact treatment. We therefore reverse Jackson's topical order, not only to better understand the physics behind these phenomena, but also to distinguish retardation from time-dilation effects. A full treatment of special relativity is given in Ch. 17.

B. Formal treatment of radiation: Green function of the wave equation.

Although it was not noted at the time, the description of spherical-wave propagation began in Chs. 4 and 5 with electro- and magnetostatics, specifically with the equations

$$\nabla^2\phi = -4\pi\rho\,, \tag{15.6a}$$

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}\,. \tag{15.6b}$$

As a reminder, the formal solution of Eqs. (15.6) is

$$(\phi,\vec{A}) = \frac{1}{c}\int_V d^3r'(c\rho,\vec{J})G(\vec{r},\vec{r}\,')\,, \tag{15.7a}$$

where the Green function $G(\vec{r},\vec{r}\,')$ is defined by

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,')\,. \tag{15.7b}$$

That Eq. (15.7a) is a solution of Eqs. (15.6) is easily verified by direct substitution. For empty space we recall that

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}\,, \tag{15.8}$$

leading to

$$(\phi, \vec{A}) = \frac{1}{c}\int_V d^3r' \frac{(c\rho, \vec{J})}{|\vec{r}-\vec{r}'|}, \tag{15.9}$$

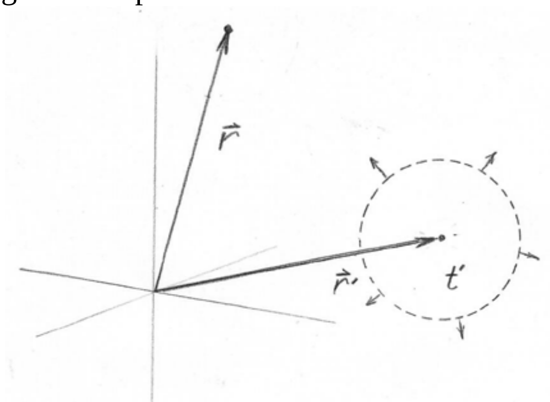
By analogy with the statics case, the formal solution of Eq. (15.2) is

$$(\phi, \vec{A}) = \frac{1}{c}\int_V d^3r' \int_{-\infty}^{\infty} dt'(c\rho, \vec{J})G(\vec{r},\vec{r}',t,t'), \tag{15.10a}$$

where the time-dependent Green function $G(\vec{r},\vec{r}',t,t')$ for $\mu=\varepsilon=1$ is defined by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},\vec{r}',t,t') = -4\pi\delta(\vec{r}-\vec{r}')\delta(t-t'). \tag{15.10b}$$

Again, this is easily verified by direct substitution. The new feature is the time operator, which describes the delay between an action at $(\vec{r}',t')$ and its perception at $(\vec{r},t)$. This delay, which results from the finite speed of light, is absent in statics, where $c\to\infty$ and the results of actions instantly propagate throughout all space.



Given the statics solution and this information, it is significantly easier to obtain $G(\vec{r},\vec{r}',t,t')$ by inspection than it is to derive it. We follow the inspection route here, leaving the derivation to Sec. C. We begin by recalling that the purpose of the Green function is to describe the consequence at $(\vec{r},t)$ of an event that occurred at $(\vec{r}',t')$ – Green functions are not called "propagators" for nothing. Now suppose that at t' a flash bulb at $\vec{r}'$ emits isotropically a large number of photons, as indicated in the diagram. In free space the photons propagate away from $\vec{r}'$ as a spherical shell whose radius increases at the speed *c*. The only change with respect to electrostatics is that now there is a propagation delay.

Thus the spatial part of $G(\vec{r},\vec{r}',t,t')$ for free-space propagation must be the same as that of its statics equivalent, Eq. (15.8). To accommodate the new requirement, we must specify the time that the shell arrives at $\vec{r}$. This is evidently

$$t = t' + \frac{1}{c}|\vec{r}-\vec{r}'|. \tag{15.11}$$

We can therefore state with some confidence that in empty space

$$G(\vec{r},\vec{r}',t,t') = G(\vec{r},\vec{r}')\delta(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|) \tag{15.12a}$$

$$= \frac{\delta(t - t' - \frac{1}{c}|\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|} \tag{15.12b}$$

Thus

$$(\phi, \vec{A}) = \frac{1}{c}\int_V d^3r' \int_{-\infty}^{\infty} dt'(c\rho, \vec{J})\frac{\delta(t - t' - \frac{1}{c}|\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|}, \tag{15.13}$$

where $(\phi, \vec{A})$ and $(c\rho, \vec{J})$ are functions of $(\vec{r}, t)$ and $(\vec{r}\,', t')$, respectively. The t' in the above equations is called the *retarded time,* since it refers to the time that the action occurred. The reduction of Eq. (15.13) to Eq. (15.9) as $c \to \infty$ is evident.

As with statics, the Green function represents the ultimate in superposition, reducing everything to point sources. Since the equations are linear, the potentials generated by these sources are superposed by integration over space and time. Here, “point" now means localized in time as well as in space. As noted in Ch. 2, integrals are far easier to deal with than the original differential equations. While evaluation may involve approximations, integrals allow these approximations to be made more intelligently and their consequences better defined. This leads to better understandings of solutions and ranges of validity. In performing $(\vec{r}, t)$ operations, for example calculating $\vec{E}$ and $\vec{H}$ from ϕ and $\vec{A}$, note that these operations act on the Green function, not on the sources themselves.

Chapter 15 is based entirely on Eq. (15.13), or more accurately, on the general equation in Sec. D that follows when the integration over t' is done explicitly. The Liénard-Wiechert potentials follow immediately. However, determining electric and magnetic fields from these potentials is more challenging, thanks to the retarded time. Nevertheless, for harmonic sources (time dependences $e^{-i\omega t}$), matters again simplify. This is covered in Sec. G. Having succeeded in getting the math right, the remaining sections are concerned with extracting the physics from the math.

Equation (15.13) shows that the leading terms in ϕ and $\vec{A}$ are of zero and first order, respectively, in inverse powers of *c*. This is consistent with the limit $c \to \infty$, where $\vec{A}$ vanishes and ϕ is reduced to electrostatics. Anticipating that radiation is of order c^{-1}, we see that $\vec{A}$ contributes to radiation directly, whereas the contribution of ϕ requires going beyond the zero-order (electrostatics) approximation. As this involves new terms, it follows that radiation cannot be described as a simple extension of electrostatics, nor can it be described entirely by $\vec{A}$.

C. Mathematical foundation of the Green function.

Because Eq. (15.13) is fundamental to radiation, scattering, and diffraction, it is worth placing it on firm mathematical grounds before proceeding further. That is the purpose of this section. However, if one is willing to accept Eq. (15.13) without proof and go directly to general properties and applications, this section can be bypassed without injury.

Derivations of Eq. (15.13) are common textbook material, but not everyone approaches it in the same way. Zangwill's derivation initially looks promising, but leads to a mathematical dead end, which Zangwill circumvents by handwaving. Jackson's approach is rigorous but not straightforward. Although it leads to the correct result, Jackson omits most of the steps. We follow the Jackson derivation, but fill in the missing details. The derivation involves three Fourier transforms. What is interesting is that it mirrors equivalently some steps that we did for electrostatics.

We start by noting that any point in empty 4-dimensional space is the same as any other point, so without loss of generality we place the origin of our coordinate system at $\vec{r}\,'$ (following the electrostatics procedure) and measure time t from $t'=0$. The quantities $(\vec{r}\,',t')$ are put back in at the end. In this coordinate system Eq. (15.10b) is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},t) = -4\pi\delta(\vec{r})\delta(t)\,. \tag{15.14}$$

Next, write $G(\vec{r},t)$ as the Fourier transform of its harmonic representation $G(\vec{r},\omega)$, specifically

$$G(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, G(\vec{r},\omega)\, e^{-i\omega t}\,. \tag{15.15a}$$

For completeness, $G(\vec{r},\omega)$ is given in terms of $G(\vec{r},t)$ by the inverse transform

$$G(\vec{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, G(\vec{r},t)\, e^{i\omega t}\,. \tag{15.15b}$$

Substituting Eq. (15.15a) into Eq. (15.14) yields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{\infty} d\omega\, G(\vec{r},\omega)\, e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega \left(\nabla^2 + k^2\right) G(\vec{r},\omega)\, e^{-i\omega t} \tag{15.16a}$$

$$= -4\pi\delta(\vec{r})\delta(t)\,, \tag{15.16b}$$

where as usual $k^2 = \omega^2/c^2$. Next, take the Fourier time transform of Eqs. (15.16) using $\int_{-\infty}^{\infty} dt\, e^{i\omega' t}$. This eliminates $\delta(t)$, and from the integral definition of the continuum delta function, converts Eq. (15.16a) into

$$\int_{-\infty}^{\infty} dt\, e^{i\omega' t}\left(\int_{-\infty}^{\infty} d\omega \left(\nabla^2 + k^2\right) G(\vec{r},\omega)\, e^{-i\omega t}\right) = 2\pi\left(\nabla^2 + k'^2\right) G\left(\vec{r},\omega'\right) \tag{15.17a}$$

$$= -4\pi\,\delta(\vec{r})\,. \tag{15.17b}$$

This completes the preliminary steps.

Next, note that the configuration is spherically symmetric, so only the radial part of ∇^2 is needed (again repeating part of the electrostatics derivation). The part of Eq. (15.17a) in brackets is therefore converted to

$$\left(\nabla^2 + k'^2\right)G(r,\omega') = \left(\frac{1}{r^2}\frac{d^2}{dr^2}(r^2 G) + k'^2\, G\right) = -2\delta(\vec{r})\,. \tag{15.18}$$

We now recall our intuitive assessment, suspecting that the solution to Eq. (15.18) might be a spherical wave centered on the origin, or

$$G(\vec{r},\omega) = A^{\pm}\frac{e^{\pm ik'r}}{r}, G(\vec{r},\omega') = A^{\pm}\frac{e^{\pm k'r}}{r} \tag{15.19}$$

where $A^{\pm}$ is a constant to be determined. This function is well defined everywhere except at the singularity $r = 0$. Substituting Eq. (15.19) into Eq. (15.18) we find that for $r \neq 0$ Eq. (15.18) is indeed satisfied, because $\delta(r) = 0$ for $r \neq 0$ as well.

Now consider the situation at $r = 0$. Rather than attempt to work with two infinities, we repeat our electrostatics procedure, integrating both sides of Eq. (15.18) over a small spherical volume of radius R centered on $\vec{r} = 0$. On the right side, this operation eliminates the delta function. On the left side, we take advantage of the fact that $\nabla^2 = \nabla \cdot \nabla$ and Gauss' Theorem to cast the expression into a surface integral in a region where everything is well defined. Doing the integral specifically we find

$$A^{\pm}\int_0^R r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \left(\nabla \cdot \nabla\left(\frac{e^{\pm ikr}}{r}\right) + k^2\,\frac{e^{\pm ikr}}{r}\right) = -2\,. \tag{15.20}$$

The term proportional to k^2 vanishes in the limit $R \to 0$. The first term has the value $(-4\pi A^{\pm})$. Thus

$$A^{\pm} = \frac{1}{2\pi}, \tag{15.21}$$

so $G(\vec{r},\omega)$ is defined. We now recover $G(\vec{r},t)$ by performing the inverse Fourier transform Eq. (15.15b):

$$G(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left(\frac{e^{\pm ik'r}}{r}\right) e^{-i\omega t} = \frac{1}{2\pi\, r}\int_{-\infty}^{\infty} d\omega\; e^{i(\pm r/c - t)\omega} \tag{15.22a}$$

$$= \frac{\delta(\pm r/c - t)}{r}\,. \tag{15.22b}$$

The sign ambiguity is resolved by appealing to causality: in our universe, at least, radiation propagates *away* from its source, so the positive sign is the appropriate one.

Returning to a general coordinate system with the origin at $\vec{r}\,'$ and a starting time t', the final expression is

$$G(\vec{r},\vec{r}',t,t') = \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}'|+t'-t)}{|\vec{r}-\vec{r}'|} \tag{15.23a}$$

$$= \frac{\delta(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}. \tag{15.23b}$$

This is what we deduced in Sec. A. $G(\vec{r},\vec{r}',t,t')$ is traditionally written as Eq. (15.23b), taking advantage of the fact that when the limits of integration are considered, the delta function is an even function of its argument.

In deriving Eq. (15.23b), we have obtained in passing the Green function of the Helmholtz Equation:

$$\left(\nabla^2 + k^2\right)G_H(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}'), \tag{15.24}$$

specifically

$$G_H(\vec{r},\vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}. \tag{15.25}$$

D. The t' integral.

Equation (15.13) can be simplified by evaluating the integral over t' explicitly. While at first sight the delta function appears to make this trivial, doing so without more thought is a standard error. The approach fails because $\vec{r}\,'$ is a function of t'. Therefore, dt' is not the differential of the argument of the delta function.

We can fix this as follows. Let the argument of the delta function be a new variable

$$u' = t - t' - \frac{1}{c}|\vec{r}-\vec{r}\,'|. \tag{15.26}$$

If we can determine how to replace dt' with du', then the δ-function integration is indeed trivial, with the result $u' = 0$. With the observer's coordinates $\vec{r}$ and t fixed,

$$du' = -dt' - \frac{1}{c}d(|\vec{r}-\vec{r}\,'|). \tag{15.27}$$

Now

$$d\,|\vec{r}-\vec{r}'| = d\left(\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}\right) = \frac{2\vec{r}'\cdot d\vec{r}' - 2\vec{r}\cdot d\vec{r}'}{2\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}} \tag{15.28a,b}$$

$$= \hat{R}\cdot d\vec{r}\,' = \hat{R}\cdot\frac{d\vec{r}\,'}{dt'}dt' = (\hat{R}\cdot\vec{v})dt'; \tag{15.28c,d,e}$$

where $\vec{R} = \vec{r} - \vec{r}\,'$. Thus

$$du' = -dt' + \frac{1}{c}\hat{R}\cdot\vec{v}\,dt', \tag{15.29}$$

or

$$dt' = -\frac{du'}{1-\frac{1}{c}\hat{R}\cdot\vec{v}}. \tag{15.30}$$

It is worth remembering operations done on $R = |\vec{r}-\vec{r}\,'|$, for this function underlies a substantial part of the rest of Ch.15.

Substituting Eq. (15.30) into Eq. (15.13), reversing the limits on the t' integration, and performing the trivial u' integration, we obtain

$$(\phi,\vec{A}) = \frac{1}{c}\int_V d^3r' \left.\frac{(c\rho,\vec{J})}{|\vec{r}-\vec{r}\,'|\,(1-\hat{R}\cdot\vec{v}'/c)}\right|_{t'=t_{ret}} \tag{15.31a}$$

and

$$t' = t_{ret} = t - \frac{1}{c}|\vec{r}-\vec{r}\,'|, \tag{15.31b}$$

where $t' = t_{ret}$ is the time that the arriving signal left $\vec{r}\,'$. Because calculations are only taken to order c^{-1}, Eq. (15.31a) can also be written

$$(\phi,\vec{A}) = \frac{1}{c}\int_V d^3r' \frac{(c\rho,\vec{J})}{|\vec{r}-\vec{r}\,'|}\left(1+\frac{1}{c}\hat{R}\cdot\vec{v}\right)_{t'=t_{ret}}. \tag{15.32}$$

Because $\vec{A}$ is already of order c^{-1}, the correction term affects only $c\rho$. Doing the math we obtain

$$(\phi,\vec{A}) = \frac{1}{c}\int_V d^3r' \left.\frac{(c\rho+\rho\hat{R}\cdot\vec{v},\vec{J})}{|\vec{r}-\vec{r}\,'|}\right|_{t'=t_{ret}}. \tag{15.33}$$

But $\rho\vec{v} = \vec{J}$, so Eq. (15.28) can also be written

$$(\phi,\vec{A}) = \frac{1}{c}\int_V d^3r' \left.\frac{(c\rho+\hat{R}\cdot\vec{J},\vec{J})}{|\vec{r}-\vec{r}\,'|}\right|_{t'=t_{ret}}. \tag{15.34}$$

If the $\vec{r}\,'$ dependence of $\hat{R}$ and $t' = t_{ret}$ can be ignored, then Eq. (15.28) reduces to

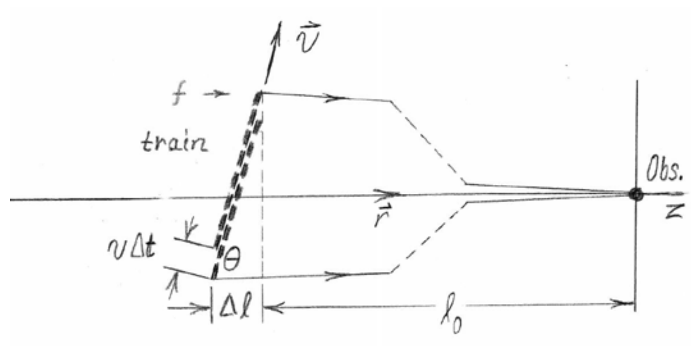
$$= \left(\phi_o(\vec{r},t') + \hat{R}\cdot\vec{A},\vec{A}\right)_{t'=t_{ret}}, \tag{15.35}$$

where $\phi_o(\vec{r},t')$ is the electrostatic solution. The scalar potential has two contributions that can be described at least approximately as the standard electrostatic expression evaluated at the retarded time, and the longitudinal projection of $\vec{A}$. The former is entirely reasonable, because the zero-order contribution should reduce to the static limit when $c \to \infty$. The second contribution, $\hat{R} \cdot \vec{A}$, is due to retardation. Its meaning is discussed in the next section.

E. Retardation physics.

Although Maxwell's Equations follow from Coulomb's Force Law and special relativity, it is useful to interpret Eq. (15.32) in classical terms, which can be defined as taking approximations to the level c^{-1}. Special relativity, as seen for example in time dilation and the apparent shortening of lengths in the direction of motion, is a c^{-2} effect. The obvious manifestation of retardation is that the $\vec{r}\,'$ integration is done with $t' = t_{ret}$, the main effect being a delay $\sim R/c$ as perceived by the observer. However, the "extra" term in the scalar potential is also a retardation effect, as we now show. The model, due to Griffith, provides additional insight into the meaning of Eq. (15.32).

Consider a train of length L_T that is moving with a velocity $\vec{v}$ in a coordinate system with a stationary observer at $\vec{r}$ (see diagram). You know that the length of the train is L_T, because you are a passenger on the train and have measured it directly. The question now is, what is the length $L_T\,'$ as perceived by the observer at $\vec{r}$?



This sounds like a particularly stupid question, and indeed it would be if *c* were infinite. However, with *c* finite and the track angle shown, the ray leaving the rear must travel farther than the ray leaving the front. Because the observer perceives the rays arriving at the same time, the ray from the rear must have left earlier. Therefore, the train is not quite in the same position when the two rays were emitted, which is the key to the calculation.

From the diagram, the extra distance traveled by the ray leaving the rear is

$$\Delta l = L_T \cos\theta . \tag{15.36}$$

The extra delay Δt caused by this extra distance is

$$\Delta t = \frac{\Delta l}{c} = \frac{L_T}{c}\cos\theta . \tag{15.37}$$

During this time the train has traveled a distance

$$\Delta L_T = v\Delta t = L_T \frac{v}{c}\cos\theta = L_T \frac{1}{c}\hat{r}\cdot\vec{v} \tag{15.38}$$

Thus the observer concludes that the length of the train is not L_T but

$$L_T\,' = L_t\left(1 + \frac{1}{c}\vec{v}\cdot\hat{R}\right). \tag{15.39}$$

When Eq. (15.39) is compared to Eq. (15.32) the meaning of the integrand, the correction term, and the potentials seen by the observer become understandable. Everything in the integrand, ρ, $\vec{J}$, $\vec{v}$, and t', is to be evaluated in the home coordinate system, i.e., "on the train." The results ϕ and $\vec{A}$ are quantities *perceived* by the observer. Although the main difference is the time delay $\sim R/c$, perception must also include any discrepancies that result from the motion of ρ and $\vec{J}$. The motional "length" discrepancy affects the volume differential d^3r', and is not accommodated by R/c. We can ignore its effect on $\vec{A}$, because $\vec{A}$ is already of order c^{-1}. However, the length correction does apply to ϕ, which is of order c^0. We shall see that this correction term is critical in far-field radiation, where it ensures that $\vec{E}$ is perpendicular to the propagation vector $\vec{k}$.

To place these results in the larger context of special relativity, Jackson does not derive Eq. (15.32) explicitly because he treats moving charges relativistically. However, Eq. (15.32) can be inferred from the exact expression he gives in Ch 12, although this does not lead to an understanding of the physics involved. However, the relativistic result does show that Eq. (15.32) is accurate to parts in c^{-3}, which can be anticipated because relativistic effects are of order c^{-2}.

F. The Liénard-Wiechert potentials; radiation fields in the c^{-1} approximation.

The simplest application of Eq. (15.34) is that of a point charge q at $\vec{r} = \vec{r}_o(t)$ moving with a velocity $\vec{v}_o = d\vec{r}_o/dt$. In this case

$$\rho(\vec{r}\,', t') = q\delta(\vec{r}\,' - \vec{r}_o(t')), \tag{15.40a}$$

$$\vec{J}(\vec{r}\,', t') = \rho\vec{v}_o = q\vec{v}_o\delta(\vec{r}\,' - \vec{r}_o(t')). \tag{15.40b}$$

Note that three different position vectors are involved: the location $\vec{r}_o(t')$ of q at time t'; the dummy variable $\vec{r}\,'$ of integration; and the location $\vec{r}$ of the observer. Putting everything together, the charge density that enters Eq. (15.34) is $\rho(r', t') = q\delta(\vec{r}\,' - \vec{r}_o(t'))$. With $\vec{r}_o$ being a function of t', Eq. (15.31) becomes a self-consistent expression connecting t' to t, but this is a manageable challenge.

The potentials themselves are given by Eqs. (15.34). Because the charge and current densities are delta functions and the arguments of the delta functions agree with the differential, the integrations are trivial. The results are

$$\phi = \left(\frac{q}{R} + \hat{R}\cdot\vec{A}\right)_{t'=t_{ret}}; \tag{15.41a}$$

$$\vec{A} = \left.\frac{q\vec{v}_o}{Rc}\right|_{t'=t_{ret}}; \tag{15.41b}$$

where $\vec{R} = \vec{r} - \vec{r}_o(t')$, $\vec{v}_o = \vec{v}_o(t')$, and $R = |\vec{R}|$. Equations (15.36) are the *Liénard-Wiechert potentials*. In this case the c^{-0} contribution to ϕ is exactly the electrostatic potential evaluated at the location of q at the time the received signal was emitted.

The calculation of $\vec{E}$ and $\vec{H}$ in the far-field range is somewhat challenging. It proceeds as follows. As usual,

$$\vec{E}(\vec{r},t) = \left(-\frac{1}{c}\frac{\partial \vec{A}(\vec{R},t')}{\partial t} - \nabla_r \phi(\vec{R},t') \right)_{t'=t-R/c} . \tag{15.42}$$

The evaluation challenge follows because $\vec{A}$ and ϕ are defined in the home frame, whereas the derivatives are defined in the perception frame. It is important that this distinction be recognized.

Evaluation of the vector-potential term is relatively straightforward. Write

$$-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\left(\frac{d}{dt'}\left(\frac{q\vec{v}_o(t')}{cR(t')} \right) \right) \frac{\partial t'}{\partial t} \tag{15.43a}$$

$$= -\frac{q}{c^2}\left(\frac{1}{R}\frac{d\vec{v}_o(t')}{dt'} - \frac{\vec{v}_o}{R^2}\frac{dR}{dt'} \right) \frac{\partial t'}{\partial t} \tag{15.43b}$$

$$\cong -\frac{q\vec{a}_o(t')}{c^2 R}\,\frac{\partial t'}{\partial t}. \tag{15.43c}$$

where $\vec{a}_o(t')$ is the acceleration of q in its home frame when the signal is emitted. The term $\sim R^{-2}$ is discarded because it vanishes relative to R^{-1} at long distances. Equation (15.qq) shows that

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \hat{R}\cdot\vec{v}_o/c} \cong 1 + \frac{\hat{R}\cdot\vec{v}_o}{c} \cong 1, \tag{15.44}$$

because Eq. (15.40) is already of order c^{-1}. The result is

$$-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \cong \left. \frac{q\vec{a}_o}{Rc} \right|_{t=t_{ret}} . \tag{15.45}$$

Calculation of the ϕ contribution is much more involved. We consider specifically the x component, assuming for the moment that an origin at $\vec{r}_o$. Then

$$-\hat{x}\frac{\partial}{\partial x}\left(\frac{q(xv_{ox} + yv_{oy} + zv_{oz}}{rc} \right) = -\hat{x}\frac{q}{r^2 c}\left(v_{ox} + x\frac{\partial v_{ox}}{\partial x} + y\frac{\partial v_{oy}}{\partial x} + z\frac{\partial v_{oz}}{\partial x} \right). \tag{15.46}$$

Now

$$\frac{\partial v_{ox}}{\partial x} = \frac{\partial}{\partial x} v_{ox}(t - \frac{1}{c}r) = \frac{dv_{ox}}{dt}\frac{\partial}{\partial x}(t - \frac{1}{c}r) = -\frac{a_{ox}}{c}\frac{\partial r}{\partial x}. \tag{15.47}$$

In Eq. (15.46) we have used the first-order difference between $\partial t_{ref} / \partial t$ and 1 to replace $\partial / \partial t$ with $\partial / \partial t_{ref}$. This is allowed at the present order of approximation, because Eq. (15.46) is proportional to c^{-1}. Also

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}. \tag{15.48}$$

Repeating the same calculations for the *y* and *z* components and combining the results yields

$$-\nabla \varphi = \frac{q\vec{r}\,(\vec{r} \cdot \vec{a}_{o,ret})}{cr^3} \tag{15.49a}$$

$$= \frac{q\hat{R}(\hat{R} \cdot \vec{a}_o)}{cR}\bigg|_{t=t_{ret}}, \tag{15.49b}$$

where Eq. (15.49b) is Eq. (15.49a) with the origin restored. Therefore, the electric field in the radiation zone is given by

$$\vec{E}(\vec{r},t) = -\frac{q}{Rc^2}(\underline{I} - \hat{R}\hat{R}) \cdot \vec{a}_o\bigg|_{t_{ret}} \tag{15.50a}$$

$$= \frac{q}{Rc^2}\left(\hat{R} \times (\hat{R} \times \vec{a}_o)\right)\bigg|_{t_{ret}}. \tag{15.50b}$$

Thus while the electric field is proportional to the acceleration of q in its home frame, only the projection of the acceleration perpendicular to the observation direction matters. The motional-distortion effect entering through ϕ is seen to be a critical part of the calculation. Similar calculations for the magnetic field yield, not surprisingly

$$\vec{H}(\vec{r},t) = -\frac{q}{Rc^2}\hat{R} \times \vec{a}_o\bigg|_{t_{ret}}. \tag{15.51}$$

Zangwill places the Liénard-Wiechert potentials in a more general context by providing several derivations, including a relativistically exact derivation by Minkowski. The Minkowski derivation is historically significant because he invented the four-dimensional description of the special theory of relativity. In deriving the far-field expressions, Jackson and Griffith base their calculations entirely on $\vec{A}$, the workaround succeeding by first calculating $\vec{H}$, then using the vector cross product to calculate $\vec{E}$. While this results in a field $\vec{E}$ perpendicular to the propagation direction, it misses the retardation-physics contribution to ϕ. This $\vec{A} \to \vec{H} \to \vec{E}$ workaround appears to be common to textbooks in general.

G. Harmonic time dependence: zones, and fields in the radiation zone.

The remaining sections deal with harmonic time dependences $e^{-i\omega t}$. This topic is covered by Jackson in Chs. 9 and 10, where zones and multipole expansions are covered in detail. As traditionally interpreted, a "harmonic time dependence" means that the time and space dependences are assumed to be independent, so

$$\rho(\vec r\,',t') = \rho(\vec r\,')e^{-i\omega t'}; \tag{15.52a}$$

$$\vec J(\vec r\,',t') = \vec J(\vec r\,')e^{-i\omega t'}, \tag{15.52b}$$

that is, time-dependent deformations do not occur. This assumption not only enormously simplifies calculations but also is widely applicable. First, it allows retarded time to be introduced at the start. Second, multipole descriptions follow. Third, in the radiation zone the locally flat approximation applies, reducing operations ∇ to $i\vec k$, a simplification familiar from plane waves. The main challenge is to keep track of the additional approximations used at different stages of development.

With the time dependence of both ρ and $\vec J$ expressed as $e^{-i\omega t}$, Eq. (15.34) becomes

$$(\phi,\vec A) = \frac{1}{c}\int_V d^3r'\,\frac{(c\rho + \hat R\cdot\vec J,\vec J)}{|\vec r - \vec r\,'|}e^{-i\omega t + (\omega/c)|\vec r - \vec r\,'|} \tag{15.53a}$$

$$= \frac{e^{-i\omega t}}{c}\int_V d^3r'\,\frac{(c\rho + \hat R\cdot\vec J,\vec J)}{|\vec r - \vec r\,'|}e^{ik|\vec r - \vec r\,'|}\,. \tag{15.53b}$$

where $\omega/c = k$, and ρ and $\vec J$ are now functions only of $\vec r\,'$. Here, $\vec r$ and $\vec r\,'$ are defined relative to a coordinate system that is more or less centered on ρ and $\vec J$. A more specific interpretation of "more or less" will be given below. To reduce presentational complexity, in the following we consider only $\vec A$ and $\vec J$, dropping the electrostatic term completely and bringing in $\hat R\cdot\vec J$ only as needed.

Generally speaking, Eqs. (15.53b) cannot be integrated in closed form, and additional approximations must be made. At the least disruptive level, we can expand

$$\frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} = \frac{1}{4\pi}\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\,, \tag{15.54}$$

where $j_l(kr)$ and $h_l^{(1)}(kr)$ are the spherical Bessel and Hankel functions, respectively, and $r_<$ and $r_>$ are the lesser and greater, respectively, of $r = |\vec r\,|$ and $r' = |\vec r\,'|$. Then

$$\vec A(\vec r,t) = \frac{e^{-i\omega t}}{c}\int_V d^3r'\,\vec J(\vec r\,')\left(\frac{1}{4\pi}\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right). \tag{15.55}$$

This is the *induction zone*, and is appropriate when the wavelength $\lambda = 2\pi/k$ is of the order of the dimensions of the system. If $k \to 0$ Eq. (15.55) reduces to the electrostatic and magnetostatic equivalents, where $j_l(kr)$ and $h_l^{(1)}(kr)$ reduce to the monomials r^l and r^{-l-1}, respectively. The induction zone is computationally intensive but of considerable practical importance, because this is the range that is relevant for the calculation of real-time critical dimensions (RT/CD) in integrated-circuit (IC) technology.

At the other extreme, if both the source is so localized and the observer so close relative to λ that $k\,|\vec{r}-\vec{r}\,'| << 1$, then $e^{ik|\vec{r}-\vec{r}\,'|}$ never differs significantly from 1 and can be ignored. In this situation Eqs. (15.) reduce to the electrostatics and magnetostatics expressions multiplied by $e^{-i\omega t}$, for example

$$\vec{A}(\vec{r},t) = \frac{e^{-i\omega t}}{c}\int_V d^3r\,'\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}. \tag{15.56}$$

This situation is very unusual, and is mentioned only for completeness.

By far the most useful approximation is that of the radiation zone, which is defined as $kr >> 1$. In this case we can expand

$$k\,|r-r\,'| = k\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'} \tag{15.57a}$$

$$\cong kr - k\hat{r}\cdot\vec{r}\,', \tag{15.57b}$$

keeping terms only to first order in r'/r. Because kr is independent of $r\,'$ we can remove it from the integral, leaving

$$\vec{A} = \frac{1}{c}e^{ikr-i\omega t}\int_V d^3r\,'\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}e^{-ik\hat{r}\cdot\vec{r}\,'}. \tag{15.58}$$

We have now realized spherical waves.

In the next approximation, we assume that *r* is much larger than the range of $\vec{r}\,'$, which is defined by $\vec{J}(\vec{r}\,')$. In that case we can ignore the presence of $\vec{r}\,'$ in the denominator and write $|\vec{r}-\vec{r}\,'| \cong r$. Equation (15.53b) is now reduced to

$$\vec{A} = \frac{e^{ikr-i\omega t}}{rc}\int_V d^3r\,'\vec{J}\,e^{-ik\hat{r}\cdot\vec{r}\,'}. \tag{15.59}$$

Next, we can *define* the origin to be located such that $\hat{k} = \hat{r}$. Then $k\hat{r} = \vec{k}$, $\hat{R} = \hat{k}$, and Eq. (15.59) is now reduced to

$$\vec{A} = \frac{e^{ikr-i\omega t}}{rc}\int_V d^3r\,'\vec{J}\,e^{-i\vec{k}\cdot\vec{r}\,'}. \tag{15.60}$$

Likewise, with $\hat{k} = \hat{R}$, the scalar potential has now been reduced to

$$\phi = \hat{k}\cdot\vec{A}. \tag{15.61}$$

Finally, if we are sufficiently far from the source, then we can apply the locally flat approximation, writing

$$e^{ikr} \cong e^{i\vec{k}\cdot\vec{r}} \,. \tag{15.62}$$

In this situation we can replace operators ∇ with $i\vec{k}$. This converts the definition of the Lorentz gauge to

$$\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=i\vec{k}\cdot\vec{A}-i\frac{\omega}{c}\phi$$

$$=ik\left(\hat{k}\cdot\vec{A}-\phi\right)=0\,. \tag{15.63}$$

It also converts the calculation of $\vec{H}$ in the far field to

$$\vec{H}=\nabla\times\vec{A}=i\vec{k}\times\vec{A}\,, \tag{15.64}$$

so $\vec{H}$ is clearly orthogonal to $\vec{k}$. Finally, $\vec{E}$ is given by

$$\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\nabla\phi=\frac{i\omega}{c}\vec{A}-\vec{k}(\hat{k}\cdot\vec{A})=ik\left(\vec{A}-\frac{c}{\omega}\vec{k}\hat{k}\cdot\vec{A}\right) \tag{15.65a}$$

$$=ik(\underline{I}-\hat{k}\hat{k})\cdot\vec{A}=-ik\,(\hat{k}\times(\hat{k}\times\vec{A}))=ik\vec{A}_{\perp}\,. \tag{15.65b}$$

Thus $\vec{E}$ is not only perpendicular to $\vec{k}$ and $\vec{H}$, but ϕ is an active participation in that it eliminates the longitudinal component of $\vec{A}$ in the calculation of $\vec{E}$.

Returning to the integral for $\vec{A}$, we have

$$\vec{A}=\frac{e^{ikr-i\omega t}}{rc}\int_V d^3r'\,\vec{J}(\vec{r}\,')e^{-i\vec{k}\cdot r'} \tag{15.66a}$$

$$=\frac{e^{ikr-i\omega t}}{rc}\vec{J}_{\vec{k}} \tag{15.66b}$$

where $\vec{J}_{\vec{k}}$ is the three-dimensional Fourier transform of $\vec{J}(\vec{r}\,')$. Thus we have not only succeeded in generating plane waves, but in so doing have reduced differential operations to algebra. The only difference with respect to previous treatments of plane waves is that the direction of $\vec{k}$ is now defined by the location of the observer.

Additional quantities of interest include the radiated intensity and the total integrated power. Working through the math, the cycle-averaged Poynting vector is

$$<\vec{S}>=\frac{1}{8\pi c\,r^2}|\,\vec{k}\times\vec{J}_{\vec{k}}\,|^2\,\hat{k}\,. \tag{15.67}$$

Supposing that $\vec{J}$, and therefore $\vec{J}_{\vec{k}}$ and $\vec{A}$, are parallel to $\hat{z}$, we obtain

$$< \vec{S} > = \frac{k^2}{8\pi c r^2} | \vec{J}_{\vec{k}} |^2 (\sin^2\theta)\ \hat{r} . \tag{15.68}$$

Integrating over the entire solid angle, the total radiated intensity is

$$I = r^2 \int_{-1}^{1} d(\cos\theta) \int_{0}^{2\pi} d\varphi \left(\frac{k^2}{8\pi cr^2} | \vec{J}_{\vec{k}} |^2 \sin^2\theta \right) \tag{15.69a}$$

$$= \frac{k^2}{6c} | \vec{J}_{\vec{k}} |^2 . \tag{15.69b}$$

These are the quantities of primary interest in far-field radiation.

H. Multipole expansions; dipole radiation.

We now return to the integral for $\vec{A}$, and look for approximations involving $\vec{J}_{\vec{k}}$. These are termed multipole expansions. Write

$$e^{-i\vec{k}\cdot\vec{r}'} = 1 - i\vec{k}\cdot\vec{r}' - \left(i\vec{k}\cdot\vec{r}'\right)^2 + \dots . \tag{15.70}$$

This assumes that charge and current distributions are limited in extent, in addition to the observer being in the far-field region. The primary interest is in the leading term, which is the dipole contribution.

As the leading term in the series, the dipole contribution is given by

$$\vec{A} = \frac{e^{ikr-i\omega t}}{rc} \int_V d^3r'\, \vec{J}(\vec{r}') . \tag{15.71}$$

This integral vanishes in magnetostatics, but with an $e^{-i\omega t}$ time dependence this is no longer the case. Following the same procedure that we did in magnetostatics, consider

$$\int_V d^3r' \nabla\cdot(x'\vec{J}(\vec{r}')) = \int_V d^3r' \left((\nabla x')\cdot\vec{J}(\vec{r}') + x'\nabla\cdot\vec{J}\right) \tag{15.72a}$$

$$= \int_V d^3r' \left(J_x(\vec{r}') + i\omega x' \rho(\vec{r}')\right) = 0 . \tag{15.72b}$$

Repeating the calculation for the other two Cartesian coordinates gives

$$\int_V d^3r'\, \vec{J}(\vec{r}') = -i\omega \int_V d^3r'\, \vec{r}'\, \rho(\vec{r}') = -i\omega\, \vec{p} , \tag{15.73}$$

where $\vec{p}$ is the dipole associated with $\rho(\vec{r})$. We have thus converted an integral over $\vec{J}$ to an integral over a charge density ρ. Taking the next steps, we have

$$\vec{A}(\vec{r},t) = (-ik\,\vec{p}) \frac{e^{ikr-i\omega t}}{r} . \tag{15.74}$$

In the locally flat approximation, it follows that in the radiation zone

$$\vec{E}(\vec{r},t) = \frac{ik}{rc}(-i\omega)\vec{p}e^{ikr-i\omega t} + \frac{i\vec{k}\cdot\vec{p}}{r} ik\hat{r}e^{ikr-i\omega t} \tag{15.75a}$$

, (15.75b)

$$= \frac{k^2}{r}(\underset{\sim}{I} - \hat{k}\hat{k}) \cdot \vec{p}\, e^{ikr - i\omega t}$$

$$= \frac{k^2}{r}\vec{p}_\perp e^{ikr - i\omega t}\,, \tag{15.75c}$$

where $\vec{p}_\perp$ is the projection of $\vec{p}$ as perceived by the viewer at $\vec{r}$. The field has its maximum strength when the observer views the full extent of the dipole, and vanishes when the dipole is viewed end-on. This is the atomic-scale explanation for Brewster's Angle for TM-polarized radiation, as discussed below. It also explains why a Brewster's Angle for TE polarization cannot exist.

The calculation of $\vec{H}$ is similarly straightforward:

$$\vec{H} = \nabla \times \vec{A} = i\vec{k} \times \vec{A} \tag{15.76a}$$

$$= \frac{k}{r}(\vec{k} \times \vec{p})e^{ikr - i\omega t}\,. \tag{15.76b}$$

$\vec{H}$ is obviously orthogonal to $\vec{A}$, and by Eq. (15.76b), to $\vec{E}$. Given $\vec{E}$ and $\vec{H}$ we can obtain $\vec{S}$. Assuming that $\vec{p}$ is real,

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H} = \frac{\omega}{4\pi r^2}(\vec{k} \times \vec{p})^2 \hat{k}\cos^2(kr - \omega t)\,. \tag{15.77}$$

The time-averaged value is obviously

As an example, let $\vec{p} = p\,\hat{z}$, then evaluate the above equations using spherical coordinates. The magnitude of $\vec{E}$ is

$$E = \left(\frac{k^2}{r}p_z \sin\theta\right)e^{ikr - i\omega t}\,, \tag{15.78}$$

which yields

$$\vec{S} = \hat{k}\frac{c}{4\pi}\frac{k^4}{r^2}p_z^2 \sin^2\theta \cos^2(kr - \omega t)\,. \tag{15.79}$$

With $k = 2\pi/\lambda$, the inverse fourth-power dependence of the dipole Poynting vector on wavelength is obvious. The time average of Eq. (15.79) can be obtained by inspection.

I. Scattering.

In scattering, radiation is a consequence of dipoles induced by incoming radiation. Thomson and Rayleigh scattering represent two levels of approximation, while Mie scattering is the exact description of scattering from a sphere. We consider details of Thomson and Rayleigh scattering in this section, and outline the general calculation for Mie scattering.

(1) Thomson scattering.

If the dipole $q\Delta\vec{r}$ is induced at a sufficiently high frequency, as for example an electron driven by hard X-rays or γ rays, then to a good approximation it does not move at all, so $\Delta\vec{r} \cong 0$. But q is accelerated, thus by Eqs. (15.50), it must radiate. This type of radiation is called Thomson scattering. We calculate it here.

Write

$$\vec{J} \cong q\vec{v}\delta(\vec{r}). \tag{15.80}$$

For a harmonic drive, from Eq. (15.41b)

$$\vec{A}(\vec{r}) = \frac{q}{rc}\vec{v}_o e^{ikr-i\omega t}. \tag{15.81}$$

From $\vec{F} = m\vec{a}$, where the incident field is $\vec{E}_o e^{-i\omega t}$,

$$\vec{v}_o = \frac{iq\vec{E}_o}{m\omega}e^{-i\omega t}. \tag{15.82}$$

Then

$$\vec{A}(\vec{r}) = i\frac{q^2\vec{E}_o}{m\omega rc}e^{ikr-i\omega t}, \tag{15.83}$$

and finally

$$\vec{E}_{rad} = ik(\underline{I} - \hat{k}\hat{k})\cdot\vec{A} \tag{15.84a}$$

$$= -\frac{1}{r}\left(\frac{q^2}{mc^2}\right)\vec{E}_{o\perp}e^{ikr-i\omega t}. \tag{15.84b}$$

Equation (15.84b) describes Thomson scattering. Its striking characteristic is the complete absence of any frequency (wavelength) dependence. The more familiar Rayleigh scattering depends on wavelength as λ^{-4}, and is discussed next. The distinction occurs because a point charge has no length reference. In contrast, the length reference for Rayleigh scattering is the dipole. The internal reference scale makes the difference.

For $q = e$ the term in the large brackets is recognized as the classical radius of the electron, $r_e = 2.82\times10^{-15}$ m, which is defined as the distance where the electrostatic self-energy e^2/r_e of the electron equals its rest-mass energy mc^2.

(2) Rayleigh (dipole) scattering.

The classic model for Rayleigh scattering is the dielectric sphere in a uniform electric field, where the dielectric function of the sphere is ε and the radius is $a << \lambda$. The use of electrostatics to calculate the induced dipole is then a reasonable approximation. From Ch. 6, the dipole is

$$\vec{p} = \frac{\varepsilon - 1}{\varepsilon + 2} a^3 \vec{E}_o e^{-i\omega t} . \tag{15.85}$$

As we found in electrostatics, this dipole is mathematically perfect; no higher-order multipole terms exist. From Eq. (15.75b), the scattered electric field is

$$\vec{E}_{scat} = \frac{k^2}{r} \left(\frac{\varepsilon - 1}{\varepsilon + 2} a^3 \right) \left((\underline{I} - \hat{k}\hat{k}) \cdot \vec{E}_o \right) e^{ikr - i\omega t} . \tag{15.86}$$

The time-averaged intensity follows from Eq. (15.71):

$$< \vec{S} > = \hat{r} \frac{ck^4 a^6}{8\pi r^2} \left(\frac{\varepsilon - 1}{\varepsilon + 2} \right)^2 | E_\perp |^2 \tag{15.87a}$$

$$= \hat{r} \frac{2\pi^3 c a^6}{r^2 \lambda^4} \left(\frac{\varepsilon - 1}{\varepsilon + 2} \right)^2 | E_\perp |^2 . \tag{15.87b}$$

In Eq. (15.72b) the inverse-fourth-power dependence of $\vec{S}$ on λ is shown explicitly.

The dependence of Rayleigh scattering on λ^{-4} explains the blue color of the sky during the day, and the red color of the sun at sunrise and sunset, where the short wavelengths are preferentially scattered away. Atmospheric scattering is primarily caused by N_2 and O_2 molecules. Scattering also affects polarization, which is investigated as a homework problem. A viewer looking straight up at sunset sees light that is linearly polarized in the direction orthogonal to the illumination direction.

(3) Mie scattering.

If the radius of the sphere is large enough so the electrostatic approximation is no longer valid, then scattering must be described by Mie theory. This was first obtained by Gustav Mie in 1908. It is surprisingly complicated, covering almost 13 pages in Born and Wolf, independent of subsequent discussion. The calculation factors into the equivalent of TM and TE polarizations for a planar interface, although here TM and TE refer to the radial coordinate. Rather than repeat 13 pages of dense mathematics, we cover only the highlights and add some remarks. The reader interested in details should consult Born and Wolf.

The Mie solution is based on the scalar potentials ${}^m\Pi$ and ${}^e\Pi$, in the notation of Born and Wolf. These are termed the Debye potentials, and $\vec{E}$ and $\vec{H}$ follow from with appropriate spatial derivatives. The Debye potentials satisfy the Helmholtz Equation, and so have series expansions that can be written in terms of the spherical Bessel functions $j_l(k_{in} r)$ and $h_l^{(1)}(k_{out} r)$ inside and outside the sphere, respectively,

Texts typically present a workaround based on replacing the Laplace Equation with the Helmholtz Equation, then repeating the electrostatic solution. This takes advantage of the spherical-Bessel-function solutions of the Helmholtz equation, and reduces correctly to the electrostatic solution for $k \rightarrow 0$. Assuming azimuthal symmetry, the expansions are

$$\phi_{in}(\vec{r}) = \sum_{l=0}^{\infty} A_l \, j_l(k_{in} r) P_l(\cos\theta), \; r \leq a \,; \tag{15.88a}$$

$$\phi_{out}(\vec{r}) = \sum_{l=0}^{\infty} B_l h_l^{(1)}(k_{out} r) P_l(\cos\theta), \; r > a \,. \tag{15.88b}$$

Given the plane-wave expansion

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) \, j_l(kr) P_l(\cos\theta), \tag{15.89}$$

one next calculates $\vec{E} = -\nabla\phi$ and $\vec{D} = \varepsilon\vec{E}$, then applies the boundary conditions on tangential $\vec{E}$ and normal $\vec{D}$. The coefficients A_l and B_l are then determined.

However, the question for which this is the answer is not clear. As we know from our discussion of far-field radiation, $\nabla\phi$ is the *longitudinal*, not transverse, contribution to the wave. This may be the contribution of the scalar-potential part of the incoming wave, but this presumes that the source dipole that generates the incoming wave is not perpendicular to the observation direction, so that a longitudinal component exists. It is usually not a good idea for the observer to impose conditions on the source.

Accordingly, the interested reader is referred to Born and Wolf. We only reproducing their figure of the exact result for the scattering cross section of droplets of water in the atmosphere, which is shown at the right.

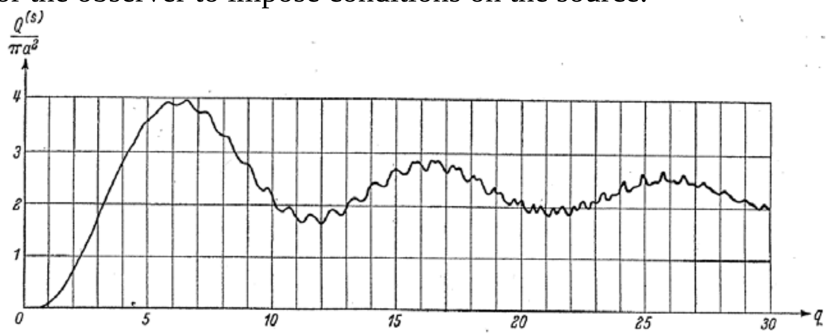


Fig. 13.14. The scattering cross-section of dielectric spheres of refractive index $n = 1{\cdot}33$ as function of the parameter $q = 2\pi a/\lambda^{(1)}$.
(After B. Goldberg, *J. Opt. Soc. Amer.*, **43** (1953), 1221.)

The cross section starts from zero, as expected from an effective-medium treatment. The fine structure is due to the excitation of resonances in the water droplet. For large radii the cross section approaches the limiting value of twice the geometric cross-sectional area of the droplet. This factor of 2 relative to the cross-sectional area occurs for any opaque object. It follows from the optical theorem, and is a general consequence of diffraction. This is discussed in more detail in a later section.

J. Constructive and destructive interference: beam steering and X-ray diffraction.

Because the equations are linear, the theory developed in the previous sections for single sources applies to multiple sources as well. The only new aspect needed is to modify the theory to locate sources at vectors $\vec{r}_0$ other than the origin. This is accomplished by replacing $\vec{r}\,' \to \vec{r}\,' + \vec{r}_o$, so $\vec{r}\,'$ is now measured with respect to $\vec{r}_0$.

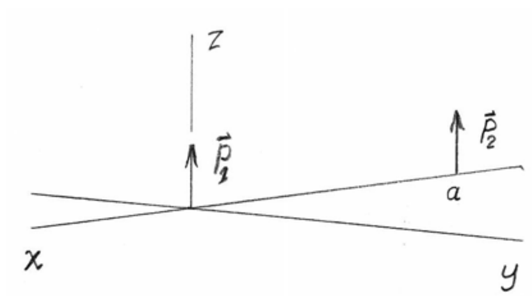
Because we wish to reference both the observer vector $\vec{r}$ and the displacement vector $\vec{r}_o$ to the common origin, the least confusing approach is to evaluate integrals as usual then displace the result by $\vec{r}_o$. In the radiation zone this generates a prefactor $e^{-i\vec{k}\cdot\vec{r}_o}$ according to

$$e^{ik|\vec{r}-\vec{r}\,'-\vec{r}_o|} \cong e^{-i\vec{k}\cdot\vec{r}_o} e^{ik|\vec{r}-r'|} . \tag{15.90}$$

The remaining phase term $e^{ik|\vec{r}-\vec{r}\,'|}$ is then treated as usual.

Because Nature adds fields, not intensities, and because fields are described by phase as well as amplitude, this means that we have control over constructive and destructive interference by placing dipoles judiciously and selecting the right parameters. This flexibility is used to increase the range of cell-phone communications in both transmission and reception modes, which is why cell-phone towers have multiple antennas. On the atomic scale, constructive interference from multiple locations is the source of the spot patterns observed in X-ray diffraction and also the "images" seen in transmission-electron microscopy (TEM). In a more basic example, the Ewald-Oseen Extinction Theorem describes reflection as constructive interference from dipoles induced in a material on the atomic scale.

The principles involved can be demonstrated by a two-antenna example that is simple enough so the calculation is straightforward, yet detailed enough so possible sources of confusion are anticipated and clarified. Consider two antennas radiating as electric dipoles oriented vertical (z) direction, as shown in the diagram. One antenna is located at the origin $\vec{r}_1 = 0$ of a local coordinate system, and the other at $\vec{r}_2 = -a\hat{x}$. Let both antennas radiate at a wavelength $\lambda = 2\pi/k$. Do conditions exist such that the radiated field is zero in the y directions and maximized in the x directions?



To investigate possibilities, start by placing the observer in the xy plane at the general point

$$\vec{r} = r(\hat{x}\cos\varphi + \hat{y}\sin\varphi) . \tag{15.91}$$

Then in the radiation zone

$$\vec{k} = k(\hat{x}\cos\varphi + \hat{y}\sin\varphi) . \tag{15.92}$$

Because the first antenna is located at the origin, its field is given directly by Eq. (15.75c), writing $p_\perp = p_1$. We use p_1 as both an amplitude and phase reference. According to the above, the field of the second antenna is given by writing $p_\perp = p_2 e^{i\phi_2}$ and multiplying Eq. (15.75c) by $e^{-i\vec{k}\cdot\vec{r}_o} = e^{+ia\vec{k}\cdot\hat{x}}$. The result is

$$\vec{E}(\vec{r},t) = \frac{k^2}{r}\hat{z}\left(p_1 + p_2 e^{i\phi_2} e^{ika\cos\varphi}\right)e^{ikr-i\omega t} \ . \tag{15.93}$$

We first select parameters to eliminate the field in the y directions. With $\vec{k} = k\hat{y}$, Eq. (15.93) becomes

$$\vec{E}_{tot}(r\hat{y},t) = \frac{k^2}{r}\hat{z}\left(p_1 + p_2 e^{i\phi_2}\right)e^{ikr-i\omega t} \ . \tag{15.94}$$

Thus we accomplish our goal $\vec{E}_{tot}(r\hat{y},t) = 0$ if $p_1 = p_2$ and $e^{i\phi_2} = -1$, that is, if the dipoles radiate with the same power and are 180^{o} out of phase. We could have deduced this by inspection, but it is important to see the mathematics in action.

Now consider an observer on the x axis. For $\vec{r} = \pm r\hat{x}$ and using the above values for p_2 and ϕ_2 we have

$$\vec{E}_{tot}(r\hat{y},t) = \frac{k^2}{r}\hat{z}p_1\left(1 - e^{-ika}\right)e^{ikr-i\omega t} \ . \tag{15.95}$$

Thus $|\vec{E}_{tot}(r\hat{y},t)|$ is maximized if $a = n\pi/k = n\lambda/2$, where $n = \pm 1, \pm 3, \pm 5,$ etc. That is, the antennas are spaced at least a half-wavelength apart. Of course, any integral multiple of wavelengths beyond the half-wavelength works as well. Under these conditions the signal from either antenna arrives in phase with the signal from the other, so the value of the field along the x axis in either direction is twice that of a single antenna.

The Poynting vector for an observer in the xy plane provides additional information. Including the above parameters, Eq. (15.9) 5becomes

$$\vec{E}_{tot}(r\hat{y},t) = \frac{k^2}{r}\hat{z}p_1\left(1 - e^{-i\pi\cos\varphi}\right)e^{ikr-i\omega t} \ . \tag{15.96}$$

Taking the real projection yields

$$\mathrm{Re}(\vec{E}_{tot}) = \frac{k^2}{r}\hat{z}p_1\left(\cos(kr-\omega t) - \cos(kr-\omega t - \pi\cos\varphi)\right). \tag{15.97}$$

Then with $\vec{\boldsymbol{H}} = \hat{\boldsymbol{k}} \times \vec{\boldsymbol{E}} = \hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}$, we obtain

$$\vec{S} = \frac{ck^4}{4\pi r^2}\hat{r}\left(\cos^2(kr-\omega t) + \cos^2(kr-\omega t - \pi\cos\varphi)\right.$$

$$\left. - 2\cos(kr-\omega t)\cos(kr-\omega t-\pi\cos\varphi)\right). \tag{15.98}$$

The time average is therefore

$$<\vec{S}> = \frac{ck^4 p_1^2}{4\pi r^2}\hat{k}\left(1 - \cos(\pi\cos\varphi)\right) \tag{15.99}$$

Thus the power radiated in the forward and backward directions along the *x* axis is four times that radiated by a single antenna, or twice that if full power had been sent to only one of the antennas. Yet the total power radiated is the same, since the second term in Eq. (15.90) averages to zero.

A linear array of four antennas spaced a quarter-wavelength apart can not only eliminate fields in the y direction but also in either the forward or reverse x direction. The solution is left as an exercise.

X-ray diffraction provides a second example of constructive and destructive interference. When an X-ray beam

$$\vec{E}(\vec{r},t) = \vec{E}_o e^{i\vec{k}_o\cdot\vec{r}-i\omega t} \tag{15.100}$$

is incident on a crystal, its electric field induces dipoles according to $\vec{F} = m\vec{a}$. If every charge in the crystal, usually an electron in a bond, has the same polarizability α, then the incoming field induces a dipole $\vec{p} = \alpha\vec{E}_{loc}$ at each charge site $\vec{R}_n$, where $\vec{E}_{loc}$ is the field at the site. Ignoring local-field corrections, this is just the field of the incoming wave at the site.

By Eq. (15.67c) and the above, the radiated field $\vec{E}_n$ from the n^{th} dipole is therefore

$$\vec{E}_n = \frac{k^2}{r}\left(\alpha\,\vec{E}_o\,e^{i\vec{k}_o\cdot\vec{R}_n}\right)e^{-i\hat{k}\cdot\vec{R}_n}\,e^{ikr-i\omega t}, \tag{15.101}$$

where the term in the parentheses describes the dipole, and the remainder of the expression describes its contribution to the emerging radiation. Summing over all sites $\vec{R}_n$ gives the scattered field

$$\vec{E}_{tot} = \frac{k^2}{r}\alpha\,\vec{E}_o\left(\sum_{\vec{R}_n} e^{i(\vec{k}_o-\vec{k})\cdot\vec{R}_n}\right)e^{ikr-i\omega t}, \tag{15.102}$$

A nonvanishing time-averaged intensity will result only if $\vec{k}_o$ and $\vec{k}$ satisfy

$$((\vec{k}_o - \vec{k})\cdot\vec{R}_n = 2\pi\nu \tag{15.103}$$

for all R_n, where ν is an integer. These diffraction spots allow the structure of the crystal to be deduced. If the sum is infinite, the result is a delta function. However, since crystals are never infinitely large, these sums are finite. If the sum is approximated by an integral, the result is a sinc function. The full-width-half-maximum of this function gives the size of the scattering region. A degenerate solution $\nu = 0$ is always obtained for forward scattering for any configuration of scatterers.

Another physical situation where the previous discussion applies is in the reflection of light from a material. This is not immediately obvious because reflection is not usually treated as an atomic-scale process. However, since we have developed the necessary machinery, we consider it here. Let Eq. (15.100) represents a light beam that is incident on a transparent material. The $\vec{F} = m\vec{a}$ argument that we just used for X-rays applies here as well: the field of the incident beam generates a dipole $q\Delta\vec{r}$ at each of the electrons in the material. Since time-varying dipoles radiate, the result is a scattered field given by Eq. (15.102). Because the dipoles are all driven by the same wave, radiation from the different dipoles adds constructively in the specular direction, but destructively otherwise.

For TM polarization, we can in principle find an angle where the induced dipoles are viewed end-on. At this angle $\vec{p}_{\perp} = 0$ and for the observer, the radiation vanishes. This is the atomic-scale basis for the Brewster Angle. In contrast, if the incoming beam is TE polarized, then the induced dipoles are parallel to the surface, and hence from the perspective of the observer always possess maximum radiation efficiency. Therefore, TE-induced dipoles never exhibit a Brewster angle. For the same reason, at any angle of incidence the reflectance for TE-polarized light will always be greater than that for TM-polarized light, because TM-induced dipoles have a component in the radiation direction.

As a historical note, the atomic-scale model of reflectance was developed by Ewald in 1912 and independently by Oseen in 1915. Ewald worked with discrete dipoles, and Oseen a continuum. The model is also known as the Ewald-Oseen Extinction Theorem, because in the mathematics the radiation from the induced dipoles cancels the incident radiation that created them and replaces it with a polarization wave that obeys Snell's Law. The Theorem is interesting for several reasons. Its derivation is a difficult mathematical challenge, covering almost 6 pages in Born and Wolff. The difficulty follows from the fact that the generated radiation has the same wavelength as the light that created it, and has comparable intensity. Thus the calculation is a difficult challenge in self-consistency. In contrast, for *nonlinear* optics self-consistency is not an issue, because the radiation here not only emerges at a different wavelength but also is orders of magnitude weaker. Here, the atomic model leads to insights that cannot be obtained by the standard phenomenological tensorial treatments. This is also one of the few cases where a nonlinear problem is simpler than its linear equivalent.

In the Ewald-Oseen model, reflected light originates from the outermost layer of dipoles. Of course, dipoles radiate in all directions, not just in the specular direction. The fact that radiation also exists in nonspecular directions seems weird, because in our intensity-dominated world there is nothing to be seen. Nevertheless, the proof that the radiation is there is quite simple. If we block off a section of the surface, then cancellation is not exact, and scattered light appears in nonspecular directions. Blocking part of the surface obviously did not create this radiation, it was there all along. One of the most interesting aspects of E&M on the quantum level is that somehow photons know where, and where not, to go.