Math 672 Lecture 23

Theorem: Let V be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and let S be a subset of V. Then:

- $S^\perp$ is a subspace of V.
- If S is a subspace of V, then $S \cap S^\perp = \{0\}$.
- If S is a subspace of V, then $(S^\perp)^\perp = S$.

Proof (third conclusion): Suppose S is a subspace of V. Now consider $S^{\perp\perp} = (S^\perp)^\perp$. By the first result, $S^{\perp\perp}$ is a subspace of V. Note that $S \subseteq S^{\perp\perp}$:

$$\langle u, w \rangle = 0 \quad \forall w \in S \quad \forall u \in S^\perp \text{ (by definition of } S^\perp\text{)}$$

$$\Rightarrow \langle w, u \rangle = 0 \quad \forall u \in S^\perp \; \forall w \in S$$

$$\Rightarrow w \in S^{\perp\perp} \quad \forall w \in S.$$

Now suppose $w \in S^{\perp\perp}$ and consider $v = P_S w$. Note that

$$v \in S \subseteq S^{\perp\perp}, \; w \in S^{\perp\perp}$$

$$\Rightarrow w - v \in S^{\perp\perp}.$$

But $w - v \in S^\perp$ by definition of P_S. Thus

$$w - v \in S^\perp \cap S^{\perp\perp} = \{0\}$$

$$\Rightarrow w - v = 0$$

$$\Rightarrow w = v$$

$$\Rightarrow w \in S.$$

Thus $S^{\perp\perp} = S.$ //

<u>Theorem</u>: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a finite-dimensional subspace of V. (Note that V need not be finite dimensional.) Then

$$V = S \oplus S^{\perp}.$$

<u>Proof</u>: By the previous theorem, $S + S^{\perp}$ is a direct sum, so it suffices to prove that every $v \in V$ can be written as

$$v = u + w, \; u \in S, w \in S^{\perp}.$$

But, by the projection theorem,

$$\langle v - P_S v, z \rangle = 0 \;\; \forall z \in S$$

$$\Rightarrow v - P_S v \in S^{\perp}$$

$$\Rightarrow v = u + w, \; u = P_S v \in S, w = v - P_S v \in S^{\perp}. //$$

<u>Theorem</u>: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a finite-dimensional subspace of V. Then

1. $P_S(u)=u \ \forall u\in S$

2. $P_S(u)=0 \ \forall u\in S^\perp$

3. $\mathcal{R}(P_S)=S$

4. $\mathcal{N}(P_S)=S^\perp$

5. $P_S^2=P_S$, that is, $P_S(P_S(u))=P_S(u) \ \forall u\in V$

6. $\|P_S u\|\le\|u\| \ \forall u\in V$.

Proof: Recall that, for all $u\in V$, $v=P_S u$ is the unique vector in S satisfying

$$\langle u-v,w\rangle=0 \quad \forall w\in S$$

$$\Longleftrightarrow \quad u-v\in S^\perp.$$

1. If u itself belongs to S, then we have

$$\langle u-u,w\rangle=\langle 0,w\rangle=0 \quad \forall w\in W$$

$$\Rightarrow u=P_S u.$$

2. If $u\in S^\perp$, then

$$\langle u,w\rangle=0 \quad \forall w\in S$$

$$\Rightarrow \langle u-0,w\rangle=0 \quad \forall w\in S$$

$$\Rightarrow 0=P_S u.$$

3. By definition of P_S, $\mathcal{R}(P_S)\subseteq S$, so #1 shows that $\mathcal{R}(P_S)=S$.

4. #2 shows that $S^\perp \subseteq \mathcal{N}(S)$. We have

$$u \in \mathcal{N}(P_S) \Rightarrow P_S u = 0$$

$$\Rightarrow \langle u-0, w\rangle = 0 \ \forall w \in S$$

$$\Rightarrow \langle u, w\rangle = 0 \ \forall w \in S$$

$$\Rightarrow u \in S^\perp.$$

Thus $\mathcal{N}(S) \subseteq S^\perp$ and hence $\mathcal{N}(S) = S^\perp$.

5. This follows from #3 and #1.

6. Let $u \in V$. Since $P_S u$ and $u - P_S u$ are orthogonal, we have

$$\|P_S u\|^2 + \|u - P_S u\|^2 = \|u\|^2 \quad \text{(Pythagorean theorem)}$$

$$\Rightarrow \|P_S u\|^2 \leq \|u\|^2$$

$$\Rightarrow \|P_S u\| \leq \|u\|. //$$

We end this chapter with two facts about earlier topics, where we get more information using an inner product.

<u>Recall</u>: If V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$, then there exists a basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular:

$$(*) \quad \begin{aligned} &\mathcal{M}_{\mathcal{B}}(T(v)) = A\,\mathcal{M}_{\mathcal{B}}(v) \ \forall v \in V, \\ &A_{ij} = 0 \text{ if } 1 \leq i,j \leq n \text{ and } i > j \quad (n = \dim(V)). \end{aligned}$$

Also recall that (*) is equivalent to

$$T(v_j) = \sum_{i=1}^{j} A_{ij} v_j, \quad j=1,2,\dots,n,$$

and hence to

$$T(v_j) \in \text{span}(v_1,\dots,v_j), \; j=1,2,\dots,n.$$

<u>Theorem</u>: Let V be an inner product space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$.

Then there exists an orthonormal basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular.

<u>Proof</u>: We know that there exists a basis $\mathcal{B}' = \{v_1,\dots,v_n\}$ of V such that $\mathcal{M}_{\mathcal{B}',\mathcal{B}'}(T)$ is upper triangular. Let $\mathcal{B} = \{u_1,\dots,u_n\}$ be the orthonormal basis of V produced from $\mathcal{B}'$ by Gram-Schmidts and recall that

$$\text{span}(u_1,\dots,u_j) = \text{span}(v_1,\dots,v_j) \quad \forall j=1,2,\dots,n.$$

It follows that

$$T(v_j) \in \text{span}(v_1,\dots,v_j) \quad \forall j=1,2,\dots,n.$$

Also, note that

$$u_j = \sum_{i=1}^{j} \alpha_{ij} v_i \quad \text{for some } \alpha_{1j},\dots,\alpha_{jj} \quad \text{(by the Gram-Schmidt algorithm)}$$

$$\Rightarrow T(u_j) = T\left(\sum_{i=1}^{j} \alpha_{ij} v_i\right) = \sum_{i=1}^{j} \alpha_{ij} T(v_i)$$

$\Rightarrow T(u_j) \in \operatorname{span}(T(v_1), \ldots, T(v_j)) \subseteq \operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(u_1, \ldots, u_j)$

This shows that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is also upper triangular. //

Recall: A linear functional on V is an element of $V' = \mathcal{L}(V, F)$. We call V' the dual space of V. We know that $\dim(V') = \dim(V)$, so $V' \cong V$. In the case of an inner product space, the isomorphism is simple.

Theorem: Let V be a finite-dimensional vector space over F ($\mathbb{R}$ or $\mathbb{C}$) and let $\varphi \in V'$. Then there exists a unique $u \in V$ such that

$$\varphi(v) = \langle v, u \rangle \quad \forall\, v \in V.$$

Proof: Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for V. Given $\varphi \in V'$, define

$$u = \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j \quad \left(\text{or just } u = \sum_{j=1}^{n} \varphi(u_j)\, u_j \text{ if } F = \mathbb{R}\right)$$

Then, for all $v \in V$,

$$\begin{aligned} \varphi(v) &= \varphi\left(\sum_{j=1}^{n} \langle v, u_j \rangle u_j\right) \\ &= \sum_{j=1}^{n} \langle v, u_j \rangle \varphi(u_j) \quad (\text{since } \varphi \text{ is linear}) \\ &= \left\langle v, \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j \right\rangle \end{aligned}$$

$$= \langle v, u \rangle.$$

This proves existence. If $w \in V$ also satisfies

$$\varphi(v) = \langle v, w \rangle \quad \forall v \in V,$$

then

$$\langle v, w \rangle = \langle v, u \rangle \quad \forall v \in V$$

$$\Rightarrow \langle v, w-u \rangle = 0 \quad \forall v \in V$$

$$\Rightarrow \langle w-u, w-u \rangle = 0$$

$$\Rightarrow w-u = 0$$

$$\Rightarrow w = u.$$

This proves uniqueness. //

The above result is (one version of) the <u>Riesz representation theorem</u>.