Classification of States

Definition: State j is said to accessible from state i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

Definition: i & j are accessible from each other are said to communicate. This is denoted by $i \leftrightarrow j$.

By definition, $i \leftrightarrow i$ $\left(P_{ii}^{(0)} = 1 \right)$.

Remark: X is irreducible if all states communicate with each other.

Definition: Let f_i denote the probability, starting in state i, the process will reenter state i.

- State i is said to be recurrent if $f_i = 1$.
- State i is said to be transient if $f_i < 1$

Let

$$I_n = \begin{cases} 1 \; ; & X_n = i \\ 0 \; ; & X_n \neq i \end{cases}$$

$\sum_{n=0}^{\infty} I_n$ represents the number of times the process visits state i. Then

$$E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=0}^{\infty} P(X_n = i \mid X_0 = i)$$

$$= \sum_{n=0}^{\infty} P_{ii}^{(n)}$$

Theorem :

i) If $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$, then state i is recurrent.

ii) If $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$, then state i is transient.

Remark : This proves that transient states can only be visited a finite number of times.

Corollary : If j is transient, then $\lim_n P_{ij}^{(n)} = 0$, $\forall i$.

Theorem : If i is recurrent, & $i \leftrightarrow j$, then j is recurrent.

Proof : Since $i \leftrightarrow j$, $\exists\, k, m$ s.t.

$$P_{ij}^{(k)} > 0 \quad \& \quad P_{ji}^{(m)} > 0.$$

For any n,

$$P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(n)} \cdot P_{ij}^{(k)}$$

since $P_{jj}^{(m+n+k)}$ is the probability of traveling from $j \to j$ in $m+n+k$ steps through any path.

However, $P_{ji}^{(m)} \cdot P_{ii}^{(n)} \cdot P_{ij}^{(k)}$ is the prob. of traveling from $j \to j$ through a specific path $j \xrightarrow[\text{steps}]{m} i \xrightarrow[\text{steps}]{n} i \xrightarrow[\text{steps}]{k} j$. Therefore,

$$\sum_{n=1}^{\infty} P_{jj}^{(m+n+k)} \geq \underbrace{P_{ji}^{(m)}}_{>0} \underbrace{P_{ij}^{(k)}}_{>0} \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

Thus, j is recurrent.

Definition: Two states that communicate with each other are in the same class.

A state that does not communicate with no other states itself is a class.

Theorem: Let C_1, C_2 be two classes. Either $C_1 = C_2$ or $C_1 \cap C_2 = \phi$.

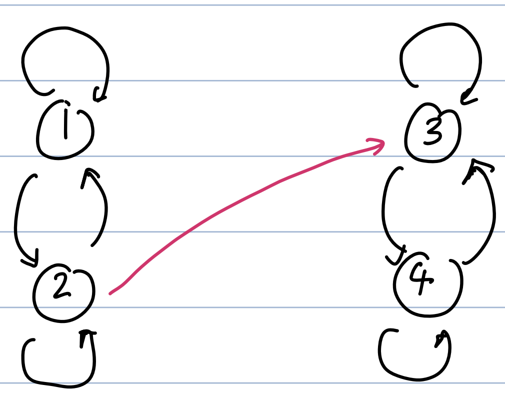
Let $k \in C_1 \cap C_2$, $i \in C_1$, $j \in C_2$.

Since $i \leftrightarrow k$ & $j \leftrightarrow k$, $i \leftrightarrow j$.

Therefore, $i \in C_2$ & $j \in C_1$. $\Rightarrow$ $C_1 = C_2$.

Ex: 1) Find all the classes of the MC represented by the following matrix:

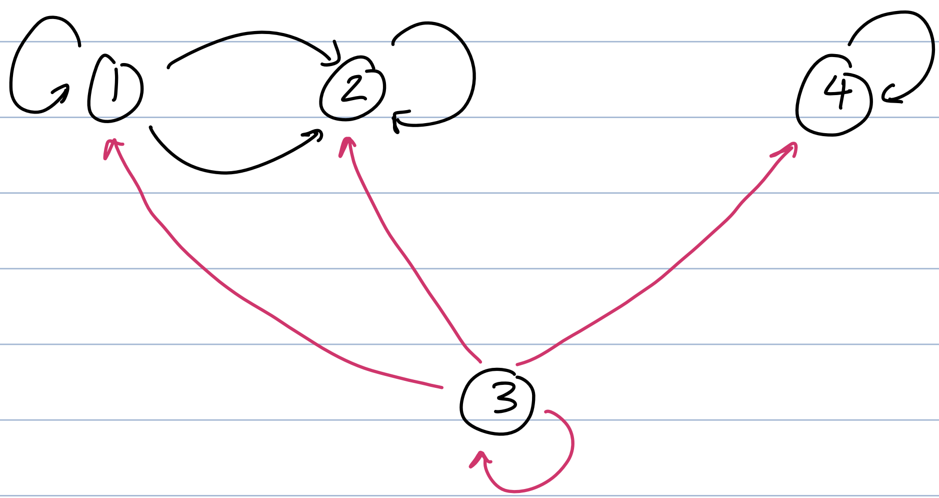
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{bmatrix} \end{array}$$



Classes:
$\{1, 2\}$, $\{3, 4\}$

②

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.5 & 0.5 & 0 & 0 \\ 2 & 0.5 & 0.5 & 0 & 0 \\ 3 & 0.25 & 0.25 & 0.25 & 0.25 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$



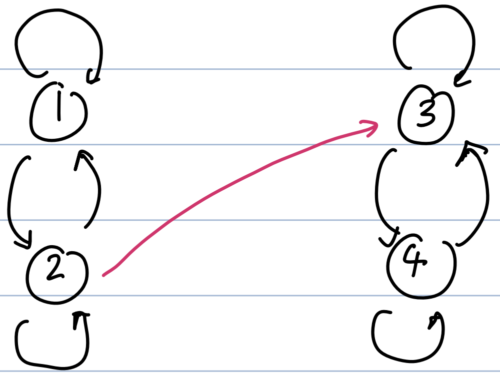
Classes :

$\{1, 2\}$, $\{3\}$, $\{4\}$.

Definition: A class C is said to be closed if $P_{ij} = 0$ $\forall i \in C$, $\forall j \notin C$.

Ex:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{bmatrix} \end{array}$$



Classes:

$$\underbrace{\{1, 2\}}_{\text{not closed}}, \underbrace{\{3, 4\}}_{\text{closed}}$$

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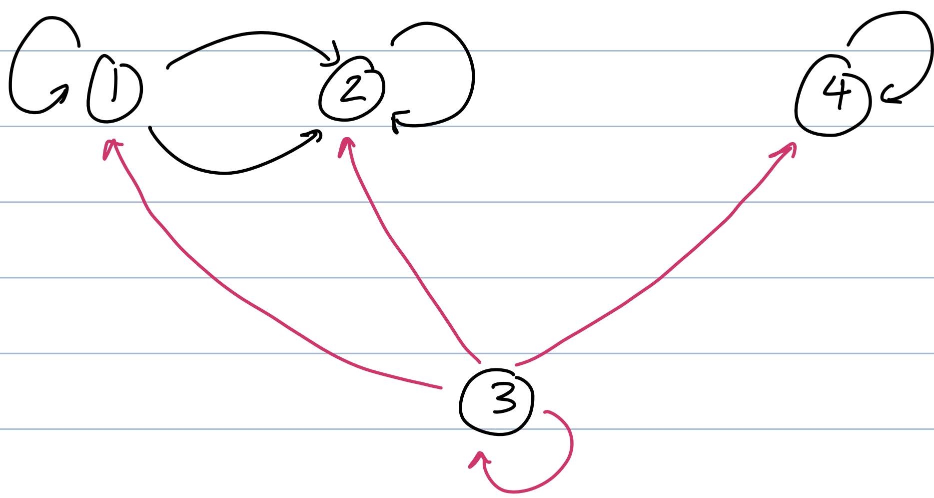
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Classes:

$$\underbrace{\{1, 2\}}_{\text{closed}}, \{3\}, \underbrace{\{4\}}_{\text{closed}}$$

Final Review

① Let $\{X_n\}$ be a sequence of independent r.v.s & $X_n \sim N\left(0, \frac{n+1}{n}\right)$. Find the limiting distribution of X_n.

Solution:

$$F_{X_n}(x) = P(X_n \le x) = P\Bigg(\underbrace{\frac{X_n - 0}{\sqrt{\frac{n+1}{n}}}}_{\sim N(0,1)} \le \frac{x}{\sqrt{\frac{n+1}{n}}}\Bigg)$$

$$= P\left(Z \le \frac{x}{\sqrt{\frac{n+1}{n}}}\right)$$

$$= \Phi\left(\frac{x}{\sqrt{\frac{n+1}{n}}}\right) \to \Phi(x) \quad \forall x \text{ where } \Phi \text{ is cts}$$

$$\Rightarrow X_n \longrightarrow Z \sim N(0,1)$$

② Let $X_1, X_2, \dots \sim N(0,1)$ & $Y_0 \sim U(0,1)$. Also, $Y_0, X_1, X_2, \dots$ are independent. Define a sequence $\{Y_n\}$ by

$$Y_n = \frac{Y_{n-1}}{2} + X_n, \quad n = 1, 2, \dots$$

Find the limiting distribution of Y_n.

Solution: Observe that

$$Y_n = \frac{Y_{n-1}}{2} + X_n = \frac{\left(\frac{Y_{n-2}}{2} + X_{n-1}\right)}{2} + X_n$$

$$= \frac{Y_{n-2}}{2^2} + \frac{X_{n-1}}{2} + X_n$$

$$= \frac{Y_{n-3}}{2^3} + \frac{X_{n-2}}{2^2} + \frac{X_{n-1}}{2^1} + X_n$$

$$\vdots$$

$$= \frac{Y_0}{2^n} + \sum_{k=0}^{n-1} \frac{X_{n-k}}{2^k}$$

$\frac{Y_0}{2^n} \to 0$ almost surely as $n \to \infty$.

Note that $\sum_{k=0}^{n-1} \frac{X_{n-k}}{2^k} \sim N(0, \sigma_n^2)$

where $\sigma_n^2 = V\left(\sum_{k=0}^{n-1} \frac{X_{n-k}}{2^k}\right) = \sum_{k=0}^{n-1} \frac{1}{2^{2k}}$

As $n \to \infty$, $\sigma_n^2 \to \sum_{k=0}^{\infty} \frac{1}{2^{2k}} = \frac{1}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3}$

Guess: $Y_n \to N(0, \frac{4}{3})$.

However, this must be proved!

Let $S_n = \sum_{k=0}^{n-1} \frac{X_{n-k}}{2^k}$

$$F_{S_n}(x) = P(S_n \leq x) = P\left(\underbrace{\frac{S_n}{\sigma_n}}_{\sim N(0,1)} \leq \frac{x}{\sigma_n}\right)$$

$$= P\left(Z \le \frac{x}{\sigma_n} \right)$$

$$= \Phi\left(\frac{x}{\sigma_n} \right)$$

Since $\sigma_n^2 \to \frac{4}{3}$, $\sigma_n \to \sqrt{\frac{4}{3}}$.

Therefore, $\Phi\left(\frac{x}{\sigma_n}\right) \to \Phi\left(\frac{x}{\sqrt{\frac{4}{3}}}\right)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4/3}}} e^{-\frac{z^2}{2}}\, dz$$

Let Let $z = \frac{y}{\sqrt{4/3}}$.

Then, $\Phi\left(x/\sqrt{\frac{4}{3}}\right) = \underbrace{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4/3}} \int_{-\infty}^{x} e^{-\frac{y^2}{2\left(\frac{4}{3}\right)}}\, dy}_{\text{Distribution function of } N\left(0, \frac{4}{3}\right)}$

$\square$