Math 600 Lecture 10

Let $C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$ and define

$$d(f,g) = \max\{|f(x)-g(x)| : x \in [a,b]\}.$$

Then $(C[a,b], d)$ is a metric space.

Weierstrass theorem: For all $f \in C[a,b]$ and for all $\varepsilon > 0$, there exists $p \in \mathcal{P}$ (a polynomial) such that

$$\max\{|p(x)-f(x)| : x \in [a,b]\} < \varepsilon.$$

What does this theorem say about $\mathcal{P}$ and $C[a,b]$?

Theorem: Let (X,d) be a metric space with the property that every infinite subset of X has a limit point in X. Then X is separable.

Proof: Choose any $\delta > 0$ and select $x_1 \in X$. Construct $x_2, x_3, \ldots$ as follows:

Given $x_1, \ldots, x_k$ such that

$$d(x_i, x_j) \geq \delta \quad \forall i,j = 1,\ldots,k,\ i \neq j,$$

choose x_{k+1} (if possible) so that

$$d(x_i, x_{k+1}) \geq \delta \quad \forall i = 1,\ldots,k.$$

We claim that this process must end after a finite number of steps. If not, we obtain a sequence $\{x_n\}$ such that

$$d(x_n, x_m) \geq \delta \quad \forall m,n \in \mathbb{Z}^+,\ m \neq n.$$

But such a sequence cannot have a limit point in X (any ball of radius δ contains at most one point in this sequence). Thus, there exists n_δ and points $x_1, \ldots, x_{n_\delta}$ such that

$$X = \bigcup_{j=1}^{n_\delta} B_\delta(x_j).$$

Write $S_\delta = \{x_1, \ldots, x_{n_\delta}\}$. Note that, for all $x \in X$, there exists some $x_j \in S_\delta$ such that $d(x, x_j) < \delta$.

Now define

$$S = \bigcup_{n=1}^{\infty} S_{1/n}.$$

Let $\varepsilon > 0$ be arbitrary and let $x \in X$. We wish to prove that there exists $y \in S$ such that $d(x,y) < \varepsilon$. But if we choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \varepsilon$, then there exists $y \in S_{1/n} \subset S$ such that

$$d(x,y) < \tfrac{1}{n} < \varepsilon.$$

Thus S is dense in X, that is, X is separable. //

Theorem: Suppose (X,d) is a metric space with the property that every infinite subset of X has a limit point in X. Then X is compact.

Proof: By the above results, X is separable, hence there exists a countable basis for X, hence every open cover for X contains a countable subcover for X. Thus it suffices to prove that if $\{G_n\}$ is a countable open cover for X, then it contains a finite subcover.

So let $\{G_n\}$ be a countable open cover for X. Define, for all $n \in \mathbb{Z}^+$,

$$F_n = \left(\bigcup_{j=1}^{n} G_j \right)^c = \bigcap_{j=1}^{n} G_j^c$$

Let us argue by contradiction and assume that $\{G_n\}$ contains no finite subcover of X. Then $\{G_1, \ldots, G_n\}$ does not cover X for all $n \in \mathbb{Z}^+$ and hence $F_n \neq \emptyset$ for all $n \in \mathbb{Z}^+$. However,

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{j=1}^{\infty} G_j^c = \left(\bigcup_{j=1}^{\infty} G_j \right)^c = X^c = \emptyset.$$

For each $n \in \mathbb{Z}^+$, let $x_n \in F_n$. By assumption, $\{x_n\}$ has a limit point $x \in X$. Note that each F_n is closed and

$$F_{n+1} \subset F_n \quad \forall n \in \mathbb{Z}^+$$

$$\Rightarrow \forall n \in \mathbb{Z}^+, \ \{x_k \mid k \geq n\} \subset F_n.$$

Thus x is a limit point of each F_n and hence, since each F_n is closed, $x \in F_n \ \forall n \in \mathbb{Z}^+$. But then $x \in \bigcap_{n=1}^{\infty} F_n$, a contradiction.

Thus $\{G_n\}$ must contain a finite subcover, and we have proved that X is compact. //

Corollary: If (X,d) is a metric space and $E \subset X$ has the property that every infinite subset of E has a limit point in E, then E is compact.

Proof: Recall that E is a compact subset of X iff (E,d) is a compact metric space. //

Connected sets

Definition: Let (X,d) be a metric space.

- We say that subsets A, B of E are <u>separated</u> iff
$$A \cap \overline{B} = \emptyset \text{ and } \overline{A} \cap B = \emptyset.$$
- We say that $E \subset X$ is <u>connected</u> if it not possible to write E as the union of two nonempty separated sets.

Examples

- The intervals $(0,1)$ and $(1,2)$ in $\mathbb{R}$ are separated:
$$(0,1) \cap \overline{(1,2)} = (0,1) \cap [1,2] = \emptyset,$$
$$\overline{(0,1)} \cap (1,2) = [0,1] \cap (1,2) = \emptyset.$$
- The intervals $[0,1]$ and $(1,2)$ are <u>not</u> separated (even though $[0,1] \cap (1,2) = \emptyset$):
$$[0,1] \cap \overline{(1,2)} = [0,1] \cap [1,2] = \{1\} \neq \emptyset.$$

Theorem: A subset E of $\mathbb{R}$ is connected iff it is an interval, that is, iff

$$(*) \qquad (x, y \in E \text{ and } x < z < y) \Rightarrow z \in E.$$

Proof: Suppose first that E is not connected, that is, that there exist nonempty separated sets $A, B \subset \mathbb{R}$ such that $E = A \cup B$. We wish to prove that $(*)$ fails. Choose $x \in A$ and $y \in B$ and assume, without loss of generality, that

$x < y$. Define

$$z = \sup \{ A \cap [x,y] \}.$$

Note that $A \cap [x,y]$ is bounded above by y, so z is well defined, and $z \in \bar{A}$ (if $z \notin A$, then z must be a limit point of A; otherwise, there would be a smaller upper bound). Since A and B are separated, $z \notin B$. In particular, $z < y$.

If $z \notin A$, then $z \notin E = A \cup B$ and $x < z < y$, so $(*)$ fails, as desired.

If $z \in A$, then $z \notin \bar{B}$, so there exists $z_1 \in (z, y)$ such that $z_1 \notin B$. But then

$$z_1 > z \Rightarrow z_1 \notin A \cap [x,y] \Rightarrow z_1 \notin A$$

and we see that

$$x < z_1 < y \text{ and } z \notin E = A \cup B.$$

Thus $(*)$ fails in this case also.

Conversely, suppose that (*) fails. Then there exist $x, y, z \in \mathbb{R}$ such that

$$x, y \in E \text{ and } z \notin E \text{ and } x < z < y.$$

Define

$$A = E \cap (-\infty, z),$$

$$B = E \cap (z, \infty).$$

Then A and B are nonempty ($x \in A, y \in B$), A and B are separated (since $A \subset (-\infty, z)$, $B \subset (z, \infty)$, and $(-\infty, z), (z, \infty)$ are separated), and $E = A \cup B$.

Thus E is separated. //