Ex: Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Define a sequence of functions $\{H_n(x)\}$ in the following manner:

$$(-1)^n \frac{d^n f(x)}{dx^n} = H_n(x) f(x). \quad ; \quad H_0 \equiv 1$$

Find $H_1(x)$, $H_2(x)$.

First, notice that $f'(x) = -x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = -x f(x)$.

$$H_{n+1}(x) f(x) = (-1)^{n+1} \frac{d^{n+1} f(x)}{dx^{n+1}}$$

$$= - \left[\frac{d}{dx} \left((-1)^n \frac{d^n f(x)}{dx^n} \right) \right]$$

$$= - \frac{d}{dx} \left[H_n(x) f(x) \right] = -\left[H_n'(x) f(x) + H_n(x) f'(x) \right]$$

$$= - \left[H_n'(x) f(x) - x f(x) H_n(x) \right]$$

$$\Rightarrow \quad H_{n+1}(x) = x H_n(x) - H_n'(x).$$

$$H_1(x) = x \cdot H_0(x) - 0 = x$$

$$H_2(x) = x H_1(x) - H_1'(x)$$

$$= x^2 - 1$$

Ex : Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ &

$$\Phi(x) = \int_{-\infty}^{x} f(u)\, du .$$

Prove that $\frac{1-\Phi(x)}{f(x)} \leq \frac{1}{x}$, $\forall x > 0$.

$$1 - \Phi(x) = \int_{x}^{\infty} f(u)\, du \leq \int_{x}^{\infty} \frac{u}{x} f(u)\, du$$

$$= \frac{1}{x} \int_{x}^{\infty} u f(u)\, du$$

$$= \frac{1}{x} \int_{x}^{\infty} -\frac{d}{du} f(u)\, du \qquad \left(f'(x) = -x f(x) \right)$$

$$= \frac{1}{x} \left(- f(u) \Big|_{x}^{\infty} \right) = \frac{1}{x} f(x)$$

$$\Rightarrow \quad \frac{1 - \Phi(x)}{f(x)} \leq \frac{1}{x}$$

$\square$

Joint Distributions

Definition : Let joint distribution function of X, Y is a function $F : \mathbb{R}^2 \to [0,1]$ given by

$$F(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{v=-\infty}^{y} \int_{u=-\infty}^{x} f(u,v)\, du\, dv$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the joint density function that satisfies :

① $f \geq 0$

② $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1.$

Marginal functions

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

For sufficiently nice $B \in \mathbb{R}^2$,

$$P((X,Y) \in B) = \iint_B f(x,y)\, dy\, dx.$$

Ex: $f(x,y) = \begin{cases} 6(1-y); & 0 \le x \le y \le 1 \\ 0 ; & \text{otherwise} \end{cases}$

i) Find f_X, f_Y

ii) $P(X+Y<1)$

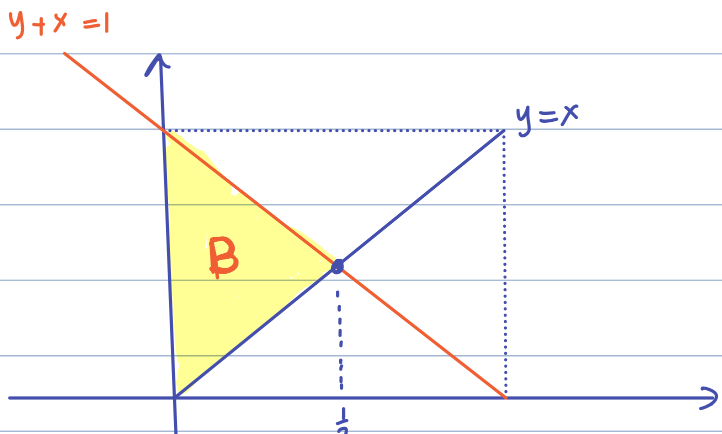
i) $$f_X(x) = \int_{y=x}^{y=1} 6(1-y)\,dy = 6\left[y - \frac{y^2}{2}\right]_x^1$$

$$= 6\left[\left(1-\frac{1}{2}\right) - \left(x - \frac{x^2}{2}\right)\right]$$

$$f_X(x) = 6\left[\frac{1}{2} - x + \frac{x^2}{2}\right] ; \quad 0 \le x \le 1$$

$$f_Y(y) = \int_{x=0}^{x=y} 6(1-y)\,dx = 6y(1-y)$$

ii)



$x=\frac{1}{2}$ $\quad y=1-x$

$$P(X+Y<1) = \iint_B f(x,y)\,dy\,dx = \int_{x=0}^{\frac{1}{2}} \int_{y=x}^{1-x} 6(1-y)\,dy\,dx$$

$$= 6 \int_{x=0}^{x=\frac{1}{2}} \left[y - \frac{y^2}{2} \right]_x^{1-x} dx$$

$$= 6 \int_0^{\frac{1}{2}} \left[(1-x) - \frac{(1-x)^2}{2} - \left(x - \frac{x^2}{2} \right) \right] dx$$

$$= 6 \int_0^{\frac{1}{2}} \left[1 - x - \frac{(1-2x+x^2)}{2} - x + \frac{x^2}{2} \right] dx$$

$$= 6 \int_0^{\frac{1}{2}} \left[1 - x - \frac{1}{2} + x - \frac{x^2}{2} - x + \frac{x^2}{2} \right] dx$$

$$= 6 \int_0^{\frac{1}{2}} \left(\frac{1}{2} - x \right) dx = 6 \left[\frac{x}{2} - \frac{x^2}{2} \right]_0^{\frac{1}{2}}$$

$$= 6 \left[\frac{1}{4} - \frac{1}{8} \right] = \frac{6}{8} = \boxed{\frac{3}{4}}$$