Special Discrete Distributions

① Bernoulli Distribution.

We say X is Bernoulli if

$$P(X=1) = p \quad , \quad P(X=0) = 1-p =: q$$

Ex : If A is an event , then 1_A is a Bernoulli RV.

$$E(X) = p$$

$$V(X) = p(1-p)$$

② Binomial RV

Let $X_1, X_2, \dots, X_n$ be independent Bernoulli RVs , Then $X = \sum_{i=1}^{n} X_i$ is called a binomial RV with parameters n, p

$X = \#$ successes of n independent Bernoulli RVs.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad , \quad k = 0, 1, 2, \dots, n$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1 \checkmark$$

Notation $X \sim B(n,p)$

Proposition : $E(X) = p$

$V(X) = np(1-p)$

Practice : $E\left(\frac{1}{1+X}\right) = \frac{1}{p(n+1)} \left(1 - (1-p)^{n+1}\right)$.

Ex: Cards are drawn one by one from a deck of 52 cards. Find the probability of drawing exactly 3 Queens if 10 cards are drawn this way.

$X \sim B(10, \frac{1}{13})$

$$P(X=3) = \binom{10}{3} \left(\frac{1}{13}\right)^3 \left(\frac{12}{13}\right)^7$$

The Poisson Distribution

$$P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!} \; ; \quad k = 0, 1, 2, \ldots, \quad \lambda > 0$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda}$$

$$= 1$$

Notation : $X \sim \text{Poisson}(\lambda)$

Ex : $X = \#$ customers arrived within a day

Proposition : $E(X) = \lambda$
$V(X) = \lambda$

Poisson Limit theorem : Let $X_n \sim B(n, \frac{\lambda}{n})$.

Then,

$$\lim_{n \to \infty} P[X_n = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad \forall k = 0, 1, 2, \ldots$$

Proof : $P(X_n = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$

$$= \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$= \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\longrightarrow 1} \cdot \frac{\lambda^k}{k!} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\longrightarrow e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\longrightarrow 1}$$

as $n \to \infty$

$$\Rightarrow \lim_{n \to \infty} P(X_n = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

Geometric

X = # trials to observe first success

$X \sim \text{Geom}(p)$

$$P(X=k) = (1-p)^{k-1}p \quad , \ k = 1, 2, \ldots$$

$P(X \geq k) = P(\text{first } k-1 \text{ trials are failures}) = (1-p)^{k-1}$

Prop: $E(X) = \dfrac{1}{p}$

Lemma: If $X \geq 0$ is integer valued,

then $E(X) = \sum_{x} P(X \geq x)$

Proof of prop:

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

$$= \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$\square$

Ex: An urn has 5 red, 3 blue, & 10 green balls. Balls are chosen one by one with replacement. What is the prob. that the first red ball is chosen in the 10 th attempt?

$$p = \frac{5}{5+3+10} = \frac{5}{18}$$

$$P(X=10) = \left(1 - \frac{5}{18}\right)^{10-1} \cdot \frac{5}{18} = \left(\frac{13}{18}\right)^{9} \cdot \frac{5}{18}$$

Practice: ① $E\left(\frac{1}{1+X}\right) = \frac{1}{p(n+1)}\left(1 - (1-p)^{n+1}\right)$

$$E\left(\frac{1}{1+X}\right) = \sum_{k=0}^{n} \frac{1}{1+k} \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(k+1)!\,(n-k)!} p^k q^{n-k}$$

$$= \frac{1}{(n+1)p} \sum_{k=0}^{\infty} \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!} p^{k+1} q^{(n+1)-(k+1)}$$

$$= \frac{1}{(n+1)p}\left[1 - q^{n+1}\right] = \frac{1}{(n+1)p}\left(1 - (1-p)^{n+1}\right)$$

(2) $X \sim \text{Poisson}(\lambda)$, $0 < \lambda < 1$.

(i) $$E(X!) = \sum_{k=0}^{\infty} \cancel{k!} \frac{e^{-\lambda} \lambda^k}{\cancel{k!}}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \lambda^k = \boxed{\frac{e^{-\lambda}}{1-\lambda}}$$

(ii) $$E(2^X) = \sum_{k=0}^{\infty} 2^k \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{k!} = e^{-\lambda} e^{2\lambda} = \boxed{e^{\lambda}}$$