Math 672 Lecture 30

Let V be a finite-dimensional complex vector space, let $T\in\mathcal{L}(V)$, and let $\lambda_1,\dots,\lambda_k$ be the distinct eigenvalues of T. Recall that

$$E(\lambda_1,T)+E(\lambda_2,T)+\cdots+E(\lambda_k,T)$$

is always a direct sum, and that T is diagonalizable iff

$$(*) \qquad V=E(\lambda_1,T)\oplus E(\lambda_2,T)\oplus\cdots\oplus E(\lambda_k,T).$$

However, $(*)$ does not hold for all T.

For each λ_j,

$$\begin{aligned} G(\lambda_j,T) &= \{v\in V \mid (T-\lambda_j I)^{\ell}(v)=0 \text{ for some } \ell\geq 1\} \\ &= \mathcal{N}((T-\lambda_j I)^{m_j}) \text{ for some } 1\leq m_j\leq n \\ &= \mathcal{N}((T-\lambda_j I)^{n}) \quad (\text{since } \mathcal{N}((T-\lambda_j I)^{\ell})=\mathcal{N}((T-\lambda_j I)^{m_j}) \\ &\qquad\qquad\qquad\qquad \forall \ell\geq m_j), \end{aligned}$$

and $G(\lambda_j,T)$ is invariant under T. Also,

$$G(\lambda_1,T)+G(\lambda_2,T)+\cdots+G(\lambda_k,T)$$

is always a direct sum. If we can prove that

$$V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_k, T),$$

then $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is block diagonal (where $\mathcal{B}$ is the union of bases for $G(\lambda_1,T), \ldots, G(\lambda_k,T)$). Moreover, if we choose the basis for $G(\lambda_j,T)$ carefully, then the block corresponding to λ_j is "almost" diagonal.

<u>Theorem</u>: Let V be a finite-dimensional complex inner product space, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

$$V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_k, T).$$

<u>Proof</u>: We argue by induction on $n = \dim(V)$. The result is obvious for $n=1$. Assume the result holds for all complex vector spaces of dimension less than n, where $n \geq 2$, and let V be a complex vector space of dimension n, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T (we know that $k \geq 1$ because V is complex). Now,

$$G(\lambda_1, T) = \mathcal{N}((T - \lambda_1 I)^{n})$$

is invariant under T. By an earlier result,

$$V = \mathcal{N}((T-\lambda_1 I)^{m_1}) \oplus \mathcal{R}((T-\lambda_1 I)^{m_1}).$$

Note that $\mathcal{R}((T-\lambda_1 I)^{m_1})$ is also invariant under T:

$$\begin{aligned} u \in \mathcal{R}((T-\lambda_1 I)^{m_1}) &\Rightarrow u = (T-\lambda_1 I)^{m_1}(v) \text{ for some } v \in V \\ &\Rightarrow T(u) = T((T-\lambda_1 I)^{m_1}(v)) \\ &\qquad = (T-\lambda_1 I)^{m_1}(T(v)) \quad \text{(since polynomials in } T \text{ commute)} \\ &\Rightarrow T(u) \in \mathcal{R}((T-\lambda_1 I)^{m_1}). \end{aligned}$$

Define $U = \mathcal{R}((T-\lambda_1 I)^{m_1})$, $S \in \mathcal{L}(U)$, $S = T|_U$. Let $\lambda_2', \dots, \lambda_\ell'$ be the distinct eigenvalues of S. By the induction hypothesis,

$$U = G(\lambda_2', S) \oplus \cdots \oplus G(\lambda_\ell', S)$$

$$\Rightarrow V = G(\lambda_1, T) \oplus G(\lambda_2', S) \oplus \cdots \oplus G(\lambda_\ell', S).$$

Now, it is clear that each λ_j' is an eigenvalue of T:

$$S(v_j) = \lambda_j' v_j \Rightarrow T(v_j) = \lambda_j' v_j.$$

Thus, $\{\lambda_2', \dots, \lambda_\ell'\} \subseteq \{\lambda_2, \dots, \lambda_k\}$. We can thus write

$$V = G(\lambda_1, T) \oplus G(\lambda_2, S) \oplus \cdots \oplus G(\lambda_k, S),$$

with the understanding that $G(\lambda_j, S)$ may be trivial for some j's.

(Actually, this is not possible, but we must prove this.)

Note that $G(\lambda_j,S) \subseteq G(\lambda_j,T)$ for each $j=2,3,\dots,k$:

$$v \in G(\lambda_j,S) \Rightarrow (S-\lambda_j I)^t(v)=0 \text{ for some } t>0$$
$$\Rightarrow (T-\lambda_j I)^t(v)=0 \text{ (since } S(v)=T(v) \text{ for all } v\in U)$$
$$\Rightarrow v \in G(\lambda_j,T).$$

We wish to show that, in fact, $G(\lambda_j,S)=G(\lambda_j,T)$ for all $j=2,\dots,k$.

Suppose $2\le j\le k$ and $v\in G(\lambda_j,T)$. Since

$$v \in V = G(\lambda_1,T) \oplus G(\lambda_2,S) \oplus \cdots \oplus G(\lambda_k,S),$$

there exist

$$v_1 \in G(\lambda_1,T),\ v_2 \in G(\lambda_2,S),\ \dots,\ v_k \in G(\lambda_k,S)$$

such that

$$v = v_1 + v_2 + \cdots + v_k.$$

But we then have $v_j \in G(\lambda_j,T)$ for all $j=1,2,\dots,k$ (since $G(\lambda_j,S)\subseteq G(\lambda_j,T)$ and hence

$$v = v_j \in G(\lambda_j,S),\ v_\ell = 0\ \forall\, \ell \ne j$$

(since otherwise we have two different representations of v as an element of $G(\lambda_1,T)\oplus\cdots\oplus G(\lambda_k,T)$). This shows that $G(\lambda_j,T)\subseteq G(\lambda_j,S)$ and hence that $G(\lambda_j,S)=G(\lambda_j,T)$.

We have thus proved that

$$V = G(\lambda_1,T)\oplus\cdots\oplus G(\lambda_k,T)$$

and the proof by induction is complete. //

<u>Corollary</u>: Let V be a finite-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Then there exists a basis for V consisting of generalized eigenvectors of T.

<u>Corollary</u>: Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

$$\sum_{j=1}^{k} \dim(G(\lambda_j, T)) = \dim(V).$$

<u>Definition</u>: Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$, and let λ be an eigenvalue of T. We call $\dim(G(\lambda, T))$ the <u>algebraic multiplicity</u> of λ and $\dim(E(\lambda, T))$ the <u>geometric multiplicity</u> of λ.

<u>Definition</u>: Let F be a field, let $\lambda \in F$, and let $t \in \mathbb{Z}^+$. We call the matrix

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in F^{t \times t}$$

a $t \times t$ Jordan Block.

Example: Suppose $\dim(V) = 12$, $T \in \mathcal{L}(V)$ has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with algebraic multiplicities 6, 4, 2, respectively and geometric multiplicities 2, 2, 1, respectively. Suppose further that

$$G(\lambda_1, T) = \mathcal{N}((T-\lambda_1 I)^3),$$
$$G(\lambda_2, T) = \mathcal{N}((T-\lambda_2 I)^2),$$
$$G(\lambda_3, T) = \mathcal{N}((T-\lambda_3 I)^2),$$

where we have chosen the smallest exponent in each case (for example, $\mathcal{N}((T-\lambda_1 I)^2) \subsetneq \mathcal{N}((T-\lambda_1 I)^3)$. How do we construct a basis $\mathcal{B}$ for V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is as close to diagonal as possible?

Since $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus G(\lambda_3, T)$ and each $G(\lambda_j, T)$ is invariant under T, $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ will be block diagonal (with block sizes 6, 4, 2) if $\mathcal{B}$ is the union of bases for $G(\lambda_1, T)$, $G(\lambda_2, T)$, $G(\lambda_3, T)$.

(1a) There exists $u_1 \in G(\lambda_1, T)$ such that

$$\{u_1, (T-\lambda_1 I)(u_1), (T-\lambda_1 I)^2(u_1)\}$$

is linearly independent. Define

$$v_1 = (T-\lambda_1 I)^2(u_1),\ v_2 = (T-\lambda_1 I)(u_1),\ v_3 = u_1.$$

Then

$$(T-\lambda_1 I)(v_1) = (T-\lambda_1 I)^3(u_1) = 0 \Rightarrow T(v_1) = \lambda_1 v_1,$$

$$(T-\lambda_1 I)(v_2) = (T-\lambda_1 I)^2(u_1) = v_1 \Rightarrow T(v_2) = \lambda_1 v_2 + v_1,$$

$$(T-\lambda_1 I)(v_3) = (T-\lambda_1 I)(u_1) = v_2 \Rightarrow T(v_3) = \lambda_1 v_3 + v_2$$

(1b) By assumption, $\dim(G(\lambda_1, T)) = 6$, so there must be another linearly independent set

$$\{u_2, (T-\lambda_1 I)(u_2), (T-\lambda_1 I)^2(u_2)\} \subseteq G(\lambda_1, T).$$

Define

$$v_4 = (T-\lambda_1 I)^2(u_2),\ v_5 = (T-\lambda_1 I)(u_2),\ v_6 = (T-\lambda_1 I)(u_3).$$

Then

$$T(v_4) = \lambda_1 v_4,\ T(v_5) = \lambda_1 v_5 + v_4,\ T(v_6) = \lambda_1 v_6 + v_5.$$

(1c) Now the block corresponding to

$$G(\lambda_1, T) = \operatorname{span}(v_1, v_2, v_3, v_4, v_5, v_6)$$

is

$$\left[\begin{array}{ccc|ccc} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_1 & 1 & \\ & & & & \lambda_1 & 1 \\ & & & & & \lambda_1 \end{array}\right] \in \mathbb{C}^{6\times 6}$$

(made up of two 3x3 Jordan blocks)

<u>Important</u>: Was there any other choice (if we insist on using Jordan blocks)? Answer: No. Since $\dim(E(\lambda_1,T))=2$ and each generalized eigenvector chain contains one eigenvector, there must be exactly two chains, so two 3x3 blocks.

2 Since $\dim(E(\lambda_2,T))=2$, $\dim(G(\lambda_2,T))=4$, and $G(\lambda_2,T)=(T-\lambda_2 I)^2$, there must be two independent generalized eigenvector chains of length 2 in $G(\lambda_2,T)$:

$$G(\lambda_2,T)=\operatorname{span}(v_7,v_8,v_9,v_{10}),$$

$$T(v_7)=\lambda_2 v_7,$$

$$T(v_8)=\lambda_2 v_8+v_7,$$

$$T(v_9)=\lambda_2 v_9,$$

$$T(v_{10})=\lambda_2 v_{10}+v_9$$

The block corresponding to $G(\lambda_2,T)=\operatorname{span}(v_7,v_8,v_9,v_{10})$ is

$$\left[\begin{array}{cc|cc} \lambda_2 & 1 & & \\ & \lambda_2 & & \\ \hline & & \lambda_2 & 1 \\ & & & \lambda_2 \end{array}\right]$$

(3) Since $\dim(E(\lambda_3, T)) = 1$ and $\dim(G(\lambda_3, T)) = 2$ (with $G(\lambda_3, T) = \mathcal{N}((T - \lambda_3 I)^2)$), there must be a single chain of length 2:

$$G(\lambda_3, T) = \operatorname{span}(v_{11}, v_{12}),$$

$$T(v_{11}) = \lambda_3 v_{11},$$

$$T(v_{12}) = \lambda_3 v_{12} + v_{11}.$$

The block is

$$\begin{bmatrix} \lambda_3 & 1 \\ & \lambda_3 \end{bmatrix}.$$

Thus, if $\mathcal{B} = \{v_1, v_2, \ldots, v_{12}\}$, then

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T) = \left[\begin{array}{ccc|ccc|cc|cc|cc}
\lambda_1 & 1 & & & & & & & & & & \\
& \lambda_1 & 1 & & & & & & & & & \\
& & \lambda_1 & & & & & & & & & \\
\hline
& & & \lambda_1 & 1 & & & & & & & \\
& & & & \lambda_1 & 1 & & & & & & \\
& & & & & \lambda_1 & & & & & & \\
\hline
& & & & & & \lambda_2 & 1 & & & & \\
& & & & & & & \lambda_2 & & & & \\
\hline
& & & & & & & & \lambda_2 & 1 & & \\
& & & & & & & & & \lambda_2 & & \\
\hline
& & & & & & & & & & \lambda_3 & 1 \\
& & & & & & & & & & & \lambda_3
\end{array}\right]$$