Arbitrage Bounds for Vanilla Options

Andrew Papanicolaou

October 2022

The following note presents some arbitrage arguments that are fundamental for our understanding of vanilla European and American options. The arguments used will rely on minimal assumptions about the definition of arbitrage, and will use no probability. The idea is to identify structural relationships between these instruments that should hold so long as the markets are liquid and frictionless.

1 Lower Bound on European Call Option

Denote a European call options as lower case c. Interest rate r, dividend yield g, strike K, maturity T, current time t, current price of the stock S_t. The payoff is

$$(S_T - K)^+ = \max(S_T - K, 0) \ .$$

We say that the European call option is **in-the-money** if $S_T > K$, **out-of-the-money** if $S_T < K$, and **at-the-money** if $S_T = K$.

Let's consider the case $g = 0$. The following structural bounds hold in a liquid and frictionless market:

1. $c \geq 0$ by no arbitrage (if $c < 0$ then let someone pay you to own a claim that cannot cost you anything).
2. $(S_T - K)^+ \geq S_T - K$, and so the value of the European call should be greater or equal to the present value of linear payoff $S_T - K$, which is $S_t - e^{-r(T-t)}K$. Hence, $c \geq S_t - e^{-r(T-t)}K$.

Putting items 1. and 2. together, we have

$$c \geq \max(0, S_t - e^{-r(T-t)}K) = \left(S_t - e^{-r(T-t)}K\right)^+ \ ,$$

which is equation (11.4) in Hull 10th Ed.

What if there is mispricing $c < S_t - e^{-r(T-t)}K$? Then do the following:

- Borrow \$$c$ at rate r
- long the call
- short a forward on S_T

It costs net zero to set up this position, but at time T we have

$$\underbrace{(S_T - K)^+}_{\text{long option}} \quad - \underbrace{ce^{r(T-t)}}_{\text{to bank}} \quad + \underbrace{S_t e^{r(T-t)}}_{\text{forward price}} \quad - \underbrace{S_T}_{\text{deliver stock}}$$

$$= \underbrace{(S_T - K)^+ - (S_T - K)}_{\geq 0} \quad + e^{r(T-t)} \underbrace{(S_t - e^{-r(T-t)}K - c)}_{>0 \text{ with mispricing}}$$

$$> 0 \ .$$

Thus there is arbitrage if $c < S_t - e^{-r(T-t)}K$.

From now on take $t = 0$ ***and denote*** $S_0 = S$***.***

Time Value of Options

The European call can be valued as

$$c = (S - K)^+ + \text{time value} \ .$$

We refer to $(S-K)^+$ as the call option's **intrinsic value.** For $g = 0$ and $r \geq 0$, a European call option always has non-negative time value,

$$c \geq (S - e^{-rT}K)^+ \geq (S - K)^+ \geq 0 \ ,$$

and for $r > 0$ we can show that an in-the-money European call option has positive time value,

$$c \geq S - e^{-rT}K > S - K \geq 0 \quad \text{if } S \geq K \ . \tag{1}$$

In general, time value of a European call option is always positive for $g = 0$ and $r \geq 0$ (for out-of-the-money call options as well as in-the-money), but this is a slightly more technical argument that we leave for later. Figures 1 and 2 illustrate how the time value of a European call option can become negative if $g > 0$ and the option is far in-the-money.

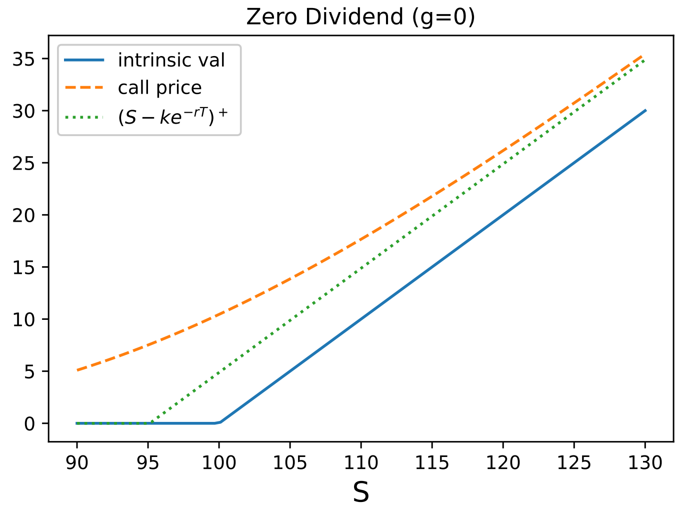

Figure 1: With $g = 0$, the European call option always has positive time value, i.e., it is worth more than its intrinsic value.

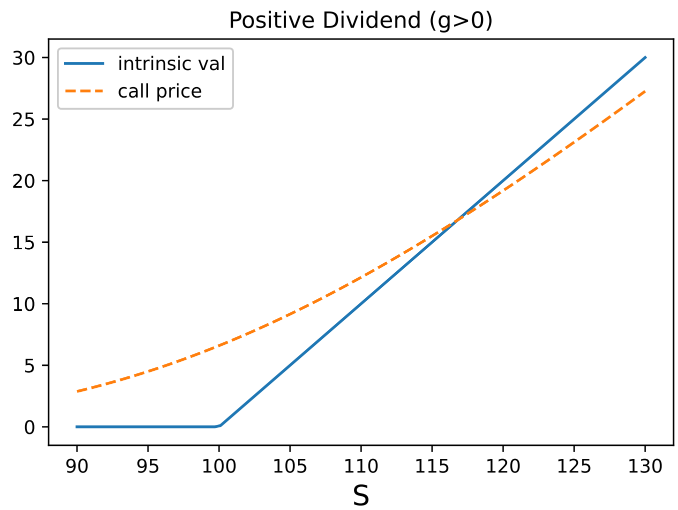

Figure 2: With $g > 0$, the European call option that is far in-the-money has negative time value, i.e., it is less than its intrinsic value.

2 American Options

An American call option is similar to its European counterpart, with the additional feature that it may be exercised at *any* time prior to T. The question to ask when valuing an

American call is *When is it optimal to exercise early?* In general it is difficult to find a precise answer to this question, but for non-dividend paying stocks we can say something concrete, namely, that the American feature is worth nothing.

Similarly for the American call, we have the decomposition of price in terms of intrinsic value and time value,

$$C = (S - K)^+ + \text{time value} \ .$$

Note that $C \geq c$, otherwise there is arbitrage (i.e., if mispricing $c > C$ then long the American, short the European, keep the difference and let the American payoff net with the European payoff at maturity). For $r > 0$ we can also show that the time value of an in-the-money American call option on a non-dividend paying stock is always positive,

$$\begin{aligned} C &= (S - K)^+ + \text{American time value} \\ &\geq c \\ &= (S - K)^+ + \text{European time value} \\ &> S - K \qquad \text{if } S \geq K \ , \end{aligned}$$

where the last inequality follows from (1). This means that

$$\text{American time value} > 0 \qquad \text{if } S \geq K \ ,$$

and hence **it is never optimal to early exercise an American call option on a non-dividend-paying stock.**[1]

The following arguments give further intuition on why early exercise is not optimal when $r > 0$.

1. If you are long the American call and want to hold the stock after exercise, then it is better to continue holding the option and collect interest on the \$$K$ you'll use to purchase (or to not accrue interest on the \$$K$ you'll borrow to exercise). In addition, holding the option avoids losses that would've otherwise occurred had the stock dropped after early exercise.

2. Suppose you are long the American call and believe the stock will go down. You intend to sell the stock immediately upon exercise, for a time-T net cash position of

$$\text{EE} = e^{rT}(S - K) \ .$$

This is not an optimal strategy. Instead, it is better to short the stock and continue

[1]Note that it is never optimal to early exercise if $S < K$. If $S < K$, then early exercise involves buying the stock at \$$K$ whereas the market offers it at a lower price of \$$S$, in which case early exercise makes no sense and is clearly a misuse of the optionality.

holding the option until maturity, for a time-T net cash position of

$$
\begin{aligned}
&\underbrace{(S_T - K)^+}_{\text{option}} \quad + \underbrace{e^{rT}S}_{\text{in bank}} \quad - \underbrace{S_T}_{\text{short}} \\
&= \underbrace{(S_T - K)^+ - (S_T - K)}_{\geq 0} \quad + \underbrace{e^{rT}S - K}_{>\text{EE}} \\
&> \text{EE} .
\end{aligned}
$$

Basically, $S > S - K$, and so the principal upon which you earn interest is greater if you short sell, and if your prediction is correct (i.e., if $S_T < K$) then you make even more money than you would've if you'd exercised early.

Early Exercise of American Put Options

European put options have the same intrinsic and time-value decomposition,

$$p = (K - S)^+ + \text{time value}.$$

For put options it is the interest r that plays the role that the dividend yield g played for call options. In particular, it may be optimal to early exercise an American put option if $r > 0$, but not if $r = 0$. Figures 3 and 4 illustrate how a European put option can have negative time value if $r > 0$, but not if $r = 0$.

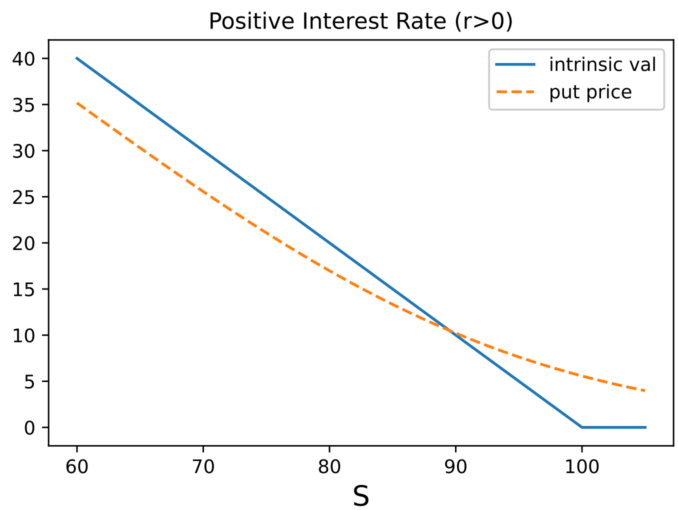

Figure 3: If $r > 0$, then there is negative time value for a European put option that is far in-the-money.

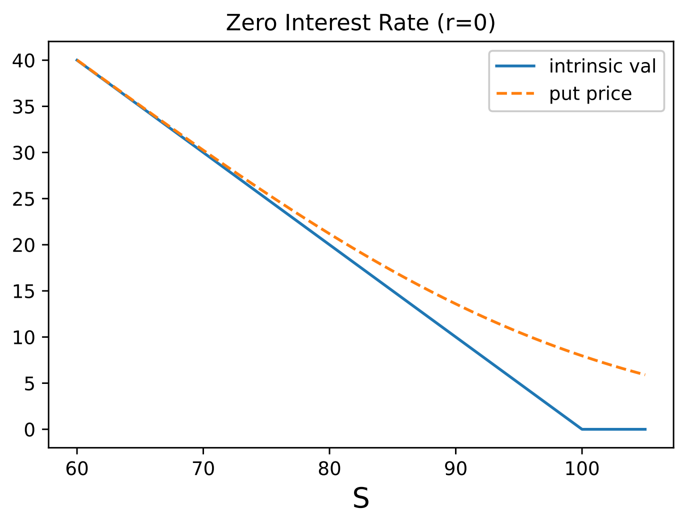

Figure 4: If $r = 0$, then a European put option always has positive time value.

3 Other Bounds

More Precise No-Arbitrage Statement

In fact, we have have a (slightly more involved) no-arbitrage bound on the European call,

$$c > 0 \ , \tag{2}$$

which follows if we define an arbitrage to be any portfolio that has zero entry cost with some chance of returning a positive profit and no possibility of returning a loss. Under this definition, $c = 0$ is arbitrage because it results in a riskless profit if $S_T > K$. From the perspective of making markets, someone who charges \$0 is completely exposed to any small chance that the option might finish in-the-money, which could be a large loss if a lot of these options are sold. Thus, market makers (similar to insurance companies) need to be compensated for rare outcomes in the option payoff, and therefore there needs to be a positive price for an option.

For $g = 0$ and $r > 0$, by combining (1) and (2) we obtain the strict inequality

$$c > (S - K)^+ \ ,$$

and thus the European call option has positive time value for non-dividend paying stocks and positive interest rate. Then the American call's time-value positivity is a simple consequence,

$$C \geq c > (S - K)^+ \ ,$$

for $g = 0$ and $r > 0$.

Case $r = 0$

The above arbitrage arguments still hold when $r = 0$. For $r = g = 0$, if $c = S - K > 0$, then the arbitrage portfolio is to borrow \$$c$, long the option, and short the forward contract. There is net-zero initial cost but at time T we have $(S_T - K)^+ - c + S - S_T = (S_T - K)^+ - (S_T - K) + (S - K - c) = (K - S_T)^+$, which is an arbitrage. Thus, $c > (S - K)^+$ even if $r = 0$. The American call's time-value positivity is again a simple consequence,

$$C \geq c > (S - K)^+ \ ,$$

therefore confirming that it is never optimal to early exercise an American call on a non-dividend paying stock — even if $r = 0$.

Put-Call Parity

A major structural equation obeyed by European vanilla options is the put-call parity, which says that

$$c + e^{-rT} K = p + e^{-gT} S \ , \tag{3}$$

or in words can be summarized as ***the call plus cash equals the put plus the stock.*** The derivation of (3) begins with the following equation,

$$S_T - K = (S_T - K)^+ - (K - S_T)^+ \ .$$

The present value of the left-hand side is $Se^{-gT} - e^{-rT}K$, and the valuation of the right-hand side should equal $c - p$. Therefore, we re-arrange to obtain (3).

Equation (3) is more aptly referred to as the *European* put-call parity, and does not hold for American options. However, there is an American version of put-call parity involving inequalities; here we derive it for the case $g = 0$. We have already seen that $C = c$ when $g = 0$ and $r \geq 0$, and we know that $P \geq p$ because the American has extra optionality. Therefore, using the European put-call parity, we have

$$S - e^{-rT}K = c - p = C - p \geq C - P \ ,$$

which we re-arange to obtain $C - P \leq S - e^{-rT}K$. Next, let us consider a portfolio containing an American call option, the amount $\$K$, and a short sale of the stock. Let $\tau \in [0, T]$ be the minimum of the optimal put exercise time or T (whichever comes first). Letting C_τ and P_τ denote the value of the American call and put at time τ, we see that the portfolio super-replicates the American put,

$$\begin{aligned} C_\tau + Ke^{r\tau} - S_\tau &\geq (S_\tau - K)^+ + Ke^{r\tau} - S_\tau \\ &= (K - S_\tau)^+ + K(e^{r\tau} - 1) \\ &= P_\tau + K(e^{r\tau} - 1) \\ &\geq P_\tau \ , \end{aligned}$$

which we discount back to time $t = 0$ and re-arrange to obtain $S - K \leq C - P$. Combining this with the other inequality on $C - P$ shown above, we have the American put-call parity (see equation (11.7) in Hull 10th Ed.),

$$S - K \leq C - P \leq S - e^{-rT}K \ .$$

A similar statement can be made for American options on a stock with dividends.