Math 600 Lecture 14

Recall: Let (X,d) be a metric space

- A sequence $\{x_n\}$ is called Cauchy iff

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z}^+ \ (m,n \geq N \Rightarrow d(x_m, x_n) < \varepsilon).$$

- If $\{x_n\}$ is Cauchy, then it is bounded.
- If $\{x_n\} \subset E$ is Cauchy, where E is compact, then $\{x_n\}$ converges to a point of E.
- If $\{x_n\}$ is convergent, then it is Cauchy.
- If $X = \mathbb{R}^k$ and $\{x_n\} \subset \mathbb{R}^k$ is Cauchy, then $\{x_n\}$ converges to a point of $\mathbb{R}^k$. (Thus, in $\mathbb{R}^k$, $\{x_n\}$ converges iff it is Cauchy.)
- Finally, X is called complete iff every Cauchy sequence in X converges to a point of X. (Thus $\mathbb{R}^k$, and in particular $\mathbb{R}$, is complete.)

Sequences of real numbers

Definition: Let $\{x_n\}$ be a sequence of real numbers. We say that $\{x_n\}$ is increasing iff $x_n \leq x_{n+1} \ \forall n \in \mathbb{Z}^+$, strictly increasing iff $x_n < x_{n+1} \ \forall n \in \mathbb{Z}^+$, decreasing iff $x_{n+1} \leq x_n \ \forall n \in \mathbb{Z}^+$, strictly decreasing iff $x_{n+1} < x_n \ \forall n \in \mathbb{Z}^+$, and monotonic iff it is either increasing or decreasing.

<u>Theorem</u>: Let $\{x_n\}$ be a sequence of real numbers.

1. If $\{x_n\}$ is increasing and bounded above, then it converges to some $x \in \mathbb{R}$.
2. If $\{x_n\}$ is decreasing and bounded below, then it converges to some $x \in \mathbb{R}$.

<u>Proof</u>: We will prove the first result; the proof of the second is analogous.

Since $\{x_n\}$ is bounded above, $x = \sup \{x_n\}$ exists. We have

$$x_n \leq x \quad \forall n \in \mathbb{Z}^+$$

and, for all $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$x_N > x - \varepsilon$$

(otherwise $x-\varepsilon$ would be an upper bound of $\{x_n\}$ less than the least upper bound).

Since $\{x_n\}$ is increasing, it follows that

$$n \geq N \Rightarrow x_n \geq x_N \Rightarrow x_n \in (x-\varepsilon, x] \subset B_\varepsilon(x).$$

This shows that $x_n \to x$. //

<u>Definition</u>: Let $\{x_n\}$ be a sequence of real numbers. We say that $\{x_n\}$ <u>diverges to ∞</u> (or $+\infty$) and write $x_n \to \infty$ (or $x_n \to +\infty$) iff

$$\forall M \in \mathbb{R} \ \exists N \in \mathbb{Z}^+ \ (n \geq N \Rightarrow x_n \geq M).$$

Analogously, we say that $\{x_n\}$ <u>diverges to $-\infty$</u> and write $x_n \to -\infty$ iff

$$\forall M \in \mathbb{R} \ \exists N \in \mathbb{Z}^+ \ (n \geq N \Rightarrow x_n \leq M).$$

Lemma: Let $\{x_n\}$ be a sequence of positive real numbers. Then $x_n \to \infty$ iff $\frac{1}{x_n} \to 0$.

Proof: Exercise. //

Lemma: Let $a \in \mathbb{R}^+$. Then $a^{1/n} \to 1$.

Proof: Suppose first that $a > 1$. Then $a^{1/n} > 1$ (since it is easy to prove by induction that $y \leq 1 \Rightarrow y^n \leq 1$). Recall that

$$x > 0 \Rightarrow (1+x)^n = 1 + nx + \cdots + x^n > 1 + nx$$

and thus

$$1 + n(a^{1/n} - 1) < (1 + a^{1/n} - 1)^n = a$$

$$\Rightarrow a^{1/n} - 1 < \frac{a-1}{n}$$

$$\Rightarrow a^{1/n} - 1 \to 0 \quad \text{(since we know that } a^{1/n} - 1 > 0\text{)}$$

$$\Rightarrow a^{1/n} \to 1.$$

Now suppose that $0 < a < 1$. Then $b = \frac{1}{a} > 1$ and we have

$$b^{1/n} \to 1$$

$$\Rightarrow \left(\frac{1}{a}\right)^{1/n} \to 1$$

$$\Rightarrow \frac{1}{\left(\frac{1}{a}\right)^{1/n}} \to 1$$

$$\Rightarrow a^{1/n} \to 1. \quad //$$

Examples

1. If $p > 0$, then $n^p \to \infty$
2. If $a > 1$, then $a^n \to \infty$.
3. If $p > 0$ and $a > 1$, then $\frac{n^p}{a^n} \to 0$.

Proof: 1. Let $M > 0$ be given. Then

$$n^p \geq M \iff n \geq M^{1/p}.$$

So choose $N \in \mathbb{Z}^+$ such that $N \geq M^{1/p}$. Then

$$n \geq N \Rightarrow n \geq M^{1/p} \Rightarrow n^p \geq M.$$

Thus $n^p \to \infty$. (Note: We are using standard properties of powers, such as the fact that x^p is well defined for all $x \in \mathbb{R}^+$ and $p \in \mathbb{R}$, and that $x \mapsto x^p$ is an increasing function of x if $p > 0$. It's a bit tedious to prove all of these properties; see Exercise 6 from Chapter 1 of Rudin for the most important of them.)

2. Let $M > 0$ be given.

$$a^n > M \iff a > M^{1/n}.$$

By the previous lemma, $M^{1/n} \to 1$, and $a > 1$ by assumption. Hence there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow M^{1/n} < a \Rightarrow a^n > M.$$

Thus $a^n \to \infty$.

3. Suppose $p > 0$ and $a > 1$. Choose $k \in \mathbb{Z}^+$ such that $k > p$. Write $a = 1 + \alpha$, where $\alpha > 0$. Note that

$$a^n = (1+\alpha)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} \alpha^j = \sum_{j=0}^{n} \binom{n}{j} \alpha^j > \binom{n}{k} \alpha^k \quad (\text{if } n \geq k).$$

Assume that $n > 2k$. Then

$$a^n > \binom{n}{k} \alpha^k = \frac{n!}{(n-k)!\,k!} \alpha^k = \frac{n(n-1)\cdots(n-k+1)}{k!} \alpha^k > \frac{\left(\frac{n}{2}\right)^k}{k!} \alpha^k$$

$$= \frac{n^k \alpha^k}{2^k k!}.$$

Thus, if $n > 2k$, we have

$$0 < \frac{n^p}{a^n} < \frac{n^p}{\frac{n^k \alpha^k}{2^k k!}} = \frac{2^k k!}{\alpha^k} n^{p-k} = (\text{const.})\, n^{p-k}.$$

Since $k > p$, $n^{p-k} \to 0$ as $n \to \infty$. Thus $\frac{n^p}{a^n} \to 0$. //

The last result is important (and a bit surprising). It shows that <u>any</u> (increasing) exponential grows faster than <u>any</u> power function.

For example,

$$\frac{n^{10^6}}{(1+10^{-6})^n} \to 0. \; (!)$$