Math672
Exam 1
Summary of material to know

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Definitions

You must be prepared to define the following terms:

1. *Subspace* of a vector space
2. *Sum* of two subspaces
3. *Direct sum* of two subspaces
4. *Span* of vectors $v_1, v_2, \ldots, v_n$
5. *Linear independence* or *linear dependence* of $v_1, v_2, \ldots, v_n$
6. *Basis* of a vector space
7. *Dimension* of a vector space
8. *Linear map*
9. *Null space* of a linear map
10. *Range* of a linear map
11. *Surjective, injective, bijective* (applied to a function)
12. *Invertible* (applied to a function)
13. *Isomorphic* (applied to a pair of vector spaces); *isomorphism*
14. *Invariant subspace* of a linear operator $T \in \mathcal{L}(V)$.
15. *Eigenvalue, eigenvector, eigenspace* of a linear operator $T \in \mathcal{L}(V)$.
16. *Singular* linear map $T \in \mathcal{L}(V, W)$.
17. *Norm* on a vector space (over $\mathbb{R}$ or $\mathbb{C}$).
18. *Inner product* on a vector space over $\mathbb{R}$.
19. *Inner product* on a vector space over $\mathbb{C}$.
20. *Orthogonal* (for a pair of vectors or a set of vectors).
21. *Orthogonal projection operator* P_S onto a finite-dimensional subspace S of an inner product space V.
22. *Orthogonal complement* $S^\perp$ of a subset S
23. *Adjoint* of $T \in \mathcal{L}(V, W)$, where V and W are inner product spaces.
24. *Generalized eigenvector* of $T \in \mathcal{L}(V)$, where V is a vector space.
25. *Algebraic multiplicity* of an eigenvalue.

26. *Geometric multiplicity* of an eigenvalue.

27. *Minimal polynomial* of $T \in \mathcal{L}(V)$, where V is a finite-dimensional complex inner product space.

28. *Characteristic polynomial* of $T \in \mathcal{L}(V)$, where V is a finite-dimensional complex inner product space.

You must also be able to define the following notation:

29. $\mathcal{M}_{\mathcal{B}} : V \to F^n$, where V is an n-dimensional vector space with basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$.

30. $\mathcal{M}_{\mathcal{B},\mathcal{C}} : \mathcal{L}(V,W) \to F^{m\times n}$, where V is an n-dimensional vector space with basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and W is an m-dimensional vector space with basis $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$.

Proofs

Be prepared to prove any of the following results:

1. Let V and W be vector spaces over a field F. Then V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

2. Let V and W be vector spaces over a field F, and assume that V is finite dimensional. If $T : V \to W$ is linear, then
$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V).$$

3. Let V and W be finite-dimensional vector spaces over a field F, let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$ be bases for V and W, respectively, and let $T \in \mathcal{L}(V,W)$. Then there exists a unique matrix $A \in F^{m\times n}$ such that
$$\mathcal{M}_{\mathcal{C}}(T(v)) = A\mathcal{M}_{\mathcal{B}}(v) \;\forall\, v \in V.$$

4. Let V be a vector space over a field F, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T with corresponding eigenvectors $v_1, v_2, \ldots, v_k$, respectively. Then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

5. Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Then T has an eigenvalue.

6. Let V be an inner product space over $\mathbb{R}$, let S be a finite-dimensional subspace of V, and let $x \in V$. Then v satisfies
$$v \in S \text{ and } \|v - x\| \le \|u - x\| \;\forall\, u \in S$$
if and only if
$$\langle v - x, w\rangle = 0 \;\forall\, w \in S.$$

7. Let V be a finite-dimensional inner product space over F ($\mathbb{R}$ or $\mathbb{C}$). Then, for all $\phi \in V'$, there exists a unique vector $u \in V$ such that
$$\phi(v) = \langle v, u\rangle \;\; \forall\, v \in V.$$

8. Let V and W be inner product spaces over F ($\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{L}(V,W)$. Then there exists a unique linear map $T^* : W \to V$ such that
$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \;\; \forall\, v \in V \,\forall\, w \in W.$$

9. Let V be a complex inner product space and let $T \in \mathcal{L}(V)$ be self-adjoint. Then every eigenvalue of T is real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

10. Let V be a finite-dimensional real inner product space and let $T \in \mathcal{L}(V)$ be self-adjoint. Then there exists an orthonormal basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal.

11. Let V be a finite-dimensional complex inner product space and let $T \in \mathcal{L}(V)$ be normal. Then there exists an orthonormal basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal.

12. Let V be a finite-dimensional vector space over a field F and let $T \in \mathcal{L}(V)$. then
$$\mathcal{N}(T^j) \subset \mathcal{N}(T^{j+1}) \;\forall j \ge 0$$
and there exists an integer m such that $0 \le m \le n = \dim(V)$ and
$$\mathcal{N}(T^j) \subsetneq \mathcal{N}(T^{j+1} \;\forall\, j < m \text{ and } \mathcal{N}(T^j) = \mathcal{N}(T^m) \;\forall\, j > m.$$

13. Let V a vector space over a field F, let $T \in \mathcal{L}(V)$, let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T, and let $v_1, v_2, \ldots, v_k$ be corresponding generalized eigenvalues of T. The $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

14. Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then
$$V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_k, T).$$

15. Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$, and let p_T be the characteristic polynomial of T. Then $p_T(T) = 0$.

16. Let V be a finite-dimensional complex vector space, let $T \in \mathcal{L}(V)$, and let m_T be the minimal polynomial of T. Then $\lambda \in \mathbb{C}$ is a an eigenvalue of T if and only if λ is a root of m_T.

Other things to know

You should be able to work out examples in $\mathbb{R}^n$ or $\mathcal{P}_n(\mathbb{R})$, where n is not very large (I will not ask about any examples in $\mathbb{C}^n$ or $\mathcal{P}_n(\mathbb{C})$ on this exam). Here are examples of what I might ask:

- Be able to apply the Gram-Schmidt process to orthogonalize a basis $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ or $\mathcal{P}_n$ (where k and n are small integers).
- Given an inner product space V and a finite-dimensional subspace S of V, be able to compute the best approximation to $x \in V$ from S. You should be able to do this using any basis for S. Of course, it is easiest if you are given an orthonormal or orthogonal basis, but if the basis is not orthogonal, you can use the Gram matrix of the basis.
- Given the Jordan form of an operator, list the dimensions of the eigenspaces and generalized eigenspaces of the operator, together with its characteristic and minimal polynomials.
- Be able to identify all possible Jordan forms of an operator given its characteristic and minimal polynomials.

The solution of any such problem must be fully justified (in effect, you have to prove that your solution is correct). You can only use results proven in this class as justifications.