Last time :

Law of total probability :

If $\bigcup_{i=1}^{n} B_i = \Omega$ & $B_i \cap B_j = \phi \ \ \forall i \neq j$ ,

$$P(A) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$$

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) P(B_j)}{P(A)}$$

$$\boxed{P(B_j|A) = \frac{P(A|B_j)\, P(B_j)}{\sum_i P(A|B_i) P(B_i)}}$$ Bayes' rule.

Ex : There are two identical urns. Urn 1 contains 2 red balls & 5 black balls. Urn 2 contains 4 red & 7 black balls. Find the probability of a randomly chosen ball being red.

| 2R 5B | 4R 7B |
|---|---|
| $u_2$ | $u_2$ |

$P(R|U_1) = \frac{2}{7}$

$P(R|U_2) = \frac{4}{11}$

$$P(R) = P(R|U_1) P(U_1) + P(R|U_2) P(U_2)$$

$$= \boxed{\frac{2}{7} \cdot \frac{1}{2} + \frac{4}{11} \cdot \frac{1}{2}}$$

Ex: A box contains 2 fair coins & 1 double-headed coin. A coin is randomly selected & flipped. Given that a head was obtained, what is the probability that a fair coin was picked?

$$P(F \mid H) = \frac{\overbrace{P(H \mid F)}^{\frac{1}{2}}\,\overbrace{P(F)}^{\frac{2}{3}}}{\underbrace{P(H \mid F)}_{\frac{1}{2}}\,\underbrace{P(F)}_{\frac{2}{3}} + \underbrace{P(H \mid D)}_{1}\,\underbrace{P(D)}_{\frac{1}{3}}}$$

# Gambler's ruin

A gambler walks into a casino with $\$N$.
He bets $\$1$ on each spin of a roulette wheel on the event that the result is red.

$$P(R) = p < \tfrac{1}{2}$$
$$P(R^c) = 1 - p = q > \tfrac{1}{2}$$

The gambler's objective is to reach $\$M \geq \$N$.
without losing all the money. Find the probability that the gambler reaches the goal.

Solution : Let $p_k$ denote the probability that the Gambler succeeds when he has $\$k$.
Condition on the first spin :

$$p_k = P(W|R)P(R) + P(W|B)P(B)$$
$$p_k = p_{k+1} \cdot p + p_{k-1} \cdot q \quad ; \quad k = 1, 2, \ldots, M-1$$

$$p_0 = 0$$
$$p_M = 1$$

$$\underbrace{(p+q)}_{=1} p_k = p p_{k+1} + q p_{k-1}$$

$$\Rightarrow \quad p_{k+1} - p_k = \left(\frac{q}{p}\right)(p_k - p_{k-1})$$

add

$$P_2 - P_1 = \left(\frac{q}{p}\right)(P_1 - \underbrace{P_0}_{=0}) = \left(\frac{q}{p}\right) P_1$$

$$P_3 - P_2 = \left(\frac{q}{p}\right)\underbrace{(P_2 - P_1)}_{\left(\frac{q}{p}\right)P_1} = \left(\frac{q}{p}\right)^2 P_1$$

$$\vdots$$

$$P_{k+1} - P_k = \left(\frac{q}{p}\right)^k P_1$$

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$$P_{k+1} - P_1 = P_1\left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^k\right]$$

$$P_{k+1} = P_1\left[1 + \left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^k\right]$$

$$P_{k+1} = P_1\left[\frac{1 - \left(\frac{q}{p}\right)^{k+1}}{1 - \left(\frac{q}{p}\right)}\right]$$

Let $k = M-1$ :

$$P_M = 1 = P_1\left[\frac{1 - \left(\frac{q}{p}\right)^M}{1 - \left(\frac{q}{p}\right)}\right]$$

$$\Rightarrow \quad P_1 = \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^M}$$

Therefore

$$\Rightarrow \quad P_{k+1} = \frac{\cancel{1 - \frac{q}{p}}}{1 - \left(\frac{q}{p}\right)^M} \cdot \frac{1 - \left(\frac{q}{p}\right)^{k+1}}{\cancel{1 - \frac{q}{p}}} = \frac{1 - \left(\frac{q}{p}\right)^{k+1}}{1 - \left(\frac{q}{p}\right)^M}$$

or

$$\boxed{P_K = \frac{1-\left(\frac{q}{p}\right)^K}{1-\left(\frac{q}{p}\right)^M}}$$

Ex: $K = \$100$, $M = \$200$, $p = 0.49$, $q = 0.51$

$$P_{100} = \frac{1-\left(\frac{0.51}{0.49}\right)^{100}}{1-\left(\frac{0.51}{0.49}\right)^{200}} \approx 0.018$$

# Random Variables

**Definition :** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random variable is a real-valued function $X : \Omega \to \mathbb{R}$ such that $\forall a \in \mathbb{R}$, $\{\omega : X(\omega) \leq a\} = X^{-1}((-\infty, a]) \in \mathcal{F}$.

**Definition :** The distribution function $F_X$ of a rv $X$ is the function $F$ defined by

$$F(x) = P\{\omega \in \Omega : X(\omega) \leq x\}, \quad \forall x \in \mathbb{R}.$$

**Proposition :** Let $X$ be a rv & Let $F$ be its distribution function. Then $\forall x, y \in \mathbb{R}$,

(i) $0 \leq F(x) \leq 1$

(ii) $x \leq y \Rightarrow F(x) \leq F(y)$

(iii) $\lim_{x \to -\infty} F(x) = 0$ & $\lim_{x \to \infty} F(x) = 1$

(iv) $\lim_{y \to x^+} F(y) = F(x)$

(v) $F(x-) := \lim_{y \to x^-} F(y)$ exists

right continuous with left limits

(vi) $F$ has at most countable number of discontinuities.

Lemma: (a) If $A_1 \subset A_2 \subset \ldots \in \mathcal{F}$, then

$$\lim_{n \to \infty} P(A_n) = P\left(\bigcup_n A_n\right)$$

(b) If $B_1 \supset B_2 \supset \ldots \in \mathcal{F}$, then

$$\lim_{n \to \infty} P(B_n) = P\left(\bigcap_n B_n\right)$$