Corollary: $P\left[\,|X - E[X]| \geq a\right] \leq \dfrac{Var(X)}{a^2}$

Ex: Let X be a rv with $E(X) < \infty$. If $V(X) = 0$, then prove that X is a constant with probability 1.

Solution: $C_n = \left\{ |X - E(X)| > \frac{1}{n} \right\}$

$$P\left(|X - E(X)| > \tfrac{1}{n}\right) \leq \frac{V(X)}{\left(\frac{1}{n}\right)^2} = 0$$

$$P(X \neq E(X)) = P\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} P(C_n) = 0$$

$\square$

Definition: We say $g: \mathbb{R} \to \mathbb{R}$ is a convex function if $\forall x, a \in \mathbb{R}$, $\exists \lambda_a \in \mathbb{R}$ s.t

$$g(x) \geq g(a) + \lambda_a (x - a)$$

Theorem: Jensen's inequality

Let X be a rv with $E(X) < \infty$. Then, for any convex function g,

$$E[g(X)] \geq g(E(X)).$$

Proof: Let g be a convex function & choose $a = E(X)$. Then $\exists\, \lambda_a \in \mathbb{R}$ s.t.

$$g(X) \geq g(E(X)) + \lambda_a (X - E(X))$$

Therefore,

$$E(g(X)) \geq E(g(E(X))) + \lambda_a \underbrace{E(X - E(X))}_{=0}$$

$$\Rightarrow E(g(X)) \geq g(E(X)) \quad \square$$

Theorem: Hölder's inequality:

If $p, q > 1$ & $\frac{1}{p} + \frac{1}{q} = 1$, then

$$E|XY| \leq (E|X^p|)^{\frac{1}{p}} (E|X|^q)^{\frac{1}{q}}$$

Theorem: If $p \geq 1$, then

$$\left(E\left[|X+Y|^p\right]\right)^{\frac{1}{p}} \leq (E|X^p|)^{\frac{1}{p}} + (E|Y^p|)^{\frac{1}{p}}$$

Convergence of Random variables

Let $\{X_n\}$ be a sequence of random variables. In this section, we aim to understand the convergence of X_n. Since $X_n : \Omega \to \mathbb{R}$, $\{X_n\}$ is a sequence of functions.

We will discuss four ways of interpreting the statement $X_n \to X$ as $n \to \infty$.

Definition: Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, P)$.

(a) Almost sure convergence: We say $X_n \xrightarrow{a.s.} X$ if

$$P\left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} = 1.$$

(b) r^{th} mean convergence: We say $X_n \xrightarrow{r} X$, if $E[|X_n|^r] < \infty \quad \forall n \in \mathbb{N}$ &

$$\lim_{n\to\infty} E|X_n - X|^r = 0. \quad (r \geq 1).$$

(c) Convergence in probability: We say $X_n \xrightarrow{P} X$ if $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} P\left\{ \omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon \right\} = 0.$$

(d) Convergence in distribution: We say $X_n \xrightarrow{D} X$ if

$$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$$

$\forall x$ s.t. $F_X(x) = P(X \leq x)$ is continuous.

Remark: These modes of convergence are NOT equivalent.

The following implications hold in general:

$$X_n \xrightarrow{a.s} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$$

$$X_n \xrightarrow{r} X \implies X_n \xrightarrow{P} X$$

Theorem: If $X_n \xrightarrow{1} X$, then $X_n \xrightarrow{P} X$.

Proof: Let $\varepsilon > 0$. Then, by Markov's inequality,

$$P\{|X_n - X| > \varepsilon\} \leq \frac{E|X_n - X|}{\varepsilon} \longrightarrow 0 \text{ as } n \to \infty$$

$\square$

MATH 630

Problems -10/30

Name:

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(1) Let $X_1, X_2, \ldots$ be a sequence of random variables with density

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)}$$

for $x \in \mathbb{R}$. Prove that $X_n \xrightarrow{P} 0$ as $n \to \infty$.

Let $\varepsilon > 0$.

$$P\left[\,|X_n - 0| > \varepsilon\,\right] = 1 - P\left[-\varepsilon < X_n < \varepsilon\right]$$

$$= 1 - \int_{-\varepsilon}^{\varepsilon} \frac{n}{\pi(1+n^2x^2)}\,dx = 1 - \frac{n}{\pi} \cdot \frac{1}{n^2} \int_{-\varepsilon}^{\varepsilon} \frac{1}{\frac{1}{n^2} + x^2}\,dx$$

$$= 1 - \frac{1}{n\pi} \cdot n \cdot \arctan(nx)\Big|_{-\varepsilon}^{\varepsilon}$$

$$= 1 - \frac{2}{\pi} \arctan(nx)\Big|_{0}^{\varepsilon} = 1 - \frac{2}{\pi} \cdot \underbrace{\arctan(n\varepsilon)}_{\to \frac{\pi}{2}} \quad \text{as } n \to \infty$$

Therefore, $P\left[\,|X_n| > \varepsilon\,\right] \to 0 \implies X_n \xrightarrow{P} 0$.

(2) Let $U \sim U(0,1)$. Consider the following sequence:

$$X_n = \begin{cases} 5 & U \leq \frac{2}{3} - \frac{1}{n} \\ 10 & \text{otherwise} \end{cases}$$

Let

$$Y = \begin{cases} 5 & U \leq \frac{2}{3} \\ 10 & \text{otherwise} \end{cases}$$

Prove that $X_n \xrightarrow{P} Y$.

Let $\varepsilon > 0$.

$$P(|X_n - Y| > \varepsilon) \leq P(X \neq Y)$$

$$= P\left(\frac{2}{3} < U \leq \frac{2}{3} - \frac{1}{n}\right)$$

$$= \frac{1}{n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$\square$.