Math 600 Lecture 18

Theorem (alternating series test) Suppose $\{x_n\}$ is a decreasing sequence of real numbers and $x_n \to 0$. Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n x_n$$

converge.

Proof: Since $-\sum_{n=1}^{\infty} (-1)^{n+1} x_n = \sum_{n=1}^{\infty} (-1)^n x_n$, it suffices to prove that the first series converges. Define

$$S_N = \sum_{n=1}^{N} (-1)^{n+1} x_n .$$

Note that, for all $k = 1, 2, \ldots$,

$$S_{2k} = S_{2k-2} + x_{2k-1} - x_{2k} \geq S_{2k-2} \quad (\text{since } x_{2k-1} \geq x_{2k})$$

and

$$S_{2k+1} = S_{2k-1} - x_{2k} + x_{2k+1} \leq S_{2k-1} \quad (\text{since } x_{2k} \geq x_{2k+1}).$$

Thus the partial sums with even indices form an increasing sequence ($S_2 \leq S_4 \leq S_6 \leq \cdots$) and those with odd indices form a decreasing sequence ($S_1 \geq S_3 \geq S_5 \geq \cdots$). Moreover, in the sequence of partial sums, every odd-indexed term is greater than every even-indexed term. For consider $S_{2k+1}, S_{2\ell}$. If $k \geq \ell$, then

$$s_{2\ell} \le s_{2k} < s_{2k} + x_{2k+1} = s_{2k+1}$$

while if $k < \ell$, then

$$s_{2\ell} < s_{2\ell} + x_{2\ell+1} = s_{2\ell+1} \le s_{2k+1}.$$

Thus

$$s_{2k} \to s_*, \quad s_{2k+1} \to s^*,$$

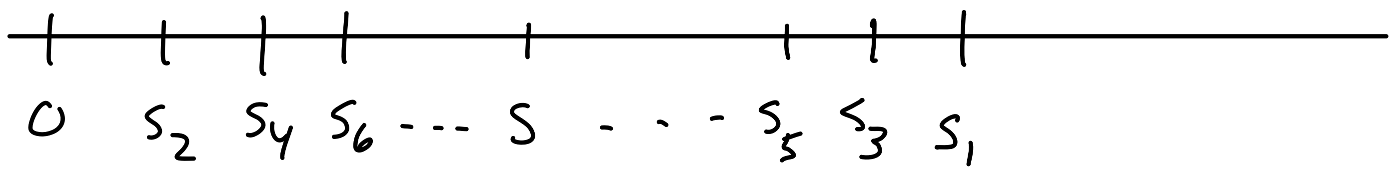
where $s_*, s^* \in \mathbb{R}$ and $s_* \le s^*$. Finally,

$$s_{2k} \le s_* \le s^* \le s_{2k+1} = s_{2k} + x_{2k+1} \quad \forall k \in \mathbb{Z}^+.$$

But $x_{2k+1} \to 0$ and hence

$$s_* = s^* = s, \quad s_{2k} \to s, \quad s_{2k+1} \to s.$$

It follows that $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ converges to s. //



Thus, for example, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, even though $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem: Let $\{x_n\}$ be a sequence of real numbers. If $\sum_{n=1}^{\infty} |x_n|$ converges, then so does $\sum_{n=1}^{\infty} x_n$.

Proof: This follows from the Cauchy criterion:

$$\sum_{n=1}^{\infty} |x_n| \text{ converges} \Rightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{Z}^+ \left(m, n \geq N \Rightarrow \left|\sum_{k=n}^{m} |x_k|\right| < \varepsilon\right)$$

$$\Rightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{Z}^+ \left(m, n \geq N \Rightarrow \left|\sum_{k=n}^{m} x_k\right| < \varepsilon\right)$$

$$\left(\text{since } \left|\sum_{k=n}^{m} x_k\right| \leq \sum_{k=n}^{m} |x_k| = \left|\sum_{k=n}^{m} |x_k|\right|\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} x_n \text{ converges.} //$$

Definition: Let $\{x_n\}$ be a sequence of real numbers. We say that $\sum_{n=1}^{\infty} x_n$ <u>converges absolutely</u> iff $\sum_{n=1}^{\infty} |x_n|$ converges. We say that $\sum_{n=1}^{\infty} x_n$ <u>converges conditionally</u> iff $\sum_{n=1}^{\infty} x_n$ converges and $\sum_{n=1}^{\infty} |x_n|$ diverges.

The following theorem is obvious, and we have already used the second conclusion.

Theorem: 1. If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers and $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ both converge, then $\sum_{n=1}^{\infty} (x_n + y_n)$ converges and

$$\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

2. If $\{x_n\}$ is a sequence of real numbers and $\sum_{n=1}^{\infty} x_n$ converges, then, for all $c \in \mathbb{R}$, $\sum_{n=1}^{\infty} c x_n$ converges and

$$\sum_{n=1}^{\infty} c x_n = c \sum_{n=1}^{\infty} x_n .$$

How do we multiply two series?

$$\left(\sum_{m=0}^{\infty} x_m\right)\left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x_m\right) y_n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x_m y_n\right) \quad (1)$$

or

$$\left(\sum_{m=0}^{\infty} x_m\right)\left(\sum_{n=0}^{\infty} y_n\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} y_n\right) x_m$$

$$= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} x_m y_n\right) \quad (2)$$

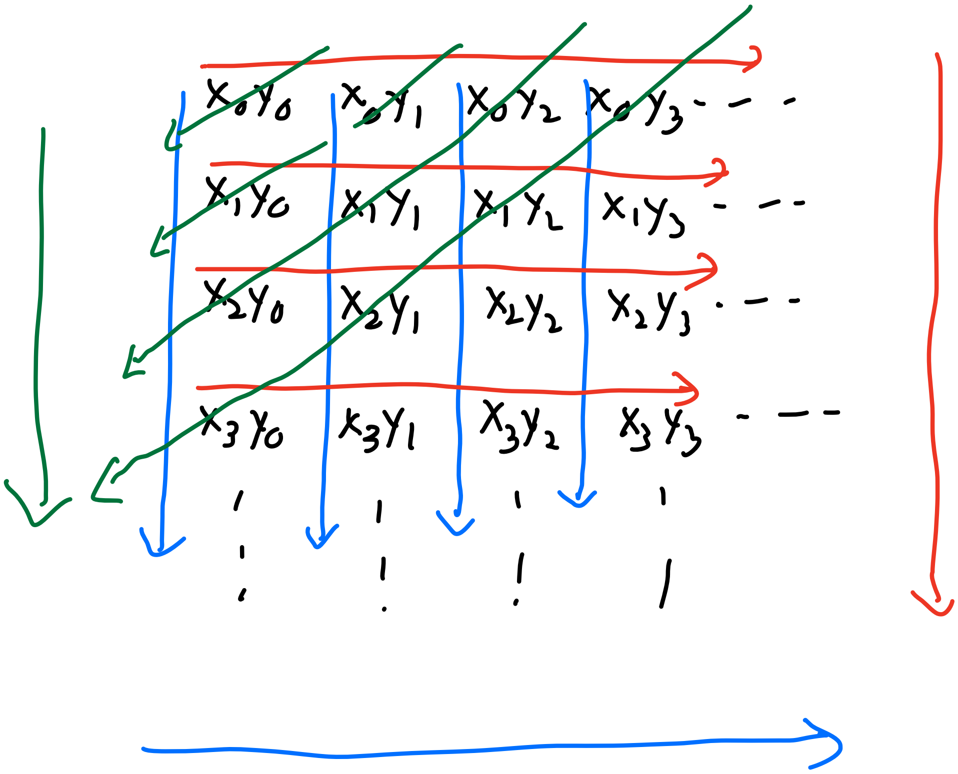
or

$$\left(\sum_{m=0}^{\infty} x_m\right)\left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x_k y_{n-k}\right) \quad (3)$$

or

?

Somehow, we must add all the products $x_m y_n$, $m, n = 0, 1, 2, \dots$, but we might not get the same result with every order.



In fact, the following theorem holds, although I won't prove it in this class. A <u>rearrangement</u> of $\sum_{n=1}^{\infty} x_n$ is a series $\sum_{k=1}^{\infty} x_{m(k)}$, where $m : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a bijection.

<u>Theorem</u> (Riemann): Let $\{x_n\}$ be a sequence of real numbers and let L be any real number or ∞ or $-\infty$. If $\sum_{n=1}^{\infty} x_n$ is conditionally convergent, then there exists a rearrangement of $\sum_{n=1}^{\infty} x_n$ satisfying

$$\sum_{k=1}^{\infty} x_{m(k)} = L.$$

Thus, we choose a certain order to define $\left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{n=0}^{\infty} y_n\right)$.

Definition: Let $\{x_n\}$ and $\{y_n\}$ $(n=0,1,2,\dots)$ be sequences of real numbers. We define the product of $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ to be the series

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x_k y_{n-k}\right).$$

(Informally,

$$(*) \qquad \left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x_k y_{n-k}\right),$$

but the product need not converge, even if both series on the left converge, so $(*)$ can be misleading.)

Before we get to the main theorem about absolute convergence, we note one of the main implications of absolute convergence.

Theorem: Let $\{x_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} x_n$ converges absolutely. Then every rearrangement of $\sum_{n=1}^{\infty} x_n$ converges, and to the same limit.

Proof: Suppose $\sum_{n=1}^{\infty} x_n = s$ and write $s_N = \sum_{n=1}^{N} x_n$, so that $s_N \to s$.

Let $m: \mathbb{Z}^+ \to \mathbb{Z}^+$ be a bijection, so that $\sum_{k=1}^{\infty} x_{m(k)}$ is a rearrangement of $\sum_{n=1}^{\infty} x_n$, and write $s_N' = \sum_{k=1}^{N} x_{m(k)}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{Z}^+$ such that

$$m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} |x_k| \right| < \varepsilon \Rightarrow \sum_{k=n}^{m} |x_k| < \varepsilon.$$

Choose M so that

$$\{1, 2, \ldots, N-1\} \subset \{m(1), m(2), \ldots, m(M-1)\}.$$

Then,

$$\begin{aligned} l \geq k \geq M &\Rightarrow m(k), m(k+1), \ldots, m(l) \geq N \\ &\Rightarrow \left| \sum_{j=k}^{l} x_{m(j)} \right| \leq \sum_{j=k}^{l} |x_{m(j)}| \\ &\leq \sum_{k=\min\{m(k),\ldots,m(l)\}}^{\max\{m(k),\ldots,m(l)\}} |x_k| \\ &< \varepsilon. \end{aligned}$$

Thus, by the Cauchy criterion,

$$\sum_{k=1}^{\infty} x_{m(k)}$$

converges. Also, for a given $l \geq M$, let us define

$$J = (J_1 \cup J_2) \setminus (J_1 \cap J_2),$$

where

$$J_1 = \{1, 2, \ldots, l\},$$
$$J_2 = \{m(1), m(2), \ldots, m(l)\}.$$

Then

$$|s_p' - s_p| = \left| \sum_{k=1}^{p} x_{m(k)} - \sum_{k=1}^{p} x_k \right|$$

$$\leq \sum_{j \in J} |x_j| < \varepsilon \quad (\text{since } \{1, 2, \dots, N\} \cap J = \emptyset).$$

Thus

$$\lim_{p \to \infty} s_p' = \lim_{p \to \infty} s_p = s. \quad //$$