Math 672 Lecture 12

Definition: Let V be a vector space over a field F.

- Each element f of $\mathcal{L}(V, F)$ is called a linear functional on V.

  [Note: When V is infinite-dimensional, $\mathcal{L}(V,F)$ usually denotes the space of continuous linear functionals on V.]

- $\mathcal{L}(V,F)$ is called the dual space of V and is denoted V'.

We know from a previous exercise that, if V is finite-dimensional, then $V' \cong V$ (because $V \cong F^n$ and $V' \cong F^n$, where $n = \dim(V)$).

The following result gives an alternate proof of this fact.

Theorem: Let V be an n-dimensional vector space over a field F with basis $\{v_1, v_2, \ldots, v_n\}$. For $j = 1, 2, \ldots, n$, define $\varphi_j : V \to F$ by

$$\varphi_j(v_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a basis for V', called the dual basis of $\{v_1, v_2, \ldots, v_n\}$.

Proof: We know from a previous theorem that each φ_j is a well-defined element of V'.

Suppose $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ satisfy

$$\sum_{j=1}^{n} \alpha_j \varphi_j = 0,$$

that is,

$$\left(\sum_{j=1}^{n} \alpha_j \varphi_j\right)(v) = 0 \quad \forall v \in V.$$

Then, for all $i = 1, 2, \dots, n$,

$$\left(\sum_{j=1}^{n} \alpha_j \varphi_j\right)(v_i) = 0$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \varphi_j(v_i) = 0$$

$$\Rightarrow \alpha_i = 0 \quad (\text{since } \varphi_j(v_i) = 0 \text{ for } j \neq i \text{ and } \varphi_i(v_i) = 1)$$

This proves that $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is linearly independent.

Now let f be an arbitrary element of V' and define

$$\alpha_i = f(v_i), \quad i = 1, 2, \dots, n.$$

We claim that

$$f = \sum_{j=1}^{n} \alpha_j \varphi_j ,$$

that is, that

$$f(v) = \left(\sum_{j=1}^{n} \alpha_j \varphi_j\right)(v) \quad \forall v \in V.$$

To prove this, let $v = \sum_{i=1}^{n} \beta_i v_i$ be an arbitrary element of V.

Then

$$f(v) = f\left(\sum_{i=1}^{n} \beta_i v_i\right) = \sum_{i=1}^{n} \beta_i f(v_i) = \sum_{i=1}^{n} \beta_i \alpha_i .$$

On the other hand, for each $j = 1, 2, \ldots, n$,

$$\varphi_j(v) = \varphi_j\left(\sum_{i=1}^{n} \beta_i v_i\right) = \sum_{i=1}^{n} \beta_i \varphi_j(v_i) = \sum_{i=1}^{n} \beta_i \delta_{ij} = \beta_j$$

and hence

$$\left(\sum_{j=1}^{n} \alpha_j \varphi_j\right)(v) = \sum_{j=1}^{n} \alpha_j \varphi_j(v) = \sum_{j=1}^{n} \alpha_j \beta_j .$$

It follows that

$$f = \sum_{j=1}^{n} \alpha_j \varphi_j$$

and we have shown that $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ spans V'. //

Note, from the preceding proof, that

$$f = \sum_{j=1}^{n} f(v_j) \varphi_j \quad \forall f \in V'$$

This is the special property of the dual basis — it's easy to represent an element of V' as a linear combination of the dual basis. With a typical basis, it's necessary to solve a system of equations to represent a vector in terms of the basis.

<u>Example</u>: Consider the vector space $V = \mathcal{P}_n(\mathbb{R})$, an $(n+1)$-dimensional vector space over $\mathbb{R}$, and let $x_1, x_2, \ldots, x_{n+1}$ be $n+1$ distinct distinct real numbers. For each $j = 1, 2, \ldots, n+1$, define $L_j \in V$ by

$$L_j(x) = \prod_{\substack{i=1 \\ i \neq j}}^{n+1} \frac{(x - x_i)}{(x_j - x_i)}.$$

For example, if $n = 2$, then

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)},$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)},$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

Note that

$i \neq j \Rightarrow L_j(x_i) = 0$ (since $(x_i - x_i)$ is one factor in the product defining $L_j(x_i)$)

and

$$L_j(x_j) = \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{(x_j - x_i)}{(x_j - x_i)} = 1.$$

Since $\dim(V) = n+1$, we can prove that $\{L_1, L_2, \ldots, L_{n+1}\}$ is a basis for V by proving that it is linearly independent.

Now,

$$\sum_{j=1}^{n+1} \alpha_j L_j = 0$$

$$\Leftrightarrow \left(\sum_{j=1}^{n+1} \alpha_j L_j\right)(x) = 0 \quad \forall x \quad (\text{since } 0 \in V \text{ is the } 0 \text{ function})$$

$$\Leftrightarrow \sum_{j=1}^{n+1} \alpha_j L_j(x) = 0 \quad \forall x$$

$$\Rightarrow \sum_{j=1}^{n+1} \alpha_j L_j(x_i) = 0 \quad \forall i = 1, 2, \ldots, n+1$$

$$\Rightarrow \alpha_i = 0 \quad \forall i = 1, 2, \ldots, n+1 \quad (\text{since } L_j(x_i) = \delta_{ij}).$$

Therefore, $\{L_1, L_2, \ldots, L_{n+1}\}$ is linearly independent and hence is a basis for V.

What is the dual basis?

Answer: $\{\varphi_1, \varphi_2, \ldots, \varphi_{n+1}\}$, where $\varphi_j : V \to \mathbb{R}$ is defined by

$$\varphi_i(p) = p(x_i) \quad \text{(evaluation at } x_i\text{)}.$$

We then have

$$\varphi_i(L_j) = L_j(x_i) = \delta_{ij},$$

as required.

Recall that the dual basis has the special property that

$$\varphi \in V' \Rightarrow \varphi = \sum_{j=1}^{n+1} \varphi(L_j)\varphi_j .$$

In this example, $\{L_1, L_2, \ldots, L_{n+1}\}$ has a similar property:

$$p \in V \Rightarrow p = \sum_{j=1}^{n+1} p(x_j) L_j \qquad \text{(Why?)}.$$

$\{L_1, L_2, \ldots, L_{n+1}\}$ is called a <u>nodal basis</u>. $L_1, L_2, \ldots, L_{n+1}$ are called <u>Lagrange polynomials</u>.

<u>Lemma</u>: Let $T \in \mathcal{L}(V,W)$. Then $T': W' \to V'$ defined by

$$T'(\varphi) = \varphi \circ T \quad \forall \varphi \in W'$$

is an element of $\mathcal{L}(W',V')$. (We call T' the <u>dual map</u> of T.)

<u>Proof</u>: Clearly $\varphi \circ T \in \mathcal{L}(V, F)$ ($T: V \to W$, $\varphi: W \to F$, the composition of linear maps is linear). Thus $T': W' \to V'$ is well defined. Let $\varphi, \psi \in W'$ and $\alpha, \beta \in F$. Then

$$\begin{aligned} T'(\alpha\varphi + \beta\psi) &= (\alpha\varphi + \beta\psi) \circ T \\ &= \alpha\varphi\circ T + \beta\psi\circ T \\ &= \alpha T'(\varphi) + \beta T'(\psi), \end{aligned}$$

since

$$\begin{aligned} ((\alpha\varphi + \beta\psi)\circ T)(v) &= (\alpha\varphi+\beta\psi)(T(v)) \\ &= \alpha\varphi(T(v)) + \beta\psi(T(v)) \\ &= \alpha(\varphi\circ T)(v) + \beta(\psi\circ T)(v) \\ &= (\alpha(\varphi\circ T) + \beta(\psi\circ T))(v). \end{aligned}$$

Thus T' is linear. //

What is the significance of T'? The answer will be clearer in an inner product space, but the main result is

$$\mathcal{M}_{\mathcal{C}',\mathcal{B}'}(T') = \left(\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)\right)^T,$$

where

$\mathcal{B},\mathcal{C}$ are bases for V, W, respectively,

$\mathcal{B}',\mathcal{C}'$ are the dual bases for V', W', respectively.