Math 672 Lecture 21

Recall that an orthogonal or orthonormal basis is convenient because we can easily express any vector as a linear combination of the basis. Specifically, if $\{u_1, \ldots, u_n\}$ is an orthogonal or orthonormal basis for V, then

$$\forall v \in V,\ v = \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \quad \text{(orthogonal case)},$$

$$\forall v \in V,\ v = \sum_{j=1}^{n} \langle v, u_j \rangle u_j \quad \text{(orthonormal case)}$$

Theorem: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$), let S be a finite-dimensional subspace of V, let $\{u_1, \ldots, u_n\}$ be an orthogonal basis for S, and let $x \in V$. Then

$$(*) \qquad v = \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} u_j \qquad \text{(Same formula as above!)}$$

is the unique solution of

$$\min_{w \in S} \|w - x\|.$$

If $\{u_1, \ldots, u_n\}$ is orthonormal, then $(*)$ simplifies to

$$v = \sum_{j=1}^{n} \langle x, u_j \rangle u_j. \qquad \text{(Ditto!)}$$

Proof: It suffices to prove that v, as defined by $(*)$, satisfies

$$(**) \qquad \langle v - x, w \rangle = 0 \quad \forall w \in S.$$

It is easy to prove that (**) is equivalent to

$$(***) \quad \langle v-x, u_i \rangle = 0 \quad \forall i = 1, 2, \ldots, n.$$

We have, for $i = 1, 2, \ldots, n$,

$$\left\langle \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} u_j - x, u_i \right\rangle = \left\langle \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} u_j, u_i \right\rangle - \langle x, u_i \rangle$$

$$= \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_i \rangle - \langle x, u_i \rangle$$

$$= \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_j \rangle - \langle x, u_j \rangle \quad \left(\begin{array}{c} \text{since } \langle u_j, u_i \rangle = 0 \\ \text{for } i \neq j \end{array} \right)$$

$$= \langle x, u_i \rangle - \langle x, u_i \rangle = 0.$$

This proves (***). //

It is important to notice that the same formula solves two problems:

- If $\{u_1, \ldots, u_n\}$ is an orthonormal basis <u>for V</u>, then

$$\sum_{j=1}^{n} \langle x, u_j \rangle u_j = x \quad \forall x \in V.$$

- If $\{u_1, \ldots, u_n\}$ is an orthonormal basis for <u>a subspace S of V</u>, then

$$\sum_{j=1}^{n} \langle x, u_j \rangle u_j$$

is the vector in S closest to x.

Example: We can use the projection theorem to compute polynomial approximations to $f \in C[a,b]$. For this example, we will use $C[-1,1]$ for convenience.

$$L^2(-1,1) \text{ inner product: } \langle f,g \rangle_{L^2} = \int_{-1}^{1} f(x)g(x)\,dx$$

$$\text{Alternate inner product: } \langle f,g \rangle_C = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$$

The second inner product might seem rather arbitrary (and strange, since the weighting function is singular at the endpoints), but notice:

$$f,g \in C[-1,1] \Rightarrow \langle f,g \rangle_C = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$$

$x = \cos\theta$

$dx = -\sin\theta\,d\theta$

$\sqrt{1-x^2} = \sqrt{1-\cos^2\theta}$

$= \sin\theta$

$$= \int_{\pi}^{0} \frac{f(\cos\theta)g(\cos\theta)}{\sin\theta}(-\sin\theta)\,d\theta$$

$x = 1 \to \theta = 0$

$x = -1 \to \theta = \pi$

$$= \int_{0}^{\pi} f(\cos\theta)g(\cos\theta)\,d\theta.$$

It follows that $\langle \cdot,\cdot \rangle_C$ is well defined on $C[-1,1]$.

Define the polynomial (!) $T_n(x) = \cos(n \arccos(x))$ for all $n \geq 0$.

We have

$$T_0(x) = \cos(0) = 1,$$

$$T_1(x) = \cos(\arccos(x)) = x,$$

and

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\Rightarrow \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\Rightarrow T_{n+1}(x) + T_{n-1}(x) = \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x))$$

$$= \cos(n\arccos(x) + \arccos(x)) + \cos(n\arccos(x) - \arccos(x))$$

$$= 2\cos(n\arccos(x))\cos(\arccos(x))$$

$$= 2x T_n(x)$$

$$\Rightarrow T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x).$$

This shows that T_n is a polynomial (of degree n) for all $n \geq 0$.

Note:

$$\langle T_n(x), T_m(x) \rangle_c = \int_0^{\pi} T_n(\cos\theta) T_m(\cos\theta)\, d\theta$$

$$= \int_0^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta$$

$$= \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

Thus $\{T_0, T_1, T_2, \ldots\}$ is an orthogonal set, and $\{T_0, T_1, \ldots, T_n\}$ is an orthogonal basis for $\mathcal{P}_n$ (regarded as a subspace of $C[-1,1]$).

See the Mathematica file ProjectionTheoremExample.nb for the rest of this example.