Ex: Let $X_1, X_2 \sim Geo(p)$ be independent

$$P(X_i = k) = (1-p)^{k-1} p \quad ; \quad k \geq 1$$

$$P(Z=k) = \sum_{n=1}^{k-1} P(X=n) \cdot P(Y=k-n)$$

$$= \sum_{n=1}^{k-1} (1-p)^{n-1} \cdot p \cdot (1-p)^{k-n-1} p$$

$$= \sum_{n=1}^{k-1} (1-p)^{k-2} p^2$$

$$= (1-p)^{k-2} p^2 \cdot (k-1) \leftarrow \text{pmf of } X_1 + X_2.$$

Quiz

Ex: Let $X_1 \sim Poisson(\lambda_1)$, $X_2 \sim Poisson(\lambda_2)$ be independent. Find $P(X_1 + X_2 = k)$ using a convolution.

$$P(X_1 + X_2 = n) = \sum_{k=0}^{n} P(X_1 = k, X_2 = n-k)$$

$$= \sum_{k=0}^{n} P(X_1 = k) \cdot P(X_2 = n-k)$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda_1} \lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2} \lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \underbrace{(\lambda_1+\lambda)^n}_{\text{Binomial theorem}}$$

Definition : The conditional distribution function of Y given $X = x$ is

$$F_{Y|X}(y|x) = P(Y \leq y \mid X = x)$$

for any x such that $P(X = x) > 0$.

The conditional mass function of Y given $X = x$ is given by

$$p_{Y|X}(y|x) = P(Y = y \mid X = x)$$

Notice that $p_{Y|X}(y|x) = \dfrac{p_{X,Y}(x,y)}{p_X(x)}$ &

$$\sum_y p_{Y|X}(y|x) = \frac{\sum_y p_{X,Y}(x,y)}{p_X(x)}$$

$$= \frac{p_X(x)}{p_X(x)} = 1.$$

Ex:

$x \backslash y$	0	1	2
1	0.1	0.1	0.2
2	0.2	0.3	0.1

(a) $$P_{X|Y}(1|0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{0.1}{0.1+0.2} = \frac{1}{3}$$

(b) $$P_{Y|X}(2|2) = \frac{P(X=2, Y=2)}{P(X=2)} = \frac{0.1}{0.1+0.1+0.2} = \frac{1}{4}$$

Ex: $X, Y \sim Geo(p)$ are independent

We know that
$$P(X+Y=k) = k \cdot p^2 (1-p)^{k-2} \quad ; \quad k \geq 2$$

Let $Z = X+Y$. Find $P_{X|Z}(x|z)$

$$P_{X|Z}(x|z) = \frac{P(X=x, Z=z)}{P(Z=z)}$$

$$= \frac{P(X=x, Y=z-x)}{P(Z=z)}$$

$$= \frac{(1-p)^{x-1} p \cdot (1-p)^{z-x-1} p}{(z-1) p^2 (1-p)^{z-2}}$$

$$P_{X|Z}(x|z) = \frac{1}{z-1}$$

Ex: Let $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ be independent. Let $Z = X + Y$.
Find the distribution of $X \mid Z$.

$$P(X = k \mid Z = n) = \frac{P(X = k, Z = n)}{P(Z = n)}$$

$$= \frac{P(X = k, X + Y = n)}{P(Z = n)} = \frac{P(X = k, Y = n - k)}{P(Z = n)}$$

$$= \frac{P(X = k) \cdot P(Y = n - k)}{P(Z = n)} = \frac{\dfrac{e^{-\lambda_1} \lambda_1^{k}}{k!} \cdot \dfrac{e^{-\lambda_2} \cdot \lambda_2^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1 + \lambda_2)} \cdot (\lambda_1 + \lambda_2)^n}{n!}}$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{k} \left(1 - \frac{\lambda_1}{\lambda_2 + \lambda_2} \right)^{n-k} \Longrightarrow Z \sim B\left(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$