Math 600 Lecture 3

Let $a,b \in \mathbb{R}$ with $a<b$, and assume that $f:(a,b)\to\mathbb{R}$ is bounded above (there exists $M\in\mathbb{R}$ such that $f(x)\le M$ for all $x\in(a,b)$). Does the LUB property of $\mathbb{R}$ imply that

$$\sup\{f(x)\mid x\in(a,b)\}$$

exists? Does it guarantee that there exists $c\in(a,b)$ such that

$$f(c)=\sup\{f(x)\mid x\in(a,b)\}?$$

Functions

Definition: Let A,B be sets. A function $f:A\to B$ is (essentially) a rule for assigning a unique element $f(a)$ in B to each $a\in A$. (To be precise, a function $f:A\to B$ is a subset S of $A\times B=\{(a,b)\mid a\in A, b\in B\}$ with the property that for each $a\in A$, there exists exactly one $(x,y)\in S$ with $x=a$ (then $y=f(a)$).)

We call A the domain of f, B the co-domain of f, and we define the range of f to be the set

$$\mathcal{R}(f)=\{f(a)\in B\mid a\in A\}.$$

Definition: Suppose A,B are sets and $f:A\to B$ is a function.

1. We say that f is injective (one-to-one) iff

$(a_1, a_2 \in A \text{ and } f(a_1) = f(a_2)) \Rightarrow a_1 = a_2$

2. We say that f is <u>surjective</u> (<u>onto</u>) iff

$$\forall b \in B \ \exists a \in A, \ f(a) = b.$$

(Equivalently: f is surjective iff $\mathcal{R}(f) = B$.)

3. We say that f is <u>bijective</u> iff it is both injective and surjective.

4. If $S \subseteq A$, then the <u>image $f(S)$ of S under f</u> is the set

$$f(S) = \{f(s) \in B \mid s \in S\}.$$

(Note that $\mathcal{R}(f) = f(A)$.)

5. If $T \subseteq B$, then the <u>inverse image $f^{-1}(T)$ of T under f</u> is the set

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}.$$

<u>Examples</u>: Which of the following functions are injective? Surjective?

- $f: \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$
- $g: \mathbb{R} \to \mathbb{R}, \ g(x) = e^x$
- $h: \mathbb{R} \to \mathbb{R}, \ h(x) = \sin(x)$

<u>Definition</u>: Let A, B be sets and let $f: A \to B$ be a function. We say that f is <u>invertible</u> iff there exists $g: B \to A$ such that

$$(*) \qquad \big(g(f(a)) = a \ \forall a \in A\big) \text{ and } \big(f(g(b)) = b \ \forall b \in B\big).$$

Theorem: Let A, B be sets and let $f: A \to B$ be a function. Then:

- f is invertible iff f is bijective.
- If f is invertible, then there is a unique function $g: B \to A$ satisfying $(*)$. In this case, we write f^{-1} in place of g and call f^{-1} the inverse of f.

Proof: Suppose first that f is bijective and define $g: B \to A$ by

$$g(b) = a \iff f(a) = b.$$

Note that since f is surjective, given $b \in B$, there exists $a \in A$ such that $f(a) = b$. Moreover, since f is injective, there is only one such a. Thus g is well defined. Then

$$a \in A \Rightarrow g(f(a)) = a \quad (\text{since } f(a) = b \Rightarrow g(f(a)) = g(b) = a)$$

and

$$b \in B \Rightarrow f(g(b)) = b \quad (\text{since } g(b) = a \Rightarrow f(g(b)) = f(a) = b).$$

Thus g satisfies $(*)$ and hence f is invertible.

Conversely, assume that f is invertible. Then, for all $b \in B$, $f(g(b)) = b$; this shows that f is surjective. Also,

$$\begin{aligned} & a_1, a_2 \in A,\ f(a_1) = f(a_2) \\ \Rightarrow\ & g(f(a_1)) = g(f(a_2)) \quad (\text{why?}) \\ \Rightarrow\ & a_1 = a_2 \quad (\text{by } (*)). \end{aligned}$$

Thus f is also injective, and we have shown that f is bijective.

Continue to assume that f is invertible, and suppose we have functions $g: B \to A$ and $h: B \to A$ satisfying

$$\left(g(f(a)) = a \ \forall a \in A\right) \text{ and } \left(f(g(b)) = b \ \forall b \in B\right)$$

and

$$\left(h(f(a)) = a \ \forall a \in A\right) \text{ and } \left(f(h(b)) = b \ \forall b \in B\right).$$

Then, for all $b \in B$,

$$f(g(b)) = b = f(h(b))$$

$$\Rightarrow g(b) = h(b) \quad \text{(since } f \text{ is injective)}.$$

This proves that $h = g$. //

<u>Definition</u>: Let A, B be sets. If there exists a bijection $f: A \to B$, we say that A and B have the same <u>cardinality</u> and write $A \sim B$.

Note that this defines an <u>equivalence relation</u>:

- For all sets A, $A \sim A$ ($\sim$ is <u>reflexive</u>)
- For all sets A, B, $A \sim B \Rightarrow B \sim A$ ($\sim$ is <u>symmetric</u>)
- For all sets A, B, C, $(A \sim B \text{ and } B \sim C) \Rightarrow A \sim C$ ($\sim$ is <u>transitive</u>)

Cardinality captures the concept of the size of a set, in the sense of the number of elements in it. We will see that this concept can be counter-intuitive.

Definition: For any set A, we say that A is

- finite iff $A=\emptyset$ or $A\sim\{1,2,\dots,n\}$ for some $n\in\mathbb{Z}^+$;
- infinite iff A is not finite;
- countably infinite iff $A\sim\mathbb{Z}^+$;
- countable iff A is finite or countably infinite.
- uncountable iff A is not countable.

Differs from Rudin's definitions

Example

Let $E=\{2k \mid k\in\mathbb{Z}\}$ = the set of even integers. Define $f:\mathbb{Z}\to E$ by $f(k)=2k$. It is straightforward to verify that f is a bijection; thus $E\sim\mathbb{Z}$. A set can have the same cardinality as one of its proper subset. (This is only possible for an infinite set. In fact, a set is infinite iff there exists a bijection from the set onto a proper subset.)

Definition: Let S be a set. A sequence in S is a function $x:\mathbb{Z}^+\to S$. However, we usually write x_n ($n\in\mathbb{Z}^+$) instead of $x(n)$, and we often refer to "a sequence $\{x_n\}$". Thus, by an abuse of notation, we identify the sequence (which is technically a function) with its range. We call $x_1,x_2,x_3,\dots$ the terms of the sequence.

Note that if S is a countably infinite set, then there exists a bijection $x : \mathbb{Z}^+ \to S$, which can then be thought of as a sequence: $S = \{x_n\}$. We can say that S can be written as a sequence.

(In this example, the terms of $\{x_n\}$ are all distinct, since x is injective. In general, if we just say that $\{x_n\}$ is a sequence in S, there is no assumption that the terms are distinct.)

<u>Theorem</u> Every subset of a countable set is countable.

<u>Proof</u>: Let S be a countable set and let $T \subset S$.

<u>Case 1:</u> S is finite, say $S = \{x_1, x_2, \ldots, x_n\}$. Then $T = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$ for some $\{j_1, j_2, \ldots, j_m\} \subset \{1, 2, \ldots, n\}$, where $m \le n$. But then $T \sim \{1, 2, \ldots, m\}$, since $y : \{1, 2, \ldots, m\} \to T$, $y(i) = x_{n_i}$, is a bijection. Thus T is finite.

<u>Case 2:</u> S is countably infinite. Then there exists a bijection $x : \mathbb{Z}^+ \to S$ and thus we can consider S to be a sequence: $S = \{x_n\}$.

If T is finite, then there is nothing to prove. Otherwise, let us describe T inductively, as follow:

- Let j_1 be the smallest positive integer such that $x_{j_1} \in T$.
- Assume that $\{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\} \subset T$, where $j_1 < j_2 < \cdots < j_k$ and

  $$j < j_k,\ j \notin \{j_1, \ldots, j_k\} \Rightarrow x_j \notin T.$$

  Let j_{k+1} be the smallest integer greater that j_k such that $x_{j_{k+1}} \in T$.

This defines a sequence $\{x_{j_k}\}$; more specifically, it defines an surjection $f: \mathbb{Z}^+ \to S$, $f(k) = x_{j_k}$. The range of f is T (i.e. $T = \{x_{j_k}\}$), which shows that T is countably infinite. (How do we know that $T = \{x_{j_k}\}$? If $y \in T$, then $y = x_n$ for some $n \in \mathbb{Z}^+$. By construction, $j_n \geq n$, so $x_n \in T$ implies that $x_n = x_{j_k}$ for some $k \leq n$.) //