Ex: There are 5 balls numbered 1-5 & five boxes numbered 1-5. The balls are randomly placed in the boxes. What is the probability that exactly one box is empty?

$|\Omega| = 5^5$

choose empty box

combine 2 balls

place 2 in one of the boxes

place other 3 balls in the other boxes.

$|A| = \binom{5}{1}\binom{5}{2}\binom{4}{1}3!$ or $5!\binom{5}{2}$

$$P(A) = \frac{5!\binom{5}{2}}{5^5}$$

Ex: A teacher returns 10 graded exams to 10 students at random. What is the probability that at least one student gets their own exam?

$A_i = \{ i\text{th student gets the right exam} \}$

$$P(A_i) = \frac{1}{10}$$

$i_1, i_2 \in \{1, 2, \dots, 10\}$

$$P(A_{i_1} \cap A_{i_2}) = \frac{8!}{10!}$$

$$P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) = \frac{7!}{10!}$$

$\vdots$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \frac{(n-k)!}{n!}$$

$\bigcup_{i=1}^{10} A_i$ is the event that at least one student gets the correct exam.

By the inclusion - exclusion principle

$$P\left(\bigcup_{i=1}^{10} A_i\right) = \sum_{i=1}^{10} \overbrace{P(A_i)}^{\frac{1}{10}} - \sum_{i<j\leq 10} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k\leq 10} P(A_i \cap A_j \cap A_k) - \dots - P\left(\bigcap_{i=1}^{10} A_i\right)$$

Notice that

$$\sum_{i_1<i_2<\dots<i_k \leq 10} \overbrace{P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k})}^{= \frac{(n-k)!}{n!}} = \binom{n}{k} \frac{(n-k)!}{n!}$$

$$= \frac{1}{k!}$$

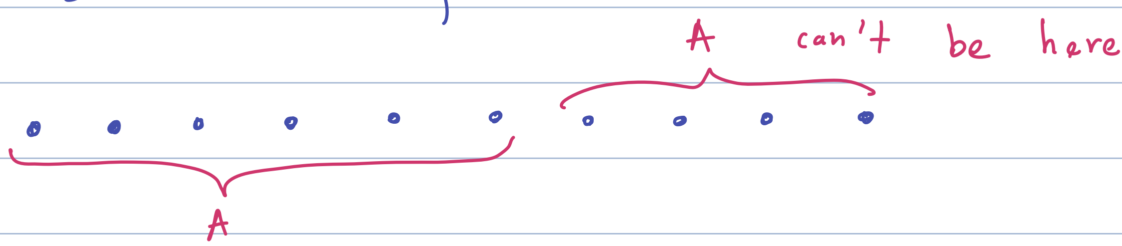
Therefore,

$$P\left(\bigcup_{i=1}^{10} A_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots - \frac{1}{10!}$$

Practice Quiz

① Suppose 10 students are standing in a row & A, B are two specific students in this group. What is the probability that exactly 3 students are standing between A & B?

If A stands ahead, there are $10 - 3 - 1 = 6$ places A can be in



Similarly, if B is ahead, B can be in 6 positions.

Due to symmetry

permutations of the other students.

$$\text{Anwer} = \frac{2 \times (10-3-1)\,(10-2)!}{10}$$

$$= \boxed{\frac{(2)(6)(8!)}{10}}$$

② A class contains exactly 5 seniors & 10 juniors. The class is divided randomly into 5 groups of three. Find the probability that each group will have exactly one senior.

$A = \{$ each group consists of exactly one senior $\}$

$$|\Omega| = \binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}$$

$$= \frac{15!}{(3!)^5}$$

$$|A| = \binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}\, 5! = \frac{10!}{(2!)^5} \cdot 5!$$

$$\boxed{P(A) = \frac{\frac{10!\,5!}{(2!)^5}}{\frac{15!}{(3!)^5}}}$$

Conditional Probability

Let $A, B \in \mathcal{F}$. The conditional probability that A occurs given that B occurred is denoted by $P(A|B)$ & is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0.$$

Ex: A fair coin is rolled twice. Find the probability that one of the numbers is a four given that the sum is even.

$B = \{\text{sum is } 7\} = \{(1,6), (6,1), (4,3), (3,4), (2,5), (5,2)\}$

$A = \{\text{one number is a four}\}$

$A \cap B = \{(4,3), (3,4)\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{6/36} = \boxed{\frac{1}{3}}$$

The Law of total probability

Let $A \in \mathcal{F}$. For every disjoint sequence $B_1, B_2, \ldots \in \mathcal{F}$ that covers Ω $\left(\bigcup_i B_i = \Omega\right)$,

$$P(A) = \sum_i P(A|B_i)\, P(B_i)$$

Proof: Notice that A be written as the union of disjoint sets :

$$A = \bigcup_i (A \cap B_i)$$

Therefore,

$$P(A) = \sum_i P(A \cap B_i)$$

$$= \sum_i P(A|B_i)\,P(B_i) \qquad \left(\text{Since } P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)} \right.$$

$\square$