Math 600 Lecture 8

Review: Let (X,d) be a metric space and let $E \subset X$.

- E is <u>open</u> iff $\forall x \in E\ \exists r>0,\ B_r(x) \subset E$.
- $y \in X$ is a <u>limit point of E</u> iff $\forall r>0$, $B_r(y)$ contains a point of E distinct from y.
- E is <u>closed</u> iff every limit point of E belongs to E. E' denotes the set of limit points of E, and the <u>closure</u> of E is $\overline{E} = E \cup E'$.
- E is closed iff E^c is open.
  E is open iff E^c is closed.
- An arbitrary union of open sets is open.
  An arbitrary intersection of closed sets is closed.
  A finite intersection of open sets is open.
  A finite union of closed sets is closed.
- If $Y \subset X$, then Y is a metric space under the same metric d.
- If $E \subset Y \subset X$, then E is <u>open relative to Y</u> iff E is open in the metric space Y. Similarly, E is <u>closed relative to Y</u> iff E is closed in the metric space Y.
- $E \subset Y \subset X$ is open relative to Y iff there exists an open set G in X such that $E = Y \cap G$.

  <u>Exercise</u>: $E \subset Y \subset X$ is closed relative to Y iff there exists a closed set F in X such that $E = Y \cap F$.

- E is <u>compact</u> iff every open cover of E has a finite subcover (i.e. iff, whenever $\{G_\alpha \mid \alpha \in A\}$ is a collection of open sets such that $E \subset \bigcup_{\alpha\in A} G_\alpha$, there exist $\alpha_1, \ldots, \alpha_n \in A$ such that $E \subset \bigcup_{j=1}^{n} G_{\alpha_j}$).
- If E is compact, then E is closed.
- If E is compact and $F \subset E$ is closed, then F is compact.
- If E is compact and $F \subset X$ is closed, then $E \cap F$ is compact
- <u>Exercise</u>: If E is compact, then E is bounded.
- $E \subset Y \subset X$ is compact relative to Y iff E is compact in the metric space Y.
- $E \subset Y \subset X$ is compact relative to Y iff E is compact relative to X.

<u>Theorem</u>: Let (X,d) be a metric space and suppose $\{E_\alpha \mid \alpha \in A\}$ is a collection of compact subsets of X with the property that the intersection of every finite subcollection of $\{E_\alpha \mid \alpha \in A\}$ is nonempty. Then

$$\bigcap_{\alpha \in A} E_\alpha \neq \emptyset.$$

<u>Proof</u>: We will prove the contrapositive. Suppose $\bigcap_{\alpha\in A} E_\alpha = \emptyset$. Choose any $\alpha_0 \in A$ and write $A' = A \setminus \{\alpha_0\}$. We have

$$E_{\alpha_0} \cap \left(\bigcap_{\alpha\in A'} E_\alpha\right) = \emptyset$$

$$\Rightarrow E_{\alpha_0} \subset \left(\bigcap_{\alpha \in A'} E_\alpha \right)^c$$

$$\Rightarrow E_{\alpha_0} \subset \bigcup_{\alpha \in A'} E_\alpha^c .$$

Since E_{α_0} is compact and E_α^c is open for all $\alpha \in A'$, there exist $\alpha_1, \dots, \alpha_n \in A'$ such that

$$E_{\alpha_0} \subset \bigcup_{j=1}^n E_{\alpha_j}^c$$

$$\Rightarrow E_{\alpha_0} \subset \left(\bigcap_{j=1}^n E_{\alpha_j} \right)^c$$

$$\Rightarrow \bigcap_{j=0}^n E_{\alpha_j} = \emptyset .$$

Thus there is a finite subcollection of $\{E_\alpha | \alpha \in A\}$ with an empty intersection //

<u>Corollary</u>: Let (X,d) be a metric space and suppose $\{E_n\}$ is a sequence of nonempty compact subsets of X such that $E_{n+1} \subset E_n \ \forall n \in \mathbb{Z}^+$. Then $\bigcap_{n \geq 1}^{\infty} E_n \neq \emptyset$.

<u>Theorem</u>: Let (X,d) be a metric space, let K be a compact subset of X, and let $E \subset K$ be an infinite set. Then E has a limit point in K.

<u>Proof:</u> We will prove the contrapositive. If no point of K is a limit point of E, then, for all $x \in K$, there exists $r_x > 0$ such that $B_{r_x}(x)$ contains at

most one point of E (specifically, $B_{r_x}(x) \cap E = \{x\}$ if $x \in E$ and $B_{r_x}(x) \cap E = \emptyset$ if $x \notin E$). But then $\{B_{r_x}(x) \mid x \in K\}$ is an open cover of K that cannot have a finite subcover (since the union of any finite subcollection cannot contain the infinite set E). Thus K is not compact. //

<u>Theorem</u>: Let $\{[a_n, b_n]\}$ be a sequence of closed intervals in $\mathbb{R}$ such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ $\forall n \in \mathbb{Z}^+$. Then $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

[Note: This does not follow from our earlier results because we don't yet know that $[a,b]$ is compact.]

<u>Proof</u>: Note that the hypothesis implies that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall n \in \mathbb{Z}^+.$$

Thus, in particular, $\{a_n\}$ is bounded above by b_1 and hence $a = \sup\{a_n\}$ exists in $\mathbb{R}$. Since $\{a_n\}$ is actually bounded above by every b_k, we see that

$$a \leq b_k \quad \forall k \in \mathbb{Z}^+$$

(a is the <u>least</u> upper bound of $\{a_n\}$). Also,

$$a_k \leq a \quad \forall k \in \mathbb{Z}^+$$

(a is an upper bound of $\{a_k\}$). Thus

$$a_k \leq a \leq b_k \quad \forall k \in \mathbb{Z}^+.$$

that is, $a \in \bigcap_{k=1}^{\infty} [a_k, b_k]$. //

<u>Definition</u>: Let $k \in \mathbb{Z}^+$. A <u>k-cell</u> is a subset of $\mathbb{R}^k$ of the form

$$\{x \in \mathbb{R}^k \mid a_j \le x_j \le b_j \ \forall j = 1, \ldots, k\},$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{R}$ are given real numbers with

$$a_j \le b_j \ \forall j = 1, \ldots, k.$$

<u>Theorem</u>: Let $k \in \mathbb{Z}^+$ and suppose $\{C_n\}$ is a sequence of k-cells satisfying $C_{n+1} \subset C_n \ \forall n \in \mathbb{Z}^+$. Then $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

<u>Proof</u>: Suppose

$$C_n = \{x \in \mathbb{R}^k \mid a_{n,j} \le x_j \le b_{n,j} \ \forall j = 1, \ldots, k\}.$$

Note that

$$C_{n+1} \subset C_n \Rightarrow a_{n,j} \le a_{n+1,j} \le b_{n+1,j} \le b_{n,j} \ \forall j = 1, \ldots, k$$

and thus

$$\forall j = 1, \ldots, k, \ \{[a_{n,j}, b_{n,j}]\} \text{ satisfies the previous theorem}$$

$$\Rightarrow \forall j = 1, \ldots, k, \ \exists x_j \in \bigcap_{n=1}^{\infty} [a_{n,j}, b_{n,j}]$$

But then

$$x \in \bigcap_{n=1}^{\infty} C_n$$

(since $a_{n,j} \le x \le b_{n,j} \ \forall j = 1, \ldots, k \ \forall n \in \mathbb{Z}^+$). //

Theorem: Let $k \in \mathbb{Z}^+$. Then every k-cell is a compact subset of $\mathbb{R}^k$.

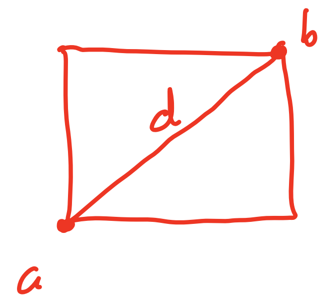
(In particular, every closed interval $[a,b]$ is a compact subset of $\mathbb{R}$.)

Proof: Let C be a k-cell and suppose

$$C = \{ x \in \mathbb{R}^k \mid a_j \le x_j \le b_j \ \forall j = 1, \dots, k \}.$$

Define

$$d = \|b - a\|_2 = \left[\sum_{j=1}^{k} |b_j - a_j|^2 \right]^{1/2}$$



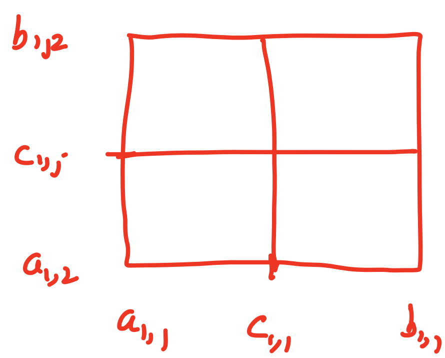
and note that

$$x, y \in C \Rightarrow \|y - x\|_2 \le d.$$

Let us argue by contradiction and assume that C is not compact. Then there exists an open cover $\{G_\alpha \mid \alpha \in A\}$ of C with no finite subcover.

Let us define

$$a_{1,j} = a_j, \ b_{1,j} = b_j, \ c_{1,j} = \frac{a_{1,j} + b_{1,j}}{2}.$$



The intervals

$$[a_{1,j}, c_{1,j}], [c_{1,j}, b_{1,j}], \ j = 1, \dots, k$$

define 2^k k-cells,

$$[a_{1,1}, c_{1,1}] \times [a_{1,2}, c_{1,2}] \times \dots \times [a_{1,k}, c_{1,k}],$$

$$[c_{1,1}, b_{1,1}] \times [a_{1,2}, c_{1,2}] \times \cdots \times [a_{1,k}, c_{1,k}],$$

$$\vdots$$

$$[c_{1,1}, b_{1,1}] \times [c_{1,2}, b_{1,2}] \times \cdots \times [c_{1,k}, b_{1,k}].$$

Each of these 2^k k-cells is covered by $\{G_\alpha \mid \alpha \in A\}$, and at least one of them cannot be covered by a finite subcollection. Call that k-cell C_1.

Now, we can subdivide C_1 in the same way into 2^k k-cells, and once again, one of them, call it C_2, cannot be covered by a finite subcollection of $\{G_\alpha \mid \alpha \in A\}$. Continuing in this way, we construct a sequence $\{C_n\}$ of k-cells with $C_{n+1} \subset C_n$. By an earlier theorem, there exists $x \in \bigcap_{n=1}^{\infty} C_n$. There must exist $\alpha' \in A$ such that $x \in G_{\alpha'}$ and, since $G_{\alpha'}$ is open, there exists $r > 0$ such that $B_r(x) \subset G_{\alpha'}$.

But, since the k-cells C_n are getting smaller and smaller — specifically, $u, v \in C_n \Rightarrow \|u - v\|_2 \leq 2^{-n} d$ — it follows that $C_n \subset B_r(x) \subset G_{\alpha'}$ for all $n \in \mathbb{Z}^+$ sufficiently large. This contradicts that no C_n can be covered by a finite subcollection of $\{G_\alpha \mid \alpha \in A\}$. //