Math 672 Lecture 7

<u>Definition</u>: Let V, W be vector spaces over F. We define

$$\mathcal{L}(V,W) = \{T: V \to W \mid T \text{ is linear}\}.$$

Given $S, T \in \mathcal{L}(V,W)$, we define $S+T: V \to W$ by

$$(S+T)(v) = S(v) + T(v) \quad \forall v \in V.$$

For $T \in \mathcal{L}(V,W)$, $\alpha \in F$, we define $\alpha T: V \to W$

$$(\alpha T)(v) = \alpha T(v) \quad \forall v \in V.$$

<u>Lemma</u>:

1. $S, T \in \mathcal{L}(V,W) \Rightarrow S+T \in \mathcal{L}(V,W)$
2. $T \in \mathcal{L}(V,W), \alpha \in F \Rightarrow \alpha T \in \mathcal{L}(V,W)$
3. $\mathcal{L}(V,W)$, with the operations defined above, is a vector space over F.

<u>Proof</u>:

1. Let $S, T \in \mathcal{L}(V,W)$. By definition, $S+T$ is a map from V into W, so we need only prove that $S+T$ is linear. We have

$$(S+T)(\alpha u + \beta v) = S(\alpha u + \beta v) + T(\alpha u + \beta v) \quad \text{(by definition of } S+T\text{)}$$

$$= \alpha S(u) + \beta S(v) + \alpha T(u) + \beta T(v) \quad \text{(since } S \text{ and } T \text{ are linear)}$$

$$= \alpha\left(S(u)+T(u)\right) + \beta\left(S(v)+T(v)\right)$$

$$= \alpha\,(S+T)(u) + \beta\,(S+T)(v) \quad \text{(by definition of } S+T\text{)}.$$

This holds for all $u, v \in V$ and all $\alpha, \beta \in F$; hence $S+T$ is linear.

2. Let $T \in \mathcal{L}(V,W)$ and $\alpha \in F$. By definition, αT maps V into W. We have

$$(\alpha T)(\beta_1 v_1 + \beta_2 v_2) = \alpha\, T(\beta_1 v_1 + \beta_2 v_2) \quad \text{(by definition of } \alpha T\text{)}$$

$$= \alpha\left(\beta_1 T(v_1) + \beta_2 T(v_2)\right) \quad \text{(since } T \text{ is linear)}$$

$$= \beta_1\left(\alpha T(v_1)\right) + \beta_2\left(\alpha T(v_2)\right) \quad \text{(various vector space operations)}$$

$$= \beta_1\,(\alpha T)(v_1) + \beta_2\,(\alpha T)(v_2) \quad \text{(by definition of } \alpha T\text{)}.$$

This holds for all $v_1, v_2 \in V$ and all $\beta_1, \beta_2 \in F$, so αT is linear.

3. All of the vector space properties are easy to verify. Note that the zero operator ($0: V \to W$, $0(v) = 0_W \ \forall v \in V$) is an element of $\mathcal{L}(V,W)$ and is the additive identity of $\mathcal{L}(V,W)$. For each $T \in \mathcal{L}(V,W)$, $-T$ is defined by $(-T)(v) = -T(v) \ \forall v \in V$. //

Note: I am following the book by defining $\mathcal{L}(V,W)$ to be the set of all linear maps $T: V\to W$, even if V and W are infinite-dimensional. However, this is not standard; usually (almost universally, I think)

$$\mathcal{L}(V,W) = \{T: V\to W \mid T \text{ is linear and continuous}\}.$$

When V, W are finite-dimensional, every linear map $T: V\to W$ is continuous, so there is no harm in omitting this condition.

In the case of $\mathcal{L}(V,V)$, we can also define multiplication:

$$ST = S\circ T \quad \forall\, S,T\in \mathcal{L}(V,V),$$

that is, ST is defined by

$$(ST)(v) = S(T(v)) \quad \forall\, v\in V.$$

Example: Multiplication on $\mathcal{L}(V,V)$ is not commutative.

Let $D: \mathcal{P}\to\mathcal{P}$ be the derivative operator and define $T:\mathcal{P}\to\mathcal{P}$ by $T(a_0+a_1 x+\cdots+a_n x^n) = a_0$. Then, if $p(x)=x$, we have

$$Dp = 1,\quad (TD)(p) = 1,$$

$$Tp = 0,\quad (DT)(p) = 0 \neq (TD)(p).$$

Lemma: Multiplication on $\mathcal{L}(V,V)$ is associative and distributive over addition:

$$(RS)T = R(ST) \quad \forall R,S,T \in \mathcal{L}(V,V),$$

$$R(S+T) = RS+RT \quad \forall R,S,T \in \mathcal{L}(V,V),$$

$$(S+T)R = SR+TR \quad \forall R,S,T \in \mathcal{L}(V,V).$$

Also, the identity operator is a multiplicative identity for $\mathcal{L}(V,V)$.

Proof: Let $R,S,T \in \mathcal{L}(V,V)$. Then, for $v \in V$,

$$((RS)T)(v) = (RS)(T(v)) = R(S(T(v))),$$

$$(R(ST))(v) = R((ST)(v)) = R(S(T(v))),$$

which shows that $((RS)T)(v) = (R(ST))(v)$ $\forall v \in V$, that is, $(RS)T = R(ST)$.

Next, again for $v \in V$,

$$\begin{aligned}(R(S+T))(v) &= R((S+T)(v)) = R(S(v)+T(v)) \\ &= R(S(v)) + R(T(v)) \quad \text{(by linearity of } R\text{)} \\ &= (RS)(v) + (RT)(v) \\ &= (RS+RT)(v).\end{aligned}$$

Thus $(R(S+T))(v) = (RS+RT)(v)$ $\forall\, v \in V$, that is,

$R(S+T) = RS+RT$.

The proof that $(S+T)R = SR+TR$ is similar (a separate proof is needed because multiplication is not commutative). //

Note that we can also define ST for $T \in \mathcal{L}(V,U)$ and $S \in \mathcal{L}(U,W)$:

$$ST \in \mathcal{L}(V,W)$$

$$(ST)(v) = S(T(v)) \quad \forall\, v \in V.$$

We still have

$$(RS)T = R(ST) \quad \forall\, T \in \mathcal{L}(V,U), S \in \mathcal{L}(U,W), R \in \mathcal{L}(W,Z),$$

$$R(S+T) = RS+RT \quad \forall\, S,T \in \mathcal{L}(V,U)\ \forall R \in \mathcal{L}(U,W),$$

$$(S+T)R = SR+TR \quad \forall\, R \in \mathcal{L}(V,U)\ \forall\, S,T \in \mathcal{L}(U,W).$$

<u>Definition</u>: Let $T \in \mathcal{L}(V,W)$.

- The <u>null space</u> (or <u>kernel</u>) of T is the set

$$\mathcal{N}(T) = \{v \in V \mid T(v) = 0\}.$$

- The <u>range</u> (or <u>image</u>) of T is the set

$$\mathcal{R}(T) = \{T(v) \mid v \in V\} = \{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

<u>Theorem</u>: Let $T \in \mathcal{L}(V,W)$. Then $\mathcal{N}(T)$ is a subspace of V and $\mathcal{R}(T)$ is a subspace of W.

<u>Proof</u>: In both cases, we must verify that the three properties of a subspace are satisfied. We know that $T(0) = 0$ $(T(0_V) = 0_W)$, which implies that $0_V \in \mathcal{N}(T)$ and $0_W \in \mathcal{R}(T)$.

Suppose $u, v \in \mathcal{N}(T)$; then we know that $T(u) = T(v) = 0$. We must show that $u+v \in \mathcal{N}(T)$, that is, that $T(u+v) = 0$. But, by linearity of T,

$$T(u+v) = T(u) + T(v) = 0 + 0 = 0.$$

Thus $u+v \in \mathcal{N}(T)$, and $\mathcal{N}(T)$ is closed under scalar multiplication.

Now suppose $u \in \mathcal{N}(T)$ and $\alpha \in F$. Then $T(u) = 0$ and we must show that $\alpha u \in \mathcal{N}(T)$, that is, that $T(\alpha u) = 0$. But

$$T(\alpha u) = \alpha T(u) = \alpha \cdot 0 = 0,$$

and thus $\alpha u \in \mathcal{N}(T)$. This shows that T is closed under scalar multiplication.

Now suppose $x, y \in \mathcal{R}(T)$. Then, by definition of $\mathcal{R}(T)$, there exist $u, v \in V$ such that $T(u) = x$ and $T(v) = y$. We must show that $x+y \in \mathcal{R}(T)$, that is, that $x+y = T(z)$ for some $z \in V$. But

$$x+y = T(u) + T(v) = T(u+v);$$

thus $x+y = T(z)$ for $z = u+v$. This shows that $\mathcal{R}(T)$ is closed under addition.

Finally, if $x \in \mathcal{R}(T)$ and $\alpha \in F$, then $x = T(u)$ for some $u \in V$, and

$$\alpha x = \alpha T(u) = T(\alpha u).$$

This shows that $\alpha x \in \mathcal{R}(T)$, and thus that $\mathcal{R}(T)$ is closed under scalar multiplication. The proof is complete. //

A simple consequence: Suppose $T: V \to W$, where V and W are finite dimensional, and $\mathcal{R}(T)$ is a proper subspace of W. Suppose we wish to solve $T(v) = w$ for $v \in V$, given some $w \in W$. Does the equation have a solution?

<u>Answer</u>: Probably not: $T(v) = w$ has a solution iff $w \in \mathcal{R}(T)$. Since $\mathcal{R}(T)$ is a proper subspace of W, it makes up a very small part of W (think of a line – a 1D subspace – in $\mathbb{R}^2$ or $\mathbb{R}^3$, or a plane – a 2D subspace – in $\mathbb{R}^3$).