# Math 672 Lecture 14

For the rest of the course, we will study linear operators — linear maps of the form $T: V \to V$ — on a finite-dimensional vector space $V$. The goal is to understand the "structure" of a linear operator in as much detail as possible.

<u>Definition</u>: Let $V$ be a vector space over a field $F$ and let $T \in \mathcal{L}(V,V)$. We say that a subspace $U$ of $V$ is <u>invariant under $T$</u> iff

$$T(u) \in U \quad \forall\, u \in U.$$

What is the significance of this concept?

Suppose $T \in \mathcal{L}(V,V)$, $U$ is a subspace of $V$ that is invariant under $T$, and $\{v_1, \ldots, v_k\}$ is a basis for $U$. Extend $\{v_1, \ldots, v_k\}$ to a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for $V$, and consider $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$.

Let $v \in V$ and define $x = \mathcal{M}_{\mathcal{B}}(v)$, $y = \mathcal{M}_{\mathcal{B}}(T(v))$:

$$v = \sum_{i=1}^{n} x_i v_i$$

$$T(v) = \sum_{i=1}^{n} y_i v_i = \sum_{i=1}^{n} (Ax)_i v_i$$

(since $\mathcal{M}_{\mathcal{B}}(T(v)) = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)\mathcal{M}_{\mathcal{B}}(v) \iff y = Ax$), or

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i$$

(since $\mathcal{M}_{\mathcal{B}}(v_j) = e_j$ and $Ae_j = A_j$, the jth column of $A$).

Thus,

$$A_{ij} \neq 0 \iff v_i \text{ is needed to represent } T(v_j)$$

In general, $A$ is "dense," meaning that most or all values $A_{ij}$ are nonzero. But since $U$ is invariant under $T$,

$$T(v_1), \ldots, T(v_k) \text{ depend only on } v_1, \ldots, v_k,$$

that is,

$$v_{k+1}, \ldots, v_n \text{ are not needed to represent } T(v_1), \ldots, T(v_k).$$

Thus

$$A_{ij} = 0 \quad \forall j = 1, \ldots, k,\ i = k+1, \ldots, n$$

and $A$ has the form

$$A = \left[\begin{array}{ccc|ccc} A_{11} & \cdots & A_{1k} & A_{1,k+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \\ A_{k1} & \cdots & A_{kk} & A_{k,k+1} & \cdots & A_{k,n} \\ \hline 0 & \cdots & 0 & A_{k+1,k+1} & \cdots & A_{k+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & A_{n,k+1} & \cdots & A_{nn} \end{array}\right] = \left[\begin{array}{c|c} A^{(1,1)} & A^{(1,2)} \\ \hline 0 & A^{(2,2)} \end{array}\right].$$

If we can arrange it that $W = \mathrm{span}(v_{k+1}, \ldots, v_n)$ is also invariant under $T$, then

$$T(v_{k+1}), \ldots, T(v_n) \text{ depend only on } v_{k+1}, \ldots, v_n,$$

that is,

$$v_1, \ldots, v_k \text{ are not needed to represent } T(v_{k+1}), \ldots, T(v_n),$$

that is,

$$A_{ij} = 0 \quad \forall\, i = 1, \ldots, k,\ j = k+1, \ldots, n$$

In this case,

$$A = \left[\begin{array}{c|c} A^{(1,1)} & 0 \\ \hline 0 & A^{(2,2)} \end{array}\right].$$

Hopefully you can see that any linear algebraic question would be easier to answer with such a matrix.

<u>Definition</u>: Let $V$ be a vector space over a field $F$, and let $T \in \mathcal{L}(V,V)$. We say that $\lambda \in F$ is an <u>eigenvalue</u> of $T$ iff there exists $v \in V$ such that

$$T(v) = \lambda v \text{ and } v \neq 0.$$

In this case, we say that $v$ is an eigenvector of $T$ corresponding to the eigenvalue $\lambda$.

Note that if $T(v)=\lambda v$, then

$$T(\alpha v) = \alpha T(v) = \alpha(\lambda v) = \lambda(\alpha v).$$

Therefore, if $v$ is an eigenvector of $T$ corresponding to $\lambda$, then so is every nonzero multiple of $v$.

Note also that

$$\begin{aligned} T(v) = \lambda v &\iff T(v) - \lambda v = 0 \\ &\iff (T-\lambda I)(v) = 0 \quad (\text{where } I: V \to V \text{ is the identity operator}) \\ &\iff v \in \mathcal{N}(T-\lambda I). \end{aligned}$$

We call

$$E(\lambda, T) = \mathcal{N}(T-\lambda I)$$

the eigenspace of $T$ corresponding to $\lambda$.

It is obviously an invariant subspace of $T$:

$$v \in \mathcal{N}(T-\lambda I) \Rightarrow T(v) = \lambda v$$
$$\Rightarrow T(T(v)) = T(\lambda v) = \lambda T(v)$$
$$\Rightarrow (T-\lambda I)(T(v)) = 0$$
$$\Rightarrow T(v) \in \mathcal{N}(T-\lambda I).$$

Note that ever vector in $E(\lambda,T)$, except $0$, is an eigenvector of $T$ corresponding to $\lambda$.

Note also that, if $\lambda$ is an eigenvalue of $T$, then $E(\lambda,T)$ is nontrivial ($\dim(E(\lambda,T)) \geq 1$).

**Lemma:** Let $T \in \mathcal{L}(V,V)$. Then $\{0\}$, $V$, $\mathcal{N}(T)$, and $\mathcal{R}(T)$ are all invariant under $T$.

**Proof:** Since $T(0)=0$, $\{0\}$ is invariant under $T$, and $V$ is invariant under $T$ since $T: V \to V$ ($T(v) \in V \ \forall v \in V$). We have

$$T(v) = 0 \Rightarrow T(T(v)) = T(0) = 0$$
$$\Rightarrow T(v) \in \mathcal{N}(T).$$

Thus $\mathcal{N}(T)$ is invariant under $T$. Finally, $T(v) \in \mathcal{R}(T)$ for all $v \in V$ and hence for all $v \in \mathcal{R}(T)$. Thus $\mathcal{R}(T)$ is invariant under $T$. //

**Definition:** Let $T \in \mathcal{L}(V,W)$. We say that $T$ is singular iff there is a nonzero solution to $T(v)=0$ (that is, iff $\mathcal{N}(T)$ is nontrivial). We say that $T$ is nonsingular iff $T(v)=0$ has only the zero solution (that is, iff $\mathcal{N}(T)$ is trivial).

**Note:** If $T \in \mathcal{L}(V,V)$ and $V$ is finite-dimensional, then

$$T \text{ is nonsingular} \iff T \text{ is invertible.}$$

This holds also if $T \in \mathcal{L}(V,W)$ and $\dim(V)=\dim(W)$.

For $T \in \mathcal{L}(V,W)$ ($\dim(W) \neq \dim(V)$), "nonsingular" and "invertible" are not equivalent.

**Theorem:** Let $T \in \mathcal{L}(V,V)$, where $V$ is finite dimensional, and let $\lambda \in F$. Then the following are equivalent:

- $\lambda$ is an eigenvalue of $T$
- $T-\lambda I$ is singular
- $T-\lambda I$ is not injective
- $T-\lambda I$ is not surjective
- $T-\lambda I$ is not invertible.