Math 672 Lecture 33

Theorem: Let V be a finite-dimensional vector space over $\mathbb{C}$, let $T \in \mathcal{L}(V)$, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T, and let

$$G(\lambda_j, T) = \mathcal{N}((T-\lambda_j I)^{r_j}),$$

where r_j is the smallest positive integer for which this holds (that is, where $\mathcal{N}((T-\lambda_j I)^{r_j-1}) \subsetneq \mathcal{N}((T-\lambda_j I)^{r_j})$). Then

$$m_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}.$$

Proof: Choose a basis $\mathcal{B}$ for V such that $J = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is in Jordan form: $J = \operatorname{diag}(J_1, J_2, \ldots, J_\ell)$, where each J_i is a Jordan block. For $p \in \mathcal{P}(\mathbb{C})$, it is easy to show that

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(p(T)) = p(J).$$

($\mathcal{M}_{\mathcal{B},\mathcal{B}}$ is a linear map, so it suffices to prove that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T^j) = J^j$ for all $j \geq 0$. But $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T^0) = \mathcal{M}_{\mathcal{B},\mathcal{B}}(I) = I = J^0$, $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T) = J$,

$$\mathcal{M}_{\mathcal{B}}(T^2(v)) = \mathcal{M}_{\mathcal{B}}(T(T(v))) = J\mathcal{M}_{\mathcal{B}}(T(v))$$
$$= J^2\mathcal{M}_{\mathcal{B}}(v) \quad \forall v \in V$$

$$\Rightarrow \mathcal{M}_{\mathcal{B},\mathcal{B}}(T^2) = J^2,$$

etc.) Also, it is easy to show that

$$p(J) = \mathrm{diag}(p(J_1), p(J_2), \ldots, p(J_\ell))$$

(similarly, it suffices to prove that $J^j = \mathrm{diag}(J_1^j, J_2^j, \ldots, J_\ell^j)$).

Now, if J_t is a Jordan block corresponding to λ_j and $\lambda \neq \lambda_j$, then $J_t - \lambda I$ is nonsingular (an upper triangular matrix with nonzeros on the diagonal); hence

$$(J_t - \lambda_i I)^c \text{ is nonsingular } \forall c \geq 0 \ \forall i \neq j.$$

Now, since λ is a root of m_T iff λ is an eigenvalue of T, and since m_T can be fully factored over $\mathbb{C}$, we have

$$m_T(x) = (x-\lambda_1)^{c_1}(x-\lambda_2)^{c_2}\cdots(x-\lambda_k)^{c_k} \text{ for some } c_1, \ldots, c_k \in \mathbb{Z}^+.$$

It follows that (still assuming that J_t corresponds to λ_j)

$$m_T(J_t) = \prod_{i=1}^{k}(J - \lambda_i I)^{c_i}$$
$$= \left(\prod_{\substack{i=1\\ i\neq j}}^{k}(J - \lambda_i I)^{c_i}\right)(J - \lambda_j I)^{c_j},$$

so

$$m_T(J_t) = 0 \iff \left(\prod_{\substack{i=1\\ i\neq j}}^{k}(J_t - \lambda_i I)^{c_i}\right)(J_t - \lambda_j I)^{c_j}$$

$$\Longrightarrow (J_t - \lambda_j I)^{c_j} = 0 \quad \Big(\text{since } \prod_{\substack{i=1 \\ i \neq j}}^{k} (J_t - \lambda_i I)^{c_i}$$

is invertible$\Big)$

$$\Longrightarrow c_j \geq r_j.$$

Since m_T has the minimal degree of any polynomial satisfying $m_T(T) = 0$, we must have $c_j = r_j$ for all j, that is,

$$m_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2} \cdots (x-\lambda_k)^{r_k}. \quad //$$

<u>Definition</u>: Let V be a finite-dimensional vector space over $\mathbb{C}$, let $T \in \mathcal{L}(V)$, let $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ be the distinct eigenvalues of T, and let m_j be the algebraic multiplicity of λ_j, $j = 1, 2, \ldots, k$ ($m_j = \dim(G(\lambda_j, T))$). The <u>characteristic polynomial</u> of T is the polynomial

$$p_T(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k}.$$

<u>Theorem</u> (the Cayley-Hamilton theorem): Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Then $p_T(T) = 0$.

<u>Proof</u>: Since $p(T) = 0$ iff $p(x)$ is a multiple of $m_T(x)$, it suffices to prove that $m_T(x) \mid p_T(x)$. But

$$m_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2} \cdots (x-\lambda_k)^{r_k},$$

$$p_T(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k},$$

and $r_j \le m_j$ for $j=1,2,\ldots,k$ (why?), and the result follows:

$$p_T(x) = m_T(x)\,(x-\lambda_1)^{m_1-r_1}(x-\lambda_2)^{m_2-r_2}\cdots(x-\lambda_k)^{m_k-r_k}. \quad //$$

(b) Suppose V is a finite dimensional complex vector space and $T : V \to V$ is linear with $(x-3)^2(x-9)$ as its minimal polynomial and $(x-3)^5(x-9)^2$ as its characteristic polynomial.

(i) [**4 points**] What are the eigenvalues of T and what are the dimensions of the corresponding generalized eigenspaces?

(ii) [**6 points**] What are the possible Jordan canonical forms of T and what does this tell us about the dimensions of the eigenspaces?

(i) The eigenvalues of T are $\lambda_1 = 3$, $\lambda_2 = 9$, with

$$\dim(G(\lambda_1, T)) = 5,$$

$$\dim(G(\lambda_2, T)) = 2.$$

(ii) We can write

$$J = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right],$$

where $B_1 \in \mathbb{C}^{5\times5}$ corresponds to $G(\lambda_1, T)$ and $B_2 \in \mathbb{C}^{2\times2}$ corresponds to $G(\lambda_2, T)$. Since $r_2 = 1$ $(m_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2} = (x-3)^2(x-9))$, B_2 must be simply $B_2 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ (the largest Jordan block is 1×1).

For B_1, there are two choices. There must be at least one 2×2 block (since $r_1 = 2$). To fill out the five dimensions, there could be a second 2×2 block and a 1×1 block, or there could be three 1×1 blocks:

$$B_1 = \left[\begin{array}{cc|cc|c} 3 & 1 & & & \\ & 3 & & & \\ \hline & & 3 & 1 & \\ & & & 3 & \\ \hline & & & & 3 \end{array}\right] \quad \text{or} \quad B_1 = \left[\begin{array}{cc|c|c|c} 3 & 1 & & & \\ & 3 & & & \\ \hline & & 3 & & \\ \hline & & & 3 & \\ \hline & & & & 3 \end{array}\right]$$

Thus

$$J = \begin{bmatrix} 3 & 1 & & & & & \\ & 3 & & & & & \\ & & 3 & 1 & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 9 & \\ & & & & & & 9 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} 3 & 1 & & & & & \\ & 3 & & & & & \\ & & 3 & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 9 & \\ & & & & & & 9 \end{bmatrix}$$

$\dim(\mathcal{N}(T-3I)) = 3$

$\dim(\mathcal{N}((T-3I)^2)) = 5$

$\dim(\mathcal{N}(T-9I)) = 2$

$\dim(\mathcal{N}(T-3I)) = 4$

$\dim(\mathcal{N}((T-3I)^2)) = 5$

$\dim(\mathcal{N}(T-9I)) = 2$

6. (a) Find all possible Jordan forms for a real matrix whose characteristic polynomial is $p(X) = (X-1)^4(X-2)^2$ and minimal polynomial is $m(X) = (X-1)^2(X-2)^2$.

We have $\lambda_1 = 4$, $\lambda_2 = 2$ with

$$\dim(G(\lambda_1, T)) = 4, \quad \dim(G(\lambda_2, T)) = 2.$$

Thus

$$J = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right] \in \mathbb{R}^{6\times 6},$$

where $B_1 \in \mathbb{R}^{4\times 4}$, $B_2 \in \mathbb{R}^{2\times 2}$. Note that $r_1 = 2$, $r_2 = 2$ (the largest Jordan block for both λ_1 and λ_2 is 2). Thus B_2 must be

$$B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

There are two possibilities for B_1:

$$B_1 = \left[\begin{array}{cc|cc} 4 & 1 & & \\ & 4 & & \\ \hline & & 4 & 1 \\ & & & 4 \end{array}\right] \quad \text{or} \quad B_1 = \left[\begin{array}{cc|c|c} 4 & 1 & & \\ & 4 & & \\ \hline & & 4 & \\ \hline & & & 4 \end{array}\right]$$

Thus

$$J = \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & 1 & & \\ & & & 4 & & \\ & & & & 2 & 1 \\ & & & & & 2 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 2 & 1 \\ & & & & & 2 \end{bmatrix}$$

$\dim(\mathcal{N}(T-4I)) = 2$

$\dim(\mathcal{N}((T-4I)^2)) = 4$

$\dim(\mathcal{N}(T-2I)) = 2$

$\dim(\mathcal{N}(T-4I)) = 3$

$\dim(\mathcal{N}((T-4I)^2)) = 4$

$\dim(\mathcal{N}(T-2I)) = 2$

Assume there exists
(a) ~~Find~~ a basis C for $\mathcal{V}$ such that the matrix associated to T is

$$[T]_C = \begin{bmatrix} 1 & & & & & \\ & 2 & 1 & & & \\ & & 2 & 1 & & \\ & & & 2 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}$$

(b) What are the characteristic polynomial and minimal polynomial for T?

(c) What are the eigenspaces and their dimensions?

$m_T(x) = (x-1)(x-2)^3(x-3)^2, \quad p_T(x) = m_T(x)$

Assuming $C = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, we have

$E(1,T) = \operatorname{span}(v_1)$

$E(2,T) = \operatorname{span}(v_2)$

$E(3,T) = \operatorname{span}(v_5)$

All three eigenspaces have dimension 1.

$$\dim(\mathcal{N}(T-I)) = 1$$
$$\dim(\mathcal{N}(T-2I)) = 1$$
$$\dim(\mathcal{N}((T-2I)^2)) = 2$$
$$\dim(\mathcal{N}((T-2I)^3)) = 3$$
$$\dim(\mathcal{N}(T-3I)) = 1$$
$$\dim(\mathcal{N}((T-3I)^2)) = 2$$