Math 600 Lecture 32

<u>Recall:</u> If $f_n : E \to Y$ $\forall n \in \mathbb{Z}^+$, $f : E \to Y$ ($E \subset X$), then $f_n \to f$ uniformly on E iff

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z}^+ \ (n \geq N \text{ and } x \in E) \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon.$$

<u>Theorem</u>: Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $E \subset X$, and assume that for all $n \in \mathbb{Z}^+$, $f_n : E \to Y$ is a continuous function. If $f_n \to f$ uniformly on E, where $f : E \to Y$, then f is continuous on E.

<u>Proof:</u> Let $x \in E$ and let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \text{ and } u \in E \Rightarrow d_Y(f_n(u), f(u)) < \frac{\varepsilon}{3}.$$

In particular,

$$\forall u \in E, \ d_Y(f_N(u), f(u)) < \frac{\varepsilon}{3}.$$

By assumption, f_N is continuous. Hence there exists $\delta > 0$ such that

$$(u \in E \text{ and } d_X(u,x) < \delta) \Rightarrow d_Y(f_N(u), f_N(x)) < \frac{\varepsilon}{3}.$$

But then

$$\begin{aligned}(u \in E \text{ and } d_X(u,x) < \delta) \Rightarrow d_Y(f(u), f(x)) &\leq d_Y(f(u), f_N(u)) + d_Y(f_N(u), f_N(x)) + \\ &\quad d_Y(f_N(x), f(x)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus f is continuous at x. Since $x \in E$ was arbitrary, it follows that f is continuous on E. //

<u>Theorem</u>: For each $n \in \mathbb{Z}^+$, let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable and suppose $f_n \to f$ uniformly on $[a,b]$, where $f : [a,b] \to \mathbb{R}$. Then f is Riemann integrable on $[a,b]$ and

$$\int_a^b f_n \to \int_a^b f.$$

<u>Proof</u>: First, we prove that f is Riemann integrable on $[a,b]$. Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \quad \forall x \in [a,b].$$

In particular,

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \quad \forall x \in [a,b].$$

Since f_N is Riemann integrable on $[a,b]$, there exists a partition $P = \{x_0, \ldots, x_n\}$ on P such that

$$U(P, f_N) - L(P, f_N) < \frac{\varepsilon}{3}.$$

But

$$U(P,f) = \sum_{j=1}^{n} \sup\{f(x) \mid x_{j-1} \leq x \leq x_j\} \Delta x_j$$

$$< \sum_{j=1}^{n} \left(\sup\{f_N(x) \mid x_{j-1} \leq x \leq x_j\} + \frac{\varepsilon}{3(b-a)} \right) \Delta x_j$$

$$= U(P,f_N) + \frac{\varepsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j$$

$$= U(P,f_N) + \frac{\varepsilon}{3},$$

Similarly,

$$L(P,f) > L(P,f_N) - \frac{\varepsilon}{3}.$$

Therefore,

$$U(P,f) - L(P,f) < \left(U(P,f_N) + \frac{\varepsilon}{3}\right) - \left(L(P,f_N) - \frac{\varepsilon}{3}\right)$$

$$= U(P,f_N) - L(P,f_N) + \frac{2\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus f is Riemann integrable on $[a,b]$.

Now we show that $\int_a^b f_n \to \int_a^b f$. Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow \left(|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall x \in [a,b] \right).$$

But then

$$n \geq N \Rightarrow \left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\varepsilon}{b-a} = \varepsilon.$$

This completes the proof. //

The above theorems show why uniform convergence is so powerful. We now give some technical results that are useful for verifying uniform convergence.

Theorem: Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $E \subset X$ and let $f_n : E \to Y$, $n = 1, 2, 3, \ldots$, and $f : E \to Y$ be given functions. Define

$$M_n = \sup\{d_Y(f_n(x), f(x)) \mid x \in E\}.$$

Then $f_n \to f$ uniformly on E iff $M_n \to 0$.

Proof: Immediate from the definition of uniform convergence. //

Theorem (the Cauchy criterion for uniform convergence): Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $E \subset X$, and let $f_n : E \to Y$, $n = 1, 2, 3, \ldots$, be given functions. Then, if $\{f_n\}$ converges uniformly on E, then

$$(*) \quad \forall \varepsilon > 0 \ \exists N \in \mathbb{Z}^+ \ (m, n \geq N \Rightarrow (d_Y(f_m(x), f_n(x)) < \varepsilon \ \forall x \in E)).$$

If Y is complete, then the converse holds.

Proof: Suppose first that $\{f_n\}$ converges uniformly on E, say $f_n \to f$ uniformly on E. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow (d_Y(f_n(x), f(x)) < \tfrac{\varepsilon}{2} \ \forall x \in E).$$

But then

$$m, n \geq N \Rightarrow (d_Y(f_m(x), f_n(x)) \leq d_Y(f_m(x), f(x)) + d_Y(f(x), f_n(x)) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \ \forall x \in E).$$

Thus $(*)$ holds.

Conversely, suppose $(*)$ holds and Y is complete. Then $(*)$ implies that $\{f_n(x)\}$ is a Cauchy sequence for each $x \in E$, and hence converges (since Y is complete). Define $f: E \to Y$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in E.$$

Let $\varepsilon > 0$ be given. By (*), there exists $N \in \mathbb{Z}^+$ such that

$$m, n \geq N \Rightarrow \left(d_Y(f_m(x), f_n(x)) < \frac{\varepsilon}{2} \ \forall x \in E \right).$$

We claim that

$$n \geq N \Rightarrow \left(d_Y(f_n(x), f(x)) < \varepsilon \ \forall x \in E \right).$$

To see this, let $n \geq N$ be fixed. For any $x \in E$, there exists $N_x \geq N$ such that

$$m \geq N_x \Rightarrow d_Y(f_m(x), f(x)) < \frac{\varepsilon}{2}.$$

But then

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f_{N_x}(x)) + d_Y(f_{N_x}(x), f(x))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (*) holds, and the proof is complete. //

A series $\sum_{n=1}^{\infty} f_n$ of (real-valued) functions converges uniformly iff the sequence $\left\{ \sum_{n=1}^{N} f_n \right\}$ converges uniformly.

<u>Theorem</u> (the Weierstrass M-test): Let $E \subset \mathbb{R}$ and suppose $f_n : E \to \mathbb{R}$ is a given function for each $n \in \mathbb{Z}^+$. If there exists a sequence $\{M_n\}$ of nonnegative real numbers such that

$$\sum_{n=1}^{\infty} M_n \quad \text{converges}$$

and

$$|f_n(x)| \leq M_n \quad \forall x \in E \quad \forall n \in \mathbb{Z}^+,$$

then

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on E.

<u>Proof:</u> Let $\{S_n\}$ denote the sequence of partial sums:

$$S_n(x) = \sum_{k=1}^{n} f_k(x) \quad \forall x \in E.$$

Let $\varepsilon > 0$ be given. Then, since $\sum_{k=1}^{\infty} M_k$ converges, there exists $N \in \mathbb{Z}^+$ such that

$$m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} M_k \right| < \varepsilon = \sum_{k=n}^{m} M_k < \varepsilon$$

(the Cauchy criterion for series). But then

$$m, n \geq N \Rightarrow |S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k < \varepsilon \quad \forall x \in E,$$

and hence $\{S_n\}$ is uniformly Cauchy on E. It follows that $\{S_n\}$

converges uniformly on E, that is, $\sum_{n=1}^{\infty} f_n$ converges uniformly on E. //