Math 600 Lecture 4

Let A, B, C be sets and consider the following:

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Are both always true? Is one of them always true?

Definitions

- If A, B are sets, then the union of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

and the intersection of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- If $A_1, A_2, \dots, A_n$ are sets, then

$$\bigcup_{j=1}^{n} A_j = \{x \mid \exists\, j \in \{1, \dots, n\},\ x \in A_j\},$$

$$\bigcap_{j=1}^{n} A_j = \{x \mid x \in A_j\ \forall j = 1, \dots, n\}.$$

- Similarly, if $\{A_n\}$ is an (infinite) sequence of sets, then

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid \exists n \in \mathbb{Z}^+, x \in A_n\},$$

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \ \forall n \in \mathbb{Z}^+\}.$$

- Finally, if A is any set and, for each $\alpha \in A$, E_α is a set, then

$$\bigcup_{\alpha \in A} E_\alpha = \{x \mid \exists \alpha \in A, x \in E_\alpha\},$$

$$\bigcap_{\alpha \in A} E_\alpha = \{x \mid x \in A_\alpha \ \forall \alpha \in A\}.$$

(This last notation allows us to refer to the union of an uncountable collection of sets.)

<u>Examples</u>

1. $\forall n \in \mathbb{Z}^+$, define $A_n = [\frac{1}{n}, \infty) = \{x \in \mathbb{R} \mid x \geq \frac{1}{n}\}$. What is

$$\bigcup_{n=1}^{\infty} A_n \ ?$$

2. $\forall n \in \mathbb{Z}^+$, define $A_n = (-\frac{1}{n}, 1+\frac{1}{n})$. What is

$$\bigcap_{n=1}^{\infty} A_n \ ?$$

<u>Theorem</u>: Let A, B, C be sets. Then

- $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

<u>Theorem</u>: Let A be a set and suppose there exists a surjection $f: \mathbb{Z}^+ \to A$. Then A is countable.

<u>Proof:</u> (Recall that countable means finite or countably infinite.)

For each $a \in A$, define $S_a = f^{-1}(\{a\}) = \{n \in \mathbb{Z}^+ \mid f(n) = a\}$. Since f is surjective, $S_a \neq \emptyset$ for all $a \in A$. Define $g: A \to \mathbb{Z}^+$ by the condition that, for all $a \in A$, $g(a)$ is the least element of S_a.

Then $\mathcal{R}(g)$ is countable (because every subset of a countable set is countable) and g is a bijection (since each $n \in \mathbb{Z}^+$ can belong to at most one S_a — otherwise, f is not a well defined function). Thus $A \sim \mathcal{R}(g)$, and hence A is countable. //

<u>Theorem</u>: The union of countably many countable sets is countable.

<u>Proof:</u> We will prove the hardest case: a countably infinite union of countably infinite sets is countably infinite. Suppose that, for each $n \in \mathbb{Z}^+$, E_n is a countably infinite set. We wish to prove that

$$E = \bigcup_{n=1}^{\infty} E_n$$

is countably infinite.

Since each E_n is countably infinite, it can be written as a sequence:

$\forall n \in \mathbb{Z}^+$, $E_n = \{x_{nj}\} = \{x_{nj} \mid j = 1, 2, 3, \ldots\}$.

Let $\{x_{nj} \mid n \in \mathbb{Z}^+, j \in \mathbb{Z}^+\}$ be enumerated in the order illustrated below:

$$\begin{array}{ccccc}
x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\
x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\
x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\
x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

That is, the elements of E are listed as follows:

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \ldots$$

Since the elements of E can be put into a sequence, E must be countably infinite. (Note: Since the sets $E_1, E_2, \ldots$ might not be disjoint, the above list might contain repetitions. So, to be precise, the list above shows how to define a surjection from $\mathbb{Z}^+$ onto E; hence, by the previous theorem, E is countable. Since, for example, E_1 is countably infinite and $E_1 \subset E$, E must be countably infinite and not finite.) //

[The previous proof might seem a bit "hand-wavy" because the surjection $f:\mathbb{Z}^+\to E$ is not defined explicitly. Here's how we can do this.

<u>Definition</u>: The <u>triangular numbers</u> $T_1, T_2, T_3, \ldots$ are defined as follows:

$$T_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \quad n=1,2,3,\ldots$$

<u>Lemma</u>: For each $n\in\mathbb{Z}^+$, there exists a unique choice of $k,j\in\mathbb{Z}$ such that

$$k\geq 1 \text{ and } 0\leq j\leq k-1 \quad \text{and} \quad n=T_k-j.$$

<u>Proof</u>: Note that $\{T_k\}$ is a strictly increase sequence of positive integers; thus, for each $n\in\mathbb{Z}^+$, there exists a unique $k\geq 1$ such that $T_{k-1}<n\leq T_k$. Then j is uniquely determined as $j=T_k-n$; since $T_k-T_{k-1}=k$, we have $0\leq j\leq k-1$. //

We can now define a surjection $f:\mathbb{Z}^+\to E$ by

$$f(T_k-j) = x_{j+1,k-j}.$$

It suffices to prove that

$$\varphi:\mathbb{Z}^+\to S, \ \varphi(T_k-j)=(j+1,k-j), \text{ where } S=\mathbb{Z}^+\times\mathbb{Z}^+,$$

is a bijection. First suppose that

$$\varphi(T_k - j) = \varphi(T_\ell - i).$$

Then

$$(j+1, k-j) = (i+1, \ell - i)$$

$$\Rightarrow j+1 = i+1 \text{ and } k-j = \ell - i$$

$$\Rightarrow i = j \text{ and } \ell = k$$

$$\Rightarrow T_\ell - i = T_k - j.$$

Thus φ is injective. Now suppose $(p,q) \in S$, that is,

$$p, q \in \mathbb{Z}^+.$$

Define $j = p-1$, $k = p+q-1$. Then

$$\varphi(T_k - j) = (j+1, k-j) = (p, p+q-1-(p-1)) = (p, q),$$

and hence φ is surjective.

It is now straightforward to show that f is a surjection. $\rbrack$

Theorem: Let A be a countable set. Then $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ factors}}$ is countable.

Proof: We argue by induction on n. Since $A^1 = A$, it is given that A^1 is countable. Now suppose A^{n-1} is countable. Then

$$A^n = \{(a_1, \dots, a_n) \mid a_j \in A \ \forall j = 1, \dots, n\}$$

$$= \bigcup_{a \in A} \{(a_1, a_2, \dots, a_{n-1}, a) \mid (a_1, \dots, a_{n-1}) \in A^{n-1}\}.$$

For each $a \in A$,

$$\{(a_1, \dots, a_{n-1}, a) \mid (a_1, \dots, a_{n-1}) \in A^{n-1}\}$$

is clearly countable. Since A is countable, we see that A^n is a countable union of countable sets, and hence is countable. //

Corollary: $\mathbb{Q}$ is countable.

Proof: Note that

$$S = \{(p, q) \in \mathbb{Z} \times \mathbb{Z} \mid q \neq 0\}$$

is countable (it's a subset of $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$, which is countable by the last theorem). Also,

$$f : S \to \mathbb{Q}, \ f(p, q) = \frac{p}{q}$$

is surjective. Thus $\mathbb{Q}$ is countable. //

Theorem: $\mathbb{R}$ is uncountable

"Proof" (Cantor's diagonalization argument) Each $x \in (0,1)$ can be written in decimal form as

$$x = 0.d_1d_2d_3\cdots \quad (d_j \in \{0,1,\ldots,9\}\ \forall j;\ \nexists n,\ d_j = 9\ \forall j \geq n).$$

Suppose $f: \mathbb{Z}^+ \to (0,1)$. We will show that f cannot be surjective; hence $(0,1)$ is uncountable; hence $\mathbb{R}$ is uncountable.

Define $x \in (0,1)$ as follows:

$$x = 0.e_1e_2e_3\cdots,$$

where

$$e_n = \begin{cases} 2 & \text{if } f(n) \neq 2, \\ 3 & \text{if } f(n) = 2. \end{cases}$$

Then $x \neq f(n)$ for all $n \in \mathbb{Z}^+$ (since the nth digit of $f(n)$ is different from the nth digit of x), and hence f is not surjective. /

The above proof is not valid in our development because we have not defined the decimal expansion of $x \in (0,1)$.