Math 600 Lecture 9

Let (X,d) be a metric space and let $E \subset X$. Define "E is bounded."

Prove: If E is unbounded, then there exists a sequence $\{x_n\}$ with no limit point.

<u>Lemma:</u> Let (X,d) be a metric space and let $E \subset X$. If E is compact, then E is bounded.

<u>Proof:</u> Choose any point $x \in E$ and note that $\{B_n(x) \mid n \in \mathbb{Z}^+\}$ is an open cover of E (every point $y \in E$ satisfies $d(y,x) < n$ for some $n \in \mathbb{Z}^+$). Thus, if E is compact, a finite subcollection $\{B_{n_1}(x), B_{n_2}(x), \ldots, B_{n_k}(x)\}$ covers E. But then

$$E \subset B_r(x),$$

where $r = \max\{n_1, n_2, \ldots, n_k\}$. Thus E is bounded. //

Thus, a compact set is closed and bounded. In $\mathbb{R}^k$, the converse is true.

<u>Theorem</u> (Heine-Borel) Let $k \in \mathbb{R}^k$ and let $E \subset \mathbb{R}^k$ be closed and bounded. Then E is compact.

<u>Proof:</u> Since E is bounded, it is a subset of some k-cell C. We have seen a k-cell is compact, and we have also seen that a closed subset of a compact set is compact. Thus E is compact. //

By an earlier result, every infinite subset of a compact set has a limit point in that set. The converse is also true in any metric space.

Theorem: Let (X,d) be a metric space and let $K \subset X$. If every infinite subset E of K has a limit point in K, then K is compact.

The proof is a bit involved and requires some additional concepts.

Definition: Let (X,d) be a metric space. We say that $E \subset X$ is <u>dense</u> in X iff for all $x \in X$ and all $r > 0$, there exists $e \in E$ such that

$$d(e,x) < r.$$

Definition: Let (X,d) be a metric space. We say that X is <u>separable</u> iff it contains a countable dense subspace.

Example: Let $k \in \mathbb{Z}^+$. Then $\mathbb{R}^k$ is separable.

Proof: Define $S = \{(r_1, r_2, \ldots, r_k) \mid r_1, \ldots, r_k \in \mathbb{Q}\} = \mathbb{Q}^k$. By an earlier theorem $\mathbb{Q}^k$ is countable. Let $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $r > 0$ be given.

For each $j = 1, \ldots, k$, there exists $r_j \in (x_j - \frac{r}{\sqrt{k}}, x_j + \frac{r}{\sqrt{k}}) \cap \mathbb{Q}$. It follows that

$$d(r,x) = \|r-x\|_2 = \left[\sum_{j=1}^{k} (r_j - x_j)^2\right]^{1/2} < \left[\sum_{j=1}^{k} \frac{r^2}{k}\right]^{1/2} = r.$$

Thus $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. //

Definition: Let (X,d) be a metric space. A collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X is called a <u>base</u> for X (or for the topology on X) iff for every open subset G of X and for every $x \in G$, there exists $\alpha' \in A$ such that $x \in U_{\alpha'} \subset G$.

Theorem: Let (X,d) be a separable metric space. Then there exists a countable base for X.

Proof: Let $S=\{x_n\}$ be a countable dense subset of X and define

$$\mathcal{U}=\{B_r(x_n) \mid n\in\mathbb{Z}^+, r\in\mathbb{Q}^+\}.$$
$$=\bigcup_{n=1}^{\infty}\{B_r(x_n) \mid r\in\mathbb{Q}^+\}$$

Then $\mathcal{U}$ is countable (since it's a countable union of countable sets).

We claim that $\mathcal{U}$ is a base for X. Let G be an open subset of X and let $x\in G$. Then there exists $r\in\mathbb{R}^+$ such that $B_r(x)\subset G$.

Choose $r'\in(0,\frac{r}{2})\cap\mathbb{Q}$ and $x_n\in S$ such that $d(x_n,x)<r'$.

Then

$$x\in B_{r'}(x_n)\subset B_r(x)\subset G.$$

(To see this, note that

$$y\in B_{r'}(x_n)\Rightarrow d(y,x)\le d(y,x_n)+d(x_n,x)<r'+r'=2r'<r.$$

Thus $B_{r'}(x_n)\subset B_r(x)$.) This completes the proof. //

Theorem: Let (X,d) be a metric space with a countable base. If $E\subset X$ is any set and $\{G_\alpha \mid \alpha\in A\}$ is an open cover for E, then there is a countable subcover (i.e. there exists a finite or countably infinite subset A' of A such that $E\subset\bigcup_{\alpha\in A'}G_\alpha$).

Proof: Let $\{U_n \mid n \in \mathbb{Z}^+\}$ be a countable base for X, let $E \subset X$, and let $\{G_\alpha \mid \alpha \in A\}$ be an open cover for E. For each $x \in E$, there exists $\alpha_x \in A$ such that $x \in G_{\alpha_x}$. Since $\{U_n\}$ is a base for the topology of X, for each $x \in E$, there exists $n_x \in \mathbb{Z}^+$ such that $x \in U_{n_x} \subset G_{\alpha_x}$. Define $B = \{U_{n_x} \mid x \in E\} \subset \{U_n\}$ and note that B is countable. By construction, for each $U \in B$, there exists G_{α_U} such that $U \subset G_{\alpha_U}$ (there may be many such G_{α_U}, but we need only one). But then

$$\{G_{\alpha_U} \mid U \in B\}$$

is a countable open cover for E. //

Theorem: Let (X,d) be a metric space with the property that every infinite subset of X has a limit point in X. Then X is separable.

Proof: Choose any $\delta > 0$ and select $x_1 \in X$. Construct $x_2, x_3, \ldots$ as follows: Given $x_1, \ldots, x_k$ such that

$$d(x_i, x_j) \geq \delta \;\; \forall i,j = 1,\ldots,k,\; i \neq j,$$

choose x_{k+1} (if possible) so that

$$d(x_i, x_{k+1}) \geq \delta \;\; \forall i = 1,\ldots,k.$$

We claim that this process must end after a finite number of steps. If not, we obtain a sequence $\{x_n\}$ such that

$$d(x_n, x_m) \geq \delta \;\; \forall m,n \in \mathbb{Z}^+,\; m \neq n.$$

But such a sequence cannot have a limit point in X (any ball of radius δ

contains at most one point in this sequence). Thus, there exists n_δ and points $x_1, \ldots, x_{n_\delta}$ such that

$$X = \bigcup_{j=1}^{n_\delta} B_\delta(x_j).$$

Write $S_\delta = \{x_1, \ldots, x_{n_\delta}\}$. Note that, for all $x \in X$, there exists some $x_j \in S_\delta$ such that $d(x, x_j) < \delta$.

Now define

$$S = \bigcup_{n=1}^{\infty} S_{1/n}.$$

Let $\varepsilon > 0$ be arbitrary and let $x \in X$. We wish to prove that there exists $y \in S$ such that $d(x,y) < \varepsilon$. But if we choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \varepsilon$, then there exists $y \in S_{1/n} \subset S$ such that

$$d(x,y) < \frac{1}{n} < \varepsilon.$$

Thus S is dense in X, that is, X is separable. //

<u>Theorem</u>: Suppose (X,d) is a metric space with the property that every infinite subset of X has a limit point in X. Then X is compact.

<u>Proof:</u> By the above results, X is separable, hence there exists a countable basis for X, hence every open cover for X contains a countable subcover for X. Thus it suffices to prove that if $\{G_n\}$ is a countable open cover for X, then it contains a finite subcover.

So let $\{G_n\}$ be a countable open cover for X. Define, for all $n \in \mathbb{Z}^+$,

$$F_n = \left(\bigcup_{j=1}^{n} G_j \right)^c = \bigcap_{j=1}^{n} G_j^c$$

Let us argue by contradiction and assume that $\{G_n\}$ contains no finite subcover of X. Then $\{G_1, \ldots, G_n\}$ does not cover X for all $n \in \mathbb{Z}^+$ and hence $F_n \neq \emptyset$ for all $n \in \mathbb{Z}^+$. However,

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{j=1}^{\infty} G_j^c = \left(\bigcup_{j=1}^{\infty} G_j \right)^c = X^c = \emptyset.$$

For each $n \in \mathbb{Z}^+$, let $x_n \in F_n$. By assumption, $\{x_n\}$ has a limit point $x \in X$. Note that each F_n is closed and

$$F_{n+1} \subset F_n \quad \forall n \in \mathbb{Z}^+$$

$$\Rightarrow \forall n \in \mathbb{Z}^+, \ \{x_k \mid k \geq n\} \subset F_n.$$

Thus x is a limit point of each F_n and hence, since each F_n is closed, $x \in F_n \ \forall n \in \mathbb{Z}^+$. But then $x \in \bigcap_{n=1}^{\infty} F_n$, a contradiction.

Thus $\{G_n\}$ must contain a finite subcover, and we have proved that X is compact. //