Math 672 Lecture 11

Theorem: Let V, W be finite-dimensional vector spaces over F with bases $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$, $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$, respectively, and let $T \in \mathcal{L}(V, W)$. Then there exists a unique matrix $A \in F^{m \times n}$ such that

$$\mathcal{M}_{\mathcal{C}}(T(v)) = A\mathcal{M}_{\mathcal{B}}(v) \quad \forall\, v \in V.$$

Moreover, the columns of A are $\mathcal{M}_{\mathcal{C}}(T(v_1)), \mathcal{M}_{\mathcal{C}}(T(v_2)), \ldots, \mathcal{M}_{\mathcal{C}}(T(v_n))$ (i.e. $A = [\mathcal{M}_{\mathcal{C}}(T(v_1)) | \mathcal{M}_{\mathcal{C}}(T(v_2)) | \cdots | \mathcal{M}_{\mathcal{C}}(T(v_n))]$),

Proof: Let us define $A \in F^{m \times n}$ by

$$A = [\mathcal{M}_{\mathcal{C}}(T(v_1)) | \mathcal{M}_{\mathcal{C}}(T(v_2)) | \cdots | \mathcal{M}_{\mathcal{C}}(T(v_n))].$$

Recall that $\mathcal{M}_{\mathcal{B}}, \mathcal{M}_{\mathcal{C}}$ are linear. Let $v \in V$ and suppose $x = \mathcal{M}_{\mathcal{B}}(v)$, that is, suppose $v = \sum_{j=1}^{n} x_j v_j$. Then

$$A\mathcal{M}_{\mathcal{B}}(v) = Ax = \sum_{j=1}^{n} x_j \mathcal{M}_{\mathcal{C}}(T(v_j))$$

$$= \mathcal{M}_{\mathcal{C}}\left(\sum_{j=1}^{n} x_j T(v_j)\right) \quad (\text{since } \mathcal{M}_{\mathcal{C}} \text{ is linear})$$

$$= \mathcal{M}_{\mathcal{C}}\left(T\left(\sum_{j=1}^{n} x_j v_j\right)\right) \quad \text{(since } T \text{ is linear)}$$

$$= \mathcal{M}_{\mathcal{C}}(T(v)).$$

Thus A satisfies

$$A\,\mathcal{M}_{\mathcal{B}}(v) = \mathcal{M}_{\mathcal{C}}(T(v)) \quad \forall v \in V,$$

as desired.

Now suppose $B \in F^{m \times n}$ also satisfies

$$B\,\mathcal{M}_{\mathcal{B}}(v) = \mathcal{M}_{\mathcal{C}}(T(v)) \quad \forall v \in V.$$

Then

$$A\,\mathcal{M}_{\mathcal{B}}(v) = B\,\mathcal{M}_{\mathcal{B}}(v) \quad \forall v \in V$$

$$\Rightarrow A\,\mathcal{M}_{\mathcal{B}}(v_j) = B\,\mathcal{M}_{\mathcal{B}}(v_j) \quad \forall j = 1, 2, \ldots, n$$

$$\Rightarrow A e_j = B e_j \quad \forall j = 1, 2, \ldots, n \quad \text{(since } \mathcal{M}_{\mathcal{B}}(v_j) = e_j \text{ by def'n)}$$

$$\Rightarrow A_j = B_j \quad \forall j = 1, 2, \ldots, n \quad \text{(that is, } A \text{ and } B \text{ have the same columns)}$$

$$\Rightarrow A = B.$$

(Here we used the fact that for any matrix $M \in F^{m \times n}$, $Me_j = M_j$ if e_j is the jth standard basis vector for F^n.)

Thus A is unique, and the proof is complete. //

We write $\mathcal{M}(T)$ or $\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)$ for the matrix A of the previous theorem, and call $\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)$ the matrix of the linear map T with respect to the bases $\mathcal{B}$ of V and $\mathcal{C}$ of W. Thus

$$\mathcal{M}_{\mathcal{C}}(T(v)) = \mathcal{M}_{\mathcal{B},\mathcal{C}}(T)\,\mathcal{M}_{\mathcal{B}}(v) \quad \forall\, v \in V.$$

$\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)$ represents T in the sense of the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\mathcal{M}_{\mathcal{B}}"'] & W \\
F^n \arrow[r, "\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)"'] & F^m \arrow[u, "\mathcal{M}_{\mathcal{C}}^{-1}"']
\end{tikzcd}$$

$$\mathcal{M}_{\mathcal{C}}T = \mathcal{M}_{\mathcal{B},\mathcal{C}}(T)\,\mathcal{M}_{\mathcal{B}} \iff T = \mathcal{M}_{\mathcal{C}}^{-1}\,\mathcal{M}_{\mathcal{B},\mathcal{C}}\,\mathcal{M}_{\mathcal{B}}$$

Example: Define $D: \mathcal{P}_3 \to \mathcal{P}_2$ by $Dp = p'$. We use the standard bases:

$$\mathcal{B} = \{1, x, x^2, x^3\} \text{ for } \mathcal{P}_3,$$

$$\mathcal{C} = \{1, x, x^2\} \text{ for } \mathcal{P}_2.$$

What is $\mathcal{M}_{\mathcal{B},\mathcal{C}}(D)$?

Solution: We have $\mathcal{P}_3 \cong \mathbb{R}^4$, $\mathcal{P}_2 \cong \mathbb{R}^3$, so

$$D: \mathcal{P}_3 \to \mathcal{P}_2 \implies \mathcal{M}_{\mathcal{B},\mathcal{C}}(D) \in \mathbb{R}^{3\times 4}.$$

We have

$$D(1) = 0 \implies \mathcal{M}_{\mathcal{C}}(D(1)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x) = 1 \implies \mathcal{M}_{\mathcal{C}}(D(x)) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x^2) = 2x \implies \mathcal{M}_{\mathcal{C}}(D(x^2)) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$D(x^3) = 3x^2 \implies \mathcal{M}_{\mathcal{C}}(D(x^3)) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Thus

$$\mathcal{M}_{\mathcal{B},\mathcal{C}}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

<u>Example</u> Define $T: \mathcal{P}_3 \to \mathcal{P}_3$ by $T(p(x)) = (x+1)p'(x)$.

We use $\mathcal{B} = \{1, x, x^2, x^3\}$ on both the domain and co-domain:

$$T(1) = 0 \Rightarrow \mathcal{M}_{\mathcal{B}}(T(1)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = (x+1)\cdot 1 = x+1 \Rightarrow \mathcal{M}_{\mathcal{B}}(T(x)) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x^2) = (x+1)\cdot 2x = 2x + x^2 \Rightarrow \mathcal{M}_{\mathcal{B}}(T(x^2)) = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$T(x^3) = (x+1)\cdot 3x^2 = 3x^2 + 3x^3 \Rightarrow \mathcal{M}_{\mathcal{B}}(T(x^3)) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}.$$

Therefore,

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

<u>Theorem</u>: Suppose V and W are finite-dimensional vector spaces over a field F, and let $\dim(V)=n$, $\dim(W)=m$. Then

$$\mathcal{L}(V,W) \cong F^{m\times n}.$$

In fact, given bases $\mathcal{B}$ for V and $\mathcal{C}$ for W, $\mathcal{M}_{\mathcal{B},\mathcal{C}}:\mathcal{L}(V,W)\to F^{m\times n}$ is an isomorphism.

<u>Proof</u>: We must prove that $\mathcal{M}_{\mathcal{B},\mathcal{C}}$ is linear and invertible.

Suppose $T,S\in\mathcal{L}(V,W)$. Then $\mathcal{M}_{\mathcal{B},\mathcal{C}}(T+S)$ is the unique matrix in $F^{m\times n}$ satisfying

$$\mathcal{M}_{\mathcal{C}}((T+S)(v)) = \mathcal{M}_{\mathcal{B},\mathcal{C}}(T+S)\,\mathcal{M}_{\mathcal{B}}(v) \quad \forall v\in V.$$

But

$$\begin{aligned}\mathcal{M}_{\mathcal{C}}((T+S)(v)) &= \mathcal{M}_{\mathcal{C}}(T(v)+S(v)) \quad (\text{by def'n of } T+S)\\ &= \mathcal{M}_{\mathcal{C}}(T(v))+\mathcal{M}_{\mathcal{C}}(S(v)) \quad (\text{since } \mathcal{M}_{\mathcal{C}} \text{ is linear})\\ &= \mathcal{M}_{\mathcal{B},\mathcal{C}}(T)\,\mathcal{M}_{\mathcal{B}}(v)+\mathcal{M}_{\mathcal{B},\mathcal{C}}(S)\,\mathcal{M}_{\mathcal{B}}(v)\\ &= \left(\mathcal{M}_{\mathcal{B},\mathcal{C}}(T)+\mathcal{M}_{\mathcal{B},\mathcal{C}}(S)\right)\mathcal{M}_{\mathcal{B}}(v) \quad \forall v\in V,\end{aligned}$$

which shows that

$$\mathcal{M}_{\mathcal{B},\mathcal{E}}(T+S) = \mathcal{M}_{\mathcal{B},\mathcal{E}}(T) + \mathcal{M}_{\mathcal{B},\mathcal{E}}(S)$$

(since $\mathcal{M}_{\mathcal{B},\mathcal{E}}(T) + \mathcal{M}_{\mathcal{B},\mathcal{E}}(S)$ is also an element of $F^{m\times n}$).

Similarly, if $T \in \mathcal{L}(V,W)$ and $\alpha \in F$, then

$$\begin{aligned}
\mathcal{M}_{\mathcal{E}}((\alpha T)(v)) &= \mathcal{M}_{\mathcal{E}}(\alpha T(v)) \\
&= \alpha \mathcal{M}_{\mathcal{E}}(T(v)) \\
&= \alpha \left(\mathcal{M}_{\mathcal{B},\mathcal{E}}(T)\mathcal{M}_{\mathcal{B}}(v)\right) \\
&= (\alpha \mathcal{M}_{\mathcal{B},\mathcal{E}}(T))\mathcal{M}_{\mathcal{B}}(v) \quad \forall v \in V,
\end{aligned}$$

and hence

$$\mathcal{M}_{\mathcal{B},\mathcal{E}}(\alpha T) = \alpha \mathcal{M}_{\mathcal{B},\mathcal{E}}(T).$$

Thus $\mathcal{M}_{\mathcal{B},\mathcal{E}}$ is linear.

Now suppose $T \in \mathcal{L}(V,W)$ and $\mathcal{M}_{\mathcal{B},\mathcal{E}}(T) = 0$. Then

$$\begin{aligned}
&\mathcal{M}_{\mathcal{E}}(T(v)) = 0\mathcal{M}_{\mathcal{B}}(v) \quad \forall v \in V \\
\Rightarrow\ &\mathcal{M}_{\mathcal{E}}(T(v)) = 0 \quad \forall v \in V \\
\Rightarrow\ &T(v) = 0 \quad \forall v \in V \text{ (since } \mathcal{M}_{\mathcal{E}} \text{ is an isomorphism)} \\
\Rightarrow\ &T = 0.
\end{aligned}$$

Thus $\mathcal{N}(\mathcal{M}_{\mathcal{B},\mathcal{C}}) = \{0\}$, which shows that $\mathcal{M}_{\mathcal{B},\mathcal{C}}$ is injective.

Finally, suppose $A \in F^{m \times n}$. Define $L : F^n \to F^m$ by

$$L(x) = Ax \quad \forall x \in F^n$$

and $T : V \to W$ by

$$(*) \qquad T = \mathcal{M}_{\mathcal{C}}^{-1} L \mathcal{M}_{\mathcal{B}}.$$

Note that $T \in \mathcal{L}(V,W)$ (the composition of linear maps is linear). Moreover,

$$\begin{aligned}
(*) &\Rightarrow \mathcal{M}_{\mathcal{C}} T = L \mathcal{M}_{\mathcal{B}} \\
&\Rightarrow (\mathcal{M}_{\mathcal{C}} T)(v) = (L \mathcal{M}_{\mathcal{B}})(v) \quad \forall v \in V \\
&\Rightarrow \mathcal{M}_{\mathcal{C}}(T(v)) = A \mathcal{M}_{\mathcal{B}}(v) \quad \forall v \in V \\
&\Rightarrow A = \mathcal{M}_{\mathcal{B},\mathcal{C}}(T).
\end{aligned}$$

This shows that $\mathcal{M}_{\mathcal{B},\mathcal{C}}$ is surjective, and the proof is complete. //

<u>Theorem</u>: Let V, W, Z be finite-dimensional vector spaces over F, and let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be bases for V, W, Z, respectively. If $T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,Z)$, then

$$\mathcal{M}_{\mathcal{B},\mathcal{D}}(ST) = \mathcal{M}_{\mathcal{C},\mathcal{D}}(S)\, \mathcal{M}_{\mathcal{B},\mathcal{C}}(T).$$

Proof: Assume $n = \dim(V)$, $m = \dim(W)$, $p = \dim(Z)$, and define

$$A = \mathcal{M}_{\mathcal{C},\mathcal{D}}(S), \quad L_A : F^m \to F^p, \; L_A(x) = Ax \quad \forall x \in F^m,$$

$$B = \mathcal{M}_{\mathcal{B},\mathcal{C}}(T), \quad L_B : F^n \to F^m, \; L_B(x) = Bx \quad \forall x \in F^n.$$

Then

$$T = \mathcal{M}_{\mathcal{C}}^{-1} L_B \mathcal{M}_{\mathcal{B}}, \quad S = \mathcal{M}_{\mathcal{D}}^{-1} L_A \mathcal{M}_{\mathcal{C}}$$

$$\Rightarrow \quad ST = \mathcal{M}_{\mathcal{D}}^{-1} L_A \mathcal{M}_{\mathcal{C}} \mathcal{M}_{\mathcal{C}}^{-1} L_B \mathcal{M}_{\mathcal{B}}$$

$$= \mathcal{M}_{\mathcal{D}}^{-1} L_A L_B \mathcal{M}_{\mathcal{B}}$$

$$= \mathcal{M}_{\mathcal{D}}^{-1} L_{AB} \mathcal{M}_{\mathcal{B}},$$

where $L_{AB} : F^n \to F^p$ ($p = \dim(Z)$), $L_{AB}(x) = (AB)x \quad \forall x \in \mathbb{R}^n$.

But this implies that

$$\mathcal{M}_{\mathcal{D}} ST = L_{AB} \mathcal{M}_{\mathcal{B}} \quad (\text{i.e. } \mathcal{M}_{\mathcal{D}}(S(T(v))) = AB\,\mathcal{M}_{\mathcal{B}}(v) \; \forall v \in V)$$

and hence that

$$AB = \mathcal{M}_{\mathcal{B},\mathcal{D}}(ST),$$

as desired. //