Math 672 Lecture 18

Definition: Let V be a vector space over $F=\mathbb{R}$ or $F=\mathbb{C}$.

A norm on V is a real-valued function $v \longmapsto \|v\|$ satisfying the following properties:

- $\|v\| \geq 0 \ \forall v \in V$ and $\|v\| = 0$ iff $v=0$;
- $\|\alpha v\| = |\alpha| \, \|v\| \ \forall v \in V \ \forall \alpha \in F$;
- $\|u+v\| \leq \|u\| + \|v\| \ \forall u, v \in V$ (the triangle inequality)

Examples: On F^n, each of following is a norm.

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$
- $\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}$

On $C[a,b]$, each of the following is a norm:

- $\|f\|_{L^1(a,b)} = \int_a^b |f(t)| \, dt$
- $\|f\|_{L^2(a,b)} = \sqrt{\int_a^b |f(t)|^2 \, dt}$

- $\|f\|_{L^\infty(a,b)} = \max\{|f(t)| : a \le t \le b\}$

Note: Given a norm $\|\cdot\|$ on V,

$$d(u,v) = \|u-v\|$$

defines a metric on V.

For our purposes, the most useful norms are those defined by inner products:

Definition: Let V be a vector space over $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle$ mapping $V \times V$ into $\mathbb{R}$ ($\langle u,v \rangle \in \mathbb{R}$ $\forall u,v \in V$) is called an inner product for V iff

- $\langle v,v \rangle \ge 0$ $\forall v \in V$ and $\langle v,v \rangle = 0$ iff $v=0$
- $\langle \alpha u + \beta v, w \rangle = \alpha \langle u,w \rangle + \beta \langle v,w \rangle$ $\forall u,v,w \in V$ $\forall \alpha,\beta \in \mathbb{R}$
- $\langle u,v \rangle = \langle v,u \rangle$ $\forall u,v \in V$

(Together, the second and third properties imply that

$$\langle w, \alpha u + \beta v \rangle = \alpha \langle w,u \rangle + \beta \langle w,v \rangle \quad \forall u,v,w \in V \ \forall \alpha,\beta \in \mathbb{R}.)$$

Examples

- The dot product, $\langle x,y\rangle_2 = x\cdot y = \sum_{i=1}^{n} x_i y_i$, is an inner product on $\mathbb{R}^n$.
- $\langle f,g\rangle_{L^2} = \int_a^b f(t)g(t)\,dt$ defines an inner product (the L^2 inner product) on $C[a,b]$.

Lemma: If V is an inner product space over $\mathbb{R}$, then

$$\langle 0,v\rangle = \langle v,0\rangle = 0 \quad \forall v\in V.$$

Proof: This follows because, for a fixed $v\in V$, $f(u)=\langle u,v\rangle$ defines a linear functional on V (and every linear functional f satisfies $f(0)=0$). //

Theorem (the Cauchy-Schwarz inequality) Let V be a vector space over $\mathbb{R}$ and let $\langle\cdot,\cdot\rangle$ be an inner product on V. Then

$$|\langle u,v\rangle| \le [\langle u,u\rangle]^{1/2}[\langle v,v\rangle]^{1/2} \quad \forall u,v\in V.$$

Moreover, equality holds iff one of u,v is a scalar multiple of the other.

Proof: If $u=0$ or $v=0$, then the result holds because both sides of the inequality are 0.

Now suppose $u \neq 0$ and $v \neq 0$. Assume $\langle u,u\rangle = \langle v,v\rangle = 1$.

Then we must prove that $|\langle u,v\rangle| \leq 1$.

We have

$$\begin{aligned}
&\langle u-v, u-v\rangle \geq 0 \\
\Rightarrow\ & \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle \geq 0 \\
\Rightarrow\ & 2 - 2\langle u,v\rangle \geq 0 \\
\Rightarrow\ & \langle u,v\rangle \leq 1
\end{aligned}$$

and

$$\begin{aligned}
&\langle u+v, u+v\rangle \geq 0 \\
\Rightarrow\ & \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle \geq 0 \\
\Rightarrow\ & 2 + 2\langle u,v\rangle \geq 0 \\
\Rightarrow\ & \langle u,v\rangle \geq -1.
\end{aligned}$$

Thus $-1 \leq \langle u,v\rangle \leq 1$, that is,

$$|\langle u,v\rangle| \leq 1.$$

Moreover, equality holds iff

$u = v$ (in which case $\langle u,v\rangle = 1$)

or

$$u=-v \quad (\text{in which case } \langle u,v\rangle=-1).$$

Finally, let u,v be any nonzero vectors in V. Define

$$x=\langle u,u\rangle^{-1/2}u, \quad y=\langle v,v\rangle^{-1/2}v.$$

Note that

$$\langle x,x\rangle=\langle \langle u,u\rangle^{-1/2}u, \langle u,u\rangle^{-1/2}u\rangle$$

$$=\langle u,u\rangle^{-1}\langle u,u\rangle=1$$

and, similarly,

$$\langle y,y\rangle=1.$$

Thus

$$|\langle x,y\rangle|\leq 1$$

$$\Rightarrow \left|\langle \langle u,u\rangle^{-1/2}u, \langle v,v\rangle^{-1/2}v\rangle\right|\leq 1$$

$$\Rightarrow \langle u,u\rangle^{-1/2}\langle v,v\rangle^{-1/2}|\langle u,v\rangle|\leq 1$$

$$\Rightarrow |\langle u,v\rangle|\leq \langle u,u\rangle^{1/2}\langle v,v\rangle^{1/2},$$

as desired. Finally, equality holds iff $x=y$ or $x=-y$, in which case u is a multiple of v:

$$x=\pm y \Rightarrow \langle u,u\rangle^{-1/2}u=\pm\langle v,v\rangle^{-1/2}v$$

$$\Rightarrow u = \pm \frac{\langle u,u\rangle^{1/2}}{\langle v,v\rangle^{1/2}} v. \quad //$$

Given an inner product $\langle \cdot,\cdot \rangle$ on V (a real vector space), we define the corresponding norm by

$$(*) \qquad \|v\| = \sqrt{\langle v,v\rangle} \quad \forall v \in V.$$

<u>Theorem</u>: Let V be a vector space over $\mathbb{R}$ and let $\langle \cdot,\cdot \rangle$ be an inner product on V. Then $(*)$ defines a norm on V.

<u>Proof</u>: By definition,

$$\langle v,v\rangle \geq 0 \;\; \forall v \in V \text{ and } \langle v,v\rangle = 0 \text{ iff } v = 0.$$

Thus $\|v\|$ is well defined for all $v \in V$, and

$$\|v\| \geq 0 \;\; \forall v \in V \text{ and } \|v\| = 0 \text{ iff } v = 0.$$

Next, if $\alpha \in \mathbb{R}$ and $v \in V$, then

$$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v\rangle} = \sqrt{\alpha^2 \langle v,v\rangle} = |\alpha|\sqrt{\langle v,v\rangle} = |\alpha| \cdot \|v\|.$$

Finally, we must prove the triangle inequality. Let $u,v \in V$. Then

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{(by the Cauchy-Schwarz inequality)}$$

$$= (\|u\| + \|v\|)^2$$

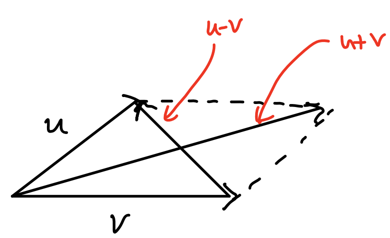
$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|.$$

This completes the proof. //

<u>Theorem</u>: Let V be a vector space over $\mathbb{R}$ and let $\|\cdot\|$ be a norm defined on V.

1. If $\|\cdot\|$ is defined by an inner product $\langle\cdot,\cdot\rangle$, then the <u>parallelogram law</u> holds:

$$\forall u,v \in V, \quad \|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$



2. If the parallelogram law holds, then there is an inner product defining $\|\cdot\|$.

<u>Proof</u>: 1. The proof is a direct calculation:

$$\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$$

$$= \langle u,u\rangle + \cancel{2\langle u,v\rangle} + \langle v,v\rangle + \langle u,u\rangle - \cancel{2\langle u,v\rangle} + \langle v,v\rangle$$

$$= 2\langle u,u\rangle + 2\langle v,v\rangle = 2\|u\|^2 + 2\|v\|^2.$$

2. If $\|\cdot\|$ is defined by $\langle\cdot,\cdot\rangle$, then

$$\|u+v\|^2-\|u-v\|^2=\|u\|^2+2\langle u,v\rangle+\|v\|^2-(\|u\|^2-2\langle u,v\rangle+\|v\|^2)$$

$$=4\langle u,v\rangle,$$

so let us <u>define</u> $\langle\cdot,\cdot\rangle$ by

$$\langle u,v\rangle=\frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2\right)\quad\forall u,v\in V.$$

We must prove that the properties of an inner product are satisfied.

We have

$$\langle u,u\rangle=\frac{1}{4}\left(\|u+u\|^2-\|u-u\|^2\right)=\frac{1}{4}\|2u\|^2=\|u\|^2\geq 0$$

and hence

$$\langle u,u\rangle=0\Leftrightarrow\|u\|=0\Leftrightarrow u=0.$$

Since

$$\|v-u\|=\|-1(u-v)\|=|-1|\,\|u-v\|=\|u-v\|,$$

we have

$$\langle v,u\rangle=\langle u,v\rangle\quad\forall u,v\in V.$$

It remains to prove that

$$\langle\alpha u,v\rangle=\alpha\langle u,v\rangle\quad\forall\alpha\in F\ \forall u,v\in V$$

and

$$\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle\quad\forall u,v,w\in V.$$

There doesn't seem to be any short proof of the above.

See

math.stackexchange.com/questions/21792.