Notes on Multivariable Calculus

MATH 503-010 — Fall 2021

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1 Leibnitz' rule

The following theorem tells how to differentiate an integral with respect to a parameter.

Theorem 1.0.1. ***Leibnitz's rule.***

Let α and β be in $C^1(a,b)$, and let $f(x,t)$ and $f_x(x,t)$ be continuous on $(a,b)\times(\alpha(x),\beta(x))$. Then

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,t)\,dt = f(x,\beta(x))\beta'(x) - f(x,\alpha(x))\alpha'(x) + \int_{\alpha(x)}^{\beta(x)} f_x(x,t)\,dt\,,$$

for $x \in (a,b)$.

Example: Let $F(x) = \displaystyle\int_{\sin x}^{2x^3} e^{x^2+t}\,dt\,.$ Find $F'(x)$. Check the answer by first doing the integration.

Here, $\alpha(x) = \sin x$, $\beta(x) = 2x^3$ and $f(x,t) = e^{x^2+t}$. Applying Leibnitz's rule, we obtain

$$\begin{aligned} F'(x) &= e^{x^2+2x^3}(6x^2) - e^{x^2+\sin x}(\cos x) + \int_{\sin x}^{2x^3} (2x)e^{x^2+t}\,dt \\ &= 6x^2 e^{x^2+2x^3} - (\cos x)e^{x^2+\sin x} + 2x \int_{\sin x}^{2x^3} e^{x^2+t}\,dt\,. \end{aligned}$$

This is the answer we sought. For this particular example, we can simplify the answer since the integral can be calculated:

$$\int_{\sin x}^{2x^3} e^{x^2+t}\,dt = e^{x^2+t}\Big|_{\sin x}^{2x^3} = e^{x^2+2x^3} - e^{x^2+\sin x}\,,$$

so that

$$F'(x) = (6x^2 + 2x)e^{x^2+2x^3} - (2x + \cos x)e^{x^2+\sin x}\,.$$

If instead we first compute the integral and then differentiate the result, we obtain from the integration above

$$F(x) = e^{x^2+2x^3} - e^{x^2+\sin x}\,,$$

and using the chain rule,

$$F'(x) = (2x + 6x^2)e^{x^2+2x^3} - (2x + \cos x)e^{x^2+\sin x}$$

in agreement with the result obtained using Leibnitz's rule.

1.1 Exercises.

1. Let $F(x) = \displaystyle\int_x^{x^2} \ln^2(x+t)dt\,,\ x > 0$. Find $F'(x)$.

2. Let $G(x,y) = \displaystyle\int_{xy}^{x+y} \frac{\sin[(x+y)t]}{t}\, dt$. Find $G_y(x,y)$.

3. Find $\displaystyle\frac{d}{dx}\int_0^{\ln x} \ln(1+xe^t)dt$, for $x > 0$.

2 Scalar functions

Notation. Consider a scalar function on a domain (open and connected set) $\Omega \subset \mathbb{R}^n$, $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The following notation will be used throughout these notes. Wherever derivatives appear, it is assumed that they exist.

$$
\begin{aligned}
\frac{\partial f}{\partial x_i} &= \partial_{x_i} f = f_{x_i} = \text{partial derivative of } f \text{ w.r. to } x_i \\
\nabla f &= \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = \text{gradient of } f \quad \text{(a vector)} \\
\|\mathbf{x}\| &= \sqrt{x_1^2 + \cdots + x_n^2} = \text{Euclidean norm of } \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} = (x_1, \ldots, x_n).
\end{aligned}
$$

2.1 Max/Min Problems

One common problem that appears in applications consists of finding local extrema of a real function of several variables. The following theorem helps with this problem.

Theorem 2.1.1. ***Taylor's Theorem.***
If f is a C^3 function in the neighborhood of $\mathbf{a} \in \mathbb{R}^n$, then the following expansion holds

$$f(\mathbf{a}+\mathbf{dx}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{dx} + \frac{1}{2}\,\mathbf{dx} \cdot \mathbf{H}(\mathbf{a})\,\mathbf{dx} + O(\|\mathbf{dx}\|^3), \tag{1}$$

where $\mathbf{H}$ is the (symmetric) Hessian matrix: $\mathbf{H} = (H_{ij}) = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Observations:

1. If f is a C^3 function, then it is also a C^2 function, so that $\mathbf{H}$ is a symmetric matrix.

2. A quadratic form $Q(\mathbf{u})$ is positive definite (respectively, negative definite) if $Q(\mathbf{u}) > 0$ (respectively, $Q(\mathbf{u}) < 0$) for all $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \in \mathbb{R}^n$.

3. A quadratic form induced by a real symmetric matrix $\mathbf{A}$, i.e.

$$Q(\mathbf{u}) = \mathbf{u} \cdot \mathbf{A}\mathbf{u} = \mathbf{u}^\top \mathbf{A}\mathbf{u}\,,$$

   is positive-definite (respectively, negative-definite) iff all the eigenvalues of $\mathbf{A}$ are positive (respectively, negative).

4. For a real symmetric matrix $\mathbf{A}$, the induced quadratic form is positive-definite iff all the leading (upper-left) principal minors of $\mathbf{A}$ are positive; it is negative-definite iff all the leading principal minors of $-\mathbf{A}$ are positive.

5. It is easy to see (taught in any multivariable calculus class) that ∇f is a vector in the direction of maximum increase of f and $-\nabla f$ is a vector in the direction of maximum decrease of f.

6. Since the goal is to find *local* extrema, i.e. max or min in the neighborhood of a point, the terms $O(\|d\mathbf{x}\|^3)$ in Taylor's expansion (1) can be neglected when deciding on whether $f(\mathbf{a}+\mathbf{dx}) > f(\mathbf{a})$ or $f(\mathbf{a}+\mathbf{dx}) < f(\mathbf{a})$ for $\|\mathbf{dx}\|$ small.

Theorem 2.1.2. ***Max/Min Theorem.***

Let f be a C^3 function in the neighborhood of $\mathbf{a} \in \mathbb{R}^n$. If $\mathbf{a}$ is a local extremum of f, then the following statements hold:

1. $\nabla f(\mathbf{a}) = \mathbf{0}$.
2. *If $\nabla f(\mathbf{a}) = \mathbf{0}$ and $Q(\mathbf{u}) = \mathbf{u} \cdot \mathbf{H}(\mathbf{a})\mathbf{u}$ is a positive-definite quadratic form (resp., negative-definite), then $\mathbf{a}$ is a local minimum (resp., maximum) point of f.*
3. *If $Q(\mathbf{u})$ is neither positive-definite, nor negative-definite, then $\mathbf{a}$ is neither a local maximum, nor a local minimum, of f.*

Examples: Find and classify the local extrema of the function in each case below.

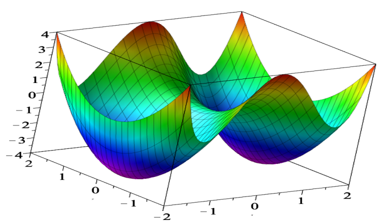
1. $f(x, y) = x^4 - 4x^2 + y^2$
2. $f(x, y) = x^2y - y^2x - x^2 - y^2$
3. $f(x, y) = e^{-(x^2+y^2)} \cos x \, \cos y \quad \text{on} \quad -\pi/2 < x, y < \pi/2\,.$

Since the functions are C^∞, i.e. their derivatives of all orders exist and are continuous, a necessary condition for local extrema is $\nabla f = \mathbf{0}$. Once the candidates for local extrema are found, in order to use Theorem 2.1.2 we need to find the Hessian of the function at those points.

1. $f_x = 4x^3 - 8x = 0,\ f_y = 2y = 0 \implies y = 0,\ 4x(x^2 - 2) = 0 \implies (0, 0), (\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are the possible extrema.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4(3x^2-2) & 0 \\ 0 & 2 \end{bmatrix}.$$

- At $(0,0)$: $H = \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -8 < 0$ and $\lambda_2 = 2 > 0$. Therefore, $(0,0)$ is neither a local max nor min (it is a saddle point).
- At $(\pm\sqrt{2}, 0)$: $H = \begin{bmatrix} 16 & 0 \\ 0 & 2 \end{bmatrix}$ has two positive eigenvalues (2 and 16). Hence the quadratic form induced by H is positive-definite, and both critical points $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are local minima. See the plot below.



2. The critical points are the solutions to

$$f_x = 2xy - y^2 - 2x = 0, \tag{2}$$

$$f_y = x^2 - 2xy - 2y = 0. \tag{3}$$

In general, solving two nonlinear equations like these is not an easy task. Use of a symbolic tool like Mathematica may help. But, in this case, we can eliminate the xy term by adding the two equations to obtain

$$x^2 - y^2 - 2x - 2y = 0 \implies (x+y)(x-y) - 2(x+y) = 0 \implies (x+y)(x-y-2) = 0.$$

Hence, critical points must lie on either line $x+y=0$ or $x-y=2$, in addition to satisfying (2) or (3) above. We look at candidates on each line:

- $x+y=0 \implies y=-x \implies x^2+2x^2+2x=0 \implies x(3x+2)=0 \implies x=0$ or $x=-2/3$. Hence, $(0,0)$ and $(-2/3,2/3)$ are critical points.
- $x-y=2 \implies x=y+2 \implies 2(y+2)y-y^2-2(y+2)=0 \implies y^2+2y-4=0$.

Therefore,

$$y=-1\pm\sqrt{5} \quad \text{and, correspondingly,} \quad x=1\pm\sqrt{5}\,.$$

Hence, $(1+\sqrt{5},-1+\sqrt{5})$ and $(1-\sqrt{5},-1-\sqrt{5})$ are also critical points. To determine the type of critical points, we calculate the Hessian of f at these critical points. Note that

$$H=\begin{bmatrix} 2(y-1) & 2(x-y) \\ 2(x-y) & -2(x+1) \end{bmatrix}.$$

- At $(0,0)$: $H=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ with negative eigenvalues $(=-2)$.
- At $(-2/3,2/3)$: $H=\begin{bmatrix} -2/3 & -8/3 \\ -8/3 & -2/3 \end{bmatrix}$. The eigenvalues satisfy the characteristic equation $\lambda^2+\frac{4}{3}\lambda-\frac{60}{3}=0$, i.e. they are

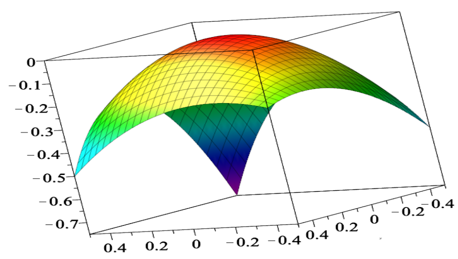
$$\lambda_{1,2}=-\frac{2}{3}\pm\frac{2\sqrt{46}}{3}\,.$$

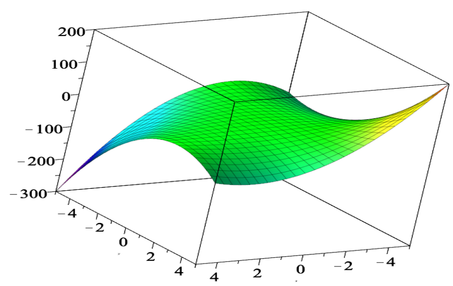
Hence, one is positive and the other is negative, and this critical point is neither a local max nor min (it is a saddle point).

• At $(1\pm\sqrt{5},-1\pm\sqrt{5})$: Note that $x-y=2$, $x+1=2\pm\sqrt{5}$ and $y-1=-2\pm\sqrt{5}$.

$$H=\begin{bmatrix}-4\pm2\sqrt{5} & 4\\ 4 & -(4\pm2\sqrt{5})\end{bmatrix}.$$

The characteristic equation in both cases $(\pm)$ is $\lambda^2+8\lambda-12=0$, and the eigenvalues are $\lambda_{1,2}=-4\pm\sqrt{28}$. Hence, one positive and one negative eigenvalue. Thus, these two critical points are neither local max nor min (they are saddle points).





3. The critical points are the solutions to

$$f_x=-e^{-(x^2+y^2)}\cos y\,(\sin x+2x\cos x)=0,$$
$$f_y=-e^{-(x^2+y^2)}\cos x\,(\sin y+2y\cos y)=0.$$

or

$$\cos y\,(\sin x+2x\cos x)=0, \tag{4}$$
$$\cos x\,(\sin y+2y\cos y)=0. \tag{5}$$

From (4), we must have $\cos y = 0$ or $\tan x = -2x$, and from (5), $\cos x = 0$ or $\tan y = -2y$.

Since we will need the Hessian, we find the 2nd order partial derivatives:

$$
\begin{aligned}
f_{xx} &= e^{-(x^2+y^2)} \cos y \,(4x^2 \cos x + 4x \sin x - 3 \cos x) \\
f_{xy} &= e^{-(x^2+y^2)}(2x \cos x + \sin x)(2y \cos y + \sin y) \\
f_{yy} &= e^{-(x^2+y^2)} \cos x \,(4y^2 \cos y + 4y \sin y - 3 \cos y)
\end{aligned}
$$

2.2 Exercises.

4. Consider the function $f(x, y, z) = xy + xz + yz$. Find all the local extrema of f and classify them.

5. For the case of a function of two variables, $f(x, y)$, derive the results that you learned in your multivariable calculus class about max/min and the second derivative test.

6. Show that ∇f is a vector in the direction of maximum increase of f and $-\nabla f$ is a vector in the direction of maximum decrease of f.

Theorem 2.2.1. ***Product rule.***
$\nabla(fg) = f\nabla g + g\nabla f$ *for any* $C^1(\Omega)$ *functions* $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

2.3 Length of a curve lying on a surface.

Consider a surface $\mathcal{S} \subset \mathbb{R}^3$ parametrized by

$$x = x(u,v)\,, \quad y = y(u,v)\,, \quad \text{and} \quad z = z(u,v)\,, \quad (u,v) \in R \subset \mathbb{R}^2.$$

Horizontal and vertical lines on the (u,v)-plane are mapped by the parametrization above to curves on $\mathcal{S}$ which have as tangent vectors

$$\boldsymbol{\tau}_u = (x_u, y_u, z_u) \quad \text{and} \quad \boldsymbol{\tau}_v = (x_v, y_v, z_v).$$

Define

$$\begin{aligned} E &= \boldsymbol{\tau}_u \cdot \boldsymbol{\tau}_u = x_u^2 + y_u^2 + z_u^2 \\ F &= \boldsymbol{\tau}_u \cdot \boldsymbol{\tau}_v = x_u x_v + y_u y_v + z_u z_v \\ G &= \boldsymbol{\tau}_v \cdot \boldsymbol{\tau}_v = x_v^2 + y_v^2 + z_v^2\,. \end{aligned}$$

Curves on $\mathcal{S}$ are images by the parametrization of curves in an open set $R \subset \mathbb{R}^2$. Let

$$\mathcal{C}\ : \quad \mathbf{u}(t) = (u(t), v(t))\,, \quad a \le t \le b$$

be the parametrization of a curve $\mathcal{C}$ in R, so that its image is a curve Γ on $\mathcal{S}$:

$$\Gamma : \quad \mathbf{x}(t) = (x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t)))\,, \quad a \le t \le b\,.$$

The element of arc length along the curve is obtained as follows

$$ds = \|\mathbf{x}'(t)\| dt \,, \quad \mathbf{x}'(t) = (x_u u' + x_v v', y_u u' + y_v v', z_u u' + z_v v')$$

$$\|\mathbf{x}'\| = \sqrt{(x_u u' + x_v v')^2 + (y_u u' + y_v v')^2 + (z_u u' + z_v v)^2} = \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}$$

$$ds = \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt \,.$$

Hence, the length of the curve Γ on $\mathcal{S}$ is

$$\boxed{\ell(\Gamma) = \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt} \,. \tag{6}$$

Examples:

1. The paraboloid of revolution $z = x^2 + y^2$ can be parametrized by

   $$x(u,v) = u \,, \quad y(u,v) = v \quad \text{and} \quad z(u,v) = u^2 + v^2 \,, \quad (u,v) \in \mathbb{R}^2 \,,$$

   and $E = 1 + 4u^2$, $F = 4uv$, and $G = 1 + 4v^2$. A curve on the paraboloid parametrized by $\mathbf{x}(u(t), v(t))$, $a \le t \le b$ has length given by

   $$\ell = \int_a^b \sqrt{(1 + 4(u(t))^2)(u'(t))^2 + 8u(t)v(t)u'(t)v'(t) + (1 + 4(v(t))^2)(v'(t))^2}\, dt$$

2. The unit sphere centered at the origin has a parametrization

   $$x(u,v) = \cos u \sin v \,, \; y(u,v) = \sin u \sin v \,, \; z(u,v) = \cos v \,, \; 0 \le u \le 2\pi \,, \; 0 \le v \le \pi \,.$$

Then

$$E = \sin^2 v\,, \quad F = 0\,, \quad G = 1\,,$$

so that a curve on the unit sphere has length

$$\ell = \int_a^b \sqrt{(u'(t))^2 \sin^2 v(t) + (v'(t))^2}\, dt\,.$$

2.3.1 Geodesics of a surface

Geodesics of a surface $\mathcal{S}$ are those curves of minimum length lying on $\mathcal{S}$, connecting points on the surface. Hence, given two points on $\mathcal{S}$, the geodesic connecting them minimizes an integral of the form

$$J[u,v] = \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt\,.$$

2.4 Exercises.

7. Consider the circle on the paraboloid $z = x^2 + y^2$ defined by

$$\mathcal{C}\ : \quad x = \sqrt{z_0}\cos\theta\,, \quad y = \sqrt{z_0}\sin\theta\,, \quad z = z_0\,, \quad 0 \le \theta \le 2\pi\,.$$

   Show that the length of the circle as given by (6) is what you expect.

8. Consider the curve on the cone $z = \sqrt{x^2 + y^2}$ given by

$$\mathcal{C}\ : \quad x = t\cos t\,, \quad y = t\sin t\,, \quad z = t\,, \quad 0 \le t \le 4\pi\,.$$

(a) Sketch the curve.

(b) Find the length of the curve.

3 Vector fields

Notation. Consider a vector function (field) on a domain (open and connected set) $\Omega \subset \mathbb{R}^n$, $\mathbf{F} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. When derivatives appear, it is assumed that they exist.

$$\mathbf{F} = (f_1, \ldots, f_m)$$
$$D\mathbf{F} = \left(\frac{\partial f_i}{\partial x_j}\right) \quad \text{is the Jacobian matrix of } \mathbf{F}.$$

Note that $D\mathbf{F}$ is an $m \times n$ matrix whose rows are $\nabla f_i\,,\ i = 1, \ldots m\,.$

3.1 The divergence and the curl.

If $m = n\,,$

$$\nabla \cdot \mathbf{F} = \mathrm{div}\mathbf{F} = \mathrm{trace}(D\mathbf{F}) = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} = \text{divergence of } \mathbf{F} \text{ (a scalar)},$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix} = (\partial_y f_3 - \partial_z f_2\,,\, \partial_z f_1 - \partial_x f_3\,,\, \partial_x f_2 - \partial_y f_1), \quad \text{where } m = n = 3,$$

is the curl of $\mathbf{F}$. In particular, if $m = n = 2$, then $f_3 = 0$ and $\partial_z = 0$, so that $\nabla \times \mathbf{F} = (0, 0, \partial_x f_2 - \partial_y f_1)$. Also, note that $\nabla \cdot \nabla \times \mathbf{F} = 0$ and $\nabla \times \nabla f = \mathbf{0}$.

3.2 Exercises.

9. Find a product rule formula for $\nabla \cdot (g\mathbf{F})$, where g is a scalar function on Ω.

10. Show that $\nabla \cdot (f \nabla g) = f \Delta g + \nabla f \cdot \nabla g$, where the Laplacian is defined in the next section.

4 The Laplacian

Given $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, a scalar function, the Laplacian of f is

$$\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \qquad (\Delta = \nabla \cdot \nabla).$$

4.1 Exercises.

11. Derive a product rule for the Laplacian, namely for $\Delta(fg)$.

12. For a given scalar function φ, find a vector field $\mathbf{F}$ such that

$$(\Delta\varphi)^2 + \nabla\varphi \cdot \nabla\Delta\varphi = \nabla \cdot \mathbf{F}\,.$$

5 Line and surface integrals

Consider a simple path C in $\mathbb{R}^n$ ($n = 2$ or 3) parametrized by

$$\mathbf{x} : [a, b] \to C \subset \mathbb{R}^n \,,$$

a smooth injective function (i.e. one-to-one, continuously differentiable). Such a parametrization defines an orientation of the path. The reverse orientation of C is the simple path $-C$ defined by a parametrization that inverts the order of points in C as t increases. For example, $\mathbf{x}^* : [a, b] \to C$ defined by $\mathbf{x}^*(t) = \mathbf{x}(a + b - t)$ parametrizes $-C$.

The line integral along C of a scalar function f defined on $\mathbb{R}^n$ ($n = 2$ or 3) is

$$\int_C f \, ds = \int_a^b f(\mathbf{x}(t)) \, \|\mathbf{x}'(t)\| \, dt \,.$$

Note that this definition is independent of the parametrization, as long as the orientation is the same. Indeed, let $\mathbf{y} : [c, d] \to C$ be another parametrization of C. Then there exists a one-to-one increasing function $\gamma : [c, d] \to [a, b]$ such that

$$\mathbf{x}(t) = \mathbf{x}(\gamma(\tau)) = \mathbf{y}(\tau) \,, \quad \mathbf{y}'(\tau) = \mathbf{x}'(\gamma(\tau))\gamma'(\tau) \,, \quad \gamma'(\tau) > 0 \,,$$

so that a change of variables shows

$$\int_c^d f(\mathbf{y}(\tau)) \, \|\mathbf{y}'(\tau)\| \, d\tau = \int_a^b f(\mathbf{x}(t)) \, \|\mathbf{x}'(t)\| \, dt \,.$$

A more general path C is made up of a finite number of simple paths with the end point of one being the starting point of the next, with compatible orientations. We denote: $C = C_1 + \cdots + C_k$. Then

$$\int_C f\, ds = \sum_{j=1}^{k} \int_{C_j} f\, ds\,.$$

The line integral of a vector field defined on C is $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} := \int_C \mathbf{F}\cdot\mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to C .

5.1 The fundamental theorem of line integrals

Theorem 5.1.1. ***The Fundamental Theorem.***

Let f be a differentiable function whose gradient, ∇f , is continuous on a smooth curve C parametrized by $\mathbf{x}(t)$, $a \le t \le b$. Then,

$$\int_C \nabla f \cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

5.2 Exercises.

13. Prove the Fundamental Theorem of Line Integrals.

Consider a surface S in $\mathbb{R}^3$ parametrized by

$$\mathbf{r} : D \to S \subset \mathbb{R}^3\,, \quad \mathbf{r}(u,v) = (x(u,v)\,,\, y(u,v)\,,\, z(u,v))\,,$$

a smooth injective function defined on an open set $D \subset \mathbb{R}^2$. The surface integral of a scalar function f defined on $\mathbb{R}^3$ on S is

$$\int_S f(\mathbf{x})\, dA = \iint_D f(\mathbf{r}(u,v)) \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv\,,$$

and the surface integral of a vector field on S is

$$\int_S \mathbf{F} \cdot d\mathbf{A} := \int_S \mathbf{F} \cdot \mathbf{n}\, dA \quad \text{(the flux of } \mathbf{F} \text{ across } S).$$

Hence,

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv\,.$$

6 Integral Theorems

6.1 The divergence theorem

Theorem 6.1.1. ***The Divergence Theorem.***

Given a domain $\Omega \subset \mathbb{R}^n$ (open and bounded), with a piecewise smooth boundary $\Gamma = \partial\Omega$, and $\mathbf{F} \in C^1(\overline{\Omega})$,

$$\int_\Omega \nabla \cdot \mathbf{F}\, dV = \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, dA\,,$$

where $\mathbf{n}$ is the unit normal vector to Γ pointing outwards Ω.

The Divergence Theorem is a version of the Fundamental Theorem of Calculus in multiple dimensions; it relates the integral of a "derivative" of the vector field in a domain to an integral of the field over the boundary of the domain.

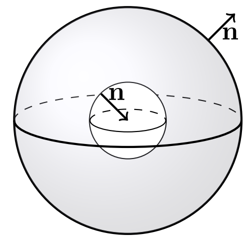
Other versions of the Fundamental Theorem of Calculus will be discussed in these notes. There are different "derivatives" of a vector field, of course.

Example. Let $\Omega \subset \mathbb{R}^3$ be a domain. Then the volume of Ω is

$$\mathrm{vol}(\Omega) = \frac{1}{3}\int_S \mathbf{x}\cdot\mathbf{n}\,dA\,,$$

where $S = \partial\Omega$ is the boundary of Ω.

Note: If there is a hole, the unit normal vector $\mathbf{n}$ points inward.



6.2 Green's theorem

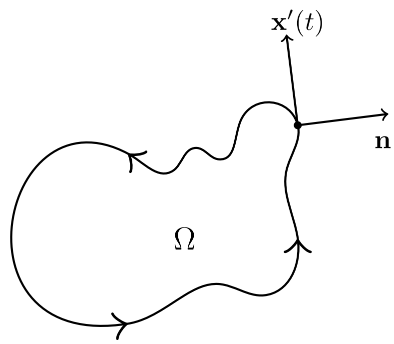
A particular case (when $n = 2$) is Green's Theorem.

Theorem 6.2.1. ***Green's Theorem.***

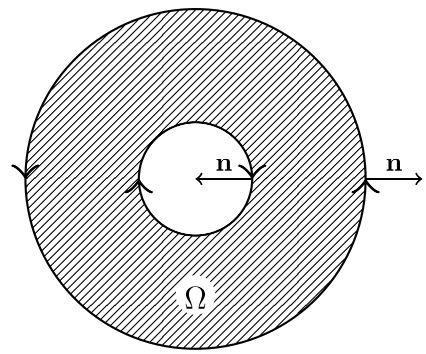
Given a domain $\Omega \subset \mathbb{R}^2$ (open and bounded), with a piecewise smooth boundary $\Gamma = \partial\Omega$, and $P, Q \in C^1(\overline{\Omega})$,

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma P\,dx + Q\,dy\,.$$

Note: $\mathbf{F} = (Q, -P)\,,\ \mathbf{x}' = (x', y')\,,\ \mathbf{n} = \dfrac{1}{\|\mathbf{x}'\|}(y', -x')\,,\ \mathbf{F}\cdot\mathbf{n} = \dfrac{1}{\|\mathbf{x}'\|}(Px' + Qy')\,,$
$\mathbf{F}\cdot\mathbf{n}\,ds = (Px' + Qy')\,dt\,.$



Note: While the above picture depicts a simply connected region, this is not necessary.
The region could look like:



6.3 An application of the divergence theorem: the continuity equation of fluid dynamics.

One application of the Divergence Theorem is to deriving the continuity equation in fluid flow, namely the differential version of conservation of mass.

Consider the flow of a fluid in a region of space where there are no sources or sinks of fluid. If we fix a "control volume" (i.e. a small fixed part of the flow region), conservation of mass states that the rate of change of mass of fluid within the control volume equals the net rate of inflow through its boundary. Let ρ be the fluid mass density, and $\mathbf{v}$ be the velocity field: $\mathbf{v}(\mathbf{x}, t) =$ the velocity of the fluid particle located at $\mathbf{x}$ at time t, and let D be the control volume. Then

$$\frac{d}{dt}\int_D \rho(\mathbf{x},t)\,dV = \text{net inflow of mass through } \partial D = -\int_{\partial D} \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}\,dA\,,$$

where $\mathbf{n}$ is the unit outward normal to D on ∂D. Note that over a very small part of the boundary, S, with area dA, along which the velocity and density are approximately constant, fluid crossing S travels a distance $\mathbf{v}\cdot\mathbf{n}$ in the direction perpendicular to S per unit of time, carrying along a mass $\rho\mathbf{v}\cdot\mathbf{n}dA$: if $\mathbf{v}\cdot\mathbf{n} > 0$, then fluid exits the control volume, while if $\mathbf{v}\cdot\mathbf{n} < 0$, then fluid enters the control volume. Adding all these contributions (as we do when defining an integral via Riemann sums), we obtain the net influx of mass across the boundary of the control volume, thus the appearance of the integral on the right-hand side of the equation above.

Since the control volume D is fixed in space and time, using the Divergence Theorem we obtain

$$\int_D \rho_t(\mathbf{x},t)\,dV = -\int_D \nabla\cdot(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t))\,dV$$

or

$$\boxed{\int_D \{\rho_t(\mathbf{x},t) + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t))\}\;dV = 0}\,.$$

This is the integral version of the equation for conservation of mass — it holds for any control volume within the flow region. Assuming that the integrand above is continuous throughout the flow region (this is a *smoothness* assumption on density and velocity), the fact that the integral is zero for all control volumes guarantees that the integrand itself must be zero (can you see why?). Hence,

$$\boxed{\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0} \tag{7}$$

is the differential version of conservation of mass, known as the *continuity equation*. Using the product rule, $\nabla \cdot (\rho \mathbf{v}) = \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}$, we obtain

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0\,,$$

where $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the *material derivative* which expresses the derivative of a quantity following the flow (e.g. following a blob of the fluid along as it flows). The flow is called *incompressible* if the density of a blob of fluid does not change as the blob moves along with the flow, i.e. $D\rho/Dt = 0$. Therefore, for incompressible flow the continuity equation reduces to

$$\boxed{\nabla \cdot \mathbf{v} = 0}\,. \tag{8}$$

6.4 Exercises.

14. Let $\mathbf{r} = \mathbf{x}$. Show that $\text{area}(\Omega) = \dfrac{1}{2} \displaystyle\int_{\partial\Omega} \mathbf{r} \cdot \mathbf{n}\, ds\,.$

15. Let $\mathbf{F}(\mathbf{x}) = \dfrac{1}{r^2}\, \mathbf{r}$.

    (a) Let $\Omega = \{\, (x,y) \,|\, x^2 + y^2 < 1 \,\}$. Compute $\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, ds$.

    (b) Let C be a positively oriented simple closed curve bounding a domain that contains the origin. Compute $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

(c) Repeat (b) above, but assuming the domain does not contain the origin.

16. Let $S = \{\, \mathbf{x} \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1, -1 \le z \le 1 \,\}$. Compute $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(\mathbf{x}) = (x + zy^2 e^y, y + z \sin x, z^2)$.

17. Let $\mathbf{G}(\mathbf{x}) = \dfrac{1}{r}\,\mathbf{r}$ and $D = \{\, \mathbf{x} \in \mathbb{R}^3 \,|\, a \le x^2 + y^2 + z^2 \le b \,\}$ $(0 < a < b)$.

    Compute $\displaystyle\int_{\partial D} \mathbf{G} \cdot \mathbf{n}\, dA$ in two different ways.

18. Let $\mathbf{F}(\mathbf{x}) = (x^3 + y \sin z, y^3 + z \sin x, z^3)$ and S be the surface of the solid bounded by $z = 0$, $z = \sqrt{1 - x^2 - y^2}$ and $z = \sqrt{4 - x^2 - y^2}$.

    Compute $\displaystyle\int_S \mathbf{F} \cdot \mathbf{dA}$.

19. Let $\mathbf{F}(\mathbf{x}) = \dfrac{1}{r^3}\,\mathbf{r}$, and S be any smooth closed surface containing the origin in its interior.

    Compute $\displaystyle\int_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{n}$ is the outward unit normal vector to S.

20. Prove that $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of path in a domain Ω iff $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ for every closed path C in Ω.

21. Prove that if $\mathbf{F} = \nabla f$ is a conservative field, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ for every closed path C in a domain Ω.

22. Prove that the area of a planar region bounded by a simple closed smooth curve $C\,:\, x = x(t),\ y = y(t),\ \ a \le t \le b$, is given by $A = \dfrac{1}{2}\displaystyle\int_a^b (x\dot{y} - y\dot{x})dt$.

23. Determine whether the following statements are true and give an explanation or counterexample.

    (a) If $\nabla \cdot \mathbf{F} = 0$ everywhere in a domain Ω, then $\mathbf{F} \cdot \mathbf{n} = 0$ at all points of $\partial\Omega$.

    (b) If $\displaystyle\int_S \mathbf{F} \cdot \mathbf{n}\, dA = 0$ on all closed surfaces $S \subset \mathbb{R}^3$, then $\mathbf{F}$ is constant.

    (c) If $\|\mathbf{F}\| < 1$, then $\displaystyle\left|\int_\Omega \nabla \cdot \mathbf{F}\, dV\right|$ is less than the area of the surface of Ω.

24. Define the **circulation** of a vector field $\mathbf{F}$ on a simple closed smooth oriented curve C as $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector consistent with the orientation of C. Prove Kelvin's Circulation Theorem.

6.5 Stokes theorem

Theorem 6.5.1. ***Stokes Theorem.***

Let $\mathbf{F}$ be a C^1 vector field in an open set containing an oriented piecewise smooth surface S bounded by a piecewise smooth curve C. Then,

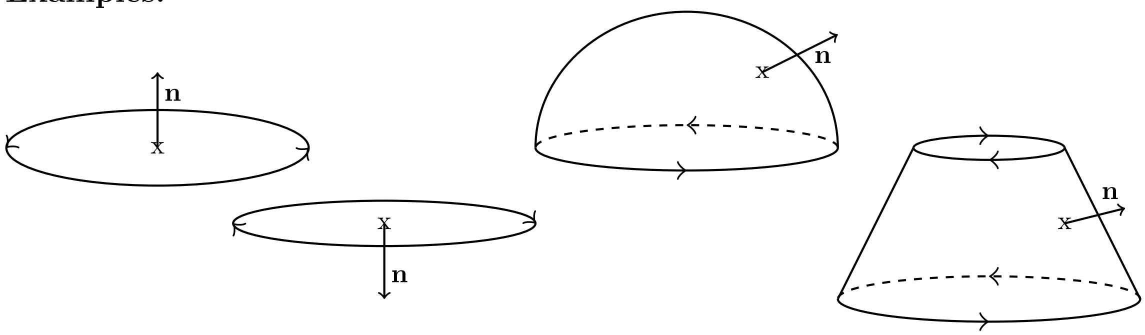
$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{s}\,,$$

where the orientation of C is the positive orientation induced by the orientation of S determined by $\mathbf{n}$.

Corollary

If S is an oriented piecewise smooth surface, then $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = 0$.

Examples:



6.6 Exercises.

25. Find the surface area of the upper spherical cap cut by $x^2 + y^2 = 1$ from the sphere $x^2 + y^2 + z^2 = 4$.

26. Let $\mathbf{F}(\mathbf{x}) = (0, x, 0)$. Compute $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$, where S is the hemisphere: $z \geq 0$, $x^2 + y^2 + z^2 = 1$, with the unit normal pointing outside the sphere.

27. Let S be the part of $z = x^2 + y^2$ lying below $z = 1$ oriented with unit normal pointing down. Let $\mathbf{F}(\mathbf{x}) = \left(yz + e^{x^3}\,,\ z^{10} - \cos y\,,\ x - y + \dfrac{1}{1 + z^4} \right)$. Compute $\displaystyle\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$.

28. Let $\phi(\mathbf{x}) = \|\mathbf{x}\|^n$ where n is sufficiently large. Compute $\displaystyle\int_B ((\Delta\phi)^2 + \nabla\phi \cdot \nabla\Delta\phi)\, dV$, where B is the unit sphere in $\mathbb{R}^3$.

6.7 Green's identities

Green's first identity is a version of integration by parts in higher dimension.

Theorem 6.7.1. ***Green's First and Second Identities.***

Given a domain $\Omega \subset \mathbb{R}^3$ (open and bounded), with a piecewise smooth boundary $\partial\Omega$, and $f, g \in C^2(\overline{\Omega})$, the following identities hold.

1. *1st identity:*

$$\int_\Omega (\nabla f \cdot \nabla g + f\Delta g)\, dV = \int_{\partial\Omega} f \nabla g \cdot \mathbf{n}\, dA.$$

2. *2nd identity:*

$$\int_\Omega (f\Delta g - g\Delta f)\, dV = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA.$$

6.8 Exercises.

29. Prove Green's 1st identity.

30. Prove Green's 2nd identity.

31. Let $\phi \in C^2(\mathbb{R}^3)$, satisfying $\phi = 0$ if $\|\mathbf{x}\| \geq M$. Prove that $\displaystyle\int_{\mathbb{R}^3} \frac{1}{4\pi r}\, \Delta\phi\, dV = -\phi(\mathbf{0})$.

    Hint: Use the the Divergence Theorem twice in a region away from the origin; then take a limit using Green's first identity.

32. Consider a body occupying a region Ω in space, with a piecewise smooth, closed and orientable boundary S. The steady-state temperature distribution in the body, $u(x, y, z)$, including a distribution of heat sinks and sources within the body, can be shown (from conservation of energy) to satisfy the BVP

$$\Delta u = F(x, y, z) \quad \text{in } \Omega, \tag{9}$$

$$u = g(x, y, z) \quad \text{in } S, \tag{10}$$

where g is the prescribed temperature on the boundary.

(a) Suppose there are two solution u_1 and u_2 of the BVP (3),(4). Define $\phi = u_1 - u_2$, and show that ϕ is a solution of the homogeneous BPV, i.e.

$$\Delta \phi = 0 \quad \text{in } \Omega,$$

$$\phi = 0 \quad \text{in } S.$$

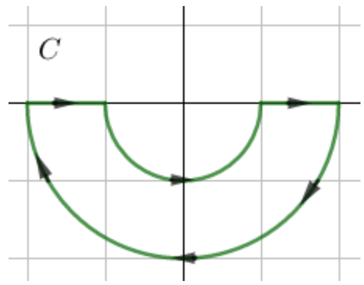
(b) Use Green's Theorem to show that $\displaystyle\iiint_R |\nabla \phi|^2 \, dV = 0$.

(c) Use the results above to show that the solution to the BVP (5),(6) is unique.

33. Show that if ϕ is harmonic, i.e. $\Delta \phi = 0$ in Ω as in Problem 32, then $\displaystyle\iint_S \frac{\partial \phi}{\partial n} \, dA = 0$. Interpret this physically.

6.9 Additional Exercises.

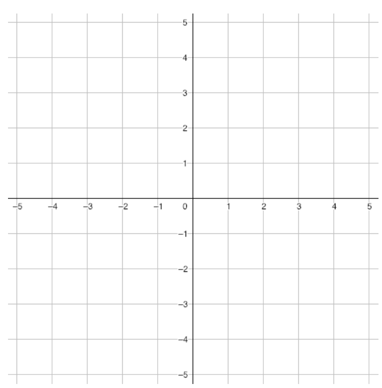
34. Let C be the clockwise-oriented boundary of the region Ω below the y-axis between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = (-2y^3 + \sin^{-1} x, 2x^3 - \cos^{-1} y)$.

35. Let C be the boundary of the rectangular region D from vertices $(4, 5)$ to $(5, 3)$ to $(1, 1)$ to $(0, 3)$ and back to $(4, 5)$.

   (a) Sketch C .



   (b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = (2y + \sin^{-1} x, 3x - \cos^{-1} y)$.

36. Let $\mathbf{F} = (P, Q)$ be a conservative field with potential ϕ, and let C be a smooth, closed curve in $\mathbb{R}^2$. Find $\int_C \mathbf{F} \cdot d\mathbf{s}$ in as many different ways as you can think of. Explain your answers.

37. Let $f(x, y, z) = z - x^2 \ln y$.

    (a) Calculate ∇f.

    (b) Let the surface S be defined by the equation $f(x, y, z) = 0$. Find an equation of the tangent plane to S at the point $(2, 1, 0)$.

38. Let $\mathbf{F}(x, y) = \left(\frac{1}{y} + \frac{y}{x^2}, 1 - \frac{x}{y^2} - \frac{1}{x} \right)$.

    (a) Show that $\mathbf{F}$ is a conservative field.

    (b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C consists of the three line segments from $(1, 1)$ to $(2, 2)$ to $(3, 2)$ to $(3, 6)$.

39. Let C be a positively oriented, piecewise-smooth, simple curve in the x, y-plane enclosing a region D of area 5. Evaluate $\int_C (3xy + \ln|y|)dx + \left(\frac{x}{5} + \frac{3x^2}{2} + \frac{x}{y} \right) dy$.