Definition: $A \subset \Omega$. The indicator function of A is the function

$$I_A(\omega) = \begin{cases} 1 \; ; & \omega \in A \\ 0 \; ; & \omega \in A . \end{cases}$$

Ex: ① Bernoulli RV

$$Y(\omega) = I_A(\omega)$$

$$P(Y = 1) = P(\{\omega : Y(\omega) = 1\})$$

$$= P(A)$$

$$P(Y = 0) = 1 - P(A)$$

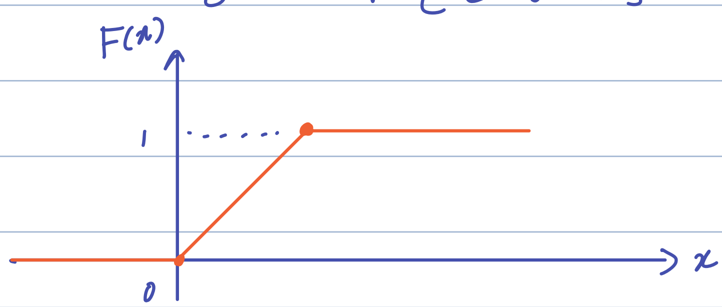
Y is called a Bernoulli RV.

② $\Omega = [0, 1]$

Let $X(\omega) = \omega \; ; \; \forall \omega \in [0, 1]$

Let the probability of A be its length.

$$P[X \leq x] = P\{[0, x]\} = x$$



X is called a Uniform rv.

Independent Random variables

Definition: Two events $A, B \in \mathcal{F}$ are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Theorem: If A, B are independent, & $P(B) > 0$, then

$$P(A \mid B) = P(A).$$

Proof: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A).$

Definition: $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of independent events if for any finite subsequence $\{i_1, i_2, \dots, i_k\} \subset \mathbb{N}$,

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$$

Ex: If two coin tosses are independent,
$P(H \cap H) = P(H) P(H) = (0.5)^2$

Ex: ① Are disjoint events independent? In general, no
$P(A \cap B) = P(A) \cdot P(B) = 0 \iff P(A) = 0$ or $P(B) = 0$.

② Could A be disjoint of itself ?

If A is independent of itself,

$$\underbrace{P(A\cap A)} = P(A)\cdot P(A)$$

$$P(A) = P(A)^2 \implies P(A) = 0, 1.$$

Independent random variables

Definition: (i) X, Y are independent if

$$\forall x, y \in \mathbb{R}, \quad P\{X \le x, Y \le y\} = P(X \le x)\, P(Y \le y)$$

Notation: $P\{X \le x, Y \le y\} = P\left[\{\omega : X(\omega) \le x\} \cap \{\omega : Y(\omega) \le y\}\right]$

(ii) A family $\{X_n\}_{n\in\mathbb{N}}$ is independent if for any finite subsequence $\{i_1, i_2, \dots, i_k\}$,

$$\forall x_{i_1}, x_{i_2}, \dots, x_{i_k} \in \mathbb{R},$$

$$P(X_{i_1} \le x_{i_1}, \dots, X_{i_k} \le x_{i_k}) = \prod_{n=1}^{k} P(X_{i_n} \le x_{i_n})$$

Theorem: If X, Y are independent, then $P(X\in A, Y \in A) = P(X\in A)\cdot P(Y\in A)$ for any set $A = (p, q]$, $B = (r, s]$.

Proof: I will prove a specific case $A = (-\infty, q]$, $B = (r, s]$,

$$P[X \in (-\infty, q], Y \in (r, s]] = P[X \le q, r < Y \le s]$$

$$= P[X \le q, Y \le s] - P[X \le q, Y \le r]$$

$$= P[X \le q] \cdot P[Y \le s] - P[X \le q] \cdot P[Y \le r]$$

$$= P[X \le q] \cdot [P[Y \le s] - P[Y \le r]]$$

$$= P[X \le q] \, P[r < Y \le s]$$

$$= P\{X \in (-\infty, q]\} \cdot P\{Y \in (r, s]\}$$

$\square$

Remark: This theorem holds for any Borel set $A, B \in \mathcal{B}(\mathbb{R})$.

Types of Distributions

Definition:

① Discrete Distributions: We say X has a discrete distribution if $\exists \{x_n\}$ of $\mathbb{R}$ s.t

$$\sum_{n=1}^{\infty} P(X = x_n) = 1.$$

② Continuous distributions: We say X has a continuous distribution if $\exists f$ on $\mathbb{R}$ s.t.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy.$$

f is called the density function.

Remark: ① If X is cts, $P(X = x) = 0$.

② $f(x) = \frac{d}{dx} F(x)$, when the derivative exists.

Definition: The probability mass function $p(x)$ of a discrete rv X is given by

$$p(x) = P(X = x), \quad x \in \mathbb{R}.$$

Remark: ① $p(x) = 0$ except for countably many points &

$$\sum_{x : p(x) > 0} p(x) = 1.$$

② $p(x) = F(x) - F(x-)$

③ $F(x) = \sum_{y \leq x} p(y)$

④ $f(x) \geq 0 \quad \forall x$

⑤ $\int_{-\infty}^{\infty} f(x)\,dx = 1$ because $\lim_{x \to \infty} F(x) = 1$.

Ex: A coin is tossed twice. Find the distribution function of the following random variables

① $X =$ # heads

② $X =$ # heads $-$ # tails.

① $\Omega = \{HH, HT, TH, TT\}$

$P(X=0) = P(TT) = \frac{1}{4} = p(0)$

$P(X=1) = P(HT \text{ or } TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = p(1)$

$P(X=2) = P(HH) = \frac{1}{4} = p(2)$

$F(0) = p(0) = \frac{1}{4}$

$F(1) = p(0) + p(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$F(2) = p(0) + p(1) + p(2) = 1$

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{1}{4} & ; \quad 0 \leq x < 1 \\ \frac{3}{4} & ; \quad 1 \leq x < 2 \\ 1 & ; \quad x \geq 2 \end{cases}$$

② $X \in \{-2, 0, 2\}$

$$P(X=-2) = P(TT) = \frac{1}{4}$$

$$P(X=0) = P(HT \text{ or } TH) = \frac{1}{2}$$

$$P(X=2) = P(TT) = \frac{1}{4}$$

$$F(x) = \begin{cases} 0 ; & x < -2 \\ \frac{1}{4} ; & -2 \le x < 0 \\ \frac{3}{4} ; & 0 \le x < 2 \\ 1 ; & x \geq 2 \end{cases}$$