Math 600 Lecture 23

<u>Definition</u>: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and let $(a,b) \subset I$. We say that that f is <u>increasing</u> on (a,b) iff

$$x_1, x_2 \in (a,b) \text{ and } x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$$

and <u>decreasing</u> on (a,b) iff

$$x_1, x_2 \in (a,b) \text{ and } x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2).$$

<u>Strictly increasing</u> and <u>strictly decreasing</u> have the obvious meanings. We say that f is <u>monotonic</u> on (a,b) iff it is increasing or decreasing on (a,b).

<u>Theorem</u>: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and suppose $(a,b) \subset I$. If f is increasing on (a,b), then, for all $c \in (a,b)$,

$$\lim_{x \to c^-} f(x) \text{ and } \lim_{x \to c^+} f(x)$$

exist, and

$$\lim_{x \to c^-} f(x) = \sup\{f(x) \mid a < x < c\} \le \inf\{f(x) \mid c < x < b\} = \lim_{x \to c^+} f(x).$$

Analogous results hold for decreasing functions.

<u>Proof</u>: Note that $S = \{f(x) \mid a < x < c\}$ is nonempty and bounded above by $f(t)$ for any $t \in (c,b)$.

Thus

$$L = \sup S$$

exists in $\mathbb{R}$. Let $\varepsilon > 0$ be given. Then there exists $s \in (a,c)$ such that

$$L - \varepsilon < f(s) \le L$$

(otherwise, $L-\varepsilon < L$ is an upper bound for S, a contradiction). Define $\delta = c - s$.

Then

$$x \in (a,c) \text{ and } |x-c| < \delta \Rightarrow s < x < c$$

$$\Rightarrow f(s) \le f(x) \le L \quad \text{(since } f \text{ is increasing)}$$

$$\Rightarrow |f(x) - L| \le |f(s) - L| < \varepsilon.$$

Thus $L = \lim\limits_{x \to c^-} f(x)$.

The proof that

$$\lim_{x \to c^+} f(x) = \inf\{f(x) \mid c < x < b\}$$

is similar, and

$$\lim_{x \to c^-} f(x) \le \lim_{x \to c^+} f(x)$$

follows immediately from

$$f(x) \le f(t) \quad \forall x \in (a,c) \ \forall t \in (c,b). //$$

<u>Corollary</u>: Let $f: I \to \mathbb{R}$ be monotonic on $(a,b) \subset I$. Then the only discontinuities of f in (a,b) are jump discontinuities.

<u>Theorem</u>: Let $f: I \to \mathbb{R}$ be monotonic on $(a,b) \subset I$. Then the set of discontinuities of f in (a,b) is countable.

<u>Proof:</u> Wlog assume that f is increasing. Let E be the set of discontinuities of f in (a,b) and, for each $x \in E$, choose $r_x \in \mathbb{Q}$ satisfying

$$\lim_{t \to x^-} f(t) < r_x < \lim_{t \to x^+} f(t).$$

Then define $\varphi: E \to \mathbb{Q}$ by $\varphi(x) = r_x \ \forall x \in E$. Clearly φ is injective and hence E is equivalent to a subset of $\mathbb{Q}$. Thus E is countable. //

<u>Note</u>: We previously defined

$$\lim_{x \to \infty} f(x), \quad \lim_{x \to -\infty} f(x).$$

The usual rules apply:

- these limits, if they exist, are unique.
- if $\lim_{x \to \infty} f(x)$, $\lim_{x \to \infty} g(x)$ exist (in $\mathbb{R}$), then

$$\lim_{x \to \infty} (f(x) \pm g(x)) = \lim_{x \to \infty} f(x) \pm \lim_{x \to \infty} g(x),$$

$$\lim_{x \to \infty} f(x)g(x) = \left(\lim_{x \to \infty} f(x)\right)\left(\lim_{x \to \infty} g(x)\right),$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)} \quad \left(\text{if } \lim_{x \to \infty} g(x) \neq 0\right).$$

These rules remain true if

$$\lim_{x \to \infty} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to \infty} g(x) = \pm\infty,$$

provided we avoid the indeterminate forms

$$\infty - \infty, \; 0 \cdot \infty, \; \frac{\infty}{\infty}.$$

(The proof is straightforward but tedious.)

Introduction to differentiation

<u>Definition</u>: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and suppose t lies in the interior of I. We say that f is <u>differentiable</u> at t iff

$$\lim_{x \to t} \frac{f(x) - f(t)}{x - t}$$

exists, in which case this limit is called the <u>derivative of f at t</u> and denoted $f'(t)$. That is,

$$f'(t) = \lim_{x \to t} \frac{f(x) - f(t)}{x - t},$$

if this limit exists.

If f is differentiable at every $t \in I$, we say that f is differentiable on I.

If $f: [a,b] \to \mathbb{R}$, we can define the (one-sided) derivatives at the endpoints as

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a},$$

$$f'(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}.$$

We don't have a special notation for the one-sided derivatives (and we don't use them much).

<u>Differentiation is linear:</u>

Suppose $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are both differentiable at t, and let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha f + \beta g)'(t) = \lim_{x \to t} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(t)}{x - t}$$

$$= \lim_{x \to t} \frac{\alpha f(x) + \beta g(x) - \alpha f(t) - \beta g(t)}{x - t}$$

$$= \lim_{x \to t} \left(\alpha \frac{f(x) - f(t)}{x - t} + \beta \frac{g(x) - g(t)}{x - t} \right)$$

$$= \alpha \lim_{x \to t} \frac{f(x) - f(t)}{x - t} + \beta \lim_{x \to t} \frac{f(x) - f(t)}{x - t}$$

$$= \alpha f'(t) + \beta f'(t).$$

<u>Differentiation defines a linear map</u>

If we define vector spaces

$$C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$$

$$C^1[a,b] = \{ f \in C[a,b] \mid f \text{ is differentiable on } [a,b] \text{ and } f' \in C[a,b] \},$$

then

$$D : C^1[a,b] \to C[a,b],$$

$$Df = f'$$

is a linear map.