Markov Chains

Definition : Let $\{X_n\}$ be a sequence of random variables taking values in a countable set S, called the state space.

If $\forall n \geq 0$,

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) \quad (*),$$

then X is said to be a Markov Chain, or to have the Markov Property.

The Markov property states that the next state X_{n+1} only depends on the current state & is independent of past states X_i, $\forall i < n$.

We write

$$p_{ik} = P(X_1 = k \mid X_0 = i)$$

$\{p_{ik},\ i, k \in S\}$, are called the transition probabilities of the chain $\{X_n\}$

The MC is called time-homogeneous if

$$P(X_{n+1} = k \mid X_n = i) = p_{ik}, \quad \forall n.$$

The Markov property ⊛ is also equivalent to statement below:

$$P(X_{n+m} = j \mid X_0 = i_0, \ldots, X_n = i) = P(X_{n+m} = j \mid X_n = i)$$

$\forall m, n \geq 0$.

Definition : $p_{ik} = P(X_{n+1} = k \mid X_n = i)$, $n \geq 0$.

A $M \times M$ matrix $Q = (p_{ij})$ is called a

stochastic matrix if : i) $p_{ij} \geq 0$, $\forall i,j$

ii) $\sum_{k \in S} p_{ik} = 1$. (row sum = 1).

In addition, if

$$\sum_{i \in S} p_{ik} = 1 ,$$

then Q is called doubly stochastic.

If $p_{ij} > 0$ $\forall i, j$, then Q is called positive. Given $X_0 = i$, the distribution of X_n is given b

$$p_{ik}^{(n)} = P(X_n = k \mid X_0 = i) .$$

Clearly , $\sum_{k \in S} p_{ik}^{(n)} = 1$. as $p_{ik}^{(n)}$ is the mass function of $X_n \mid X_0 = i$

Ex. Let $\{X_n\}$ be a MC on the state space S_X.

Show that $Y_n = (X_n, X_{n+1})$; $n \geq 0$ is a MC.

Let $S_Y = \{ (s_1, s_2) : s_1, s_2 \in S_X \}$

$P(Y_{n+1} = (j,k) \mid Y_0, Y_1, \dots, Y_n)$

$= P(X_{n+2} = k, X_{n+1} = j \mid X_0, \dots, X_{n+1})$

$= P(X_{n+2} = k, X_{n+1} = j \mid X_{n+1}, X_n)$ (Since X a is MC)

$= P(Y_{n+1} = (j,k) \mid Y_n)$.

Ex: If $\{X_n\}$ is a MC, then prove that $\{X_{2n}\}$ is a MC.

Solution: Let $Y_n = X_{2n}$.

$$P(Y_{n+1} = j \mid Y_0, Y_1, \dots, Y_n = i)$$

$$= P(X_{2n+2} = j \mid X_0, X_2, \dots, X_{2n} = i)$$

By the Markov property,

$$= P(X_{2n+2} = j \mid X_{2n} = i)$$

$$= P(Y_{n+1} = j \mid Y_n = i) \quad \square$$

Ex: Let $\{X_n\}$ be a MC with a transition matrix

$$Q = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} \end{array} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

Then,

$$Q^{(2)} = Q \times Q = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix} \end{array} = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{pmatrix}$$

$P_{ij}^{(2)}$ is the (i,j) entry of Q^2

$$P_{12}^{(2)} = 5/9 = P(X_2 = 2 \mid X_0 = 1)$$

$$P_{22}^{(2)} = 7/12 = P(X_2 = 2 \mid X_0 = 2)$$

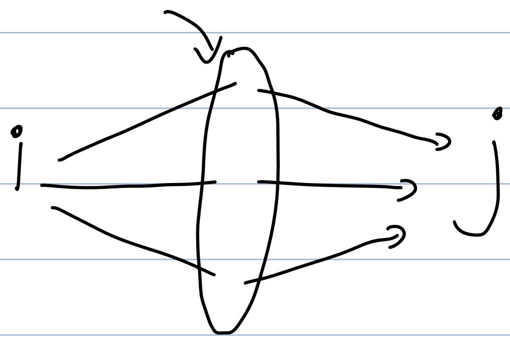
Notice that

$$P_{11}^{(2)} = (P_{11})(P_{11}) + (P_{12})(P_{21})$$

$$P_{22}^{(2)} = (P_{21})(P_{12}) + (P_{22})(P_{22})$$

In general,

$$P_{ij}^{(2)} = \sum_k (P_{ik})(P_{kj})$$



Theorem: The Chapman - Kolmogorov equations

Let $Q = [p_{ij}]$, $i, j \in S$. Then

$\forall m, n \in \mathbb{N}$,

$$\boxed{p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} \cdot p_{kj}^{(n)}}$$

Also,

$$p_{ij}^{(n+1)} = \sum_{j_1 \in S} \cdots \sum_{j_n \in S} p_{ij_1} \cdot p_{j_1 j_2} \cdots \cdot p_{j_n k}$$

Proof:

$$p_{ij}^{(m+n)} = P(X_{m+n} = j \mid X_0 = i)$$

$$= P\left(\bigcup_{k \in S} \{X_{m+n} = j, X_n = k\} \mid X_0 = i \right)$$

$$= \sum_{k \in S} P(X_{m+n} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j \mid X_n = k, X_0 = i) \, P(X_n = k \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j \mid X_n = k) \cdot P(X_n = k \mid X_0 = i)$$

$$= \sum_{k \in S} p_{kj}^{(m)} \cdot p_{ik}^{(n)}$$

$$= \sum_{k \in S} p_{ik}^{(n)} \cdot p_{kj}^{(m)}$$

Let $Q^{(n)}$ denote the matrix of the n-step transition probabilities $P_{ij}^{(n)}$.

Then, $Q^{(n+m)} = Q^{(m)} \cdot Q^{(n)}$.

In particular,

$$Q^{(2)} = Q^{(1+1)} = Q \cdot Q = Q^2$$

In general,

$$Q^{(n)} = Q^n .$$

← The n-step matrix can be obtained by multiplying the matrix Q by itself n times.

Let $i \in S = \{1, 2, \dots, M$

Define $\alpha_i = P(X_0 = i)$.

Therefore, $\sum_{i \in S} \alpha_i = 1$.

If $\alpha_j^{(n)} = P(X_n = j)$, then

$$\alpha_j^{(n)} = P(X_n = j) = \sum_i \underbrace{P(X_n = j \mid X_0 = i)}_{P_{ij}^{(n)}} \underbrace{P(X_0 = i)}_{\alpha_i}$$

$$\Rightarrow \quad \alpha_j^{(n)} = \sum_i \alpha_i \cdot P_{ij}^{(n)}$$

Ex: Let $Q = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}$, i.e. $Q = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

& $P(X_0 = 0) = \frac{1}{4}$, $P(X_0 = 1) = \frac{3}{4}$.

Find $P(X_3 = 0)$.

$$Q^2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$Q^3 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{array}{c|cc} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 1 & \frac{7}{8} & \frac{1}{8} \end{array}$$

We are given that

$$\alpha^{(0)} = (\alpha_0 \quad \alpha_1) = \left(\frac{1}{4} \quad \frac{3}{4}\right)$$

Therefore, $\alpha_0^{(3)} = P(X_3 = 0) = \sum \alpha_i \cdot P_{i0}^{(3)}$

$$= \frac{1}{4} \cdot P_{00}^{(3)} + \frac{3}{4} P_{10}^{(3)}$$

$$= \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{7}{8} = \frac{1}{4} + \frac{21}{32} = \frac{29}{32}$$

If we let $\alpha^{(n)} = (\alpha_0^{(n)} \quad \alpha_1^{(n)} \quad \cdots \quad \alpha_M^{(n)})$, then $$\boxed{\alpha^{(n)} = \alpha^{(0)} \cdot Q^{(n)}}$$

If we apply this to the previous example,

$$\alpha^{(3)} = \alpha^{(0)} \cdot Q^{(3)}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$$

$$\alpha^{(3)} = \begin{pmatrix} \frac{29}{32} & \frac{3}{32} \end{pmatrix} \implies \alpha_0^{(3)} = P(X_3 = 0) = \frac{29}{32}$$

$$\alpha_1^{(3)} = P(X_3 = 1) = \frac{3}{32}$$

<u>Definition</u>: Let $S = \{1, 2, \ldots, M\}$. We say a MC X on S is regular if $\exists\, n_0 < \infty$ such that

$$p_{ij}^{(n_0)} > 0 \quad \forall i, j \in S.$$

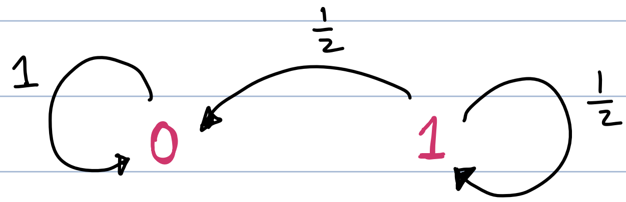
Ex: ① $$Q = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$Q^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} > 0$$

$\implies Q$ is regular.

② $$Q = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$ is <u>not</u> regular

since $p_{01}^{(n)} = 0 \quad \forall n.$



Some chains satisfy the following weaker condition.

<u>Definition</u> : X is irreducible if $\forall i, j \in S$,

$\exists\, n_0 < \infty$ s.t $P_{ij}^{(n_0)} > 0$.