Math 672 Lecture 17

Recall:

- Given $T \in \mathcal{L}(V)$, where V is a finite-dimensional complex vector space, there exists a basis $\mathcal{B}$ of V such that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is upper triangular.

- Given a finite-dimensional vector space over any field F, if $T \in \mathcal{L}(V)$ and $\mathcal{B}$ is a basis for V consisting of eigenvectors of T, then $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal.

The converse of the last result is true.

Theorem: Let V be a finite-dimensional vector space over a field F, let $T \in \mathcal{L}(V)$, and let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then $\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal iff v_j is an eigenvector of T for each $j = 1, 2, \ldots, n$.

Proof: We have already proven the "if" direction. Suppose

$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)$ is diagonal, say $A = \mathcal{M}_{\mathcal{B},\mathcal{B}}(T) = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Then, for each $j = 1, 2, \dots, n$,

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i = \lambda_j v_j \quad \left(\text{since } A_{ij} = \begin{cases} \lambda_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right).$$

We know that $v_j \neq 0$ because $\{v_1, v_2, \dots, v_n\}$ is linearly independent. Thus v_j is an eigenvector for T. //

<u>Corollary</u>: Let $T \in \mathcal{L}(V)$, where V is an n-dimensional vector space over a field F. If T has n distinct eigenvalues in F, then T is diagonalizable.

<u>Proof</u>: Let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues of T and let $v_1, \dots, v_n$ be corresponding eigenvectors. From a previous result, we know that $\{v_1, \dots, v_n\}$ is linearly independent and hence (since $\dim(V) = n$) is a basis for V. But then V has a basis consisting of eigenvectors of T, which implies that T is diagonalizable. //

Lemma: Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space over a field F, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of T. Then

$$E(\lambda_1, T) + E(\lambda_2, T) + \cdots E(\lambda_m, T)$$

is a direct sum.

Proof: Recall that it suffices to prove that

$$u_j \in E(\lambda_j, T) \ \forall j = 1, 2, \ldots, m \text{ and } \sum_{j=1}^{m} u_j = 0$$

$$\Rightarrow u_j = 0 \ \forall j = 1, 2, \ldots, m.$$

We will prove the contrapositive. Suppose $u_j \in E(\lambda_j, T)$ for $j = 1, 2, \ldots, n$ and not all u_j equal the zero vector. Let $\{j_1, \ldots, j_k\} = \{ j \in \{1, 2, \ldots, m\} \mid u_j \neq 0 \}$. Then $\{u_{j_1}, u_{j_2}, \ldots, u_{j_k}\}$ is linearly independent (since these vectors correspond to distinct eigenvalues) and hence

$$\sum_{i=1}^{k} u_{j_i} \neq 0.$$

But this implies that

$$\sum_{j=1}^{m} u_j = \sum_{i=1}^{k} u_{j_i} \neq 0,$$

as desired. This completes the proof. //

Corollary: Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space over a field F, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of T. Then

$$\sum_{j=1}^{m} \dim(E(\lambda_j, T)) \leq \dim(V)$$

Theorem Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space over a field F, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be (all of) the distinct eigenvalues of T. Then T is diagonalizable iff

$$E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T) = V,$$

that is, iff

$$\sum_{j=1}^{m} \dim(E(\lambda_j, T)) = \dim(V).$$

Proof: Suppose first that

$$\sum_{j=1}^{m} \dim(E(\lambda_j,T)) = n = \dim(V).$$

Write $k_j = \dim(E_{\lambda_j}, T)$ and let $\{v_1^{(j)}, \ldots, v_{k_j}^{(j)}\}$ be a basis for $E(\lambda_j, T)$.

We know that

$$\mathcal{B} = \bigcup_{j=1}^{m} \{v_1^{(j)}, \ldots, v_{k_j}^{(j)}\}$$

is linearly independent. Since $\mathcal{B}$ contains n vectors, it is a basis for V. Thus there exists a basis for V consisting of eigenvectors of T, and hence T is diagonalizable.

Conversely, suppose T is diagonalizable. It follows that there exist a basis $\mathcal{B}$ of V consisting of eigenvalues of T. Note that each $v \in \mathcal{B}$ lies in $E(\lambda_j, T)$ for some $j \in \{1, 2, \ldots, m\}$. Define

$$\mathcal{B}_j = \mathcal{B} \cap E(\lambda_j, T).$$

Since $\mathcal{B}_j$ is linearly independent,

$$|\mathcal{B}_j| \le \dim(E(\lambda_j, T)) \quad \forall j = 1, 2, \ldots, m$$

(where $|\mathcal{B}_j|$ is the number of elements of $\mathcal{B}_j$). Therefore,

$$n = \sum_{j=1}^{m} |\mathcal{B}_j| \le \sum_{j=1}^{m} \dim(E_{\lambda_j}, T) \le n$$

(the last inequality holds because $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum). But this implies that

$$\dim(E(\lambda_1,T)+\cdots+E(\lambda_m,T)) = \sum_{j=1}^{m}\dim(E(\lambda_j,T)) = n$$

and hence that

$$E(\lambda_1,T)+\cdots+E(\lambda_m,T)=V. //$$

Even if V is a complex finite-dimensional vector space, it is not the case that every $T\in\mathcal{L}(V)$ is diagonalizable.

Before we give an example, we point out the following: If $T: F^n \to F^n$ is defined by $T(x)=Ax$ for all $x\in F^n$, where $A\in F^{n\times n}$ is a given matrix, and if $\mathcal{B}=\{e_1,e_2,\ldots,e_n\}$ (the standard basis for F^n), then

$$\mathcal{M}_{\mathcal{B},\mathcal{B}}(T)=A.$$

This follows because $\mathcal{M}_{\mathcal{B}}(v)=v$ for all $v\in F^n$ (that is, $\mathcal{M}_{\mathcal{B}}$ is the identity operator on F^n).

<u>Example</u>: Define $T:\mathbb{C}^3\to\mathbb{C}^3$ by $T(x)=Ax$ for all $x\in\mathbb{C}^3$, where

$$A=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}.$$

Since A is upper triangular, the eigenvalues of A are the diagonal entries of A. Thus 1 is the only eigenvalue of A.

We find the eigenvectors by solving $(A-I)x=0$:

$$A-I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-I)x = 0 \iff \begin{aligned} x_2 + x_3 &= 0 \\ x_3 &= 0 \end{aligned} \iff x_2 = x_3 = 0 \quad (x_1 \text{ is free})$$

Thus

$$E(1,T) = \mathcal{N}(A-I) = \operatorname{span}(e_1)$$

We see that $\mathbb{C}^3$ does not have a basis consisting of eigenvectors of T, and therefore T is not diagonalizable.