Math 672 Lecture 22

<u>Definition</u>: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a finite-dimensional subspace of V. Define $P_S : V \to V$ by the condition that $P_S(x)$ is the best approximation to x from S. We called P_S the <u>orthogonal projection</u> (operator) onto S.

<u>Note</u>: P_S is well defined by the projection theorem.

<u>Lemma</u> Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a finite-dimensional subspace of V. Then $P_S \in \mathcal{L}(V)$.

<u>Proof</u>: Let $x, y \in V$ and $\alpha, \beta \in F$. Note that $P_S(x)$ is the unique element of S satisfying

$$\langle x - P_S(x), w \rangle = 0 \quad \forall w \in S,$$

and similarly for $P_S(y)$, $P_S(\alpha x + \beta y)$. But then

$$\begin{aligned}
&\langle \alpha x + \beta y - (\alpha P_S(x) + \beta P_S(y)), w \rangle \\
&= \langle \alpha (x - P_S(x)) + \beta (y - P_S(y)), w \rangle \\
&= \alpha \langle x - P_S(x), w \rangle + \beta \langle y - P_S(y), w \rangle = 0 + 0 \quad \forall w \in S,
\end{aligned}$$

which shows that $P_S(\alpha x + \beta y) = \alpha P_S(x) + \beta P_S(y)$. //

Theorem (the Gram-Schmidt orthogonalization procedure): Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$), let $\{v_1, \ldots, v_n\} \subseteq V$ be linearly independent, and define

$$S_k = \operatorname{span}(v_1, \ldots, v_k), \quad k = 1, 2, \ldots, n.$$

Define $u_1, \ldots, u_n \in V$ as follows:

$$u_1 = v_1,$$

$$u_{k+1} = v_{k+1} - P_{S_k} v_{k+1}, \quad k = 1, \ldots, n-1.$$

Then, for each $k = 1, 2, \ldots, n$, $\{u_1, \ldots, u_k\}$ is an orthogonal basis for S_k.

Proof: We argue by induction on k. For $k=1$, we see that $\{u_1\} = \{v_1\}$ is a basis for $S_1 = \operatorname{span}(v_1)$, and it is (vacuously) orthogonal.

Suppose $\{u_1, \ldots, u_k\}$ is an orthogonal basis for $S_k = \operatorname{span}(v_1, \ldots, v_k)$. Note that

$$P_{S_k} v_{k+1} = \sum_{j=1}^{k} \alpha_j v_j$$

for a certain choice of $\alpha_1, \ldots, \alpha_k$. It follows that

$$u_{k+1} = v_{k+1} - P_{S_k} v_{k+1} \in \operatorname{span}(v_1, \ldots, v_{k+1});$$

Moreover,

$$\langle u_{k+1}, u_j \rangle = \langle v_{k+1} - P_k v_{k+1}, u_j \rangle = 0 \quad \forall j = 1, 2, \ldots, k \quad (\text{since } u_j \in S_k \text{ for } j = 1, \ldots, k)$$

and hence $\{u_1, \ldots, u_{k+1}\}$ is an orthogonal subset of S_{k+1}. It follows that

$\{u_1, \ldots, u_{k+1}\}$ is an orthogonal basis for S_{k+1}. This completes the proof by induction. //

<u>Corollary</u>: Let V be a finite-dimensional inner product space over F ($\mathbb{R}$ or $\mathbb{C}$). Then V has an orthogonal (or orthonormal) basis.

<u>Recall</u>: If $\{u_1, \ldots, u_k\}$ is an orthogonal basis for S_k, then

$$P_{S_k} v_{k+1} = \sum_{j=1}^{k} \frac{\langle v_{k+1}, u_k \rangle}{\langle u_k, u_k \rangle} u_k.$$

<u>Example</u>: Let us compute an orthogonal basis for $\mathcal{P}_3$, regarded as a subspace of $C[0,1]$ (under the $L^2(0,1)$ inner product).

The standard basis for $\mathcal{P}_3$ is $\{p_0, p_1, p_2, p_3\}$, where $p_j(x) = x^j$. Let us write $\{q_0, q_1, q_2, q_3\}$ for the orthogonal basis.

<u>Step 1</u>: $q_0 = p_0 \Rightarrow q_0(x) = 1$. Write $S_0 = \text{span}(p_0) = \text{span}(q_0)$.

<u>Step 2</u>: $q_1 = p_1 - P_{S_0} p_1$

$$P_{S_0} p_1 = \frac{\langle p_1, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 \qquad \langle p_1, q_0 \rangle = \int_0^1 x \cdot 1 \, dx = \frac{1}{2}$$

$$\langle q_0, q_0 \rangle = \int_0^1 1^2 dx = 1$$

$$= \frac{1}{2} q_0$$

$$\Rightarrow q_1(x) = p_1(x) - \frac{1}{2} q_0(x) = x - \frac{1}{2} \qquad \text{Write } S_1 = \operatorname{span}(p_0, p_1)$$

<u>Step 3:</u> $q_2 = p_2 - P_{S_1} p_2$

$$P_{S_1} p_2 = \frac{\langle p_2, q_0\rangle}{\langle q_0, q_0\rangle} q_0 + \frac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1$$

$$= \frac{1/3}{1} q_0 + \frac{1/12}{1/12} q_1$$

$$= \frac{1}{3} q_0 + q_1$$

$$\Rightarrow q_2(x) = p_2(x) - \frac{1}{3} q_0(x) - q_1(x)$$

$$= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}$$

$$\langle p_2, q_0\rangle = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3}$$

$$\langle q_0, q_0\rangle = 1$$

$$\langle p_2, q_1\rangle = \int_0^1 x^2\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$\langle q_1, q_1\rangle = \int_0^1 (x - 1/2)^2 dx$$

$$= \int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

<u>Step 4:</u> $q_3 = p_3 - P_{S_2} p_3$

$$P_{S_2} p_3 = \frac{\langle p_3, q_0\rangle}{\langle q_0, q_0\rangle} q_0 + \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 + \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2$$

$$\langle p_3, q_0\rangle = \int_0^1 x^3 \cdot 1\, dx = \frac{1}{4} \qquad \boxed{\frac{\langle p_3, q_0\rangle}{\langle q_0, q_0\rangle} = \frac{1}{4}}$$

$$\langle q_0, q_0\rangle = 1$$

$$\langle p_3, q_1\rangle = \int_0^1 x^3\left(x - \frac{1}{2}\right) dx = \int_0^1 \left(x^4 - \frac{1}{2}x^3\right) dx$$

$$= \frac{1}{5} - \frac{1}{8} = \frac{3}{40}$$

$$\langle q_1, q_1 \rangle = \frac{1}{12}$$

$$\boxed{\frac{\langle p_3, q_1 \rangle}{\langle q_1, q_1 \rangle} = \frac{9}{10}}$$

$$\langle p_3, q_2 \rangle = \int_0^1 x^3 \left(x^2 - x + \frac{1}{6}\right) dx$$

$$= \int_0^1 \left(x^5 - x^4 + \frac{1}{6}x^3\right) dx$$

$$= \frac{1}{6} - \frac{1}{5} + \frac{1}{24}$$

$$= \frac{20 - 24 + 5}{120} = \frac{1}{120}$$

$$\langle q_2, q_2 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{36 - 90 + 80 - 30 + 5}{180} = \frac{1}{180}$$

$$\boxed{\frac{\langle p_3, q_2 \rangle}{\langle q_2, q_2 \rangle} = \frac{3}{2}}$$

Thus

$$q_3(x) = p_3(x) - \frac{1}{4} q_0(x) - \frac{9}{10} q_1(x) - \frac{3}{2} q_2(x)$$

$$= x^3 - \frac{1}{4} - \frac{9}{10}\left(x - \frac{1}{2}\right) - \frac{3}{2}\left(x^2 - x + \frac{1}{6}\right)$$

$$= x^3 - \frac{1}{4} - \frac{9}{10}x + \frac{9}{20} - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{4}$$

$$= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

An orthogonal basis is

$$\left\{1,\ x-\frac{1}{2},\ x^2-x+\frac{1}{6},\ x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20}\right\}.$$

Why not compute an orthonormal basis as we go along?

Answer: It's probably a bit harder. For example, we would have

$$q_0(x) = 1$$

$$q_1(x) = \frac{1}{\sqrt{1/12}}\left(x-\frac{1}{2}\right) = 2\sqrt{3}\left(x-\frac{1}{2}\right)$$

$$q_2(x) = \frac{1}{\sqrt{3/2}} = \frac{\sqrt{2}}{\sqrt{3}}\left(x^2-x+\frac{1}{6}\right),$$

etc.

I think that carrying around the square roots is inconvenient.

Definition: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a subset of V. The orthogonal complement $S^\perp$ of S is the set

$$S^\perp = \{u \in V \mid \langle u, w\rangle = 0\ \forall w \in S\}.$$

Theorem: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let S be a subset of V. Then:

- $S^\perp$ is a subspace of V.
- If S is a subspace of V, then $S \cap S^\perp = \{0\}$.
- If S is a subspace of V, then $(S^\perp)^\perp = S$.

Proof: To prove that $S^\perp$ is a subspace, we just verify the three necessary properties:

- $0 \in S^\perp$ because 0 is orthogonal to every vector, and hence to every vector in S.
- Suppose $u, v \in S^\perp$. Then

$$\langle u, w\rangle = 0 \text{ and } \langle v, w\rangle = 0 \quad \forall w \in S$$

$$\Rightarrow \langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = 0+0 = 0 \quad \forall w \in S$$

$$\Rightarrow u+v \in S^\perp.$$

  Thus $S^\perp$ is closed under addition.

- Suppose $u \in S^\perp$ and $\alpha \in F$. Then

$$\langle u, w\rangle = 0 \quad \forall w \in S$$

$$\Rightarrow \langle \alpha u, w\rangle = \alpha\langle u, w\rangle = \alpha \cdot 0 = 0 \quad \forall w \in S$$

$$\Rightarrow \alpha u \in S^\perp.$$

  Thus $S^\perp$ is closed under scalar multiplication, and we have verified that $S^\perp$ is a subspace of V.

Next, suppose S is a subspace of V. If $w \in S \cap S^\perp$, then $\langle w, w \rangle = 0$ (since $w \in S^\perp$ and $w \in S$), which implies that $w = 0$.

Since obviously $0 \in S \cap S^\perp$, we see that $S \cap S^\perp = \{0\}$.

Now consider $S^{\perp\perp} = (S^\perp)^\perp$. By the first result, $S^{\perp\perp}$ is a subspace of V. Note that $S \subseteq S^{\perp\perp}$:

$$\langle u, w \rangle = 0 \;\; \forall w \in S \;\; \forall u \in S^\perp \text{ (by definition of } S^\perp)$$

$$\Rightarrow \langle w, u \rangle = 0 \;\; \forall u \in S^\perp \; \forall w \in S$$

$$\Rightarrow w \in S^{\perp\perp} \;\; \forall w \in S.$$

Now suppose $w \in S^{\perp\perp}$ and consider $v = P_S w$. Note that

$$v \in S \subseteq S^{\perp\perp}, \; w \in S^{\perp\perp}$$

$$\Rightarrow w - v \in S^{\perp\perp}.$$

But $w - v \in S^\perp$ by definition of P_S. Thus

$$w - v \in S^\perp \cap S^{\perp\perp} = \{0\}$$

$$\Rightarrow w - v = 0$$

$$\Rightarrow w = v$$

$$\Rightarrow w \in S.$$

Thus $S^{\perp\perp} = S.$ //