MATH 630

Exam 1

Review Session

Fall 2024

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(1) Suppose $P\left(\left[0, \frac{8}{4+n}\right]\right) = \frac{2+e^{-n}}{6}$ for all $n \in \mathbb{N}$. Find $P(\{0\})$.

Let $A_n = \left[0, \frac{8}{4+n}\right]$. $\{A_n\}$ is a decreasing sequence of sets & $\bigcap A_n = \{0\}$. By continuity of P,

$$P(\{0\}) = P\left(\bigcap A_n\right) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \frac{2+e^{-n}}{6} = \frac{1}{3}$$

(2) Suppose $P([0,\infty)) = 1$. Prove that there exists $n \in \mathbb{N}$ such that $P([0,n)) > 0.99$.

Assume $P([0,n)) \leq 0.99 \quad \forall n \in \mathbb{N}$.

Let $A_n = [0,n)$. $\{A_n\}$ is increasing &

$$\bigcup A_n = [0,\infty)$$

$$1 = P([0,\infty)) = P\left(\bigcup A_n\right) = \lim_{n\to\infty} P(A_n)$$

$$= \lim_{n\to\infty} \underbrace{P([0,n))}_{\leq 0.99} \leq 0.99.$$

This a contradition. Thus, our assumption is false.

(3) A flight has 100 seats, and there are 100 passengers, each with an assigned seat. Due to a mix-up, each passenger randomly selects a seat when boarding the plane. What is the probability that exactly five passenger sit in their assigned seat?

Method I: We want 5 people to select their correct seats. We can do this $\binom{100}{5}$ ways.

The remaining 95 must select the wrong sets.

$A_i = \{$ i th person selects the correct sets$\}$; $i = 1, 2, \ldots, 95$
(when 5 have selected the correct seat)

$$P(A_1^c \cap \ldots \cap A_{95}^c) = 1 - P\left(\bigcup_{i=1}^{95} A_i\right)$$

$$= 1 - \left[\sum P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \right.$$

$$\left. - \cdots + (-1)^{95+1} P(A_1 \cap A_2 \cap \ldots \cap A_{95}) \right]$$

$$= 1 - \left[\sum_{i=1}^{95} \frac{1}{95} - \overbrace{\sum_{i_1 < i_2 \leq 95}}^{\binom{95}{2}} \frac{(95-2)!}{95!} + \overbrace{\sum_{i_1 < i_2 < i_3 \leq 95}}^{\binom{95}{3}} \frac{(95-3)!}{95!} \right.$$

$$\left. - \cdots + \frac{1}{95!} \right]$$

$$= 1 - \left[1 - \binom{95}{2} \frac{93!}{95!} + \binom{95}{3} \frac{92!}{95!} - \cdots + \frac{1}{95!} \right]$$

$$= \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{95!} \Rightarrow \text{Ans: } \boxed{\binom{100}{5}\left(\frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{95!}\right)}$$

Method II

$$P(X \geq 5) = \frac{\binom{100}{5} 95!}{100!}$$

$$P(X \geq 6) = \frac{\binom{100}{6} 94!}{100!}$$

$$P(X = 5) = P(X \geq 5) - P(X \geq 6)$$

$$= \boxed{\frac{\binom{100}{5} 95! - \binom{100}{6} 94!}{100!}}$$

(4) Let X be a discrete random variable with a probability mass function

$$p(k) = \frac{-p^k}{k\ln(1-p)}$$

for $k = 1, 2, 3, ...$ and $0 < p < 1$. Find the following expected values:

(a) $E(X)$

(b) $E\left(\frac{1}{1+X}\right)$

(a) $$E[X] = -\sum_{k=1}^{\infty} k \cdot \frac{p^k}{k\ln(1-p)}$$

$$= \frac{1}{\ln(1-p)} \sum_{k=1}^{\infty} p^k$$

$$= \frac{1}{\ln(1-p)} \cdot \frac{p}{1-p}$$

(b) $$E\left[\frac{1}{X+1}\right] = -\sum_{k=1}^{\infty} \frac{1}{k+1} \cdot \frac{p^k}{k\ln(1-p)}$$

$$= -\frac{1}{\ln(1-p)} \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right] p^k$$

$$= -\frac{1}{\ln(1-p)} \left[\sum_{k=1}^{\infty} \frac{p^k}{k} - \sum_{k=1}^{\infty} \frac{p^k}{k+1}\right]$$

$$= -\frac{1}{\ln(1-p)} \left[\underbrace{\sum_{k=1}^{\infty} \frac{p^k}{k}}_{-\ln(1-p)} - \frac{1}{p} \underbrace{\sum_{k=1}^{\infty} \frac{p^{k+1}}{k+1}}_{(-\ln(1-p) - p)}\right]$$

(5) Let A, B, and C be some events. Prove that

$$|P(A \cap B) - P(A \cap C)| \leq P(B \Delta C).$$

$$P(A \cap B) = P(A \cap B \cap C) + P(A \cap B \cap C^c)$$

$$P(A \cap C) = P(A \cap C \cap B) + P(A \cap C \cap B^c)$$

$$|P(A \cap B) - P(A \cap C)| = |P(\cancel{A \cap B \cap C}) + P(A \cap B \cap C^c) - (P(\cancel{A \cap C \cap B}) + P(A \cap C \cap B^c))|$$

$$= |P(A \cap B \cap C^c) + P(A \cap C \cap B^c)|$$

$$\leq P(A \cap B \cap C^c) + P(A \cap C \cap B^c)$$

$$\leq P(B \cap C^c) + P(B^c \cap C) \quad (*)$$

$$= P(B \Delta C) \quad \square$$

$(*)$ $A \cap B \cap C^c \subset B \cap C^c$

$A \cap C \cap B^c \subset C \cap B^c$

(6) Let $Z = \sum_{i=1}^{n} X_i$, where $X_1, ..., X_n \sim Poisson(\lambda)$ are independent. Using a convolution, prove that

$$P(Z = k) = \frac{e^{-n\lambda}(n\lambda)^k}{k!}$$

for $k = 0, 1, 2, ...$.

The result is true when $n=1$

$$P(X_1 = k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{because} \quad X_1 \sim Poisson(\lambda).$$

Assume $P\left(\sum_{i=1}^{n} X_i = k\right) = \frac{e^{-\lambda n}(\lambda n)^k}{k!}$. Then

$$P\left(\sum_{i=1}^{n+1} X_i = k\right) = P\left(\sum_{i=1}^{n} X_i + X_{n+1} = k\right)$$

$$= \sum_{j=0}^{k} P\left(\sum_{i=1}^{n} X_i = j,\ X_{n+1} = k-j\right)$$

$$= \sum_{j=0}^{k} P\left(\sum_{i=1}^{n} X_i = j\right) \cdot P(X_{n+1} = k-j)$$

$$= \sum_{j=0}^{k} \frac{e^{-\lambda n}(\lambda n)^j}{j!} \cdot \frac{e^{-\lambda}\lambda^{k-j}}{(k-j)!}$$

$$= e^{-\lambda(n+1)} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (\lambda n)^j \lambda^{k-j}$$

$$= \frac{e^{-\lambda(n+1)}}{k!} \underbrace{\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (\lambda n)^j \lambda^{k-j}}_{(\lambda n + \lambda)^k} = \frac{e^{-\lambda(n+1)}(\lambda(n+1))^k}{k!} \quad \square$$

$$\overbrace{}^{\text{Total} = 44}$$

(7) A school has 24 seniors and 20 juniors. The students are randomly placed in 6 distinct classrooms numbered from 1 to 6. What is the probability that 4 of the classrooms contain 4 seniors and 3 juniors each, while the remaining 2 classrooms contain 4 seniors and 4 juniors each?

4 class rooms
4 seniors & 3 juniors each

2 classrooms
4 seniors & 4 juniors each.

Select 4 classrooms

Select 16 seniors to be placed in 4 classrooms

Select 12 juniors

Divide seniors into 4 groups of 4

Divide juniors in to 4 groups of 3

Divide the rest into 2 classrooms

$$\frac{\binom{6}{4}\binom{24}{16}\binom{20}{12}\dfrac{16!}{(4!)^4}\cdot\dfrac{12!}{(3!)^4}\cdot\dfrac{8!}{(4!)^2}\cdot\dfrac{8!}{(4!)^2}}{6^{44}}$$

(8) Let $X \sim Poisson(10)$. Find $E\left(\frac{1}{(X+1)(X+2)(X+3)}\right)$.

$$E\left[\frac{1}{(X+1)(X+2)(X+3)}\right] = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)(k+3)} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k+3)!}$$

$$= \frac{e^{-\lambda}}{\lambda^3} \sum_{k=0}^{\infty} \frac{\lambda^{k+3}}{(k+3)!}$$

$$= \boxed{\frac{e^{-\lambda}}{\lambda^3}\left[e^{\lambda} - 1 - \lambda - \frac{\lambda^2}{2}\right]}$$

(9) Let $X \geq 0$ be an integer-valued random variable and $\{a_n\}$ be a non-negative sequence. Prove that

$$\sum_{i=1}^{\infty}(a_1 + ... + a_i)P(X = i) = \sum_{i=1}^{\infty} a_i P(X \geq i)$$

$$\sum_{i=1}^{\infty} (a_1 + \dots + a_i) P(X=i) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} a_j \quad P(X=i)$$

$$= \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_j P(X=i)$$

$$= \sum_{j=1}^{\infty} a_j \underbrace{\sum_{i=j}^{\infty} P(X=i)}_{P(X=j) + P(X=j+1) + \dots}$$

$$= \sum_{j=1}^{\infty} a_j P(X \geq j) \qquad \square$$