Math 600 Lecture 24

Theorem: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and assume that t lies in the interior of I. If f is differentiable at t, then f is continuous at t.

Proof: It suffices to prove that

$$f(x) \to f(t) \text{ as } x \to t,$$

that is, that

$$f(x) - f(t) \to 0 \text{ as } x \to t.$$

But

$$f(x) - f(t) = \frac{f(x)-f(t)}{x-t}(x-t) \to f'(t) \cdot 0 = 0 \text{ as } x \to t$$

(since $\frac{f(x)-f(t)}{x-t} \to f'(t)$ and $x-t \to 0$). This completes the proof. //

The product and chain rules

Note: If f is differentiable at t, then

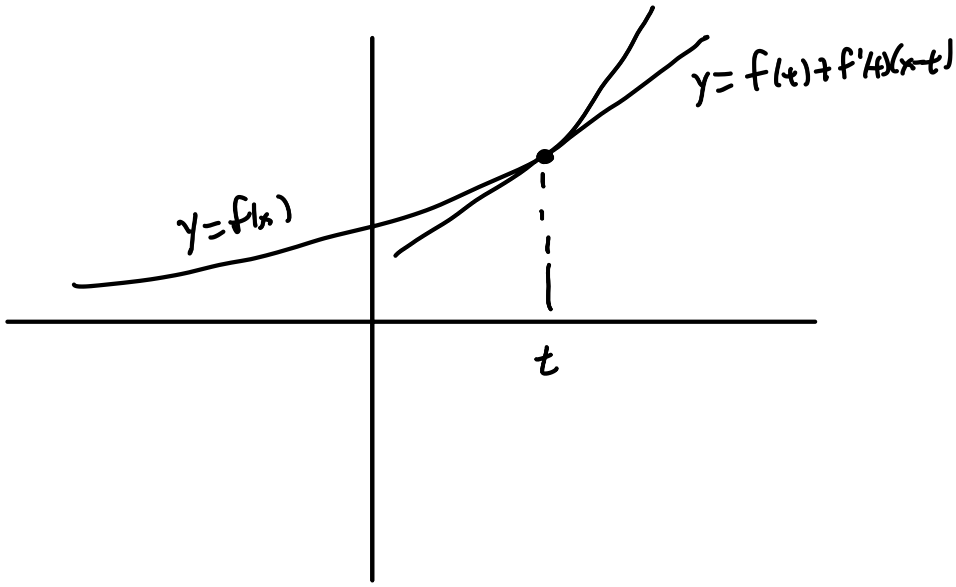
$$\frac{f(x)-f(t)}{x-t} \to f'(t) \text{ as } x \to t$$

$$\Rightarrow \frac{f(x)-f(t)}{x-t} - f'(t) \to 0 \text{ as } x \to t$$

$$\Rightarrow \frac{f(x)-f(t)-f'(t)(x-t)}{x-t} \to 0 \text{ as } x \to t$$

$$\Rightarrow \frac{f(x)-(f(t)+f'(t)(x-t))}{x-t} \to 0 \text{ as } x-t \to 0$$

You should recognize the expression $f(t)+f'(t)(x-t)$: $y=f(t)+f'(t)(x-t)$ is the tangent line to $y=f(x)$ at $x=t$:



The condition

$$\frac{f(x)-(f(t)+f'(t)(x-t))}{x-t} \to 0 \text{ as } x-t\to 0$$

Says something about how well f is approximated by the tangent line approximation: The error

$$f(x)-(f(t)+f'(t)(x-t))$$

is small compared to $|x-t|$ as $|x-t|\to 0$. We write

$$f(x)-(f(t)+f'(t)(x-t)) = o(|x-t|) \qquad (\text{or } f(t+h)-(f(t)+f'(t)h) = o(h))$$

where $o(h)$ ("little-oh" of h) denotes a quantity that satisfies

$$\frac{o(h)}{h} \to 0 \text{ as } h\to 0.$$

We can use the converse: If

$$f(x) = f(t)+m(x-t)+o(|x-t|)$$

(or $f(t+h) = f(t)+mh+o(h)$), where $m\in\mathbb{R}$, then m must equal $f'(t)$.

This often allows us to deduce the derivative.

Examples

1. Suppose $h(x) = f(x)g(x)$ and f, g are differentiable at t. Then

$$h(t+h) = f(t+h)g(t+h) = \left(f(t) + f'(t)h + o(h)\right)\left(g(t) + g'(t)h + o(h)\right)$$

$$= f(t)g(t) + f'(t)g(t)h + f(t)g'(t)h + f(t)o(h) + g(t)o(h)$$

$$f'(t)g'(t)h^2 + f'(t)h\,o(h) + g'(t)h\,o(h) + o(h)^2$$

$$= h(t) + \left(f'(t)g(t) + f(t)g'(t)\right)h + o(h)$$

(note that the last six terms are all $o(h)$, so their sum is $o(h)$).

Thus $h'(t)$ must equal $f'(t)g(t) + f(t)g'(t)$ (the product rule).

2. Suppose $h(x) = f(g(x))$, g is differentiable at t, and f is differentiable at $f(t)$. Then

$$h(t+h) = f(g(t+h)) = f(g(t) + g'(t)h + o(h))$$

$$= f(g(t)) + f'(g(t))(g'(t)h + o(h)) + o(g'(t)h + o(h))$$

$$= f(g(t)) + f'(g(t))g'(t)h + o(h)$$

Thus $h'(t)$ must equal $f'(g(t))g'(t)$ (the chain rule).

3. Let $\varphi(x) = x^{-1}$. Then

$$\varphi'(t) = \lim_{x \to t} \frac{\varphi(x) - \varphi(t)}{x - t} = \lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{t+h} - \frac{1}{t}}{h}$$

$$= \lim_{h\to 0} \frac{t-t-h}{ht(t+h)}$$

$$= \lim_{h\to 0} -\frac{h}{ht(t+h)}$$

$$= \lim_{h\to 0} -\frac{1}{t(t+h)} = -\frac{1}{\lim_{h\to 0} t(t+h)} = -\frac{1}{t^2}$$

4. Suppose $h(x) = \dfrac{f(x)}{g(x)}$. Then

$$h(x) = f(x)\varphi(g(x)) \qquad \left(\varphi(x) = \frac{1}{x}\right)$$

$$\Rightarrow h'(x) = f'(x)\varphi(g(x)) + f(x)\varphi'(g(x))g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Local extrema and stationary points

Definition: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and let a be an interior point of I.

We say that a is a local minimizer (and $f(a)$ a local minimum) of f iff

there exists $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subset I$ and

$f(a) \le f(x) \quad \forall x \in (a-\varepsilon, a+\varepsilon).$

Local maximizer and local maximum are defined analogously.

Theorem (Fermat): Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and let a be an interior point of I. If a is a local minimizer or local maximizer of f and f is differentiable at a, then $f'(a) = 0$.

Proof: Suppose a is a local minimizer of f and $f'(a)$ exists. Then

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \ge 0 \quad (\text{since } f(a+h) - f(a) \ge 0 \text{ and } h > 0)$$

and

$$f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \le 0 \quad (\text{since } f(a+h) - f(a) \ge 0 \text{ and } h < 0).$$

But then $f'(a) = 0$ must hold.

The proof in the case of a local maximizer is similar. //

Definition: Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, and let a be an interior point of I. If f is differentiable at a and $f'(a) = 0$, we call the point a a stationary point of f.

What's so significant about Fermat's theorem? It is that it is usually impossible to verify that a satisfies the definition of local minimizer ($f(a) \le f(x)$ for all x near a), much less to find a from the definition. But $f'(a) = 0$ is easy to verify and also gives a method for finding candidates for local minimizers and local maximizers.