Math 672 Lecture 24

Recall: If $T \in \mathcal{L}(V,W)$, then $T' \in \mathcal{L}(W',V')$ is defined by

$$T'(\psi') = \psi \circ T,$$

that is,

$$(*) \quad (T'(\psi'))(v) = \psi(T(v)) \ \forall v \in V.$$

Now suppose V and W are inner product spaces. (In this case we write $\langle \cdot,\cdot \rangle_V$ and $\langle \cdot,\cdot \rangle_W$ for the inner products.)

The Riesz representation theorem implies that we can think of an inner product space as its own dual:

$$\forall \varphi \in V' \ \exists u \in V, \ \varphi(v) = \langle v, u \rangle \ \forall v \in V.$$

If we think of $w \in W$ as being an element of W' (instead of representing an element of W', as is more correct), then $(*)$ becomes

$$\langle v, T'(w) \rangle_V = \langle T(v), w \rangle_W \ \forall v \in V \ \forall w \in W.$$

This inspires the following theorem/definition (which is slightly different from $(*)$ in the complex case).

Theorem: Let V and W be finite-dimensional inner product spaces over F ($\mathbb{R}$ or $\mathbb{C}$), and let $T \in \mathcal{L}(V,W)$. Then there exists a unique $T^* \in \mathcal{L}(W,V)$, called the adjoint of T, satisfying

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \quad \forall v \in V \ \forall w \in W.$$

Proof: Let $w \in W$ be fixed and define $\varphi : V \to F$ by

$$\varphi(v) = \langle T(v), w\rangle_W \quad \forall v \in V.$$

Then $\varphi \in V'$:

$$\begin{aligned}
\varphi(\alpha u + \beta v) &= \langle T(\alpha u + \beta v), w\rangle_W \\
&= \langle \alpha T(u) + \beta T(v), w\rangle_W \quad \text{(since } T \text{ is linear)} \\
&= \alpha \langle T(u), w\rangle_W + \beta \langle T(v), w\rangle_W \quad \text{(by properties of } \langle \cdot,\cdot\rangle_W\text{)} \\
&= \alpha\varphi(u) + \beta\varphi(v).
\end{aligned}$$

It follows from the Riesz representation theorem that there exists a unique element of V, which we call $T^*(w)$, such that

$$\varphi(v) = \langle v, T^*(w)\rangle_V \quad \forall v \in V$$

$$\Leftrightarrow \langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \quad \forall v \in V.$$

This defines $T^* : W \to V$. It remains only to show that T^* is linear. By the Riesz representation theorem, there is only one vector x ($x = T^*(w)$) satisfying

$$\langle T(v), w\rangle_W = \langle v, x\rangle_V \quad \forall v\in V.$$

We have

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \quad \forall v\in V,$$

$$\langle T(v), z\rangle_W = \langle v, T^*(z)\rangle_V \quad \forall v\in V$$

$$\Rightarrow \bar{\alpha}\langle T(v), w\rangle_W + \bar{\beta}\langle T(v), z\rangle_W$$

$$= \bar{\alpha}\langle v, T^*(w)\rangle_V + \bar{\beta}\langle v, T^*(z)\rangle_V \quad \forall v\in V$$

$$\Rightarrow \langle T(v), \alpha w+\beta z\rangle_W = \langle v, \alpha T^*(w)+\beta T^*(z)\rangle_V \quad \forall v\in V$$

$$\Rightarrow T^*(\alpha w+\beta z) = \alpha T^*(w)+\beta T^*(z) \quad \forall v\in V.$$

Thus $T^*\in\mathcal{L}(W,V)$, as desired. //

Example: Consider the follow subspace of $C^\infty[0,1]$ (under the L^2 inner product):

$$S = \{f\in C^\infty[0,1] : f(0)=f(1)=0\}.$$

Define $D: S\to S$ by $D(f)=f'$. What is D^*?

Solution: Let f, g be arbitrary elements of S. Then

$$\langle D(f), g\rangle = \int_0^1 f'(x)g(x)\,dx = f(x)g(x)\Big|_0^1 - \int_0^1 f(x)g'(x)\,dx \quad \left(\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}\right)$$

$$= -\int_0^1 f(x)g'(x)\,dx \quad (\text{since } f(0)=f(1)=0)$$

$$= \langle f, -g' \rangle.$$

Thus D^* is defined by $D^*(g) = -g'$. //

<u>Theorem</u>: Let U, V, W be finite-dimensional inner product spaces over F ($\mathbb{R}$ or $\mathbb{C}$).

1. $\forall S, T \in \mathcal{L}(V,W)$, $(S+T)^* = S^* + T^*$
2. $\forall T \in \mathcal{L}(V,W)$, $\lambda \in F$, $(\lambda T)^* = \bar{\lambda} T^*$ (if $F = \mathbb{C}$) or $(\lambda T)^* = \lambda T^*$ (if $F = \mathbb{R}$).
3. $\forall T \in \mathcal{L}(V,W)$, $(T^*)^* = T$
4. $\forall S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(U,V)$, $(ST)^* = T^* S^*$
5. $I^* = I$, where $I: V \to V$ is the identity operator.

<u>Proof</u>: These proofs all depend on the definition of the adjoint, to wit, if $w \in W$ and $u \in V$ satisfies

$$\langle T(v), w \rangle_W = \langle v, u \rangle_V \quad \forall v \in V,$$

then $u = T^*(w)$.

1. We have

$$\langle (S+T)(v), w \rangle_W = \langle S(v) + T(v), w \rangle_W$$

$$= \langle S(v), w \rangle_W + \langle T(v), w \rangle_W$$

$$= \langle v, S^*(w) \rangle_V + \langle v, T^*(w) \rangle_V$$

$$= \langle v, S^*(w) + T^*(w) \rangle_V$$

$$= \langle v, (S^* + T^*)(w) \rangle_V \quad \forall v \in V \ \forall w \in W$$

$\Rightarrow (S+T)^* = S^* + T^*$.

2. Next,

$$\langle (\lambda T)(v), w \rangle_W = \langle \lambda T(v), w \rangle_W = \lambda \langle T(v), w \rangle_W$$

$$= \lambda \langle v, T^*(w) \rangle_V$$

$$= \langle v, \bar{\lambda} T^*(w) \rangle_V$$

$$= \langle v, (\bar{\lambda} T^*)(w) \rangle_V \quad \forall v \in V \ \forall w \in W$$

$\Rightarrow (\lambda T)^* = \bar{\lambda} T^*$.

3. We have

$$\langle T^*(w), v \rangle_V = \overline{\langle v, T^*(w) \rangle_V}$$

$$= \overline{\langle T(v), w \rangle_W}$$

$$= \langle w, T(v) \rangle_W \quad \forall v \in V \ \forall w \in W$$

$\Rightarrow (T^*)^* = T$.

4. Suppose $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Then

$$\langle (ST)(u), w \rangle_W = \langle S(T(u)), w \rangle_W$$

$$= \langle T(u), S^*(w) \rangle_V$$

$$= \langle u, T^*(S^*(w)) \rangle_U$$

$$= \langle u, (T^*S^*)(w) \rangle_U \quad \forall u \in U \ \forall w \in W$$

$$\Rightarrow (ST)^* = T^*S^*.$$

5. Finally,

$$\langle I(v), u \rangle_V = \langle v, u \rangle_V = \langle v, I(u) \rangle_V \quad \forall v \in V \ \forall u \in V$$

$$\Rightarrow I^* = I. \quad //$$

Given $T \in \mathcal{L}(V,W)$, the four subspaces

$$\mathcal{N}(T),\ \mathcal{R}(T),\ \mathcal{N}(T^*),\ \mathcal{R}(T^*)$$

are often called the <u>fundamental subspaces</u> defined by T. We already know that

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V),$$

$$\dim(\mathcal{N}(T^*)) + \dim(\mathcal{R}(T^*)) = \dim(W).$$

<u>Theorem</u>: Let V, W be finite-dimensional inner product spaces over F ($\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{L}(V,W)$. Then

$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*) \text{ and } \mathcal{N}(T^*)^\perp = \mathcal{R}(T),$$

$$\mathcal{N}(T)^\perp = \mathcal{R}(T^*) \text{ and } \mathcal{R}(T^*)^\perp = \mathcal{N}(T).$$

Proof: We have

$$
\begin{aligned}
w \in \mathcal{R}(T)^{\perp} &\iff \langle z, w\rangle_W = 0 \quad \forall z \in \mathcal{R}(T)\\
&\iff \langle T(v), w\rangle_W = 0 \quad \forall v \in V\\
&\iff \langle v, T^*(w)\rangle_V = 0 \quad \forall v \in V\\
&\iff T^*(w) = 0\\
&\iff w \in \mathcal{N}(T^*)
\end{aligned}
$$

Thus $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^*)$. It follows that

$$\mathcal{R}(T)^{\perp\perp} = \mathcal{N}(T^*)^{\perp} \Rightarrow \mathcal{R}(T) = \mathcal{N}(T^*)^{\perp}.$$

Also, since $T^{**} = T$, we have

$$\mathcal{R}(T^*)^{\perp} = \mathcal{N}(T^{**}) = \mathcal{N}(T)$$

and

$$\mathcal{R}(T^*) = \mathcal{N}(T^{**})^{\perp} = \mathcal{N}(T)^{\perp}. //$$

Theorem: Let V, W be finite-dimensional inner product spaces over F ($\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{L}(V,W)$. Then

$$\dim(\mathcal{R}(T^*)) = \dim(\mathcal{R}(T)).$$

Proof: Choose $v_1, \ldots, v_n \in V$ such that $\{T(v_1), \ldots, T(v_n)\}$ is a basis for $\mathcal{R}(T)$. It suffices to prove that $\{T^*(T(v_1)), \ldots, T^*(T(v_n))\}$ is a basis for $\mathcal{R}(T^*)$. First,

$$\alpha_1 T^*(T(v_1)) + \cdots + \alpha_n T^*(T(v_1)) = 0$$

$$\Rightarrow \quad T^*(\alpha_1 T(v_1) + \cdots + \alpha_1 T(v_n)) = 0$$

$$\Rightarrow \quad \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) \in \mathcal{N}(T^*) = \mathcal{R}(T)^\perp.$$

But

$$\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \in \mathcal{R}(T),$$

and therefore

$$\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0 \quad (\text{since } \mathcal{R}(T) \cap \mathcal{R}(T)^\perp = \{0\})$$

$$\Rightarrow \quad \alpha_1 = \cdots = \alpha_n = 0 \quad (\text{since } \{T(v_1), \ldots, T(v_n)\} \text{ is linearly independent}).$$

Thus $\{T^*(T(v_1)), \ldots, T^*(T(v_n))\}$ is linearly independent.

We can prove directly that $\{T^*(T(v_1)), \ldots, T^*(T(v_n))\}$ spans $\mathcal{R}(T^*)$ (see below). But here's a simpler way to finish the proof:

We just showed that $\dim(\mathcal{R}(T^*)) \geq \dim(\mathcal{R}(T))$. This result applies to all linear maps, so it applies to T^*:

$$\dim(\mathcal{R}(T^{**})) \geq \dim(\mathcal{R}(T^*))$$

$$\Rightarrow \quad \dim(\mathcal{R}(T)) \geq \dim(\mathcal{R}(T^*)).$$

Thus $\dim(\mathcal{R}(T^*)) = \dim(\mathcal{R}(T))$, as desired.

(Direct proof that $\{T^*(T(v_1)), \ldots, T^*(T(v_n))\}$ spans $\mathcal{R}(T^*)$.)

Now suppose $v \in \mathcal{R}(T^*)$, say $v = T^*(w)$ for $w \in W$. We have

$$W = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp = \mathcal{R}(T) + \mathcal{N}(T^*)$$

and hence there exist $y \in \mathcal{R}(T)$, $z \in \mathcal{N}(T^*)$ such that $w = y + z$.

Since $y \in \mathcal{R}(T)$, there exist $\alpha_1, \dots, \alpha_n \in F$ such that

$$y = \alpha_1 T(v_1) + \dots + \alpha_n T(v_n).$$

But then

$$\begin{aligned} v = T^*(w) &= T^*(\alpha_1 T(v_1) + \dots + \alpha_n T(v_n) + z) \\ &= \alpha_1 T^*(T(v_1)) + \dots + \alpha_n T^*(T(v_n)) + \cancelto{0}{T^*(z)} \\ &\in \operatorname{span}(T^*(T(v_1)), \dots, T^*(T(v_n))). \end{aligned}$$

Thus $\{T^*(T(v_1)), \dots, T^*(T(v_n))\}$ spans $\mathcal{R}(T^*)$, and we have shown that $\{T^*(T(v_1)), \dots, T^*(T(v_n))\}$ is a basis. //