Chapter 11: Conduction, induction, and magnetodynamics (06 Feb 2021)

A. Perspective.

In Ch. 2, we show how Maxwell's Equations follow from the Coulomb and Ampère force laws together with charge conservation and Faraday's Law of Induction. Although the complete set of Maxwell's Equations is derived in Ch. 2, discussion of time-dependent phenomena was deferred. Current flow, induction, and related phenomena provide excellent opportunities for applying problem-solving strategies that start from basics, identify the points on which the problem turns, identify the relevant variables, get the math right, then extract all the physics from the math. These phenomena involve only first derivatives with respect to time, and hence are less complicated than wave phenomena, where second derivatives are involved. Wave phenomena are covered in Chs. 12-14.

We begin with the microscopic theory of conductivity, which shows that for systems in motion, charge carriers are essentially transported with the object in which they reside. The description of field-driven motion within a material leads to drift velocity, Ohm's Law, and distributed forces that act throughout the material. In these calculations electric and magnetic fields are assumed to be applied externally, and the currents and voltages that result are the responses. Examples of eddy currents in conducting plates highlight the role of the vector potential, and applications of the conduction equation show how the (conservative) scalar potential adjusts to satisfy conditions. Nature is well aware of the conservative nature of the scalar potential – spark plugs work.

Inductors and transformers represent a second category, where the magnetic fields are generated by the currents themselves. In treating these configurations, the need to use the complete Faraday expression to define electromotive force (EMF) is made evident. Inductors are components that store energy and consequently, as noted in Ch. 2, can function as either sources or loads. Transformers are essentially coupled inductors, but the relatively large number of small equations needed to describe them drives home the importance of bookkeeping, properly defining parameters, identifying relevant variables, and drawing correct diagrams to avoid errors, particularly sign errors. We emphasize the time domain instead of the usual Fourier (frequency) domain, because the time domain better illustrates the physics involved and present pulse-and-digital applications require this more general approach. The treatment also picks up the physics that is missing in the standard abbreviated treatments.

The coupling of Ampère's Force Law and Faraday's Law of Induction to describe the interaction of mechanical motion and electrical phenomena provides an excellent opportunity not only to pull everything together, but also to describe the physics underlying induction motors of the type used in many present-generation electric vehicles. We conclude with discussions of the Hall effect, historically significant in the early phases of semiconductor technology, and the van der Pauw configuration, where an ingenious application of the theory of complex variable provides an unexpected route for determining conduction parameters of materials.

B. Microscopic theory of conductivity: the physical basis of Ohm's Law.

Ohm's Law, the macroscopic equation written as $\vec{J} = \sigma \vec{E}$ or $V = IR$, follows from a combination of classical physics and statistical mechanics. On a microscopic scale, carriers *j* (electrons or holes) are treated classically as localized independent particles of charge *q* and mass *m* that are moving with thermal velocities $\vec{v}_{oj}$, and are accelerated ballistically between collisions by an externally applied electric field $\vec{E}$. Collisions with defects in the host lattice transfer the energy and momentum obtained by acceleration to the host lattice, and the process repeats. When averaged over carriers and time, as shown below, the result is Ohm's Law.

The procedure for calculating σ is analogous to that used in Ch. 7 to obtain ε, except that viscous friction is replaced by collision statistics and impulse forces. For simplicity, we assume that the mass *m* of the carriers is isotropic. Between collisions the carriers move freely and independently. Using the j^{th} carrier as an example, its ballistic motion is described by

$$\vec{F} = m\vec{a} = m\frac{d\,\vec{v}_j}{dt} = q\vec{E}\,. \tag{11.1}$$

The solution of Eq. (10.1) is

$$\vec{v}_j(t) = \vec{v}_{j0} + \frac{q\vec{E}}{m}t\,, \tag{11.2}$$

where $\vec{v}_{j0}$ is velocity of the carrier just after the last collision, which we assume occurs at $t = 0$. For particles moving ballistically, $\vec{v}_j(t)$ therefore consists of two parts: the initial velocity $\vec{v}_{j0}$, whose direction is assumed randomized by collisions, and an acceleration term $q\vec{E}t/m$, which is independent of history and the same for all carriers.

Statistical mechanics enters next. We assume Poisson statistics, where the probability of a collision in a time increment *dt* is independent of the time *t* since the last collision. From a physics perspective, this is a good approximation if the change of velocity $\Delta\vec{v}_j$ acquired during ballistic motion is small compared to $\vec{v}_{j0}$ itself, an assumption that we will check as soon as the derivation is completed and we have an expression to evaluate. Then given *N* carriers, the number dN that is lost in a time interval *dt* is

$$\frac{dN}{N} = -\frac{dt}{\tau}\,, \tag{11.3}$$

where τ is a lifetime parameter to be determined. By integrating Eq. (11.3), the number of carriers $N(t)$ in an initial group N_o that remain unscattered after a time *t* is

$$N(t) = N_o e^{-t/\tau} . \tag{11.4}$$

In calculating time averages, the probability function $e^{-t/\tau}$ must be normalized to 1 when integrated from $t = 0$ to ∞. The normalization factor is $1/\tau$, so the normalized probability function is $e^{-t/\tau}/\tau$.

Therefore, the time average of $\vec{v}_j(t)$ is

$$< \vec{v}_j > = \int_0^\infty dt \left(\vec{v}_{oj} + \frac{q\vec{E}}{m} t \right) \frac{1}{\tau} e^{-t/\tau} = \vec{v}_{oj} + \frac{q\tau}{m} \vec{E} , \tag{11.5}$$

valid for any *j*. To obtain the drift velocity $< \vec{v} >$, we average over *j*, or alternatively, over all past and future collisions of the j^{th} carrier. Because there is no current for $\vec{E} = 0$, the ensemble average eliminates the first term on the right. The second term is then the drift velocity $< \vec{v} >$. The macroscopic current density is therefore

$$\vec{J} = nq < \vec{v} > = \frac{nq^2\tau}{m} \vec{E} = \sigma \vec{E} . \tag{11.6a}$$

Thus $\sigma = n q^2 \tau / m$ in atomic-scale parameters, allowing τ to be determined.

As the accelerated carriers scatter, they transfer momentum to the lattice. By rearranging the drift term in Eq. (11.5), the average transfer per collision is

$$< \Delta\vec{p} > = m < \vec{v} > = q\tau\vec{E} . \tag{11.6b}$$

The average momentum density transferred is therefore

$$n < \Delta\vec{p} > = n\, m < \vec{v} > = n\, q\tau\vec{E} . \tag{11.6c}$$

The force density follows as

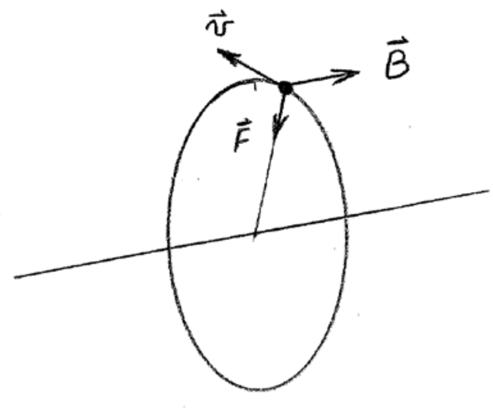
$$n < \vec{F} > = n < \frac{d\vec{p}}{dt} > = n \frac{m < \vec{v} >}{\tau} = n\, q\vec{E} . \tag{11.6d}$$

Thus the average force delivered to the lattice by each carrier is the same as if the carrier were fixed to the lattice. This is what we expect: there should be no free lunch. For electrically neutral materials this force is cancelled by the average force on the ion cores, which by charge neutrality have the same average charge density but of opposite sign. Thus electrically neutral materials experience no net electric force, which agrees with our experience.

To assess the validity of the above model, we need to show that $| \Delta\vec{v}_j | << | \vec{v}_{j0} |$, which ensures that the incremental collision probability is independent of *t*. Let $m = m_e$, where m_e is the free-electron mass. Then at 20 ^{o}C

$$v_o \approx \sqrt{3kT/m_e} = \sqrt{(25\ meV)/(9.11 \times 10^{-31}\ kg)}$$

$$= 1.15 \times 10^5 \, m/s \, . \tag{11.7}$$



To estimate $|\Delta \vec{v}_j|$, consider a 14-gauge (1.63 mm dia.) copper wire carrying its maximum code-rated current of $I = 15\,\mathrm{A}$. Then $J = 7.2 \times 10^6 \ \mathrm{A/m^2}$. Because $n_{Cu} = 8.49 \times 10^{28} \ \mathrm{m^{-3}}$, we find $|\Delta \vec{v}_j| = 0.053 \ \mathrm{cm/s}$, a very leisurely 2 m per hour. With a cushion of nearly 9 orders of magnitude, Poisson statistics is clearly valid. Curiously, even though the magnetic field arising from this slow drift is in principle a relativistic quantity, it is easily measured. The electric field driving this current is a similarly modest $E = \rho J = 1.24 \ \mathrm{mV/cm}$. For achievable fields in metals, in most cases σ can clearly be treated as a constant.

Continuing, the resistivity $\rho_{Cu} = 1/\sigma_{Cu} = 1.72 \times 10^{-6} \, \Omega \ \mathrm{cm}$ of annealed copper at 20 $^{\mathrm{o}}$C leads to $\tau = 2.44 \times 10^{-14} \ s$, again assuming $m = m_e$. The classical mean free path $\Lambda = |\vec{v}_o| \tau$ between collisions is therefore 2.8 nm. However, classical concepts break down at this point: the actual value is about an order of magnitude larger. The correct calculation requires quantum mechanics, which shows that only carriers within kT of the Fermi surface contribute to conduction. Nevertheless, a full quantum mechanical treatment yields the classical result. This still leaves the above calculations well within the range of validity of the assumptions. The results of averaging are unaffected because all equations are linear in $< \vec{v} >$.

If a magnetic field is also present, the results are more complicated. With both $\vec{E}$ and $\vec{B}$ included, Eq. (11.1) becomes

$$m \frac{d\vec{v}_j}{dt} = q\vec{E} + \frac{q}{c} \vec{v}_j \times \vec{B} \, . \tag{11.8}$$

The general solution is better understood by first considering the homogeneous ($\vec{E} = 0$) version of Eq. (11.8), then adding $\vec{E}$ later. To do this write

$$\vec{v} = \hat{x} v_x + \hat{y} v_y + \hat{z} v_z \, , \tag{11.9}$$

then substitute Eq. (11.9) in Eq. (11.8) with $\vec{E} = 0$. If $\vec{B} = B\hat{z}$, then $dv_z/dt = 0$ for the component v_z of $\vec{v}$ along z, so $v_z = v_{zo}$ is constant. The equations for v_x and v_y are

$$m \frac{dv_x}{dt} = \frac{qB}{c} v_y \, ; \tag{11.10}$$

$$m \frac{dv_y}{dt} = -\frac{qB}{c} v_x \, . \tag{11.11}$$

Differentiating either equation with respect to time and substituting the result for the first derivative in the other equation yields

$$\frac{d^2 v_x}{dt^2} = -\frac{qB}{mc} v_x , \tag{11.12}$$

with a similar equation for v_y. The solutions are

$$v_x = v_{xo} \cos(\omega_B t + \theta_x) ; \quad v_y = -v_{xo} \sin(\omega_B t + \theta_x) , \tag{11.13}$$

where v_{xo} and θ_x are constants and $\omega_B = qB/mc$ is the *cyclotron resonance* frequency. The motion for a general initial velocity follows the well-known helical path that spirals around the central line of flux.

Now introduce $\vec{E}$. For $\vec{E} \parallel \vec{B}$ the motion is the same as if $\vec{B}$ were not present. If $\vec{E}$ has a component perpendicular to $\vec{B}$, the motion is more complicated. However, if collisions dominate to the extent that only a small fraction of the helical orbit is ever realized, then to an adequate approximation the equations can be solved by iteration. In the first iteration we substitute the zero-order solution Eq. (11.2) into Eq. (11.8), obtaining

$$m\frac{d\vec{v}_j}{dt} = q\vec{E} + \frac{q}{c}\left(\vec{v}_{jo} + \frac{q\vec{E}}{m}t\right) \times \vec{B} . \tag{11.14}$$

By integrating both sides we obtain the next term as part of a power series in *t*:

$$\vec{v}_j(t) \approx \vec{v}_{jo} + \frac{q\vec{E}}{m}t + \frac{q}{mc}(\vec{v}_{jo} \times \vec{B})t + \frac{q^2 t^2}{2m^2 c}\vec{E} \times \vec{B} . \tag{11.15}$$

Performing the time average yields

$$< \vec{v}_j > = \vec{v}_{jo} + \frac{q\tau}{m}\vec{E} + \frac{q\tau}{mc}(\vec{v}_{jo} \times \vec{B}) + \frac{q^2\tau^2}{m^2 c}\vec{E} \times \vec{B} . \tag{11.16a}$$

After averaging over *j* we obtain

$$< \vec{v} > = \frac{q\tau}{m}\vec{E} + \frac{q\tau^2\,\omega_B}{m}\vec{E} \times \hat{B} . \tag{11.16b}$$

Supposing *n* carriers per unit volume, the resulting average current density $< \vec{J} >$ is

$$< J > = \sigma\left(\vec{E} + \omega_B \tau \vec{E} \times \hat{B}\right) \tag{11.17}$$

Thus the current has a component perpendicular to the main axis. If the current is laterally constrained, the deflected charge accumulates at the sidewalls, generating a field that opposes the magnetic term. This is the basis of the Hall effect, which is discussed in the last section of this chapter.

Equation (11.16b) shows that the transferred momentum also has a component perpendicular to $\vec{E}$. This is the mechanism by which torque is generated in motors, as discussed in Sec. I. Being delivered by collisions, note these are distributed forces that act throughout the material, as opposed to the more usual contact forces typically encountered in mechanics. Although electric forces cancel on the average in neutral material, the internal forces acting on carriers have at least one negative effect, electromigration. Collisions are occasionally energetic enough to displace atoms. This eventually opens gaps in conductors at the high local current densities used in integrated-circuits technology, and the device fails.

Magnetoresistance is described by the next term in the iterative solution. Substituting Eq. (11.15) into Eq. (11.8), performing the integration, then averaging the result over time, we obtain

$$\vec{J} = nq < \vec{v} > = \sigma\left(\vec{E} + \omega_B \tau \vec{E} \times \hat{B} + (\omega_B \tau)^2 (\vec{E} \times \hat{B}) \times \hat{B}\right) \quad (11.18a)$$

$$= \sigma\left(\vec{E} + \omega_B \tau \vec{E} \times \hat{B} - (\omega_B \tau)^2 \vec{E}\right). \quad (11.18b)$$

Magnetoresistance is of second order in $\omega_B \tau$ and acts to oppose current flow. Derivation of Eq. (11.18b) is left as a homework assignment.

Before we place too much confidence in the above, we must verify that $\omega\tau << 1$, so that q indeed only traverses a small segment of the helical path. Inserting the above values for Cu in a 1 T magnetic field, $\omega_B \tau = \frac{qB\tau}{mc} = 4.3 \times 10^{-3}$, to be compared to 2π. Thus on the average charges travel only about 0.07% of a complete orbit, so an iterative solution is justified.

With expressions for $< \vec{v} >$ with electric and magnetic fields both present, it is useful to investigate the average electric and magnetic forces within a conductor. Suppose that a 1 T field is applied perpendicular to a Cu wire carrying its full rated current. The electric-force density on the electrons is

$$F_{elec}/vol = nqE = (13{,}000\frac{coul}{cm^3})(1.0 \times 10^{-3}\frac{V}{cm}) \approx 1400\frac{N}{cm^3}. \quad (11.19)$$

The corresponding magnetic force density is

$$F_{mag}/vol = (nq < v >)B = (580\frac{A}{cm^2})(1\frac{Vs}{m^2}) = 5.8\frac{N}{cm^3}. \quad (11.20)$$

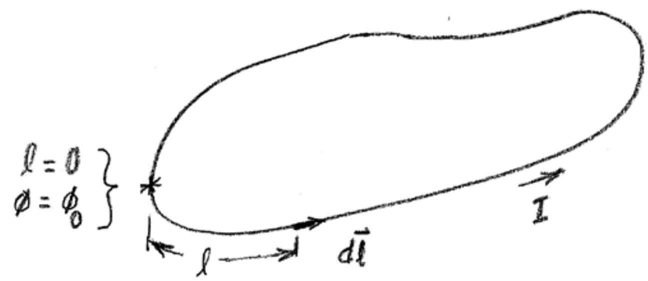
Thus electric forces are over 200 times stronger than the magnetic forces, an imbalance noted in Ch. 1. Why then do motors rely on magnetic forces? The reason is that electric forces apply equally to the carriers and the background atomic charge, and hence cancel. For the magnetic force, the background charge is fixed, so the entire contribution is due to the drift velocity of the mobile charge. This unbalanced nature is what gives the magnetic force practical value.

Regarding magnetic forces, Eq. (11.8) shows that if c were twice as large, motors would only develop half as much torque for the same current. The fundamental constants of our universe are clearly working in our favor.

C. Induced currents: Faraday's Equation and conduction in one dimension.

The treatments of field-induced currents and the responses of scalar potentials in wires provide excellent opportunities to illustrate the advantages of starting from basics, establishing constraints, identifying relevant variables, getting the math right, then extracting the physics from the math. One-dimensional problems (wires) differ qualitatively from problems in two and three dimensions (plates and bulk material, respectively), and provide an interesting perspective on the function of the scalar potential. Wires are covered in this section, and two-dimensional configurations in the next. Magnetic fields are taken to be independent, i.e., supplied externally, as opposed to inductors and transformers, where the relevant magnetic fields are generated by the currents themselves, and hence become part of the solution.

The basic one-dimensional configuration is the wire loop shown in the figure, and the basic equation to be solved is $\vec{J} = \sigma \vec{E}$. Assume that the wire has a length l, a cross-sectional area A_w, and a conductivity σ (resistivity $\rho = 1/\sigma$). We assume that the magnetic flux threading the loop is changing either because the magnetic field is changing with time or the loop is moving or changing shape. Then $V_{emf} \neq 0$, and from Eq. (2.46a),



$$\vec{J} = \sigma \vec{E} = \sigma \left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \phi + \frac{1}{c} \vec{v} \times \vec{B} \right) \tag{11.21}$$

everywhere in the wire.

While this equation is correct, it is too general and therefore not particularly useful for solving specific problems. We need to examine constraints and identify relevant variables. Starting with charge conservation, in the quasistatic approximation $\partial \rho / \partial t = 0$, so one constraint is clearly $\nabla \cdot \vec{J} = 0$. We now apply Gauss' Theorem

$$\int_V d^3 r \, \nabla \cdot \vec{J} = \int_S d^2 r \, \hat{n} \cdot \vec{J} = 0 \tag{11.22}$$

to any segment of the wire. Let the surface S of the segment be divided into three parts, the two ends and the side. Let $\hat{n}$ at one end be parallel to $\vec{J}$, and at the other antiparallel. Because no current emerges from the side of the wire, $\hat{n} \cdot \vec{J} = 0$ there. Therefore, the surface integral reduces to

$$\int_{end\,1} d^2 r \, \hat{l}_1 \cdot \vec{J} + \int_{end\,2} d^2 r \, (-\hat{l}_2) \cdot \vec{J} = I_1 - I_2 = 0 . \tag{11.23}$$

Hence the current I_1 entering one end of the segment must equal the current I_2 emerging at the other. While this is so obvious that the above appears to be significant overkill, it does prove that

$$I_1 = I_2 = I \tag{11.24}$$

is a system invariant. Thus we replace $\vec{J}$, which depends on A_w according to $\vec{J} = I/A_w$, with *I*, which does not. Defining the resistivity $\rho = 1/\sigma$, which in principle can also vary along the wire, Eq. (11.21) becomes

$$0 = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi + \frac{1}{c}\vec{v}\times\vec{B} - \frac{\rho I}{A_w}\hat{l}\,, \tag{11.25}$$

where $\hat{l}$ is the local direction of $d\vec{l}$ in Eq. (2.46b). *I* is positive or negative depending on whether the current flows with or opposite $d\vec{l}$.

We now have something with which we can work. Equation (11.25) describes the behavior of the one-dimensional conductor on a point-by-point basis. Continuing the mathematical development, we now consider the line integral of Eq. (11.25), starting at a predefined location $l = 0$ of the loop. For simplicity we assume that A_w and ρ are the same everywhere on the wire, although this does not have to be the case. Integrating along the wire and noting that $d\vec{l} \parallel \vec{l}$, the line integral becomes

$$0 = \int_0^l d\vec{l}\cdot\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi + \frac{1}{c}\vec{v}\times\vec{B} - \frac{\rho I}{A}\hat{l}\right) \tag{11.26a}$$

$$= \int_0^l d\vec{l}\cdot\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\vec{v}\times\vec{B}\right) - \phi(l) + \phi(0) - \frac{\rho l}{A_w}I\,. \tag{11.26b}$$

We have taken advantage of the fact that $d\vec{l}\cdot\nabla\phi$ is an exact differential to express $\phi(l)$ explicitly. With ρ and A_w independent of l, the fourth term can also be integrated exactly with the result shown. We have therefore found ϕ at any point *l* along the wire:

$$\phi(l) = \phi(0) - \frac{\rho l}{A_w}I + \int_0^l d\vec{l}\cdot\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\vec{v}\times\vec{B}\right). \tag{11.27}$$

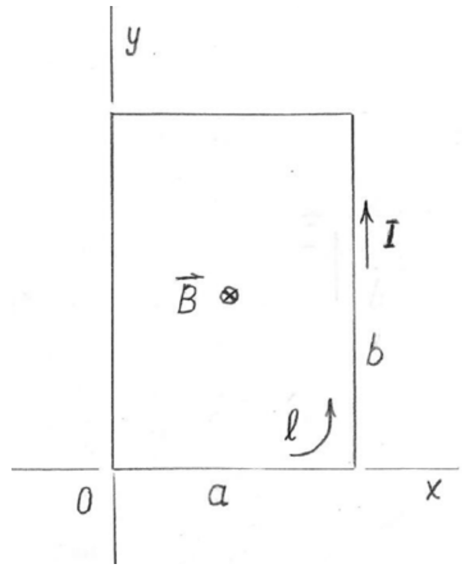
Hence the purpose of ϕ is to provide the degree of freedom necessary to solve Eq. (11.26a). Note that this equation differs from Faraday's Law of Induction in that the integral is *open*-loop, as opposed to *closed*-loop. The advantages of converting the loop integral in Faraday's Law to a line integral is now evident. It allows $\vec{E}_{emf}$ to be separated into its physical constituents, yielding significantly more information than the loop integral from which it started.

However, we still need to determine *I*. To do this we apply the second constraint, which is that $\phi(l)$ is conservative. Then if the length of the wire is *L*, $\phi(L) = \phi(0)$. This provides the information needed to determine *I* as

$$V_{emf} = \int_0^{L_w} d\vec{l}\cdot\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\vec{v}\times\vec{B}\right) = IR\,, \tag{11.28}$$

where $R = \rho L / A_w$ is the resistance of the loop, and V_{emf} is the voltage of the "battery" that drives the current.

Drawing some conclusions about physics, we note that if $I = 0$, caused for example by a gap in the wire, then Eq. (11.27) shows that $\phi(l)$ tracks the source terms exactly. But since $\phi(L) = \phi(0)$, the entire potential V_{emf} developed along the wire must appear across the gap. Nature is well aware of the conservative nature of ϕ: spark plugs in internal-combustion engines work.



These points are illustrated by considering a specific example: a rectangular loop of wire of conductivity σ, cross-sectional area A_w, and sides of length *a* and *b* parallel to the *x* and *y* axes, respectively, as shown in the diagram. The loop is in a uniform magnetic field $\vec{B} = -B\hat{z}$ generated by the vector potential

$$\vec{A}(\vec{r},t) = -\hat{y}\, x\, B\,, \tag{11.29}$$

where $B = B(t)$ is increasing with time. The objective is to calculate $\phi = \phi(l)$ around the loop.

Let the lower left corner be the reference point $l = 0$ at potential $\phi(0) = 0$, and let the line integral follow the right-hand rule, i.e., *l* increasing counterclockwise when viewed from above. As drawn, $\vec{I}$ is in the same direction as $\vec{l}$, so $d\vec{l} \cdot \hat{l} = dl$ and the resistance term in Eq. (11.25) has the correct sign. With $\vec{v} = 0$, it is only necessary to evaluate the path integral over $\vec{A}$. With $\vec{A} \cdot d\vec{l} = 0$ except for the side $x = a$, the result of the emf integration is

$$\phi_{emf}(l) = 0\,, \qquad 0 \le l \le a\,; \tag{11.30a}$$

$$= \frac{a(l-a)}{c}\frac{dB}{dt}\,, \qquad a \le l \le a+b\,; \tag{11.30b}$$

$$= \frac{ab}{c}\frac{dB}{dt}\,, \qquad a+b \le l\,. \tag{11.30c}$$

Setting $\phi_{emf}(l_w) = \phi(2(a+b)) = 0$ yields $I = V_{emf} / R$, as expected.

The complete solution is obtained by adding the resistive term to ϕ_{emf}:

$$\phi(l) = \phi_{emf} - \frac{l}{l_w} V_{emf}\,. \tag{11.31}$$

As a check, V_{emf} agrees with the value calculated from the change of flux through the loop, noting that $\hat{n} = \hat{z}$. Thus

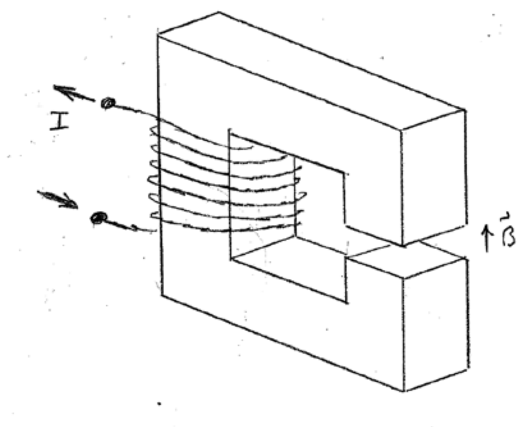
$$-\frac{1}{c}\frac{d\Phi_m}{dt} = \frac{ab}{c}\frac{dB}{dt}\,, \tag{11.32}$$

which is Eq. (11.30c).

This result is also consistent with Lenz' Law, because the magnetic field generated by I opposes the change of $\vec{B}$ that created it. This consistency is no accident: it follows from a careful definition of all quantities and the use of a right-handed coordinate system.

A question arose in class about whether self-inductance is relevant here. This would be the situation if we were considering transients. What we solved here is the quasistatics case where $d\vec{B}/dt = \text{constant}$.

D. Eddy currents from a time-dependent magnetic field.



Consider a two-dimensional example consisting of a large plate of conductivity σ and thickness d located in the xy plane. Let a circular region of radius $\rho \le a$ exist in the plane, where the flux density $\vec{B} = B\hat{z}u(a-\rho)$ is uniform and increasing linearly with time, generated for example by an electromagnet, as shown in the figure. We seek to determine ϕ and $\vec{J}$ in the plate.

There are two ways to solve this problem, and we consider both. First, apply Faraday's Law of Induction directly. At any radius $\rho \le a$ we have

$$\int_C \vec{E}\cdot d\vec{l} = 2\pi\rho E_\varphi = -\frac{1}{c}\frac{\partial\Phi_m}{\partial t} = -\frac{\pi\rho^2}{c}\frac{dB}{dt}\,. \tag{11.33}$$

Hence

$$\vec{J} = \sigma E_\varphi\hat{\varphi} = \sigma\frac{\pi\rho}{2c}\frac{dB}{dt}(-\hat{\varphi})\,, \tag{11.34}$$

in a right-handed coordinate system with $\vec{B} = B\hat{z}$. For $\rho \ge a$

$$\int_C \vec{E}\cdot d\vec{l} = 2\pi\rho E_\varphi = -\frac{1}{c}\frac{\partial\Phi_m}{\partial t} = -\frac{\pi a^2}{c}\frac{dB}{dt}\,, \tag{11.35}$$

so

$$\vec{J} = \sigma E_\varphi\hat{\varphi} = \sigma\frac{\pi a^2}{2\rho c}\frac{dB}{dt}(-\hat{\varphi})\,. \tag{11.36}$$

It is seen that there is no need to introduce ϕ, hence we can simply set $\phi = 0$.

The second approach takes advantage of $\vec{A}$. With no motion, Eq. (11.25) becomes

$$-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\nabla\phi-\frac{\vec{J}}{\sigma}=0\,. \qquad (11.37)$$

In the same local coordinate system $\vec{A}$ is given by Eqs. (9.82):

$$\vec{A}=A\hat{\varphi}=\frac{B\rho}{2}\hat{\varphi} \qquad \text{for } \rho\le a\,; \qquad (9.82a)$$

$$=\frac{Ba^2}{2\rho}\hat{\varphi} \qquad \text{for } \rho>a\,. \qquad (9.82b)$$

ϕ is eliminated because $\phi(0)=\phi(2\pi\rho)$ and the configuration is cylindrically symmetric, so at best ϕ can only be a constant. Hence $\nabla\phi=0$, and therefore

$$J=-\frac{\sigma}{c}\frac{\partial\vec{A}}{\partial t} \qquad (11.38)$$

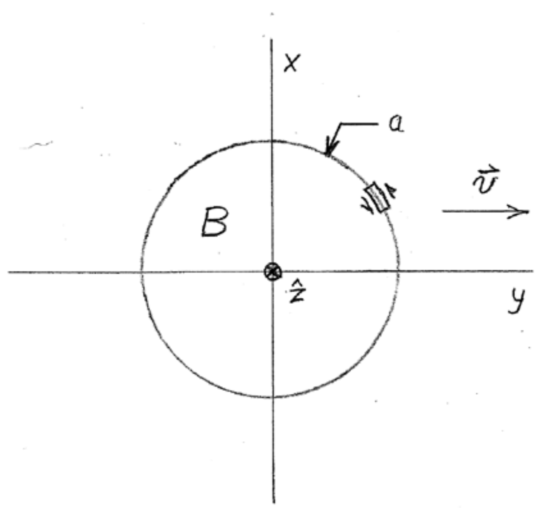
with $\vec{A}$ given by Eqs. (9.82). The result is the same.

The energy dissipated in the plate can be calculated from the power density $dU/dt=\vec{J}\cdot\vec{E}$, then integrating over the volume. The calculation is straightforward, and is left as a homework assignment.

E. Eddy currents from motion.

The fundamental equations require a different approach when motion is involved. Here, basics are illustrated in a different way, but again, when the math is finished, everything is used and nothing is left over. Consider a large plate of thickness *d* and conductivity σ located in the *xy* plane. As shown in the diagram, let the plate be pulled at a constant velocity $\vec{v}=v\hat{y}$ through a region containing a uniform magnetic flux density given by $\vec{B}=B\hat{z}$ for $\rho\le a$ and zero elsewhere, or

$$\vec{B}=B\hat{z}u(a-\rho)\,. \qquad (11.39)$$



With the *z* axis directed into the paper the drive is provided by the Lorentz force $(1/c)(\vec{v}\times\vec{B})$ acting on the mobile carriers. The result is a current density $\vec{J}$.

In turn, $\vec{J}$ interacts with $\vec{B}$ to generate a $\vec{v}\times\vec{B}$ force that resists the motion. The objective is to determine $\vec{J}$, ϕ, the force required to drag the plate through the field, and to verify that energy is conserved in the process. We can anticipate that ϕ must exist, because it creates the field necessary for $\rho>a$ to sweep away carriers that are driven to the boundary $\rho = a$ by the Lorentz force. We work in the laboratory coordinate system rather than that the plate because first, the solution is stationary in the lab frame, and second, $(\partial \vec{A}/\partial t) = 0$ there. We also assume that $|\vec{v}| << c$, so the configuration is in quasiequilibrium.

With $\partial \vec{A}/\partial t$ discarded, Eq. (11.25) reduces to

$$\frac{1}{c}\vec{v} \times \vec{B} - \nabla\phi - \frac{1}{\sigma}\vec{J} = 0 . \tag{11.40}$$

Now $(1/c)(\vec{v} \times \vec{B})$ is constant for both $\rho < a$ and $\rho > a$. Since $\nabla \cdot \vec{J} = 0$, it follows first that the normal component of $\vec{J}$ is continuous across the $\rho = a$ boundary, and second, because $\vec{J} = \sigma\vec{E} = -\sigma\nabla\phi$, then

$$\nabla^2\phi = 0. \tag{11.41}$$

We are well familiar with Eq. (11.41) from electrostatics. The relevant expansions are

$$\phi_{in}(\vec{r}) = \sum_{l=0}^{\infty} A_l \rho^l \cos l\varphi , \ \rho \le a ; \tag{11.42a}$$

$$\phi_{out}(\vec{r}) = \sum_{l=0}^{\infty} B_l \rho^{-l} \cos l\varphi , \ \rho > a . \tag{11.42b}$$

The sine terms vanish by symmetry.

The second boundary condition is $\phi_{in} = \phi_{out}$. Applying this condition yields

$$A_1 a = B_1/a . \tag{11.43}$$

The first condition yields

$$\hat{\rho} \cdot \left(-\nabla\phi_{in} + (\hat{\rho}\cos\varphi - \hat{\varphi}\sin\varphi)\frac{vB}{c} \right) = -\hat{\rho} \cdot \nabla\phi_{out} . \tag{11.44}$$

Combining the two yields

$$A_1 = \frac{vB}{2c} \tag{11.45}$$

and therefore

$$\vec{J}_{in} = \sigma\vec{E}_{emf,in} = \sigma\left(-\frac{vB}{2c} + \frac{vB}{c} \right)\hat{x} = \sigma\frac{vB}{2c}\hat{x} . \tag{11.46}$$

Thus for $\rho < a$ the force due to $\nabla\phi$ cancels exactly half the drive due to $(1/c)(\vec{v} \times \vec{B})$.

For $\rho > a$ the current is

$$\vec{J}_{out} = \sigma \vec{E}_{out} = \sigma \frac{B_1}{\rho^2} \hat{\rho}\cos\varphi + \hat{\varphi}\sin\varphi) \tag{11.47a}$$

$$= \sigma \frac{vB}{2c}\left(\frac{a^2}{\rho^2}\right)(\hat{\rho}\cos\varphi + \hat{\varphi}\sin\varphi)\,. \tag{11.47b}$$

We note that at $\varphi = \pm\pi/2$ the currents on either side of the boundary are equal in magnitude but flow in opposite directions.

We now consider force and energy. The basic equation is

$$\frac{\vec{F}}{(vol)} = -\rho\nabla\phi + \frac{\rho_c}{c}\vec{v}\times\vec{B} = \frac{1}{c}\vec{J}\times\vec{B}\,, \tag{11.48}$$

where ρ_c is the density of the mobile charge. The term involving $\nabla\phi$ vanishes because $\rho = 0$ (the plate is neutral). Integrating over $\rho < a$ we obtain

$$\vec{F} = -\hat{y}\,\frac{\sigma v B^2}{2c^2}(\pi a^2 d)\,, \tag{11.49}$$

where the term in parenthesis is the relevant volume. The force opposes that pulling the plate and is proportional to *v*, i.e., acts as viscous friction. The power necessary to move the plate at the speed *v* is therefore

$$P = |\vec{F}\cdot\vec{v}| = \frac{\sigma v^2 B^2}{2c}(\pi a^2 d)\,. \tag{11.50}$$

By energy conservation, this must equal the resistive loss I^2R, because no other dissipative forces are present. From Ch. 2 the power dissipated per unit volume is $\vec{J}\cdot\vec{E}$. Considering first $\rho < a$ we obtain

$$P_{in} = (\vec{J}_{in}\cdot\vec{E}_{in})(\pi a^2 d) = \sigma\,\vec{E}_{in}^2(\pi a^2 d) = \frac{\sigma v^2 B^2}{4c^2}(\pi a^2 d)\,. \tag{11.51}$$

This is exactly half the power generated by the force moving the plate. For $\rho > a$ the equivalent dissipation is

$$P_{out} = \sigma\left(\frac{vBa}{2c}\right)^2 \int_a^\infty \rho\, d\rho \int_0^{2\pi} d\varphi \frac{d}{\rho^4} = 2\pi\,\sigma\left(\frac{vBa}{2c}\right)^2 \int_a^\infty \frac{d\rho}{\rho^3} d = \frac{\sigma v^2 B^2}{4c^2}(\pi a^2 d)\,, \tag{11.52}$$

which is the same as the dissipation for $\rho < a$. Thus the two together account exactly for the power delivered to the plate.

We can cross-check the calculation by considering the power delivered to the system by the current that results from $(1/c)(\vec{v}\times\vec{B})$. This is

$$P_{in} = (\pi a^2 d)\, J_{in} E_{in} = \sigma(\pi a^2 d)\left(\frac{vB}{2c}\right)\left(\frac{vB}{c}\right) = \frac{\sigma v^2 B^2}{2c^2}(\pi a^2\, d)\,, \tag{10.53}$$

which agrees with the power from the applied force. Thus everything is consistent.

F. Induction, inductors, and inductance. Lenz' Law.

Practical applications of Faraday's Law of Induction are legion, ranging from inductors, transformers, and ac induction and dc motors, to name a few. While the examples that follow may seem mundane, they cover some physics aspects that are not well known, or at least not well appreciated, and provide unique opportunities to make sign errors. Signs are a challenge, and to obtain correct results, accurate diagrams are essential. In this section we develop these topics for inductors, leaving transformers and motors to Secs. G and H, respectively. Further applications are left to problems.

Inductors, transformers, and motors all involve Faraday's Law of Induction, which is usually written

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi}{dt}, \tag{2.36a}$$

where C is any closed path, $q\vec{E}_{emf}$ is the electromotive force (EMF), and Φ is the magnetic flux threading C. Inductors and transformers involve inductances L. In engineering terms, L is defined as

$$V = L\frac{dI}{dt}, \tag{11.54}$$

where I is the current in the winding and V is the voltage across the terminals. The sign is defined to be positive if the current increases when a positive voltage is applied to the "positive" terminal of the passive component, i.e., when the inductor is functioning as a load.

The applied-physics equivalent of Eq. (11.54) is

$$V = \frac{N}{c}\frac{d\Phi}{dt}, \tag{11.55}$$

where the sign is again defined to be positive if Φ increases when a positive voltage is applied to the positive terminal. If L is a constant and both Φ and I are initially zero, then Eqs.(11.54) and (11.55) can be integrated to connect I and Φ according to

$$\Phi = \frac{cL}{N}I. \tag{11.56}$$

Equation (11.56) is essential in describing the physics of transformers, because it relates the current in a particular winding to its contribution to the total flux in the core.

We consider L next. It can be calculated in several ways. The most convenient approach is based on stored energy, and starts with the expression for power P. Because electrical power is given by $P = IV$, with the inductor functioning as a load we have

$$\frac{dW}{dt} = P = IV = IL\frac{dI}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2\right). \tag{11.57a}$$

$$= \frac{d}{dt}\left(\frac{1}{2}LI^2\right). \tag{11.57b}$$

Equation (11.57b) assumes that L is constant, which is almost always the case. Integrating these exact differentials from 0 to t yields

$$W = \frac{1}{2}LI^2. \tag{11.58}$$

Thus L is positive definite.

As an example of determining L, consider the inductance of a bar of magnetic material of permeability μ, circular cross section of radius a, and length l, which has been bent in a circle to form a toroidal core. If N turns of wire are wrapped around the core, then to within minor geometric corrections, Stokes' Theorem and the Ampère-Maxwell Equation yield

$$H = \frac{4\pi NI}{lc}. \tag{11.59}$$

The energy stored in the core is

$$W = \frac{\vec{H}\cdot\vec{B}}{8\pi}\left(\pi a^2 l\right) \tag{11.60a}$$

$$= \frac{2\pi^2 \mu a^2 N^2 I^2}{lc^2} = \frac{1}{2}LI^2. \tag{11.60b,c}$$

Consequently

$$L = \frac{4\pi^2 \mu a^2 N^2}{lc^2} \tag{11.61}$$

is the inductance of the configuration. It is seen that L is determined entirely by geometry, as was capacitance C. In particular, L is proportional to the square of the number of turns N. This dependence of L on N is important in treating transformers.

A more general demonstration of the geometric origin of L follows from

$$V_{emf} = -\int_0^l d\vec{l}\cdot\left(-\frac{1}{c}\frac{d\vec{A}}{dt}\right) = \frac{1}{c}\frac{d}{dt}\int_0^l \vec{A}\cdot d\vec{l}, \tag{11.62}$$

where we have taken the configuration to be stationary, and are treating the inductor as a load ($\vec{E}_{emf}$ and $d\vec{l}$ point in opposite directions, see below). Then by the definition of $\vec{A}$:

$$V_{emf} = \frac{d}{dt}\left(\frac{1}{c}\int_0^l d\vec{l}\cdot\frac{1}{c}\int_V d^3r'\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) \tag{11.63a}$$

$$= \frac{d}{dt}\left(\frac{1}{c}\int_0^l d\vec{l}\cdot\frac{1}{c}\int_V d\vec{l}\,'\frac{I(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) \tag{11.63b}$$

$$= \left(\frac{1}{c^2} \int_0^l \int_0^l \frac{d\vec{l} \cdot d\vec{l}\,'}{|\vec{r} - \vec{r}\,'|} \right) \frac{dI}{dt} \,, \tag{11.63c}$$

where we assume that $\vec{J}(\vec{r})$ is confined to a wire of negligible cross section. Unfortunately, this double integral diverges, a consequence of assuming that the radius of the wire is zero. This divergence is analogous to that which arises in calculating the energy of a collection of point charges.

The divergence does not occur if L is obtained from stored energy. Start by writing

$$W = \frac{1}{8\pi} \int_V d^3r' \left(\vec{B} \cdot \vec{H} \right) = \frac{1}{8\pi} \int_V d^3r' \left(\vec{H} \cdot \nabla \times \vec{A} \right). \tag{11.64}$$

Next, use the vector identity

$$\nabla \cdot (\vec{H} \times \vec{A}) = \vec{A} \cdot \nabla \times \vec{H} - \vec{H} \cdot \nabla \times \vec{A} \,. \tag{11.65}$$

Because $\vec{H}$ and $\vec{A}$ are localized, the divergence term on the left vanishes when integrated over V. Next, take advantage of the Ampère-Maxwell Equation to replace $\nabla \times \vec{H}$, obtaining

$$W = \frac{1}{8\pi} \frac{4\pi}{c} \int_V d^3r \, \vec{A} \cdot \vec{J} \tag{11.66a}$$

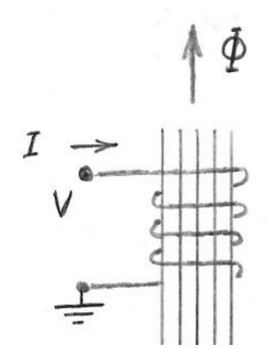
$$= \frac{1}{2c^2} \int_V d^3r \int_V d^3r' \frac{\vec{J}(\vec{r}) \cdot \vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \tag{11.66b}$$

$$= \frac{1}{2} L I^2 \,. \tag{11.66c}$$

Therefore,

$$L = \frac{1}{c^2 I^2} \int_V d^3r \int_V d^3r' \frac{\vec{J}(\vec{r}) \cdot \vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \,. \tag{11.67}$$

The advantage of Eq. (11.67) relative to Eq. (11.63c) is not only that it converges, but also that it incorporates the concept of mutual inductance when two or more independent current distributions are involved. Mutual inductance refers to the coupling of one current distribution to the magnetic field created by another. This is the physics of transformers, the topic that we discuss in Sec. H.

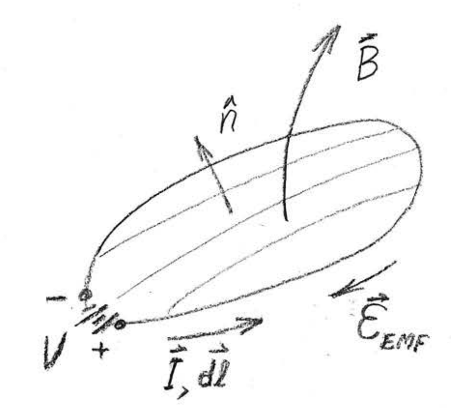


We now consider Eqs. (2.36a), Eqs. (11.54), and (11.55) in detail, noting that the first includes a negative sign whereas the other two do not. This apparent inconsistency is resolved by considering the equations in detail, and underscores the importance of diagrams. Consider first the engineering and applied-physics definitions, Eqs. (11.54) and (11.55) applied to the inductor shown in the diagram. Φ is the result of integrating $\hat{n} \cdot \vec{B}$ across the core, so a positive Φ requires a positive $\hat{n} \cdot \vec{B}$. Now the relative directions of $\vec{B}$ and $\vec{I}$ follow unambiguously from the Ampère-Maxwell Equation relating

$\vec{H}$ and $\vec{J}$, which connects the two by the right-hand rule. Thus to be consistent with Eqs. (11.54) and (11.55), the winding must be wrapped around the core with right-handed parity, as shown.

Next, consider Eq. (2.36a), which involves a line integral around a closed trajectory *C*. The problem with this equation is that, as written, it is ambiguous. The direction of Φ is not defined. This ambiguity is removed by writing Eq. (2.36a) as

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S d^2r\,\hat{n}\cdot\vec{B}, \tag{11.68}$$



where *S* is defined by its perimeter *C*. Now by Stokes' Theorem, the direction of $\hat{n}$ follows from the direction of $d\vec{l}$ according to the right-hand rule. Thus to be consistent with the definition of positive Φ as laid out in the diagram above, the direction of $d\vec{l}$ must be that of $\vec{I}$, as shown in the diagram of the one-turn inductor on the right.

Now, let the inductor function as a load, i.e., taking energy from the source *V*. In this situation $\vec{I}$ and Φ both increase with time, so the time derivative is positive. Since we have shown that $\vec{I}$ and $d\vec{l}$ must point in the same direction, then Eq. (11.68) shows that $\vec{E}_{EMF}$ and $d\vec{l}$ must point in *opposite* directions. Thus the EMF developed is a back-EMF opposing the voltage *V* that is causing *I* to increase. This is exactly what we expect if the source is delivering power to the inductor. If we replace the voltage source with a resistor, so power is dissipated and the flux decreases with time, the equations show that $\vec{E}_{EMF}$ and $d\vec{l}$ now point in the *same* direction, so the voltage developed acts to keep the current flowing. The result is again consistent with expectations. We conclude that the sign in Eq. (11.68) is the manifestation of Lenz' Law, and that there is no inconsistency among Eqs. (2.36a), (11.54), and (11.55) as long as the diagrams are constructed with care.

The analogy of viewing the EMF as a lumped-circuit equivalent is incorrect in one sense: the EMF has no "terminals". It is defined as a closed-loop integral, with no beginning and no end. It represents a nonconservative potential that drives a current but cannot be broken into parts. In fact Eq. (11.62) is inadequate in another sense: it fails to indicate how or where the EMF is developed, giving only the integrated result. Opening the loop to turn it into an incomplete line integral fixes the problem, as discussed in Sec. C.

Given the above, if diagrams are consistently right-handed, then we can dispense with signs and use the engineering definitions without apology. However, before considering transformers, several loose ends need to be fixed.

G. Green functions of circuits: a series *RL* example.

Many applications of inductors today, for example pulse and digital circuits, involve responses of inductors to voltages of arbitrary time dependences. In these applications, standard Fourier (harmonic) analysis is not adequate. More general approaches are needed. Developing a Green-function approach for a simple *RL* circuit provides an opportunity to review the Green-function mathematics discussed in previous chapters in a different context.

We consider the series *L-R* circuit shown in the diagram. The current is given by

$$\left(L\frac{d}{dt}+R\right)I(t)=V(t)\,. \tag{11.69}$$

Following previous procedures, we propose that

$$I(t)=\int_{-\infty}^{\infty}dt'G(t,t')V(t')\,, \tag{11.70}$$

where the Green function $G(t,t')$ is to be determined. If Eq. (11.70) is truly a solution, then

$$\left(L\frac{d}{dt}+R\right)\int_{-\infty}^{\infty}dt'G(t,t')V(t')$$

$$=\int_{-\infty}^{\infty}dt'\left(\left(L\frac{d}{dt}+R\right)G(t,t')\right)V(t')=V(t)\,. \tag{11.71}$$

If the left side is to equal $V(t)$, then

$$\left(L\frac{d}{dt}+R\right)G(t,t')=\delta(t-t')\,. \tag{11.72}$$

Since the δ-function is mostly zero, we examine the solution of the homogeneous equation. This is easily seen to be

$$\left(L\frac{d}{dt}+R\right)I_o e^{-t/\tau}=0\,, \tag{11.73}$$

where $\tau=L/R$. Accordingly, let

$$G(t,t')=C_o e^{-(t-t')/\tau}u(t-t')\,. \tag{11.74}$$

The unit step function serves two purposes: first, to set $G(t,t')=0$ for $t<t'$, and second, to generate $\delta(t-t')$ upon differentiating with respect to *t*. In contrast to the versions we encountered previously, which were appropriate to second-derivative equations, the delta function here results from a discontinuity in value rather than slope. Substituting Eq. (11.74) into Eq. (11.72) and integrating over the singularity we find that $C_o=1/L$, so

$$G(t,t')=\frac{1}{L}e^{-(t-t')/\tau}u(t-t')\,. \tag{11.75}$$

As an example, let $V(t)=V_o u(t-t_o)$. Then

$$I(t)=\int_{-\infty}^{\infty}dt'\frac{1}{L}e^{-(t-t')/\tau}u(t-t')V_o u(t'-t_o) \tag{11.76a}$$

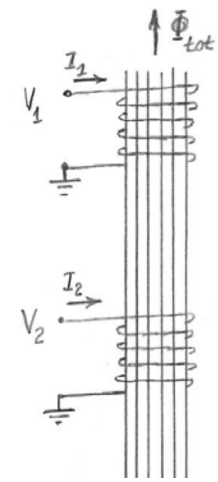
$$= \frac{V_o}{L} \int_{t_o}^{t} dt' e^{-(t-t')/\tau} = \frac{V_o \tau}{L} \left(1 - e^{-(t-t_o)/\tau}\right) u(t - t_o) \tag{11.76b,c}$$

$$= \frac{V_o}{R} \left(1 - e^{-(t-t_o)/\tau}\right) u(t - t_o) . \tag{11.76d}$$

The operatory equation for a series RLC circuit has both first and second derivatives, so its Green function is a closer match to those discussed in orthogonal-function expansions. We leave the derivation of this Green function as a homework assignment.

H. Transformers.

A transformer is a passive linear component that consists of two or more coils (windings) wrapped around a common core. The windings are independent except that they share the total flux Φ_{tot} in the core, as illustrated in the figure. In practice the core is wrapped back to form a complete loop (not shown). In the ideal transformer, flux leakage is nonexistent: the same Φ_{tot} passes through all windings. For high-permeability core material this is an excellent approximation. We assume this in the following.



The transformer provides a good example of how to set up a problem, identify relevant variables, determine fundamental constraints, and obtain a solution. Although the standard approach is to Fourier-transform the basic equations and work in frequency space, we retain time dependence for generality, recognizing that many applications today involve pulse and digital circuits where harmonic descriptions are inadequate.

The problem can be stated as follows. Let winding 1 (the primary) be driven by a voltage $V_0(t)$ through a series resistance R_1. Let the other windings be connected to resistors R_j such that the currents flowing in these windings are given by $I_j = V_j / R_j = V_j G_j$. Although other elements can be used as loads, we restrict discussion to resistors because these generate enough complexity to illustrate necessary points without being overly complicated.

The question is, what are the currents in the windings as a function of time? The solution begins by describing the configuration. The winding *j* is defined by 6 parameters: its terminal voltage V_j, current I_j, number of turns N_j, handedness h_j, its contribution Φ_j to the total flux Φ_{tot} in the core, and its inductance L_j. Note that L_j measured with all other windings open-circuited. As found with the solenoid, L_j is proportional to N_j^2. The constant of proportionality is determined by the geometry of the core, and in principle is the same for all windings. Diagrams are critical, and the above shows a right-handed pair of windings consistent with the discussion of the previous section.

Next, establish the connection between the current in a winding and its contribution to Φ_{tot}. This is given by Eq. (11.56).

The next step in solving the problem is to determine how the terms can be linked. Faraday's Law shows that in operation the EMF of the j^{th} winding is

$$V_j = \frac{N_j}{c}\frac{d\Phi_{tot}}{dt}, \tag{11.77}$$

where

$$\Phi_{tot} = \sum_{j=1}^{n} \Phi_j \tag{11.78}$$

is the total flux, which threads all windings. Equations (11.78) and (11.79) are additional conditions. Because Φ_{tot} is common to all windings, we immediately have the general result that for any two windings *j* and *k*

$$\frac{V_j}{V_k} = \frac{N_j}{N_k}. \tag{11.79}$$

At this point it is convenient to denote winding $j = 1$ as the primary, with the remaining windings as secondaries. The primary winding is generally the one that functions as a load, accepting power from an outside source, with the secondaries performing power distribution and hence functioning as sources for the loads attached to them. Accordingly, let the primary potential be V_1, in which case the secondary potentials are

$$V_j = \frac{N_j}{N_1} V_1. \tag{11.80}$$

Depending on the loads attached to the secondaries, the V_j generate currents I_j that through Eq. (11.77d) become fluxes Φ_j. For example, if a resistor $R_2 = 1/G_2$ is connected across the terminals of winding 2, then

$$I_2 = -G_2 V_2; \quad \Phi_2 = \frac{cL_2}{N_2} I_2 = -\frac{cL_2}{N_2} G_2 V_2. \tag{11.81a,b,c}$$

The minus sign connecting I_j to V_j follows from the right-hand direction conventions of the diagram.

Accordingly,

$$V_1 = \frac{N_1}{c}\frac{d\Phi_{tot}}{dt} = \frac{N_1}{c}\frac{d}{dt}\left(\Phi_1 + \Phi_2 + \Phi_3 + ...\right) \tag{11.82a}$$

$$= \frac{N_1}{c}\frac{d}{dt}\left(\frac{cL_1}{N_1} I_1 + \frac{cL_2}{N_2} I_2 + \frac{cL_3}{N_3} I_3 + ...\right) \tag{11.82b}$$

$$= L_1 \frac{d}{dt}\left(I_1 + \frac{N_1 L_2}{N_2 L_1} I_2 + \frac{N_1 L_3}{N_3 L_1} I_3 + ...\right) \tag{11.82c}$$

$$= L_1 \frac{d}{dt}\left(I_1 + \frac{N_2}{N_1} I_2 + \frac{N_3}{N_1} I_3 + ...\right) \tag{11.82d}$$

$$= L_1 \frac{d}{dt}\left(I_1 - \left(\frac{N_2^2}{N_1^2} G_2 \right) V_2 - \left(\frac{N_3^2}{N_1^2} G_3 \right) V_3 + ... \right). \tag{11.82e}$$

Equation (11.82d) follows from Eq. (11.82c) noting that $L_j \sim N_j^2$, with the constant of proportionality being the same for all windings. Thus Eqs. (11.82c-e) show that V_1 views I_1 as arising from L_1 as if the remaining windings were not there, and the currents in the secondaries according to

$$N_1 I_1 = N_j I_j . \tag{11.83}$$

This is the usual rule for currents. Equation (11.82e) follows from Eq. (11.82d) using Eq. (11.82a), showing that conductances and resistances transform as turns ratios squared. These voltage, current, and impedance relations are usually derived for harmonic time dependences ωt, but as seen from the above, are correct in general. Also, the net result is a linear superposition of scaled admittances of the separate windings. There is no crosstalk. This is consistent with the transformer behaving as a linear device.

However, the original problem has not yet been solved. To proceed further, we note that all voltages V_j are written as first derivatives of Φ_{tot}, all currents are determined by the V_j and the attached impedances, all fluxes are determined by the currents, and Φ_{tot} itself is the sum of all fluxes. We have therefore identified the common variable as Φ_{tot}, which plays the part of the current *I* in the expressions for the wire. By collecting all terms, we obtain (with resistive loads) a first-order differential equation for Φ_{tot}. Once Φ_{tot} is obtained, the remaining parameters can be determined.

As a simple example, let a voltage V_o be applied at $t = 0$ to the primary of a two-winding transformer through a resistor R_1, and let the secondary be short-circuited. Then by the above equations

$$V_2 = \frac{N_2}{c} \frac{d\Phi_{tot}}{dt} = 0 , \tag{11.84}$$

so assuming that $\Phi_{tot} = 0$ when V_o is applied, then $\Phi_{tot} = 0$ forever. Then $I_1 = V_o / R_1$, and by Eq. (11.82c) and (11.83) we have

$$I_2 = -\frac{N_1}{N_2} I_1 = -\frac{N_1}{N_2} \frac{V_o}{R_1} . \tag{11.85}$$

We have realized the impossible: a dc transformer! Of course, all secondaries have some resistance, so this benign situation is only temporary.

As a more complicated example, consider the above transformer with a resistor $R_2 = 1/G_2$ in the secondary. The equations for Φ_1 and Φ_2 are

$$\Phi_1 = \frac{cL_1}{N_1} I_1 = \frac{cL_1}{N_1 R_1} (V_0 - V_1) = \frac{c\tau_1}{N_1} \left(V_0 - \frac{N_1}{c} \frac{d\Phi_{tot}}{dt} \right); \tag{11.86a,b,c}$$

$$\Phi_2 = \frac{cL_2}{N_2} I_2 = -\frac{cL_2}{N_2 R_2} V_2 = -\frac{c\tau_2}{N_2} \frac{N_2}{c} \frac{d\Phi_{tot}}{dt} = -\tau_2 \frac{d\Phi_{tot}}{dt}; \tag{11.86d,e,f,g}$$

where $\tau_1 = L_1/R_1$ and $\tau_2 = L_2/R_2$ are the time constants of the primary and secondary, respectively, viewed as independent circuits. The equation to be solved is therefore

$$\left(1 + (\tau_1 + \tau_2)\frac{d}{dt}\right)\Phi_{tot} = \frac{c\tau_1}{N_1} V_0 . \tag{11.87}$$

Assuming that $\Phi_{tot} = 0$ at $t = 0$, the solution is

$$\Phi_{tot} = \frac{c\tau_1}{N_1} V_0 \left(1 - e^{-t/(\tau_1+\tau_2)}\right). \tag{11.88}$$

Given Eq. (11.88), the various parameters of the model can be evaluated. Explicit values include

$$V_2 = \frac{N_2 \tau_1}{N_1(\tau_1 + \tau_2)} V_0 e^{-t/(\tau_1+\tau_2)}; \tag{11.89a}$$

$$I_2 = \frac{N_2 \tau_1}{N_1 R_2 (\tau_1 + \tau_2)} V_0 e^{-t/(\tau_1+\tau_2)}; \tag{11.89b}$$

$$V_1 = \frac{N_1}{N_2} V_2 = \frac{\tau_1}{(\tau_1 + \tau_2)} V_0 e^{-t/(\tau_1+\tau_2)}; \tag{11.89c}$$

$$I_1 = \frac{1}{R_1}(V_0 - V_1) = \frac{V_0}{R_1}\left(1 - \frac{\tau_1}{(\tau_1 + \tau_2)} e^{-t/(\tau_1+\tau_2)}\right). \tag{11.89d}$$

The results exhibit expected but also some unexpected behavior. If $R_2 \to \infty$ then $\tau_2 \to 0$. Then $I_2 = 0$, and

$$V_1 = \frac{N_1}{N_2} V_2 = V_0 \, e^{-t/\tau_1} . \tag{11.90}$$

At the other extreme $R_2 \to 0$, so the secondary is a perfect short-circuit. Then the equation reduces to the previous solution above.

The main applications of transformers involve harmonic time dependences, where everything varies according to $e^{-i\omega t}$. In this case all time derivatives d/dt reduce to $(-i\omega)$, and the above differential equations reduce to algebra. This is the limit usually discussed. It is straightforward enough so no additional treatment is necessary.

One further example: we return to Sec. D and evaluate the effective resistance associated with the eddy currents induced in the conducting plate. The instantaneous power density $\vec{J} \cdot \vec{E}$ dissipated at any point in the plate follows from Eq. (10.20) if we first take real projections, then do the time average. From Eqs. (11.32) and (11.34),

$$< \vec{J}\cdot\vec{E} > = \frac{\sigma\,\rho^2}{8\pi^2a^2c^2}\left|\frac{d\Phi_m}{dt}\right|^2 \qquad \text{for } \rho \le a\,; \tag{11.91a}$$

$$= \frac{\sigma\,a^4}{8\pi^2a^2c^2\rho^2}\left|\frac{d\Phi_m}{dt}\right|^2 \qquad \text{for } \rho > a\,. \tag{11.91b}$$

Integrating this over the volume of the plate, assuming that it has a thickness *d* and outer radius *b* (to avoid a logarithmic divergence), gives

$$< P_{tot} > = \frac{\sigma\,d}{4\,c^2\pi\,a^4}\left|\frac{d\Phi_m}{dt}\right|^2\left(\frac{1}{4}a^4 + \ln\left(\frac{b}{a}\right)\right) \tag{11.92a}$$

$$= \frac{\sigma\,d}{16\,\pi\,c^2}\left|\frac{d\Phi_m}{dt}\right|^2\left(1 + 4\ln\left(\frac{b}{a}\right)\right) \tag{11.92b}$$

Now

$$\left|\frac{d\Phi_m}{dt}\right|^2 = \frac{c^2V^2}{N_1^2}\,. \tag{11.93}$$

From the perspective of the input terminals, average losses are described by $V^2/(2R)$. Doing the math, the plate appears as if it were a parallel resistor of magnitude

$$R = \frac{4\pi\,N_1^2}{\sigma\,d\,(1+4\ln(b/a))}\,. \tag{11.94}$$

The effective resistance is inversely proportional to the conductivity of the plate and its thickness, as we might expect. The factor N_1^2 in the numerator simply reflects the transformation of the impedance from a one-turn secondary to an N_1-turn primary.

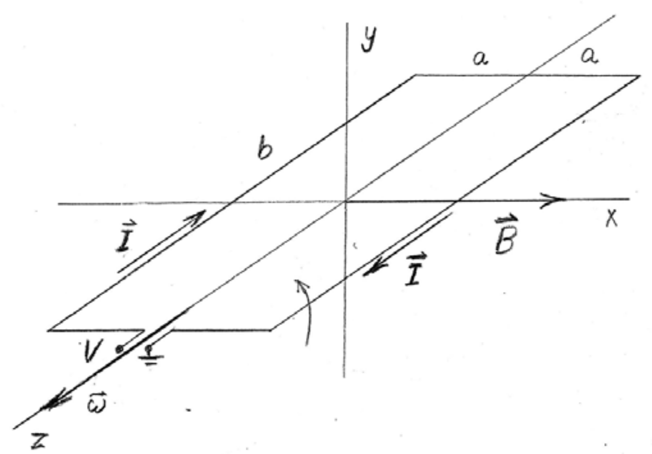
I. Motors and generators.

Why would anyone include a section on motors and generators in a graduate-level text on E&M? For additional practice in setting up and solving equations, in addition to general interest. In addition motors require that Ampère's Force Law be combined with the Faraday Law of Induction, providing experience not only in the interaction of mechanical motion with electrical induction, but also excellent examples of how energy transfers from electrical to mechanical form and *vice versa*. Induction motors provide a particularly good opportunity to apply the principles developed earlier in this chapter. As these are a common technology in most electric cars, the topic is relevant.

The tools that we use here are partly from mechanics and partly from electrodynamics. From mechanics we use power $P = \frac{dW}{dt} = \frac{\vec{F}\cdot d\vec{l}}{dt} = \vec{F}\cdot\vec{v}$ and torque $\vec{N} = \vec{r}\times\vec{F}$, where $\vec{r}$ is the lever arm. From electrodynamics we use $P = I\,V$, Ampère's differential force relation $d\vec{F} = \frac{Id\vec{l}\times\vec{B}}{c}$, and Faraday's Law of Induction $V_{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi}{dt}$, with $\Phi = \int_S d^2r\,\hat{n}\cdot\vec{B}$.

The minus sign that goes with Faraday's Law is usually built into the diagrams. Note that Ampère's force relation is just $\vec{F} = \frac{q}{c}\vec{v} \times \vec{B}$ developed as $q\vec{v} = (\rho\, d^3r)\vec{v} = \vec{J}\, d^3r = \frac{I}{A} A d\vec{l} = I d\vec{l}$. Typical parameters to be determined include torque, back-EMF, and power flow from both mechanical and electrical perspectives.

Methods are best illustrated by example. We consider first direct-current (dc) motors and generators, since these are easiest to understand and provide a good review of how configurations should be analyzed. dc motors are found in heavy-duty applications such as cranes, diesel-electric locomotives, and rolling mills, among others, where speed control is critical and full torque is required at zero rpm. These applications follow by noting that for a given magnetic field the torque developed is linear in the current, and that there is no frequency dependence. Hence dc motors can exert full torque at zero speed, in contrast to internal-combustion engines, which must rotate to function. Steam engines can also develop full torque at zero rpm, but do so at significantly lower efficiency. Because full torque at zero rpm with reasonable efficiency are required characteristics for locomotives, diesel-electric versions, where a diesel engine drives a generator that drives a motor, now dominate this segment of technology. Second, by controlling the input current, rotation speeds can be controlled accurately. This is an important factor in rolling mills that convert steel ingots into wire. Here, as the ingot elongates, the speeds of succeeding pairs of rollers must coordinate with little room for error. In applications where electrical power is supplied as a voltage, Eq. (10.96) shows that the back emf can also determine the speed at which the motor can operate.



The basic configuration is shown in the figure. An external magnetic field is established using either permanent magnets or coils (not shown). The rotor consists of loops of wire that we idealize as a single rectangular loop. In practice these loops are wrapped around a solid core, or armature. In motors with commutators, the loops are equally spaced in azimuth. As the armature rotates, the loop that is closest to the *xz* plane is connected to the electrical source and carries the current *I*. This is the loop that is shown in the diagram.

As usual, we start the analysis with a diagram, ensuring that all equations are consistent with it. Let $\vec{B} = B\hat{x}$ be parallel to the *x* axis, the *y* axis be normal to the loop, and the *z* axis be the axis of rotation ($\vec{\omega} = \omega\hat{z}$). The loop has ends of length $2a$ parallel to *x*, sides of length *b* parallel to *z*, and zero resistance. We assume that the current, $\vec{I} = I\hat{z}$ and $\vec{I} = -I\hat{z}$ on the front and back sides, respectively, is driven externally by a voltage source (not shown.) What is shown as $V = V_{emf}$ is the back EMF that develops in response to $\vec{B}$, the angular velocity $\vec{\omega}$, and the dimensions of the configuration.

From Ampère's differential force equation, the force developed on the front side is

$$\vec{F} = \frac{1}{c}\int_0^b I d\vec{l} \times \vec{B} = \frac{1}{c}\int_0^b I dz\, \hat{z} \times B\hat{x} = \frac{IbB}{c}\hat{y}\,. \tag{11.95}$$

The force on the back side has the same magnitude but the opposite direction.

With a lever arm of $\vec{r} = a\hat{x}$ on the front side and $\vec{r} = -a\hat{x}$ on the back, the total torque developed is

$$\vec{N} = 2\frac{bIB}{c} a\hat{x} \times \hat{y} = \frac{2abBI}{c}\hat{z}\,. \tag{11.96}$$

The torque acts to turn the loop in the positive φ direction. Taking $\omega > 0$, the mechanical power delivered by the motor to the load is

$$P_{mech} = \vec{N} \cdot \vec{\omega} = \frac{2abBI\omega}{c}\,. \tag{11.97}$$

We now consider how the source provides this power. The key is the back emf developed by rotation. We calculate this by evaluating the path integral of $\vec{E} \cdot d\vec{l}$ starting with the grounded end of the loop. With the armature turning at an angular velocity $\vec{\omega} = \omega\hat{z}$, the field created on the front side is

$$\vec{E}_{emf} = \frac{1}{c}\vec{v} \times \vec{B} = \frac{1}{c}\omega a\hat{y} \times B\hat{x} \tag{11.98a}$$

$$= \frac{a\omega B}{c}(-\hat{z})\,. \tag{11.98b}$$

The direction is reversed on the back side. With $d\vec{l} = \hat{z}dl$ on the back side and $(-\hat{z}dl)$ on the front side, the line integral yields

$$V_{emf} = \frac{2ab\omega B}{c}\,, \tag{11.99}$$

which is positive, as expected for a back-EMF. V_{emf} can also be calculated as

$$V_{emf} = -\frac{1}{c}\frac{d\Phi_m}{dt} = -\frac{2abB}{c}\frac{d}{dt}(-\sin\omega t)\bigg|_{t=0} = \frac{2ab\omega B}{c}\,, \tag{11.100}$$

although, as we usually note, the sign in this case needs to be checked against that obtained by line integration.

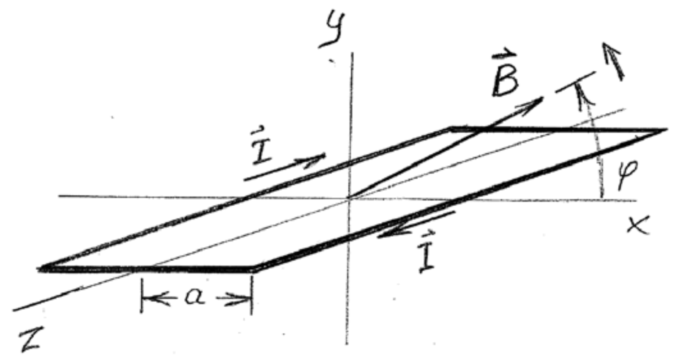
The external source that drives the current *I* works against this positive potential. Consequently, the loop behaves as a load, as it must if electrical energy is being converted to mechanical energy. The other characteristic of a load, that the directions of $\vec{E}$ and $\vec{I}$ are opposite, is also satisfied. Multiplying Eq. (11.103) by *I*, we see that the electrical power IV_{emf} delivered by the source is exactly that given by Eq. (11.101). If the loop has a resistance *R*, then the back voltage has a resistive component $V_{res} = IR$, and the total power required is

$$P_{tot} = -\frac{abIB}{c}\omega + I^2 R\,. \tag{11.101}$$

Thus all energy expenditures are taken into account.

The inverse process is generation, where mechanical power is converted to electrical power. For a given $\vec{\omega}$, Eq. (11.105) shows that the flow of mechanical power is reversed by reversing the torque, and that this is accomplished by reversing the direction of the current. Again, this is consistent with electrical principles: with the field and current now pointing in the same direction, the configuration functions as a source of electrical energy. The motor and generator hardware is identical, so whether a configuration acts as a motor or a generator depends on whether a source or load is connected to its terminals.

We now consider induction motors. From a physics perspective, these are considerably more interesting than their dc equivalents, motors, but their treatment is more challenging. All fractional-horsepower ac motors fall into this category, as do motors in many electric and hybrid cars. As with the dc motor described above, the armature consists of conducting coils, but these are short-circuited so that an induced current can flow without need for external connections. The key feature is that the magnetic field rotates around the armature, thereby generating by induction the current needed to develop torque.



The configuration is shown in the figure. All quantities are considered positive when they have the directions shown in the diagram. z is again the rotation axis, now for both armature and field. Let

$$\vec{B} = B(\vec{x}\cos\Delta\omega t + \hat{y}\sin\Delta\omega t), \tag{11.102}$$

where $\Delta\omega = \omega_B - \omega_A$ is the difference between the rotation speeds ω_B and ω_A of $\vec{B}$ and the armature, respectively. Repeating the calculation for the dc motor, the voltage induced in the loop follows from Faraday's Law:

$$V = \frac{1}{c}\frac{d\Phi_m}{dt} = \frac{1}{c}\frac{d}{dt}\int_S d^2r\,\hat{n}\cdot\hat{B} = \frac{1}{c}\frac{d}{dt}(2a\,l\,B\sin\Delta\omega t) \tag{11.103a}$$

$$= 2\frac{\Delta\omega}{c}a\,l\,B\cos\Delta\omega t = I\,R_c\,. \tag{11.103b}$$

The current is determined by the resistance R_c of the loop.

We now calculate the force generated by the interaction of the current and the magnetic field. The parts of the loop that are relevant are clearly those of length b. Evaluating Ampère's basic equation for a single turn, we find that

$$\vec{F} = \frac{1}{c}I\,l\,\hat{z}\times B(\hat{x}\cos\Delta\omega t + \hat{y}\sin\Delta\omega t) \tag{11.104a}$$

$$= \frac{I\,l\,B}{c}\left(\hat{y}\cos\Delta\omega t - \hat{x}\sin\Delta\omega t\right). \tag{11.104b}$$

An equal force of opposite direction is generated on the back side of the coil, leading to an overall torque of

$$\vec{N} = 2a\,\hat{x}\times\vec{F} \tag{11.105a}$$

$$= \frac{2\,a\,l\,B}{c}\frac{V}{R_c}\hat{z}\cos\Delta\omega t\,. \tag{11.105b}$$

$$= \frac{4\,a^2\,l^2\,B^2}{c^2 R_c}\Delta\omega\,\hat{z}\cos^2\Delta\omega t\,. \tag{11.105c}$$

Thus even though Eq. (10.105b) happears to average to zero, $\vec{I}$ and $\vec{B}$ interact to give a net torque on the loop.

The interesting feature is that the torque is proportional to $\Delta\omega$. Thus if the rotation speed of $\vec{B}$ exceeds that of the armature, electrical power is converted to mechanical power and the car accelerates or climbs the hill. If $\Delta\omega < 0$, then mechanical power is converted to electrical power, and the battery is regenerated. This can be viewed as a mechanical version of Lenz' Law: a difference in azimuth results in a torque that works to align the loop with $\vec{B}$. In short, the loop is always attempting to catch up to $\vec{B}$. In fractional-horsepower motors, power flow is exclusively from electrical to me5hanical. These cannot run synchronously $(\Delta\omega = 0)$, because by Eq. (10.105c) no torque is developed when the rotation is synchronous.

While the above shows that mechanical and electrical power are connected in an induction motor, it does not describe how power is transferred. Being a short circuit, the loop can only dissipate energy, not feed it back to the source. The answer lies in the magnetic field created by the current flowing in the loop. This generates an EMF in the stator coils producing $\vec{B}$, allowing the coils to function either as a load or source depending on the sign of $\Delta\omega$.

The question remains as to how magnetic fields can be made to rotate. Three-phase power accomplishes this by orienting field coils 120^o apart, then phasing the associated ac voltages with 120^o delays. For example, let

$$\vec{B} = B_o\left(\hat{x}\cos\omega t + \left(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)\cos\left(\omega t - \frac{2\pi}{3}\right) + \left(-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right)\cos\left(\omega t - \frac{4\pi}{3}\right)\right)$$

$$= B_o\frac{3}{2}\left(\hat{x}\cos\omega t + \hat{y}\sin\omega t\right) = \frac{3B}{2}\hat{\rho}(t)\,. \tag{11.106}$$

Although two phases are in principle sufficient, magnetic fields are not generally uniform enough to provide a smooth rotation in this case. For electric cars, the control electronics makes rotation/speed decisions based on the measured ground speed of the vehicle and whether the operator wants the car to speed up or slow down. For single-phase fractional horsepower ac motors, rotation is effectively accomplished by the system factoring a unidirectionally oscillating magnetic field into a superposition of two fields that rotate in opposite directions:

$$B\,\hat{x}\cos\omega t = \frac{B}{2}\big((\hat{x}\cos\omega t + \hat{y}\sin\omega t) + (\hat{x}\cos\omega t - \hat{y}\sin\omega t)\big). \qquad (11.107)$$

In this case a high-current temporary "starter" coil with an ideally 90° phase shift is used to initiate rotation. Equation (10.107) shows that single-phase drive develops fields with the "right" and "wrong" rotations. The coil in the armature is designed with a sufficiently large inductance so that the current generated by the "wrong" direction, for which the frequency is necessarily much higher, is small.

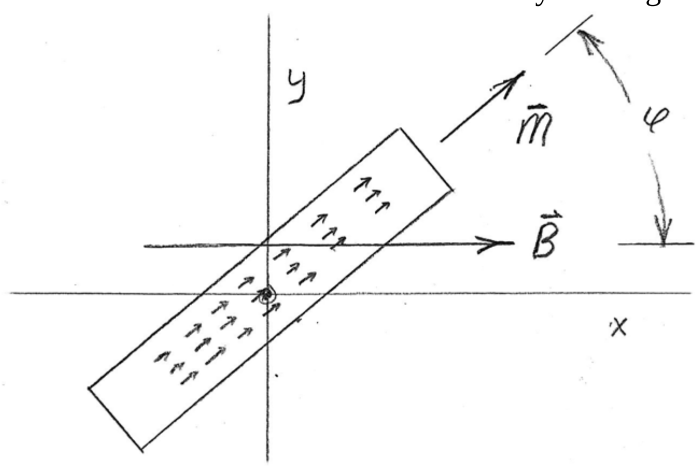
The last configuration considered is the permanent-magnet motor, which is the type found in most electric cars today. The diagram is shown on the right. The armature consists of a permanent magnet, usually $Nd_2Fe_{14}B$, located in a strong magnetic field $\vec{B}$. The magnetic field is created by currents passing through coils, whose phases are controlled electronically to match rotation speed and torque to demand. The permanent magnet consists of domains of net magnetization $\vec{M}$, as shown in the figure. Because the energy of a single dipole $\vec{m}$ is given by

$$dW = -\vec{m}\cdot\vec{B}, \qquad (11.108)$$

the dipole experiences a torque $d\vec{N} = \vec{m}\times\vec{B}$. Taken together, the overall result is a torque

$$\vec{N} = (vol)\vec{M}\times\vec{B}, \qquad (11.109)$$

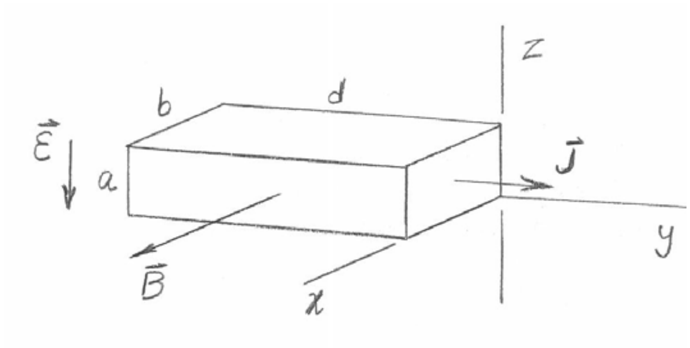
where (vol) is the volume of the magnet. The back EMF in this case is created by the magnetic field of the permanent magnet sweeping the fixed coils as it goes past.



A physics reason for discussing this configuration deals with the treatment of the magnetic flux. The fact that this penetrates the magnet to act directly on the domain-size dipoles seems to contradict everything what we learned about magnetic materials in Ch. 10, where we showed that ferromagnets with high permeability shielded their interiors from external fields. In the present situation the penetration of $\vec{B}$ occurs for two reasons. First, under maximally efficient operation, $\vec{B}$ is perpendicular to the dipoles. Hence the dipoles cannot act to screen the field. Second, a high permeability is the result of an external field easily reorienting dipoles to line up with it. In the present case the orientation of the dipoles is locked in the material, so they cannot change direction. Thus the permeability is basically that of empty space. The field penetrates, torque is generated, and by properly controlling angular velocities and phases, power is either delivered to the mechanical part of the configuration or removed from it depending on whether $\vec{B}$ leads or lags $\vec{M}$. The extremely high efficiencies of electric vehicles follows by recycling the energy, minimizing waste, and not using any energy when the vehicle is stationary.

J. The Hall effect and the van der Pauw configuration.

The Hall effect describes the potential difference induced across a bar that is carrying a current density $\vec{J}$ perpendicular to a magnetic flux density $\vec{B}$, as shown in the figure. The force $(q/c)(\vec{v} \times \vec{B})$ deflects the carriers. The deflection is countered by the electric field $\vec{E}$ that results from the surface charge density that accumulates at the side of the bar, so the current continues straight down the bar. The sign of the potential depends on the sign of the carriers. Accordingly, the Hall effect is historically important because it proved that semiconductors could carry current by either electrons or holes.



The equation describing the Hall effect is

$$\vec{F} = q\vec{E} + \frac{q}{c} < \vec{v} > \times \vec{B} = 0, \tag{11.110}$$

where $< \vec{v} >$ is the drift velocity. Because q cancels in Eq. (11.110), it is not immediately obvious how the direction of $\vec{E}$ can be related to the sign of q. The solution of the dilemma is that the direction of $\vec{v}$ also depends on the sign of the carriers, so if the direction is referenced to the direction of current flow $\vec{J} = nq < \vec{v} >$, then the deflection is downward in the above diagram for carriers of either sign. The result is

$$q\vec{E} + \frac{1}{nc} \vec{J} \times \vec{B} = 0. \tag{11.111}$$

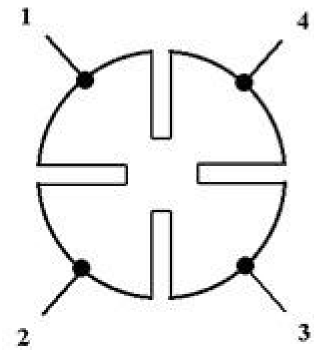
Thus in the above diagram the deflected charge always appears on the bottom surface. Expressing Eq. (11.111) in terms of the observables $I = abJ$ and $V = -Ea$ gives

$$n = \frac{IB}{qbcV_H}, \tag{11.112}$$

where the Hall voltage V_H is the potential of the top surface if the bottom is grounded.

The appearance of the Hall voltage in the denominator shows why measurements of the Hall effect had to wait until the advent of semiconductor technology. Using the data of Sec. B, we find that the voltage across a 5 mm wide Cu block operating at full rated current density in a transverse field of 1 T is 2.6 μV. This value would be very difficult to measure. However, respectable values are reached with moderately doped semiconductors, which have carrier concentrations orders of magnitude less.

Because these notes deal with mathematics as well as physics, it would be a disservice not to include a discussion of the van der Pauw configuration. In 1958 van der Pauw, who at the time was working for Philips, published a technical report where he used the theory of complex variables to show that the resistivity and carrier concentration of a suitably cut planar sample of material of thickness *d* could be obtained from a combination of resistance and voltage measurements, some of which are done in a transverse magnetic field.



Ideally, the sample is cut approximately in the form of a 4-leaf clover with the idea of locating the contacts as far as possible from the active region, the cross in the center, as shown in the figure. To determine the resistivity, a current is passed between two adjacent contacts, say 1 and 2, and the resulting voltage is measured between the two remaining adjacent contacts, say 3 and 4. The ratio of measured voltage to current is defined as a resistance

$$R_{12,34} = \frac{V_{34}}{I_{12}} \tag{11.113}$$

The measurement is then repeated with the group of contacts rotated by one, leading to a second ratio $R_{23,41}$. The resistivity ρ is then given by

$$\rho = R_s d\ , \tag{11.114}$$

where

$$e^{R_{12,34}/R_s} + e^{R_{23,41}/R_s} = 1\,. \tag{11.115}$$

To determine the carrier concentration and sign, a magnetic field is applied perpendicular to the sample in the positive *z* direction and a constant current is applied across alternate pairs of contacts, for example 1, 3. The voltage across the remaining two contacts, for example 2, 4, is recorded as $V_{13;P}$. The measurement is then repeated with the magnetic field reversed, yielding $V_{13,N}$. The difference $V_{13} = V_{13,P} - V_{13,N}$ is then calculated. The process continues until all four combinations V_{13}, V_{24}, V_{31}, and V_{42} have been determined. The carrier type and concentration follows from

$$n = \frac{IB}{qV_H}\ , \tag{11.116}$$

where

$$V_H = \left(V_{13} + V_{24} + V_{31} + V_{24}\right)/8\,. \tag{11.117}$$

Yes, believe it or not, it works. There is something to be said about the theory of complex variables, as we discovered when we considered causality.