Conditional expectation

The conditional expectation of X given $Y=y$ is defined as follows:

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$$

where $f_{X|X}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

Ex: $f(x,y) = \begin{cases} 6(1-y) \ ; & 0 < x \leq y \leq 1 \\ 0 \ ; & \text{otherwise} \end{cases}$

1) Find $E[X \mid Y=y]$

$$f_Y(y) = \int_{x=0}^{x=y} 6(1-y)\, dy = 6y(1-y)$$

$$f_{X|Y}(x|y) = \frac{6(1-y)}{6y(1-y)} = \frac{1}{y} \ ; \quad 0 < x \leq y \leq 1$$

$$E[X \mid Y=y] = \int_{x=0}^{x=y} x \cdot \frac{1}{y}\, dx$$

$$E[X \mid Y=y] = \frac{1}{y} \cdot \frac{x^2}{2} \Big|_0^y = \frac{1}{2y}$$

ii) Find $E[X]$

$$E[X] = E[E[X|Y]]$$

$$= \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\, dy$$

$$= \int_{y=0}^{y=1} \frac{1}{2y} \cdot 6y(1-y)\, dy$$

$$= 3\left[y - \frac{y^2}{2}\right]_0^1 = \boxed{\frac{3}{2}}$$

Ex: Let $X \sim B(n, Y)$ & $Y \sim U(0,1)$.

i) Find $E[X]$

$$E[X] = E[E[X|Y]] = E[nY]$$

$$= n\,E[Y] = \frac{n}{2}$$

ii) $E[XY]$

$$E[XY] = E[E[XY|Y]]$$

$$= E[Y\, E[X|Y]]$$

$$= E[Y \cdot nY]$$
$$= n \cdot E[Y^2]$$

$$= n\left[V(Y) + E(Y)^2\right]$$

$$= n\left[\frac{1}{12} + \frac{1}{4}\right] = \boxed{\frac{n}{3}}$$

iii) Find $\operatorname{cov}(Y, E(X \mid Y))$.

$$\operatorname{Cov}(Y, E(X \mid Y)) = E[Y\, E(X \mid Y)] - E[Y]\, E[E[X \mid Y]]$$

$$= E[Y \cdot nY] - E[Y] \cdot E[X]$$
$$= n E[Y^2] - E[Y] E[X].$$
$$= n \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{n}{2}$$

$$= \frac{n}{3} - \frac{n}{4} = \boxed{\frac{n}{12}}$$

Ex : $f(x,y) = \begin{cases} 30xy^2 \; ; & x-1 \leq y \leq 1-x, \quad 0 \leq x \leq 1. \\ 0 \; ; & \text{otherwise.} \end{cases}$

Find $E[Y \mid X = \frac{1}{2}]$.

Inequalities

Theorem: Let $X \geq 0$. (i.e. $f(x) = 0$ when $x < 0$).
Then, $E[X] \geq 0$.

Proof: $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$= \int_{0}^{\infty} x f_X(x)\, dx \geq 0$ $\square$

Theorem: If $X \leq Y$, then $E[X] \leq E[Y]$.

Proof: $Z = Y - X \geq 0$

$0 \leq E(Z) = E(Y - X) = E(Y) - E(X)$

$\Rightarrow \quad E[Y] \geq E[X]$ $\square$

Theorem: Let h be a non-negative function & $a > 0$. Then,

$$P(h(X) \geq a) \leq \frac{E(h(X))}{a}$$

Proof: Let $A = \{x \mid h(x) \leq a\}$

$$I_A(x) = \begin{cases} 1 & ; x \in A \\ 0 & ; x \in A^c \end{cases}$$

Notice that $h(X) \geq a I_A$.

$$\Rightarrow \quad E[h(X)] \geq a\, E[I_A]$$

$$\Rightarrow \quad E[h(X)] \geq a \cdot P(h(X) \geq a)$$

$$\Rightarrow \quad P(h(X) \geq a) \leq \frac{E[h(X)]}{a} \qquad \square$$

Theorem : Markov's inequality

$\forall\, a > 0$,

$$P[\,|X| \geq a\,] \leq \frac{E(|X|)}{a}$$

Proof : Follows from the previous theorem when $h(x) = |x|$.

Theorem : Chebyshev's inequality.

$$\forall\, a > 0, \quad P(|X| \geq a) \leq \frac{E[X^2]}{a^2}$$

Proof : $|X| \geq a \iff X^2 \geq a^2$.

Therefore,

$$P(|X| \geq a) = P(X^2 \geq a^2) \leq \frac{E[X^2]}{a^2} \qquad \square$$

Corollary: $P\left[\,|X - E[X]| \geq a\,\right] \leq \dfrac{Var(X)}{a^2}$