Theorem: Let V be a finite-dimensional vector space over a field \bar{f} and let $N \in \mathcal{L}(V)$ be nilpotent. Then there exist namegative integer $m_1, ..., m_k$ and vectors $v_1, ..., v_k \in V$ such that

{\(\nabla_1, \nabla_1, \ldots, \nabla_1 \ldots_1, \nabla_2, \nabla_1 \ldots_1, \nabla_1 \nabla_1, \nabla_1 \nab

 $N^{m_j+1}(v_j) = O \quad \forall j=1,2,...,k.$

Proof: We argue by induction on n=dim(V), If n=1, then the only nilpotent operator is the zero operator and we can take $m_1=0$ and v_i to be any nonzero vector in V. (Why must N be the zero operator? Suppose V=Span(v) and $N(V)\neq 0$. The N(V) must equal λv for some narrow λ (some V=Span(v)) and hence $N(v)\neq 0$ $\forall k \geq 1$, a contradiction.)

Now suppose the result holds for all vector spaces over F with dimension at most n-1, where $n \ge 2$. Let V be a vector space over F having dimension n and let $N \in \mathcal{L}(V)$ be nilpotents. Note that N is not injective (because it is nilpotent) and hence N is not surjective. It follows that, if $U = \mathcal{R}(N)$, then $\dim(U) < n$. Recall that

Q(N) is invariant under N, so we can define $S \in \mathcal{L}(U)$ by S = NI. Then S is nilpotent and, by the induction hypothesis, U

there exist nonregative notigers mi, , --, mi and no --, ut EU such that

 $u_1, N(u_1), -, N^{m'_1}(u_1), -, N^{m'_2}(u_1)$ (using the form a basis for U. Moreover, each $u_j \in R(N)$, so S=N(U)

there exist v,,-, vt EV such that uj=N(vj) for j=1,-, t.

Define mj=mj+1 and consider

$$\begin{split} \mathcal{B}' &= \left\{ V_{1}, \mathcal{N}(u_{1}), \dots, \mathcal{N}^{m_{1}'}(u_{1}), \dots, V_{t}, \mathcal{N}(u_{t}), \dots, \mathcal{N}^{m_{t}'}(u_{t}) \right\} \\ &= \left\{ V_{1}, \mathcal{N}(v_{1}), \dots, \mathcal{N}^{m_{1}'}(v_{1}), \dots, V_{t}, \mathcal{N}(v_{t}), \dots, \mathcal{N}^{m_{t}'}(u_{t}) \right\}. \end{split}$$

We will prove that B' is linearly subgrandows and that B' can be extended to a basis B for V of the type described in the theorem.

Suppose dijEF, OLIEmj, Lijet, satisty

$$\sum_{j=1}^{t} \sum_{i=0}^{m_{ij}} N^{i}(r_{j}) = 0,$$

Applying N to both sides yields

$$\sum_{j=1}^{t} \sum_{i=0}^{m_{j}} \alpha_{ij} N^{i+1}(v_{j}) = 0$$

$$\implies \sum_{j=1}^{\ell} \alpha_{m_{j,j}} N^{m_{j}}(v_{j}) = 0$$

$$\Rightarrow \sum_{j=1}^{t} \alpha_{m_{j},j} N^{m_{j}}(u_{j}) = 0$$

Thus $\alpha_{ij}=0$, $0 \le i \le m_j$, $1 \le j \le t$, and we have shown that B^l is linearly independent.

If |B'| < n, then extend B' to a basis B'' of V by adding vectors W_{t+1}, \ldots, W_k . (Otherwise, just define k=t and the proof is complete). For each $l=t+1,\ldots,k$,

$$\mathcal{N}(\omega_{\mathbf{e}}) \in \mathcal{R}(\mathcal{N})$$

$$= \sum_{j=1}^{t} \sum_{i=1}^{m_j} N^i (v_j) \text{ for some } w_{ij} \in F$$

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$$\Longrightarrow$$
 $\mathcal{N}(v_{\ell})=0$, where $v_{\ell}=w_{\ell}-\sum_{j=1}^{t}\sum_{i=1}^{m_{j}}v_{ij}N^{i-1}(v_{j})$

Now define

 $\mathbb{G} = \left\{ v_1, \mathcal{N}(v_t), \dots, \mathcal{N}^{\mathsf{m}}(v_1), \dots, v_t, \mathcal{N}(v_t), \dots, \mathcal{N}^{\mathsf{n}_t}(v_t), v_{t+\nu}, \dots, v_k \right\}.$

Then B spans V, since $B' \subseteq B$ and $w_e \in span(B)$ for d = t+1,...,n. Thus, since $|B| = |B^u|$, we see that B is a basis for V, and B has the desired form (note that $m_j = 0$ and $N^{otl}(v_j) = 0$ for j = t+1,...,n). This completes the proof by induction f

- · V = G(x,,T) @ -- + G(x,,T)
- · Each G(xj,T) is invariant under T.
- · (T-1)] (G(x;) is nilpotent.

Let us assume that m; is the smallest positive integer such that $\left[\left(T-\lambda_{j} I \right) \right]_{6/\lambda_{j}, T}^{m_{j}} = 0.$

Suppose Sj, t; are the geometric and algebraic multiplication of dj:

Si=din(Elijit)(Elijit)(Elijit)=tj There is a basis for $G(\lambda_j,T)$ consisting of S_j generalized eigenvector chains (some of which may contain only one vector, which would be an eigenvector):

$$\mathcal{B}_{j} \left\{ \left(T - \lambda_{j} I \right)^{r_{ij} - l} (v_{ij})_{j} \left(T - \lambda_{j} I \right)^{v_{ij} - 2} (v_{ij})_{j}, \dots, v_{ij}, \right. \\ \left. \left(T - \lambda_{j} I \right)^{r_{2j} - l} (v_{2j})_{j} \left(T - \lambda_{j} I \right)^{r_{2j} - 2} (v_{2j})_{j}, \dots, v_{2j}, \right. \\ \left. \left(T - \lambda_{j} I \right)^{r_{2j} - l} (v_{2j})_{j} \left(T - \lambda_{j} I \right)^{r_{2j} - 2} (v_{2j})_{j}, \dots, v_{2j}, \right. \\ \left. \left. \left(T - \lambda_{j} I \right)^{r_{2j} - l} (v_{2j})_{j} \left(T - \lambda_{j} I \right)^{r_{2j} - 2} (v_{2j})_{j}, \dots, v_{2j}, \right. \right\}$$

(this follows from the previous theorem). If we define

$$\mathcal{B} = \bigcup_{\bar{J}=1}^{k} \mathcal{B}_{\bar{J},1}$$

Then $A = 9M_{B,B}(T)$ is block diagonal, where the jth block is itself block diagonal, with s. Jordan blocks (some of which may be 1×1 . When B is chosen this way, A is called the Jordan form (or Jordan canonical form or Jordan normal form) of I.

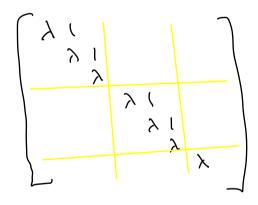
Nutes:

- . The Jordan form is generally not unique (since the block can be reordered by reordering the chains).
- · The values of k, si, iz, -, i, mi, -, nea, ti,-, ta, and si,-, sh may or may not uniquely (up to reorder my) define the Jardan form.

There must be one church of lasth 3:

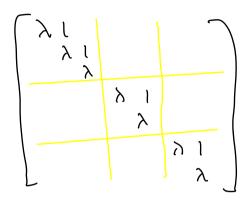
There could be a second chan of lagth 3, plus a chan of lagth 1:

In this case, the Jordan form is



Or (in addition to the first chain of leigth 3), we could have two charms of leigth 2;

Then the Jordan forn would be



What information unsigned determines the Jordan form (agash, up to recordering)?

Answer: din (M(IT-NI)), 14jem.

([ase]	Case 2
dim (n(T-20))	3	3
din (n((T-NE)2))	5	6
dim(91((J-, xJ)3))	7	7