Math 672 Lecture 4

<u>Definition</u>: Let $v_1, v_2, \ldots, v_n$ be vectors in V. We say that $\{v_1, v_2, \ldots, v_n\}$ is a <u>basis</u> for V iff $\{v_1, v_2, \ldots, v_n\}$ is linearly independent and spans V.

<u>Theorem</u>: Let $v_1, v_2, \ldots, v_n$ be vectors in V. Then $\{v_1, v_2, \ldots, v_n\}$ is a basis for V iff each $u \in V$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$:

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

<u>Proof</u>: Suppose first that each $u \in V$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$. Then $\{v_1, v_2, \ldots, v_n\}$ spans V (since each $u \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$) and is linearly independent (since $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ is the only way to write 0 as a linear combination of $v_1, v_2, \ldots, v_n$). Thus $\{v_1, v_2, \ldots, v_n\}$ is a basis of V.

Conversely, suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of V. Then each $u \in V$ can be written as

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ (since $\{v_1, v_2, \ldots, v_n\}$ spans V) and this representation is unique by the second theorem of Lecture 3. //

We now present two of the basic facts about bases:

- Every spanning set can be reduced to a basis (by discarding unneeded vectors).
- Every linearly independent set can be extended to a basis (by adding vectors if needed).

Theorem: Let $\{v_1, v_2, \ldots, v_m\}$ span V, where V is a nontrivial vector space ($V \neq \{0\}$). Then some subset of $\{v_1, v_2, \ldots, v_m\}$ is a basis for V.

Proof: Write $B_0 = \{v_1, v_2, \ldots, v_m\}$. If B_0 is linearly independent, then B_0 is a basis for V and the proof is complete (B_0 is a subset of itself). Otherwise, some vector $v_{j_1} \in B_0$ is a linear combination of the rest of the vectors in B_0. Define

$$B_1 = B_0 \setminus \{v_{j_1}\}.$$

Then, by an earlier lemma, $\operatorname{span}(B_1) = \operatorname{span}(B_0) = V$.

If B_1 is linearly independent, then it is a basis for V, and the proof is complete. Otherwise, continuing removing vectors, one at a time, to produce smaller and smaller spanning sets. Eventually, B_k must be a linearly independent spanning set (a basis) for some $k < m$. Otherwise, we obtain that $B_m = \emptyset$ spans V, a contradiction. //

Corollary: Every nontrivial finite-dimensional vector space contains a basis.

Proof: By definition, a finite-dimensional vector space contains a spanning set and, since V is nontrivial, the previous theorem guarantees that this spanning set contains a basis. //

Theorem: Let $\{v_1, v_2, \ldots, v_k\}$ be a linearly independent set in V, where V is a finite-dimensional vector space. Then either $\{v_1, v_2, \ldots, v_k\}$ is a basis for V or there exist vectors $v_{k+1}, \ldots, v_n \in V$ such that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

Proof: Since V is finite-dimensional, it contains a spanning set; let the number of vectors in this set be m. Recall that no linearly independent set in V can contain more than m vectors.

Now, if $\{v_1, v_2, \dots, v_k\}$ spans V, then it is a basis for V and the proof is complete. Otherwise, there exists $v_{k+1} \in V$ such that $v_{k+1} \notin \text{span}(v_1, v_2, \dots, v_k)$. By an earlier exercise (2A/11), $\{v_1, v_2, \dots, v_{k+1}\}$ is linearly independent. We continue adding vectors in this way to produce larger and larger linearly independent sets until we obtain a linearly independent spanning set (a basis) $\{v_1, v_2, \dots, v_n\}$. Note that $\{v_1, v_2, \dots, v_n\}$ must span V for some $n \leq m$, since V contains at most m linearly independent vectors. //

Recall that, for subspaces U, W of V,

$$U + W = \{u + w \mid u \in U \text{ and } w \in W\}.$$

Also, we say that $U+W$ is a direct sum (and write it as $U \oplus W$) iff each $x \in U+W$ can be written uniquely as $x = u + w$, where $u \in U$ and $w \in W$.

<u>Theorem</u>: Let V be a finite-dimensional vector space and let U be a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$.

Proof: First note that if $U = V$, then we can take $W = \{0\}$, while if $U = \{0\}$, we can take $W = V$. Thus, in the rest of the proof, we can assume that U is a nontrivial, proper subspace of V. By earlier results, we know that there exists a basis $\{u_1, \dots, u_k\}$ of U. By the above theorem, we can extend this to a basis $\{u_1, \dots, u_k, u_{k+1}, \dots, u_n\}$ of V. We have

$$U = \operatorname{span}(u_1, \dots, u_k),$$

and we define

$$W = \operatorname{span}(u_{k+1}, \dots, u_n).$$

We claim that $V = U + W$ and that $U + W$ is a direct sum. Since $\{u_1, \dots, u_n\}$ is a basis for V, for all $v \in V$, there exist $\alpha_1, \dots, \alpha_n$ such that

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_n v_n \\ &= (\alpha_1 v_1 + \dots + \alpha_k v_k) + (\alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n) \\ &\in U + W. \end{aligned}$$

This shows that $V = U + W$ (actually, it shows that $V \subseteq U + W$, but $U + W \subseteq V$ holds by definition).

To show that $U + W$ is a direct sum, it suffices to show that

$U \cap W = \{0\}$. Suppose $x \in U \cap W$. Then

$$x \in U \Rightarrow x = \alpha_1 v_1 + \cdots + \alpha_k v_k \text{ for some } \alpha_1, \ldots, \alpha_k \in F$$

and

$$x \in W \Rightarrow x = \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n \text{ for some } \alpha_{k+1}, \ldots, \alpha_n \in F.$$

But then

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n$$

$$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_k v_k - \alpha_{k+1} v_{k+1} - \cdots - \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_k = \alpha_{k+1} = \cdots = \alpha_n = 0$$ (since $v_1, \ldots, v_n$ are linearly independent).

Thus $x = 0v_1 + \cdots + 0v_k = 0$, and we have shown that $U \cap W = \{0\}$. //

Example

Working out an example of these concepts usually involves solving a system of linear algebraic equations and properly interpreting the results.

- We know that $\mathbb{R}^3$ is spanned by three vectors, so any set of more than three vectors in $\mathbb{R}^3$ is linearly dependent. Show that

$$\{(1,0,1), (1,3,0), (2,3,1), (1,-2,2), (4,-1,5)\} = \{v_1, v_2, v_3, v_4, v_5\}$$

spans $\mathbb{R}^3$ and find a subset that is a basis.

Solution: We begin by solving

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 + \alpha_5 v_5 = x,$$

where $x = (x_1, x_2, x_3)$ is an arbitrary element of $\mathbb{R}^3$:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \alpha_5 \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 4\alpha_5 &= x_1 \\ 3\alpha_2 + 3\alpha_3 - 2\alpha_4 - \alpha_5 &= x_2 \\ \alpha_1 \qquad + \alpha_3 + 2\alpha_4 + 5\alpha_5 &= x_3 \end{aligned}$$

Solution by Gaussian elimination with back substitution in augmented matrix form:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 4 & x_1 \\ 0 & 3 & 3 & -2 & -1 & x_2 \\ 1 & 0 & 1 & 2 & 5 & x_3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 4 & x_1 \\ 0 & 3 & 3 & -2 & -1 & x_2 \\ 0 & -1 & -1 & 1 & 1 & x_3 - x_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 4 & x_1 \\ 0 & -1 & -1 & 1 & 1 & x_3 - x_1 \\ 0 & 3 & 3 & -2 & -1 & x_2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 4 & x_1 \\ 0 & -1 & -1 & 1 & 1 & x_3 - x_1 \\ 0 & 0 & 0 & 1 & 2 & 3x_3 + x_2 - 3x_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & 2 & -3x_3 - x_2 + 4x_1 \\ 0 & -1 & -1 & 0 & -1 & -2x_3 - x_2 + 2x_1 \\ 0 & 0 & 0 & 1 & 2 & 3x_3 + x_2 - 3x_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & -5x_3-2x_2+6x_1 \\ 0 & -1 & -1 & 0 & -1 & -2x_3-x_2+2x_1 \\ 0 & 0 & 0 & 1 & 2 & 3x_3+x_2-3x_1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 6x_1-2x_2-5x_3 \\ 0 & 1 & 1 & 0 & 1 & -2x_1+x_2+2x_3 \\ 0 & 0 & 0 & 1 & 2 & -3x_1+x_2+3x_3 \end{array}\right]$$

The reduced (and equivalent) system is

$$\begin{aligned} \alpha_1 + \alpha_3 + \alpha_5 &= 6x_1-2x_2-5x_3 \\ \alpha_2+\alpha_3 + \alpha_5 &= -2x_1+x_2+2x_3 \\ \alpha_4+2\alpha_5 &= -3x_1+x_2+3x_3 \end{aligned}$$

For each $x\in\mathbb{R}^3$, a solution is

$$\alpha_1 = 6x_1-2x_2-5x_3,\ \alpha_2=-2x_1+x_2+2x_3,\ \alpha_4=-3x_1+x_2+3x_3$$

$$\alpha_3=\alpha_5=0.$$

That is, we can write x as

$$x=\alpha_1v_1+\alpha_2v_2+\alpha_4v_4 \quad \text{(with the above values of } \alpha_1,\alpha_2,\alpha_4)$$

which shows that $\{v_1,v_2,v_4\}$ spans $\mathbb{R}^3$. We also see that if we eliminate v_3, v_5 (i.e. require that $\alpha_3=\alpha_5=0$), then

$$\alpha_1v_1+\alpha_2v_2+\alpha_4v_4=x$$

has a unique solution for each $x\in\mathbb{R}^3$. Thus $\{v_1,v_2,v_4\}$ is linearly independent.