Math 672 Lecture 15

Recall: $\lambda \in F$ is an eigenvalue of $T \in \mathcal{L}(V)$ iff there exists $v \in V$, $v \neq 0$, such that $T(v) = \lambda v$. Equivalently, λ is an eigenvalue of T iff $T - \lambda I$ is singular ($\mathcal{N}(T-\lambda I)$ is nontrivial).

Theorem: Let $T \in \mathcal{L}(V)$ have distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m \in F$, with corresponding eigenvectors $v_1, v_2, \ldots, v_m \in V$. Then $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Proof: We argue by contradiction and assume that $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent. Then there exists $k > 1$ such that $\{v_1, \ldots, v_{k-1}\}$ is linearly independent and $v_k \in \text{span}(v_1, \ldots, v_{k-1})$, say

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1}, \quad \alpha_1, \ldots, \alpha_{k-1} \in F.$$

Then

$$T(v_k) = T(\alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1})$$

$$\Rightarrow \lambda_k v_k = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{k-1} \lambda_{k-1} v_{k-1}$$

$$\Rightarrow \alpha_1 \lambda_k v_1 + \cdots + \alpha_{k-1} \lambda_k v_{k-1} = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{k-1} \lambda_{k-1} v_{k-1}$$

$$\Rightarrow \alpha_1(\lambda_k-\lambda_1)v_1 + \cdots + \alpha_{k-1}(\lambda_k-\lambda_{k-1})v_{k-1} = 0$$

$$\Rightarrow \alpha_1(\lambda_k-\lambda_1) = \cdots = \alpha_{k-1}(\lambda_k-\lambda_{k-1}) = 0$$ (since $\{v_1,\ldots,v_{k-1}\}$ is linearly independent)

$$\Rightarrow \alpha_1 = \cdots = \alpha_{k-1} = 0$$ (since $\lambda_k - \lambda_j \neq 0$ for $j=1,\ldots,k-1$ by assumption)

$$\Rightarrow v_k = 0.$$

But $v_k \neq 0$ because v_k is an eigenvector. This contradiction completes the proof.//

Note that the above is a "genuine" proof by contradiction. The theorem is $P \Rightarrow Q$, where

P = $v_1, v_2, \ldots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$

Q = $\{v_1, v_2, \ldots, v_m\}$ is linearly independent

I assumed P and $\sim Q$ and <u>used both assumptions</u> to derive a contradiction. The contradiction is $P \wedge (\sim P)$, but this cannot be recast as a proof by contrapositive (again, both P and $(\sim Q)$ were used to prove $\sim P$).

<u>Corollary</u>: Let $T \in \mathcal{L}(V)$, where $\dim(V) = n$. Then T has at most n distinct eigenvalues.

Polynomials and eigenvalues

As we know, multiplication (i.e. composition) of operators is not commutative:

$$S,T \in \mathcal{L}(V) \Rightarrow ST = TS \quad \text{is false!}$$

However, if m,n are nonnegative integers, then

$$T^m T^n = T^n T^m$$

because both equal T^{m+n}. Here,

$$T^n = \underbrace{TT\cdots T}_{n \text{ factors}},$$

$$T^0 = I.$$

Given $T \in \mathcal{L}(V)$ and

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}(F) \quad (\text{so that } a_0, \ldots, a_n \in F),$$

we define

$$p(T) = a_0 I + a_1 T + \cdots + a_n T^n.$$

It then follows that

$$(*) \qquad p(T)q(T) = q(T)p(T) \quad \forall\, p,q \in \mathcal{P}(F).$$

(These special operators do commute.)

Also, if $p, q, r \in \mathcal{P}(F)$, then

$$(**) \quad p(x)q(x) = r(x) \Rightarrow p(T)q(T) = r(T).$$

For instance,

$$\begin{aligned}(x+2)(3x+1) &= x(3x+1) + 2(3x+1) \\ &= 3x^2 + x + 6x + 2 \\ &= 3x^2 + 7x + 2,\end{aligned}$$

$$\begin{aligned}(T+2I)(3T+I) &= T(3T+I) + 2I(3T+I) \\ &= 3T^2 + T + 6T + 2I \\ &= 3T^2 + 7T + 2I.\end{aligned}$$

(We could write a formal proof of $(*)$ and $(**)$, but it doesn't seem worthwhile.)

<u>Lemma</u>: Let $S, T \in \mathcal{L}(V)$ be given. Then ST is singular iff (T is singular or S is singular).

<u>Proof:</u> If T is singular, then there exists $v \in V$ such that $v \neq 0$ and $T(v) = 0$. But then $(ST)(v) = S(T(v)) = S(0) = 0$ and hence ST is singular. If T is nonsingular and S is singular, then T is invertible and there exists $v \in V$ such that $v \neq 0$ and

$S(v)=0$. But then $S(T(T^{-1}(v)))=S(v)=0$ and hence $(ST)(T^{-1}(v))=0$. Since $T^{-1}(v)\neq 0$ (because $v\neq 0$ and T^{-1} is nonsingular), this shows that ST is singular.

Thus

$$(T \text{ is singular or } S \text{ is singular}) \Longrightarrow ST \text{ is singular.}$$

Conversely, if both T and S are nonsingular, then both are invertible and hence ST is invertible and thus nonsingular. //

By induction, it is easy to extend the previous result to any number of operators.

<u>Corollary</u>: Let $S_j \in \mathcal{L}(V)$ for $j=1,2,\dots,k$. Then

$S_1 S_2 \cdots S_k$ is singular iff there exists $j\in\{1,2,\dots,k\}$ such that S_j is singular.

The above is all we need to prove that every operator $T\in\mathcal{L}(V)$, <u>where V is a vector space over $\mathbb{C}$</u>, has at least one eigenvalue.

<u>Theorem</u>: Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T\in\mathcal{L}(V)$. Then T has an eigenvalue.

Proof: Suppose $\dim(V)=n$ and let v be any nonzero vector in V. The set

$$\{v, T(v), T^2(v), \dots, T^n(v)\}$$

has $n+1$ elements and hence is linearly dependent. Therefore, there exist $a_0, a_1, \dots, a_n \in \mathbb{C}$ such that

$$a_0 v + a_1 T(v) + \dots + a_n T^n(v) = 0$$

$$\Longleftrightarrow (a_0 I + a_1 T + \dots + a_n T^n)(v) = 0$$

$$\Longleftrightarrow p(T)v = 0,$$

where $p(x) = a_0 + a_1 x + \dots + a_n x^n$. Note that $1 \le \deg(p(x)) \le n$; say $\deg(p(x)) = m$. By the fundamental theorem of algebra there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{C}$ such that

$$p(x) = a_m (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m).$$

We thus have

$$a_m (T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_m I)v = 0$$

$$\Rightarrow (T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_m I) \text{ is singular}$$

$$\Rightarrow \exists\, j \in \{1,2,\dots,m\},\ T-\lambda_j I \text{ is singular} \quad \text{(by the Lemma)}$$

$$\Rightarrow \lambda_j \text{ is an eigenvalue of } T. //$$

The preceding result is not true if V is a vector space over $\mathbb{R}$.

Example: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x) = (x_2, -x_1).$$

Then

$$T(x) = \lambda x \text{ and } x \neq 0$$

$$\Rightarrow (x_2, -x_1) = (\lambda x_1, \lambda x_2) \text{ and } x \neq 0$$

$$\Rightarrow x_2 = \lambda x_1 \text{ and } -x_1 = \lambda x_2 \text{ and } (x_1 \neq 0 \text{ or } x_2 \neq 0)$$

$$\Rightarrow -x_1 = \lambda(\lambda x_1) \text{ and } x_1 \neq 0 \quad (\text{since } x_1 = 0 \Rightarrow x_2 = 0)$$

$$\Rightarrow \lambda^2 = -1$$

Thus T has no eigenvalue in $\mathbb{R}$ ($\lambda^2 = -1$ has no real solution).

Complexification

From undergraduate linear algebra, you may be used to thinking that $\pm i$ are eigenvalues of the above operator (which is defined by the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$).

Here is the formal process by which we can say that $T \in \mathcal{L}(V)$ can have complex eigenvalues, when V is a vector space over $\mathbb{R}$.

1. Define the <u>complexification</u> of V:

$$V_{\mathbb{C}} = \{u + iv \mid u, v \in V\}$$

$V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$ under the obvious operations.

Note that the complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$!

2. Define the complexification of T:

$$T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$$

$$T_{\mathbb{C}}(u+iv) = T(u) + iT(v) \quad \forall u+iv \in V_{\mathbb{C}}.$$

Then, by an abuse of terminology, we say that any eigenvalue of $T_{\mathbb{C}}$ is an eigenvalue of T. Note that if $T_{\mathbb{C}}$ has an eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then any corresponding eigenvector v lies in $V_{\mathbb{C}} \setminus V$.

Interesting fact: If V is a vector space over $\mathbb{R}$ with basis $\{v_1, v_2, \ldots, v_n\}$, then $\{v_1, v_2, \ldots, v_n\}$ is also a basis for $V_{\mathbb{C}}$ (as a vector space over $\mathbb{C}$). Thus

$$\dim(V_{\mathbb{C}}) = \dim(V).$$

vector space over $\mathbb{C}$

vector space over $\mathbb{R}$.