MATH 630

FINAL EXAM

Study Guide

Fall 2024

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(1) **What topics will be covered in the exam?**

- Combinations and permutations
- Axioms and properties of probability measures
- The distribution function
- Conditional probability and independence
- Discrete random variables and expected values
- Special discrete distributions:(Bernoulli Binomial, Poisson, Geometric).
- Joint probability distributions and sums of random variables.
- Continuous random variables
- Special continuous distributions (Normal, exponential, uniform)
- Conditional Distributions and Conditional expectation.
- Inequalities (Markov's inequalities, Chebyshev's inequality, Cauchy-Schwartz inequality, Jensen's inequality).
- Convergence of random variables (Almost sure convergence, r-th mean convergence, convergence in probability, convergence in distribution)
- The strong and weak law of large numbers.
- The central limit theorem.
- Markov chains
  - Markov's property and its applications
  - Irreducible Markov chains
  - Regular Markov chains
  - Chapman-Kolmogorov equation
  - Stationary distributions
  - Mean first passage/ return times
  - Recurrent and transient states

(2) **How should I prepare for the exam?**

- Review the following homework problems:
  - HW 4: 6-8
  - HW 5: 2-5
  - HW 6: 2-5, 7
- Review problems 2-5 of the take-home exam.
- Review exam 1.
- Review all the problems attached to this study guide. Some of these problems will be discussed in class and the solutions for the rest will be uploaded to Canvas.
- This final exam is cumulative, designed to assess your comprehensive understanding of the material covered throughout the semester. When you encounter a topic in any question, please review your notes and refer to related problems discussed during the relevant lectures.
- Not every question will be identical/ similar to what is mentioned on this study guide. Therefore, it's important that you do not just memorize these problems.

(3) Can you give any hints?

- I will ask you to prove at least one theorem related to Markov chains. Note that this might not be the only theorem you'll need to prove, so familiarize yourself with proving multiple theorems from other topics
- There will be no questions related to the inclusion-exclusion principal.
- I will ask you to verify whether a given sequence is a Markov chain. We did similar problems in class.
- I will ask you to find the stationary distribution/ mean return times of a given MC. Be able to find different classes.
- I will ask you to verify whether a given sequence obeys WLLN/ SLLN (Know the four theorems).

(4) **What is the structure of the exam?** The final exam will include no more than 12 problems, some of which will have multiple parts. As discussed in class, while the exam is structured to be completed within two hours, you will be given three hours to finish.

(5) **Am I allowed to bring a cheat sheet to the exam?** Yes, <u>**with some restrictions**</u>.

- You may ONLY include formulas, theorems, and definitions.

- TWO sheets of paper (front and back) is allowed.
- No proofs, examples, solutions (or parts of them), or hints.
- The formula sheet must be turned in with the exam.

(6) **Am I allowed to use a calculator?** No, and you won't need one.

(7) VERY IMPORTANT:

- I am preparing three distinct versions of the final exam to accommodate different groups of students taking the exam at various times and dates due to university-approved reasons. It is crucial that you maintain the confidentiality of the exam questions and do not discuss them with your peers. Although the exams will differ, it's essential that no student group has access to extra resources.
- To ensure equity, I will not be holding any office hours after December 16th. However, I will provide ample additional office hours before this date, specifically on December 16th and earlier. Students scheduled to take the exam later are encouraged to utilize these sessions.
- The purpose of this policy is to prevent any unintended advantage for those taking the exam later. Each version of the exam will be different but will be structured to reflect the same level of difficulty and comprehension, based on the material covered in this study guide.

*I will notify you if I make any changes to this study guide.

Practice Problems

Note: Some of these problems will be discussed in class, and solutions for the remaining ones will be uploaded to Canvas. I have already covered some of these questions during lectures. I may also add a few more questions to this list later.

(1) A biased coin lands on heads with probability 0.6. The coin is flipped repeatedly. What is the probability of observing a run of five consecutive heads before a run of 5 consecutive tails?

(2) Let $X_n \sim B(n, \frac{\lambda}{n})$ for $n \in \mathbb{N}$. Prove that $X_n \to X \sim Poisson(\lambda)$ in distribution as $n \to \infty$.

(3) Give a probabilistic justification for the following limit:

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

(4) Let $\{Y_n\}$ be independent $Poisson(1)$ random variables. Set

$$X_n = \frac{\sqrt{n}(\sum_{k=1}^{n} Y_k - n)}{\sum_{k=1}^{n} Y_k^2}$$

Find the limiting distribution of X_n.

Hint: If $X_n \to X$ in distribution and $Y_n \to c \neq 0$ in probability, then $X_n/Y_n \to X/c$ in distribution (This theorem is called Cramer's theorem/ Slutsky's theorem. We did not discuss this theorem in class, but this is very useful).

(5) The joint density function of X and Y is given below:

$$f(x,y) = \begin{cases} cx^3y & \text{if } x, y \geq 0 \text{ and } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y|X = x)$

(6) You are give that $X|Y = y \sim U(-y, y)$ and $Y \sim U(0, 1)$. Find $V(X)$.

(7) There are 100 coins in a box. The probability that the ith coin flipped is a head is $i/99$, where $i = 0, 1, 2, ..., 99$. A randomly selected coin is flipped repeatedly. If the first n flips are all heads, what is the conditional probability that the next $(n + 1)$ flip is also a head?

(8) Let $X, Y \sim U(0, 1)$ be two random variables not necessarily independent. Prove that $E|X - Y| \leq \frac{1}{2}$.

(9) X_n are standard normal and $Y_0 \sim U(0,1)$. $Y_0, X_1, X_2, ...$ are independent. Define the sequence Y_n via the relation

$$Y_n = \frac{Y_{n-1}}{2} + X_n$$

for $n = 1, 2, 3, ...$Prove that Y_n convergence in distribution to some random variable.

(10) Consider a Markov chain on the state space $\{1, 2, 3\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find the stationary distribution π.

(11) Let $X = \{X_n\}$ be a MC. Prove that $Z_n = (X_n, X_{n+1})$ is a MC.

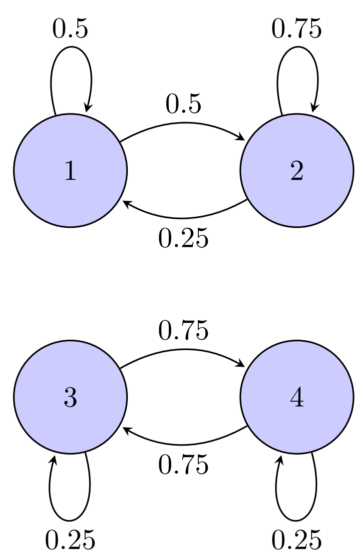
(12) Let $\{X_n\}$ and $\{Y_n\}$ be independent regular MCs. Prove that $Z_n = (X_n, Y_n)$ is regular.

(13) If i is recurrent and i communicates with j, then prove that j is also recurrent.

(14) Prove the Chapman-Kolmogorov equation.

(15) Let $X_1, X_2, ...$ be independent and $P(X_n = 1) = a_n$ and $P(X_n = 0) = 1 - a_n$. If $a_n \to 0$, then prove that X_n converges in probability.

(16) Consider the following transition diagram:



This is an example of a reducible MC (because the chain can be reduced to two distinct MCs).

(a) State the transition matrix of this MC.

(b) Find the stationary distribution π.

(17) Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

(18) On a given day a stock either goes up (1), remains constant (2), or goes down (3). Let $X = \{X_n\}$ be a three-state MC that denotes the behavior of the stock (1,2, or 3) on the nth day. The transition matrix of X is given below:

$$\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

(a) Find π. Interpret its meaning in the context of this problem.

(b) Suppose the stock is currently up. What is the probability that it does not go down on any of the following three days?

(19) Prove WLLN. (You are not required to know the proof of SLLN).

(20) If g is a continuous function and $X \to k$ in probability, prove that $g(X) \to g(k)$ in probability.

(21) Let $X \sim Poisson(\lambda)$ and $\lambda \sim U(1, 2)$. Compute the following:

(a) $E(X|\lambda = 1.5)$

(b) $P(E(X|\lambda) > 1.5)$

(22) Prove that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^c)P(C^c|B)$$

(23) Let $Z \sim N(0, 1)$. Find $E(\max\{Z - c, 0\})$ where c is some constant.

(24) Let X be a non-negative rv. Prove that

$$E(X) = \int_0^\infty P(X > t)dt.$$

(25) Let X and Y be random variables. Prove that

$$E(X) = E((X|Y)).$$

(26) Let $\{X_n\}$ be a sequence of independent random variables and $X_n \sim N(c^n, \frac{\sqrt{n}}{2})$ where $0 < c < 1$. Does $\{X_n\}$ obey WLLN? SLLN?

(27) Let $\{X_n\}$ be a sequence of random variables with finite mean and $V(X_k) = \sigma^2 < \infty$ for all k. If $cov(X_i, X_j) \leq 0$ for all $i \neq j$, prove that

$$Y_n/n \xrightarrow{2} 0$$

where $Y_n = \sum_{i=1}^{n}(X_i - E(X_i))$.

(28) You are given the following information about $\{X_n\}$.

- For some $M > 0$, $V(X_n) \leq M$ for all $n = 1, 2, ...$
- $cov(X_i, X_j) < 0$ for all $i \neq j$.

Prove that $\{X_n\}$ obeys WLLN.

(29) Let $\{X_n\}$ be a sequence of random variables with finite mean and $V(X_k) = \sigma^2 < \infty$ for all k. If $cov(X_i, X_j) \leq 0$ for all $i \neq j$, prove that

$$Y_n/n \xrightarrow{2} 0$$

where $Y_n = \sum_{i=1}^{n}(X_i - E(X_i))$.