Chapter 10. Magnetostatics (11 Nov 2020).

A. Overview.

Magnetostatics is less familiar to physicists than electrostatics, not just because it follows electrostatics in the curriculum, but also because its vector-calculus demands are greater. However, magnetism is involved in many common phenomena that provide instructive examples of how Maxwell's Equations and appropriate mathematical theorems can be applied to understand physical properties. It also offers an excellent opportunity to see how Maxwell's Equations follow from experiment, in magnetostatics from Ampère's Force Law. Aside from their vector character, the associated derivations will look much like a review.

Some aspects of magnetostatics overlap directly with electrostatics. For example, the macroscopic equations

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \varepsilon\vec{E} \,; \tag{10.1a,b}$$

$$\vec{B} = \vec{H} + 4\pi\vec{M} = \mu\vec{H} \,. \tag{10.1c,d}$$

have the same form even though the physics leading to them is entirely different. Also, for $\vec{J} = 0$, Ampère's Equation in the statics limit reduces to

$$\nabla \times \vec{H} = 0 \,. \tag{10.2a}$$

With no source terms present $\vec{H}$ can be written

$$\vec{H} = -\nabla\phi_m \,, \tag{10.2b}$$

where ϕ_m is the magnetic scalar potential. From $\nabla \cdot \vec{B} = \nabla \cdot (\mu\vec{H}) = 0$ it follows that for materials with a uniform permeability μ, the magnetic scalar potential ϕ_m satisfies Laplace's Equation. In this homogeneous-equations case, magnetostatics maps directly onto electrostatics. For example, for spherical geometries we can use series expansions of spherical harmonics, or Legendre polynomials if the configuration is azimuthally symmetric.

Nevertheless, there are significant differences. Magnetic charges (monopoles) are expected to have been created in large quantities in the “big bang”, but despite many and often heroic efforts, none have ever been found. The accepted explanation is that during the exponential expansion phase of the universe from about 10^{-36} to about 10^{-32} s, their concentration was diluted to insignificance. Hence, at least in our part of the universe, magnetic flux lines never terminate, but form closed loops. Therefore, flux lines are much more fundamental in magnetostatics than in electrostatics.

Also, with no magnetic charge, magnetic fields arise either from moving electric charges or magnetic dipoles associated with spin or orbital angular momentum of electronic wavefunctions. The former mechanism is unquestionably relativistic. The latter clearly quantum-mechanical, although a case can be made for relativity as well, given that the concept of spin follows from the relativistic Dirac equation. In cgs units the distinction between the two origins is clear: if the prefactor of a term is proportional to $(1/c)$ its origin is relativistic, whereas if the $(1/c)$ is missing, its origin is quantum mechanical. This division is not generally accepted; the writer of the Wikipedia article on magnetism considers the orbital angular momentum of electrons to be a current rather than an angular momentum. Although the developments in Secs. C and D are based on current, Sec. E develops magnetostatics with the $\vec{F} = m\vec{a}$ approach, which connects to angular momentum.

A second major difference is that when source terms are considered, the potential that describes magnetism is a vector, $\vec{A}$, not a scalar, ϕ. Until the Aharonov-Bohm effect was verified experimentally, many physicists were not convinced that $\vec{A}$ was anything more than a mathematical construct introduced to solve equations. The Aharonov-Bohm effect is the change of phase of the wave function of an electron traversing an otherwise field-free region. This effect is particularly important in physics because it is purely quantum-mechanical: it cannot be described either classically or by special relativity. As such, its significance is much deeper than something that simply contributes to line integrals. As Feynman notes, it demonstrates incontrovertibly that potentials are more fundamental than fields. We discuss this in detail in Sec. G.

However, assigning $\vec{A}$ purely to mathematics was already short-sighted, because as engineers are well aware, eddy currents exist. Eddy currents are driven by $\partial\vec{A}/\partial t$, which describes the EMF developed by magnetic fields that are changing in time. The importance of $\vec{A}$ is no longer an issue.

Properties of magnetic materials are also qualitatively different from those of dielectrics. The permeabilities μ of natural materials range from slightly less than 1 for diamagnets up to values of the order of $10^3 - 10^6$ for ferromagnets. These latter values are far in excess of permittivities encountered in electrostatics, and lead to phenomena such as flux trapping that have no electrostatic analogs. In addition, owing to domain dynamics, values of μ in ferromagnets are history-dependent and intrinsically nonlinear. Consequently, Green functions cannot be used, and while frequency dependences $\mu(\omega)$ exist, they cannot be predicted. Therefore, for magnetics, sum rules and Kramers-Kronig relations have no useful equivalents.

Magnetic materials can be grouped into several categories depending on the atomic-scale physics. Diamagnetism is a property of materials with electrons paired in bonds. With no net spins to line up to strengthen the applied field, by Faraday's Law atomic-scale reaction currents arise to oppose the field that created them. The results are values of μ less than 1. Diamagnetic effects are typically very small. For example, the permeability of water is $\mu = 0.999837$,

Paramagnetism is a property of elements or molecules with parallel spins. Parallel spins are preferred for quantum-mechanical reasons. Wave functions associated with fermions must be antisymmetric, so if the spin states of a pair of electrons are symmetric, then the spatial part of their wavefunction must be antisymmetric. This better separates the two charges, thereby reducing their repulsive Coulomb interaction and hence the overall energy of the system (Hund's Rule). Perhaps the most spectacular example of paramagnetism is that of liquid oxygen, where the parallel spins are those of the two 2p lone-pair electrons. These do not participate in the $\mathrm{O_2}$ triple bond (think $\mathrm{N_2}$ with two leftover electrons). With $\mu = 1.0967$, liquid O_2 is sufficiently magnetic so it can be suspended between the pole pieces of a permanent magnet (see figure. Numerous examples can be found on-line; this more professional example is from Jefferson Laboratories.) Paramagnetism is the net result of the competition between the applied field working to align dipoles and entropy to disorder them. If entropy were not so effective, liquid oxygen would be ferromagnetic.

Ferrimagnetism is a phenomenon partway between paramagnetism and ferromagnetism, in that the magnetic properties of ferrimagnets are spin-based, but the spins in a unit cell are aligned antiparallel as well as parallel. The group includes antiferromagnets. Magnetite $(\mathrm{Fe_3O_4})$ is probably the best example. At $\mu = 1.045$, its permeability is about half that of liquid O_2. Most garnets are ferrimagnets, although their magnetism is quite anisotropic thanks to their complex silicate structure. Nevertheless, the permeability of gemstone garnets is high enough so they are attracted by strong magnets, a test that is often used to identify them.

Ferromagnetism is probably the most well-known magnetic phenomenon, and the one that everyone thinks of when the subject is brought up. Ferromagnetism is due to the spin of electrons in partially filled bands coupled by the exchange interaction, with minor contributions from their angular momenta. The classic example is iron, which has permeabilities μ ranging up to 50,000 depending on how the material is processed. Permanent magnets provide the opportunity to apply various fundamental equations to configurations where magnetic fields do not average to zero. These remanences (fields remaining when the magnetization fields are removed) are typically about 4 orders of magnitude larger than the 1 Gauss field of the earth, for example up to 1.2 T for Alnico and 1.4 T for $Nd_2Fe_{14}B$ alloys. These are strong enough to be dangerous, not because the fields affect us directly, but because the attractive force between two permanent magnets can be strong enough to cause serious damage to fingers if you are careless enough to let one get caught between. Yet these fields are insignificant relative to the ca. 10^{11} T fields that are predicted to occur at the surfaces of some neutron stars. Fields of these magnitudes would be unimaginably dangerous, but fortunately are never encountered on earth.

Chapter 10 proceeds as follows. We begin after Sec. D of Ch. 2, where the microscopic version of Maxwell's Equations is derived starting with Ampère's force law. The path from there to the macroscopic magnetic equations is more convoluted than what one would expect from our experience with electrostatics. It starts by determining the contribution of a pre-existing magnetic dipole $\vec{m}$ to $\vec{A}$, which in Sec. B is done by a multipole expansion, and in Sec. D by relating atomic-scale angular momenta to $\vec{m}$ through an application of $\vec{F} = m\vec{a}$. Although the $\vec{F} = m\vec{a}$ approach is more physical, it requires performing a time average, an operation that will be important in remaining chapters but one that we have not discussed up to now.

In Sec. C the dipoles $\vec{m}$ are turned into the macroscopic Ampère Equation. Although the math is more challenging, many parts of the path will be familiar from the derivations done in Ch. 7. Jackson also uses the multipole-expansion route in his Secs. 5.6 and 5.8, although many steps are missing in his derivation. We supply the missing steps here. Jackson alludes to force treatments in his Sec. 6.6, but these are related to the Lorentz force and are not relevant to the present topic.

Much of the remainder of Ch. 9 consists of examples, many of which involve ferromagnets. Ferromagnets are not usually discussed in textbooks because the nonlinear and history-dependent responses of ferromagnets limit what can be done relative to linear systems. Energy also has some unique aspects in ferromagnets, and an understanding of internal energy in these materials is essential for calculating face pressures of magnets, as done in Sec. H. Practical considerations such as magnetic analogs of electric circuits are covered in Sec. J. Dynamic situations, including Faraday's Equation are left to Ch. 10 and subsequent chapters.

One loose end left over from Ch. 2 is the expression for the total force between two loops carrying currents I_1 and I_2. With no good place to put it, we include it in this section. Start with Ampère's Law for the differential force between two current elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$:

$$d\vec{F}_{12} = \frac{I_1 d\vec{l}_1 \times [I_2 d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)]}{c^2 \, |\vec{r}_1 - \vec{r}_2|^3}. \tag{10.3}$$

Using the bac-cab theorem of vector products, this can be written

$$\vec{F}_{12} = \frac{I_1 I_2}{c^2} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \times [d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)]}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$= \frac{I_1 I_2}{c^2} \oint_{C_1} \oint_{C_2} \frac{(d\vec{l}_1 \cdot (\vec{r}_1 - \vec{r}_2))\, d\vec{l}_2 - (d\vec{l}_1 \cdot d\vec{l}_2)(\vec{r}_1 - \vec{r}_2)}{|\,\vec{r}_1 - \vec{r}_2\,|^3} \,. \tag{10.4}$$

We recognize immediately

$$\frac{\vec{r}_1 - \vec{r}_2}{|\,\vec{r}_1 - \vec{r}_2\,|^3} = -\nabla_{\vec{r}_1} \left(\frac{1}{|\,\vec{r}_1 - \vec{r}_2\,|} \right). \tag{10.5}$$

Thus the integral over $d\vec{l}_1 = d\vec{r}_1$ is

$$\oint_C d\vec{r}_1 \cdot \nabla_{\vec{r}_1} \frac{1}{|\,\vec{r}_1 - \vec{r}_2\,|} = \int_S d^2 r_1 \hat{n} \cdot \nabla_{\vec{r}_1} \times \nabla_{\vec{r}_1} \frac{1}{|\,\vec{r}_1 - \vec{r}_2\,|} = 0 \,. \tag{10.6}$$

Therefore

$$\vec{F}_{12} = -\frac{I_1 I_2}{c^2} \oint_{C_1} \oint_{C_2} \frac{(d\vec{l}_1 \cdot d\vec{l}_2)(\vec{r}_1 - \vec{r}_2)}{|\,\vec{r}_1 - \vec{r}_2\,|^3} \,. \tag{10.7}$$

Except for the sign, this expression is obviously symmetric in the two loops, as expected.

A second loose end is the definition of magnetostatics. In a truly static or more accurately quasistatic configuration, all macroscopic time derivatives are zero. In particular the condition $\partial < \vec{E} > / \partial t = 0$ reduces Ampère's Equation to

$$\nabla \times < \vec{B} > = \frac{4\pi}{c} < \vec{J} > . \tag{10.8}$$

Because $\nabla \cdot (\nabla \times < \vec{B} >) = 0$, it follows that one definition of magnetostatics is

$$\nabla \cdot < \vec{J} > = 0 \,. \tag{10.9}$$

This is consistent with the charge-conservation equation provided

$$\partial < \rho > / \partial t = 0 \,.$$

A second definition of magnetostatics is

$$\nabla \cdot < \vec{A} > = 0 \,. \tag{10.10}$$

This follows by noting that $\partial \phi / \partial t = 0$ reduces the Lorentz gauge to the Coulomb gauge. In Sec. E we derive the magnetization contribution $c \nabla \times \vec{M}$ to $< \vec{J} >$ working with time on the atomic scale, where $\vec{M}$ is the magnetization (magnetic dipole density) analogous to $\vec{P}$ in electrostatics. However, the final result is quasistatic due to an average over time. A full time-dependent treatment of magnetization is done in Ch. 11.

The preceding paragraph uses the $< >$ notation to designate macroscopic quantities. However, unless specifically designated otherwise, from now on all quantities are considered macroscopic, so the brackets are not shown to simplify notation.

B. Macroscopic magnetostatics I: magnetic moments via a multipole expansion.

We begin by expanding the expression for of $\vec{A}$ in a Cartesian-multipole series. Through the dipole approximation

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{d^3r'\,\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{10.11a}$$

$$= \frac{1}{c}\int d^3r'\,\vec{J}(\vec{r}\,')\left(\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \ldots\right). \tag{10.11b}$$

Recall that the expansion assumes the range of $|\vec{r}\,'|$ about the origin is small compared to the distance r to the observer. The steps that follow will eventually circle back to Eq. (10.11a), first converting the dipole integral over $\vec{J}$ to a dipole moment $\vec{m}$, then converting an integral over $\vec{m}$ to the term $c\nabla\times\vec{M}$ to be added to $\vec{J}$. The final step is familiar: capitalize on the $\vec{r}$ dependence of $1/|\vec{r}-\vec{r}\,'|$ by applying the Lorentzian operator to the result, generating a delta function that eliminates the integral. The result is the macroscopic Ampère equation.

Given the absence of magnetic charge, we can anticipate that the leading term in Eq. (10.11b) is zero. To prove this, consider

$$\int_V d^3r'\,\nabla_{\vec{r}\,'}\cdot(x'\,\vec{J}(\vec{r}\,')) = \int_S d^2r'\,\hat{n}\cdot(x'\,\vec{J}(\vec{r}\,')) = 0\,. \tag{10.12}$$

The surface integral vanishes because $\vec{J}(\vec{r}\,')$ is localized. Now

$$\nabla_{\vec{r}\,'}\cdot(x'\,\vec{J}(\vec{r}\,')) = (\nabla_{\vec{r}\,'}x')\cdot\vec{J} + x'\nabla\cdot\vec{J}\,. \tag{10.13}$$

But in magnetostatics $\nabla\cdot\vec{J}=0$. Using in addition $\nabla_{\vec{r}\,'}x' = \hat{x}$, Eq. (10.12) reduces to

$$\int_V d^3r'\,\nabla_{\vec{r}\,'}\cdot(x'\,\vec{J}(\vec{r}\,')) = \int_S d^3r'\,J_x(\vec{r}\,')) = 0\,. \tag{10.14}$$

Thus the *x* component of the contribution of $\vec{J}$ vanishes. By repeating the procedure for J_y and J_z, we find that the entire leading term vanishes, as expected.

The second term in the expansion can be cast into a much more useful form by following a similar but more complicated procedure. Including the terms that Jackson leaves out, we start with

$$\int_V d^3r'\,\nabla\cdot(x'^2\,\vec{J}(\vec{r}\,')) = \int_V d^3r'\left(2x'\,\hat{x}\cdot\vec{J} + x'^2\,\nabla\cdot\vec{J}\right)$$

$$= 2\int_V d^3r'\,x'\,J_x = 0\,, \tag{10.15}$$

then examine

$$\int_V d^3r'\nabla_{\vec{r}'}\cdot(x'y'\vec{J}(\vec{r}')) = \int_V d^3r'\left(x'\hat{y}\cdot\vec{J} + y'\hat{x}\cdot\vec{J}\right)$$

$$= \int_V d^3r'(x'J_y + y'J_x) = 0\,. \tag{10.16}$$

Similar expressions are obtained by considering y'^2 and z'^2 in Eq. (10.15), and $y'z'$ and $z'x'$ in Eq. (10.16). Next, write the second expansion term in Eq. (10.11b) as

$$\vec{r}\cdot\int_V d^3r'\vec{r}'\vec{J}(\vec{r}') = x\int_V d^3r'x'\left(\hat{x}J_x + \hat{y}J_y + \hat{z}J_z\right)$$

$$+ y\int_V d^3r'y'\left(\hat{x}J_x + \hat{y}J_y + \hat{z}J_z\right)$$

$$+ z\int_V d^3r'z'\left(\hat{x}J_x + \hat{y}J_y + \hat{z}J_z\right). \tag{10.17}$$

The “diagonal” terms in the integrals in Eq. (10.17) can be eliminated by Eq. (10.15). By Eq. (10.16), each of the remaining “off-diagonal” terms can be antisymmetrized. The result of these operations is

$$\vec{r}\cdot\int_V d^3r'\vec{r}'\vec{J}(\vec{r}') = -\frac{1}{2}x\int_V d^3r'\left(\hat{y}(y'J_x - x'J_y) + \hat{z}(z'J_x - x'J_z)\right)$$

$$-\frac{1}{2}y\int_V d^3r'\left(\hat{z}(z'J_y - y'J_z) + \hat{x}(x'J_y - y'J_x)\right)$$

$$-\frac{1}{2}z\int_V d^3r'\left(\hat{x}(x'J_z - z'J_x) + \hat{y}(y'J_z - z'J_y)\right). \tag{10.18}$$

Now, note that

$$\vec{r}'\times\vec{J} = \hat{x}(y'J_z - z'J_y) + \hat{y}(z'J_x - x'J_z) + \hat{z}(x'J_y - y'J_x)\,, \tag{10.19}$$

so

$$\vec{r}\times(\vec{r}'\times\vec{J}) = \hat{x}\left(y(x'J_y - y'J_x) - z(z'J_x - x'J_z)\right)$$

$$+ \hat{y}\left(z(y'J_z - z'J_y) - x(x'J_y - y'J_x)\right)$$

$$+ \hat{z}\left(z(y'J_z - z'J_y) - y(y'J_z - z'J_y)\right). \tag{10.20}$$

By comparing Eqs. (10.18) and (10.20), we conclude that

$$\vec{A}(\vec{r}) = -\frac{1}{r^3}\vec{r}\times\left(\frac{1}{2c}\int_V d^3r'\left(\vec{r}'\times\vec{J}(\vec{r}')\right)\right) \tag{10.21a}$$

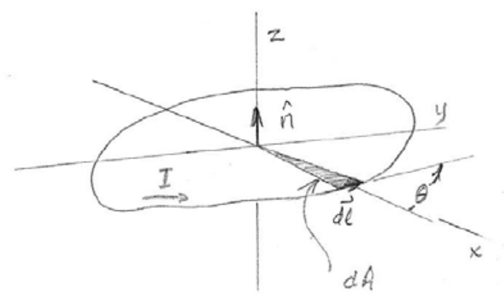
$$= +\frac{\vec{m}\times\vec{r}}{r^3}\,, \tag{10.21b}$$

where the integral in the large parentheses in Eq. (10.21a) is the *magnetic moment*

$$\vec{m} = \frac{1}{2c}\int_V d^3r'\left(\vec{r}'\times \vec{J}(\vec{r}')\right). \tag{10.22}$$

The minus sign in Eq. (10.21a) has been absorbed in Eq. (10.21b) by inverting the order of $\vec{r}$ and $\vec{m}$. We have therefore defined the *magnetic dipole density* or *magnetization* $\vec{M}$ as

$$\vec{M}(\vec{r}) = \frac{1}{2c}(\vec{r}\times\vec{J}(\vec{r})). \tag{10.23}$$



We have also accomplished the critically important step of establishing a mechanism to introduce magnetic dipoles into our calculations whether or not they originate with currents.

As an aside, if the configuration is planar and $\vec{J}(\vec{r}')$ is confined to a wire of negligible diameter, then Eq. (10.22) takes on a particularly simple form. Referring to the diagram, integrate $d^3r'\vec{J}(\vec{r}')$ over the cross section, converting it to

$$d^3r'\vec{J}(\vec{r}') = \vec{I}dl = Id\vec{l}, \tag{10.24}$$

where in the last step we move the direction attribute from I to dl. Now note that the differential area indicated in the diagram is $dA_l' = (1/2)r'\sin\theta' dl'$ and that $\vec{r}'\times d\vec{l}' = r'dl'\sin\theta'$, where θ' is the angle between the radius line and the local direction of the path, Eq. (10.22) reduces to

$$\vec{m} = \frac{I}{2c}\oint_C \vec{r}'\times d\vec{l}' = \frac{I}{2c}\hat{n}\int_0^{2\pi} d\theta'\sin\theta'\left(r(\theta')\right)^2$$

$$= \frac{IA_l}{c}\hat{n}, \tag{10.25}$$

where A_l is the area enclosed by the current path and $\hat{n}$ is the unit normal vector in the direction given by the right-hand rule. The magnitude of the magnetic moment is therefore the product of the current and the area of the loop divided by *c*.

In contrast to the electric dipole $\vec{p}$, the treatment of $\vec{m}$ differs in cgs and SI conventions, causing possible confusion. For the electric case the SI dipole is

$$\vec{p}_{SI} = \int_V d^3r'\vec{r}'\rho_{SI}(\vec{r}'). \tag{10.26a}$$

The cgs version is identical:

$$\vec{p}_{cgs} = \int_V d^3r'\,\vec{r}\,'\rho_{cgs}(\vec{r}\,')\,. \tag{10.26b}$$

In both systems $\vec{p}$ has the dimensions (charge)×(length).

In the magnetic case, the SI version (Jackson's Eq. (5.54)) is

$$\vec{m}_{SI} = \frac{1}{2}\int_V d^3r'\,\vec{r}\,' \times \vec{J}_{SI}(\vec{r}\,')\,. \tag{J.5.54}$$

The cgs version includes a c^{-1} in the prefactor:

$$\vec{m}_{cgs} = \frac{1}{2c}\int_V d^3r'\,\vec{r}\,' \times \vec{J}_{cgs}(\vec{r}\,')\,. \tag{10.27}$$

Because $\vec{J} = \rho\vec{v}$, the cgs dimension of $\vec{m}$ remains (charge)×(length). In contrast, the SI dimension is (current)×(area). The SI system therefore favors magnetic moments associated with current loops, whereas the cgs system is more suited to atomic-scale moments arising from orbital angular momentum and spin. The cgs convention also preserves the same dimensions for $\vec{E}$ and $\vec{B}$.

It might seem strange that $\vec{m}$ should have dimensions of (charge)×(length) when magnetic charges do not exist. Accepting this representation may be less of a struggle if one recalls that an electric dipole also has no net charge. The cgs rationale is that if magnetic charges did exist, then the dimensions of the magnetic dipole must be the same as those of the electric dipole. We recall that interface screening charge in electrostatics is quite real, and gives a useful microscopic perspective of screening. The same applies to magnetic screening. This is as close as we can come to realizing magnetic charge.

C. Macroscopic magnetostatics II: magnetization and permeability.

With the magnetic effect of localized currents represented as dipoles, the second step is to describe magnetic effects that originate with macroscopic collections of atomic-scale moments. Let a collection of dipoles $\vec{m}_j$ be located at sites $\vec{r}_j$. Then Eq. (10.27) becomes

$$\vec{A}(\vec{r}) = \sum_j \frac{\vec{m}_j \times (\vec{r} - \vec{r}_j)}{|\vec{r} - \vec{r}_j|^3}\,. \tag{10.28}$$

For simplicity assume that all $\vec{m}_j = \vec{m}$ are identical. If they effectively form a continuum, then Eq. (10.28) can be written as

$$\vec{A}(\vec{r}) = \int_V d^3r'\,\frac{n\,\vec{m} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \int_V d^3r'\,\frac{\vec{M} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\,, \tag{10.29}$$

where *n* is the volume density of magnetic dipoles. The magnetization $\vec{M}$ is defined as $\vec{M} = n\vec{m}$.

Recalling our experience with electrostatics, the next step is obvious: we need to convert the denominator $|\vec{r}-\vec{r}\,'|^3$ into $|\vec{r}-\vec{r}\,'|$ so a Laplacian in $\vec{r}$ can turn it into a delta function, in which case the integral is trivial. To do this rewrite Eq. (10.29) as

$$\vec{A}(\vec{r}) = \int_V d^3r\,'\vec{M}\times\nabla_{\vec{r}\,'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \tag{10.30}$$

then use

$$\int_V d^3r'\nabla_{\vec{r}\,'}\times\left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = \int_S d^2r'\hat{n}\times\left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = 0\,, \tag{10.31}$$

which assumes that the region of space containing the dipoles is finite. By expanding the curl operation, Eq. (10.31) becomes

$$\nabla_{\vec{r}\,'}\times\left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = \left(\nabla_{\vec{r}\,'}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\times\vec{M} + \frac{1}{|\vec{r}-\vec{r}\,'|}\nabla_{\vec{r}\,'}\times\vec{M}\;. \tag{10.32}$$

Combining Eqs. (10.30) and (10.32) gives the desired result:

$$\int_V d^3r'\frac{\vec{M}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \int_V d^3r'\frac{\nabla_{\vec{r}\,'}\times M(r')}{|\vec{r}-\vec{r}\,'|}\,. \tag{10.33}$$

If a current density $\vec{J}$ is also present, Eqs. (10.11) show that $\vec{A}$ is

$$\vec{A}(\vec{r}) = \frac{1}{c}\int_V d^3r'\frac{\vec{J}(\vec{r}\,')+c\nabla_{\vec{r}\,'}\times M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,. \tag{10.34}$$

Thus $c\nabla\times\vec{M}$ is completely equivalent to a current density. Equation (10.34) can be compared to the scalar-potential equivalent:

$$\phi(\vec{r}) = \int_V d^3r'\frac{\rho(\vec{r}\,')-\nabla\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,. \tag{10.35}$$

We are now ready to pull some physics out of the math. Noting that $\vec{B}=\nabla\times\vec{A}$, let's evaluate

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}\,. \tag{10.36}$$

We already showed that for magnetostatics $\nabla\cdot\vec{A}=0$. We now see the point of converting Eq. (10.33) to Eq. (10.34): the Laplacian operating on Eq. (10.34) generates a delta function, leading immediately to the result

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \nabla\times\vec{M}\;. \tag{10.37}$$

Following the same path that led to $\vec{D}$, we combine the two vector operations, yielding

$$\nabla \times (\vec{B} - 4\pi \vec{M}) = \frac{4\pi}{c} \vec{J} , \tag{10.38}$$

where we define the quantity in parentheses in Eq. (10.38) as the *magnetic field intensity* $\vec{H}$, or

$$\vec{H} = \vec{B} - 4\pi \vec{M} . \tag{10.39}$$

$\vec{H}$ is therefore seen to satisfy the equation

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} . \tag{10.40}$$

As with $\vec{D}$ for electrostatics, $\vec{H}$ is independent of "polarization charge" (here the magnetic version), depending only on the external current $\vec{J}$. We can also introduce a scaling factor μ, writing

$$\vec{B} = \mu \vec{H} . \tag{10.41}$$

As with ε, μ can be used as a phenomenological parameter, as well as calculated from first principles.

This completes the magnetostatic development. However, for the moment let's bring time back into the picture. Then by the macroscopic version of the Lorentz gauge, $\nabla \cdot \vec{A} = -(\mu\varepsilon/c)(\partial\phi/\partial t)$. Because $\vec{E} = -\nabla\phi$, we see that the first term on the right of Eq. (10.36) does not vanish, but changes Eq. (10.38) to

$$\nabla \times \vec{H} = \frac{4\pi\mu}{c} \vec{J} + \frac{\mu}{c} \frac{\partial \vec{D}}{\partial t} , \tag{10.42}$$

and Ampère's macroscopic equation is complete. In short, if Ampère had been more careful, he would have picked up the term that is now credited to Maxwell. The lesson here is that if you're going to discard terms in a derivation, be sure you have a good reason to do so.

Equations (10.1a,b) and (10.1c,d) have the same form, but the underlying physics is different enough to warrant some discussion. In electrostatics, it is easy to accept that $\vec{E}$, not $\vec{D}$, is the fundamental quantity, not only because the Lorentz and Coulomb force laws are written in terms of $\vec{E}$, but also because $q\vec{E}$ is the force that drives the induced polarization charge density $\vec{P}$. Further, electric fields inside materials are orders of magnitude larger than those that can be achieved by static external fields. Consequently, $\vec{P}$ can be calculated by a perturbation approach. In most cases $\vec{P}$ is accurately proportional to $\vec{E}$, leading directly to a history-independent static dielectric constant ε.

In magnetics, even though the Lorentz and Ampère force laws show that the fundamental quantity is $\vec{B}$, the driving field equivalent to $\vec{E}$ is $\vec{H}$, and for initially unmagnetized materials the responses are $\vec{M}$ and $\vec{B}$. Thus $\vec{B} = \mu \vec{H}$ is a measure of how thoroughly $\vec{H}$ has aligned the microscopic dipoles, with the dynamic permeability reaching $\mu = 1$, or $\Delta\vec{B} = \Delta\vec{H}$, after all dipoles are aligned. As a source term $\vec{H}$ is usually generated by $\vec{J}$, although permanent magnets are an exception, as shown by several examples below. Because the alignment of magnetic dipoles is usually nonlinear and history-dependent, only general conclusions about μ can be drawn.

In his Sec. 6.6 Jackson cites literature results that are nominally also done by the weighting-function approach, but these are special cases that have no connection to localized moments.

D. The $\vec{F} = m\vec{a}$ approach to magnetostatics.

We now repeat Secs. C and D using $\vec{F} = m\vec{a}$. Here, the magnetic dipoles are specifically occupied electronic wavefunctions on individual atoms. Although the configuration is purely quantum-mechanical, a classical-physics analog can be defined by noting that the q_i experience atomic forces that are orders of magnitude larger than anything that can be applied externally. Then in classical terms the q_i travel in circular paths. In this picture the quantum and classical worlds are connected through the angular momentum $\vec{L}$.

We start with the definition of $\vec{J}$ for a collection of point charges q_i moving with a velocity $\vec{v}_i = d\vec{r}_i/dt$:

$$\vec{J}(\vec{r}) = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i) \,. \tag{10.43}$$

In contrast to electrostatics, $\vec{J} = \vec{J}(\vec{r},t)$ must be averaged over time as well as position. Thus

$$< \vec{J}(\vec{r}) > = \frac{1}{T}\int_0^T dt \int_V d^3r' W(\vec{r} - \vec{r}\,') \sum_i q_i \vec{v}_i \delta(\vec{r}\,' - \vec{r}_i) \,, \tag{10.44a}$$

$$= \frac{1}{T}\int_0^T dt \sum_i q_i \vec{v}_i \, W(\vec{r} - \vec{r}_i) \,, \tag{10.44b}$$

$$= \frac{1}{T}\int_0^T dt \int_V d^3r' \rho \vec{v}\,' W(\vec{r} - \vec{r}\,') = \rho < \vec{v} > , \tag{10.44c}$$

where $T \to \infty$. If the force equation contains only inertial and scattering terms, then $< \vec{v} >$ is the drift velocity, and the result is Ohm's Law expressed in terms of atomic-scale parameters. This limit will be discussed in Ch. 10.

As noted above, we are interested in the limit where atomic forces dominate and all external interactions can be neglected. Let the q_i be located at $\vec{r} = \vec{r}_i + \Delta\vec{r}_i$, where $\vec{r}_i$ is the center of the circular motion, and $\Delta\vec{r}_i = \Delta\vec{r}_i(\vec{r},t)$ describes the motion itself. The macroscopic current density $<\vec{J}>$ is then given by

$$<\vec{J}> = \frac{1}{T}\int_0^T dt \int_V d^3r' \left(\sum_i q_i \vec{v}_i \delta(\vec{r}' - \vec{r}_i - \Delta\vec{r}_i(\vec{r}_i,t)) \right) W(\vec{r}-\vec{r}')$$

$$= \sum_i q_i \frac{1}{T}\int_0^T dt\, \vec{v}_i\, W(\vec{r}-\vec{r}_i-\Delta\vec{r}_i)$$

$$\cong \sum_i q_i \frac{1}{T}\int_0^T dt\, \vec{v}_i (1 - \Delta\vec{r}_i \cdot \nabla_{\vec{r}}) W(\vec{r}-\vec{r}_i)$$

$$= -\sum_i q_i \frac{1}{T}\int_0^T dt\, \vec{v}_i (\Delta\vec{r}_i \cdot \nabla_{\vec{r}}) W(\vec{r}-\vec{r}_i)\,. \tag{10.45}$$

The third line follows from the second because $|\Delta\vec{r}_i|$ is assumed to be small compared to the characteristic dimension of $W(\vec{r}-\vec{r}')$. The fourth line follows from the third because that the net current is assumed to be zero.

We now replace the sum according to $\sum_i f_i = \int_V d^3r'\, n\, f(\vec{r}')$, where n is the volume density of charges q_i. Then assuming that the circular motion is in the *xy* plane,

$$\Delta\vec{r}(\vec{r}',t) = \Delta r(\hat{x}\cos\omega t + \hat{y}\sin\omega t\,. \tag{10.46}$$

Then Eq. (10.za) becomes

$$<\vec{J}(\vec{r})> = \frac{q}{T}\int_0^T dt \int_V d^3r'\, n\omega\Delta r(\hat{x}\sin\omega t - \hat{y}\cos\omega t)$$

$$\times\left(\Delta r\left((\cos\omega t)\frac{\partial}{\partial x} + \sin\omega t \frac{\partial}{\partial y}\right) W(\vec{r}-\vec{r}')\right). \tag{10.47}$$

We now evaluate the time average explicitly. The result is

$$<\vec{J}(\vec{r})> = \frac{n\,q\,\omega\Delta r^2}{2}\int_V d^3r' \left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right) W(\vec{r}-\vec{r}')\,. \tag{10.48}$$

The factor $(1/2)$ in Eq. (10.48) is seen to arise from the temporal average. Writing the above in matrix form gives

$$<\vec{J}> = \frac{q}{2}\int_V d^3r'\, n\omega\Delta r^2 \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} W(\vec{r}-\vec{r}')\,. \tag{10.49}$$

Now with the motion in the *xy* plane, the angular momentum of q_i is

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = m\vec{r}_i \times \vec{v}_i = -m\omega\,\Delta r_i^2(-\hat{z}\sin^2\omega t - \hat{z}\cos^2\omega t) \tag{10.50a}$$

$$= m\omega\Delta r_i^2\hat{z}\,. \tag{10.50b}$$

The obvious extension of the above to an arbitrary direction of $\vec{L}$ is

$$< \vec{J} > = \frac{q}{2m}\int_V d^3r'\,n\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ L_x & L_y & L_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} W(\vec{r}-\vec{r}\,')\,. \tag{10.51a}$$

$$= \frac{q}{2m}\int_V d^3r'\,n\,\vec{L}\times\nabla_{\vec{r}}W(\vec{r}-\vec{r}\,')\,. \tag{10.51b}$$

We now define the magnetic moment $\vec{m}$ as

$$\vec{m} = \frac{q}{2mc}\vec{L}\,. \tag{10.52}$$

Equation (10.52) therefore can be written

$$< \vec{J} > = c\int_V d^3r'(n\,\vec{m})\times\nabla_{\vec{r}}W(\vec{r}-\vec{r}\,')$$

$$= c\int_V d^3r'\vec{M}\times\nabla_{\vec{r}}W(\vec{r}-\vec{r}\,')\,, \tag{10.53}$$

where $\vec{M} = n\,\vec{m}$ is the magnetization, i.e., the volume density of magnetic dipoles.

We now capitalize on the identity

$$\int_V d^3r'\nabla_{\vec{r}\,'}\times\left(\vec{M}\,W(\vec{r}-\vec{r}\,')\right) = \int_V d^3r'\left(W(\vec{r}-\vec{r}\,')\nabla_{\vec{r}\,'}\times\vec{M} + \vec{M}\times\nabla_{\vec{r}\,'}W(\vec{r}-\vec{r}\,')\right)$$

$$= \int_S d^2r'\,\hat{n}\times\left(\vec{M}\,W(\vec{r}-\vec{r}\,')\right) = 0\,. \tag{10.54}$$

This cannot be applied directly to Eq. (10.53) because the operator $\nabla_{\vec{r}}$ there acts on $\vec{r}$, not $\vec{r}\,'$. However, the functional form of $W(\vec{r}-\vec{r}\,')$ allows this change to be made with the introduction of a minus sign, so the identity remains applicable. We therefore have the result

$$< \vec{J} > = -c\int_V d^3r'\vec{M}\times\nabla_{\vec{r}}W(\vec{r}-\vec{r}\,') = c\nabla_{\vec{r}}\times < \vec{M} >\,. \tag{10.55}$$

Combining this with the contribution of an externally applied current density $\vec{J}$, we have finally

$$\vec{A} = \frac{1}{c}\int_V d^3r' \frac{J + c\nabla\times\vec{M}}{|\vec{r}-\vec{r}'|}. \tag{10.56}$$

Although the result is the same as that obtained in Sec. C, the derivation in this section provides a direct physical description of the origin of magnetic moments on the atomic scale. It also connects these moments to angular momenta, bringing quantum mechanics into the picture. It also shows that a calculation of the macroscopic current density requires an average over time as well as space, even though the result is quasistatic.

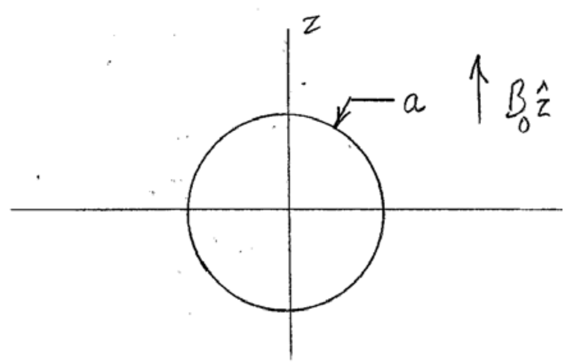
E. Boundary-value problems in magnetostatics: examples.

The various aspects of magnetics are best illustrated by examples, which also allow us to better appreciate the use of Maxwell's Equations in boundary-value problems than resorting to discussion. Section E is basically a collection of problems that I assigned as homework or put on exams on various occasions. These examples also bring out the phenomenon of flux trapping and illustrate other characteristics that result from permeabilities $\mu >> 1$, which have no analog in electrostatics.

Example 1. Magnetic scalar potential, magnetic plasmons, and the Meissner effect.

With $\vec{J} = 0$ we can take advantage of Eq. (10.40) and solve boundary-value problems using a magnetic scalar potential. The following example is from the 2015 qualifying exam.

An iron cannonball 8 inches (20 cm) in diameter, left over from the US Civil War, rests on the bottom of the Atlantic Ocean in the otherwise uniform magnetic field of the Earth. For this problem, assume that the radius of the ball is $r = a$, that its permeability is μ, and that everything else in the universe consists of material with $\mu = 1$. Also, if the ball were not there, the field would be $\vec{B} = B_o\hat{z}$. Determine:



(a) the magnetic flux density $\vec{B}$ at all points in space;

(b) repeat (a) for $\vec{H}$; and

(c) the limits of (a) and (b) for $\mu \to \infty$.

(d) Viewing the question as a mathematician rather than a physicist, is there a value of μ such that the magnetic equivalent of a plasmon could exist? If so, what is the value of μ?

(e) Evaluate the fields supposing that the cannonball is a superconductor (Meissner effect), for which $\mu = 0$.

Solution: Except for terminology, this is the problem of a dielectric sphere in a uniform electric field. From Ampère's Law, we have

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} = 0 . \tag{10.57}$$

As noted above, with $\vec{J} = 0$ we can write $\vec{H} = -\nabla\phi_m$, where ϕ_m is a magnetic scalar potential. With $\vec{B} = \mu\vec{H}$, μ uniform in both the cannonball and its surroundings, and $\nabla \cdot \vec{B} = 0$, it follows that $\nabla^2\phi = 0$, so we can use the machinery that we developed for electrostatics.

Given the rotational symmetry of the configuration about the z axis, ϕ_m can be expanded as an infinite series of Legendre polynomials. However, we can discard all but the $l = 1$ expansion term, writing

$$\phi_{m,in} = -A_1 z = -A_1 r \cos\theta ; \tag{10.58a}$$

$$\phi_{m,out} = B_1 r^{-2}\cos\theta - B_o z = B_1 r^{-2}\cos\theta - B_o r\cos\theta . \tag{10.58b}$$

We will also need the magnetic fields, so applying the gradient operator in spherical coordinates:

$$\vec{H}_{in} = -\nabla\phi_{m,in} = A_1\hat{z} = A_1(\hat{r}\cos\theta - \hat{\theta}\sin\theta) ; \tag{10.59a}$$

$$\vec{H}_{out} = -\nabla\phi_{m,out} = B_o(\hat{r}\cos\theta - \hat{\theta}\sin\theta) + \frac{B_1}{r^3}(2\hat{r}\cos\theta + \hat{\theta}\sin\theta) . \tag{10.59b}$$

The boundary condition tangential $\vec{H}$ continuous is equivalent to ϕ_M continuous. This leads to

$$-A_1 a = \frac{B_1}{a^2} - B_o a . \tag{10.60}$$

We verify this by comparing the tangential components of $\vec{H}$ on either side of the interface. The boundary condition that normal $\vec{B}$ is continuous leads to

$$\mu A_1 = B_o + \frac{2B_1}{a^3} . \tag{10.61}$$

Doing the math we get

$$A_1 = \frac{3}{2+\mu}B_o ; \tag{10.62a}$$

$$B_1 = \frac{\mu - 1}{\mu + 2}B_o . \tag{10.62b}$$

Then

$$\vec{H}_{in} = \frac{3B_o}{2+\mu}\hat{z} ; \tag{10.63a}$$

$$\vec{B}_{in} = \frac{3\mu B_o}{2+\mu}\hat{z} ; \tag{10.63b}$$

$$\vec{H}_{out} = \vec{B}_{out} = B_o\hat{z} + \frac{\mu - 1}{\mu + 2}\left(\frac{a^3}{r^3}\right)\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right); \quad (10.63c)$$

which are the solutions to parts (a) and (b). Taking the limit as $\mu \to \infty$:

$$\vec{H}_{in} = 0; \quad (10.64a)$$

$$\vec{B}_{in} = 3B_o\hat{z}; \quad (10.64b)$$

$$\vec{H}_{out} = \vec{B}_{out} = B_o\hat{z} + B_o\left(\frac{a^3}{r^3}\right)\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right); \quad (10.64c)$$

which is the answer to part (c). Note that in this limit $\vec{H}$ vanishes inside the cannonball, but $\vec{B}$ does not. A magnetic plasmon would exist if $\mu = -2$, answering part (d). A negative value of μ cannot occur with natural materials, although artificial structures (metamaterials) can be fabricated that simulate this property at specific frequencies. This will be discussed in detail next semester. This case is exactly parallel to the electrostatic problem covered in the last chapter, down to the concept of an interface magnetic charge.

A second special case is the Meissner effect. With $\mu = 0$:

$$\vec{H}_{in} = \frac{3}{2}\vec{B}_o; \quad (10.65a)$$

$$\vec{B}_{in} = 0; \quad (10.65b)$$

$$\vec{H}_{out} = \vec{B}_{out} = B_o\hat{z} - \frac{1}{2}\left(\frac{a^3}{r^3}\right)\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right); \quad (10.65c)$$

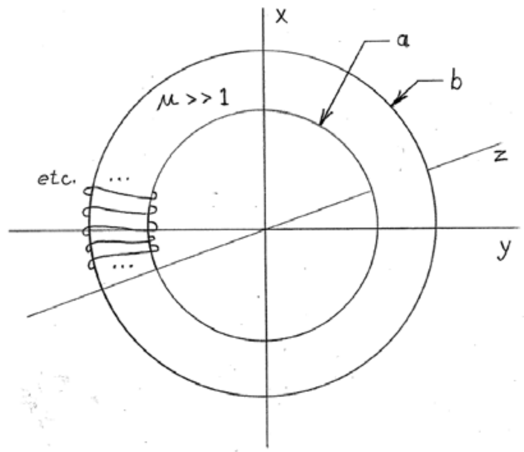
which is the result requested in part (e). Here $\vec{B}$ vanishes inside the cannonball, but $\vec{H}$ does not. The same behavior occurs with $\vec{D}$ and $\vec{E}$ in electrostatics in the corresponding limits of ε. Note that in both cases the energy density inside the sphere is zero.

Example 2. Magnetized sphere.

Suppose the sphere in the previous example has a magnetization $\vec{M} = M\hat{z}$. If no external field is applied field, find $\vec{B}$ and $\vec{H}$ everywhere in space.

There is no difference formally between this configuration and that of an electrically polarized sphere, which we have done already as a homework assignment. Accordingly, this is left as a homework assignment as well.

The next four examples deal with a configuration that can be solved analytically. At the same time they provide an opportunity to define flux lines, determine why flux lines concentrate in ferromagnetic materials, and address flux leakage in general. We discuss these aspects as four separate problems.



Example 3: Torus with a distributed winding. Suppose that a bar of circular cross section and $\mu >> 1$ is bent in the form of a circle of inner radius *a* and outer radius *b*, then centered on the origin, as shown in the diagram. N turns of wire carrying a current *I* are wrapped uniformly around the bar, such that the current effectively forms a continuous sheet over the entire bar (only a portion of the windings are shown). For the rest of the problem, simplify the math by considering only quantities in the *xy* plane.

(a) By choosing appropriate integration paths, determine the fields $\vec{B}$ and $\vec{H}$ everywhere in the *xy* plane, both inside and outside the bar.

(b) By choosing an integration path straddling a boundary between the inside and the outside of the bar, i.e., located partly inside and outside the bar, show that flux leakage does not occur in this configuration.

Solution:

(a) We can solve this using a combination of Stokes' Theorem and Ampère's Law, where the surface is a disc of radius ρ centered on the *z* axis. Our first task is to define $\vec{J}$. Assuming that the current emerges at the outer $r = b$ surface in the diagram, in the *xy* plane $\vec{J}$ is

$$\vec{J} = \frac{NI}{2\pi a}\hat{z}\delta(\rho - a) - \frac{NI}{2\pi b}\hat{z}\delta(\rho - b) \tag{10.52a}$$

$$= \frac{NI}{2\pi}\hat{z}\left(\frac{1}{a}\delta(\rho - a) - \frac{1}{b}\delta(\rho - b)\right). \tag{10.52b}$$

Since the configuration and $\vec{J}$ are independent of φ, at any given radius $\vec{H} = H_\varphi\hat{\varphi}$ will also be independent of φ except for direction. Applying Ampère's magnetostatics equation and Stokes' Theorem yields

$$\int_S d^2r'\hat{n}\cdot(\nabla\times\vec{H}) = \oint_C \vec{H}\cdot d\vec{l} = 2\pi\rho H_\varphi = \frac{4\pi}{c}\int_S d^2r'\hat{n}\cdot\vec{J}(\vec{r}\,'). \tag{10.53}$$

Then

$$H_{\varphi} = 0 \text{ if } \rho < a \text{ or } \rho > b; \tag{10.54a}$$

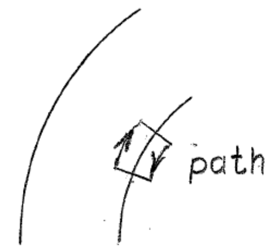
$$= \frac{2NI}{\rho c} \text{ if } a \le \rho \le b. \tag{10.54b}$$

Inside the bar

$$\vec{B} = \mu \vec{H} = \mu H_\varphi \hat{\varphi}. \tag{10.55}$$

Outside the bar $\vec{B} = \vec{H}$, so $\vec{B}$ vanishes there.

(b) To investigate flux leakage, use a path that follows constant-radius lines inside and outside the bar, and is connected along radius lines at the ends (see figure). If the path subtends a segment of arc $\Delta\varphi$, we obtain



$$\oint \vec{H} \cdot d\vec{l} = \frac{2NI}{\rho_1 c} \rho_1 \Delta\varphi + 0 + 0 + 0 = \frac{2NI}{c} \Delta\varphi$$

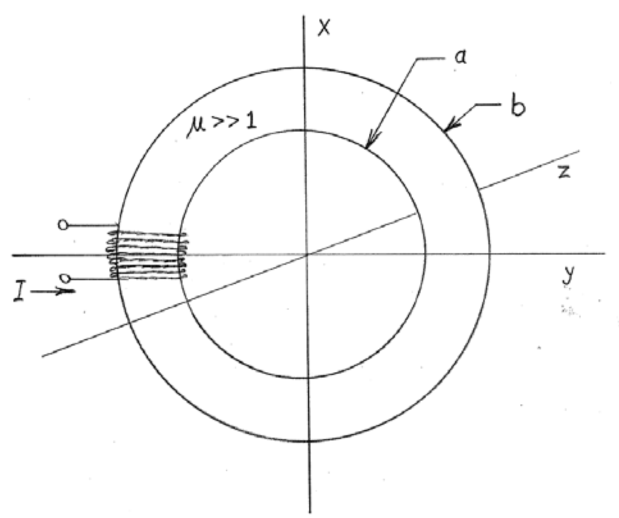
$$= \frac{4\pi}{c} \int_S d^2 r' \hat{n} \cdot \vec{J} = \frac{4\pi}{c} I \, \Delta N = \frac{4\pi}{c} NI \left(\frac{\Delta N}{N} \right) = \frac{4\pi \, NI}{c} \left(\frac{\Delta\varphi}{2\pi} \right)$$

$$= \frac{2NI}{c} \Delta\varphi. \tag{10.56}$$

The top and third lines agree. Because nothing is left over, field leakage does not occur. This identity between the path integral and the enclosed current is true for *any* path partly inside and outside the bar, showing that there are no magnetic fields outside the bar.

Example 4. Torus with a localized winding; flux lines and flux trapping. Repeat the calculations of Example 3, but with the N turns of wire covering only a small fraction of the bar (see diagram).

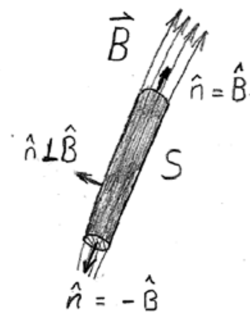
The immediate problem is to determine the path for evaluating the line integral. Should this be the path of Example 3, which assumes that the magnetic flux stays entirely within the bar, or should this be a shorter path that is only big enough to include the current? The difference will obviously be significant. The question can be resolved by examining the difficulty of removing flux lines from material with a large permeability. If the flux lines remain trapped in the material, then the appropriate path is the original one. We examine trapping by determining the angles that a flux line makes with the interior and exterior of the bar as it emerges from the surface, which highlights the



roles of $\vec{B}$ and $\vec{H}$ in the process. We find that restricting the windings to a fraction of the toroid makes flux leakage inevitable, but for materials with large permeabilities, this leakage is insignificant.

We consider first flux lines. Treatments of magnetostatics often begin with this topic, but we have gotten this far into Ch. 10 without considering them in detail. The first step is to define a flux line mathematically. This was already done in Ch. 1 sec, F, but the main points are repeated here. Suppose a small region of space contains a magnetic flux density $\vec{B}$. Applying Gauss' Theorem to $\nabla \cdot \vec{B} = 0$ yields

$$\int_V d^3r\, \nabla \cdot \vec{B} = 0 = \int_S d^2r\, \hat{n} \cdot \vec{B}\,, \qquad (10.57)$$



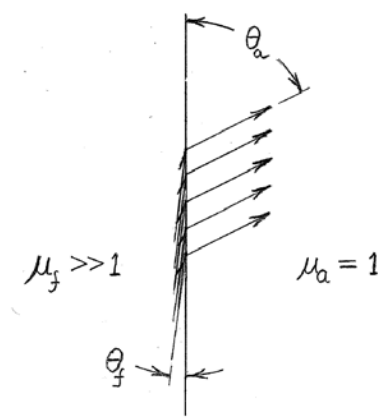
where S is defined to be an approximately cylindrical surface with sides and end caps whose normal vectors are perpendicular and parallel, respectively, to $\vec{B}$. Thus the sides contribute nothing to the integral, and with $\nabla \cdot \vec{B} = 0$, the flux Φ_m entering and leaving the end caps is the same:

$$\Phi_M = \int_{end\ 1} d^2r\, \hat{B}_1 \cdot \vec{B}_1 = \int_{end\ 2} d^2r\, \hat{B}_2 \cdot \vec{B}_2\,. \qquad (10.58)$$

This of course is nothing more than the statement that there is no magnetic charge.

With the result independent of length, either the ends recede to infinity, or the surface must be closed. Since the first option is physically impossible, the surface, and therefore the flux lines, form a continuous loop. The flux line is then defined by reducing the cross-sectional area of the tube until it contains only one quantum $h/2e$ of magnetic flux.

We now consider conditions for which a flux line can leave a ferromagnetic material. Except for scale, the calculation repeats that which was done for electric flux lines leaving a dielectric. Let the flux line arrive at the boundary from the ferromagnet side at an angle θ_f and emerge in air $(\mu_a = 1)$ at an angle θ_a, as shown in the diagram at the right. We assume that the extent is small enough so the boundary can be considered locally flat. To connect θ_f and θ_a define field components on the air side:



$$\begin{aligned}\vec{H}_a &= \hat{x}H_{ax} + \hat{z}H_{az} \\ &= \vec{B}_a = \hat{x}B_{ax} + \hat{z}B_{az}\,;\end{aligned} \qquad (10.59)$$

and corresponding components on the ferromagnet side:

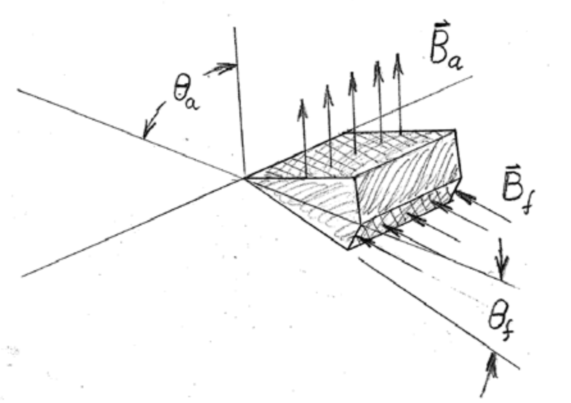
$$\begin{aligned}\vec{H}_f &= \hat{x}H_{fx} + \hat{z}H_{fz} \\ &= \vec{B}_f/\mu = \hat{x}(B_{fx}/\mu) + \hat{z}(B_{fz}/\mu)\,.\end{aligned} \qquad (10.60)$$

With $\nabla \cdot \vec{B} = 0$ it follows that $B_{fz} = B_{az} = H_{az}$, and with $\nabla \times \vec{H} = 0$, that $H_{fx} = H_{ax}$. Then

$$\tan(\theta_f) = \frac{H_{fz}}{H_{fx}} = \frac{B_{fz}/\mu}{H_{ax}} = \frac{B_{az}}{\mu H_{ax}} = \frac{H_{az}}{\mu H_{ax}}$$

$$= \frac{1}{\mu}\tan(\theta_a)\,. \tag{10.61}$$

For values of $\mu \sim 10^4 - 10^6$, it is seen that the only flux lines that can escape from the ferromagnet are those in a very narrow angular range $\Delta\theta \sim (1/\mu)$ with respect to the surface. Thus it is very difficult to get a flux line out of the ferromagnet, except at normal incidence, where $\tan\theta_a$ and $\tan\theta_f$ are both infinite so the value of μ is irrelevant. Thus the appropriate path for the line integral in Eq. (10.53) is the internal path within the bar, i.e., that of Exercise 3, and to an excellent approximation, all flux lines generated by the current stay in the ferromagnet.



Yet since the parallel component of $\vec{H}$ must be continuous across the boundary, so with localized windings some leakage must occur. This is a critical issue in transformer design, where the magnetic flux generated by one winding must pass through the other windings.

The calculation can be taken one step further by determining the cross section for flux leakage in a ferromagnet. To do this construct another surface having side walls with normal vectors perpendicular to the flux lines and ends parallel or antiparallel to them, as shown in the figure at the bottom of the previous page. The "ends" where the flux enters and leaves share a common width, and the ratio of their heights is equal to the ratio $(\sin\varphi_f/\sin\varphi_a)$, since these share a common hypotenuse. Substituting these relations yields

$$\frac{A_f}{A_a} = \frac{\sin\theta_f}{\sin\theta_a} = \sqrt{\left(\frac{\tan^2\theta_f}{1+\tan^2\theta_f}\right)\left(\frac{1+\tan^2\theta_a}{\tan^2\theta_a}\right)}$$

$$= \sqrt{\frac{1+\tan^2\theta_a}{\mu^2+\tan^2\theta_a}} = \frac{1}{\sqrt{1+(\mu^2-1)\cos^2\theta_a}}\,. \tag{10.62}$$

This ratio is approximately proportional to $1/(\mu\cos\theta_a)$ except at $\theta_a = \theta_f = \pi/2$, where it is equal to 1.

To get the complete cross section in the ferromagnet, we integrate over angles:

$$A_f = A_a \int_0^{\theta_{a,\max}} d\theta \frac{1}{\sqrt{1+(\mu^2-1)\cos^2\theta}}$$

$$\approx \frac{A_a}{\mu < \cos\theta >}, \tag{10.63}$$

where $\theta_{a,\max}$ is an effective upper limit of the emergence angle on the air side of the interface and $< \cos\theta >$ is an effective average. In the worst case A_a will be of the order of the external area of the bar. Hence most of the flux lines stay in the bar. Thus for ferromagnets of moderate cross sections and large permeabilities, leakage is a relatively minor effect.

Example 5. Electromagnets.

Here, we repeat Example 4 assuming that a small gap of length l_g exists in the bar while the current in the wires is kept the same. We assume no flux leakage and negligible fringing in the gap, thereby restricting results to separations that are relatively small compared to the lateral dimensions of the bar.

To simplify the presentation, we consider $\vec{B}$ and $\vec{H}$ only in the *xy* plane, assume that $\vec{B}$ and $\vec{H}$ are uniform over the cross sectional area *A* of the bar, and take the effective lengths of the bar and gap to be be *l* and l_g, respectivcely. Then $\nabla \cdot \vec{B} = 0$ requires the normal component of $\vec{B}$ to be continuous at the faces of the gap. Thus

$$B_{in} = \mu H_{in} = B_{out} = H_{out}. \tag{10.64}$$

Next, since

$$\oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} NI = H_{in} l + H_{out} l_g, \tag{10.65}$$

it follows that

$$B_{in} = B_{out} = H_{out} = \frac{4\pi \, \mu \, NI}{c(l + \mu l_g)}. \tag{10.66}$$

Thus the *effective* length of the gap is μl_g, or μ times its metric length l_g. For highly permeable materials this is a huge effect: even a very small gap causes a very large reduction in the generated magnetic flux.

A practical manifestation of this effect is found in electromagnetic clutches in power machinery. Comments follow in Sec. I.

Example 6. Example 5 for a permanent magnet.

If the material in the bar has a remanent magnetization $\vec{M} = M\hat{\varphi}$, the source of magnetization is $\vec{M}$, not *I*, so Eq. (10.1c) applies. With the normal component of $\vec{B}$ continuous and the loop integral of $\vec{H}$ zero, the equations now become

$$\oint_C \vec{H} \cdot d\vec{l} = H_{in} l + H_{out} l_g = 0$$

$$= (B - 4\pi M) l + B l_g \, . \tag{10.67}$$

The solution is

$$B_{in} = B_{out} = H_{out} = 4\pi M \left(\frac{l}{l + l_g}\right); \tag{10.68a}$$

$$H_{in} = -4\pi M \frac{l_g}{l + l_g} \, . \tag{10.68b}$$

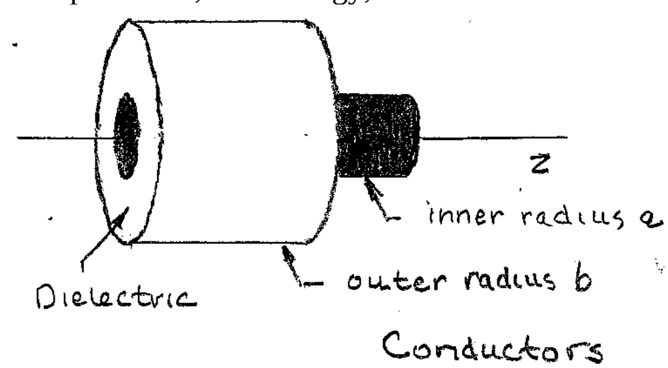
Thus in the magnet $\vec{H}$ and $\vec{\boldsymbol{B}}$ point in opposite directions. Also, in contrast to the electromagnet case, the gap now has a relatively small effect. This is why permanent magnets exert a strong pull even if the spacing between pole faces is large.

This striking difference in air-gap sensitivity is worth a comment about the physics behind the effect. In an electromagnet $\vec{J}$ generates $\vec{H}$, which in turn aligns atomic-scale dipoles to generate a net field $\vec{B}$. In a permanent magnet the dipoles are already aligned, and because flux lines must go somewhere, the effect of the gap is greatly reduced.

The face pressure can be calculated. However, this requires a discussion of how energy is stored in a ferromagnet. This is done in Sec. H.

Example 7: The coaxial cable: vector potentials, field energy, and inductance.

A coaxial cable consists of an inner wire of radius *a* and an outer sheath of radius *b*, as shown in the diagram. The thickness of the outer sheath can be taken to be zero. The inner and outer conductors carry currents of $(+I)$ and $(-I)$, respectively, with densities distributed uniformly over their respective cross sections. The inner and outer conductors are separated by a volume filled with an insulating dielectric with $\mu = 1$ and $\varepsilon > 1$. We assume that the region of interest is far enough away from the ends of the cable so a two-dimensional calculation is an adequate approximation (no dependence on the axial dimension *z*.) The objectives are to:



(a) Write the expressions for the current densities $\vec{J}(\rho)$ for both inner and outer conductors;

(b) Determine $\vec{H}$ and $\vec{B}$ everywhere;

(c) Determine the vector potential $\vec{A}$, again everywhere; and

(d) Determine the inductance per unit length of the coaxial cable.

We start by defining the current densities:

$$\text{inner:} \quad \vec{J}_{in} = \frac{I}{\pi a^2}\hat{z}\,u(a-\rho) \tag{10.69a}$$

$$\text{outer:} \quad \vec{J}_{out} = -\frac{I}{2\pi b}\delta(\rho - b)\,\hat{z}\,. \tag{10.69b}$$

These are easily verified by integrating over their cross sections to obtain the total current flowing in each. Note that both are dimensionally correct, but for different reasons.

We next use Ampère's Law and Stokes' Theorem to calculate $\vec{H}$, evaluating

$$\int_S d^2r'\,\hat{n}\cdot\nabla\times\vec{H} = \oint_C \vec{H}\cdot d\vec{l} = \frac{4\pi}{c}\int_S d^2r'\,\hat{n}\cdot\vec{J}\ , \tag{10.70}$$

where $\hat{n} = \hat{z}$ and S is a disc of radius ρ with $\hat{n} = \hat{z}$ and centered on the symmetry axis. Then doing the math:

$$\vec{H} = H_\varphi\hat{\varphi} = \frac{2\pi\rho J}{c}\hat{\varphi} = \frac{2\rho I}{ca^2}\hat{\varphi} \qquad \text{for } \rho < a\,; \tag{10.71a}$$

$$= \frac{2I}{\rho c}\hat{\varphi} \qquad \text{for } a \le \rho \le b\,; \tag{10.71b}$$

$$= 0 \qquad \text{for } \rho > b\,. \tag{10.71c}$$

Because the permeability is everywhere equal to 1, $\vec{B} = \vec{H}$.

$\vec{A}$ can also be obtained with Stokes' Theorem. Rewrite $\vec{B} = \nabla\times\vec{A}$ as $\nabla\times\vec{A} = \vec{B}$, and compare to $\nabla\times\vec{H} = (4\pi/c)\vec{J}$, which we used above. Thus $\vec{B}$ can be interpreted as the source for $\vec{A}$, if we can identify some properties of $\vec{A}$ and define an appropriate surface/path integral. To do this, consider

$$\vec{A}(\vec{r}) = \frac{1}{c}\int_V d^3r'\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,. \tag{10.11a}$$

$\vec{J}$ is entirely in the z direction, so $\vec{A} = A_z\hat{z}$ must be entirely in the z direction as well. Further, if A_z were a function of φ, it would not be single-valued. If it were a function of z, it would violate the assumption that the system is independent of z. Hence $\vec{A} = \hat{z}A_z = \hat{z}A_z(\rho)$ must be a function only of ρ.

Next, establish the appropriate integration path. With $\vec{H}$ as the source, the surface normal must be $\hat{n} = \hat{H} = \hat{\varphi}$. We let S be a rectangle of length Δz with two sides parallel to the z axis and two sides along ρ. Because the functional form of $\vec{H}$ changes at $\rho = a$ and again at $\rho = b$, three rectangles are needed: an inner one for $0 \le \rho \le a$, a midrange surface for $a \le \rho \le b$, and an outer one for $\rho \ge b$. Considering first the inner rectangle, we have

$$\int_S d\rho dz \hat{\varphi} \cdot \nabla \times \vec{A} = \oint_C \vec{A} \cdot d\vec{l} = \Delta z \left(A_z(\rho) - A_z(0) \right) \tag{10.72a}$$

$$= \Delta z \int_0^\rho d\rho \, \frac{2\rho I}{ca^2} = \Delta z \, \frac{\rho^2 I}{ca^2}, \quad 0 \le \rho \le a, \tag{10.72b}$$

or

$$A_z(\rho) = \frac{\rho^2 I}{ca^2} + A_z(0), \quad 0 \le \rho \le a. \qquad \text{(10.72b).}$$

We don't yet know $A_z(0)$, so for now we leave it undefined. The paths along the radii are orthogonal to z and hence contribute nothing.

For the midrange rectangle, a similar calculation yields

$$A_z(\rho) - A_z(a) = \frac{2I}{c} \ln\left(\frac{\rho}{a}\right), \qquad a \le \rho \le b. \tag{10.73}$$

For the outer rectangle:

$$A_z(\rho) - A_z(b) = 0, \qquad \rho \ge b. \tag{10.74}$$

But we know that $\vec{A} = 0$ as $\rho \to \infty$. Therefore $A_z(b) = 0$, and from Eq. (10.73)

$$A_z(a) = -\frac{2I}{c} \ln \frac{b}{a}. \tag{10.75}$$

Therefore,

$$A_z(\rho) = \frac{2I}{c}\left(\ln\left(\frac{\rho}{b}\right) - \ln\left(\frac{b}{a}\right) \right) = \frac{2I}{c} \ln \frac{\rho}{b}, \quad a \le \rho \le b. \tag{10.76}$$

Equating $A_z(a)$ for the two regions defines $A_z(0)$. The result is

$$A_z(\rho) = -\frac{I}{c}\left(2\ln\frac{b}{a} + 1 - \frac{\rho^2}{a^2}\right), \quad 0 \le \rho \le a\ . \tag{10.77}$$

This completes the solution.

In the above, we tacitly assume that the tangential component of $\vec{A}$ is continuous at any interface. This is easily proven by applying Stokes' Theorem to a loop straddling the interface and taking the limit as its radial dimension approaches zero. For completeness, Gauss' Theorem and the relation $\nabla \cdot \vec{A} = 0$, which is valid for magnetostatics, shows that the normal component of $\vec{A}$ is also continuous at an interface.

To evaluate the inductance, we jump ahead to Ch. 10 because the inductance L has not yet been defined. However, if we are willing to accept that one of the definitions of L is

$$\text{Stored energy} = W = \frac{1}{2}L I^2, \tag{10.78}$$

we can proceed. Using the expression for energy density from Ch. 1 we have

$$W = \frac{1}{2}L I^2 = \int_V d^3r'\left(\frac{1}{8\pi}\vec{B}\cdot\vec{H}\right). \tag{10.79}$$

Doing the integration yields

$$\int_V d^3r'\left(\frac{1}{8\pi}\vec{B}\cdot\vec{H}\right) = \frac{1}{8\pi}\frac{4I^2}{c^2}2\pi\,\Delta z\left(\int_0^a \rho\, d\rho\frac{\rho^2}{a^4} + \int_a^b \rho\, d\rho\frac{1}{\rho^2}\right)$$

$$= \frac{I^2\Delta z}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right) = \frac{1}{2}\Delta L\, I^2 \tag{10.80}$$

so

$$\frac{\Delta L}{\Delta z} = \frac{1}{c^2}\left(\frac{1}{2} + 2\ln\frac{b}{a}\right). \tag{10.81}$$

The first term is associated with the wire and the second with the insulation.

G. Aharonov-Bohm effect.

In Ch. 2 we showed that in the EMF line integral, $\vec{A}$ defines how ϕ changes along the integration path. And as also noted above, the time rate of change of $\vec{A}$ drives eddy currents. Nevertheless, many physicists did not consider $\vec{A}$ to be anything more than a mathematical aid until the Aharonov-Bohm effect was demonstrated. This deals with the change of phase of the wave function of an electron traveling parallel to an ideal solenoid, a region where all fields vanish and only $\vec{A}$ remains. We provide the mathematical foundation here.

We know that in magnetostatics $\vec{E}=0$ and for an ideal solenoid, $\vec{B}=0$ outside the solenoid. Hence our task reduces to determining $\vec{A}$ outside the solenoid. Following the logic of the last example, $\vec{J}\sim\hat{\varphi}$ so $\vec{A}\sim\hat{\varphi}$ is expected as well. For an ideal solenoid of radius *a*, $\vec{B}=B_o\hat{z}$ for $\rho<a$ and zero for $\rho>a$. By symmetry $\vec{A}$ cannot be a function of φ, or it would be multivalued. Also, it cannot be a function of *z*, because the configuration is translationally invariant in the *z* direction. Thus $\vec{A}$ can only be a function of ρ.

Applying Stokes' Theorem to $\nabla\times\vec{B}=\vec{A}$ and using as a surface a disc of radius ρ centered on the *z* axis, and perpendicular to the *z* axis, we find

$$\vec{A}=\frac{B_o\rho}{2}\hat{\varphi} \quad \text{for } \rho<a; \tag{10.82a}$$

$$=\frac{B_o a^2}{2\rho}\hat{\varphi} \quad \text{for } \rho\geq a. \tag{10.82b}$$

Thus $\vec{A}$ is continuous at $\rho=a$, as required, and vanishes as $\rho\to\infty$. To show that the solution is consistent with $\vec{B}=0$ for $\rho>a$, evaluate

$$\nabla\times\vec{A}=\hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)+\hat{\varphi}\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)+\hat{z}\,\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho\, A_\varphi)-\frac{\partial A_\rho}{\partial\varphi}\right)$$

$$=\vec{0}\,. \tag{10.83}$$

Thus the proposition is proved. The vector character of the curl operation ensures that $\vec{B}$ is indeed zero for $\rho>a$, even though $\vec{A}\neq 0$ there. This is in contrast to the coaxial cable, where everything is zero outside the cable.

Measurements of the properties of an electron moving parallel to a long solenoid with negligible flux leakage, with the electron far enough away so no part of its wave function penetrated the solenoid core, reveal its change in quantum-mechanical phase. With $\vec{E}$ and $\vec{B}$ both zero, the electron is traveling in a rigorously field-free region. According to the Lorentz force law, its properties cannot be affected by the solenoid. The only way that an effect can occur is if the electron is interacting in some way with the vector potential. This result is in agreement with the predictions of Ehrenberg and Siday, predictions that were later rediscovered independently by Aharonov and Bohm. Corrections due to fringing magnetic fields and the finite probability of finding the electron in the core can be calculated, and the data corrected for these effects.

Returning to the Lorentz force law, has this created a contradiction? No, but we need to invoke some deep physics to resolve the dilemma. The vector potential appears in the canonical momentum of quantum mechanics as

$$\vec{p}_{tot}=\vec{p}_{mech}+\frac{e}{c}\vec{A}. \tag{10.84}$$

More generally, the Aharonov-Bohm effect demonstrates that potentials (as used in quantum mechanics) are more fundamental than fields (as used in classical E&M). Feynman covers this in vol. 2 of his lectures. One of his comments is worth repeating: "$\vec{E}$ and $\vec{B}$ are slowly disappearing from the modern expression of physical laws; they are being replaced by $\vec{A}$ and ϕ." Thinking back on what we have studied so far, you will realize that nearly all of our work in this and previous chapters is based on potentials. We will also find that potentials are much more efficient than fields at describing radiation, scattering, and diffraction. The only exception to this rule occurs in calculating reflection and transmission at planar interfaces, where the boundary conditions are represented most efficiently in terms of fields. Incidentally, special relativity is no help either, because with the electron moving perpendicular to $\vec{A}$, the Lorentz transformation leaves $\vec{A}$ unchanged in the rest frame of the electron. The result is about as definitive as one can get.

H. Energy and force in ferromagnets.

Force calculations are based on energy, and treatments of energy for a given configuration start by identifying how the configuration can store energy. Starting from Ampère's Law, we derived the expression for the energy density in Ch. 1:

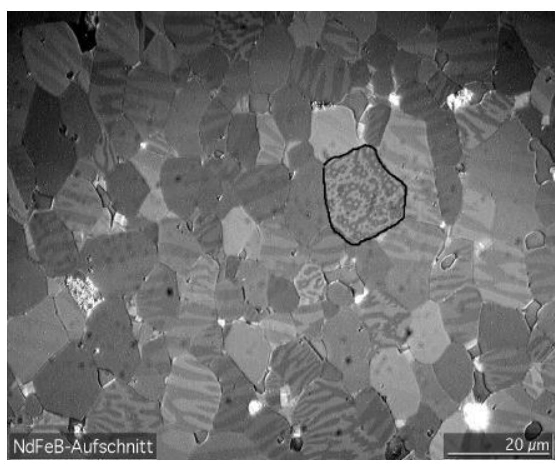
$$\frac{1}{(vol)}\frac{dW}{dt} = \frac{dU_f}{dt} = -\nabla\cdot\left(\frac{c}{4\pi}\vec{E}\times\vec{H}\right) - \frac{1}{4\pi}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}. \tag{1.44}$$

As noted in Ch. 1 sec. G, this is a *differential* relation, and consequently can only describe *changes* rather than absolute values. For nonlinear systems such as ferromagnets, absolute values are obtained by starting from appropriate initial conditions and doing integrations explicitly. The common expressions

$$U_{mf} = \frac{1}{8\pi}\vec{H}\cdot\vec{B} = \frac{\mu}{8\pi}\vec{H}^2 = \frac{1}{8\pi\mu}\vec{B}^2 \tag{10.85}$$

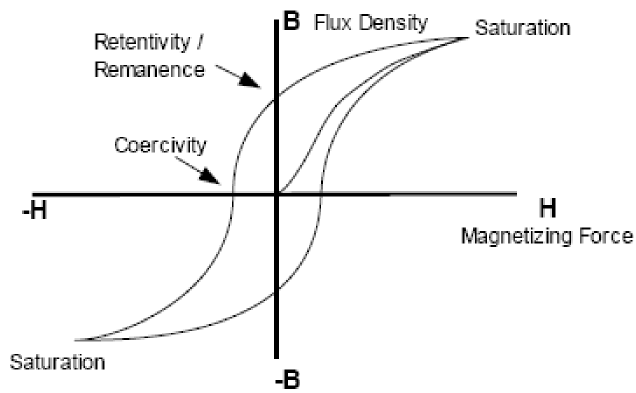
obtained assuming μ is constant, does not work for ferromagnets. Instead, Eq. (1.32) must be integrated explicitly.

We first consider the physics of ferromagnets, which begins with a consideration of their internal structure, and the fact that they consist of pre-existing dipoles, rather than dipoles that are induced. The energy of a dipole in a magnetic field is $-\vec{m}\cdot\vec{B}$, which converts to an internal energy density of $-\vec{M}\cdot\vec{B}$. To minimize energy ferromagnets with no net magnetization organize into domains, where dipoles in any given domain all point in the same direction, but the magnetization directions of adjacent domains are oriented such that return paths



for flux lines are entirely internal. By inhibiting external paths, energy is therefore minimized. The figure, taken from Wikipedia: Magnetic Domains, is an image of a cross-section of a Nd magnet taken with a polarizing microscope, and illustrates this domain structure directly. The domains are separated by "Bloch walls" where the orientation is changing from one direction to another. These interface regions are typically several nm wide depending on the exchange interaction (the value for iron is 10 nm).

If an external field $\vec{H}$ is applied to an unmagnetized ferromagnet, energy is minimized by aligning a greater fraction of the internal dipoles with the external field. Individual domains can suddenly snap into alignment, or domain walls can migrate laterally. The former mechanism generates electrical noise ("Barkhausen steps") in a coil surrounding the ferromagnet. The latter situation is generally easier to realize. For this reason permanent-magnet alloys always include elements such as boron to pin domain walls and therefore prevent unintended demagnetization. When all dipoles are aligned $\vec{B} = \vec{H} + 4\pi\vec{M}$, and any further increase in $\vec{H}$ increases $\vec{B}$ with an incremental permeability $\mu = 1$.



The net flux density $\vec{B}$ resulting from the application of an external magnetic field intensity $\vec{H}$ is described by a hysteresis curve, where the macroscopic average $\vec{B}$ is given as a function of $\vec{H}$ with $\vec{H}$ varying cyclically, as shown in the figure. The initial trajectory starting at $\vec{B} = \vec{H} = 0$ is the magnetization curve. Special points include the value of $\vec{B}$ on either return path when $\vec{H} = 0$. This is the remanence, or the capacity of the material to retain residual magnetism once $\vec{H}$ is removed. Coercivity is the value of $\vec{H}$ when $\vec{B} = 0$. With behavior being both nonlinear and history-dependent, ferromangnetism cannot generally be described by Green functions and permeabilities $\mu(\omega)$, although linearity may be a good approximation if the fields and frequencies are sufficiently small. With all this variety in mechanisms and responses, it is perhaps not surprising that even the mention of magnetostatics can cause apprehension and concern in physicists.

As an aside, the best-known models of hysteresis are those of Preisach and Jiles-Atherton (F. Preisach, Z. Phys. **94**, 277-302 (1935); D. C. Jiles and D. L. Atherton, J. Magnetism and Magnetic Materials **61**, 48-60 (1986)). The Preisach model is purely phenomenological, describing hysteresis in terms of "hysterons", domains that flip their magnetization once the magnetic energy gets high enough to overcome the magnetization barrier. The Jiles-Atherton model uses a mean-field approach that you would recognize now that you know something about the Clausius-Mossotti model and effective-medium theory. A search of the literature on magnetism yields hundreds of papers, indicating the complexity of the topic and the general absence of a universal description.

Despite this complexity, simple energy calculations are still possible. For example, given a hysteresis loop, we can determine the energy lost per cycle. To do this, multiply Eq. (1.32) by a time increment Δt, converting the equation into increments of energy density. The result is the magnetic part of Eq. (1.45):

$$\frac{dU_m}{dt}\Delta t = \Delta U_M \tag{10.86a}$$

$$= \frac{1}{4\pi}\vec{H}\cdot\left(\frac{\partial \vec{B}}{\partial t}\right)\Delta t = \frac{1}{4\pi}\vec{H}\cdot\Delta\vec{B}\,. \tag{10.86b}$$

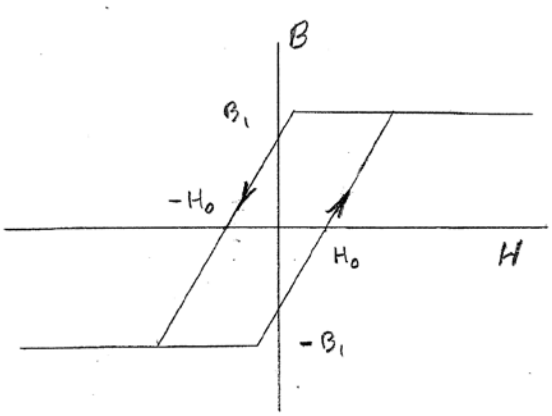
Equation (10.86b) can be integrated analytically in simple cases, and numerically if necessary. In geometric terms, this corresponds to evaluating the area under a $\vec{H}$ vs. $\vec{B}$ curve, instead of the usual $\vec{B}$ vs. $\vec{H}$ curve.

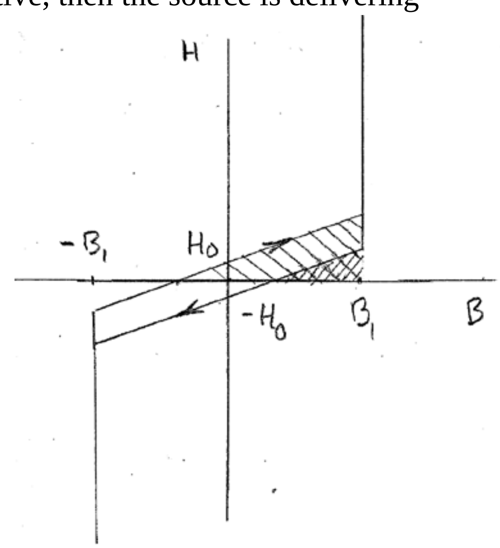
For example, consider a material of volume V with a piecewise-linear hysteresis curve as shown in the figure on the next page. Let the loop saturate at values *H* axis at values $\pm H_o$. At saturation the slope changes to $\mu = 1$. The objective is to determine the energy dissipated per cycle. To solve the problem, replot the hysteresis curve as shown at the right. The upper and lower branches are

Upper branch: $H = H(B) = H_o + \dfrac{1}{\mu}B$; (10.87a)

Lower branch: $H = H(B) = -H_o + \dfrac{1}{\mu}B$. (10.87b)

The signs of *H* and *ΔB* determine which parts of the diagram represent energy transfer to the material and from the material. If *HΔB* is positive, then the source is delivering energy to the material. The area under the *H* vs. *B* curve is the energy density delivered. In the rising part of the loop in the diagram, this area is





denoted as the hatched triangle. If $H\Delta B$ is negative, the material is returning energy to the source. The energy density delivered is the area between the curve and the *H* axis. For the falling part of the curve, this is the much smaller double-cross-hatched area. The inverse is seen to occur in the region where B is negative.

From the above, we conclude that the energy density dissipated per cycle is simply the area enclosed by the H vs. B loop. In the example

$$\int_{-B_1}^{B_1} dB\left(H_o + \frac{1}{\mu}B - (-H_o) - \frac{1}{\mu}B\right) = 4H_o B_1. \tag{10.88}$$

Thus the energy dissipated per cycle is

$$\Delta W = \frac{4H_o B_1}{4\pi}V = \frac{H_o B_1}{\pi}V, \tag{10.89}$$

where V is the volume of the material. More complicated loops would generally have to be treated numerically, but the conclusion is the same: the energy lost per cycle is the area of the hysteresis loop scaled in cgs units by $1/(4\pi)$.

In the above calculation, nothing is said about the magnetization energy $U_m = -\vec{M}\cdot\vec{B}$, even though the macroscopic value of $\vec{B}$ is changing with $\vec{H}$. The reason is that we assume that all dipoles remain parallel to the local field $\vec{B}_{loc}$, so up to saturation $U_d = -4\pi\vec{M}^2$. Thus the energy delivered by the magnetization source goes entirely into the field. The new aspect relative to unpolarized material is the magnetic field intensity $\vec{H}$.

I. Examples.

Example 1: Determine the energy stored in the toroid of Example 5 of Sec. E.

We suppose that the core is made of material with no net magnetization, and that the applied field $\vec{H}$ is small enough so the macroscopic relation $\vec{B} = \mu\vec{H}$ is valid, where μ is approximately constant. If the toroid has an effective length l, then by Ampère's Equation

$$H = \frac{B}{\mu} = \frac{4\pi NI}{cl}. \tag{10.90}$$

Since B is proportional to H, the relevant field energy density up to saturation is given by

$$U = \frac{1}{8\pi}BH = \frac{\mu}{8\pi}\left(\frac{4\pi NI}{cl}\right)^2 = 2\pi\mu\left(\frac{NI}{lc}\right)^2. \tag{10.91}$$

If the effective cross-sectional area of the bar is A, the total energy stored in the field is

$$W = 2\pi\mu Al\left(\frac{NI}{lc}\right)^2. \tag{10.92}$$

The dipole energy does not enter due to the ferromagnetic approximation that the local fields in the domains are already at their maximum value.

Example 2. If the above toroid has a gap of effective width l_g, determine the pressure on the ends of the bar on either side of the gap.

The condition on normal $\vec{B}$ ensures that the magnitude $|\vec{B}|$ is the same in the gap as in the material. As shown in Sec. E,

$$\frac{4\pi NI}{c} = H_{in} l + H_g l_g = H_{in} l + \mu H_{in} l_g ; \tag{10.93a}$$

so

$$H_{in} = \frac{4\pi NI}{c(l+\mu l_g)} ; \tag{10.93b}$$

$$B_{in} = B_g = H_g = \mu H_{in} = \frac{4\pi \mu NI}{c(l+\mu l_g)} ; \tag{10.93c}$$

again in the limit that the applied fields are weak enough so μ is constant and the dipole contribution to the total energy is unchanged.

To find the pressure on the ends, evaluate the energy W_f stored in the fields, then apply conservation of energy to a small increase in the width of the gap:

$$\Delta W_f = \vec{F} \cdot \Delta \vec{l}_g = \frac{dW_f}{dl_g} \Delta l_g . \tag{10.94}$$

Therefore

$$F = \frac{dW_f}{dl_g} . \tag{10.95}$$

Now

$$W_f = \frac{A}{8\pi}\left(\mu l H_{in}^2 + l_g H_g^2\right) = \frac{A}{8\pi} \mu (l + \mu l_g) H_{in}^2 ;$$

$$\frac{dW_f}{dl_g} = \frac{A}{8\pi} H_g^2 + \frac{A}{4\pi} \mu (l+\mu l_g) H_{in} \frac{dH_{in}}{dl_g} \tag{10.96a}$$

$$= \frac{A}{8\pi} H_g^2 - \frac{A}{4\pi} H_g^2 , \tag{10.96a}$$

where the first term is due to the increased gap, and the second to the reduced energy of the field as a result of the with the increased gap. The face pressure is therefore

$$P = \frac{1}{A}\frac{dW_f}{dl_g} = -\frac{1}{8\pi} H_g^2 = -\frac{1}{8\pi} B^2 . \tag{10.97}$$

This is not good: the negative sign of the result shows that the field acts to increase the gap, in contrast to what we know from experience. Something is wrong.

The answer is found in Faraday's Law of Induction. We have not yet discussed induction, so at this stage the following must be accepted on faith. The above equations show that increasing the width of the gap reduces the flux in the core. This in turn generates an EMF (voltage) given by

$$V_{EMF} = -\frac{N}{c}\frac{d\Phi}{dt} \tag{10.98}$$

where $\Phi_m = BA$ is the magnetic flux in the core. Since $dW/dt = V_{EMF} I$, if I is held constant by an external source, a time integration yields

$$\Delta W = -\frac{N}{c}\Delta\Phi_m . \tag{10.99}$$

Evaluating Eq. (10.99) reveals that this contribution to the pressure is exactly twice that of the negative term and of opposite sign.

Thus work is done in forcing the pole faces apart. Energy is delivered to the field in the gap and to the current source, and some energy is extracted from the field in the magnet. The situation is completely analogous to the electrostatics problem of removing the dielectric from a parallel-plate capacitor that is connected to a battery. As the dielectric is removed the stored energy is reduced, suggesting that the capacitor tends to be blown apart, but the energy going to the battery is twice the reduction. The capacitor remains stable. There is no explosion.

An examination of the above equations shows that for energy storage, magnet strength, and other purposes, the fact that the gap length l_g is multiplied by μ shows that even a small gap greatly reduces performance of electromagnets. We find below that this is not true of permanent magnets. As a practical matter, a common technology based on electromagnetism occurs in electromagnetic clutches on power machinery, for example lawn mowers. Anyone who has worked with these clutches knows that the spacing of their face plates must be adjusted to fractions of a millimeter, or the clutches won't activate.

Example 3. Repeat Example 1 assuming that the toroid material is fully magnetized.

In this case Eqs. (10.65) and (10.66) reduce to

$$H = 0 ; \tag{10.100a}$$

$$B = 4\pi M ; \tag{10.100b}$$

$$W = -Al\vec{M}\cdot\vec{B} = -4\pi A l M^2 . \tag{10.100c}$$

The relation $H = 0$ can be clarified. With a static configuration and no current, $\nabla\times\vec{H} = 0$. With a uniform magnetization $\nabla\cdot\vec{M} = 0$ so $\nabla\cdot\vec{H} = 0$. Thus $\vec{H}$ is at most a constant. Since $\vec{H}$ must vanish at infinity, its value must be zero.

Example 4. Repeat Example 2 with the toroid fully magnetized.

Example 2 shows that when a gap is introduced, B is reduced. A fully magnetized ferromagnet consists of a single domain, so $U_m = -\vec{M}\cdot\vec{B}$ is now a function of B, and must therefore be included explicitly in energy calculations.

The calculation is interesting for several reasons. First, with a fully magnetized ferromagnet, all dipoles are already oriented parallel to $\vec{B}$, so $\vec{B}_{loc} = \vec{B}$. Thus as the gap is opened, $(-\vec{M}\cdot\vec{B})$ becomes less negative, requiring work. Second, $\mu = 1$ is constant only in the gap. The concept of μ does not apply to the ferromagnet, so U_f here must be obtained by integration. field energy in the magnet is proportional is not simply proportional to the field, so the equations describing the situation for a constant μ are not relevant. Third, the simple thermodynamic expression $\vec{F} = -\nabla U_m$, where U_m is the magnetic energy density, is incomplete because energy is stored in both fields and dipoles. Finally, the magnetic fields are found to be much less sensitive to the gap than in the electromagnet case.

Start by calculating the internal and gap fields. From Eqs. (10.65) and (10.66)

$$H_m l + H_g l_g = 0; \tag{10.101a}$$

$$B_m = B_g = H_g = H_m + 4\pi M\,. \tag{10.101b}$$

The solution is

$$H_m = -4\pi M\,\frac{l_g}{l+l_g}; \tag{10.102a}$$

$$B_m = B_g = H_g = 4\pi M\,\frac{l}{l+l_g}\,. \tag{10.102b}$$

Since all dipoles experience this field, the energy stored as the dipole-field interaction is

$$W_d = -(\vec{M}\cdot\vec{B}_m)\times(\text{volume}) = -4\pi M^2 (Al)\frac{l}{l+l_g}\,. \tag{10.103}$$

Thus the presence of the gap has made the dipole energy less negative, so work will probably be required to open the gap.

Now

$$\vec{F} = -\nabla W_{tot} = -\frac{d}{dl_g}(W_d + W_m + W_g) \tag{10.104}$$

where W_d is given above. With $\mu = 1$, the gap energy is given by

$$W_g = \frac{Al_g}{8\pi} H_g^2. \tag{10.105}$$

The contribution to $\vec{F}$ from the field in the magnet material is

$$\frac{dW_m}{dl_g} = \frac{Al}{4\pi} H_m \frac{dB_m}{dl_g}, \tag{10.106}$$

which can be obtained without integration. Putting the pieces together:

$$\frac{1}{A}\frac{dW_d}{dl_g} = \frac{4\pi M^2 l^2}{(l+l_g)^2}; \tag{10.107a}$$

$$\frac{1}{A}\frac{dW_m}{dl_g} = -\frac{4\pi M^2 l l_g}{(l+l_g)^2}; \tag{10.107b}$$

$$\frac{1}{A}\frac{dW_g}{dl_g} = \frac{2\pi M^2 l^2}{(l+l_g)^2} - \frac{4\pi M^2 l l_g}{(l+l_g)^2}. \tag{10.107c}$$

In the limit $l_g \to 0$ the expressions reduce to

$$\frac{1}{A}\frac{dW}{dl_g} = 6\pi M^2. \tag{10.108}$$

The physics is straightforward. Equation (10.107a) describes the work necessary to raise dipoles from their minimum energy as a result of the decrease of *B*. This is the thermodynamic result, and would be the complete picture if additional energy storage were not involved. Equations (10.107b) and the second term in (10.107c) follow from the reduction in field energy due to an opening of the gap, and act to decrease the pressure pulling the two surfaces together across the gap. The first term in Eq. (10.107c) is the energy density associated with the field in the gap, which did not exist prior to the formation of the gap. In the limit of zero thickness, this is half the contribution of the dipoles, leading to Eq. (10.108). Evaluation of Eq. (10.108) for a flux density $B = 1.8\,\mathrm{T}$ is left for a problem assignment.

The above calculation highlights some properties of permanent magnets in general. First, opening gaps or removing "keeper" elements across the pole faces decreases $\vec{B}$, increases $\vec{H}$, and in particular generates magnetic fields outside the magnetized material, all of which reduce the (negative) energy of the system and make it thermodynamically less stable. While the present calculation was done for a particularly simple system, the same general characteristics apply to any permanent magnet.

J. Magnetic circuits.

The concept of magnetic circuits is extremely useful in engineering in the design of transformers, inductors, motors, and generators. We base the discussion on configurations similar to that used in Sec. E, again assuming that μ is large enough so leakage is not a factor. Applying Stokes' Theorem to Ampère's Equation, we have

$$\int_S d^2r'\hat{n}\cdot\nabla\times H = \oint_C \vec{H}\cdot d\vec{l} = H_\varphi \ell = \frac{4\pi}{c} N I\ , \tag{10.109}$$

where for a given path $l = 2\pi\,\rho$, independent of where the windings are placed on the bar. In the language of magnetic circuits, the technical term for this integral is *magnetomotive force*, or MMF, in parallel with its electric equivalent, the EMF. More generally, ℓ is a generalized distance through the magnetic core. The magnetic flux in the core is given by

$$\Phi_m = \int_S \vec{B}\cdot n\, d^2r = H\,\frac{\mu A_b}{\ell}\,, \tag{10.110}$$

where A_b is the cross-sectional area of the core and ℓ is the generalized length.

We can compare this to the expression for a current flowing in the same core, assuming that the core contains an internal electric field $\vec{E}$, which could be generated for example by moving the bar in a uniform external magnetic field. The expression is

$$I = \int_S \vec{J}\cdot n\, d^2r = E\,\frac{\sigma A_b}{\ell} = \frac{E}{R}, \tag{10.111}$$

where R is the resistance. The parallel is obvious. It allows the magnetic reluctance R_m to be defined as

$$R_m = \frac{\ell}{\mu A_b}. \tag{10.112}$$

The correspondences are summarized in the following table:

Electric:	Magnetic:
$\oint_C \vec{E}\cdot d\vec{l}$ = EMF	$\oint_C \vec{H}\cdot d\vec{l}$ = MMF
Electric current I	Magnetic flux Φ_m
Conductivity σ	Permeability μ
EMF = V_{EMF}	MMF = $\frac{4\pi}{c} N I$ (for example)
Resistance $R = \frac{\rho\ell}{A_w} = \frac{\ell}{\sigma A_w}$	Reluctance $R_m = \frac{\ell}{\mu A_b}$
Ohm's Law $V = IR$	Hopkinson's Law: MMF $= \Phi_m R_m$.

For example, we can form parallel and series combinations of reluctances as well as resistances. If one is interested in maximizing Φ_m to optimize the coupling of the primary and secondary windings in a transformer, for example, then it is necessary to pay attention to details such as this. Maximizing flux means minimizing the reluctance as much as possible for a given MMF. Thus shell transformers, where the return path completely encircles the windings, provide better coupling than yoke transformers, where the cage is reduced to two paths. Of course, this is achieved at the cost of increased complexity and more expensive manufacturing.

Where do the parallels break down? First, current leakage is generally not a factor in electric circuits, whereas at least some flux leakage is unavoidable in magnetic circuits. Second, electric current is carried by charged particles, typically electrons. Magnetic circuits have no such analog. Third, electric circuits are generally linear. Magnetic circuits are always nonlinear to a certain extent, and may be highly nonlinear. An interesting discussion of magnetic circuits can be found on Wikipedia.

K. Numbers.

One of the challenges of dealing with practical calculations involving magnets is units. The purpose of this section is to demonstrate that units can be understood, and that the results on magnetic forces etc. agree with everyday experience.

Start by considering whether observed remanences make sense when compared to atomic-scale dipoles. A typical permanent-magnet alloy is Alnico, which as its name suggests is a mixture of aluminum, nickel, and cobalt. Here, remanences reach 1.2 T, about four orders of magnitude larger than the 1 Gauss (10^{-4} T) magnetic field of the earth. The strongest ferromagnets in current general use are $Nd_2Fe_{14}B$ alloys, with remanences of about 1.4 T. The presence of nonmagnetic elements such as Al in Alnico and B in “neodymium” magnets may seem surprising, but is essential for pinning domain walls and thereby preventing unintended demagnetization. The fields produced are strong enough to be dangerous, not because we are affected by them directly, but because two Nd magnets can attract each other sufficiently to cause serious damage to fingers if you are careless and get one caught between.

At the other end of the scale is a neutron star. The magnetic dipole moment of a neutron is $-9.662\times10^{-27}\ \mathrm{A\,m^2}$, about 3 orders of magnitude less than that of an electron. As a point of physics, until the discovery of quarks, the fact that the neutron had a magnetic moment was a complete mystery, because objects with no charge should also have no magnetic properties, even if they have spin. Now the density ρ_n of a neutron star is of the order of $5\times10^{44}\ \mathrm{kg\,m^{-3}}$. Putting the numbers together shows that the magnetic field at the surface of a neutron star should be of the order of 10^{11} – $10^{12}\ \mathrm{T}$. Fields of these magnitudes would be unimaginably dangerous, but fortunately are never encountered on earth.

Regarding calculations and units, we use iron as the example. The first question: what is the theoretical maximum remanence of iron at maximum density if all atomic-scale magnetic dipoles $\vec{m}$ are aligned? In cgs units the equation to be evaluated is

$$\vec{B} = 4\pi\vec{M} \ , \tag{10.113}$$

where the magnetization $\vec{M} = n\vec{m}$ is the volume density of dipoles $\vec{m}$. Data are the volume density of iron atoms,

$$n = \frac{7.86\,\mathrm{g/cm^{-3}}}{(55.847)(1.66\times 10^{-24}\,g)} = 8.48\times 10^{28}\ \mathrm{m^{-3}}, \tag{10.114}$$

and the magnetic moment $2.2\mu_B$ per Fe atom. Here

$$\mu_B = \frac{e\hbar}{2m_e} = 9.274009994(57)\times 10^{-24}\,\frac{J}{T}\,. \tag{10.115}$$

An immediate difficulty presents itself: the data are given in SI units, whereas Eq. (10.xx) is in cgs units. To fix this, convert Eq. (10.xx) to SI with the help of Appendix 2:

$$\vec{B}_{cgs} = \sqrt{\frac{4\pi}{\mu_o}}\vec{B}_{SI} = 4\pi\vec{M}_{cgs} = 4\pi n\vec{m}_{cgs} = 4\pi n\sqrt{\frac{\mu_o}{4\pi}}\vec{m}_{SI}\,. \tag{10.116}$$

Thus

$$\vec{B}_{SI} = \mu_o n\vec{m}_{SI}\,. \tag{10.117}$$

Evaluating Eq. (10.116) with the above parameters we find $\vec{B}_{\max,Fe} = 2.17\ \mathrm{T}$. This is larger than the 1.2 T value cited above, but not so large as to shake our confidence in the calculation.

The second question is more interesting. What cross-sectional area is necessary to generate a single quantum of flux, $\Phi_0 = \dfrac{\hbar}{2e} = 2.067833831(13)\times 10^{-15}\ \mathrm{weber}$? A caution here: my theorist colleague notes that the flux quantum is relevant only in superconductivity, but let's proceed anyway. The density of flux lines for our Fe magnet is

$$\frac{\#\,\mathrm{lines}}{\mathrm{area}} = \frac{2.174\,\mathrm{w\,m^{-2}}}{2.068\times 10^{-15}\ \mathrm{w}} = 1.051\times 10^{15}\ \mathrm{m^{-2}}\,. \tag{10.118}$$

The number of dipoles needed to generate one flux quantum is therefore

$$(\#/\mathrm{area})/n^{2/3} = 1.84\times 10^{4}\,. \tag{10.119}$$

It seems odd that we should need ~10^4 moments to generate a single quantum of flux, so let's ask a 3rd question: if each electron generates one quantum of flux, how much actually escapes the electron beyond a distance a of closest approach? The dipole moment of an electron in empty space

$$\vec{m} = g_e \mu_B = g_e \frac{e\hbar}{2m_e} . \tag{10.120}$$

Classical physics has $g_{e,cl} = -1$, quantum mechanics says $g_{e,qm} = -2$, and quantum electrodynamics yields $g_{e,qed1} = -2.0023228$ to first order in vertex corrections involving virtual photons and $-2.002\ 319\ 304\ 363\ 287(538)$ to fifth order in vertex corrections. The experimental value, $g_{e,\exp} = -2.002\ 319\ 304\ 361\ 46(26)$ agrees with the quantum-electrodynamics prediction to 10 significant figures, making it historically the most accurately verified prediction in the history of physics.

Returning to question 3, we answer it by integrating the dipole expression

$$\vec{B} = \frac{3(\vec{r}\cdot\vec{m}) - \vec{r}^{\,2}\vec{m}}{r^5} \tag{10.121}$$

over the bisector plane from a to infinity, since the return flux is normal to the bisector plane. In SI units the relevant SI expression is

$$\Phi = \int_a^\infty \rho d\rho \int_0^{2\pi} d\varphi \frac{-\rho^2(\hat{z}\cdot\vec{m})}{\rho^5} = -\frac{2\pi}{a}\frac{\mu_o}{4\pi}\frac{g_e e\hbar}{2m_e} \tag{10.122a}$$

$$= \Phi_o = \frac{\hbar}{2e} , \tag{10.122b}$$

which gives

$$a = 3.53\times 10^{-14}\ \text{m} . \tag{10.123}$$

At least this is greater than the classical radius of the electron,

$$a_c = \frac{e^2}{mc^2} = 2.818\times 10^{-15}\ \text{m} . \tag{10.124}$$

To the extent that this calculation has any meaning, it shows that on the interatomic scale of $\sim 10^{-10}\ \text{m}$ most of the flux cycles back to the electron and never emerges from its vicinity.

Following through for iron, we have $a \sim (8.48\times 10^{28}\ \text{m}^{-3})^{-1/3} = 2.28\times 10^{-10}\ \text{m}$, whence

$$\Phi/\Phi_o = \frac{2.28\times 10^{-10}}{3.53\times 10^{-14}} = 6460 , \tag{10.125}$$

which can be compared to our above estimate that $\sim 18{,}000$ dipoles are needed to generate a single line of magnetic flux. Again, the agreement is not exact, but also not so far off that the model should be discarded.