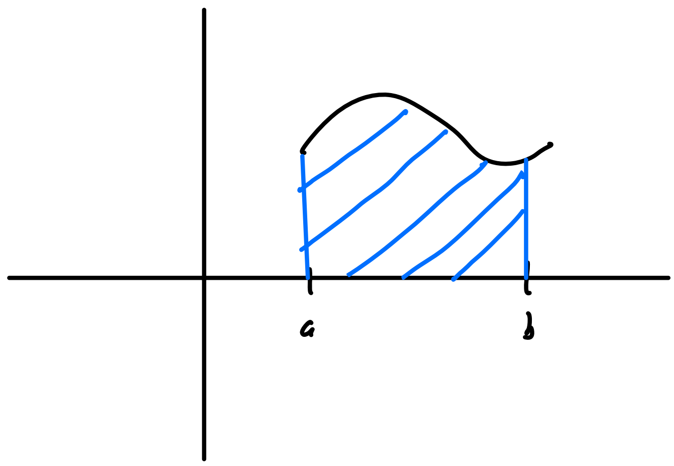
Math 600 Lecture 27

Introduction to Riemann integration

Consider $f: [a,b] \to \mathbb{R}$ and assume, for convenience, that $f(x) > 0$ $\forall x \in [a,b]$:



We often wish to compute the area between $y=0$ and $y=f(x)$ on $[a,b]$.

(Why? Computing areas is important in geometry, but the real reason is that if $y=f(t)$ represents the instantaneous rate of change of some quantity, w.r.t. to time, at time $t \in [a,b]$, then this area is – numerically – the change in that quantity from $t=a$ to $t=b$.)

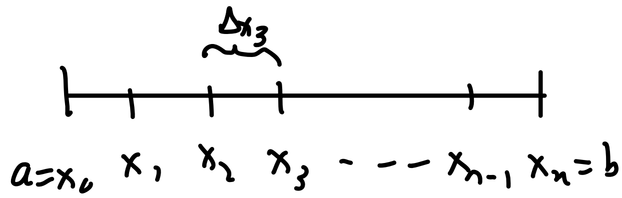
The following analysis should be familiar from calculus class.

<u>Definition</u>: A <u>partition</u> of $[a,b]$ is a set $P = \{x_0, x_1, \dots, x_n\} \subset [a,b]$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

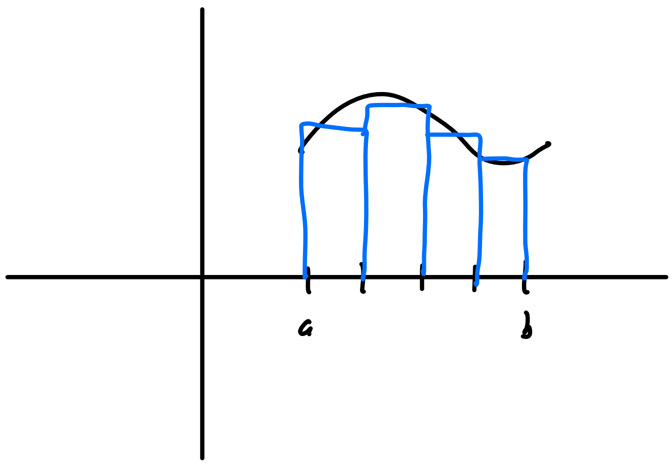
We write $\Delta x_j = x_j - x_{j-1}$, $j = 1, 2, \dots, n$. The <u>mesh size</u> of P is

$$|P| = \max\{\Delta x_j \mid j = 1, 2, \dots, n\}.$$



("Partition" is a poor choice of word, since partition has another well-established meaning in mathematics.)

Given a partition $\{x_0, x_1, \dots, x_n\}$ of $[a,b]$, we approximate the desired area as the sum of the areas of rectangles, one on each subinterval $[x_{i-1}, x_i]$:



The height of the rectangle on $[x_{i-1}, x_i]$ is $f(x_i^*)$, where x_i^* is any point in $[x_{i-1}, x_i]$:

$$A \approx \sum_{j=1}^{n} f(x_j^*) \Delta x_j \qquad (x_j^* \in [x_{j-1}, x_j] \ \forall j = 1, \dots, n).$$

The expression $\sum_{j=1}^{n} f(x_j^*) \Delta x_j$ is called a <u>Riemann sum</u>.

One then (in calculus class) takes the limit as $\|P\| \to 0$; if this limit exists, it is called the (Riemann) integral of f on $[a,b]$:

$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{j=1}^{n} f(x_j^*)\Delta x_j .$$

However, this is not a very convenient definition; indeed, this is a new kind of limit and would require a new definition. We avoid this by taking a different approach.

Upper and lower (Darboux) sums

Given $f:[a,b]\to\mathbb{R}$ and a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$, we define the upper (Darboux) sum of f on P by

$$U(P,f)=\sum_{j=1}^{n} M_j\,\Delta x_j, \quad M_j=\sup\{f(x)\mid x_{j-1}\le x\le x_j\},\ j=1,\dots,n$$

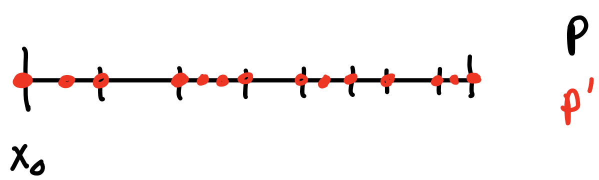
and the lower (Darboux) sum of f on P by

$$L(P,f)=\sum_{j=1}^{n} m_j\,\Delta x_j, \quad m_j=\inf\{f(x)\mid x_{j-1}\le x\le x_j\},\ j=1,\dots,n.$$

(Then every Riemann sum for f relative to P satisfies

$$L(P,f)\le \sum_{j=1}^{n} f(x_j^*)\Delta x_j \le U(P,f).)$$

<u>Definition</u>: Let P, P' be partitions of $[a,b]$. We say that P' is a <u>refinement</u> of P iff $P\subset P'$.



Lemma: If P, P' are partitions of $[a,b]$ and P' is a refinement of P, then

$$L(P,f) \le L(P',f) \le U(P',f) \le U(P,f).$$

Proof: The inequality $L(P',f) \le U(P',f)$ is obvious from the definition.

Let us prove that $L(P,f) \le L(P',f)$. It suffices to prove this inequality in the case that $P' = P \cup \{x_\ell'\}$, where $x_\ell \in (x_{\ell-1}, x_\ell)$. (If P' contains m more points than P, we can define $P_1', P_2', \ldots, P_m' = P'$, where P_1' has one more point than P and P_{j+1}' has one more point than P_j', $j = 1, \ldots, m-1$. Then we would have

$$L(P,f) \le L(P_1',f) \le \cdots \le L(P_m',f) = L(P',f).)$$

So assume that $P' = P \cup \{x_\ell'\}$, $x_{\ell-1} < x_\ell' < x_\ell$. Then

$$L(P',f) - L(P,f) = \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell'\}(x_\ell' - x_{\ell-1}) + \inf\{f(x) \mid x_\ell' \le x \le x_\ell\}(x_\ell - x_\ell')$$

$$- \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\}(x_\ell - x_{\ell-1}).$$

Since

$$\inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell'\} \ge \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\}$$

and

$$\inf\{f(x) \mid x_\ell' \le x \le x_\ell\} \ge \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\},$$

we have

$$L(P',f) - L(P,f) \ge \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\}(x_\ell' - x_{\ell-1}) + \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\}(x_\ell - x_\ell')$$

$$- \inf\{f(x) \mid x_{\ell-1} \le x \le x_\ell\}(x_\ell - x_{\ell-1}) = 0$$

(since $x_\ell' - x_{\ell-1} + x_\ell - x_\ell' = x_\ell - x_{\ell-1}$).

This completes the proof.

The proof that $U(P', f) \leq U(P, f)$ is similar. //

<u>Definition</u>: Let $f : [a,b] \to \mathbb{R}$ and let $\mathcal{P}$ be the set of all partitions on $[a,b]$.

The <u>upper</u> and <u>lower Riemann integrals</u> of f on $[a,b]$ are

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \mathcal{P}\},$$

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \in \mathcal{P}\},$$

respectively. If

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

then we say that f is <u>Riemann integrable</u> on $[a,b]$ and define the <u>Riemann integral</u> of f on $[a,b]$ to be this common value:

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$