Math 600 Lecture 13

True or false: Let (X,d) be a metric space and suppose that, for each $n \in \mathbb{Z}^+$, $E_n \subset X$ is compact. Then $\bigcap_{n=1}^{\infty} E_n$ is compact.

__Theorem__: Let (X,d) be a metric space and let $E \subset X$. Then E is compact iff every sequence contained in E has a subsequence that converges to a point in E.

__Proof:__ Suppose first that E is compact. Then, by an earlier theorem, each infinite subset of E has a limit point in E. If the set $\{x_n \mid n \in \mathbb{Z}^+\}$ has infinitely many elements, then it has a limit point $x \in E$ and, by an earlier lemma, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. If the set $\{x_n \mid n \in \mathbb{Z}^+\}$ has only finitely many elements, then there exists some $x \in E$ and an increasing sequence $\{n_k\}$ in $\mathbb{Z}^+$ such that $x_{n_k} = x$ for all $k \in \mathbb{Z}^+$. But then $x_{n_k} \to x$.

Conversely, suppose every sequence in E has a subsequence that converges to a point of E. Let S be any infinite subset of E. Then there exists a sequence $\{x_n\}$ in S with distinct terms. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \to x$. It follows immediately that x is a limit point of S. Since every infinite subset of E has a limit point in E, it follows that E is compact. //

Theorem: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X. Then the set of all subsequential limits of $\{x_n\}$ is closed.

Proof: Let S be the set of all subsequential limits of $\{x_n\}$ and suppose $x \in S^c$. Then, since x is not a subsequential limit of $\{x_n\}$, there exists $r>0$ such that $B_r(x)$ contains no term of $\{x_n\}$ other than possibly x itself. But then, since $B_r(x)$ is open, for each $y \in B_r(x)$, there exists $r_y > 0$ such that $B_{r_y}(y) \subset B_r(x)$, and hence $B_{r_y}(y)$ contains no term of $\{x_n\}$. Thus each $y \in B_r(x)$ is not a subsequential limit of $\{x_n\}$, and it follows that $B_r(x) \subset S^c$. Therefore, S^c is open and hence S is closed. //

Cauchy sequences

The following definition describes a sequence that "ought" to converge or "looks like" it is converging. The only question is whether the limit exists in X.

Definition: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is <u>Cauchy</u> iff

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{Z}^+ \; (m,n \geq N \Rightarrow d(x_m, x_n) < \varepsilon).$$

<u>Example</u>: Let α_n equal the rational number defined by the first n digits of π:

$$x_1 = 3.1,\ x_2 = 3.14,\ x_3 = 3.141,\ x_4 = 3.1415, \ldots$$

Note that

$$m, n \geq N \Rightarrow |x_m - x_n| < 10^{-N}.$$

It follows that $\{\alpha_n\}$ is Cauchy. If we regard $\{\alpha_n\}$ as a sequence in $\mathbb{Q}$, it is Cauchy but not convergent. Clearly, though, in an intuitive sense, $\{\alpha_n\}$ "acts like" a convergent sequence.

<u>Lemma</u>: Let (X,d) be a metric space and let $\{x_n\}$ be a Cauchy sequence in X. Then $\{x_n\}$ is bounded.

<u>Proof</u>: Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}^+$ such that

$$m, n \geq N \Rightarrow d(x_m, x_n) < 1.$$

In particular,

$$n \geq N \Rightarrow d(x_n, x_N) < 1.$$

Define

$$R = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}.$$

Then

$$d(x_n, x_N) \leq R \quad \forall n \in \mathbb{Z}^+$$

and hence $\{x_n\}$ is bounded. //

Theorem:

1. If (X,d) is a metric space and $\{x_n\}$ is a convergent sequence in X, then $\{x_n\}$ is Cauchy.
2. If (X,d) is a compact metric space and $\{x_n\}$ is a Cauchy sequence in X, then $\{x_n\}$ converges to a point of X.
3. If $k \in \mathbb{Z}^+$ and $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^k$, then $\{x_n\}$ converges to a point of $\mathbb{R}^k$.

Proof:

1. Suppose $x_n \to x$ in X. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}.$$

But then

$$m,n \geq N \Rightarrow d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) \quad \text{(by the triangle inequality)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{x_n\}$ is Cauchy.

2. Now suppose that X is compact and $\{x_n\}$ is Cauchy. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x \in X$. (Either $\{x_n | n \in \mathbb{Z}^+\}$ is infinite, in which case it has a limit point in X, or it is finite, in which case $x_{n_k} = x$ for some $x \in X$ and infinitely many $k \in \mathbb{Z}^+$.) Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}^+$ such that

$$m,n \geq N' \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2}$$

Since $x_{n_k} \to x$ and $n_k \to \infty$, there exists $K \in \mathbb{Z}^+$ such that

$$k \geq K \Rightarrow \left(d(x_{n_k}, x) < \frac{\varepsilon}{2} \text{ and } n_k \geq N\right).$$

For any $n \geq N$, we have

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $x_n \to x$.

3. If $\{x_n\} \subset \mathbb{R}^k$ is Cauchy, then it is also bounded and hence belongs to a k-cell C. Since every k-cell is compact, it follows from part 2 that $\{x_n\}$ converges. //

<u>Definition</u>: Let (X,d) be a metric space. We say that X is <u>complete</u> iff every Cauchy sequence in X converges to a point of X.

By the previous theorem, $\mathbb{R}^k$ is complete for all $k \in \mathbb{Z}^+$; in particular, $\mathbb{R}$ is complete. Obviously $\mathbb{Q}$ (or $\mathbb{Q}^k$) is not complete.