Math 600 Lecture 38

<u>Theorem</u>: Let E be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$, let $f: E \to \mathbb{R}^n$ be differentiable on E, and assume that Df is continuous on E. Suppose that $(x_0, y_0) \in E$ satisfies

$$f(x_0, y_0) = 0,$$

$$D_y f(x_0, y_0) \text{ is nonsingular (invertible).}$$

Then exist open sets $U \subset \mathbb{R}^m$, $V \in \mathbb{R}^n$ such that

$$x_0 \in U,\ y_0 \in V,\ U \times V \subset E$$

and $\psi: U \to V$ such that

$$f(x, \psi(x)) = 0 \quad \forall x \in U.$$

Moreover, for all $x \in U$, $y = \psi(x)$ is the only point in V satisfying

$$f(x, y) = 0.$$

Finally, ψ is continuously differentiable and $\psi'(x) = -D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))$.

<u>Proof:</u> We will prove existence and uniqueness by setting up a contractive mapping. Define $\varphi: B_\varepsilon(x_0) \times B_\delta(y_0) \to B_\delta(y_0)$ by

$$\varphi(x, y) = y - D_y f(x_0, y_0)^{-1} f(x, y),$$

where $\varepsilon > 0$, $\delta > 0$ are to be specified. Note that

$$\varphi(x, y_1) - \varphi(x, y_2) = y_1 - D_y f(x_0, y_0)^{-1} f(x, y_1) - y_2 + D_y f(x_0, y_0)^{-1} f(x, y_2)$$

$$= y_1 - y_2 - D_y f(x_0, y_0)^{-1}(f(x, y_1) - f(x, y_2)).$$

Now,

$$f(x, y_1) - f(x, y_2) = \int_0^1 D_y f(x, y_2 + t(y_1 - y_2))(y_1 - y_2)\,dt \qquad (\text{why?})$$

$$= \left(\int_0^1 D_y f(x, y_2 + t(y_1 - y_2))\,dt\right)(y_1 - y_2)$$

$$\Rightarrow D_y f(x_0, y_0)^{-1}(f(x, y_1) - f(x, y_2)) = \left(\int_0^1 D_y f(x_0, y_0)^{-1} D_y f(x, y_2 + t(y_1 - y_2))\,dt\right)(y_1 - y_2)$$

$$\Rightarrow y_1 - y_2 - D_y f(x_0, y_0)^{-1}(f(x, y_1) - f(x, y_2))$$

$$= y_1 - y_2 - \left(\int_0^1 D_y f(x_0, y_0)^{-1} D_y f(x, y_2 + t(y_1 - y_2))\,dt\right)(y_1 - y_2)$$

$$= \left(I - \int_0^1 D_y f(x_0, y_0)^{-1} D_y f(x, y_2 + t(y_1 - y_2))\,dt\right)(y_1 - y_2).$$

Now choose $\varepsilon' > 0$, $\delta' > 0$ such that $B_{\varepsilon'}(x_0) \times B_{\delta'}(y_0) \subset E$ and define $A : B_{\varepsilon'}(x_0) \times B_{\delta'}(y_0) \times B_{\delta'}(y_0) \to \mathcal{L}(\mathbb{R}^n)$ by

$$A(x, y_1, y_2) = I - \int_0^1 D_y f(x_0, y_0)^{-1} D_y f(x, y_2 + t(y_1 - y_2))\,dt.$$

Since the mapping $L \mapsto L^{-1}$ is continuous, we can choose ε', δ' sufficiently small that $D_y f(x, y)$ is invertible for all $x, y \in B_{\varepsilon'}(x_0) \times B_{\delta'}(y_0)$.

Note that A is continuous (why?) and $A(x_0, y_0, y_0) = 0$.

Therefore, there exists $\varepsilon \in (0, \varepsilon']$ and $\delta \in (0, \delta']$ such that

$$(x, y_1, y_2) \in B_\varepsilon(x_0) \times B_\delta(y_1) \times B_\delta(y_2) \implies \|A(x, y_1, y_2)\| \le \frac{1}{2}.$$

We have

$$\begin{aligned}
\varphi(x,y) - y_0 &= y - y_0 - D_y f(x_0,y_0)^{-1} f(x,y) \\
&= y - y_0 - D_y f(x_0,y_0)^{-1}\left(f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)\right) \\
&= A(x,y,y_0)(y-y_0) + \left(\int_0^1 D_y f(x_0,y_0)^{-1} D_x f(x_0 + t(x-x_0), y_0)\,dt\right)(x-x_0)
\end{aligned}$$

$$\begin{aligned}
\implies \|\varphi(x,y) - y_0\| &\le \|A(x,y,y_0)\|\,\|y-y_0\| + \left\|\int_0^1 D_y f(x_0,y_0)^{-1} D_x f(x_0+t(x-x_0),y_0)\,dt\right\| \|x-x_0\| \\
&\le \frac{1}{2}\delta + \left\|\int_0^1 D_y f(x_0,y_0)^{-1} D_x f(x_0+t(x-x_0),y_0)\,dt\right\| \varepsilon.
\end{aligned}$$

Since $D_x f(x, y_0)$ depends continuously on x, we can reduce ε, if necessary, to ensure that the second term is less than $\frac{1}{2}\delta$ for all $x \in B_\varepsilon(x_0)$, and hence that

$$\|\varphi(x,y_0) - y_0\| < \delta \quad \forall x \in B_\varepsilon(x_0).$$

Thus, for all $x \in B_\varepsilon(x_0)$, $\varphi(x,\cdot)$ maps $B_\delta(y_0)$ into $B_\delta(y_0)$.

Next,

$$(x, y_1, y_2) \in B_\varepsilon(x_0) \times B_\delta(y_1) \times B_\delta(y_2)$$

$$\begin{aligned}
\implies \|\varphi(x,y_1) - \varphi(x,y_2)\| &= \|A(x,y_1,y_2)(y_1-y_2)\| \\
&\le \|A(x,y_1,y_2)\|\,\|y_1-y_2\| \\
&\le \lambda \|y_1 - y_2\|,
\end{aligned}$$

and thus $\varphi(x,\cdot)$ is a contractive mapping. Therefore, for each $x \in B_\varepsilon(x_0)$, there exists a unique $y \in B_\delta(y_0)$ such that

$$\varphi(x,y) = y$$

$$\iff y - D_y f(x_0,y_0)^{-1} f(x,y) = y$$

$$\iff -D_y f(x_0,y_0)^{-1} f(x,y) = 0$$

$$\iff f(x,y) = 0.$$

Define $U = B_\varepsilon(x_0)$, $V = B_\delta(y_0)$, and $\psi : U \to V$ by the condition that $\psi(x) = y$, where y is the unique point in V such that $f(x,y) = 0$.

This proves the existence and uniqueness of ψ.

We will save the rest of the proof for the next lecture. //

7. **[12 points]** Let $(u(x,y), v(x,y))$ be the unique simultaneous solution of the equations

$$\begin{cases} xu^3 + (y+1)uv = 6 \\ yu^2 + v^2 + xy = 9, \end{cases}$$

for (x,y) near $(0,0)$ and (u,v) near $(2,3)$. Compute u_x, u_y, v_x and v_y at the point $(x,y) = (0,0)$. Clearly state every theorem that you use.

Define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f((x,y),(u,v)) = \begin{bmatrix} xu^3 + (y+1)uv - 6 \\ yu^2 + v^2 + xy - 9 \end{bmatrix}$$

Note that

$$f((0,0),(2,3)) = \begin{bmatrix} 0 + 1\cdot 6 - 6 \\ 0 + 9 + 0 - 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$f'_{(u,v)}((x,y),(u,v)) = \begin{bmatrix} 3xu^2 + (y+1)v & (y+1)u \\ 2yu + x & 2v \end{bmatrix},$$

$$f'_{(u,v)}((0,0),(2,3)) = \begin{bmatrix} 0 + 1\cdot 3 & 1\cdot 2 \\ 0+0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 6 \end{bmatrix}.$$

Since $f((0,0),(2,3)) = 0$ and $f'_{(u,v)}((0,0),(2,3))$ is nonsingular, the implicit function theorem applies. There exist open sets U, V in $\mathbb{R}^2$ such that $(0,0) \in U$, $(2,3) \in V$ and $\psi : U \to V$ such that

$$f((x,y), \psi(x,y)) = 0 \quad \forall (x,y) \in \mathcal{U}$$

and $(u,v) = \psi(x,y)$ is the unique solution of

$$f((x,y),(u,v)) = 0$$

that lies in V. Also,

$$\psi'(x,y) = -f'_{(u,v)}((x,y), \psi(x,y))^{-1} f'_{(x,y)}((x,y), \psi(x,y)).$$

$$\Rightarrow \psi'(0,0) = -f'_{(u,v)}((0,0),(2,3))^{-1} f'_{(x,y)}((0,0),(2,3)).$$

We computed $f'_{(u,v)}((0,0),(2,3))$ above. We have

$$f((x,y),(u,v)) = \begin{bmatrix} xu^3 + (y+1)uv - 6 \\ yu^2 + v^2 + xy - 9 \end{bmatrix}$$

$$\Rightarrow f'_{(x,y)}((x,y),(u,v)) = \begin{bmatrix} u^3 & uv \\ y & u^2 + x \end{bmatrix}$$

$$\Rightarrow f'_{(x,y)}((0,0),(2,3)) = \begin{bmatrix} 8 & 6 \\ 0 & 4 \end{bmatrix}$$

Thus

$$\psi'(0,0) = -\begin{bmatrix} 3 & 2 \\ 0 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 8 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -8/3 & -14/9 \\ 0 & -2/3 \end{bmatrix}$$

Since

$$\Psi'(0,0) = \begin{bmatrix} u_x(0,0) & u_y(0,0) \\ v_x(0,0) & v_y(0,0) \end{bmatrix},$$

we see that

$$u_x(0,0) = -8/3, \quad u_y(0,0) = -14/9$$

$$v_x(0,0) = 0, \quad v_y(0,0) = -2/3.$$