Moment Generating Functions

Definition: The MGF $M_X(t)$ of X is given by
$$M_X(t) = E(e^{tX})$$
for all t for which the expectation is finite.

Theorem: $E(X^n) = M^{(n)}(0)$ when $M(t) < \infty$ over some neighborhood containing 0.

Proof: The Taylor expansion of $M(t)$ about 0 is
$$M(t) = \sum_{n=0}^{\infty} \frac{M^{(n)}(0)}{n!} t^n$$

Also,
$$M(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{X^n t^n}{n!}\right]$$
$$= \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n \implies M^{(n)}(0) = E(X^n).$$

Theorem: Let X, Y be rvs having mgfs M_X & M_Y. If $\exists a > 0$ such that $M_X(t) = M_Y(t)$ $\forall t \in (-a, a)$, then X & Y have the same distribution.

Theorem: <u>Continuity theorem</u>

Let $\{X_n\}$ be a sequence of rvs & $\{F_n\}$ be the corresponding distribution functions & $\{M_n\}$ be the mgfs. Suppose $\exists a > 0$ s.t $M_n(t) \to M_X(t)$ as $n \to \infty$ $\forall t \in (-a, a)$ where $M_X(t)$ denotes the mgf of some rv X. Then $F_n(x) \to F(x)$ $\forall x$ at which F is continuous.

Theorem: $X \sim N(\mu, \sigma^2)$. Then

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad \forall t \in \mathbb{R}.$$

<u>The central limit theorem</u>

Consider a sample of random quantities $X_1, \ldots, X_n$. Define the sample mean $\overline{X}_n$ by

$$\overline{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}.$$

When $X_1, \ldots, X_n$ are i.i.d & $E(X_i) = \mu$, $V(X_i) = \sigma^2$,

$$E(\overline{X}_n) = \mu \quad \& \quad V(\overline{X}_n) = \frac{\sigma^2}{n}.$$

The purpose of this section is to investigate the long run behavior of $\overline{X}_n$ as $n \to \infty$.

Theorem: The central limit theorem.

Let $X_1, X_2, \ldots$ be i.i.d & $E(X_i) = \mu < \infty$, $V(X_i) = \sigma^2 < \infty$. Then,

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} Z \sim N(0,1) \quad \text{as } n \to \infty.$$

$$\left(\text{or} \quad \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{D} Z \sim N(0,1) \quad \text{as } n \to \infty \right).$$

Proof: Assume $\mu = 0$, $\sigma^2 = 1$. & that the mgf of X_i, $M(t)$, is finite on some neighborhood $(-a, a)$ of 0.

The mgf of $X_i/\sqrt{n}$ is $M\left(\frac{t}{\sqrt{n}}\right)$

Therefore, $M_{\frac{\sum X_i}{\sqrt{n}}}(t) = \left[M\left(\frac{t}{\sqrt{n}}\right) \right]^n$.

Let $L(t) = \ln(M(t))$.

Note that $L(0) = E[e^0] = 1$

$$L'(0) = \frac{M'(0)}{M(0)} = \frac{E(X_i)}{1} = 0$$

$$L''(0) = \frac{\underbrace{M(0)}_{=1}\,\underbrace{M''(0)}_{E(X^2)} - \underbrace{[M'(0)]^2}_{=0}}{\underbrace{[M(0)]^2}_{=1}}$$

$$L''(0) = M''(0) = E(X^2) = 1$$

Our goal is to prove that $\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n \to e^{t^2/2}$ or

$$\lim_{n\to\infty} n\,L\left(\frac{t}{\sqrt{n}}\right) = \lim_{n\to\infty} \ln\left[M\left(\frac{t}{\sqrt{n}}\right)^n\right] = \frac{t^2}{2}.$$

Note that

$$\lim_{n\to\infty} n\,L\left(\frac{t}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{L\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{L'\left(\frac{t}{\sqrt{n}}\right)\left(-\frac{1}{2}n^{-\frac{3}{2}}t\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{L\left(\frac{t}{\sqrt{n}}\right)t}{2n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{L''\left(\frac{t}{\sqrt{n}}\right)\cdot\left(-\frac{1}{2}t\,n^{-\frac{3}{2}}\right)t}{-n^{-\frac{3}{2}}}$$

$$= \lim_{n\to\infty} \left[L''\left(\frac{t}{\sqrt{n}}\right)\cdot\frac{t^2}{2}\right] = L''(0)\cdot\frac{t^2}{2}$$

$$= \frac{t^2}{2}$$

Therefore,

$$M_{\frac{\sum X_i}{\sqrt{n}}}(t) \to e^{\frac{t^2}{2}}.$$

By the continuity theorem,

$$\frac{\sum X_i}{\sqrt{n}} \xrightarrow{D} Z \sim N(0,1) \qquad \square$$

Ex: (1) Let $X_1, X_2, \ldots \sim B(m,p)$ be independent.

Let $\overline{X}_n = \dfrac{\sum^n X_i}{n}$

Then $$\frac{\overline{X}_n - mp}{\sqrt{\frac{mp(1-p)}{n}}} \xrightarrow{D} N(0,1)$$

(2) Let $X_1, \ldots, X_{100} \sim U(0,1)$. be independent.

$E(X_i) = \frac{1}{2}$, $V(X_i) = \frac{1}{12}$. Then,

$$P\left[\sum_{i=1}^{100} X_i \leq 55\right] = P\left[\frac{\sum_{i=1}^{100} X_i}{100} \leq 0.55\right]$$

$$= P\left[\frac{\frac{\sum_{i=1}^{100} X_i}{100} - \frac{1}{2}}{\sqrt{\frac{1}{12(100)}}} \leq \frac{0.55 - \frac{1}{2}}{\sqrt{\frac{1}{12(100)}}}\right] \approx \Phi\left(\frac{0.55 - \frac{1}{2}}{\sqrt{\frac{1}{12(100)}}}\right) \approx$$

Markov Chains

Definition : Let $\{X_n\}$ be a sequence of random variables taking values in a countable set S, called the state space.

If $\forall n \geq 0$,

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i),$$

then X is said to be a Markov Chain, or to have the Markov Property.

The Markov property states that the next state X_{n+1} only depends on the current state & is independent of past states X_i, $\forall i < n$.

We write

$$p_{ik} = P(X_1 = k \mid X_0 = i)$$

$\{p_{ik},\ i, k \in S\}$, are called the transition probabilities of the chain $\{X_n\}$

The MC is called time-homogeneous if

$$P(X_{n+1} = k \mid X_n = i) = p_{ik}, \quad \forall n.$$