Math 600 Lecture 34

Definition Let (X,d) be a metric space and let $E\subset X$ be compact.

We define
$$C(E)=\{f:E\to\mathbb{R}\mid f \text{ is continuous on } E\}.$$

For all $f\in C(E)$, we define
$$\|f\|_\infty=\max\{|f(x)|:x\in E\}.$$

It can be verified that $\|\cdot\|_\infty$ defines a norm on $C(E)$; the corresponding metric on $C(E)$ is
$$d_\infty(f,g)=\|f-g\|\quad\forall f,g\in C(E).$$

Theorem: Let (X,d) be a metric space and let $E\subset X$ be compact. Then $(C(E),d_\infty)$ is a complete metric space.

Proof: Suppose $\{f_n\}\subset C(E)$ is Cauchy. Let $\varepsilon>0$ be given. Then there exists $N\in\mathbb{Z}^+$ such that
$$m,n\ge N\Rightarrow \|f_m-f_n\|<\varepsilon,$$
that is,
$$m,n\ge N\Rightarrow\left(|f_m(x)-f_n(x)|<\varepsilon\ \forall x\in E\right).$$

This shows that $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ for all $x\in E$ and hence, since $\mathbb{R}$ is complete,
$$\lim_{n\to\infty}f_n(x)$$
exists. Define $f:E\to\mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in E.$$

Again, let $\varepsilon > 0$ be given and let $N \in \mathbb{Z}^+$ satisfy

$$m, n \geq N \Rightarrow \left(|f_m(x) - f_n(x)| < \tfrac{\varepsilon}{2} \ \forall x \in E \right)$$

For each $x \in E$, there exists $N_x \in \mathbb{Z}^+$, $N_x \geq N$, such that

$$n \geq N_x \Rightarrow |f_n(x) - f(x)| < \tfrac{\varepsilon}{2}.$$

But then

$$n \geq N \Rightarrow \Big(\forall x \in E,\ |f_n(x) - f(x)| \leq |f_n(x) - f_{N_x}(x)| + |f_{N_x}(x) - f(x)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \Big)$$

$$\Rightarrow \|f_n - f\| < \varepsilon.$$

Thus $f_n \to f$ in $C(E)$ (i.e. $f_n \to f$ uniformly on E), and we have shown that $C(E)$ is complete. //

We now wish to show that certain subsets of $C(E)$ are compact. More specifically, we wish to derive conditions on $\{f_n\} \subset C(E)$ guaranteeing that $\{f_n\}$ has a subsequence converging in $C(E)$ (i.e. converging uniformly on E).

<u>Definition</u>: Let (X, d) be a metric space, let $E \subset X$, and let $f_n : E \to \mathbb{R}$ for all $n \in \mathbb{Z}^+$. We say that $\{f_n\}$ is <u>pointwise bounded</u> on E iff

$$\forall x \in E\ \exists M > 0 \ \left(|f_n(x)| \leq M \ \forall n \in \mathbb{Z}^+ \right)$$

and $\{f_n\}$ is uniformly bounded on E iff

$$\exists M > 0 \ \left(|f_n(x)| \leq M \ \forall x \in E \ \forall n \in \mathbb{Z}^+ \right).$$

__Definition__: Let (X,d) be a metric space, let $E \subset X$, and let $\mathcal{F}$ be any set of functions of the type $f: E \to \mathbb{R}$. We say that $\mathcal{F}$ is __equicontinuous__ ("uniformly uniformly continuous") iff

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \big(x,y \in E \text{ and } d(x,y) < \delta\big) \Rightarrow \big(|f(x) - f(y)| < \varepsilon \;\; \forall f \in \mathcal{F}\big).$$

Note that if $\mathcal{F}$ is equicontinuous, then every $f \in \mathcal{F}$ is uniformly continuous.

__Theorem__: Let (X,d) be a metric space, let $E \subset X$ be compact, and let $\{f_n\} \subset C(E)$. If $\{f_n\}$ converges in $C(E)$ (i.e. if $\{f_n\}$ converges uniformly on E), then $\{f_n\}$ is equicontinuous.

__Proof__: Suppose $f_n \to f$ uniformly on E, and let $\varepsilon > 0$ be given. We know that f is continuous (the uniform limit of continuous functions is continuous) and hence uniformly continuous (since E is compact). Hence there exists $\delta_0 > 0$ such that

$$\big(x,y \in E \text{ and } d(x,y) < \delta_0\big) \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3},$$

Since $f_n \to f$ uniformly on E, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3} \;\; \forall x \in E.$$

Suppose $n \geq N$. Then

$$\begin{aligned}\big(x,y \in E \text{ and } d(x,y) < \delta_0\big) \Rightarrow |f_n(x) - f_n(y)| &\leq |f_n(x) - f(x)| + |f(x) - f(y)| + \\ &\qquad |f(y) - f_n(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since each f_n is continuous on E and hence uniformly continuous on E, for each

$n=1,2,\ldots,N-1$, there exists $\delta_n>0$ such that

$$(x,y\in E \text{ and } d(x,y)<\delta_n)\Rightarrow |f_n(x)-f_n(y)|<\varepsilon.$$

Therefore, if $\delta=\min\{d_0,d_1,\ldots,d_{N-1}\}$, then

$$(n\in\mathbb{Z}^+ \text{ and } x,y\in E \text{ and } d(x,y)<\delta)\Rightarrow |f_n(x)-f_n(y)|<\varepsilon.$$

It follows that $\{f_n\}$ is equicontinuous. //

<u>Lemma:</u> Let E be a countable set and suppose $f_n:E\to\mathbb{R}$ for all $n\in\mathbb{Z}^+$. If $\{f_n\}$ is pointwise bounded on E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x\in E$.

<u>Proof</u>: Let $E=\{x_n\}$. Since $\{f_n(x_1)\}$ is bounded, by the Heine-Borel theorem, there exists a subsequence of $\{f_n(x_1)\}$ that converges; let us call this subsequence $\{f_{1,k}(x_1)\}$.

Now suppose we have identified subsequences $\{f_{1,k}\},\{f_{2,k}\},\ldots,\{f_{\ell,k}\}$ such that $\{f_{j,k}(x_\ell)\}$ converges for all $j=1,2,\ldots,\ell$ and $\{f_{j+1,k}\}$ is a subsequence of $\{f_{j,k}\}$ for all $j=1,2,\ldots,\ell-1$. Consider $\{f_{\ell,k}(x_{\ell+1})\}$. This sequence is bounded (since it's a subsequence of $\{f_n(x_{\ell+1})\}$) and hence it has a subsequence $\{f_{\ell+1,k}(x_{\ell+1})\}$ that converges in $\mathbb{R}$.

In this way, we have constructed a sequence of subsequences:

$$\{f_{\ell,k}\},\ \ell=1,2,3,\ldots$$

Now define $\{f_{n_k}\}$ by $f_{n_k}=f_{k,k}\ \forall k\in\mathbb{Z}^+$. For each $j\in\mathbb{Z}^+$,

$\{f_{n_k}(x_j)\}$ is a subsequence of $\{f_{j,k}(x_j)\}$ and hence $\{f_{n_k}(x_j)\}$ converges. This completes the proof. //

__Theorem__ (Arzela-Ascoli theorem): Let (X,d) be a metric space, let $E \subset X$ be compact, and let $f_n \in C(E)$ for all $n \in \mathbb{Z}^+$. Suppose $\{f_n\}$ is pointwise bounded and equicontinuous on E. Then $\{f_n\}$ is uniformly bounded on E and $\{f_n\}$ contains a uniformly bounded subsequence.

__Proof:__ Let $\varepsilon > 0$ be given; then there exists $\delta > 0$ such that

$$\forall n \in \mathbb{Z}^+ \ \forall x,y \in E \ \left(d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| \right).$$

Note that $\{B_\delta(x) \mid x \in E\}$ is an open cover for E; since E is compact, there exist $p_1, \ldots, p_k \in E$ such that

$$E \subset \bigcup_{j=1}^{k} B_\delta(p_j).$$

For each $j = 1, \ldots, k$, there exists $M_j > 0$ such that

$$\forall n \in \mathbb{Z}^+, \ |f_n(p_j)| \le M_j.$$

$$\Rightarrow \forall n \in \mathbb{Z}^+ \ \forall j = 1,2,\ldots,k, \ |f_n(p_j)| \le M = \max\{M_1, \ldots, M_k\}.$$

But then, for all $x \in E$, there exists $j \in \{1, \ldots, k\}$ such that $x \in B_\delta(p_j)$ and hence $d(x,p_j) < \delta$, which implies that

$$\left(|f_n(x) - f_n(p_j)| < \varepsilon \ \forall n \in \mathbb{Z}^+ \right) \Rightarrow \left(|f_n(x)| < |f(p_j)| + \varepsilon \le M + \varepsilon \ \forall n \in \mathbb{Z}^+ \right)$$

That is,

$$\forall n \in \mathbb{Z}^+ \ \forall x \in E, \ |f_n(x)| \leq M + \varepsilon.$$

Thus $\{f_n\}$ is uniformly bounded on E.

Now, since E is compact, it contains a countable dense subset S (see Lecture 9). By the previous lemma, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in S$. We will prove that $\{f_{n_k}\}$ converges uniformly on E.

Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$\forall n \in \mathbb{Z}^+ \ \forall x, y \in E \ \left(d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}\right).$$

Note that

$$E \subset \bigcup_{x \in S} B_\delta(x)$$

(since S is dense in E, for all $y \in E$, there exists $x \in S$ such that $d(y,x) < \delta$). Therefore, since E is compact, there exist $x_1, \ldots, x_n \in S$ such that

$$E \subset \bigcup_{j=1}^{n} B_\delta(x_j).$$

We know that $\{f_{n_k}(x_j)\}$ converges for each $j = 1, \ldots, n$, so there exists $N \in \mathbb{Z}^+$ such that

$$i, j \geq N \Rightarrow \left(|f_{n_i}(x_\ell) - f_{n_j}(x_\ell)| < \frac{\varepsilon}{3} \ \forall \ell = 1, \ldots, n\right).$$

Now let $x \in E$ be arbitrary. Then $x \in B_\delta(x_\ell)$ for some $\ell \in \{1, \ldots, n\}$, and hence

$$|f_{n_k}(x) - f_{n_k}(x_\ell)| < \frac{\varepsilon}{3} \quad \forall k \in \mathbb{Z}^+.$$

But then

$$\begin{aligned}(i, j \geq N \text{ and } x \in E) \implies |f_{n_i}(x) - f_{n_j}(x)| &\leq |f_{n_i}(x) - f_{n_i}(x_\ell)| + |f_{n_i}(x_\ell) - f_{n_j}(x_\ell)| \\ &\quad + |f_{n_j}(x_\ell) - f_{n_j}(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus, by the Cauchy criterion, $\{f_{n_k}\}$ converges uniformly. //