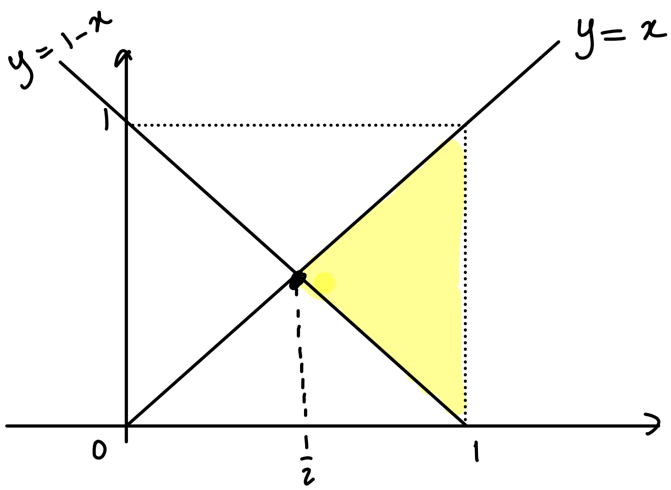
# Quiz 4

$$f(x) = \begin{cases} kxy & ; \ 0 < y < x < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Find $P(X+Y>1)$.

$$1 = \int_{x=0}^{1}\int_{y=0}^{y=x} kxy \, dy \, dx = k\int_{x=0}^{1} x \cdot \frac{y^2}{2}\Big|_0^x = k\int_0^1 \frac{x^3}{2}\, dx = k \cdot \frac{x^4}{8}\Big|_0^1$$

$$\Rightarrow \quad k = 8$$




$$P(X+Y>1) = \int_{x=\frac{1}{2}}^{x=1}\int_{y=1-x}^{y=x} 8xy\, dy\, dx = 8\int_{\frac{1}{2}}^{1} x \cdot \frac{y^2}{2}\Big|_{1-x}^{x} dx$$

$$= 4\int_{\frac{1}{2}}^{1} x\left[x^2 - (1-x)^2\right] dx = 4\int_{\frac{1}{2}}^{1} x\left[2x-1\right] dx$$

$$= 4\int_{\frac{1}{2}}^{1}\left[2x^2 - x\right] dx = 4\left[\frac{2x^3}{3} - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = 4\left[\left(\frac{2}{3}-\frac{1}{2}\right) - \left(\frac{1}{12}-\frac{1}{8}\right)\right]$$

$$= 4\left[\frac{1}{6} + \frac{1}{24}\right] = 4 \cdot \frac{5}{24} = \boxed{\frac{5}{6}}$$

# Independent Random variables

**Definition:** $X, Y$ are independent if

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y) \quad \forall x, y$$

or
$$F(x, y) = F_X(x) \cdot F_Y(y).$$

**Theorem:** $X, Y$ are independent if & only if
$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \forall x, y.$$

**Theorem:** Suppose $X$ & $Y$ are continuous random variables & let $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx.$$

If $X, Y$ are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z - x)\, dx.$$

**Proof:**
$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f(x, y)\, dy\, dx.$$

Let $y = w - x$. Then,

$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z} f(x, w - x)\, dw\, dx.$$

Differentiating both sides w.r.t $z$ yields the desired result.

Ex: Let $X, Y \sim N(0,1)$ be independent.
Find the density function of $Z = X + Y$.

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}} \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-[x^2 + (z-x)^2]} \, dx$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{[2x^2 - 2zx]}{2}} \, dx$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-(x^2 - zx)} \, dx$$

Now complete the square:

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-(x - \frac{z}{2})^2 + \frac{z^2}{4}} \, dx$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-(x - \frac{z}{2})^2} \, dx$$

Let $u = \left(x - \frac{z}{2}\right)\sqrt{2}$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \cdot \frac{du}{\sqrt{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \cdot \frac{1}{\sqrt{2}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \, du}_{= \sqrt{2\pi}}$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} \implies Z \sim N(0, 2).$$

## Conditional Distributions

The conditional distribution function of $X$ given $Y = y$ is denoted by $F_{X|Y}(\cdot \mid y)$ & is defined by

$$F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y) = \int_{-\infty}^{x} \frac{f(u, y)\, du}{f_Y(y)}$$

for every $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

$$f_{X|Y}(x \mid y) := \frac{f(x \mid y)}{f_Y(y)}$$ is called the conditional density function of $X$ given $Y = y$.

$f_{X|Y}(\cdot \mid y)$ is a valid density function since

$$\int_{-\infty}^{\infty} \frac{f(x, y)}{f_Y(y)}\, dx = \frac{1}{f_Y(y)} \underbrace{\int_{-\infty}^{\infty} f(x, y)\, dx}_{f_Y(y)} = 1.$$

Ex: $f(x,y) = \begin{cases} 6(1-y) & ; \ 0 \le x \le y \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

Find $P(Y \le \frac{1}{2} \mid X = \frac{1}{4})$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{y=x}^{y=1} 6(1-y)\,dy = 6\left[y - \frac{y^2}{2}\right]_x^1$$

$$= 6\left(\frac{1}{2} - x + \frac{x^2}{2}\right)$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6(1-y)}{6\left(\frac{1}{2} - x + \frac{x^2}{2}\right)}$$

$$= \frac{1-y}{\frac{1}{2} - x + \frac{x^2}{2}} \quad ; \quad 0 \le x \le y \le 1$$

$$f_{Y|X}(y|\tfrac{1}{4}) = \frac{1-y}{\frac{1}{2} - \frac{1}{4} + \frac{1}{32}} \quad ; \quad \frac{1}{4} \le y \le 1$$

$$f_{Y|X}(y|\tfrac{1}{4}) = \frac{32}{9}(1-y) \quad ; \quad \frac{1}{4} \le y \le 1$$

$$P(Y \le \tfrac{1}{2} \mid X = \tfrac{1}{4}) = \int_{-\infty}^{\frac{1}{2}} f_{Y|X}(y|\tfrac{1}{4})\,dy = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{32}{9}(1-y)\,dy$$

$$= \frac{32}{9} \left[ y - \frac{y^2}{2} \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{32}{9} \left[ \left( \frac{1}{2} - \frac{1}{8} \right) - \left( \frac{1}{4} - \frac{1}{32} \right) \right]$$

$$= \frac{32}{9} \left[ \frac{3}{8} - \frac{7}{32} \right] =$$

$$= \frac{32}{9} \cdot \frac{5}{32} = \boxed{\frac{5}{9}}$$