Math 672 Lecture 5

Recall:

- A basis for V is a linearly independent spanning set for V.
- Fundamental facts about bases:
  - If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then each $u \in V$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$.
  - Assuming V is nontrivial and linearly independent,
    - every linearly independent set in V is contained in a basis;
    - every spanning set for V contains a basis.

Here's another fundamental fact about bases:

Theorem: Any two bases for V contain the same number of vectors.

Proof: We have seen that every spanning set for V contains at least as many vectors as any linearly independent set. Let B_1 and B_2 be bases for V. Then the number of elements of B_1 (a linearly independent set) is less than or equal to the number of elements of B_2 (a spanning set). But exactly the same reasoning shows that the number of elements of B_2 is less than or equal to the number of elements of B_1. Therefore

$$|B_1| \leq |B_2| \text{ and } |B_2| \leq |B_1| \Rightarrow |B_1| = |B_2|. //$$

<u>Definition</u>: Let V be a finite-dimensional vector space. If V is nontrivial, then the <u>dimension</u> of V is the number of elements in a basis for V. If V is trivial, then $\dim(V) = 0$.

<u>Examples</u>

- $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a basis for F^4, so $\dim(F^4) = 4$. Similarly, $\dim(F^n) = n$ $\forall n \in \mathbb{Z}^+$.
- $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{P}_n$, so $\dim(\mathcal{P}_n) = n+1$ $\forall n \in \mathbb{Z}^+$.

<u>Theorem</u>: If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$.

<u>Proof</u>: Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for U. Then $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set in V and hence is contained in some basis $\{v_1, v_2, \ldots, v_n\}$ for V. It follows that $\dim(U) = k \leq n = \dim(V)$. //

To prove that a given set is a basis, we must prove linear independence and spanning. The following theorem reduces the work in proving that a set is a basis, provided we know the dimension of the space.

<u>Theorem</u>: Let V be an n-dimensional vector space (where $n \in \mathbb{Z}^+$).

1. If $\{v_1, v_2, \ldots, v_n\} \subseteq V$ is linearly independent, then it is a basis for V.
2. If $\{v_1, v_2, \ldots, v_n\} \subseteq V$ spans V, it is a basis for V.

<u>Proof</u>: We know that every basis of V contains exactly n vectors

1. If $\{v_1, v_2, \ldots, v_n\} \subseteq V$ is linearly independent, then we know that is a subset of some basis. But if $\{v_1, v_2, \ldots, v_n\}$ is a proper subspace of that basis, then the basis contains more than n vectors, which is impossible. Thus $\{v_1, v_2, \ldots, v_n\}$ itself must be a basis for V.

2. If $\{v_1, v_2, \ldots, v_n\} \subseteq V$ spans V, then a subset of $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. But if the basis is a proper subset of $\{v_1, v_2, \ldots, v_n\}$, then it contains fewer than n vectors, which is impossible. Thus $\{v_1, v_2, \ldots, v_n\}$ itself must be a basis for V. //

<u>Example</u>: Consider $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

To prove that $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$, we take advantage of the fact that $\dim(\mathbb{R}^3) = 3$, and we have 3 vectors, so we don't have to check both linear independence and spanning, but only one. Linear independence is easier:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\iff \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_3 \\ 2\alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 &= 0 \\ \alpha_1 - \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

At this point, we already know that the only sol'n is $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Longleftrightarrow \quad \begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

Thus $\{v_1, v_2, v_3\}$ is linearly independent and hence a basis.

<u>Theorem</u>: Let V be a finite-dimensional vector space and let U_1, U_2 be subspaces of V. Then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

<u>Proof:</u> Let $\{u_1, \dots, u_j\}$ be a basis for $U_1 \cap U_2$ (in the special case that $U_1 \cap U_2 = \{0\}$, we replace this by the empty set and take $j=0$).

Since $U_1 \cap U_2$ is a subspace of U_1, we can extend $\{u_1, \dots, u_j\}$ to a basis $\{u_1, \dots, u_j, v_1, \dots, v_k\}$ of U_1, and since $U_1 \cap U_2$ is a subspace of U_2, we can extend $\{u_1, \dots, u_j\}$ to a basis $\{u_1, \dots, u_j, w_1, \dots, w_\ell\}$ of U_2. Then

$$\dim(U_1 \cap U_2) = j,$$

$$\dim(U_1) = j + k,$$

$$\dim(U_2) = j+l$$

$$\Rightarrow \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$

$$= (j+k) + (j+l) - j = j+k+l.$$

It suffices to prove that $\{u_1, \ldots, u_j, v_1, \ldots, v_k, w_1, \ldots, w_l\}$ is a basis for $U_1 + U_2$, since then

$$\dim(U_1 + U_2) = j+k+l = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Let us first prove that $\{u_1, \ldots, u_j, v_1, \ldots, v_k, w_1, \ldots, w_l\}$ is linearly independent. Suppose

$$(*) \quad \alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k + \gamma_1 w_1 + \cdots + \gamma_l w_l = 0,$$

and note that

$$\alpha_1 u_1 + \cdots + \alpha_j u_j \in U_1 \cap U_2,$$

$$\beta_1 v_1 + \cdots + \beta_k v_k \in U_1,$$

$$\gamma_1 w_1 + \cdots + \gamma_l w_l \in U_2.$$

Since $U_1 \cap U_2 \subseteq U_1$, it follows that

$$\alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k \in U_1.$$

But $(*)$ implies that

$$\alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k = -\gamma_1 w_1 - \cdots - \gamma_l w_l \in U_2.$$

Thus

$$\alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k \in U_1 \cap U_2$$

$$\Rightarrow \alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k = \delta_1 u_1 + \cdots + \delta_j u_j \text{ for some } \delta_1, \ldots, \delta_j \in F$$

$$\Rightarrow \delta_1 u_1 + \cdots + \delta_j u_j + \gamma_1 w_1 + \cdots + \gamma_\ell w_\ell = 0$$

$$\Rightarrow \delta_1 = \cdots = \delta_j = \gamma_1 = \cdots = \gamma_\ell = 0 \quad (\text{since } \{u_1, \ldots, u_j, w_1, \ldots, w_\ell\} \text{ is linearly independent})$$

$$\Rightarrow \alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k = 0$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_j = \beta_1 = \cdots = \beta_k = 0 \quad (\text{since } \{u_1, \ldots, u_j, v_1, \ldots, v_k\} \text{ is linearly independent}).$$

Thus we have proven that $\{u_1, \ldots, u_j, v_1, \ldots, v_k, w_1, \ldots, w_\ell\}$ is linearly independent.

Now we prove that $\{u_1, \ldots, u_j, v_1, \ldots, v_k, w_1, \ldots, w_\ell\}$ spans $U_1 + U_2$.

Let $w \in U_1 + U_2$; then $w = u + v$ for some $u \in U_1$, $v \in U_2$.

There exist $\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k \in F$ such that

$$u = \alpha_1 u_1 + \cdots + \alpha_j u_j + \beta_1 v_1 + \cdots + \beta_k v_k$$

and $\gamma_1, \ldots, \gamma_j, \delta_1, \ldots, \delta_\ell \in F$ such that

$$v = \gamma_1 u_1 + \cdots + \gamma_j u_j + \delta_1 w_1 + \cdots + \delta_\ell w_\ell.$$

But then

$$w = u + v = (\alpha_1 + \gamma_1) u_1 + \cdots + (\alpha_j + \gamma_j) u_j + \beta_1 v_1 + \cdots + \beta_k v_k + \delta_1 w_1 + \cdots + \delta_\ell w_\ell$$

$\Rightarrow\ w \in \operatorname{span}(u_1, \dots, u_j, v_1, \dots, v_k, w_1, \dots, w_\ell)$.

Thus $\{u_1, \dots, u_j, v_1, \dots, v_k, w_1, \dots, w_\ell\}$ spans $U_1 + U_2$, and the proof is complete. //