Math 672 Lecture 29

Recall: Let V be a vector space over F, let $T \in \mathcal{L}(V)$, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. For each $\ell = 1, \ldots, k$, there exists m_ℓ satisfying

Note: $m_\ell \geq 1$ because λ_ℓ is an eigenvalue.

$$1 \leq m_\ell \leq n = \dim(V), \quad \mathcal{N}((T-\lambda_\ell I)^j) \subsetneq \mathcal{N}((T-\lambda_\ell I)^{j+1}), \; j = 0, \ldots, m-1,$$

$$\mathcal{N}((T-\lambda_\ell I)^j) = \mathcal{N}((T-\lambda_\ell I)^{m_\ell}) \quad \forall j \geq m_\ell.$$

We call $G(\lambda_\ell, T) = \mathcal{N}((T-\lambda_\ell I)^{m_\ell})$ the <u>generalized eigenspace</u> of T corresponding to λ_ℓ.

<u>Lemma</u>: Let V be a vector space over F, let $T \in \mathcal{L}(V)$, let $\lambda \in F$, and let $m \in \mathbb{Z}^+$. Then $\mathcal{N}((T-\lambda I)^m)$ is invariant under T.

<u>Proof:</u> Let $v \in \mathcal{N}((T-\lambda I)^m)$. Then

$$\begin{aligned}(T-\lambda I)^m(T(v)) &= T((T-\lambda I)^m(v)) \quad \text{(since polynomials in } T \text{ commute)}\\ &= T(0)\\ &= 0,\end{aligned}$$

and hence $T(v) \in \mathcal{N}((T-\lambda I)^m)$. //

<u>Theorem</u>: Let V be a vector space over F, let $T \in \mathcal{L}(V)$, let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T, and let $v_1, \ldots, v_k$ be generalized eigenvalues of T corresponding to $\lambda_1, \ldots, \lambda_k$, respectively. Then $\{v_1, \ldots, v_k\}$ is linearly independent.

<u>Proof:</u> Suppose $\alpha_1, \ldots, \alpha_k \in F$ satisfy

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$$

and let j satisfy $1 \le j \le k$. Since j is arbitrary, it suffices to prove that $\alpha_j = 0$. Let $t \in \mathbb{Z}^+$ be the largest nonnegative integer such that

$$(T - \lambda_j I)^t (v_j) \neq 0.$$

It follows that, if $w = (T - \lambda_j I)^t (v_j)$, then $w \neq 0$ and

$$(T - \lambda_j I)(w) = 0 \Rightarrow T(w) = \lambda_j w.$$

Thus w is an eigenvector of T corresponding to λ_j, and

$$(T - \lambda_\ell I)^n (v_j) = (\lambda_j - \lambda_\ell)^n v_j \neq 0 \quad \forall\, \ell \neq j$$

(here $n = \dim(V)$). Note that

$$(T - \lambda_\ell I)^n (v_\ell) = 0 \quad \forall\, \ell = 1, 2, \ldots, k$$

(since $G(\lambda_\ell, T) = \mathcal{N}((T - \lambda_\ell)^{m_\ell}) = \mathcal{N}((T - \lambda_\ell I)^n)$ because $m_\ell \le n$).

We now have

$$\left((T-\lambda_j I)^t \prod_{\substack{l=1 \\ l\neq j}}^{k} (T-\lambda_l I)^n \right) \left(\sum_{i=1}^{k} \alpha_i v_i \right)$$

$$= (T-\lambda_j I)^t \left(\prod_{\substack{l=1 \\ l\neq j}}^{k} (\lambda_l - \lambda_j)^n \alpha_j v_j \right) \quad \left(\text{since } (T-\lambda_l I)^n (v_l) = 0 \ \forall l\right)$$

$$= \alpha_j \prod_{\substack{l=1 \\ l\neq j}}^{k} (\lambda_l - \lambda_j)^n \, (T-\lambda_j I)^t (v_j)$$

$$= \alpha_j \prod_{\substack{l=1 \\ l\neq j}}^{k} (\lambda_l - \lambda_j)^n \, w$$

and hence

$$\sum_{j=1}^{k} \alpha_j v_j = 0 \implies \alpha_j = 0 \quad (\text{since } \lambda_l - \lambda_j \neq 0 \text{ for } l \neq j \text{ and } w \neq 0).$$

This completes the proof. //

<u>Definition</u>: Let V be a vector space over F and let $T \in \mathcal{L}(V)$. We say that T is <u>nilpotent</u> iff there exists $k \in \mathbb{Z}^+$ such that $T^k = 0$.

<u>Theorem</u>: Let V be a nontrivial finite-dimensional vector space over F, let $T \in \mathcal{L}(V)$ be nilpotent, and let $k \in \mathbb{Z}^+$ be the smallest positive integer such that $T^k = 0$. Then:

- 0 is an eigenvalue of T.
- T has no nonzero eigenvalues.
- If $v \in V$ satisfies $T^{k-1}(v) \neq 0$ and $T^{(k)}(v) = 0$, then $\{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly independent.

<u>Proof</u>: Since k is the smallest positive integer such that $T^k = 0$, there exists $v \in V$ such that $T^{k-1}(v) \neq 0$. But then

$$T\left(T^{k-1}(v)\right) = T^k(v) = 0 \text{ and } T^{k-1}(v) \neq 0,$$

which shows that 0 is an eigenvalue of T with eigenvector $T^{k-1}(v)$.

If λ is any eigenvalue of T with eigenvector u, then

$$\begin{aligned} T(u) = \lambda u &\Rightarrow T^k(u) = \lambda^k u \\ &\Rightarrow \lambda^k u = 0 \quad (\text{since } T^k = 0) \\ &\Rightarrow \lambda^k = 0 \quad (\text{since } u \neq 0) \end{aligned}$$

$$\Rightarrow \lambda = 0.$$

Finally, suppose $v \in V$ has the property that $T^{k-1}(v) \neq 0$.

If $\{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly dependent, then there exist $l \in \mathbb{Z}^+$ such that $\{v, T(v), \ldots, T^{l-1}(v)\}$ is linearly independent and

$$T^l(v) \in \operatorname{span}(v, T(v), \ldots, T^{l-1}(v)).$$

Say

$$T^l(v) = \alpha_0 v + \alpha_1 T(v) + \cdots + \alpha_{l-1} T^{l-1}(v),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ are not all 0 (since $T^l(v) \neq 0$).

It follows that

$$T^{k-1}(T^l(v)) = \alpha_0 T^{k-1}(v) + \alpha_1 T^k(v) + \cdots + \alpha_{l-1} T^{k+l-2}(v)$$

$$\Rightarrow 0 = \alpha_0 T^{k-1}(v)$$

$$\Rightarrow \alpha_0 = 0$$

$$\Rightarrow T^l(v) = \alpha_1 T(v) + \cdots + \alpha_{l-1} T^{l-1}(v)$$

$$\Rightarrow T^{k-2}(T^l(v)) = \alpha_1 T^{k-1}(v) + \cdots + \alpha_{l-1} T^{k+l-3}(v)$$

$$\Rightarrow 0 = \alpha_1 T^{k-1}(v)$$

$$\Rightarrow \alpha_1 = 0$$

Continuing in this fashion, we can prove that $\alpha_j = 0$ for $j = 0, 1, \ldots, l-1$, a contradiction. Thus $\{v, T(v), \ldots, T^{k-1}(v)\}$ must be linearly independent. //

Lemma: Let V be a vector space over F, let $T \in \mathcal{L}(V)$, and let $\lambda \in F$ be an eigenvalue of T. Then $(T-\lambda I)\big|_{G(\lambda,T)}$ is nilpotent.

Proof: There exists $m \in \mathbb{Z}^+$ such that

$$\mathcal{N}((T-\lambda I)^j) = \mathcal{N}((T-\lambda I)^m) \quad \forall j \geq m.$$

By definition, $v \in G(\lambda,T)$ iff there exists $j \in \mathbb{Z}^+$ such that $v \in \mathcal{N}((T-\lambda I)^j)$. But $\mathcal{N}((T-\lambda I)^j) \subseteq \mathcal{N}((T-\lambda I)^m)$ $\forall j \in \mathbb{Z}^+$, and hence

$$v \in \mathcal{N}((T-\lambda I)^m) \quad \forall v \in G(\lambda,T)$$

$$\iff (T-\lambda I)^m(v) = 0 \quad \forall v \in G(\lambda,T)$$

$$\iff (T-\lambda I)^m\big|_{G(\lambda,T)} = 0$$

$$\iff (T-\lambda I)\big|_{G(\lambda,T)} \text{ is nilpotent.} \; //$$

Let's consider a special case: $T \in \mathcal{L}(V)$, $\lambda \in F$ is an eigenvalue of T, $m \geq 1$ is the smallest positive integer such that $\mathcal{N}((T-\lambda I)^j) = \mathcal{N}((T-\lambda I)^m)$ $\forall j \geq m$ (so that $G(\lambda,T) = \mathcal{N}((T-\lambda I)^m)$), and $\dim(G(\lambda,T)) = m$ (this last condition need not hold, which is why this is a special case). From above, we know that $G(\lambda,T)$ is invariant under T and there exists $v \in G(\lambda,T)$,

$v \neq 0$, such that $\{v, (T-\lambda I)(v), \ldots, (T-\lambda I)^{m-1}(v)\}$ is linearly independent. Thus $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$, where $v_1 = (T-\lambda I)^{m-1}(v)$, $v_2 = (T-\lambda I)^{m-2}(v), \ldots, v_m = v$ is a basis for $G(\lambda, T)$.

Define $S \in \mathcal{L}(G(\lambda,T))$ by $S = T|_{G(\lambda,T)}$. What is $\mathcal{M}_{\mathcal{B},\mathcal{B}}(S)$?

Recall that we just need to identify $T(v_j)$ for each j.

We have

$$(T-\lambda I)(v_1) = (T-\lambda I)^m(v) = 0 \Rightarrow T(v_1) = \lambda v_1,$$

$$(T-\lambda I)(v_2) = (T-\lambda I)^{m-1}(v) = v_1 \Rightarrow T(v_2) = \lambda v_2 + v_1$$

$$(T-\lambda I)(v_3) = (T-\lambda I)^{m-2}(v) = v_2 \Rightarrow T(v_3) = \lambda_3 v_3 + v_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$(T-\lambda I)(v_m) = (T-\lambda I)(v) = v_{m-1} \Rightarrow T(v_m) = \lambda v_m + v_{m-1}$$

It follows that $\mathcal{M}_{\mathcal{B},\mathcal{B}}(S) = A \in F^{m \times m}$, where

$$A_{ij} = \begin{cases} \lambda & \text{if } j = i, \\ 1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

that is

$$A = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \ddots \\ & & & & \lambda \end{bmatrix}.$$