Math 600 Lecture 39

<u>Theorem</u>: Let E be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$, let $f: E \to \mathbb{R}^n$ be differentiable on E, and assume that Df is continuous on E. Suppose that $(x_0, y_0) \in E$ satisfies

$$f(x_0, y_0) = 0,$$

$$D_y f(x_0, y_0) \text{ is nonsingular (invertible).}$$

Then exist open sets $U \subset \mathbb{R}^m$, $V \in \mathbb{R}^n$ such that

$$x_0 \in U,\ y_0 \in V,\ U \times V \subset E$$

and $\psi: U \to V$ such that

$$f(x, \psi(x)) = 0 \quad \forall x \in U.$$

Moreover, for all $x \in U$, $y = \psi(x)$ is the only point in V satisfying

$$f(x, y) = 0.$$

Finally, ψ is continuously differentiable and $\psi'(x) = -D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))$.

<u>Proof</u> (continued): We have shown that there exist open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ and $\psi: U \to V$ such that

$$x_0 \in U,\ y_0 \in V,\ U \times V \subset E,\ f(x, \psi(x)) = 0 \ \forall x \in U,$$

and $y = \psi(x)$ is the only solution of $f(x,y) = 0$ that lies in V. We must show that ψ is continuously differentiable and that $\psi'(x) = -D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))$ for all $x \in U$.

Note that we can assume that $\overline{U} \times \overline{V} \subset E$ (by reducing ε' and δ', if necessary, earlier in the proof) and hence that $\|D_x f(x,y)\|$ is uniformly bounded for $(x,y) \in U \times V$. This will be needed below.

We begin by showing that ψ is Lipschitz continuous on U: There exist $C > 0$ such that

$$\|\psi(x_1) - \psi(x_2)\| \leq C\|x_1 - x_2\| \quad \forall x_1, x_2 \in U.$$

Let $x_1, x_2 \in U$. Then

$$\psi(x_1) - \psi(x_2) = \varphi(x_1, \psi(x_1)) - \varphi(x_2, \psi(x_2))$$

$$= \varphi(x_1, \psi(x_1)) - \varphi(x_2, \psi(x_1)) + \varphi(x_2, \psi(x_1)) - \varphi(x_2, \psi(x_2))$$

$$\Rightarrow \|\psi(x_1) - \psi(x_2)\| \leq \|\varphi(x_1, \psi(x_1)) - \varphi(x_2, \psi(x_1))\| + \|\varphi(x_2, \psi(x_1)) - \varphi(x_2, \psi(x_2))\|.$$

Now, for any $y \in V$ (including $y = \psi(x_1)$),

$$\varphi(x_1, y) - \varphi(x_2, y) = y - D_y f(x_0, y_0)^{-1} f(x_1, y) - y + D_y f(x_0, y_0)^{-1} f(x_2, y)$$

$$= D_y f(x_0, y_0)^{-1} (f(x_2, y) - f(x_1, y))$$

$$= D_y f(x_0, y_0)^{-1} \int_0^1 D_x f(x_1 + t(x_2 - x_1), y)(x_2 - x_1)\, dt$$

$$= \left(\int_0^1 D_y f(x_0, y_0)^{-1} D_x f(x_1 + t(x_2 - x_1), y)\, dt \right) (x_2 - x_1)$$

$$\Rightarrow \|\varphi(x_1, y) - \varphi(x_2, y)\| \leq \left\| \int_0^1 D_y f(x_0, y_0)^{-1} D_x f(x_1 + t(x_2 - x_1), y)\, dt \right\| \|x_2 - x_1\|$$

$$\leq \int_0^1 \|D_y f(x_0, y_0)^{-1}\| \|D_x f(x_1 + t(x_2 - x_1), y)\|\, dt \; \|x_2 - x_1\|$$

$$\leq C' \|x_2 - x_1\|$$

(note that I am using a version of the triangle inequality for integrals that I haven't formally proven before; also, as noted above, $\|D_x f(x,y)\|$ is uniformly bounded for $(x,y) \in U \times V$).

We previously proved that $\varphi(x,\cdot)$ is a contraction, and hence

$$\|\varphi(x_2, \psi(x_1)) - \varphi(x_2, \psi(x_2))\| \leq \lambda \|\psi(x_1) - \psi(x_2)\| \quad (0 < \lambda < 1).$$

Therefore,

$$\|\psi(x_1) - \psi(x_2)\| \leq \|\varphi(x_1, \psi(x_1)) - \varphi(x_2, \psi(x_1))\| + \|\varphi(x_2, \psi(x_1)) - \varphi(x_2, \psi(x_2))\|$$

$$\leq C' \|x_1 - x_2\| + \lambda \|\psi(x_1) - \psi(x_2)\|$$

$$\Rightarrow (1-\lambda) \|\psi(x_1) - \psi(x_2)\| \leq C' \|x_1 - x_2\|$$

$$\Rightarrow \|\psi(x_1) - \psi(x_2)\| \leq \frac{C'}{1-\lambda} \|x_1 - x_2\| = C \|x_1 - x_2\|, \quad C = \frac{C'}{1-\lambda}.$$

Now we prove that ψ is differentiable at $x \in U$:

$$f(x+p, \psi(x+p)) = 0 \quad \forall p \in \mathbb{R}^n \text{ sufficiently small}$$

$$\Rightarrow f(x, \psi(x)) + Df(x, \psi(x))(p, \psi(x+p) - \psi(x)) + o(\|(p, \psi(x+p) - \psi(x))\|) = 0.$$

Note that

$$\|(p, \psi(x+p) - \psi(x))\| = \sqrt{\|p\|^2 + \|\psi(x+p) - \psi(x)\|^2}$$

$$\leq \sqrt{\|p\|^2 + C^2 \|p\|^2} \quad \text{(since } \psi \text{ is Lipschitz continuous)}$$

$$= \sqrt{1+C^2} \|p\|$$

and hence

$$o(\|(p, \psi(x+p)-\psi(p)\|) = o(\|p\|).$$

Also,

$$f(x, \psi(x)) = 0,$$

$$Df(x, \psi(x))(p, \psi(x+p)-\psi(x)) = D_x f(x, \psi(x))p + D_y f(x, \psi(x))(\psi(x+p)-\psi(x))$$

Thus, we obtain

$$D_x f(x, \psi(x))p + D_y f(x, \psi(x))(\psi(x+p)-\psi(x)) + o(\|p\|) = 0$$

$$\Rightarrow D_y f(x, \psi(x))(\psi(x+p)-\psi(x)) = -D_x f(x, \psi(x))p + o(\|p\|)$$

$$\Rightarrow \psi(x+p) = \psi(x) - D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))p + D_y f(x, \psi(x))^{-1} o(\|p\|)$$

$$= \psi(x) - D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))p + o(\|p\|).$$

This proves that ψ is differentiable at x and that

$$D\psi(x)p = -D_y f(x, \psi(x))^{-1} D_x f(x, \psi(x))p. //$$

The inverse function theorem

Suppose $f : E \to \mathbb{R}^n$, where $E \subset \mathbb{R}^n$ is open, f is differentiable at $x_0 \in E$, and $Df(x)$ is invertible. Let us define $F : \mathbb{R}^n \times E \to \mathbb{R}^n$ by

$$F(y, x) = f(x) - y.$$

Then, if $y_0 = f(x_0)$, we have

$$F(y_0, x_0) = 0,$$

$D_x F(y_0, x_0)$ is nonsingular.

Hence, the implicit function theorem applies, and there exist open sets U, V in $\mathbb{R}^n$ and $\psi: U \to V$ such that

$$y_0 \in U,\ x_0 \in V,\ V \subset E,\ F(y, \psi(y)) = 0 \quad \forall y \in U,$$

and $x = \psi(y)$ is the unique solution of $F(y, x) = 0$ that lies in V. But

$$F(y, \psi(y)) = 0 \iff y - f(\psi(y)) = 0 \iff f(\psi(y)) = y.$$

Thus

$$f(\psi(y)) = y \quad \forall y \in U.$$

Note that f maps $\psi(U)$ onto U, and $f|_{\psi(U)}$ is injective. Thus $f|_{\psi(U)}$ is invertible. Moreover,

$$\psi(U) = (f|_{\psi(U)})^{-1}(U),$$

which shows that $\psi(U)$ is open. We can thus redefine V to be $\psi(U)$; then $\psi: U \to V$ is the inverse of $f|_V$.

The implicit function theorem guarantees that ψ is C^1. We can compute $D\psi$ by implicit differentiation:

$$f(\psi(y)) = y$$

$$\Rightarrow Df(\psi(y))\, D\psi(y) = I$$

$$\Rightarrow D\psi(y) = Df(\psi(y))^{-1}.$$

Example: Define $f:\mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x) = \begin{bmatrix} x_1x_2 + x_2^2 \\ x_1^2 + x_1x_2 \end{bmatrix}.$$

Then

$$f'(x) = \begin{bmatrix} x_2 & x_1+2x_2 \\ 2x_1+x_2 & x_1 \end{bmatrix}$$

If $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$f'(x_0) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},$$

which is clearly nonsingular. What is $(f^{-1})'(y_0)$, where

$$y_0 = f(x_0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}?$$

Answer:

$$(f^{-1})'(y_0) = f'(x_0)^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix}.$$

(Note: We should write $(f|_U)^{-1}$, not f^{-1}! There is no reason to think that f itself is invertible. Also, note that the theorem gives no information about how big the set U is.)