Chapter 4: Formal solution of boundary-condition problems in electrostatics (25 Sep 2021)



## A. Formal solution of boundary-condition problems in electrostatics.

We have already made great progress from our starting point, Coulomb's force law between two point charges. If our only goal were to develop Maxwell's Equations, we could have stopped at Ch. 2. If we only cared about empty space, we could have stopped at Eq. (2.9), which gives $\phi(\vec{r})$ everywhere as a weighted integral over a charge density $\rho(\vec{r})$.

However, nearly all applications of E&M involve boundaries, which imply *finite* regions of space. Green's monumental contribution was to develop a formalism for calculating $\phi(\vec{r})$ not only for $\rho(\vec{r})$ in bounded regions but also when $\phi(\vec{r})$ or its normal derivative is specified on a finite boundary. The formal solution of this qualitatively different problem and the general implications of the solution are the subjects of Ch. 4. This Green-function formalism leads to various basic theorems, such as the mean-value, Earnshaw's, and Thomson's theorems, and allows us to draw some fairly broad conclusions with unanticipated applications. For example, Earnshaw's Theorem explains why highly ionic materials like NaCl lack structural rigidity. However, we will also consider the microscopic perspective, where all charges, including interface charges, are mapped back into empty space, thus closing the loop with Eq. (2.9). These topics continue into Ch. 5, and provide an interesting alternative view of electrostatics that is not found in undergraduate texts, or for that matter in Jackson.

The basis for these advances is Green's Theorem. Like Gauss, George Green came out of nowhere. He had only one year of formal schooling, which ended at age 8 when he went to work in his father's mill. However, he liked mathematics, and when he was 30, he joined the Nottingham Subscription Library where he had access to information about the subject. Five years later he attempted to publish a short work "An Essay on the Application of Mathematical Analysis to Electricity and Magnetism", where he presented his results. Since he was not affiliated with an institution of higher learning, he could not get it published through standard channels so he published it himself. Only 51 copies were sold. In 1841 William Thomson (Lord Kelvin) learned of Green's work, and finally obtained a copy in 1845, two years after Green's death. Thomson had the book republished as a series of articles, and posthumous credit to Green was established.

As discussed in Ch. 3 for one-dimensional configurations, Green's Theorem converts Poisson's Equation, Eq. (2.13), to an integral expression, eliminates the operator $\nabla^2$, introduces boundary conditions, and eliminates the need to use solutions of the homogeneous equation to satisfy boundary conditions. All four aspects represent huge steps forward. Incorporating boundary conditions via a surface integral introduces these conditions directly. Converting $\nabla^2$ to an integral relation eliminates the need to evaluate a three-dimensional differential equation recursively by successive approximations. Thus, as will become apparent in this and the next three chapters, electrostatics owes a huge debt of gratitude to Green.

In the following, rather than simply derive equations, I also indicate aspects of these equations that are important, and why. The intent is to provide more insight into the derivations, the final expressions, and what they accomplish. These insights assist in achieving our main goals: get the math right, then extract all the physics from the math.

B. Green's First and Second Identities; Green's Theorem.

As noted in Ch. 2, Green functions provide solutions to inhomogeneous equations such as Poisson's Equation $\nabla^2\phi(\vec{r}) = -4\pi\rho(\vec{r})$ by finding a function $G(\vec{r},\vec{r}\,')$ such that

$$\nabla^2_{\vec{r}} G(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,'). \tag{4.1}$$

If such a function exists, then

$$\phi(\vec{r}) = \int d^3r'\,\rho(\vec{r}\,')G(\vec{r},\vec{r}\,') \tag{4.2}$$

as can be verified trivially, noting that an operator that works on $\vec{r}$ treats $\vec{r}\,'$ as a constant. Our objective is to turn Eq. (4.2) into something that can be used with finite regions, and in particular finite regions where the potential or its normal derivative (normal component of the electric field) is specified on the boundary of the region. We can guess that Gauss' Theorem will play a central role, because it expresses a volume integral as an equivalent surface integral.

This is the problem solved by Green. Starting with Gauss' Theorem,

$$\int_V d^3r'\nabla\cdot\vec{A} = \int_S d^2r'\hat{n}\cdot\vec{A}, \tag{4.3}$$

Green's insight was to let $\vec{A}$ take the special form $\phi(\vec{r}\,')\nabla_{\vec{r}'}G(\vec{r}\,')$. At this stage $\phi(\vec{r}\,')$ and $G(\vec{r}\,')$ should be interpreted as any two functions $f_1(\vec{r}\,')$ and $f_2(\vec{r}\,')$ whose only constraint is that they are differentiable. However, they will later be identified as the scalar potential and Green function, so we anticipate this development by using the notation $\phi$ and $G$ for $f_1(\vec{r}\,')$ and $f_2(\vec{r}\,')$. At this stage everything is a function of the dummy variable of integration $\vec{r}\,'$; the location $\vec{r}$ of the observer has yet to appear. Hence, to simplify notation, until further notice $\vec{r}\,'$ is understood. Substituting $\phi\nabla G$ in Eq. (4.3), we arrive at

$$\int_V d^3r'\nabla\cdot\left(\phi\nabla G\right)=\int_V d^3r'\left(\nabla\phi\cdot\nabla G+\phi\nabla^2 G\right) \tag{4.4a}$$

$$=\int_S d^2r'\phi\frac{\partial G}{\partial n'}. \tag{4.4b}$$

The notation

$$\hat{n}\cdot\nabla G=\frac{\partial G}{\partial n'} \tag{4.5}$$

is standard. As a reminder, $\hat{n}'=\hat{n}(\vec{r}\,')$ is the unit normal vector of *S* at $\vec{r}\,'$ that points *away* from *V*.

Equations (4.4) are *Green's First Identity*, which we will use later to prove the *uniqueness theorem*. This theorem, covered in Sec. F, states that any solution of the scalar potential $\phi(\vec{r})$ that satisfies Laplace's Equation $\nabla^2\phi(\vec{r})=0$ and a prescribed set of boundary conditions in a charge-free region is unique; we need look no further.

Continuing, Green repeated the calculation with $\phi$ and *G* interchanged, and subtracted the result from Eqs. (4.4). The terms $\nabla\phi\cdot\nabla G=\nabla G\cdot\nabla\phi$ cancel, leaving

$$\int_V d^3r'\left(\phi\nabla^2 G-G\nabla^2\phi\right)=\int_S d^2r'\left(\phi\frac{\partial G}{dn'}-G\frac{\partial\phi}{dn'}\right). \tag{4.6}$$

This is *Green's Second Identity,* or *Green's Theorem*. Note that everything is still a function of the dummy integration variable $\vec{r}\,'$.

As written, Eq. (4.6) is only a self-consistency relation involving $\phi$, $G$, *V*, and *S*, but it has obvious possibilities. Recalling Poisson's Equation $\nabla^2\phi=-4\pi\rho$, the Laplacian of $\phi$ in Eq. (4.6) can be replaced with $-4\pi\rho(\vec{r}\,')$, eliminating one Laplacian operator. Now if we can find a function $G=G(\vec{r},\vec{r}\,')$ that satisfies Eq. (4.1), *and* there is only *one* such singularity in *V*, then the first term in the integral on the left side becomes

$$\phi\nabla^2_{\vec{r}'}G=-4\pi\phi(\vec{r}\,')\delta(\vec{r}-\vec{r}\,'). \tag{4.7}$$

This eliminates the second Laplacian operator, and the resulting integral is trivial. Summarizing and rearranging terms, we find

$$\phi(\vec{r})=\int_V d^3r'\rho(\vec{r}\,')G(\vec{r},\vec{r}\,')-\frac{1}{4\pi}\int_S d^2r'\left(\phi\frac{dG}{dn'}-G\frac{d\phi}{dn'}\right). \tag{4.8}$$

Note that the observer's location $\vec{r}$ has finally appeared.

This result is nothing short of astounding. It not only includes our previously discussed volume integral, Eq. (4.2), but also expresses $\phi(\vec{r})$ in terms of its value $\phi(\vec{r}_s)$ or its normal derivative $\hat{n}\cdot\nabla_{\vec{r}'}\phi=-\hat{n}\cdot\vec{E}(\vec{r}_s)$ on the closed surface *S*. We have thus not only recovered our formal solution involving a charge density $\rho(\vec{r}\,')$ for regions more restrictive than empty space, but also added the capability of specifying conditions on *S*.

Our goal now reduces to finding functions $G(\vec{r},\vec{r}\,')$ that satisfy Eq. (4.1), avoid possible contradictions between potential and field on *S*, and exhibit only one delta-function singularity in *V*. These are the constraints that define $G(\vec{r},\vec{r}\,')$. It is worth pointing out that $G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r})$, which follows most easily (although not rigorously) from Eq. (4.1) noting that $\delta(\vec{r}-\vec{r}\,') = \delta(\vec{r}\,'-\vec{r})$. A formal proof is given in App. 1.

C. Dirichlet and Neumann Green functions.

We have already had some experience with Green functions, proving in Ch. 3 that

$$\nabla^2_{\vec{r}'}\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}\,'). \tag{4.9}$$

This is the Green function of empty space, with *S* being the spherical shell of radius $R\to\infty$. At infinity both $\phi$ and $\hat{n}\cdot\nabla\phi$ vanish, so the surface integral in Eq. (4.8) contributes nothing. Hence the calculation of $\phi(\vec{r})$ in infinite free space reduces to the weighted average of $\rho(\vec{r}\,')$, as discussed in Ch. 2.

In finite spaces, attempts to specify both $\phi$ and $d\phi/dn'$ on *S* lead immediately to a contradiction, because as we find below the Green functions in the two cases are different. We have seen the one-dimensional equivalent already in the last chapter. Accordingly, the search for appropriate Green functions centers on eliminating this dual-specification conflict. Since up to now *G* is undefined except for Eq. (4.1), one easy way to eliminate the problem is to eliminate the surface term containing the field. We do this simply by requiring

$$G(\vec{r},\vec{r}_s\,') = 0\,, \tag{4.10}$$

where $\vec{r}_s\,'$ is on *S*. Then $d\phi/dn'$ is free to assume any value at all, eliminating any possibility of conflict. In fact it follows from the solution of the problem.

Equations (4.10) and (4.1) define the *Dirichlet* Green function $G(\vec{r},\vec{r}\,') = G_D(\vec{r},\vec{r}\,')$. The corresponding expression for $\phi(\vec{r})$ is

$$\phi(\vec{r}) = \int_V d^3r'\,\rho(\vec{r}\,')G_D(\vec{r},\vec{r}\,') - \frac{1}{4\pi}\int_S d^2r'\,\phi\frac{dG_D}{dn'}. \tag{4.11}$$

Boundary conditions have now entered explicitly: $\phi(\vec{r})$ is determined not only by $\rho(\vec{r})$ in *V*, but also by its value on *S*. The normal component $d\phi/dn'$ of the field at *S* emerges from the solution for $\phi$.

Given the similarity between Eq. (4.1) and the equation $\nabla^2_{\vec{r}}\phi(\vec{r}) = -4\pi q\delta(\vec{r}-\vec{r}_q)$ for the potential of a point charge *q* located at $\vec{r}_q$, one obvious way to find $G_D(\vec{r},\vec{r}\,')$ is to solve the electrostatics problem with *q* at $\vec{r}\,' = \vec{r}_q$ and the boundary *S* at zero potential, then set $q=1$ and replace $\vec{r}_q$ with $\vec{r}\,'$. This is the classic image-charge method of obtaining

$G_D(\vec{r},\vec{r}')$. We use this extensively in the next two chapters. Rigorously, this requires that

$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r}), \tag{4.12}$$

which has already been mentioned and is proven in Appendix 1.

The second alternative leads to the *Neumann* Green function $G(\vec{r},\vec{r}') = G_N(\vec{r},\vec{r}')$. Our success with the Dirichlet approach suggests that we set

$$\frac{\partial G_N(\vec{r},\vec{r}_s')}{\partial n'} = 0, \tag{4.13}$$

again requiring that only one singularity exist in *V*. While this appears to be an excellent idea, it is inconsistent with Eq. (4.1). To show this, consider the special case where $\phi(\vec{r}_s') = \phi_o$ is a constant. Then applying the divergence theorem in reverse to the corresponding surface integral in Eq. (4.8):

$$-\frac{1}{4\pi}\int_S d^2r'\phi_o\hat{n}\cdot\nabla_{\vec{r}'}G_N(\vec{r},\vec{r}') = -\frac{\phi_o}{4\pi}\int_V d^3r'\nabla^2_{\vec{r}'}G(\vec{r},\vec{r}'),$$

$$= \phi_o\int_V d^3r'\delta(\vec{r}-\vec{r}') = \phi_o \tag{4.14}$$

by Eq. (4.1). This is in direct conflict with Eq. (4.13). We must find an alternative.

But before proceeding further, let us back up a minute to appreciate what just happened. The surface integral on the left side of Eq. (4.14) certainly exists, and it is immediately evident is obtained if we set

$$\hat{n}\cdot\nabla_{\vec{r}'}G_N(\vec{r},\vec{r}')\big|_{\vec{r}'=\vec{r}_s} = -\frac{4\pi}{A_S}, \tag{4.15}$$

where $A_s$ is the area of *S*. With this substitution the corresponding surface integral reduces to

$$-\frac{1}{4\pi}\int_S d^2r'\phi(\vec{r}')(\hat{n}\cdot\nabla\phi) = -\frac{1}{4\pi}\left(\frac{-4\pi}{A_S}\right)\int_S d^2r'\phi(\vec{r}') \tag{4.16a}$$

$$= \frac{1}{A_S}\int_S d^2r'\phi(\vec{r}') \tag{4.16b}$$

$$= <\phi>_S, \tag{4.16c}$$

where $<\phi>_S$ is the average value of $\phi(\vec{r}')$ on *S*.

It is worth spending a minute to appreciate what just happened. In the Neumann case, we specify the normal component of the electric field everywhere on *S*. However, since $\vec{E} = -\nabla\phi$, information about the reference value of $\phi$, being a constant, is lost. Since mathematics done correctly does not lose information, vector calculus forces the initial

apparent inconsistency to ensure that we do not lose the reference value of $\phi(\vec{r}\,')$. Putting everything together we have

$$\phi(\vec{r}) = \int_V d^3r'\,\rho(\vec{r}\,')G_N(\vec{r},\vec{r}\,') + <\phi>_S + \frac{1}{4\pi}\int_S d^2r'\,G_N\frac{d\phi}{dn'}. \tag{4.17}$$

Thus in the continuing tradition of mathematics, everything is used, and nothing is left over. We note that if part or all of *S* is at infinity, then the boundary condition reduces to $\partial G_N/\partial n' = 0$ everywhere on *S*.

More generally, Green functions are a class of solutions of linear inhomogeneous second-order differential equations of the form

$$O(\vec{r})G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}'), \tag{4.18}$$

where $O(\vec{r})$ is any *linear* second-order operator. In electro- and magnetostatics, we consider only the Laplacian. The other operator we use in E&M is the D'Alembertian, written $\Box^2$, which is the operator of the wave equation. This involves time and was already introduced in Ch. 1:

$$\Box^2(\vec{r},t) = \nabla^2_{\vec{r}} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{4.19}$$

In the general time-dependent formulation this operator acts on $\phi$ and $\vec{A}$, and is used to describe propagation, radiation, diffraction, and scattering. As another example, you are already familiar with the one-electron Hamiltonian from quantum mechanics:

$$O_Q(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}). \tag{4.20}$$

where $V(\vec{r})$ is the one-electron potential. However, we will not be using this equation in E&M.

The physical meaning of $G(\vec{r},\vec{r}\,')$ will be better appreciated when we discuss the time-dependent Green function of the wave equation, where the propagation of information occurs at the finite speed *c* instead of $c\to\infty$ as in electrostatics. However, Eq. (4.8) already shows that $G(\vec{r},\vec{r}\,')$ describes how information about a source $\rho(\vec{r})$ at $\vec{r}=\vec{r}\,'$ reaches an observer at $\vec{r}$. For this reason, in quantum mechanics Green functions are commonly termed propagators. Also, the volume integral in Eq. (4.8) represents the ultimate in linear superposition: it gives the effect seen by an observer at $\vec{r}$ as a superposition of contributions of differential charges $d^3r'\rho(\vec{r}\,')$ at $\vec{r}\,'$.

As a more general remark, the solution of an inhomogeneous linear second-order differential equation is relatively straightforward in two limits. If the driving function is a plane wave, then the solution is also a plane wave or a superposition of plane waves. The problem then reduces to Fourier analysis, and finding the solution reduces to algebra, possibly followed by a Fourier inversion to return the solution to the space domain. The second case occurs if the driving function is an impulse (delta) function. In that case, we

use the Green-function approach outlined above, making sure first that our solution is indeed a solution.

Next, because the delta function is mostly zero, the inhomogeneous equation reduces to the homogeneous equation at all points except at the singularity. We show later that if we have the solutions of the homogeneous equation, then we can always use them to construct a Green function. The process is analogous to that followed for one dimension in Ch. 3. Hence given the homogeneous solutions, the solutions to the corresponding inhomogeneous equation follow directly. Thus obtaining the solution of an inhomogeneous second-order differential equation is in principle no more difficult than obtaining the solution of the corresponding homogeneous equation.

A comment on direct numerical solutions of Laplacian equations: these almost always involve considerable wasted effort. In older texts, including the one that I used as an undergraduate in electrical engineering, the two-dimensional approach consists of drawing field maps around conductors such that the field and equipotential lines are orthogonal, then subdividing and subdividing the regions until they approach squares ("method of curvilinear squares"). In the initial pass they never did, so the original lines had to be repositioned and the calculations repeated. The method is clearly one of trial-and error. Although computers now do trial-and-error tasks rapidly, the results are not analytic. Integral representations are considerably more efficient because even in the worst cases they can be approximated to whatever level of accuracy is desired. Jackson introduces finite-element analysis in his Ch. 2 Sec. 12, and while I may add a section later, for now we'll go the Green-Theorem route.

D. The Mean Value Theorem.

In this and the next two sections, we consider some general results that follow from Gauss' Theorem, Green's First Identity, and Green's Theorem. We start with the Mean Value Theorem, which is a direct consequence of Green's Theorem.

The Mean Value Theorem states that in any charge-free region of space, the scalar potential at the *center* of a *spherical* shell is the arithmetic average of the potential over the surface of the shell. There are several ways to prove it. However, consistent with our emphasis on a "basics" approach to problem-solving, we use the Dirichlet Green function for a spherical surface *S* of radius *a* centered on $\vec{r} = 0$ with $\phi(\vec{r})$ specified everywhere on *S*. If the charge density $\rho = 0$ in *V*, then Eq. (4.11) reduces to

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int_S d^2r'\phi(\vec{r}')\hat{r}'\cdot\nabla_{\vec{r}'}G_D(\vec{r},\vec{r}'), \tag{4.21}$$

which is to be evaluated at $\vec{r} = 0$.

To do this, we use $G_D(\vec{r},\vec{r}')$ for a spherical shell of radius *a*, from either our upcoming Ch. 7 or Jackson Ch. 3:

$$G_D(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{a}{r'|\vec{r} - \frac{a^2}{r'^2}\vec{r}'|}. \tag{4.22}$$

This leads to

$$\hat{r}\cdot\nabla_{\vec{r}'}G_D(\vec{r},\vec{r}') = \frac{\partial}{\partial r'}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{a}{\sqrt{a^4+r^2r'^2-2a^2rr'\cos\gamma}}\right)$$

$$= -\frac{r'-r\cos\gamma}{(r^2+r'^2-2rr'\cos\gamma)^{3/2}} + ar\frac{rr'-a^2\cos\gamma}{(a^4+r^2r'^2-2a^2rr'\cos\gamma)^{3/2}}, \tag{4.23}$$

where

$$\cos\gamma = \hat{r}\cdot\hat{r}' = (\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi) + \hat{z}\cos\theta)\cdot(\hat{x}\sin\theta'\cos\varphi' + \hat{y}\sin\theta'\sin\varphi' + \hat{z}\cos\theta'$$

$$= \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta' \tag{4.24}$$

where $(\theta,\varphi)$ and $(\theta',\varphi')$ are the polar angles locating $\hat{r}$ and $\hat{r}'$.

We can evaluate $\phi(\vec{r})$ at the origin by simply setting $r=0$, but let's be more general and evaluate $\phi(\vec{r})$ everywhere in *V* assuming that $|\vec{r}|=a$. The result is

$$\phi(\vec{r}) = \frac{1}{4\pi a^2}\int_S a^2 d\Omega'\phi(\Omega')\frac{a(a^2-r^2)}{(a^2+r^2-2ar\cos\gamma)^{3/2}}. \tag{4.25}$$

Setting $r=0$ yields

$$\phi(0) = \frac{1}{4\pi}\int_S d\Omega'\phi(a,\Omega')$$

$$= <\phi>_S, \tag{4.26}$$

which was to be shown. Although the above approach is cumbersome, it is the most general way of obtaining $\phi(\vec{r})$ at any $|\vec{r}|<a$ given $\phi(\vec{r}_s)$, and can be used even when the charge density $\rho$ in the sphere is nonvanishing although in this case the mean-value result does not apply. Note that the radius *a* is arbitrary, meaning that the average potential of *any* concentric shell about the selected origin in the charge-free region is the same.

The mean-value theorem can also be proven with $\vec{r}$ set equal to zero directly in Eq. (4.22). Then

$$G_D(0,\vec{r}') = \frac{1}{r'} - \frac{1}{a}, \tag{4.27}$$

and Eq. (4.26) is obtained with fewer steps. This is left as a homework exercise. As a cross-check, Eq. (4.27) shows that $G_D(0,a\hat{r})=0$, as expected for the Dirichlet Green function.

The derivation of Eq. (4.26) depends on every point on the shell being equivalent to every other point on the shell, which is true for the spherical shell but not for other configurations. For more general surfaces, even if central symmetry points exist, the result is a weighted average, not an arithmetic average.

E. Earnshaw's Theorem.

A practical consequence of the mean-value theorem is Earnshaw's Theorem, which states that a charged particle cannot be held in equilibrium by electrostatic forces alone. Earnshaw's Theorem follows directly from the mean-value theorem because Eq. (4.26) shows that the potential is either constant throughout the spherical region, and hence has no internal minima, or else any positive values of $\phi$ on the boundary must be balanced by negative values on the boundary. Since the radius *a* is arbitrary, it follows that any point in the volume of interest can be no better than an unstable equilibrium for any point charge *q*. This is why highly ionic materials such as NaCl are so soft. They are saved from being liquids at room temperature by a the small amount of covalency obtained via quantum mechanics.

While the result holds only for a given radius *a*, and therefore in principle does not preclude the existence of "pockets" inside the sphere where the potential could have a local minimum, we can easily eliminate this possibility. Suppose that there exists such a local minimum, where $\phi$ at the center is less than any local value of $\phi$ around it. Then a new origin can be defined at the center of the pocket. We find that the existence of such a pocket would violate the conclusion that the center value of $\phi$ is the average of the values of $\phi$ in the immediate vicinity of the pocket. Therefore, no such pockets can exist. Because electrostatic confinement requires $\phi > 0$ everywhere surrounding the central point, it follows that electrostatic confinement itself cannot exist.

A more physically evident picture can be obtained by expanding $\phi(\vec{r})$ in a Cartesian Taylor series about a point at the origin of a small charge-freeregion of space in a Cartesian Taylor series as follows:

$$\phi(\vec{r}) = a_o + a_x x + a_y y + a_z z + a_{xy} xy + a_{yz} yz + a_{zx} zx$$
$$+ a_{xx} x^2 + a_{yy} y^2 + a_{zz} z^2 + (higher - order - terms)\,. \qquad (4.28)$$

The top row satisfies $\nabla_{\vec{r}}^2 \phi(\vec{r}) = 0$ automatically, but the second row includes higher-order terms that vanish only if a special condition applies, specifically $a_{xx} + a_{yy} + a_{zz} = 0$. This condition defines a *minimax* surface where $\phi(\vec{r})$ has a minimum in one or two directions and a maximum in two or one. Therefore, the point of lowest potential must lie on *S*, and not in the interior of the region. Again, even though we did our calculation for regions small enough to ignore higher-order terms, any finite region can be subdivided into regions of this type. Hence Earnshaw's Theorem holds in general.

F. Uniqueness Theorems.

The Uniqueness Theorem that we prove here states that any solution of Laplace's Equation within a volume *V* where *V* is charge-free and the potential is specified on *S* is unique. This is an example of the use of Green's First Identity. If only $\hat{n} \cdot \nabla \phi$ is specified on *S*, then the best that can be done is to show that $\phi$ is unique to within an additive constant, although both Green's First Identity and Green's Theorem both show that the functional dependence $\phi(\vec{r})$ within *V* is unique. Finally, uniqueness can be proven for

the most general case where a charge density also present in *V*, but the proof is significantly more difficult, placing it beyond our reach (see, e.g., J. Buchanan, The Existence and Uniqueness of Solutions to Differential Equations, http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Buchanan.pdf. I appreciate Ali Eloki for pointing out this manuscript.)

In our case, we start with Green's First Identity:

$$\int_V d^3r'\left(\nabla\phi\cdot\nabla\psi+\phi\nabla^2\psi\right)=\int_S d^2r'\phi n'\cdot\nabla\psi \tag{4.29}$$

To prove uniqueness, let $\phi=\psi=(\phi_1-\phi_2)$, where $\phi_1$ and $\phi_2$ are nominally two different solutions of Poisson's Equation that are both solutions of Laplace's Equation and have the same values on *S*. Then Eq. (4.29) becomes

$$\int_V d^3r'\left(\nabla(\phi_1-\phi_2)\cdot\nabla(\phi_1-\phi_2)+\phi\nabla^2(\phi_1-\phi_2)\right)=\int_S d^2r'(\phi_1-\phi_2)n'\cdot\nabla(\phi_1-\phi_2) \tag{4.30}$$

Because *V* is charge-free, then $\nabla^2\phi_1=\nabla^2\phi_2=0$ in *V* and the second term in the volume integral vanishes. Because $\phi_1=\phi_2=V(\vec{r}_s)$ at every point on *S*, the surface integral vanishes as well. We are left with

$$\int_V d^3r'|\nabla(\phi_1-\phi_2)|^2=0\,. \tag{4.31}$$

This can be valid only if $\phi_1$ and $\phi_2$ differ by no more than a constant. But since $\phi_1=\phi_2$ on *S*, the constant must be zero. Hence the solution to the Dirichlet problem is unique.

In the Neumann case, we assume both solutions have the same value of $\hat{n}\cdot\nabla_{\vec{r}}\phi(\vec{r})$ on the surface, and write $\phi_1=\phi_2+C$, where *C* is a constant. Without going into details, *C* cancels throughout when evaluating both Green's First Identity and Green's Theorem, although both show that other than *C*, $\phi_1$ and $\phi_2$ must have the same functional dependence on $\vec{r}$. Hence in the Neumann case, $\phi_1$ and $\phi_2$ are unique in this sense. Since the constant vanishes in calculating $\vec{E}$, it does not affect any field-dependent quantity, such as force or energy density. Details are left as a homework assignment.

## G. Thomson's Theorem.

Thomson's Theorem (Thomson = Lord Kelvin) states that, for a given set of conductors each at a fixed location and each carrying a fixed amount of charge, the energy in the surrounding electrostatic field is minimized when the surfaces of the conductors are equipotentials, i.e., conductors. This is an example of a theorem that is proved with Gauss' Theorem alone. Neither Green's First Identity nor Green's Theorem get involved.

We already know for kinematic reasons that the surface of a conductor in equilibrium cannot have a tangential component of an electric field. Thomson's Theorem shows that this is consistent with minimizing the energy contained in the associated electric fields.

Thus we have two independent reasons why the surface of a conductor must be an equipotential.

The proof is as follows. Suppose we have a set of conductors $j = 1, 2, \ldots N$ with charges $Q_j$ on each. The volume of interest $V$ is *not* the space within the conductors, but the (empty) space in which the conductors are embedded. Suppose further that we have two possible field configurations in $V$: $\vec{E}(\vec{r})$ and $\vec{E}_o(\vec{r})$, where $\vec{E}_o(\vec{r})$ is the field for which the surface of each conductor is an equipotential, and $\vec{E}(\vec{r})$ is a more general field. The proposition is proved if we can show that the total electrostatic energy of the two fields satisfies the inequality

$$W = \int_V d^3r' \left( \frac{1}{8\pi} \vec{E}^2 \right) \geq W_o = \int_V d^3r' \left( \frac{1}{8\pi} \vec{E}_o^2 \right), \tag{4.32}$$

where $W$ and $W_o$ are the total energies of the two configurations.

To prove the theorem, let $\vec{E} = \vec{E}_o + \Delta\vec{E}$, where $\Delta\vec{E} = \Delta\vec{E}(\vec{r})$ is the difference between $\vec{E}$ and $\vec{E}_o$. We next formally express the difference in the energies of the two field configurations as a volume integral over their energy densities:

$$\begin{aligned} W - W_o &= \int_V d^3r' \frac{1}{8\pi} \left( (\vec{E}_o + \Delta\vec{E})^2 - \vec{E}_o^2 \right) \\ &= \int_V d^3r' \frac{1}{8\pi} \left( \Delta\vec{E}^2 + 2\vec{E}_o \cdot \Delta\vec{E} \right). \end{aligned} \tag{4.33}$$

The first term in the integral is automatically positive definite, so the integral will be as well. We hold this integral in reserve. Accordingly, we consider the second integral. Ignoring the numerical prefactors we can write this as

$$\int_V d^3r' \vec{E}_o \cdot \Delta\vec{E} = \int_V d^3r' \nabla\phi_o \cdot \nabla(\Delta\phi), \tag{4.34}$$

where $\vec{E}_o = -\nabla\phi_o$ and $\Delta\vec{E} = -\nabla(\Delta\phi)$. We can transform the latter integral into something more useful by invoking Gauss' Theorem with the combination of functions used to derive Green's First Identity:

$$\int_V d^3r' \nabla \cdot \left( \phi_o \nabla(\Delta\phi) \right) = \int_V d^3r' \left( \nabla\phi_o \cdot \nabla(\Delta\phi) + \phi_o \nabla^2(\Delta\phi) \right) \tag{4.35}$$

We can discard the last term in the integral on the right because by assumption there are no charges in $V$:

$$\nabla^2(\Delta\phi) = 0. \tag{4.36}$$

Using Gauss' Theorem, we convert the integral on the left side of Eq. (4.35) to

$$\int_V d^3r' \nabla \cdot \left( \phi_o \cdot \nabla(\Delta\phi) \right) = \int_S d^2r' \hat{n} \cdot \left( \phi_o \cdot \nabla(\Delta\phi) \right) = -\sum_{j=1}^{N} \int_{S_j} \phi_{oj} \hat{n}_j \cdot (\vec{E} - \vec{E}_o), \tag{4.37}$$

where we assume that *S* consists of a number of conductors *j* each of which is at a constant potential $\phi_j$. Because the $\phi_j$ are constants, they can be taken out of the integrals. Now $\hat{n}_j \cdot (\vec{E} - \vec{E}_o) = 4\pi(\sigma_j - \sigma_{jo})$ per conductor *j*, where $\sigma_j$ and $\sigma_{jo}$ are the surface charge densities on the conductor associated with the fields $\vec{E}$ and $\vec{E}_o$. The integrals over these charge densities yield the total charges $Q_j$ on the different conductors. We therefore obtain

$$-\sum_{j=1}^{N} \int_{S_j} \phi_{oj} \hat{n}_j \cdot (\vec{E} - \vec{E}_o) = -\sum_{j=1}^{N} \phi_j (Q_j - Q_{jo}) = 0 , \tag{4.38}$$

since by assumption the total charge on each conductor is a constant. Hence the overall integral is zero, and we have shown that

$$W - W_o = \int_V d^3 r' (\Delta \vec{E})^2 \geq 0 . \tag{4.39}$$

Finally, this integral is zero if $\Delta \vec{E} = 0$. But this says that the energy is minimized for the field where all conductors are equipotentials. Our conclusion is consistent with what we deduced from applying a field to a conductor: the tangential component of the field at the surface of the conductor must be zero, i.e., the conductor is an equipotential.

A closely related result is that the energy of a system is reduced if an uncharged conductor is inserted into a region containing a field created by other conductors, if the charge on each other conductor is held constant. This is Jackson's prob. 1.16. The proof is similar to the above, and is left as a homework assignment.

H. Capacitance.

The Dirichlet Green function can be used to show that capacitances are geometric, independent of potential and stored charge. The procedure to describe capacitance geometrically follows a logical progression. Consider the field in the region between a series of conductors, all of which are at a potential $\phi = 0$ except for one that is at a potential *V*. Write $Q$ on this conductor as the integral of its surface charge density $\sigma$; write $\sigma$ in terms of the electric field $\vec{E}$ at its surface; write $\vec{E}$ as the gradient of the scalar potential $\phi$; write $\phi$ in terms of the potential *V* of the equipotential surface; then use the expression $Q = CV$ to identify the capacitance in the result. We will make similar geometric arguments for inductors in Ch. 11.

Accordingly, consider a configuration consisting of two or more separate conductors *j*, where conductor 1 is at a uniform potential $\phi_1 = V$ and all others are at $\phi_j = 0$. The partial capacitance $C_1$ of conductor 1 is defined as

$$Q_1 = C_1 V , \tag{4.40}$$

where $Q_1$ is the charge on $S_1$. $Q_1$ is obviously given by

$$Q_1 = \int_{S_1} d^2r'\sigma_1(r') = \frac{1}{4\pi}\int_{S_1} d^2r'\hat{n}_1 \cdot \vec{E}_1 , \tag{4.41}$$

taking advantage of Poisson's Equation and Gauss' Theorem, which shows that

$$\hat{n}_1 \cdot \vec{E}_1 = 4\pi\sigma_1 \tag{4.42}$$

on $S_1$, given that the field inside the conductor is zero. Now $\vec{E}_1$ follows from

$$\vec{E}_1 = -\nabla_{\vec{r}}\phi(\vec{r})\big|_{S_1} \tag{4.43}$$

where $\phi(\vec{r})$ is the potential in the region between conductors, or

$$\phi(\vec{r}) = \sum_j \left( -\frac{1}{4\pi}\int_{S_j} d^2r'\phi_j\hat{n}_j \cdot \nabla_{\vec{r}'}G_D(\vec{r},\vec{r}') \right). \tag{4.44}$$

Now all the conductors are at potential $\phi_j = 0$ except for $\phi_1 = V$. Accordingly, Eq. (4.44) reduces to

$$\phi(r) = -\frac{V}{4\pi}\int_{S_1} d^2r'\hat{n}'\cdot\nabla_{\vec{r}'}G_D(\vec{r},\vec{r}') . \tag{4.45}$$

Putting everything together we have finally

$$Q_1 = \frac{V}{16\pi^2}\int_{S_1} d^2r\int_{S_1} d^2r'(\hat{n}\cdot\nabla_{\vec{r}})(\hat{n}'\cdot\nabla_{\vec{r}'})G_D(\vec{r},\vec{r}') . \tag{4.46}$$

Therefore

$$C_1 = \frac{1}{16\pi^2}\int_{S_1} d^2r\int_{S_1} d^2r'(\hat{n}\cdot\nabla_{\vec{r}})(\hat{n}'\cdot\nabla_{\vec{r}'})G_D(\vec{r},\vec{r}') . \tag{4.47}$$

This expression depends only on geometry, and is independent of the potentials on the different conductors. It can be noted that if the system is overall neutral, the sum of the charges on the other conductors being held at zero potential is simply $-Q_1$. The charge on the $j^{\text{th}}$ conductor is obtained by Eq. (4.46) but with one surface integral over conductor 1 and the other over conductor *j*.

It is useful to consider specific geometries. For the standard parallel-plate capacitor with plates of area *A* separated by a dielectric of dielectric constant $\varepsilon$ and thickness *d*, the result is

$$Q = \sigma A = \frac{\varepsilon_r E}{4\pi}A = \frac{\varepsilon_r A}{4\pi}\frac{V}{d} = \frac{\varepsilon_r A}{4\pi d}V . \tag{4.48}$$

The SI version follows by starting with Poisson's Equation in the form $\nabla\cdot\vec{D} = \rho$. The result is

$$Q = \frac{\varepsilon_r \varepsilon_o A}{d} V \,. \tag{4.49}$$

For the sphere of radius *a* in empty space, we have

$$\phi(a) = V = \frac{Q}{a}; \tag{4.50}$$

hence

$$C = a \,. \tag{4.51}$$

It might be surprising that the capacitance of the sphere is its radius, but this can be made more plausible noting that for a given charge density $\sigma$, the charge Q increases as $a^2$ whereas the voltage increases as *a*. Thus the capacitance must be proportional to the radius. In SI units the capacitance of the sphere in empty space is the more plausible

$$C_{sphere} = 4\pi\varepsilon_o a \,. \tag{4.52}$$

Capacitance can also be defined by stored energy. Start with power delivered,

$$dW/dt = V I \,, \tag{4.53}$$

where *W* is the stored energy, then note that $I = dQ/dt$. Equation (4.55) becomes

$$\frac{dW}{dt} = CV\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2}CV^2\right). \tag{4.54}$$

From Ch. 1,

$$W = \frac{1}{8\pi}\int_V d^3r' \, \vec{E}^2(\vec{r}') \,, \tag{4.55}$$

where $\vec{E}(\vec{r}) = -\nabla\phi(\vec{r})$.

The calculation proceeds most efficiently using Green's First Identity, with $\vec{A} = \phi\nabla\phi$ in Eq. 1. With the help of Poisson's Equation and using $\rho = 0$, we find

$$W = \frac{1}{8\pi}\int_V d^3r \,|\nabla\phi|^2 = \frac{V}{8\pi}\int_S d^2r \phi(\hat{n}\cdot\nabla_{\vec{r}})\phi(\vec{r}) \tag{4.56a}$$

$$= \frac{V^2}{32\pi^2}\int_{S_1} d^2r \int_{S_1} d^2r'(\hat{n}\cdot\nabla_{\vec{r}})(\hat{n}'\cdot\nabla_{\vec{r}'})G_D(\vec{r},\vec{r}') \,. \tag{4.56b}$$

Hence the same result is obtained. $C_1$ is again defined in terms of the geometry of the configuration.