Chapter 2: Maxwell's Equations via force laws and induction (02 Feb 2019)

A. Perspective.

In Ch. 1, Maxwell's Equations are listed in both microscopic and macroscopic forms without considering their origin. We then introduce potentials, which allow two of the equations to be satisfied directly, then derived expressions giving electric and magnetic energy densities from these equations. This was accomplished with minimal use of vector calculus.

Physics nominally works the other way, going from data to equations to better equations. In Ch. 2 we follow this route. We restrict calculations to the microscopic level. Macroscopic electrostatics and magnetostatics are left to Chs. 5 and 9, respectively, because macroscopic equations involve the physics of materials. We find that all of Maxwell's Equations except the term $(1/c)(\partial B/\partial t)$ in the Faraday-Maxwell Equation can be derived from two force laws, Coulomb's and Ampère's, together with charge conservation. The remaining term is obtained with the "new physics" that resulted from Faraday's observations.

Derivations starting from forces do not appear to be very exciting because we already know how the story ends. However, the interesting part is how the story evolves, which also demonstrates that the math, if done consistently, leads to unambiguous conclusions with everything being used and nothing left over. As a byproduct, the Lorentz gauge is found to be the only one that is consistent with both force laws and charge conservation. As a second byproduct, the derivation of the Faraday-Maxwell Equation from Faraday's Law of Induction highlights a restriction that is not usually noted. Also, by converting the flux-density integral in Faraday's Law of Induction to line integrals, additional insight is obtained.

Specifically, the Coulomb force law gives rise to the electric field $\vec{E}$, the scalar potential ϕ, and after some math, Gauss' Equation $\nabla \cdot \vec{E} = 4\pi\rho$, Poisson's Equation $\nabla^2\phi = -4\pi\rho$, and the time-independent version $\nabla \times \vec{E} = 0$ of the Faraday-Maxwell Equation. Ampère's force law yields the magnetic flux density $\vec{B}$, the vector potential $\vec{A}$, the magnetic equivalent $\nabla \cdot \vec{B} = 0$ of Gauss' Equation, and, with the help of charge conservation, the full Ampère Equation $\nabla \times \vec{B} = (4\pi/c)\,\vec{J} + (1/c)(\partial\vec{E}/\partial t)$. The Coulomb and Ampère force laws together show the need to use the Lorentz gauge.

Because the Coulomb and Ampère force laws describe static or quasistatic (steady-state) phenomena, Faraday's Law of Induction is needed to complete the picture. As shown in Sec. E, this leads to the $-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ term in the Faraday-Maxwell Equation and its vector-potential representation $-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. For configurations in motion, we find that Faraday's Law also leads to the $\frac{1}{c}\vec{v}\times\vec{B}$ term of the Lorentz force, as required to be consistent the Ampère Law of Force. Because ϕ is conservative, the electrostatic field $-\nabla\phi$ contributes nothing to the closed-loop integral, but must be added for time-dependent boundary-value problems. By rearranging these terms, the canonical momentum $(\vec{p}+q\vec{A}/c)$ of quantum mechanics follows.

Using the Lorentz gauge, the potentials ϕ and $\vec{A}$ reduce Maxwell's Equations to a single 4-vector wave equation. Because this equation is covariant (has the same form in any inertial system), we must conclude that *c* has the same value in all inertial systems. This in turn leads to special relativity. Not a bad track record for two force laws and some measurements of induced voltages!

Summarizing all this in a single chapter is efficient, but a chapter consisting only of formalism is guaranteed to bore students and hence makes for bad instruction. When I wrote the original version of these notes I followed the Ch. 2 development, but later interrupted its thread at the end of Sec. C to go to Ch. 3, electrostatics in one dimension, since the necessary background for Ch. 3 is already developed in Sec. C. One-dimensional electrostatics is its own topic, since its math is different from that of boundary problems in two and three dimensions. One-dimensional electrostatics also has a wide range of practical applications ranging from piezoelectricity through screening, electrochemistry, digital cameras, and high-voltage failure of electrical insulation.

I then returned to Sec. D, which is close enough to Sec. C in both time and content so what was learned previously can be reinforced before it is forgotten. Faraday's contribution and its implications, covered in Sec. E, were deferred to the start of the second semester. An optional discussion of energy from the perspective of Coulomb's Law completes the chapter. At the beginning of the second semester, Ch. 2 also serves as a review before proceeding to specific topics.

The formal discussion of Green functions, necessary for dealing with electrostatics in two and three dimensions, is covered in Ch. 4 along with various theorems that derive naturally from it. Surprisingly, it is easier to understand the Green function of the wave equation than its electrostatics analog, but we leave that for later in the course when we discuss radiation, scattering, and diffraction. We do these topics using potentials, a much more efficient approach than the field approach used by Jackson. Electrostatics in two and three dimensions is covered in Chs. 5 and 6, with Cartesian configurations in the former and cylindrical and spherical configurations in the latter. The idea is to establish procedures for solving boundary-value problems using more familiar coordinates before advancing to the less familiar cylindrical- and spherical-coordinate systems.

Jackson approaches E&M somewhat differently, focusing on electrostatics, then introducing magnetics in Ch. 5, and time-dependent phenomena, including conservation laws, in Ch. 6. Zangwill also approaches E&M differently, emphasizing mathematics in Ch. 1, summarizing Maxwell's Equations in Ch. 2, then concentrating on electrostatics in Chs. 3 and 4. Green functions wait until Ch. 8, and magnetics until Ch. 10. However, in the meantime Zangwill provides considerable information about materials, which makes his version of E&M far more relevant to current interests. In addition, he includes considerable historical material, giving life to the main contributors to E&M. Our treatment of materials gets serious in Ch. 7, where we cover dielectric response theory, followed by sections on magnetic materials in Ch. 9. However, we take the opportunity to introduce elementary plasmonics in Chs. 5 and 6. Additional current topics such as negative-index materials, vortex beams, and electromagnetic-induced transparency are covered in later chapters.

It is evident that anyone who has ever attempted to write notes for a grad-level E&M course has his or her own ideas as to what is important and how the discipline should be developed. I prefer starting with the force laws as soon as practical, and building on them. This is how physics proceeds, and gives the students confidence in what follows, since they know how these results are based on experiment.

B. The Lorentz force law.

The Lorentz force law is brought up in Ch. 1. The statics version follows from the Coulomb and Ampère laws. It conveniently summarizes results and so is worth discussing first. We take advantage of it to justify our use of cgs units, and to set the relative scales of electric and magnetic phenomena. The equations in this section are given in both cgs and SI forms, although for reasons of physics we later switch to cgs units without apology.

For a point charge q moving with a velocity $\vec{v}$ in a region of space containing an electric field $\vec{E}$ and a magnetic flux density $\vec{B}$, the Lorentz force is

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B} \quad \text{(cgs)}; \qquad \vec{F} = q\vec{E} + q\vec{v}\times\vec{B}; \quad \text{(SI)}; \tag{2.1a,b}$$

where in electrostatics $\vec{E} = -\nabla\phi$ and in electrodynamics $\vec{E} = -(1/c)(\partial\vec{A}/\partial t) - \nabla\phi$.

Equations (2.1) provide an excellent summary of many aspects of the physics of classical E&M. We learn:

(1) At the classical level, the world interacts through fields, not potentials.

(2) The fundamental fields are $\vec{E}$ and $\vec{B}$, not $\vec{D}$ and $\vec{H}$.

(3) The magnetic term depends on $\vec{v}$, and therefore requires a reference frame (coordinate system) in which $\vec{v}$ can be measured. The choice of units either does (cgs) or does not (SI) reveal the relative magnitudes of the two force terms, as discussed in Ch. 1.

(4) The material properties that interact with $\vec{E}$ and $\vec{B}$ are the charge q and current $q\vec{v}$, respectively, or the charge density ρ and the current density $\vec{J} = \rho\vec{v}$, respectively, in the continuum representation.

(5) The equation is linear in q, $\vec{E}$, and $\vec{B}$, which means that we can use superposition.

As demonstrated by the Aharonov-Bohm effect, point (1) is not true at the quantum level, where $\vec{A}$ affects the phase of electronic wave functions in otherwise field-free regions.

Point (3) provides the opportunity to reiterate some of the comments made in Ch. 1.

Because $\vec{F}$ cannot depend on the parameters of the observer, we conclude that:

(a) $\vec{E}$ and $\vec{B}$ are not independent, but under appropriate conditions can transform at least partially into each other. That is, they must be interconnected parts of a larger entity. From special relativity we know that this larger entity is the second-rank field tensor.

(b) Given (a), the units of $\vec{E}$ and $\vec{B}$ logically should be the same, as in the cgs system.

(c) If $\vec{E}$ and $\vec{B}$ have the same units, then given (b), we must normalize the magnetic term by a speed to get "spelling" right. In our universe, at least, the logical choice for this is the universal speed constant *c*.

In fact, the choice of *c* is not arbitrary: it is required by special relativity. It shows in addition that the magnetic field, and therefore magnetism in general, is a consequence of relativity, and that magnetism can therefore be considered a purely relativistic phenomenon. Further, for ordinary "laboratory" speeds *v*/*c* is small, so we can expect magnetic effects to be intrinsically much smaller than electric effects. We begin our more detailed treatment of E&M with electric-field effects partly for that reason. In classical E&M, represented for example the Liénard-Wiechert potentials, everything is treated to first order in $1/c$ and hence is more appropriately termed retardation physics, although the foundation remains relativistic. Hence for our purposes we treat time as a Newtonian invariant. Any phenomenon that involves time dilation, such as synchrotron radiation, cannot be so described. We remedy this by covering special relativity in Ch. 15.

Fortunately for E&M, the value of *c* is independent of the motion of source and observer. No one knows why, but the evidence supporting this conclusion, most recently from the LIGO experiment, is insurmountable. Possibly *c* is quantized, and under ordinary conditions we are only able to access its lowest value. For whatever reason, this confers a huge practical advantage. One can imagine how complicated it would be to describe electromagnetic phenomena if we had to deal with a force law where *c* were a function of the speed of the source and/or the observer.

C. Coulomb's force law: electric fields, the scalar potential, and the first Maxwell Equation.

We begin with Coulomb's force law, which describe the interaction of two point charges q_1 and q_2. The objective is to determine how far this will take us into the complete set of Maxwell's Equations by capitalizing in addition on superposition, the continuous-function limit, and vector calculus.

When investigating interactions among charged objects, Coulomb found that his observations could be described quantitatively if the force between two point charges q_1 and q_2 is given by

$$\vec{F}_{12} = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \quad \text{(cgs);} \qquad \vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_o |\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \quad \text{(SI);} \qquad (2.2\text{a,b})$$

where $\vec{r}_1$ and $\vec{r}_2$ are the locations of q_1 and q_2, respectively, in some laboratory frame of reference, and $\vec{F}_{12}$ is the force on q_1 due to q_2. A significant amount of physics is already contained in these expressions. By examining them we see that:

(1) Equations (2.2) are bilinear in q_1 and q_2. This means that we can use superposition. We can add as many charges to the system as we please, and the same functional form will hold for the force between any two of them.

(2) The force acts along the line joining the two charges. We are dealing with a central-force problem. There are no torques.

(3) Given that there are two types of charge, positive and negative, the force can be repulsive or attractive. The sign of Eqs. (2.2) is determined by the fact that like charges repel and opposite charges attract.

(4) The magnitude of the force depends only on the distance between q_1 and q_2. This is consistent with other evidence, all of which indicates that the universe has no preferred origin.

(5) This is an inverse-square-law relation, considering that one of the powers in the denominators of Eqs. (2.2) is used to convert the position difference in the numerator into a unit vector. The inverse-square-law functional dependence leads to various conservation relations.

(6) Although other physical phenomena determine q more accurately, Eqs. (2.2) define charge in terms of force, and therefore connect electricity and magnetism to mechanics. The "natural" set of units that does this with a prefactor 1 is the cgs set. However, cgs units are not practical for everyday calculations, but we use them here because they provide better insight into the basic physics, in particular the role of special relativity. The choice of units should cause no difficulty. Anyone skilled in E&M should be able to move seamlessly between the cgs and SI systems (see Appendix 2).

We now take advantage of the bilinear nature of Eqs. (2.2) by considering the force on $q_1 = q$, which we place at $\vec{r}$, due to a collection of charges q_j at locations $\vec{r}_j$. Using cgs units from now on, unless otherwise specified, Eq. (2.2a) becomes

$$\vec{F}_q = q\sum_j \frac{q_j}{|\vec{r} - \vec{r}_j|^3}(\vec{r} - \vec{r}_j) \qquad (2.3\text{a})$$

$$= q\vec{E}(\vec{r}), \tag{2.3b}$$

where Eq. (2.3b) defines a new quantity, the electric field

$$\vec{E}(\vec{r}) = \sum_i q_i \frac{(\vec{r} - \vec{r}_i)}{|r - \vec{r}_i|^3}, \tag{2.4}$$

which represents action at a distance. For SI units, we divide the right-hand side of Eq. (2.4) by $4\pi\varepsilon_o$. This equation summarizes the influence of the other charges on q, which with Eq. (2.3b) gives the first term in the Lorentz force law in the electrostatics limit. It also gives physical meaning to $\vec{E}(\vec{r})$ as an entity in its own right, because the same value of $\vec{E}(\vec{r})$ can be realized at any given point in many different ways by appropriately arranging different charges q_j.

We can take this one step further if the configuration consists of large numbers of charges separated by distances that are much smaller than laboratory dimensions, for example electrons and nuclei in solids. In this case we can replace the sum over discrete charges with a volume integral over a continuum charge density $\rho(\vec{r}\,')$. The q_i are replaced by charge elements $dq = \rho dV = \rho d^3r'$, and the sum by a volume integral over $\vec{r}\,'$. We obtain

$$\vec{E}(\vec{r}) = \int_V \frac{d^3r'\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{|r - \vec{r}\,'|^3}. \tag{2.5}$$

This is a big step forward, because integrals are much easier to deal with than sums. Also, by transforming Eq. (2.4) to continuous functions, we can apply the various mathematical theorems of vector calculus. We can always recover the sum over point charges by substituting

$$\rho(\vec{r}) = q\delta(\vec{r} - \vec{r}\,'), \tag{2.6}$$

which is the charge density of a point charge q at $\vec{r} = \vec{r}\,'$.

We now put vector calculus to work. From the inside front cover of Jackson and Eq. (A1.22a), we have

$$\nabla_{\vec{r}}\left(\frac{d^3r'\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\right) = \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\nabla_{\vec{r}}(d^3r') + \frac{d^3r'}{|\vec{r} - \vec{r}\,'|}\nabla_{\vec{r}}(\rho(\vec{r}\,')) + d^3r'\rho(\vec{r}\,')\nabla_{\vec{r}}\frac{1}{|\vec{r} - \vec{r}\,'|} \tag{2.7a}$$

$$= 0 + 0 - \frac{d^3r'\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}, \tag{2.7b}$$

taking advantage of the fact that the operator $\nabla_{\vec{r}}$ treats functions of $\vec{r}\,'$ as constants. Thus Eq. (2.5) can be written

$$\vec{E}(\vec{r}) = -\nabla_{\vec{r}}\left(\int_V \frac{d^3r'\rho(\vec{r}\,')}{|r - \vec{r}\,'|}\right) = -\nabla_{\vec{r}}\phi, \tag{2.8}$$

where

$$\phi = \phi(\vec{r}) = \int_V \frac{d^3r'\,\rho(\vec{r}\,')}{|r - \vec{r}\,'|} \tag{2.9}$$

is the scalar potential ϕ.

We have therefore converted a force law involving a sum over vectors into an integral over scalars. This transformation is made possible because of the inverse-square-law dependence of the force between two charges. We have also shown that the electrostatic field can be written as the gradient of a scalar function. Equation (2.8) therefore gives us the electrostatic version of the Faraday-Maxwell Eq. (1.3b):

$$\nabla \times \vec{E} = 0. \tag{2.10}$$

In our next step, we introduce another relation from vector calculus, specifically

$$\nabla_{\vec{r}}^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi\delta(\vec{r} - \vec{r}\,') \qquad (2.11).$$

The proof of this relation is given in Appendix 1. By taking the divergence of Eq. (2.8) and using Eq. (2.11), we obtain

$$\nabla_{\vec{r}} \cdot \vec{E}(\vec{r}) = -\nabla^2\phi \tag{2.12a}$$

$$= -\nabla_{\vec{r}}^2 \int_V \frac{d^3r'\,\rho(\vec{r}\,')}{|r - \vec{r}\,'|} = -\int_V d^3r'\,\rho(\vec{r}\,')\nabla_{\vec{r}}^2 \frac{1}{|r - \vec{r}\,'|} \tag{2.12b}$$

$$= 4\pi \int_V d^3r'\,\rho(\vec{r}\,')\delta(r - \vec{r}\,') \tag{2.12c}$$

$$= 4\pi\,\rho(\vec{r}), \tag{2.12d}$$

which is Gauss' Eq. (1.3a). In Eq. (2.12b), note that the Laplacian $\nabla_{\vec{r}}^2$ does not act on $\rho(\vec{r}\,')$. Summarizing the result, we have

$$\nabla_{\vec{r}}^2\phi(\vec{r}) = -4\pi\rho(\vec{r}), \tag{2.13}$$

which is Poisson's Equation.

Converting divergence and curl operations to Laplacians is a crucial step, because Laplacians have Green-function (integral) solutions. We will appreciate this more in later chapters. For example, Eq. (2.13) is what we will use to solve boundary-value problems in Chs. 4 and 5.

D. Ampère's force law: magnetic fields, the vector potential, the Lorentz gauge, and two additional Maxwell Equations.

In working with forces between wires carrying currents, Ampère found that, except for their vector character, these forces are described by an equation that has the same form as Coulomb's Law between two point charges. Specifically, given two current loops 1 and 2 carrying currents I_1 and I_2, respectively, the forces between them act as if the force between current elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$ of the respective wires is given by the double differential

$$d\vec{F}_{12} = \frac{I_1 d\vec{l}_1 \times [I_2 d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)]}{c^2 \, |\, \vec{r}_1 - \vec{r}_2 \,|^3}. \tag{2.14}$$

As with point charges, Eq. (2.14) is bilinear in the current elements and satisfies an inverse-square-law relationship, so the comments made following Eq. (2.2) can be repeated. The prefactor $(1/c^2)$ indicates that we are once again dealing with a relativistic phenomenon. One of the prefactors $(1/c)$ goes with $I_1 d\vec{l}_1$ and the other with $I_2 d\vec{l}_2$, noting that a current is a charge times a velocity. Here, the direction attributes of the velocities are assigned to the differential length elements.

Following the same path that we used for electrostatics, we now investigate how far we can get into the set of Maxwell's Equations starting with Eq. (2.14). We first suppose that $I_1 d\vec{l}_1$ is a test element, and interpret the rest of the expression as something that gives rise to a field at $I_1 d\vec{l}_1$. This defines the differential magnetic flux density $d\vec{B}(\vec{r}, \vec{r}\,')$ as

$$d\vec{B}(\vec{r}, \vec{r}\,') = \frac{I_2(\vec{r}\,') d\vec{l}_2 \times (\vec{r} - \vec{r}\,')}{c \,|\, \vec{r} - \vec{r}\,'|^3}. \tag{2.15}$$

Once again, we interpret $\vec{r}$ as the location of the observer and $\vec{r}\,'$ as the location of the source. Since the expression is linear in I_2, we can again invoke superposition and convert to a continuum representation, obtaining

$$\vec{B}(\vec{r}) = \int d\vec{B}(\vec{r}, \vec{r}\,') \, d^3 r' = \frac{1}{c} \int \frac{I(\vec{r}\,') \, d\vec{l} \times (\vec{r} - \vec{r}\,')}{|\, \vec{r} - \vec{r}\,'|^3}. \tag{2.16}$$

But we can do better. We start by writing the above as a volume, not a line, integral. This takes advantage of the fact that *I* is the integral of a current density $\vec{J}(\vec{r})$ over the cross-section of a wire. This converts $I d\vec{l}$ into the product of $\vec{J}(\vec{r}\,')$ and a volume element $d^3 r'$. The direction attribute transfers to $\vec{J}(\vec{r}\,')$. With these changes and with the help of Eq. (A1.22a) of Appendix A, Eq. (2.16) becomes

$$\vec{B}(\vec{r}) = \frac{1}{c} \int_V \frac{d^3 r' \, \vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\, \vec{r} - \vec{r}\,'|^3} \tag{2.17a}$$

$$= -\frac{1}{c} \int_V d^3 r' \, \vec{J}(\vec{r}\,') \times \nabla_{\vec{r}} \left(\frac{1}{|\, \vec{r} - \vec{r}\,'|} \right). \tag{2.17b}$$

Because $\vec{J} = \rho \vec{v}$, the prefactor c^{-1} ensures that $\vec{B}$ and $\vec{E}$ have the same dimensions.

Now by analogy to what we did with the derivation of the scalar potential, consider

$$\nabla_{\vec r} \times \left(d^3r' \frac{J(\vec r\,')}{|\vec r - \vec r\,'|} \right)$$

$$= \frac{1}{|\vec r - \vec r\,'|} \nabla_{\vec r} (d^3r') \times \vec J(\vec r\,') + d^3r' \left(\nabla_{\vec r} \frac{1}{|\vec r - \vec r\,'|} \right) \times \vec J(\vec r\,') + \frac{d^3r'}{|\vec r - \vec r\,'|} \nabla_{\vec r} \times \vec J(\vec r\,')$$

$$= 0 + d^3r' \left(\nabla_{\vec r} \frac{1}{|\vec r - \vec r\,'|} \right) \times \vec J(\vec r\,') + 0, \tag{2.18}$$

because d^3r' and $\vec J(\vec r\,')$ are functions of $\vec r\,'$ not $\vec r$. We again see why it is important to identify the variable that is the target of the operator. Therefore

$$\vec B(\vec r) = \nabla_{\vec r} \times \frac{1}{c} \int \frac{d^3r' \vec J(\vec r\,')}{|\vec r - \vec r\,'|}$$

$$= \nabla_{\vec r} \times \vec A(\vec r), \tag{2.19}$$

where $\vec A(\vec r)$ is defined as

$$\vec A(\vec r) = \frac{1}{c} \int \frac{d^3r' \vec J(\vec r\,')}{|\vec r - \vec r\,'|}. \tag{2.20}$$

On a component-by-component basis, $\vec A(\vec r)$ has the same form as the quasistatic expression for $\phi(\vec r)$, but with $\vec J/c$ replacing ρ. This is consistent with the conclusion in Ch. 1 that ϕ and $\vec A$ are simply the scalar and vector parts of a larger entity, the 4-potential (ϕ, $\vec A$) of special relativity. This makes $(c\rho, \vec J)$ the 4-current. We provide further comments on this in a later chapter.

Since the divergence of a curl is zero, we immediately have the magnetic Gauss Equation:

$$\nabla \cdot \vec B = 0. \tag{1.1d}$$

Because Eqs. (1.1c,d) follow directly from Ampère's force law, this law is consistent with the fact that no evidence of a magnetic charge (magnetic monopole) has ever been found. As a result, Eqs. (1.1c,d) have absolute validity, i.e., they will not change when time is brought into the picture, or in the transition to macroscopic electrodynamics.

The calculation leading to the Ampère-Maxwell Eq. (1.1g,h) is similar to that which leads to Eq. (1.1a,b) but more challenging, and requires the use of the charge-conservation Eq. (1.4). However, it also establishes that the Lorentz gauge as the only one consistent with the Coulomb and Ampère force laws when they are considered together. We start by taking the curl of $\vec B$, writing $\vec B$ in terms of $\vec A$, then invoking the vector-calculus identity

$$\nabla_{\vec r} \times \vec B(\vec r) = \nabla_{\vec r} \times (\nabla_{\vec r} \times \vec A(\vec r))$$

$$= \nabla_{\vec r} (\nabla_{\vec r} \cdot \vec A) - \nabla_{\vec r}^2 \vec A. \tag{2.21}$$

Our experience with Coulomb's force law makes the second term easy to evaluate:

$$-\nabla_{\vec r}^2 \vec A = -\nabla_{\vec r}^2 \frac{1}{c}\int d^3r' \frac{\vec J(\vec r\,')}{|\vec r - \vec r\,'|} = -\frac{1}{c}\int d^3r'\, \vec J(\vec r\,')\nabla_{\vec r}^2 \frac{1}{|\vec r - \vec r\,'|}$$

$$= +\frac{4\pi}{c}\int d^3r'\, \vec J(\vec r\,')\delta(\vec r - \vec r\,')$$

$$= \frac{4\pi}{c}\vec J(\vec r)\,. \tag{2.22}$$

If we now assume the Coulomb gauge $\nabla\cdot\vec A = 0$, we obtain Ampère's Law as Ampère wrote it:

$$\nabla\times\vec B = \frac{4\pi}{c}\vec J\,. \tag{2.23}$$

But we can do better. Instead of making assumptions, we can evaluate $(\nabla\cdot\vec A)$ directly from Eq. (2.20):

$$\nabla_{\vec r}\cdot\vec A = \frac{1}{c}\int_V d^3r'\nabla_{\vec r}\cdot\left(\frac{\vec J(\vec r\,')}{|\vec r - \vec r\,'|}\right). \tag{2.25}$$

This is a volume integral over a divergence. This form suggests that we use Gauss' Theorem to convert the right side to a surface integral over a localized integrand, which vanishes thereby establishing $\nabla\cdot\vec A = 0$. However, this operation is illegal because $\nabla_{\vec r}$ is a function of $\vec r$ whereas d^3r' is a function of $\vec r\,'$. We must first convert the divergence to an operator in $\vec r\,'$.

To do this, consider $\nabla\cdot(\psi\,\vec a) = \psi\,\nabla\cdot\vec a + \vec a\cdot\nabla\psi$. Given that in magnetostatics $\nabla_{\vec r}\cdot\vec J(\vec r\,') = 0$,

$$\nabla\cdot\vec A = \frac{1}{c}\int_V d^3r'\vec J(\vec r\,')\cdot\nabla_{\vec r}\left(\frac{1}{|\vec r - \vec r\,'|}\right). \tag{2.26}$$

Next, capitalize on the functional form of $|\vec r - \vec r\,'|$ to change the variable of the gradient operator from $\vec r$ to $\vec r\,'$. This introduces a sign change,

$$\nabla\cdot\vec A = -\frac{1}{c}\int_V d^3r'\vec J(\vec r\,')\cdot\nabla_{\vec r\,'}\left(\frac{1}{|\vec r - \vec r\,'|}\right), \tag{2.27}$$

but more important, provides a path to the use of Gauss' Theorem. Rewrite Eq. (2.27) as

$$\nabla\cdot\vec A = -\frac{1}{c}\int_V d^3r'\nabla_{\vec r\,'}\cdot\left(\frac{\vec J(\vec r\,')}{|\vec r - \vec r\,'|}\right) + \frac{1}{c}\int_V d^3r'\frac{\nabla_{\vec r\,'}\cdot\vec J(\vec r\,')}{|\vec r - \vec r\,'|} \tag{2.28a}$$

$$= 0 + \frac{1}{c}\int_V d^3r'\frac{\nabla_{\vec r\,'}\cdot\vec J(\vec r\,')}{|\vec r - \vec r\,'|}\;. \tag{2.28b}$$

Finally, use charge conservation to convert Eq. (2.28b) to

$$\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial}{\partial t}\int_V d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= -\frac{1}{c}\frac{\partial \phi}{\partial t}. \qquad (2.29)$$

Thus the combination of Ampère's Force Law and charge conservation leads unambiguously to the Lorentz gauge. No assumptions are needed.

Since $\vec{E} = -\nabla\phi$, putting everything together yields

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \qquad (2.30)$$

This is the microscopic version of Eq. (1.3c), Ampère's Law with Maxwell's extension. Although the same answer is obtained by initially assuming the Lorentz gauge, this derivation makes a much stronger statement. By taking the divergence of Eq. (2.30), we also obtain the usual definition of magnetostatics, i.e., $\nabla \cdot \vec{J} = 0$.

To resolve a question that came up in class, Eq. (2.30) is obtained from expressions that deal exclusively with statics or quasistatics. The statics approximation appears in two places. First, we use $\nabla^2\vec{A} = -(4\pi/c)\vec{J}$ instead of its value from the wave equation:

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J} + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \qquad (2.31)$$

Second, we use $\vec{E} = -\nabla\phi$ instead of

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi. \qquad (2.32)$$

It is easily verified that the result does not change when these electrodynamic contributions are included. However, we exclude them at this stage because they do not follow from the statics equations from which we started.

Historically, the term $(1/c)(\partial\vec{E}/\partial t)$ was identified by Maxwell because he noted that Ampère's Law, as originally proposed by Ampère, was inconsistent with charge conservation. Maxwell showed that this inconsistency could be eliminated by adding $(1/c)(\partial\vec{E}/\partial t)$ to the original right-hand side. This was a particularly surprising result at the time, because there was no reason to suppose that the time derivative of $\vec{E}$ in empty space could possibly be equivalent to a current. However, experimental proof that this is correct follows from the observation of electromagnetic radiation. Without this term, electromagnetic radiation would not exist, and the world would be dark indeed. Jackson discusses the derivation of Ampère's equation in Sec. 4 of Ch. 5, where the topic is magnetostatics. Hence Jackson makes the magnetostatics approximation $\nabla \cdot \vec{J} = 0$, which eliminates the time derivative from Ampère's Equation as well.

There are three lessons to be learned from this derivation. First, Ampère could certainly have done this calculation himself, and probably did, thereby beating Maxwell to the time-derivative term in Eq. (1.3c). We can only speculate why Ampère did not point out the existence of this term. Maybe he did not carry the calculation through to completion, or maybe he decided at the beginning to assume the Coulomb gauge $\nabla \cdot \vec{A} = 0$. On the other hand, he may well have discovered this term, but concluded that it led to phenomena that appeared to be unreasonable at the time. In fairness, it would have been impossible for Ampère to confirm or disprove the existence of this term with the available equipment (note the prefactor $1/c$.)

The second lesson is that the Lorentz gauge is the only one consistent with both the Coulomb and Ampère Force Laws, and therefore is the only gauge directly consistent with electromagnetic phenomena. This goes beyond fixing an inconsistency in the equation that Ampère proposed. Although we could have saved some work by substituting $\nabla \cdot \vec{A} = -(1/c)(\partial\phi/\partial t)$ directly into Eq. (2.21), this would have made the Lorentz gauge an assumption, not a mathematical conclusion.

The third lesson to be learned follows from the negative experience of Ampère. If you're going to drop terms from an equation, you should have a very good reason to do so. Math is consistent, if nothing else, and no doubt Maxwell was grateful to Ampère for Ampère's oversight.

One loose end remains, which is to demonstrate that Ampère's force law can be reduced to the $\vec{v} \times \vec{B}$ term in the Lorentz force for point charges. With the definition of $\vec{B}$, Eq. (2.14) becomes

$$d\vec{F} = \frac{1}{c} I d\vec{l} \times \vec{B} . \tag{2.33}$$

We can rewrite this as

$$I\, d\hat{l} = \vec{J}\, A dl = \rho \vec{v} A dl = \vec{v}(\rho\, d^3 r) . \tag{2.34}$$

We now imagine shrinking the volume element $d^3 r$ down to where it includes only a single charge q. In that case the above reduces to

$$\vec{F} \to d\vec{F} = \frac{q}{c} \vec{v} \times \vec{B} , \tag{2.35}$$

where $d\vec{F} \to \vec{F}_q$ is the differential force on q.

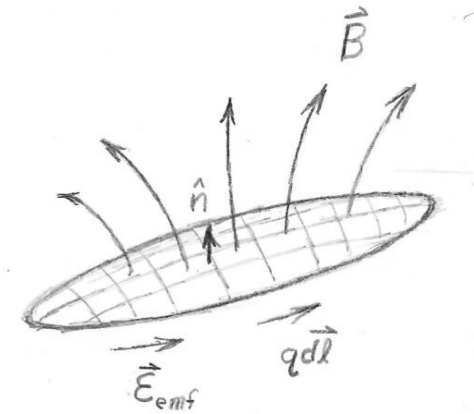
E. Faraday's Law of Induction: the Faraday-Maxwell Equation.

Maxwell's Equations are now almost complete. They are missing only the term $-(1/c)(\partial\vec{B}/\partial t)$ in the Faraday-Maxwell Equation. The data that demonstrate the existence of this term were obtained by Faraday. By moving magnets near coils, modifying the shapes of wire loops near magnets, and placing coils near other coils that were carrying currents that were changing with time, Faraday found that his results could be described by

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi_m}{dt}, \tag{2.36a}$$

where $\vec{E}_{emf}$ is the *electromotive force* associated with the time rate of change of the magnetic flux

$$\Phi_m = \int_S d^2r' \hat{n} \cdot \vec{B} \tag{2.36b}$$



that passes through a surface *S* whose perimeter is defined by *C*. *C* is any *closed* curve, and does not need to follow a physical object such as a wire. The time derivative is a *total* derivative, acting on everything in the integral that can change with time, including in principle *C*, *S*, and $\hat{n}$, as well as $\vec{B}$. One possible configuration is shown in the diagram.

The sign of Eq. (2.36a) follows from energy conservation, as seen by considering a special case. Let *C* be a circular loop of wire of radius *a* that lies in the *xy* plane and is centered on the *z* axis of a local coordinate system. We suppose that the wire carries a positive current *I* that flows in the $d\vec{l}$ direction, that is, $\vec{I} \sim \hat{\varphi}$. By the right-hand rule, the direction of $\vec{B}$ passing through the *xy* plane is $\vec{B} = B\hat{z}$. Choosing *S* to be the disc whose perimeter is *C*, the right-hand rule shows that $\hat{n} = \hat{z}$, so Φ_m is positive. Therefore, the sign of the right side of Eq. (2.36a) – and therefore the left side – is determined by whether Φ_m is increasing or decreasing with time.

On the left side of Eq. (2.36a), let E_{emf} be the magnitude of $\vec{E}_{emf}$. Then since dl is positive, the sign of the line integral depends on whether $\vec{E}_{emf}$ and $d\vec{l} \sim I$ are in the same or opposite directions. If the directions are the same, then the configuration is functioning as a source (battery) that is delivering energy to a load, for example the resistance of the wire. Because this energy comes from the magnetic field, for $\vec{E}_{emf} \parallel d\vec{l}$ we see that Φ_m must decrease with time. This is consistent with the sign of Eq. (2.36a). On the other hand, if $\vec{E}_{emf}$ and $\vec{I}$ are in opposite directions, then the loop is acting as a load, accepting energy from an external source. Φ_m must now be increasing with time, again consistent with the sign in Eq. (2.36a). Note that this is also Lenz' Law: if the flux through a closed loop is changing, then an $\vec{E}_{emf}$ appears that attempts to drive a current whose magnetic field opposes the change of Φ_m.

Potentials associated with inductors can be confusing because their terminal voltages V_{emf} are determined by line integration. With the direction of $\vec{I}$ locked to $\vec{B}$, V_{emf} is the physical quantity that must change sign if the function of the inductor changes from source to load. In contrast, the terminal voltage of a capacitor has the same sign in either case. Because we tend to be more conversant with voltages than currents, capacitors are more "logical" components than inductors, especially when transients are involved. However, not all situations concerning inductors involve line integration. For example, if Φ_m arises from *I*, we can safely conclude that if *I* increases then Φ_m must also increase, regardless of the sign of Eq. (2.36a). We return to this in Ch. 10, where various examples are given.

While Eqs. (2.36) are correct, their present form is not very useful. However, we can fix this by converting the surface integral of Eq. (2.36b) to a line integral. This conversion generates two terms and is consistent with the introduction of a third. Once this conversion is complete, the closed-loop restriction can be eliminated, and the result used to solve a wide variety of time-dependent boundary-value problems, for example eddy currents, transformers, motors, and other configurations involving conductors as discussed in Ch. 10. But because the conversion itself is fundamental, we consider it next.

We begin by taking advantage of physics to simplify the calculation. The absence of magnetic charge shows that the only relevant time dependences in Eq. (2.36b) are those of $\vec{B}$ and *C*. Those of *S* and $\hat{n}$, if any, need not be considered. You will prove this in a homework assignment later this semester by applying Gauss' Theorem to a volume *V* defined by two surfaces S_1 and S_2 that share a common perimeter *C*, then using $\nabla \cdot \vec{B} = 0$. In physical terms, this independence is a consequence of the fact that since magnetic charge does not exist, the flux passing through S_1 must equal that passing through S_2. Therefore, details of *S* and $\hat{n}$, including possible time dependences, do not matter.

Now, suppose that $\vec{B}$ is a function of time but *C* is independent of time. This occurs for example if a permanent magnet is moved past a stationary wire loop. Writing Eqs. (2.36) as

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S d^2r'\hat{n}\cdot\vec{B} = \int_S d^2r'\hat{n}\cdot\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right), \tag{2.37}$$

we apply Stokes' Theorem to the left side. This converts Eq. (2.37) to

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = \int_S d^2r'\hat{n}\cdot\nabla\times\vec{E}_{emf} = \int_S d^2r'\hat{n}\cdot\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right). \tag{2.38}$$

Because *S* is arbitrary except for *C* being fixed,

$$\nabla\times\vec{E}_{emf} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0. \tag{2.39}$$

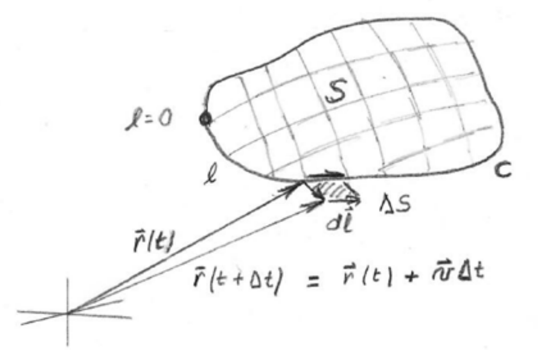
This equation is certainly familiar, and represents "new physics" in the sense that that it cannot be deduced from the Coulomb and Ampère force laws alone. The above also reveals that the Faraday-Maxwell Equation comes with a constraint: no motion is involved. For moving configurations additional terms appear, as discussed below.

Next, replace $\vec{B}$ with $\vec{A}$ using $\vec{B} = \nabla \times \vec{A}$. Applying Stokes' Theorem to the result, the right side of Eq. (2.37) becomes

$$\int_S d^2r'\hat{n} \cdot \left(-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \right) = \oint_C \left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) \cdot d\vec{l} \ . \tag{2.40}$$

We now have the first of our three line-integral terms.

Next, suppose that C is a function of time but $\vec{B}$ is independent of time. This describes a configuration where an object is moving relative to the "rest" (defining) frame of the magnetic field. Let S be defined by its perimeter C, which in turn is defined by a vector function $\vec{r}(l,t)$, where l is the distance along C measured from a reference point $l = 0$ (see diagram). At an incrementally later time $t + \Delta t$, the point $\vec{r}(l,t)$ has moved to a new position



$$\begin{aligned} \vec{r}(l,t+\Delta t) &= \vec{r}(l,t) + \frac{d\vec{r}}{dt}\Delta\vec{r} \\ &= \vec{r}(l,t) + \vec{v}\Delta t \ . \end{aligned} \tag{2.41}$$

We can evaluate the time derivative working on C by calculating the difference between $S(t+\Delta t)$ and $S(t)$, then dividing by Δt. From the diagram, the local change ΔS in S is the double differential

$$\Delta S = |\vec{v}\Delta t\,||\,d\vec{l}\,|\sin\theta, \tag{2.42}$$

where θ is the angle between $\Delta\vec{r}$ and the path increment $d\vec{l}$. But this is simply the magnitude of

$$\hat{n}\Delta S = (\vec{v}\Delta t) \times d\vec{l} \ , \tag{2.43}$$

where $\hat{n}$ is the normal to ΔS (note that the cross-product operation gives $\hat{n}$ for free) The direction of $\hat{n}$ is determined by the right-hand rule and is consistent with ΔS being positive as drawn in the diagram.

With only the perimeter involved, the contribution of the change of S with t to Eq. (2.36b) is

$$\oint_C \vec{E}_{emf} \cdot d\vec{l} = -\frac{1}{c\,\Delta t}\oint_C \vec{B} \cdot (\vec{v}\Delta t) \times d\vec{l} \tag{2.44a}$$

$$= +\oint_C d\vec{l} \cdot \left(\frac{1}{c} \vec{v} \times \vec{B} \right). \tag{2.44b}$$

In Eq. (2.44b) we have reversed the order of the cross product, interchanged dot and cross products, and cancelled the common factor Δt. This is our second line-integral term. By multiplying Eqs. (2.44) by a point charge *q*, we recognize this as the $\frac{q}{c}\vec{v} \times \vec{B}$ term of the Lorentz force. Because this term is also a direct consequence of Ampère's Force Law, it does not represent "new" physics in the sense of the Faraday-Maxwell Equation. However, it shows that the Faraday and Ampère Laws are consistent. We note that the derivation uses the *local* value of $\vec{v}$ at every point, hence Eq. (2.44b) applies whether motion occurs as a rigid translation or a local distortion.

The third line-integral term is $(-\nabla \phi)$, which arises from electrostatics. When multiplied by *q*, it also appears in the Lorentz force. The existence of this term cannot be deduced from Eqs. (2.36), but is consistent with them because the integral of the gradient of a scalar function around a loop is zero:

$$\oint_C (\nabla \phi) \cdot d\vec{l} = \int_S d^2 r\, \hat{n} \cdot \nabla \times (\nabla \phi) = 0. \tag{2.45}$$

Thus the final line-integral version of Eqs. (2.36) is

$$\oint_C \vec{E} \cdot d\vec{l} = \oint_C \left(-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \phi + \frac{1}{c} \vec{v} \times \vec{B} \right) \cdot d\vec{l}. \tag{2.46a}$$

With everything expressed as line integrals, we can open the loop, writing more generally

$$\int_0^l \vec{E} \cdot d\vec{l} = \int_0^l \left(-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \phi + \frac{1}{c} \vec{v} \times \vec{B} \right) \cdot d\vec{l}. \tag{2.46b}$$

In applications to conductors we replace $\vec{E}$ on the left side with $\sigma^{-1}\vec{J}$, generating a fourth term that describes resistive loss. We cover this in detail in Ch. 10.

Although $(-\nabla \phi)$ contributes nothing to Eq. (2.46a), its presence in Eq. (2.46b) is mandatory: ϕ provides the degree of freedom necessary to be able to solve time-dependent boundary-value problems on a point-by-point basis when resistive loss is included. Unfortunately, it is easy to forget that the essential property of ϕ that allows it to be introduced in the first place is that it is conservative. Consequently, in any application it must be treated as such.

Regarding notation, the subscript *emf* refers to the magnetic contribution to the total electric field $\vec{E}$, or

$$\vec{E}_{emf} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \frac{1}{c} \vec{v} \times \vec{B}. \tag{2.47}$$

While $(-\nabla\phi)$ is conservative, these two terms are not. Thus going around the loop twice doubles their contribution:

$$V_{emf} = \oint_C \vec{E}_{emf} \cdot d\vec{l}\ , \tag{2.48}$$

although ϕ is left unchanged. Equation (2.48) represents the physics underlying transformers, which are also covered in Ch. 10.

As noted above, as usually written the Faraday-Maxwell Equation is valid if the configuration is stationary. The above derivation allows us to determine the modifications that appear when the configuration is moving. Taking the curl of the integrands of Eq. (2.46b) yields

$$\nabla \times \vec{E} = \nabla \times \left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi + \frac{1}{c}\vec{v} \times \vec{B} \right) \tag{2.49a}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{A}) + 0 + \frac{1}{c}\nabla \times (\vec{v} \times \vec{B}) \tag{2.49b}$$

$$= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \frac{1}{c}\left(\vec{v}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{v}) + (\vec{B} \cdot \nabla)\vec{v} - (\vec{v} \cdot \nabla)\vec{B} \right). \tag{2.49c}$$

The first term in large parentheses in Eq. (2.49c) vanishes identically under all conditions, but the rest generally do not. However, if we are dealing with the special case of a rigid translation, Eq. (2.49c) reduces to

$$\nabla \times \vec{E} = -\frac{1}{c}\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right)\vec{B} = -\frac{1}{c}\frac{d\vec{B}}{dt}. \tag{2.50}$$

The term in brackets is the *convective derivative*, which takes into account the fact that *t* appears in the spatial coordinate as $\vec{r} = \vec{r}_o + \vec{v}t$, specifically $\vec{B}(\vec{r}, t) \to \vec{B}(\vec{r}(t), t)$. The typical case occurs for an object defined in one coordinate system, for instance its rest frame, and its definition there is taken over to another. The convective derivative is commonly used in hydrodynamics.

Finally, Eq. (2.46b) contains additional physics, which can be extracted by assuming the presence of a point charge *q*. Let

$$\vec{F}_q = -\frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\vec{v} \times \vec{B} - q\nabla\phi. \tag{2.51}$$

Now the mechanical force is the time derivative of the mechanical momentum,

$$\vec{F}_q = \frac{d\vec{p}_q}{dt}, \tag{2.52}$$

so $\vec{F}_q$ can be replaced with $d\vec{p}_q/dt$. Combining the two time derivatives of Eq. (2.51) yields

$$\frac{d}{dt}\left(\vec{p}_q + \frac{q}{c}\vec{A}\right) = -q\nabla\phi + \frac{q}{c}\vec{v}\times\vec{B}\,. \tag{2.53}$$

We recognize the term in the large parentheses on the left as the *canonical momentum* of quantum mechanics. The result is physically significant because it confirms that electromagnetic fields possess momentum, and that the momentum is carried by $\vec{A}$.

As discussed in Ch. 9, the implications of Eq. (2.53) go well beyond the definition of the canonical momentum. The Aharonov-Bohm effect describes the change of phase of the wave function of an electron as it passes near a solenoid carrying a steady electric current. Both electric and magnetic fields vanish outside the solenoid, but the vector potential does not. If fields were the entire story, as implied by the Coulomb, Ampère, and Faraday relations, no effect should be seen. That an effect is seen proves that potentials are more fundamental than fields. Of course, this was already known from the existence of eddy currents, and in principle from the canonical momentum.

F. Coulomb's force law and energy.

In Ch. 1 we considered energy starting from the mechanical definition of work, and went straight to Maxwell's Equations to derive expressions for the electric and magnetic energy densities and the intensity. Here, we investigate what we can learn starting with Coulomb's Law. We begin by calculating the energy expended in placing a point charge q_1 at $\vec{r}$, then bringing a second charge q_2 of the same sign from infinity to $\vec{r}\,'$. The incremental work that we do in moving q_2 by an incremental distance $d\vec{l}$ is

$$\begin{aligned} dW &= \vec{F}\cdot d\vec{l} \\ &= -q_1\vec{E}\cdot d\vec{l} = q_1\left(\nabla_{\vec{r}\,'}\phi(\vec{r},\vec{r}\,')\right)\cdot d\vec{l} \end{aligned} \tag{2.54}$$

The minus sign appears in Eq. (2.54) because in an adiabatic process the force resulting from $\vec{E}$ opposes the force that we are applying to move q_2. As we do positive work bringing the two charges together, the (q_1, q_2) system does negative work, i.e., its energy increases by the same amount. The total work necessary to bring q_2 to its final position, i.e., the total energy stored in the (q_1, q_2) system, is therefore

$$\int_0^W dW = W = q_2\int_\infty^{\vec{r}_2} \nabla\phi\cdot d\vec{l} = q_2\phi(\vec{r}_1,\vec{r}_2) = \frac{q_1 q_2}{|\,\vec{r}_1-\vec{r}_2\,|}\,. \tag{2.55}$$

Again, if we are using SI units, we divide the term on the right by $4\pi\varepsilon_o$. Because the force is the gradient of a scalar function, the result is independent of the path taken. Thus we have also shown that ϕ is a conservative potential.

Since the equations are bilinear in the charges, the introduction of a third charge q_3 results in a total system energy of

$$W = \frac{q_1 q_2}{|\,\vec{r}_1-\vec{r}_2\,|} + \frac{q_1 q_3}{|\,\vec{r}_1-\vec{r}_3\,|} + \frac{q_1 q_3}{|\,\vec{r}_2-\vec{r}_3\,|}\,. \tag{2.56}$$

We can see two patterns developing. If we bring in N charges, the first pattern expresses the change in system energy as

$$W = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{q_i q_j}{| r_i - r_j |}. \tag{2.57a}$$

The second pattern gives

$$W = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{| r_i - r_j |}. \tag{2.57b}$$

Since each of the two expressions covers all possible combinations, they are equal.

Jackson's Eq. (1.50) differs slightly from both, although he also avoids the self-energy terms $i = j$, which would result in unphysical infinities.

In the limit of very large numbers N we can ignore the small difference between $(N-1)$ and N and the differences in the start of the counting operations, and so combine the two representations as

$$W = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{q_i q_j}{| r_i - r_j |}. \tag{2.58}$$

If the charges are close enough on the laboratory scale to be considered a continuum, we can replace the discrete charges q_i and q_1 with the continuum representations $\rho(\vec{r}\,')\, d^3r'$ and $\rho(\vec{r}\,'')\, d^3r''$, respectively, and the sums with integrals. We obtain

$$W = \frac{1}{2} \int_V d^3r' \int_V d^3r'' \frac{\rho(\vec{r}\,')\rho(\vec{r}\,'')}{| \vec{r}\,' - \vec{r}\,''|} = \frac{1}{2} \int_V d^3r' \rho(\vec{r}\,')\, \phi(\vec{r}\,'), \tag{2.59}$$

where in the second version we have used the definition of ϕ. It is worth pointing out that the self-energy infinities that plagued the discrete-charge summations are now gone, as can be seen by setting $\vec{r}\,'' = 0$ and integrating d^3r' over a small volume centered on the singularity. The result is

$$\int_{\Delta V} d^3r' \frac{\rho(\vec{r}\,')\rho(0)}{r} = \int_{\Delta V} r'^2\, dr'\, d\Omega' \frac{\rho(\vec{r}\,')\rho(0)}{r} = \int_{\Delta V} r'\, dr'\, d\Omega' \rho(\vec{r}\,')\rho(0), \tag{2.60}$$

which for finite ρ is clearly well-behaved. This of course is a consequence of the fact that the differential charge $dq = \rho\, d^3r$ vanishes in the limit that the volume element d^3r vanishes. Thus the transformation from discrete charges with infinite charge densities to distributions of finite charge densities has eliminated the self-energy divergence that occurs with discrete charges, allowing us to incorporate self-energy contributions automatically.

When we discussed Coulomb's Law, we introduced at this point various mathematical theorems that led to expressions involving boundaries. This gave us the capability of addressing problems that could not be solved using discrete charges alone. The same happens here as well. Recalling that $\nabla^2 \phi = -4\pi\rho$, we convert the above to

$$W = -\frac{1}{8\pi}\int_V d^3r'\phi(\vec{r}\,')\nabla^2_{\vec{r}\,'}\phi(\vec{r}\,') \,. \tag{2.61}$$

We now invoke Green's First Identity from Appendix 1 in the form

$$\int_V d^3r'\nabla\cdot(\phi\nabla\phi) = \int_V d^3r'(\nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi) = \int_S d^2r'\phi\,(\hat{n}\cdot\nabla\phi)\,, \tag{2.62}$$

which converts the above integral to

$$W = \frac{1}{8\pi}\int_V d^3r'\vec{E}(\vec{r}\,')\cdot\vec{E}(\vec{r}\,') - \frac{1}{8\pi}\int_S d^2r'\phi(\vec{r}\,')\frac{d\phi(\vec{r}\,')}{dn'} \,. \tag{2.63}$$

We thus identify $\frac{\vec{E}\cdot\vec{E}}{8\pi}$ as an energy density. Note that this is precisely the same expression that we derived in Ch. 1 under the assumption that ε is independent of $\vec{E}$.

Using the differential form of Coulomb's Law, Eq. (1.3a), we can convert the surface integral into an integral over the surface charge density times the potential, yielding

$$W = \frac{1}{8\pi}\int_V d^3r'\vec{E}(\vec{r}\,')\cdot\vec{E}(\vec{r}\,') + \frac{1}{2}\int_S d^2r'\phi(\vec{r}\,')\sigma(\vec{r}\,') \,. \tag{2.64}$$

If S is the surface of a conductor at a potential V_o, we can extract $\phi = V_0$ from the integral. The result is

$$W = \frac{1}{8\pi}\int_V d^3r'\vec{E}(\vec{r}\,')\cdot\vec{E}(\vec{r}\,') + \frac{1}{2}V_0 Q \,, \tag{2.65}$$

where Q is the total charge on the conductor. This is consistent with the expression from which we started, although the factor of $1/2$ in Eq. (2.61) requires some explanation. This will be given later. Jackson does not mention the surface integral because he assumes that S is at infinity. But as seen in a previous section, if nothing else mathematics is consistent, so we retain it and use the more general interpretation.