Math 600 Lecture 20

<u>Definition</u>: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subset X$, and let $f: E \to Y$. If $p \in X$ is a limit point of E, then we say that

$$\lim_{x \to p} f(x) = q \quad \text{or} \quad f(x) \to q \text{ as } x \to p \qquad (q \in Y)$$

iff

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; (x \in E \text{ and } 0 < d_X(x,p) < \delta) \Rightarrow d_Y(f(x), q) < \varepsilon.$$

Note the significance of the condition

$$0 < d_X(x,p) < \delta;$$

it means that we only consider $x \neq p$. If p belongs to E, the value of $f(p)$ is unimportant.

<u>Theorem</u>: Let (X, d_X), (Y, d_Y) be metric spaces, $E \subset X$, $f: E \to Y$, and $p \in X$ be a limit point of E. Then

$$(*) \qquad \lim_{x \to p} f(x) = q$$

iff

$$(**) \qquad \forall \{p_n\} \subset E \; ((p_n \neq p \; \forall n \in \mathbb{Z}^+) \text{ and } p_n \to p) \Rightarrow f(p_n) \to q.$$

<u>Proof</u>: Suppose first that

$$\lim_{x \to p} f(x) = q.$$

Let $\{p_n\} \subset E$ satisfy $p_n \neq p$ $\forall n \in \mathbb{Z}^+$ and $p_n \to p$. We wish to prove that $f(p_n) \to q$. Let $\varepsilon > 0$ be given. Since $(*)$ holds, there exists $\delta > 0$ such that

$$(x \in E \text{ and } 0 < d_X(x,p) < \delta) \Rightarrow d_Y(f(x),q) < \varepsilon.$$

Since $p_n \to p$ and $p_n \neq p$ $\forall n$, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow 0 < d_X(p_n,p) < \delta.$$

But then

$$n \geq N \Rightarrow d_Y(f(p_n),q) < \varepsilon,$$

and hence $f(p_n) \to q$.

Conversely, suppose $(*)$ fails. Then there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \ \exists\, x \in E \cap (B_\delta(p) \setminus \{p\}),\ d_Y(f(x),q) \geq \varepsilon.$$

In particular,

$$\forall n \in \mathbb{Z}^+ \ \exists\, p_n \in E \cap (B_{1/n}(p) \setminus \{p\}),\ d_Y(f(p_n),q) \geq \varepsilon.$$

But then

$$\{p_n\} \subset E,\ p_n \neq p\ \forall n \in \mathbb{Z}^+,\ p_n \to p$$

and yet

$$f(p_n) \not\to q.$$

Thus $(**)$ fails. //

Corollary: Let (X,d_X), (Y,d_Y) be metric spaces, $E\subset X$, p a limit point of E, and let $f:E\to Y$. If $\lim\limits_{x\to p} f(x)$ exists, it is unique.

(This is a corollary of the previous theorem because it follows from the corresponding fact about limits.)

Note that if $f:X\to Y$, $g:X\to Y$, and Y is an abstract metric space, then it makes no sense to refer to $f\pm g$, fg, or f/g. But if $f:X\to\mathbb{R}$, $g:X\to\mathbb{R}$, then we define

$$f+g:X\to\mathbb{R} \text{ by } (f+g)(x)=f(x)+g(x)\ \forall x\in X,$$

$$f-g:X\to\mathbb{R} \text{ by } (f-g)(x)=f(x)-g(x)\ \forall x\in X,$$

$$fg:X\to\mathbb{R} \text{ by } (fg)(x)=f(x)g(x)\ \forall x\in X,$$

$$f/g:X\to\mathbb{R} \text{ by } (f/g)(x)=\frac{f(x)}{g(x)}\ \forall x\in X$$

and, for $c\in\mathbb{R}$,

$$cf:X\to\mathbb{R} \text{ by } (cf)(x)=cf(x)\ \forall x\in X.$$

Theorem Let (X,d) be a metric space, let $E\subset X$, let $f:X\to\mathbb{R}$, $g:X\to\mathbb{R}$, and let p be a limit point of E such that

$$\lim_{x\to p} f(x) \text{ and } \lim_{x\to p} g(x)$$

exist. Then

$$\lim_{x\to p} (f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x),$$

$$\lim_{x\to p} (f-g)(x) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x),$$

$$\lim_{x\to p} (fg)(x) = \left(\lim_{x\to p} f(x)\right)\left(\lim_{x\to p} g(x)\right)$$

$$\lim_{x\to p} (f/g)(x) = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)} \quad \left(\text{provided } \lim_{x\to p} g(x) \neq 0\right)$$

and, for all $c \in \mathbb{R}$,

$$\lim_{x\to p} (cf)(x) = c \lim_{x\to p} f(x).$$

<u>Proof:</u> Follows immediately from the previous theorem and the corresponding limit laws. //

<u>Definition:</u> Suppose (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $p \in E$, and $f: E \to Y$. We say that f is <u>continuous at p</u> iff

$(*)$ $\quad \forall \varepsilon > 0 \ \exists \delta > 0 \ (x \in E \text{ and } d_X(x,p) < \delta) \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$

We say that f is <u>continuous</u> iff f is continuous at every p in its domain E.

Theorem: Suppose (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $p \in E$, and $f: E \to Y$. If p is a limit point of E, then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Proof: This is obvious upon comparing $(*)$ with the definition of $\lim_{x \to c} f(x) = f(p)$. //

Theorem: Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces, $E \subset X$, $F \subset Y$, and suppose $f: E \to Y$, $g: F \to Z$, where $\mathcal{R}(f) \subset F$. Define $h: E \to Z$ by $h = g \circ f$ (i.e. $h(x) = g(f(x))\ \forall x \in E$). If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p.

Proof: Let $\varepsilon > 0$. Since g is continuous at $f(p)$, there exists $\delta > 0$ such that

$$(y \in F \text{ and } d_Y(y, f(p)) < \delta) \Rightarrow d_Z(g(y), g(f(p))) < \varepsilon.$$

Since f is continuous at p, there exists $\gamma > 0$ such that

$$(x \in E \text{ and } d_X(x,p) < \gamma) \Rightarrow d_Y(f(x), f(p)) < \delta.$$

But then

$$\begin{aligned}(x \in E \text{ and } d_X(x,p) < \gamma) &\Rightarrow d_Y(f(x), f(p)) < \delta \\ &\Rightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon \\ &\Rightarrow d_Z(h(x), h(p)) < \varepsilon.\end{aligned}$$

Thus h is continuous at p. //

Theorem: Let (X,d) be a metric space, let $E \subset X$, and let $f: E \to \mathbb{R}$, $g: E \to \mathbb{R}$ be continuous at $p \in E$. Then $f \pm g$ and fg are continuous at p and, if $g(p) \neq 0$, then f/g is continuous at p. Also, for all $c \in \mathbb{R}$, cf is continuous at p.

Proof: By the earlier theorem (f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$), and by the earlier theorem about limits,

$$\lim_{x \to p} (f+g)(x) = \lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) = f(p) + g(p)$$
$$= (f+g)(p).$$

Thus $f+g$ is continuous at p. The proofs for fg, f/g, cf are similar. //