Math 600 Lecture 31

<u>Theorem</u>: Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable, $R(f)\subset[A,B]$, and $\varphi:[A,B]\to\mathbb{R}$ is continuous. Then $h=\varphi\circ f$ is Riemann integrable on $[a,b]$.

<u>Proof:</u> Note that h is bounded on $[a,b]$; let's say that $|h(x)|\le K\ \forall x\in[a,b]$. Let $\varepsilon>0$ be given; we must show that there exists $P\in\mathcal{P}$ such that

$$U(P,h)-L(P,h)<\varepsilon.$$

Since φ is uniformly continuous on $[A,B]$, there exist $\delta\in(0,\varepsilon^{*})$ such that

$$s,t\in[A,B],\ |s-t|<\delta\Rightarrow|\varphi(s)-\varphi(t)|<\varepsilon^{*}.$$

Since f is Riemann integrable on $[A,B]$, there exists $P=\{x_0,x_1,\dots,x_n\}\in\mathcal{P}$ such that

$$U(P,f)-L(P,f)<\delta^2.$$

Define, for $j=1,\dots,n$,

$$M_j=\sup\{f(x)\mid x_{j-1}\le x\le x_j\},$$

$$m_j=\inf\{f(x)\mid x_{j-1}\le x\le x_j\},$$

and let M_j',m_j' be the analogous numbers for h. Let

$$\{1,2,\dots,n\}=J_1\cup J_2,$$

$$j\in J_1\Leftrightarrow M_j-m_j<\delta,$$

$$j\in J_2\Leftrightarrow M_j-m_j\ge\delta.$$

Then

$$U(P,h) - L(P,h) = \sum_{j \in J_1} (M_j' - m_j')\Delta x_j + \sum_{j \in J_2} (M_j' - m_j')\Delta x_j .$$

We have

$$j \in J_1 \Rightarrow M_j' - m_j' < \varepsilon \quad (\text{since } s,t \in [m_j, M_j] \Rightarrow |s-t| < \delta \Rightarrow |\varphi(s) - \varphi(t)| < \varepsilon)$$

and thus

$$\sum_{j \in J_1} (M_j' - m_j')\Delta x_j < \varepsilon \sum_{j \in J_1} \Delta x_j \le \varepsilon (b-a).$$

Also,

$$j \in J_2 \Rightarrow M_j' - m_j' \le 2K$$

and

$$U(P,f) - L(P,f) < \delta^2$$

$$\Rightarrow \sum_{j \in J_2} (M_j - m_j)\Delta x_j < \delta^2$$

$$\Rightarrow \delta \sum_{j \in J_2} \Delta x_j < \delta^2 \quad (\text{since } M_j - m_j \ge \delta \ \forall j \in J_2)$$

$$\Rightarrow \sum_{j \in J_2} \Delta x_j < \delta < \varepsilon \quad (\text{recall, we chose } \delta \in (0,\varepsilon)).$$

Thus

$$\sum_{j \in J_2} (M_j' - m_j')\Delta x_j \le 2K \sum_{j \in J_2} \Delta x_j < 2K\varepsilon .$$

We obtain

$$U(P,h) - L(P,h) < \varepsilon (b-a) + \varepsilon \cdot 2K = \varepsilon \left[b - a + 2K \right]$$

It follows that h is Riemann integrable on $[a,b]$ (since we could replace ε by $\frac{\varepsilon}{b-a+2k}$ above $*$). //

<u>Corollary:</u> Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$. Then fg is Riemann integrable on $[a,b]$.

<u>Proof:</u> Note that

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

and

$$\begin{aligned} f,g \text{ Riemann integrable} &\Rightarrow f+g,\ f-g \text{ Riemann integrable} \\ &\Rightarrow (f+g)^2, (f-g)^2 \text{ are Riemann integrable (by the previous theorem, with } \varphi(t)=t^2) \\ &\Rightarrow (f+g)^2-(f-g)^2 \text{ is Riemann integrable} \\ &\Rightarrow fg \text{ is Riemann integrable.} \end{aligned}$$ //

<u>Theorem:</u> Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and assume that f' and g' are Riemann integrable on $[a,b]$. Then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b)-f(a)g(a) - \int_a^b g(x)f'(x)\,dx$$

<u>Proof:</u> By the previous theorem, $fg'+f'g$ is Riemann integrable on $[a,b]$,

and

$$h = fg \Rightarrow h' = fg' + f'g.$$

Thus

$$\int_a^b h' = h(b) - h(a)$$

$$\Rightarrow \int_a^b (fg' + f'g) = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g. \quad //$$

Sequences of functions

Definition: Let (X,d), (Y,d) be metric spaces and let $E \subset X$. For each $n \in \mathbb{Z}^+$, let $f_n : E \to \mathbb{R}$ be a function, and let $f : E \to \mathbb{R}$ be another function.

We say that $f_n \to f$ pointwise on E iff

$$\forall x \in E,\ f_n(x) \to f(x) \text{ as } n \to \infty,$$

that is, iff

$$\forall x \in E\ \forall \varepsilon > 0\ \exists N \in \mathbb{Z}^+\ (n \geq N \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon).$$

We say that $f_n \to f$ uniformly on E iff

$$\forall \varepsilon > 0\ \exists N \in \mathbb{Z}^+\ (n \geq N \text{ and } x \in E) \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon.$$

Some questions:

- If f_n is continuous $\forall n$ and $f_n \to f$, is f necessarily continuous?
- (When $E = [a,b] \subset \mathbb{R}$ and $Y = \mathbb{R}$) If f_n is differentiable $\forall n$ and $f_n \to f$, is

f necessarily differentiable? If so, does $f_n' \to f'$?

- (Ditto) If f_n is Riemann integrable $\forall n$ and $f_n \to f$, is f necessarily Riemann integrable? If so, does $\int_a^b f_n \to \int_a^b f$?

Notes:

1. f is continuous at x=a iff

$$\lim_{x\to a} f(x) = f(a)$$

$$\iff \lim_{x\to a}\left(\lim_{n\to\infty} f_n(x)\right) = f(a)$$

$$\overset{?}{\iff} \lim_{n\to\infty}\left(\lim_{x\to a} f_n(x)\right) = f(a)$$

$$\iff \lim_{n\to\infty} f_n(a) = f(a) \checkmark$$

2. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\lim_{n\to\infty} f_n(x+h) - \lim_{n\to\infty} f_n(x)}{h}$$

$$= \lim_{h\to 0}\left(\lim_{n\to\infty} \frac{f_n(x+h)-f_n(x)}{h}\right)$$

$$\overset{?}{=} \lim_{n\to\infty}\left(\lim_{h\to 0} \frac{f_n(x+h)-f_n(x)}{h}\right) = \lim_{n\to\infty} f_n'(x).$$

3. $\displaystyle\int_a^b f = \int_a^b \left(\lim_{n\to\infty} f_n\right) \stackrel{?}{=} \lim_{n\to\infty} \int_a^b f_n$

In each case, the answer to the question comes down to the validity of interchanging two limits (i.e. changing the order of two limit operations)

This is a delicate question.

<u>Example</u> Let $f : (0,1)\times(0,1) \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x}{x+y}.$$

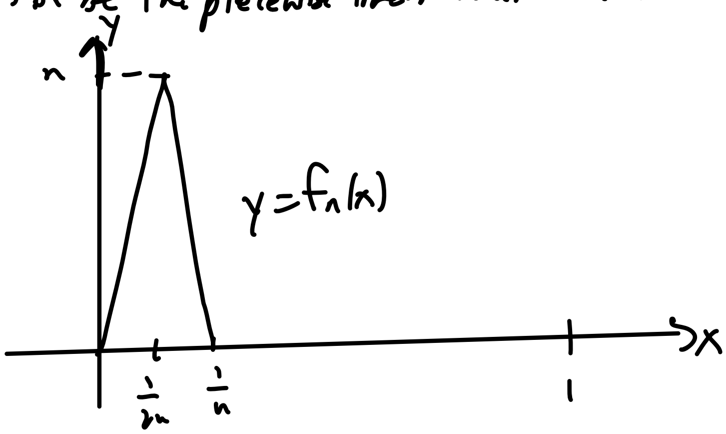
Then

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} 0 = 0,$$

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} 1 = 1$$

<u>Example</u> For each n, let $f_n : [0,1] \to \mathbb{R}$ be the piecewise linear function defined by

$$f_n(0) = 0$$
$$f_n\left(\tfrac{1}{2n}\right) = n$$
$$f_n\left(\tfrac{1}{n}\right) = 0$$
$$f_n(1) = 0$$



Then $f_n \to f$ pointwise, where $f : [0,1] \to \mathbb{R}$ is the zero function (Why?).

But

$$\int_0^1 f_n = 1 \quad \forall n \in \mathbb{Z}^+,$$

$$\int_0^1 f = 0,$$

so

$$\int_0^1 f_n \not\to \int_0^1 f.$$

<u>Example</u>: For each $n \in \mathbb{Z}^+$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then $f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

Each f_n is continuous on $[0,1]$, but $\lim_{n\to\infty} f_n$ is not.

In many cases, uniform convergence will allow the interchange of limit operations.

(But not always: If $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$, then $f_n \to f$, $f \equiv 0$, uniformly on any interval. But

$$f_n'(x) = \sqrt{n}\cos(nx)$$

and $f_n' \not\to f'$ (e.g. $f_n'(0) = \sqrt{n} \to \infty$, $f'(0) = 0$).)