Math 600 Lecture 30

Theorem: Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$ and let $c\in(a,b)$. Then f is Riemann integrable on $[a,c]$ and $[c,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: Let P be any partition on $[a,b]$, let $P' = P\cup\{c\}$, and define

$$P_1 = P'\cap[a,c],\quad P_2 = P'\cap[c,b].$$

Then P_1 is a partition on $[a,c]$ and P_2 is a partition on $[c,b]$. We then have

$$U(P',f)-L(P',f)\le U(P,f)-L(P,f)$$

and

$$\begin{aligned}U(P',f)-L(P',f) &= U(P_1,f)+U(P_2,f)-(L(P_1,f)+L(P_2,f))\\ &= (U(P_1,f)-L(P_1,f))+(U(P_2,f)-L(P_2,f)).\end{aligned}$$

Since

$$U(P_1,f)-L(P_1,f)\ge 0,\quad U(P_2,f)-L(P_2,f)\ge 0,$$

we see that

$$U(P_1,f)-L(P_1,f)\le U(P,f)-L(P,f),$$

$$U(P_2,f)-L(P_2,f)\le U(P,f)-L(P,f)$$

Since the above holds for all $P \in \mathcal{P}$, it follows that f is Riemann integrable on $[a,c]$ and on $[c,b]$ and also that

$$\int_a^b f = \int_a^c f + \int_c^b f. \quad //$$

The fundamental theorem

Theorem (FTOC, version 1): Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$ and define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \int_a^x f.$$

(Note that F is well defined by the previous theorem; also, $F(a)$ is understood to be 0.)

Then F is uniformly continuous and, if f is continuous at $x\in[a,b]$, then F is differentiable at x, with $F'(x) = f(x)$.

Proof: Note that, since f is Riemann integrable on $[a,b]$, it is bounded on $[a,b]$, say $|f(x)|\le M$ $\forall x\in[a,b]$. Let $x,y\in[a,b]$ and assume $x>y$. Then

$$\begin{aligned}|F(x)-F(y)| &= \left|\int_a^x f - \int_a^y f\right| \\ &= \left|\int_a^y f + \int_y^x f - \int_a^y f\right| \\ &= \left|\int_y^x f\right| \le \int_y^x M = M(x-y).\end{aligned}$$

Thus, for any $\varepsilon>0$,

$$x, y \in [a,b],\ |x-y| < \delta = \frac{\varepsilon}{M} \Rightarrow |F(x)-F(y)| < \varepsilon.$$

This shows that F is uniformly continuous on $[a,b]$.

Now suppose f is continuous at $x \in [a,b]$. For $h>0$, we have

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f - \int_a^{x} f\right] = \frac{1}{h}\int_x^{x+h} f$$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt$$

$$= \frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt$$

Let $\varepsilon>0$ be given. Then there exists $\delta>0$ such that

$$t \in (x, x+\delta) \Rightarrow |f(t)-f(x)| < \varepsilon.$$

But then

$$h \in (0,\delta) \Rightarrow \left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt$$

$$< \frac{1}{h}\int_x^{x+h} \varepsilon\,dt = \frac{1}{h}\cdot \varepsilon h = \varepsilon$$

This shows that

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x).$$

The proof that

$$\lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = f(x)$$

is similar. //

Theorem (FTOC, version 2): Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable and there exists $F:[a,b]\to\mathbb{R}$ such that F is continuous on $[a,b]$ and differentiable on (a,b), with $F'(x)=f(x)$ for all $x\in(a,b)$. Then

$$\int_a^b f = F(b)-F(a).$$

Proof: Let $P=\{x_0,x_1,\ldots,x_n\}$ be any partition on $[a,b]$. By the MVT, for all $j=1,\ldots,n$, there exists $t_j\in(x_{j-1},x_j)$ such that

$$F'(t_j)\Delta x_j = F(x_j)-F(x_{j-1})$$

$$\Rightarrow f(t_j)\Delta x_j = F(x_j)-F(x_{j-1})$$

$$\Rightarrow \sum_{j=1}^{n} f(t_j)\Delta x_j = \sum_{j=1}^{n}(F(x_j)-F(x_{j-1})) = F(x_n)-F(x_0) \quad \text{(telescoping sum)}$$

$$= F(b)-F(a).$$

But

$$L(P,f) \le \sum_{j=1}^{n} f(t_j)\Delta x_j \le U(P,f)$$

and thus

$$L(P,f)\le F(b)-F(a)\le U(P,f) \quad \forall P\in\mathcal{P}.$$

This is possible only if

$$\int_a^b f = F(b)-F(a). //$$

Theorem (change of variables): Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and suppose $\varphi:[A,B]\to\mathbb{R}$ maps $[A,B]$ onto $[a,b]$ with $\varphi(A)=a$, $\varphi(B)=b$, φ is differentiable on $[A,B]$, and φ' is continuous on $[A,B]$. Then

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(t))\varphi'(t)\,dt.$$

Proof: Define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \int_a^x f \quad \forall x\in[a,b]$$

and $G:[A,B]\to\mathbb{R}$ by $G(t)=F(\varphi(t))$. Then

$$F'(x)=f(x) \quad \forall x\in[a,b]$$

and

$$G'(t)=F'(\varphi(t))\varphi'(t)=f(\varphi(t))\varphi'(t) \quad \forall t\in[A,B].$$

But then

$$\int_a^b f = F(b)-F(a)$$

and

$$\begin{aligned}\int_A^B f(\varphi(t))\varphi'(t)\,dt = \int_A^B G'(t)\,dt &= G(B)-G(A)\\ &= F(\varphi(B))-F(\varphi(A))\\ &= F(b)-F(a)\\ &= \int_a^b f,\end{aligned}$$

as desired. //