Math 600 Lecture 12

<u>True or False</u>: Let (X,d) be a metric space, let $\{x_n\}$ be a sequence in X, and let x be a limit point of the set $E=\{x_n \mid n\in\mathbb{Z}^+\}$. Then $x_n\to x$.

<u>Theorem</u>: Let $\{s_n\},\{t_n\}$ be sequences in $\mathbb{R}$, and suppose $s_n\to s$ and $t_n\to t$. Then:

1. $s_n\pm t_n\to s\pm t$
2. For all $c\in\mathbb{R}$, $cs_n\to cs$ and $c+s_n\to c+s$.
3. $s_nt_n\to st$
4. $\dfrac{s_n}{t_n}\to\dfrac{s}{t}$ provided $t\neq 0$

<u>Proof:</u>

1. Exercise
2. Exercise
3. Let $\varepsilon>0$ be given. We must show that there exists $N\in\mathbb{Z}^+$ such that

$$n\geq N\Rightarrow |s_nt_n-st|<\varepsilon.$$

Note that

$$s_nt_n-st = s_nt_n-st_n+st_n-st$$
$$= (s_n-s)t_n+s(t_n-t).$$

There exist $N_1\in\mathbb{Z}^+$ such that

$$n\geq N_1\Rightarrow |t_n-t|<1\Rightarrow |t_n|=|t_n-t+t|\leq |t_n-t|+|t|<|t|+1.$$

Since $s_n \to s$, there exists $N_2 \in \mathbb{Z}^+$ such that

$$n \geq N_2 \Rightarrow |s_n - s| < \frac{\varepsilon}{2(|t|+1)}$$

and since $t_n \to t$, there exists $N_3 \in \mathbb{Z}^+$ such that

$$n \geq N_3 \Rightarrow |t_n - t| < \frac{\varepsilon}{2|s|}$$

(take $N_3 = 1$ if $s = 0$). Thus, with $N = \max\{N_1, N_2, N_3\}$, we have

$$\begin{aligned} n \geq N \Rightarrow |s_n t_n - st| &\leq |s_n - s|\,|t_n| + |s|\,|t_n - t| \\ &< \frac{\varepsilon}{2(|t|+1)} \cdot (|t|+1) + |s| \frac{\varepsilon}{2|s|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $s_n t_n \to st$.

#4. Exercise. //

Note that, in #4 above, it is understood that

$$\begin{aligned} t_n \to t,\ s \neq 0 &\Rightarrow t_n \neq 0 \quad \forall n \in \mathbb{Z}^+ \text{ sufficiently large} \\ &\Rightarrow \frac{s_n}{t_n} \text{ is well defined for all } n \in \mathbb{Z}^+ \text{ sufficiently large.} \end{aligned}$$

But we may have to ignore a finite number of terms in $\left\{\frac{s_n}{t_n}\right\}$ that are undefined (t_n may equal 0 for a finite number of values of n).

Theorem: Let $k \in \mathbb{Z}^+$ and let $\{x_n\}$ be a sequence in $\mathbb{R}^k$, where we write

$$x_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n}).$$

Then $x_n \to x = (x_1, x_2, \ldots, x_k)$ iff

$$\forall j = 1, 2, \ldots, k, \; x_{j,n} \to x_j.$$

Proof: First, suppose $x_n \to x$, which means that

$$\|x_n - x\|_2 \to 0 \quad (d(x_n, x) = \|x_n - x\|_2 \; \forall n).$$

We have

$$|x_{j,n} - x_j| \le \|x_n - x\|_2 \quad \forall j = 1, \ldots, k \;\; \forall n \in \mathbb{Z}^+$$

and hence

$$\|x_n - x\|_2 \to 0 \Rightarrow |x_{j,n} - x_j| \to 0$$
$$\Rightarrow x_{j,n} \to x_j.$$

(Given $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$n \ge N \Rightarrow \|x_n - x\|_2 < \varepsilon.$$

But then, for each $j = 1, \ldots, k$,

$$n \ge N \Rightarrow |x_{j,n} - x_j| < \varepsilon.$$

Thus, for each $j = 1, \ldots, k$, $x_{j,n} \to x_j$.)

Conversely, suppose

$$\forall j = 1, \ldots, k, \; x_{j,n} \to x_j.$$

Let $\varepsilon > 0$ be given. Then, for each $j = 1, \ldots, k$, there exists $N_j \in \mathbb{Z}^+$

such that

$$n \geq N_j \Rightarrow |x_{j,n} - x_j| < \frac{\varepsilon}{\sqrt{k}}.$$

Define $N = \max\{N_1, \dots, N_k\}$. Then

$$n \geq N \Rightarrow |x_{j,n} - x_j| < \frac{\varepsilon}{\sqrt{k}} \quad \forall j = 1, \dots, k$$

$$\Rightarrow \|x_n - x\|_2 = \left[\sum_{j=1}^{k} |x_{j,n} - x_j|^2\right]^{1/2} < \left[\sum_{j=1}^{k} \frac{\varepsilon^2}{k}\right]^{1/2}$$

$$= \varepsilon.$$

This shows that $x_n \to x$. //

<u>Corollary</u>: Let $k \in \mathbb{Z}^+$, let $\{x_n\}, \{y_n\}$ be sequences in $\mathbb{R}^k$, and suppose $x_n \to x$ and $y_n \to y$. Then:

- $x_n \pm y_n \to x \pm y$
- $x_n \cdot y_n \to x \cdot y$

If $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that $\beta_n \to \beta$, then

$$\beta_n x_n \to \beta x.$$

<u>Definition</u>: Let (X, d) be a metric space, let $\{x_n\}$ be a sequence in X, and let $\{n_k\}$ be a strictly increasing sequence in $\mathbb{Z}^+$ (that is, $n_1 < n_2 < n_3 < \cdots$). The sequence $\{x_{n_k}\}$ is called a <u>subsequence</u> of $\{x_n\}$. If $x_{n_k} \to x$, then x is called a <u>subsequential limit</u> of $\{x_n\}$.

Theorem: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X. Then $x_n \to x \in X$ iff every subsequence of $\{x_n\}$ converges to x.

Proof: HW. //

Lemma: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X. If $x \in X$ is a limit point of $\{x_n\}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to x.

Proof: Suppose x is a limit point of $\{x_n\}$. We construct a subsequence of $\{x_n\}$ converging to x as follows:

- Choose any $x_{n_1} \in B_1(x)$.
- $B_{1/2}(x)$ contains infinitely many elements of $\{x_n\}$, so choose x_{n_2} such that $x_{n_2} \in B_{1/2}(x)$ and $n_2 > n_1$.
- Continuing by induction, given $x_{n_1}, \dots, x_{n_k}$ such that $x_{n_j} \in B_{1/j}(x)$ $\forall j = 1, \dots, k$ and $n_1 < n_2 < \dots < n_k$, choose $x_{n_{k+1}}$ such that $x_{n_{k+1}} \in B_{1/k+1}(x)$ and $n_{k+1} > n_k$.

It then follows immediately that $x_{n_k} \to x$: If $\varepsilon > 0$ is given, choose $l \in \mathbb{Z}^+$ such that $\frac{1}{l} < \varepsilon$; then

$$k \geq l \Rightarrow d(x_{n_k}, x) < \frac{1}{l} < \varepsilon. //$$

Note that the converse is not true. Why?