Ex: Let $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ be independent. Let $Z = X+Y$. Find the distribution of $X \mid Z$.

$$P(X=k \mid Z=n) = \frac{P(X=k, Z=n)}{P(Z=n)}$$

$$= \frac{P(X=k, X+Y=n)}{P(Z=n)} = \frac{P(X=k, Y=n)}{P(Z=n)}$$

$$= \frac{P(X=k) \cdot P(Y=n)}{P(Z=n)} = \frac{\frac{e^{-\lambda_1} \lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^n}{n!}}{\frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(1 - \frac{\lambda_1}{\lambda_2+\lambda_2}\right)^{n-k} \Rightarrow Z \sim B\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

Ex: A bird lays N eggs, where $N \sim \text{Poisson}(\lambda)$. The probability that an individual egg develops is p. Assume independence of eggs. Find the probability that k eggs develop.

Notice that $D \mid N=n \sim B(n, p)$

$$P(D=k) = \sum_{n=k}^{\infty} P(D=k \mid N=n) \, P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{1}{(n-k)!\,k!} p^k (1-p)^{n-k} e^{-\lambda} \lambda^n$$

$m = n-k$

$$= \sum_{m=0}^{\infty} \frac{1}{m!\,k!} \cdot p^k (1-p)^m e^{-\lambda} \lambda^{m+k}$$

$$= \frac{e^{-\lambda} \lambda^k}{k!} \sum_{m=0}^{\infty} \frac{((1-p)\lambda)^m}{m!}$$

$$= \frac{e^{-\lambda} (\lambda p)^k}{k!} \cdot e^{(1-p)\lambda} = \frac{e^{-\lambda p} \cdot (\lambda p)^k}{k!}$$

Definition : The conditional expectation of Y given $X = x$ is

$$E[Y \mid X = x] = \sum_y y \, p_{Y|X}(y|x)$$

$E[Y|X]$ is called the conditional expectation of Y given X.

Theorem : Let $\psi(X) = E(Y|X)$. Then

$$E(\psi(X)) = E(Y)$$

Proof : $E(\psi(X)) = \sum_x \psi(x) \, p_X(x)$

$$= \sum_x \sum_y y \cdot p_{Y|X}(y|x) \cdot p_X(x)$$

$$= \sum_x \sum_y y \cdot \frac{p_{X,Y}(x,y)}{\cancel{p_X(x)}} \cancel{p_X(x)}$$

$$= \sum_y y \sum_x p_{X,Y}(x,y)$$

$$= \sum_y y \, p_Y(y) = E(Y)$$

$\square$

Remark : $E(Y) = \sum_x E(Y \mid X = x) \, P(X = x)$

Theorem :

(i) $E(a \mid Y) = a$

(ii) $E(aX + bY \mid Z) = a\,E(X \mid Z) + b\,E(Y \mid Z)$

(iii) $E(X \mid Y) = E(X)$ if X & Y are independent

(iv) $E(X\,g(Y) \mid Y) = g(Y)\,E(X \mid Y)$

(v) Tower property : $E[E[X \mid Y, Z] \mid Y] = E[X \mid Y]$

Proof: (iv) $E(X\,g(Y) \mid Y = y)$

$$= \sum_x x\,g(y) \cdot p_{X|Y}(x \mid y)$$

$$= \sum_x x\,g(y) \cdot \frac{p(x,y)}{p_X(x)}$$

$$= g(y) \sum_x x \cdot p_{X|Y}(x \mid y)$$

$$= g(y)\,E(X \mid Y = y)$$

(iii) $$E[X \mid Y = y] = \sum_x x \cdot p_{X|Y}(x \mid y)$$

$$= \sum_x x \cdot \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \sum_x x \cdot \frac{p_X(x) \cdot \cancel{p_Y(y)}}{\cancel{p_Y(y)}} = E(X)$$

□

Ex: You toss a fair coin repeatedly. What is the expected number of flips required to observe HH.

Condition on first flip:

$$E(HH) = E(HH \mid H_1)\, P(H_1) + E(HH \mid T_1)\, P(T_1)$$
$$= 0.5\, E(HH \mid H_1) + (1 + E(HH))\, 0.5 \quad \text{——(1)}$$

Let Z be the outcome of flip #2. Then,

$$E[HH \mid H_1] = E\big[\, E[HH \mid H_1] \mid Z \,\big]$$

$$= E\big[E[HH \mid H_1] \mid H_2 \big]\, P(H_2) + E\big[E[HH \mid H_1] \mid T_2 \big]\, P(T_2)$$

$$= \underbrace{E[HH \mid H_1, H_2]}_{2}\, P(H_2) + E[HH \mid H_1, T_2]\, P(T_2)$$

$$= 2(0.5) + (2 + E(HH))(0.5) \quad \text{——(2)}$$

By (1), (2),

$$E(HH) = 0.5\big[2(0.5) + (2 + E(HH))(0.5) \big] + (1 + E(HH))(0.5)$$

$$E(HH) = 0.5 + 0.5 + (0.5)^2 E(HH) + 0.5 + 0.5\, E(HH)$$

$$\Rightarrow \boxed{E(HH) = 6}$$

Theorem: Let $h(Y)$ be any function of & $E[h(Y)^2] < \infty$.

Then

$$E\left((X - h(Y))^2\right) \geq E\left((X - E(X|Y))^2\right)$$

Further, if $h(Y)$ is any function s.t

$$E\left((X - h(Y))^2\right) = E\left((X - E(X|Y))^2\right),$$ then

$$E\left((h(Y) - E(X|Y))^2\right) = 0$$