Black-Scholes Theory

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The Black-Scholes model is simply a geometric Brownian motion, but the ramifications of pricing theory under this model are far reaching across the field of finance. Among other things, the Black-Scholes model teaches use that volatility is paramount for the valuation of options, and expected returns are of lesser importance.

Valuation in the Black-Scholes framework amounts to solving a partial differential equation (PDE) for the price of a derivative. Derivation of the Black-Scholes PDE is the continuous-time equivalent of Δ hedging like we did for binomial trees. Continuous-time hedging is unrealistic because of transaction costs and basic logistics, but it is useful to see simplified expressions for derivative valuation. The resulting partial differential equations can be applied for

1 The Black-Scholes Model

Let us start with a geometric Brownian motion for the price of the a stock,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \ ; \tag{1}$$

this is the Black-Scholes model with drift rate μ and volatility σ. Equation (1) is intuitive because it gives us the relative returns of the stock over a short time internal. Equation (1) is also convenient for applying Itô's lemma to $\log(S_t)$, from which we get

$$d\log(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \ .$$

The drift μ in (5) is expected return over a short period of time, whereas the rate $\mu - \frac{1}{2}\sigma^2$ is the return over a longer horizon. Indeed,

$$X_T = \frac{1}{T}\log(S_T/S_0) = \mu - \frac{1}{2}\sigma^2 + \frac{\sigma}{T}W_T =_d \mu - \frac{1}{2}\sigma^2 + \frac{\sigma}{\sqrt{T}}Z \ ,$$

where $Z \sim \text{normal}(0,1)$ and $=_d$ denotes equality in distribution. The short-term average is more like an arithmetic average of returns, whereas the long-term average is more like a geometric average. For $t_i = i\Delta t$ for $i = 0, 1, \ldots, n$ with $\Delta t = 1$ year, denote annual relative returns as

$$r_i = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \ .$$

The arithmetic average of these returns is

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i \approx_d \mu + \frac{\sigma}{\sqrt{n}} Z \ .$$

The geometric mean on the other hand, is

$$\bar{r}^g = \left(\prod_{i=1}^{n}(1+r_i)\right)^{1/n} - 1 \ ,$$

and is associated with the average log return,

$$\frac{1}{n}\log(S_{t_n}/S_0) = \frac{1}{n}\log\left(\prod_{i=1}^{n}(1+r_i)\right) = \log(1+\bar{r}^g) \ .$$

Using Jensen's inequality we see that the geometric mean is lesser than the arithmetic,

$$\begin{aligned}
\bar{r}^g &= e^{\frac{1}{n}\sum_{i=1}^{n}\log(1+r_i)} - 1 \\
&< e^{\log\left(\frac{1}{n}\sum_{i=1}^{n}(1+r_i)\right)} - 1 \\
&= \bar{r} \ .
\end{aligned}$$

The geometric mean is a more conservative estimator of returns, and is often times preferred for computing Sharpe ratios. In general, μ is considered a difficult variable to estimate because (a) it is susceptible to outliers (a robust predictor of returns is the median), and (b) it requires a tremendous amount of data to obtain tight confidence bounds (e.g., over a thousand years of daily data to find μ within .01 of the true value).

The volatility σ is easier to estimate. Using r_i (either log-returns or relative returns), a simple volatility estimate is

$$\hat{\sigma} = \frac{\mathrm{std}(r_i)}{\sqrt{\Delta t}} \ ,$$

where Δ is the time step between r_i's. Other estimations of σ involve ARCH and GARCH models, but such models have heteroscedasticity, which is something being the simple setting of the basic Black-Scholes model (see chapter 23 of Hull 10th Ed.).

Example 1.1 (Estimating μ)**.** *Let S_t be Geometric Brownian motion observed at time $t_i = i\Delta t$ with $\Delta t = 1/252$ (i.e., daily observations). Denote these observations as*

$$R_i = \log(S_{t_i}/S_{t_{i-1}}) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}}) \ .$$

Given the value of σ amd n days of observations, an estimator of μ is

$$\hat{\mu} = \frac{1}{n\Delta t}\sum_{i=1}^{n}\left(R_i + \frac{1}{2}\sigma^2\Delta t\right) \ .$$

Given σ we can determine how large n needs to be so that the true value μ is in $(\hat{\mu}-.01, \hat{\mu}+.01)$ with 95% probability. Indeed,

$$\frac{\hat{\mu}-\mu}{\sigma} = \frac{1}{n\sigma\Delta t}\sum_{i=1}^{n}\left(R_i - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t\right) = \frac{1}{n\Delta t}\sum_{i=1}^{n}\Delta W_i =_d \frac{1}{\sqrt{n\Delta t}}Z \ ,$$

where $Z \sim \phi(0,1)$. For $\sigma = .2$, a 95% confidence bond is

$$\begin{aligned} .95 &= \mathbb{P}(-.01 < \hat{\mu}-\mu < .01) \\ &= \mathbb{P}\left(-\frac{.01\sqrt{\frac{n}{252}}}{.2} < Z < \frac{.01\sqrt{\frac{n}{252}}}{.2}\right) \\ &= \mathbb{P}\left(-1.96 < Z < 1.96\right) \ , \end{aligned}$$

from which we deduce

$$n = 252\left(\frac{1.96\times .2}{.01}\right)^2 = 387,234 \ \textit{days},$$

or 1,536.64 years.

2 Δ Hedging

Let $r \geq 0$ be the risk-free interest rate, and let $f(t, S_t)$ be the price of a derivative (e.g., a call option). Using Itô's lemma,

$$df(t,S_t) = \left(\frac{\partial f}{\partial t} + \mu S_t\frac{\partial f}{\partial s} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2}\right)dt + \sigma S_t\frac{\partial f}{\partial s}dW_t \ . \tag{2}$$

Now, let's setup a hedging portfolio where we are

- short the derivative
- hedge with amount $\frac{\partial f}{\partial s}$ in the underlying.

Let Π_t denote the value of this portfolio. The profit and loss (PnL) for Π_t is

$$d\Pi_t = -df(t,S_t) + \frac{\partial f}{\partial s}dS_t + r\left(\Pi_t + f(t,S_t) - \frac{\partial f}{\partial s}S_t\right)dt \ , \tag{3}$$

where $\Pi_t + f(t, S_t) - \frac{\partial f}{\partial s} S_t$ is the value of the portfolio's holdings in the risk-free bank account. From (2) we see that the dW_t terms in (3) cancel, thereby eliminating the riskiness of the position. Now, if there is to be no arbitrage, then for this riskless position it must be the case that

$$d\Pi_t = r\Pi_t dt \ . \tag{4}$$

If not, then a simple by-low-sell-dear arbitrage portfolios can be constructed over short increments of time. Therefore, combining (2), (3) and (4), we have a PDE for $f(t, s)$,

$$\frac{\partial f}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} - rf = 0 \ . \tag{5}$$

Notice that in combining (2), (3) and (4) to obtain (5) there was a cancellation of all terms involving μ, which means that the value of μ is irrelevant when pricing derivatives in the Black-Scholes model.

3 Girsanov Theorem

One way to motivate the Girsanov change of measure is with the importance sampling technique from Monte Carlo methods (see [Glasserman (2004)]). Suppose that generate several thousand independent draws from some density $p(x)$,

$$X^i \sim p(x) \ .$$

We can use these samples to estimate the expected value of some function $g(x)$,

$$\mathbb{E}g(X) = \int g(x)p(x)dx \approx \frac{1}{n}\sum_{i=1}^{n} g(X^i) \ .$$

This estimator will converge to the true expectation as $n \to \infty$ by the law of large numbers. Importance sampling is a technique by where an estimated expectation is computed using samples drawn from a different distribution. Suppose there is another density $q(x)$ for which we want to estimate $\mathbb{E}^Q g(X) = \int g(x)q(x)dx$, but we only have our samples drawn from density $p(x)$. By re-weighting these samples we have an estimate,

$$\mathbb{E}^Q g(X) = \int g(x)q(x)dx = \int \frac{q(x)}{p(x)} g(x)p(x)dx \approx \frac{1}{n}\sum_{i=1}^{n} \omega(X^i)g(X^i) \approx \mathbb{E}w(X)g(X) \ ,$$

where the importance weights $\omega(x)$ are the ratio of probability densities,

$$\omega(x) = \frac{q(x)}{p(x)} \ .$$

The density $p(x)$ is called *the proposal* and density $q(x)$ is called *the target.* A required condition selecting a proposal is to make sure that $p(x)$ and $q(x)$ are *equivalent* (i.e., $p(x) > 0$ if and only if $q(x) > 0$ for all x).

In finance, equivalent probability measures are motivated by arbitrage theory rather than sampling constraints. A thorough treatment of these arbitrage arguments is not presented here, but can be summarized by the 1st Fundamental Theorem of Asset Pricing (FTAP), which states that a market has no arbitrage if and only if there exists an equivalent measure under which discount prices are martingales. An example where FTAP presents itself is the binomial tree model.

Consider the following arithmetic Brownian motion,

$$dX_t = \mu dt + dW_t \quad \text{for } 0 \leq t \leq T \ ,$$

with $X_0 = 0$. Of interest to us in finance is a change of measure under which X_t is standard Brownian motion. We accomplish this the following density ratio,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T = e^{-\mu W_T - \frac{\mu^2 T}{2}} \ .$$

To gain an idea of how to construct such a change of measure, the following calculations should help:

$$\begin{aligned}
\mathbb{E}g(W_T) &= \frac{1}{\sqrt{2\pi T}} \int e^{-\frac{x^2}{2T}} g(x)dx \\
&= \frac{1}{\sqrt{2\pi T}} \int e^{-x\mu + \frac{\mu^2 T}{2}} e^{-\frac{(x-\mu T)^2}{2T}} g(x)dx \\
&= \mathbb{E}\left[e^{-X_T\mu + \frac{\mu^2 T}{2}} g(X_T)\right] \\
&= \mathbb{E}\left[e^{-W_T\mu - \frac{\mu^2 T}{2}} g(X_T)\right] \\
&= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T g(X_T)\right] \\
&= \int \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T g(x) d\mathbb{P}_{X_T}(x) \\
&= \int g(x) d\mathbb{Q}_{X_T}(x) \\
&= \mathbb{E}^Q g(X_T) \ ,
\end{aligned}$$

which shows that X_T's $\mathbb{Q}$-measure distribution is equal to W_T's $\mathbb{P}$-measure distribution. It is important to emphasize that his change of measure makes the *path* $(X_t)_{0\leq t \leq T}$ a Brownian motion. Indeed, for times $t_i = i\Delta t$ with $\Delta t \leq T/n$, for a function $g(x_0, x_1, \ldots, x_n)$ over the discrete path taken by X_t, we have

$$
\begin{aligned}
&\mathbb{E}^Q g(X_{t_1}, \dots, X_{t_n}) \\
&= \frac{1}{(2\pi\Delta t)^{n/2}} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\frac{(x_i - x_{i-1})^2}{2\Delta t}} g(x_1, \dots, x_n) dx_1 dx_2, \dots dx_n \\
&= \frac{1}{(2\pi\Delta t)^{n/2}} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\mu(x_i - x_{i-1} - \mu\Delta t) + \frac{\mu^2 \Delta t}{2}} e^{-\frac{(x_i - x_{i-1} - \mu\Delta t)^2}{2\Delta t}} g(x_1, \dots, x_n) dx_1 dx_2, \dots dx_n \\
&= \mathbb{E}\left[\prod_{i=1}^{n} e^{-\mu(X_{t_i} - X_{t_{i-1}}) + \frac{\mu^2 \Delta t}{2}} g(X_{t_1}, \dots, X_{t_n})\right] \\
&= \mathbb{E}\left[e^{-\mu W_{t_n} - \frac{\mu^2 t_n}{2}} g(X_{t_1}, \dots, X_{t_n})\right] \\
&= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{t_n} g(X_{t_1}, \dots, X_{t_n})\right] \ .
\end{aligned}
$$

Formal Statement of Theorem

For a stochastic process Y_t there is the so-called Doléans-Dade exponential,

$$\mathcal{E}_t(Y) = e^{Y_t - \frac{1}{2}[Y]_t}$$

where $[Y]_t$ denotes quadratic variation. The general statement of Girsanov's theorem is as follows: if $\mathcal{E}_T(Y)$ is a martingale on $[0,T]$ then $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T = \mathcal{E}_T(Y)$ is constitutes a change of measure, such that for any $\mathbb{P}$ martingale M_t, the process $M_t^Q := M_t - \int_0^t dM_t dY_t$ is a $\mathbb{Q}$-martingale. For further reading in Girsanov, see [Karatzas & Shreve (1991)] and [Musiela & Rutkowski (2006)].

Remark 3.1 (Finite Time)**.** *It is important to point out that change of measure cannot be applied over infinite time. An example why this cannot be done occurs with arithmetic Brownian motion with $\mu > 0$, for which $\mathbb{P}(\liminf_{t\to\infty} X_t < 0) = 0$ but we would want $\mathbb{Q}(\liminf_{t\to\infty} X_t = -\infty) = 1$ in order to be standard Brownian motion, and so there is no way to have equivalent probability measures.*

Remark 3.2 (Martingale Property)**.** *The Doléans-Dade exponential is not generally a martingale. There are many examples of continuous stochastic processes for which $\mathcal{E}_t(Y)$ is not a martingale.*

Example 3.1 (Changing Arithmetic to Standard Brownian Motion)**.** *For X_t being the arithmetic Brownian motion considered above, we constructed the change of measure with density $\mathcal{E}_t(-\mu W)$, and we saw that $W_t^Q := W_t + \mu t$ is a $\mathbb{Q}$-measure Brownian motion.*

4 European Options and Risk-Neutral Pricing

Let's return to our market with an asset price given by Geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \ ,$$

and with a risk-free bank account paying rate r. Consider a contract that pays $\psi(S_T)$ at time T, where $\psi(s)$ is a known function, and where the contracts has no possibility of early exercise. As per (5), the Black-Scholes PDE is

$$\frac{\partial f}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} - rf = 0$$
$$f(T,s) = \psi(s) \ .$$

Now, we can apply Itô's lemma in the following way,

$$\begin{aligned}
&e^{-rT}\psi(S_T)\\
&= f(0,S_0) + \int_0^T e^{-rt}\left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2} + \mu S_t\frac{\partial f}{\partial s} - rf\right)dt\\
&\qquad\qquad + \sigma\int_0^T e^{-rt}S_t\frac{\partial f}{\partial s}dW_t\\
&= f(0,S_0) + \int_0^T e^{-rt}\underbrace{\left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2} + rS_t\frac{\partial f}{\partial s} - rf\right)}_{=0}dt\\
&\qquad\qquad + \sigma\int_0^T e^{-rt}S_t\frac{\partial f}{\partial s}\left(\frac{\mu - r}{\sigma}dt + dW_t\right)\\
&= f(0,S_0) + \sigma\int_0^T e^{-rt}S_t\frac{\partial f}{\partial s}dW_t^Q \ ,
\end{aligned}$$

where $W_t^Q = \frac{\mu - r}{\sigma}t + W_t$ is Brownian motion under a Girsanov change of measure with

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T = \mathcal{E}_T\left(\frac{r-\mu}{\sigma}W\right) \ .$$

Thus, $\int_0^T e^{-rt}S_t\frac{\partial f}{\partial s}dW_t^Q$ is a $\mathbb{Q}$-martingale and we gave the derivative price given by a risk-neutral expectation of the payoff,

$$f(0,S_0) = e^{-rT}\mathbb{E}^Q\psi(S_T) \ ,$$

where $dS_t = rS_t dt + \sigma S_t dW_t^Q$. These calculations are a derivation of the Feynman-Kac formula and the valuation of derivatives via expectations under a change of measure.

The Feynman-Kac Formula

Consider any PDE of the form

$$\frac{\partial f}{\partial t} + \alpha(t,s)\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 f}{\partial s^2} - r(t,s)f + c(t,s) = 0$$

with terminal condition $f|_T = \psi(s)$. The solution can be represented as

$$f(t,s) = \mathbb{E}_{t,s}\left[\int_t^T e^{-\int_t^v r(u,S_u)du} c(v,S_v)dv + e^{-\int_t^T r(v,S_v)dv}\psi(S_T)\right] .$$

where $dS_t = \alpha(t,S_t)dt + \sigma(t,S_t)dW_t$, and $\mathbb{E}_{t,s}$ denotes expectation conditional on $S_t = s$.

5 Examples

The Black-Scholes PDE in (5) applies generally to all derivatives. Solving the PDe will depend on the specifications of the contract. European call and put options have explicit formulae with features that are broadly applicable to other derivative. We begin with the European call and put prices, and then move to other examples.

Example 5.1 (European Call Option)**.** *Perhaps the widely known formula in financial mathematics is the Black-Scholes call option formula. let the price of a non-dividend paying stock be a given by the Geometric Brownian motion in* (1)*. The option price is given by the Black-Scholes PDE in* (5)

$$\frac{\partial c}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 c}{\partial s^2} + rs\frac{\partial c}{\partial s} - rc = 0 ,$$

with terminal condition $c(T,s) = (s-K)^+$. Using the Feynman-Kac formula, we can represent the solution of the PDE as the risk-neutral expectation of the payoff,

$$c(t,s) = e^{-r(T-t)}\mathbb{E}^Q_{t,s}(S_T - K)^+ ,$$

where $dS_t = rS_t dt + \sigma S_T dW_t^Q$. The Feynman-Kac can be explicitly computed (without loss of generality take $t = 0$). Let $p_T(x)$ denote the risk-neutral density for $\log(S_T/S_0)$,

$$p_T(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} .$$

In terms of $p_T(x)$, we have

$$\begin{aligned} e^{-rT}\mathbb{E}^Q(S_T-K)^+ &= e^{-rT}\int_{-\infty}^{\infty}(S_0e^x - K)^+ p_T(x)dx \\ &= e^{-rT}\int_{\kappa}^{\infty}(S_0e^x - K)p_T(x)dx \end{aligned}$$

where $\kappa = \log(K/S_0)$. *Now, we have two integrals to compute:*

$$\begin{aligned}
&Ke^{-rT}\int_{\kappa}^{\infty} p_T(x)dx \\
&= \frac{Ke^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_{\kappa}^{\infty} e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}}dx \\
&= \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{v^2}{2}}dv \\
&= Ke^{-rT}(1-N(-d_2)) \\
&= Ke^{-rT}N(d_2)\ ,
\end{aligned}$$

where we used the change of variable $v = \frac{x-\left(\mu-\frac{1}{2}\sigma^2 T\right)}{\sigma\sqrt{T}}$, *denoted* $d_2 = \frac{\log(S_0/K)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, *and where* Φ *denotes the standard normal cumulative distribution function (CDF). For the other integral we have,*

$$\begin{aligned}
&S_0e^{-rT}\int_{\kappa}^{\infty} e^x p_T(x)dx \\
&= \frac{S_0e^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_{\kappa}^{\infty} e^x e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}}dx \\
&= \frac{S_0}{\sqrt{2\pi\sigma^2 T}}\int_{\kappa}^{\infty} e^{-\frac{\left(x-\left(r+\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}}dx \\
&= \frac{S_0}{\sqrt{2\pi}}\int_{-d_1}^{\infty} e^{-\frac{v^2}{2}}dv \\
&= S_0(1-N(-d_1)) \\
&= S_0N(d_1)\ ,
\end{aligned} \tag{6}$$

where $d_1 = \frac{\log(S_0/K)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$.[1] *This yields the celebrated Black-Scholes call option formula,*

$$c(t,s) = sN(d_1) - Ke^{-r(T-t)}N(d_2)\ , \tag{7}$$

where

$$\begin{aligned}
d_1 &= \frac{\log(s/K)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}\ .
\end{aligned}$$

[1] An important step in (6) is the completion of the square in the exponent,

$$-rT + \frac{2\sigma^2 Tx - \left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T} = -rT + \frac{2\sigma^2 T^2 r - \left(x-\left(r+\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T} = \frac{-\left(x-\left(r+\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}\ ,$$

which is how we get from the 2nd to the 3rd line in (6).

Example 5.2 (European Put Option)**.** *The European put option can be computed a fashion similar to how we computed European call option in the previous example, or put-call parity can be used to compute from the call price. For a non-dividend paying stock,*

$$c(t,s) + Ke^{-r(T-t)} = p(t,s) + s \ ,$$

where $c(t,s)$ is the European call price and given by (7), and $p(t,s)$ denotes the European put price. Re-arranging terms, we have the Black-Scholes put option formula,

$$p(t,s) = Ke^{-r(T-t)}N(-d_2) - sN(-d_1) \ , \tag{8}$$

with d_1 and d_2 as they were given in (7).

Example 5.3 (European Options on Dividend-Paying Stocks)**.** *For a stock with dividend yield $\delta > 0$ the SDE is*

$$\frac{dS_t}{S_t} = (\mu - \delta)dt + \sigma dW_t \ ,$$

and the Δ-hedging portfolio has PnL

$$d\Pi_t = -dc(t,S_t) + \frac{\partial c}{\partial s}dS_t + r\left(\Pi_t + c(t,S_t) - \frac{\partial c}{\partial s}S_t\right)dt + \delta\frac{\partial c}{\partial s}S_t dt = r\Pi_t dt \ .$$

From this equation for Π_t we an write a Black-Scholes PDE whoe solution is the Feynmam-Kac formula. For the call option, the price is $c(t,s) = e^{-r(T-t)}\mathbb{E}^Q_{t,s}(S_T - K)^+$ with risk-neutral SDE

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW^Q_t \ .$$

An explicit formula can be found by exploiting the calculations made for the non-dividend paying call,

$$\begin{aligned} c(t,s) &= e^{r(T-t)}\mathbb{E}^Q_{t,s}(S_T - K)^+ \\ &= e^{-\delta(T-t)}\left(e^{-(r-\delta)(T-t)}\mathbb{E}^Q_{t,s}(S_T - K)^+\right) \\ &= e^{-\delta(T-t)}\left(sN(d_1) - Ke^{-(r-\delta)(T-t)}N(d_2)\right) \\ &= se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \ , \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(s/K) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \ . \end{aligned}$$

Put-call parity can be used to find the European put option formula.

Example 5.4 (FOREX Options). *Let r denote the risk-free rate in the domestic economy and let r_f denote the risk-free rate in a foreign economy. Let the risk-neutral SDE for the exchange rate (i.e., the price to obtain 1 unit of foreign currency in exchange for the domestic currency) be*

$$\frac{dS_t}{S_t} = (r - r_f)dt + \sigma dW_t^Q \ ,$$

where W_t^Q is a risk-neutral Brownian motion for assets in the domestic economy. A European call option for purchasing the foreign currency at strike K is price using the Black-Scholes formula for a stock with dividend yield,

$$c(t,s) = se^{-r_f(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \ ,$$

where

$$\begin{aligned} d_1 &= \frac{\log(s/K) + \left(r - r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \ . \end{aligned}$$

To price a call option for purchasing the domestic currency in terms of the foreign currency involves a change of measure. The exchange rate from the foreign economy is $1/S_t$, and using Itô's lemma we see that

$$d\left(\frac{1}{S_t}\right) = -\frac{dS_t}{S_t^2} + \frac{(dS_t)^2}{S_t^3} = \frac{r_f - r - \sigma^2}{S_t}dt - \frac{\sigma}{S_t}dW_t^Q = \frac{r_f - r}{S_t}dt - \frac{\sigma}{S_t}dW_t^{Q_f} \ ,$$

where $W_t^{Q_f} = W_t^Q + \sigma t$ is Brownian motion under a change of measure with

$$\frac{d\mathbb{Q}_f}{d\mathbb{Q}} = \mathcal{E}_t(-\sigma W^Q) = e^{-\sigma W_t^Q - \frac{\sigma^2 t}{2}} \ .$$

Letting $X_t = 1/S_t$, the call option to buy domestic currency from the foreign economy is

$$c^f(t,s) = e^{-r_f(T-t)}\mathbb{E}_{t,x}^{Q_f}(X_T - K)^+ = xe^{-r(T-t)}N(d_1) - Ke^{-r_f(T-t)}N(d_2) \ ,$$

where

$$\begin{aligned} d_1 &= \frac{\log(x/K) + \left(r_f - r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \ . \end{aligned}$$

Example 5.5 (Options on Futures). *Consider future contract with delivery at time T, with price given by the SDE*

$$\frac{dF_t}{F_t} = \mu dt + \sigma dW_t \qquad \textit{for } t \leq T \ ,$$

where $\mu > 0$ if the futures market is in backwardation and $\mu < 0$ if the market is in contango. Let us derive a Black-Scholes PDE for a European option on this future. For a hedging portfolio using futures there is daily settlement, and so the value of futures positions do not affect PnL as they did in (3)*, and we have*

$$d\Pi_t = -dc(t, F_t) + \frac{\partial c}{\partial F} dF_t + r\left(\Pi_t + c(t, F_t)\right) dt \ ,$$

and thus the Black-Scholes PDE is

$$\frac{\partial c}{\partial t} + \frac{\sigma^2 F^2}{2}\frac{\partial^2 c}{\partial F^2} - rc = 0 \ ,$$

with terminal condition $c|_T = (F - K)^+$. The Feynman-Kac formula for the solution to this PDE is

$$c(t, F) = e^{-r(T-t)} \mathbb{E}^Q_{t,F}(F_T - K)^+ \ ,$$

where $dF_t = \sigma F_t dW_t^Q$. There is an explicit Black-Scholes formula for this call option, named ***Black's formula:***

$$c(t, F) = FN(d_1) - Ke^{-r(T-t)} N(d_2) \ ,$$

where

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t} \ .$$

Example 5.6 (Corporate Debt)**.** *The original papers of [Black & Scholes (1973)] and [Merton (1974)] did not focus on stock options or individual securities, but instead presented a theory for valuing corporate debt. Let A_t denote the value of a company's total assets at time t, let E_t denote the value of equity, and let D_t denote the value of debt, so that*

$$A_t = E_t + D_t \ .$$

Suppose that the debt was originally issued as a zero-coupon bond with face value K. The value to the bond holder(s) as time T is

$$D_T = \min(A_T, K) = K - (K - A_T)^+ \ ,$$

thus showing the value of bond will equal the discounted value of a zero-coupon bond minus a European put. If

$$\frac{dA_t}{A_t} = (\mu - \delta)dt + \sigma dW_t \ ,$$

where $\delta > 0$ is the company's dividend yield, then the corporate bond's values is

$$D_t = Ke^{-r(T-t)}\Phi(d_2) + A_t e^{-\delta(T-t)}N(-d_1) \ ,$$

where

$$\begin{aligned} d_1 &= \frac{\log(A_t/K) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \ . \end{aligned}$$

Example 5.7 (Hull-White Method)**.** *Suppose that we have a short-rate model,*

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma(r_t)dW_t \ ,$$

and also two zero-coupon bonds B^1 and B^2 with maturities T_1 and T_2, respectively, with $T_1 < T_2$. We can hedge B^1 with B^2 with a hedge ration $h_t = \frac{\partial B^1(t,r_t)}{\partial r} / \frac{\partial B^2(t,r_t)}{\partial r}$,

$$\begin{aligned} &d\Pi_t \\ &= -dB^1(t,r_t) + \frac{\frac{\partial B^1(t,r_t)}{\partial r}}{\frac{\partial B^2(t,r_t)}{\partial r}} dB^2(t,r_t) + r_t\left(\Pi_t + B^1(t,r_t) - \frac{\frac{\partial B^1(t,r_t)}{\partial r}}{\frac{\partial B^2(t,r_t)}{\partial r}} B^2(t,r_t)\right)dt \\ &= -\left(\frac{\partial B^1(t,r_t)}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial B^1(t,r_t)}{\partial r} + \frac{\sigma^2(r_t)}{2}\frac{\partial^2 B^1(t,r_t)}{\partial r^2}\right)dt \\ &\qquad + \frac{\frac{\partial B^1(t,r_t)}{\partial r}}{\frac{\partial B^2(t,r_t)}{\partial r}}\left(\frac{\partial B^2(t,r_t)}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial B^2(t,r_t)}{\partial r} + \frac{\sigma^2(r_t)}{2}\frac{\partial^2 B^2(t,r_t)}{\partial r^2}\right)dt \\ &\qquad\qquad + r_t\left(\Pi_t + B^1(t,r_t) - \frac{\frac{\partial B^1(t,r_t)}{\partial r}}{\frac{\partial B^2(t,r_t)}{\partial r}} B^2(t,r_t)\right)dt \\ &= r_t\Pi_t dt \ , \end{aligned}$$

where the final equality follows from absence of arbitrage in the overnight interest rate markets. Next, cancel the $r\Pi_t$ on both sides of the equation, divide through by $\frac{\partial B^1(t,r_t)}{\partial r}$ and collect B^1 terms on one side and B^2 terms on the other,

$$\begin{aligned} &\frac{1}{\frac{\partial B^1(t,r_t)}{\partial r}}\left(\frac{\partial B^1(t,r_t)}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial B^1(t,r_t)}{\partial r} + \frac{\sigma^2(r_t)}{2}\frac{\partial^2 B^1(t,r_t)}{\partial r^2} - r_t B^1(t,r_t)\right) \\ &= \frac{1}{\frac{\partial B^2(t,r_t)}{\partial r}}\left(\frac{\partial B^2(t,r_t)}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial B^2(t,r_t)}{\partial r} + \frac{\sigma^2(r_t)}{2}\frac{\partial^2 B^2(t,r_t)}{\partial r^2} - r_t B^2(t,r_t)\right) \\ &=: -\theta(t,r_t) \ , \end{aligned}$$

where $\theta(t,r)$ is the risk premium that does not depend in maturity; ***this is the Hull-White method.*** *From here, we can have a PDE for the price of any zero-coupon bond,*

$$\frac{\partial B}{\partial t} + \frac{\sigma^2(r)}{2}\frac{\partial^2 B}{\partial r^2} + \left(\lambda(\bar{r} - r) + \theta(t,r)\right)\frac{\partial B}{\partial r} - rB = 0 \ , \tag{9}$$

with terminal condition $B|_T$ = face value. The solution to (9) *is represented with the Feynman-Kac formula,*

$$B(t,r) = FV \times \mathbb{E}^Q_{t,r} e^{-\int_t^T r_s ds} \ , \tag{10}$$

with $dr_t = \left(\lambda(\bar{r} - r_t) + \theta(t,r_t)\right) dt + \sigma(r_t) dW_t^Q$, where FV denotes face value.

Example 5.8 (The Vasicek Model)**.** *The formula in* (10) *can used to price in the Vasicek short-rate model,*

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma dW_t^Q \ .$$

In this case $\int_t^T r_s ds$ is normally distributed with mean and variance explicitly computable, and thus the bond price is explicit too. We could also compute the value of call option on the bond using a Black-Scholes-type formula.

Example 5.9 (The CIR Model)**.** *The PDE in* (9) *can used to price in the Cox-Ingersol-Ross (CIR) model,*

$$dr_t = \lambda(\bar{r} - r_t)dt + \sigma\sqrt{r_t} dW_t^Q \ .$$

In the case the PDE is

$$\frac{\partial B}{\partial t} + \frac{\sigma^2 r}{2}\frac{\partial^2 B}{\partial r^2} + \left(\lambda(\bar{r} - r) + \theta(t,r)\right)\frac{\partial B}{\partial r} - rB = 0 \ ,$$

which can be solve with ansatz $B(t,r) = e^{a(t)r + b(t)}$, with $a(t)$ satisfying an explicitly solvable Riccati equation and $b(t)$ being the integral of $a(t)$ plus a constant.

Example 5.10 (American Options)**.** *Let use consider the risk-neutral SDE for a non-dividend paying stock,*

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^Q \ ,$$

where $r > 0$ is the risk-free rate. An American put option is the optimal over stopping times,

$$P(t,s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^Q_{t,s} \left(K - S_{T \wedge \tau}\right)^+ \ .$$

There is a PDE for this option, but it has a a free boundary,

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 P}{\partial s^2} + rs\frac{\partial P}{\partial s} - rP = 0 \quad &\textit{for } \frac{\partial P}{\partial s} > -1 \\ P(t,s) = k - s \quad &\textit{for } \frac{\partial P}{\partial s} = -1 \\ P(T,s) = (k - s)^+ \ . & \end{aligned}$$

The boundary is considered 'free' because it depends on the solution. In other words, the optimal exercise rule of the American option is hard to find, but if we knew it (i.e., if we knew $s^*(t)$ such that $P(t,s) > k - s$ for all $s > s^*(t)$) then solving this PDE would be an easier problem. However, binomial trees are a reliable way to solve numerically, but each tree only solves for a single initial value S_0.

Example 5.11 (The Heat Equation)**.** *This is a non-finance example, but which can serve to help better understand the relationship between Brownian motion and PDEs. Consider the PDE*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

with terminal condition $u|_T = g$.The Feynman-Kac formula is

$$u(t,x) = \mathbb{E}_{t,x} g(X_T) \ ,$$

where $dX_t = dW_t$. We can express this solution with a density,

$$u(t,x) = \int p(T-t, y-x) g(y) dy \ ,$$

where

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \ .$$

Using the PDE for u, we have

$$-\frac{\partial}{\partial t}\int p(T-t, x-y)g(y) = -\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) = \frac{1}{2}\int \frac{\partial^2}{\partial x^2}p(T-t, y-x)g(y)dy \ ,$$

and because this integral equation holds for any $g(x)$ we deduce an equation for p,

$$\frac{\partial}{\partial t}p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x) \ ,$$

with initial condition $p(0,x) = \delta(x)$, that is, the forward heat equation is the density for a Brownian motion particle.

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