Chapter 6. Boundary-condition problems in two and three dimensions: Cylindrical coordinates. (25 Jan 2020)

A. Perspective.	1
B. Cylindrical coordinates (ρ,θ,φ).	2
C. Laplacian in cylindrical coordinates: general properties and Bessel functions.	2
D. The right-circular cylinder: examples.	7
E. Nontraditional configurations: the three-dimensional wedge.	10
F. Two-dimensional configurations.	12
G. Image-charge solution in two dimensions.	16

A. Perspective.

In Ch. 5, we developed methods of solving boundary-value problems in electrostatics with analytic functions and infinite series, and applied these methods to configurations defined by constant Cartesian coordinates. In order of increasing computational power, closed-form approaches include the image-charge method of determining potentials and Green functions, and conformal mapping in two dimensions. Series approaches include direct expansion of the potential where each term satisfies Laplace's Equation, expansion of the Green function by similar series, and expansion of the Green function where each term is a product of orthogonal functions.

The same approaches apply to configurations defined by constant cylindrical and spherical coordinates. The radial eigenfunctions of the cylindrical-coordinate Laplacian are Bessel functions. These eigenfunctions connect to those of the azimuthal and longitudinal coordinates of the Laplacian operator through Bessel's Equation. In the two-dimensional limit, the cylindrical Bessel functions reduce to monomials, as do the spherical Bessel functions of the wave equation.

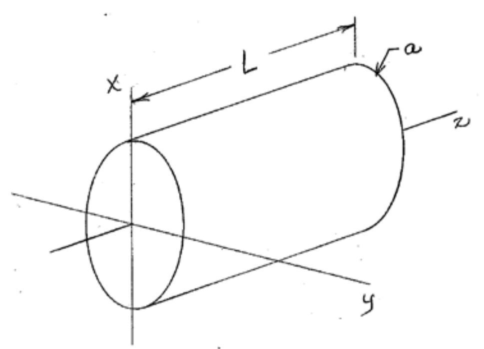
In his Chs. 2 and 3, Jackson presents material on configurations that involve both cylindrical and spherical coordinates, with emphasis on mathematics. Thus it appears that little can be added. However, Jackson's emphasis on math tends to obscure the real goal, which is how to solve boundary-value problems, not necessarily only those that involve elementary functions. Also, Jackson's mathematics would benefit by being placed in the larger context of the Bessel and Legendre functions that form the basis of these solutions. Consequently, there are some advantages to organizing the material differently, especially since our objective is to emphasize methods over math.

Finally, there is no reason to restrict configurations to those whose solutions are expressed in elementary or even familiar functions. We consider several where monomials are raised to imaginary powers. However, we do avoid configurations that require Bessel functions of imaginary order or Legendre functions, leaving them to the next generation. Imaginary-order Bessel functions have only recently been studied. See for example T. M. Dunster, SIAM J. Math Anal. **21**, 995-1018 (1990).

B. Cylindrical coordinates (ρ,θ,φ).

Cylindrical coordinates provide a more useful starting point than spherical coordinates for several reasons. First, the cylindrical-coordinate Laplacian includes the familiar $\partial^2/\partial z^2$ operator from Ch. 5. Second, the radial solution depends on the functions used for φ and *z*, illustrating an interdependence that will appear again in solutions of the wave equation in spherical coordinates. Third, the two-dimensional limit of cylinders of circular cross section is particularly useful, applicable for example to waveguides and optical fibers, both of which will be discussed next semester. Fourth, cylindrical configurations defined by (coordinate = constant) have a wide variety of shapes and hence are more interesting than those defined in either Cartesian or spherical systems. We will also consider wedges, including potentials specified on the sides of a wedge. In cylindrical coordinates the mathematical challenges arise from the radial coordinate, whereas in spherical coordinates the polar angle provides the complication.

The classic configuration is the "tin can" shown in the figure. The surface consists of three separate facets: a cylindrical shell of length *L* and circular cross section of radius *a* centered on the *z* axis, and two end caps that are perpendicular to this axis. The symmetry is too low to allow an image-charge solution unless $L \to \infty$, which reduces the Laplacian to two dimensions. Thus analysis is restricted to series solutions. The tin can is a useful start, but as physicists interested in methods of solving problems as well as solutions we can do better than that. This will take us to functions not normally encountered in physics, or even in mathematics.



Regarding series solutions, we focus on expansions of $\phi(\vec{r})$ done one facet at a time, requiring $\phi(\vec{r})$ to vanish on the other two. We know that this ensures that the solutions for all three facets are independent. We also know that expansions of $\phi(\vec{r})$ are sufficient, because we can use these to construct double-summation Green functions directly from these terms. We follow the recipeprovided in Ch. 5, starting with the Laplacian, listing its eigenfunctions and eigenvalues, then assembling these into direct expansions of the potential according to the configuration being considered.

C. Laplacian in cylindrical coordinates: general properties and Bessel functions.

The Laplacian in cylindrical coordinates (ρ, φ) is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}, \tag{6.1}$$

where $\rho = \sqrt{x^2 + y^2}$, $x = \rho\cos\varphi$, and $y = \rho\sin\varphi$. Although the eigenfunctions $Z(z)$ and $Q(\varphi)$ of $\partial^2/\partial z^2$ and $\partial^2/\partial\varphi^2$, respectively, are obvious, we list them anyway:

$\partial^2/\partial z^2$, negative curvature: $e^{\pm ikz}$ and linear combinations $\cos(kz)$ and $\sin(kz)$, eigenvalue $-k^2$;

$\partial^2/\partial z^2$, positive curvature: $e^{\pm kz}$ and linear combinations $\cosh(kz)$ and $\sinh(kz)$, eigenvalue $+\kappa^2$;

$\partial^2/\partial\varphi^2$, negative curvature: $e^{\pm i\nu\varphi}$ and linear combinations $\cos(\nu\varphi)$ and $\sin(\nu\varphi)$, eigenvalue $-\nu^2$;

$\partial^2/\partial\varphi^2$, positive curvature: $e^{\pm\nu\varphi}$ and linear combinations $\cosh(\nu\varphi)$ and $\sinh(\nu\varphi)$, eigenvalue $+\nu^2$.

Then writing $\phi(\vec{r}) = R(\rho)Q(\varphi)Z(z)$, the equation for $R(\rho)$ reduces to

$$\nabla^2 R(\rho) = \left(\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) \mp \frac{\nu^2}{\rho^2} \mp k^2\right)R(\rho) = 0 \tag{6.2a}$$

$$= \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} \mp \frac{\nu^2}{\rho^2} \mp k^2\right)R(\rho) = 0\,, \tag{6.2b}$$

where in Eq. (6.2b) we expanded the derivative operator. For our initial examples, ν is an integer, dictated by the fact that these examples require $e^{\pm i\nu\varphi}$ to be single-valued.

Equation (6.2b) can be compared to Bessel's Equation

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - \nu^2 + z^2\right)w(z) = 0\,, \tag{6.3a}$$

which has solutions $w(z) = J_\nu(z)$ and $Y_\nu(z)$, or the alternate version

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - \nu^2 - z^2\right)w(z) = 0\,, \tag{6.3b}$$

which has solutions $w(z) = I_\nu(z)$ and $K_\nu(z)$. In Eqs. (6.3) z is a dimensionless parameter that may be complex. Note that these functions $w(z)$ are defined with the *negative*-curvature eigenfunctions $e^{\pm i\nu\varphi}$, and hence represent only half the possibilities. We pick up the reasons for the missing four functions below.

Accepting that for now we'll use the negative-curvature functions for $Q(\varphi)$, we can convert Eq. (6.2b) into Bessel's Equations by multiplying through by ρ^2. The result is

$$\left(\rho^2\frac{d^2}{d\rho^2} + \rho\frac{d}{d\rho} - \nu^2 \pm (k\rho)^2\right)R(\rho) = 0\,, \tag{6.4}$$

We see by inspection that $z = k\rho$, which applied throughout given that any scaling factor of ρ cancels in the first two terms. It is also consistent with the requirement that z in Eqs. (6.3) is dimensionless. Using $J_\nu(z) = J_\nu(k\rho)$ as an example, we also see that the ν

of ν^2 is the ν of $e^{\pm i\nu\varphi}$, and the *k* of $\pm k^2\rho^2$ is the *k* of $e^{\pm ikz}$ (negative sign) or $e^{\pm kz}$ (positive sign). Thus in contrast to the situation for Cartesian coordinates, where the functions are independent except through their eigenvalue connection, the radial function here is intimately connected to the behavior of $Q(\varphi)$ and $Z(z)$.

We can now provide some intuitive insight into the behavior of the solutions of Eq. (6.4). Suppose first that $\rho \to \infty$. Then the first and last terms will dominate. Cancelling the common term ρ^2 and taking the positive sign, the equation for $R(\rho)$ reduces to

$$\left(\frac{d^2}{d\rho^2} + k^2\right) R(\rho) = 0 . \tag{6.5}$$

The solutions here are obviously $e^{\pm ik\rho}$ or the linear combinations $\cos(kr)$ and $\sin(kr)$, whereas those with the negative sign approach the real exponentials $e^{\pm k\rho}$. Thus we can conclude that $J_\nu(k\rho)$ and $Y_\nu(k\rho)$ are oscillatory, with negative eigenvalues, while $I_\nu(k\rho)$ and $K_\nu(k\rho)$ are monotonic, with positive eigenvalues.

We now consider the limit $\rho \to 0$. Considering Eq. (6.4) with either sign of the fourth term, we can suppose that in this case the last term becomes arbitrarily small with respect to the first three. Alternatively, we can simply let $k \to 0$, which is equivalent to eliminating the *z* dependence of Eq. (6.1). The equation is now reduced to two dimensions. Reconstructing the derivatives as

$$\left(\rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho}\right) - \nu^2\right) R(\rho) = 0 , \tag{6.6}$$

we use as a trial solution the monomial $R(\rho) = \rho^l$. Substituting this into Eq. (6.6) yields

$$\left(\rho \frac{d}{d\rho}\left(l\rho\,\rho^{l-1}\right) - \nu^2\rho^2\right) = \left(l^2 - \nu^2\right)\rho^2 = 0 , \tag{6.7}$$

or $l = \pm\nu$. The functions $J_\nu(k\rho)$ and $I_\nu(k\rho)$ exhibit $(k\rho)^\nu$ behavior for small $(k\rho)$ and hence converge as $\rho \to 0$, whereas $Y_\nu(k\rho)$ and $K_\nu(k\rho)$ exhibit $(k\rho)^{-\nu}$ behavior for small $(k\rho)$, and hence diverge as $\rho \to 0$. We note that the small-*z* limit also follows from letting $k \to 0$; the result is the set of equations for two-dimensional cylindrical symmetry. The table on the next page provides a useful summary.

$J_\nu(k\rho)$	$Y_\nu(k\rho)$	Oscillatory (negative eigenvalues); possesses orthogonality relations	
$I_\nu(k\rho)$	$K_\nu(k\rho)$	Monatonic (positive eigenvalues); no orthogonality relations	
Converges/diverges as $(k\rho)^\nu$ for small/large $k\rho$	Diverges/converges as $(k\rho)^{-\nu}$ for small/large $k\rho$		

Specifically, for small $k\rho = z$

$$J_\nu(z) \to \frac{z^\nu}{2^\nu \nu!}; \tag{6.8a}$$

$$Y_0(z) \to \frac{2}{\pi}\left(\ln(\frac{z}{2}) + \gamma\right); \quad Y_\nu(z) \to -\frac{1}{\pi}\left(\frac{2}{z}\right)^\nu \text{ for } \nu \geq 1; \tag{6.8b,c}$$

$$I_\nu(z) \to \left(\frac{z}{2}\right)^\nu; \tag{6.8d}$$

$$K_\nu(z) \to \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu. \tag{6.8e}$$

where $\gamma = \lim_{m\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{m} - \ln(m)\right) = 0.57721...$ is Euler's constant.

For large $k\rho = z$:

$$J_\nu(z) \to \sqrt{\frac{2}{\pi z}}\cos(z - \nu\pi/2 - \pi/4); \tag{6.9a}$$

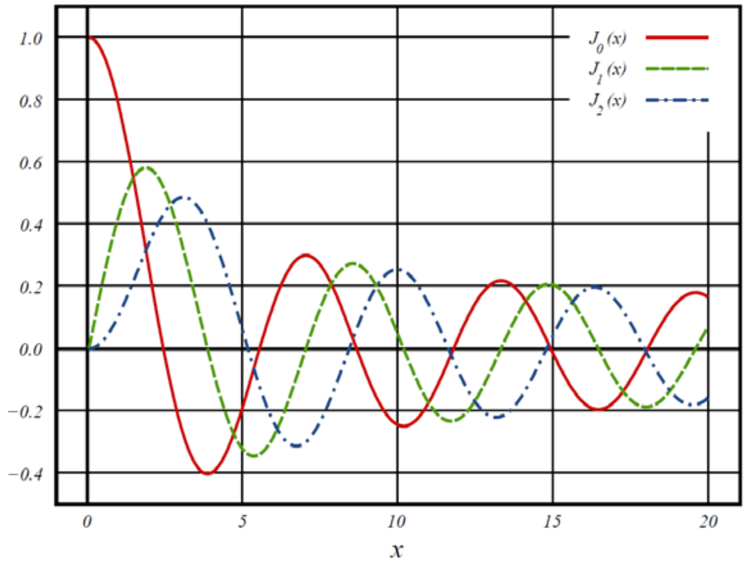
$$Y_\nu(z) \to \sqrt{\frac{2}{\pi z}}\sin(z - \nu\pi/2 - \pi/4). \tag{6.9b}$$

$$I_\nu(z) \to \frac{1}{\sqrt{2\pi z}}e^z; \tag{6.9c}$$

$$K_\nu(z) \to \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{6.9d}$$

The large-z expressions will be particularly relevant next semester when we discuss optical fibers, because their outer regions are formed from concentric shells.

Many books have been written about Bessel functions. In the boundary-condition problems discussed in Ch. 6, we'll use mainly the functions $J_\nu(x)$ and $I_\nu(x)$ that are regular at $x = 0$. In particular, The functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ are shown in the figure. As expected for functions with negative eigenvalues, the functions are oscillatory. The figure and Eq. (6.9a) show that the $J_\nu(x)$ are qualitatively the cylindrical versions of $\sin(x)$ and $\cos(x)$.



However, there are two significant differences. First, while there is only one $\sin(x)$ and one $\cos(x)$, every $J_\nu(x)$ is different. Second, while the zeroes of $\sin(x)$ and $\cos(x)$ are equally spaced by $\Delta x = \pi$, the figure shows that the zeroes of $J_\nu(x)$ are not. In solving boundary-condition problems, we will encounter situations where the series terms must vanish on the side of the cylinder, or $\phi(a,\varphi,z) \sim J_\nu(ka) = 0$. This is accomplished by defining a set of real numbers $\{x_{\nu n}\}$ such that

$$J_\nu(x_{\nu n}) = 0, \tag{6.10}$$

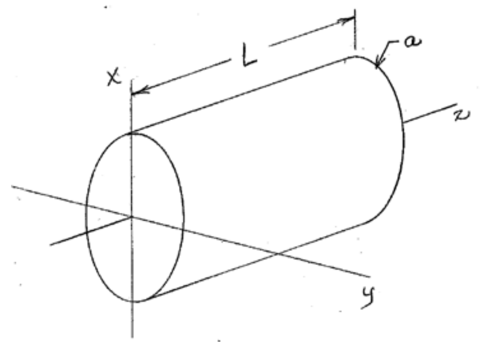
where the ν of $x_{\nu n}$ refers to the order ν of the Bessel function, and *n* to its n^{th} zero. Thus by setting $J_\nu(k\rho) = J_\nu(k_{\nu n}a) = J_\nu(x_{\nu n}) = 0$, we ensure that $\phi = 0$ on the sidewall. For example, as seen in the figure, $x_{01} = 2.405$ and $x_{12} = 7.016$.

As noted above, the $J_\nu(kr)$ satisfy an orthogonality relation. Expressed in variables that will be most useful to us, this is

$$\int_0^a \rho\, d\rho\, J_\nu(x_{\nu n}\rho/a)\, J_\nu(x_{\nu n'}\rho/a) = \frac{a^2}{2}\left(J_{\nu+1}(x_{\nu n})\right)^2 \delta_{nn'}\,. \tag{6.11}$$

It may seem strange that the orthogonality relation involves two functions $J_\nu(x)$ of the same ν but different $x_{\nu n}$. This creates no difficulty in practice, because when orthogonality is applied to $e^{\pm i(\nu-\nu')\varphi}$, it ensures that the order ν of the two Bessel functions is the same. In his Sec. 3.7 Jackson uses a page to prove Eq. (6.11), but I prefer to cite this as a reference for the proof and simply use the result. You can do the proof yourselves, since it relies on Green's Theorem, which you already know. In the meantime Eq. (6.11) is a good example of why reference books are needed.

More generally, it is not necessary that ν be an integer, or even real, although Bessel functions with complex values of ν have not been thoroughly studied. On the other hand, the spherical Bessel functions, where ν is half-integral, and Airy functions, where $\nu = \pm 1/3$, are well known. If ν is real, then $\nu^2 = (-\nu)^2$, so it follows that $J_\nu(z) = J_{-\nu}(z)$, where $J_\nu(z)$ is any of the Bessel functions. This allows us to use linear combinations such as $\sin(\nu\varphi)$ and $\cos(\nu\varphi)$ without complications.



D. Right circular cylinder: examples.

We consider first the situation where the potential $\phi(\rho,\varphi,0) = V(\rho,\varphi)$ is given on the $z = 0$ end cap of a right circular cylinder. The configuration is shown in the above diagram. The two end caps are separated by a distance *L* and connected by a cylinder of radius *a*. Because each term must vanish at $\rho = a$ and $z = L$, the appropriate expansion is

$$\phi(\vec{r}) = \sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{\nu n} J_\nu(k_{\nu n}\rho) e^{i\nu\varphi} \sinh\big(k_{\nu n}(L-z)\big) \tag{6.12}$$

where $k_{\nu n} = x_{\nu n}/a$. Equation (6.15) satisfies Laplace's Equation on a term-by-term basis, assigns the orthogonal functions to the $z = 0$ surface where $V(\rho,\varphi)$ is provided, and vanishes on the sidewall and the endplate at $z = L$. Equation (6.12) also provides an excellent example of how the parameters interact. The order ν of $J_\nu(k_{\nu n}\rho)$ is the ν of $e^{i\nu\varphi}$. The scaling factor $k_{\nu n}$ of $\sinh\big(k_{\nu n}(L-z)\big)$ is quantized by the requirement that $J_\nu(k_{\nu n}a) = 0$. Because a complete expansion requires $-\infty < \nu < \infty$, the importance of the relation $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$ becomes clear. If we prefer, this also allows us to replace $e^{i\nu\varphi}$ with the real functions $\cos(\nu\varphi)$ or $\sin(\nu n)$, and restrict the sum over ν to $0 \le \nu < \infty$. Thus all parameters except $A_{\nu n}$ are defined. These are obtained in the usual way from $\phi(\rho,\varphi,0) = V(\rho,\varphi)$, by orthogonality using the well-known relation for $e^{i(\nu-\nu')\varphi}$ and Eq. (6.11).

Pushing the example further, let $V(\rho,\varphi) = V$ be constant on the $z = 0$ face. The Fourier transform of V in the φ dimension vanishes unless $\nu = 0$. Then using the relation

$$\frac{1}{z}\frac{d}{dz}\big(z^\nu J_\nu(z)\big) = z^{\nu-1} J_{\nu-1}(z), \tag{6.13}$$

we have

$$V\int_0^a \rho\, d\rho\, J_0(k_{0n}\rho) = \frac{V}{k_{0n}}\rho J_1(k_{0n}\rho)\Big|_0^a = \frac{V}{k_{0n}} a J_1(k_{0n}a) \tag{6.14a}$$

$$= \frac{a^2}{x_{0n}} J_1(x_{0n}) \tag{6.14b}$$

so

$$A_{0n} = \frac{2V}{x_{0n} J_1(x_{0n}) \sinh(x_{0n} L/a)} . \tag{6.15}$$

Therefore

$$\phi(\vec{r}) = \sum_{n=1}^{\infty} \frac{2VJ_0(x_{0n}\rho/a) \sinh\left(x_{0n}(L-z)/a\right)}{x_{0n} J_1(x_{0n}) \sinh(x_{0n} L/a)} . \tag{6.16}$$

If the potential *V* had been given on the end cap $z = L$, then $\sinh(k_{0n}(L-z))$ would be replaced by $\sinh(k_{0n}z)$ and the same process followed. Details are left as a homework assignment. Note that the penetration of the n^{th} component into the configuration decreases exponentially with a characteristic length a/x_{0n}, showing again that high harmonics cut off very close to their facet.

Next, suppose that the potential $\phi(a,\varphi,z) = V(\varphi,z)$ is specified on the side. Then the appropriate expansion is

$$\phi(\vec{r}) = \sum_{\nu=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{\nu n} I_\nu(k_n\rho) e^{i\nu\varphi} \sin(k_n z) , \tag{6.17}$$

where $k_n = \pi n/L$. The parameter k is now determined by $\sin(k_n z)$, and sets the scale of $I_\nu(k_n\rho)$. Using again the example of constant *V*, orthogonality in φ requires $\nu = 0$, so only $I_0(k_n\rho)$ need be retained. Applying orthogonality in the *z* direction gives

$$A_{0n} = A_n = \frac{4}{\pi n I_0(k_n a)} , \tag{6.18}$$

for odd *n*, so the interior potential is

$$\phi(\vec{r}) = \sum_{\substack{n=1 \\ odd\ n}}^{\infty} \frac{4VI_o(k_n\rho)\sin(k_n z)}{\pi n I_o(k_n a)} . \tag{6.19}$$

Again, details are left as a homework assignment.

Equations (6.16) and (6.19) are connected in an interesting way. Suppose all surfaces are at a potential *V*. Then the interior potential everywhere inside the configuration is *V*. Now add the negative of Eq. (6.19) to *V*, bringing the potential of the end caps to zero and reproducing the conditions leading to Eq. (6.14). Again, the influence of the end caps decreases exponentially as one moves to the interior of the cylinder. If the cylinder is long enough, $k_{0n}\rho = x_{0n}\rho/L \to 0$ and the deep interior can be treated as a two-dimensional problem independent of the end caps.

The above deal with expansions of the potential. As we saw in the previous chapter, expansions of the Green function can be done either with a triple-summation series based entirely on orthogonal functions, or a double-summation version where each term is a solution of Laplace's Equation.

We consider the triple-summation version first. This is

$$G_D(\vec{r},\vec{r}\,') = \sum_{\nu=-\infty}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{\nu nm} J_\nu(k_{\nu n}\rho) e^{i\nu\varphi}\sin(k_m z)\,, \tag{6.20}$$

where

$$k_{\nu n} = \frac{x_{\nu n}}{a}\,; \quad k_m = \frac{\pi m}{L}\,; \tag{6.21a,b}$$

since each term must vanish on all surfaces. The $A_{\nu nm}$ are found in the usual way. Equation (6.20) is inserted into

$$\nabla^2 G_D(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') = -4\pi\delta(r-r')\frac{1}{r'}\delta(\varphi-\varphi')\delta(z-z')\,, \tag{6.22}$$

and orthogonality invoked:

$$\nabla_{\vec{r}}^2 G_D(\vec{r},\vec{r}\,') = \nabla_{\vec{r}}^2\left(\sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{\nu nm} J_\nu(k_{\nu n}\rho) e^{i\nu\varphi}\sin(k_m z)\right) \tag{6.23a}$$

$$= \sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-k_{\nu n}^2 - k_m^2\right) A_{\nu nm} J_\nu(k_{\nu n}\rho) e^{i\nu\varphi}\sin(k_m z) \tag{6.23b}$$

$$= -4\pi\,\delta(\rho-\rho')\,\frac{1}{\rho}\delta(\varphi-\varphi')\,\delta(z-z')\,. \tag{6.23c}$$

Operating on both sides with

$$\int_0^a \rho\, d\rho \int_0^{2\pi} d\varphi \int_0^L dz\, J_{\nu'}(x_{\nu'n'}\rho/a) e^{-i\nu'\varphi}\sin(k_{m'} z) \tag{6.24}$$

and doing some algebra yields

$$G_D(\vec{r},\vec{r}\,') = \sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{8\, J_\nu(k_{\nu n}\rho') J_\nu(k_{\nu n}\rho)\, e^{i\nu(\varphi-\varphi')}\,\sin(k_m z')\sin(k_m z)}{a^2 L(k_{\nu n}^2 + k_m^2)[J_{|\nu|+1}(x_{\nu n})]^2}\,. \tag{6.25}$$

$G_D(\vec{r},\vec{r}\,')$ is symmetric in $\vec{r}$ and $\vec{r}\,'$, as expected.

The problem with Eq. (6.25)is convergence: for any application it achieves its value slowly. Again, as in Cartesian coordinates, a better expansion for $G_D(\vec{r},\vec{r}\,')$ is obtained from a series where all terms satisfy Laplace's Equation except at the singularity $\vec{r} = \vec{r}\,'$. In the present case the special coordinate must be *z*, because it is the only one where the condition (coordinate = constant) defines two surfaces, and hence where the relevant eigenfunction with positive eigenvalue can be divided into two parts such that the product is zero at both ends.. Thus we write

$$G_D(\vec{r},\vec{r}\,') = \sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{\nu n} J_\nu(k_{\nu n}\rho) e^{i\nu\varphi} \sinh(k_{\nu n} z_<) \sinh(k_{\nu n}(L - z_>)). \tag{6.26}$$

This function meets all the requirements: it is zero on all surfaces, and each term satisfies Laplace's Equation except at the singularity, where $\partial^2/\partial z^2$ is undefined.

Substituting Eq. (6.26) in the defining equation, we obtain

$$\nabla_{\vec{r}}^2 G_D(\vec{r},\vec{r}\,') = \sum_{\nu=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{\nu n}\left(-k_{\nu n}^2 + \frac{\partial^2}{\partial z^2}\right) J_\nu(k_{\nu n}\rho) e^{i\nu\varphi} \sinh(k_{\nu n} z_<) \sinh(k_{\nu n}(L - z_>))$$

$$= -4\pi\delta(\rho - \rho')\frac{1}{\rho}\delta(\varphi - \varphi')\delta(z - z'). \tag{6.27}$$

Applying orthogonality yields

$$A_{\nu n}\left(-k_{\nu n}^2 + \frac{\partial^2}{\partial z^2}\right)\frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2 \; 2\pi \sinh(k_{\nu n} z_<) \sinh(k_{\nu n}(L - z_>))$$

$$= -4\pi\delta(z - z') J_\nu(k_{\nu n}\rho') e^{-i\nu\varphi'}. \tag{6.28}$$

Integrating *z* over a vanishingly small range about z' eliminates the term $(-k_{\nu n}^2)$, and leaves a first derivative that is to be evaluated at the end points of the integration range. The first derivative is well defined in these regions, so we find

$$\pi a^2 k_{\nu n} A_{\nu n} [J_{\nu+1}(x_{\nu n})]^2 \sinh(k_{\nu n} L) = -4\pi, \tag{6.29}$$

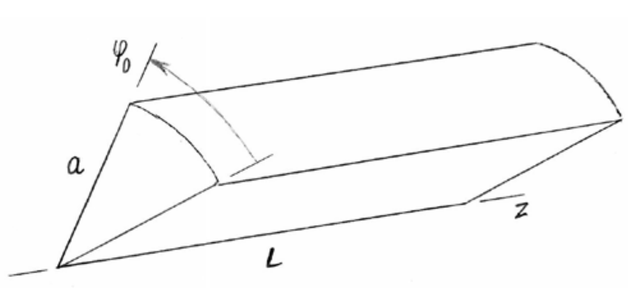
or

$$A_{\nu n} = \frac{4 J_\nu(k_{\nu n}\rho') e^{-i\nu\varphi'}}{a^2 k_{\nu n}[J_{\nu+1}(x_{\nu n})]^2 \sinh(k_{\nu n} L)}, \tag{6.30}$$

which is substituted in Eq. (6.26). The series converges exponentially, a clear advantage relative to that in Eq. (6.25).

E, Nontraditional expansions of $\phi(\vec{r})$: the 3D wedge.

Because our primary task is setting up solutions, I consider two more examples based on the wedge shown in the figure at the right. The apex is on the *z* axis, the radius is *a*, and the opening angle is φ_o. Can we obtain the series expansion for $\phi(\vec{r})$ if $\phi(a,\varphi,z) = V(\varphi,z)$ on the top surface? The answer is yes, if we follow basic principles.



Five facets are now involved. The conditions that must be met are $\phi(\vec{r}) = 0$ for the $z = 0$, $z = L$, $\varphi = 0$, and $\varphi = \varphi_o$ facets, and $\nabla^2\phi(\vec{r}) = 0$ on a term-by-term basis. The eigenfunctions for φ and *z* must therefore be sine functions, which also fits the requirement that orthogonal functions in φ and *z* are needed to represent $V(\varphi, z)$. Thus the radial function must have a positive eigenvalue and be regular at $\rho = 0$, or $I_\nu(k\rho)$.

We first construct the expansion of ϕ, starting with the requirement that $\phi = 0$ for all facets except the top. Write

$$\phi(\vec{r}) = \sum_{l,m} A_{lm} R_{lm}(\rho) Q_{lm}(\varphi) Z_{lm}(z), \tag{6.31}$$

where *l* and *m* are integers to be determined. Clearly

$$Z(z) = Z_n(z) = \sin\left(\frac{\pi n}{L} z\right) = \sin(k_{nz} z), \tag{6.32}$$

where *n* is an integer. Likewise

$$Q(\varphi) = Q_\nu(z) = \sin\left(\frac{\pi \nu}{\varphi_o} \varphi\right) = \sin(\alpha_{\nu\varphi} \varphi), \tag{6.33}$$

where ν is an integer. Therefore

$$\phi(\vec{r}) = \sum_{\nu,n} A_{\nu n} R_{\nu n}(\rho) \sin(\alpha_{\nu\varphi}\varphi) \sin(k_{nz} z), \tag{6.34}$$

Next,

$$\nabla^2\phi = \sum_{\nu,n} A_{\nu n} \left\{ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{\alpha_{\nu\varphi}^2}{\rho^2} + k_{nz}^2 \right\} R(\rho) \sin(\alpha_{\nu\varphi}\varphi) \sin(k_{nz} z) = 0\,. \tag{6.35}$$

Multiplying through by ρ^2 and discarding the two sine functions, the equation for $R(\rho)$ is

$$\nabla^2\phi = \sum_{\nu,n} A_{\nu n} \left\{ \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \alpha_{\nu\varphi}^2 + (k_{nz}\rho)^2 \right\} R_{\nu n}(\rho) = 0\,. \tag{6.36}$$

or

$$\left\{ \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \alpha_{\nu\varphi}^2 - k_{nz}^2\rho^2 \right\} R_{\nu n}(\rho) = 0\,. \tag{6.37}$$

Comparing this to Bessel's Equation for $I_\nu(z)$ we find

$$\left\{ z\frac{d}{dz}\left(z\frac{d}{dz}\right) - \nu^2 - z^2 \right\} I_\nu(z) = 0\,. \tag{6.38}$$

It follows immediately that the order ν of I_ν is a noninteger $\pi\nu/\varphi_o$, and its argument is $z = k_{nz}\rho = (\pi n/L)\rho$. Thus the solution for the ν, n term is

$$\phi_{\nu n}(\vec{r}) = I_{\pi\nu/\varphi_o}(\frac{\pi n}{L}\rho)\sin(\frac{\pi\nu}{\varphi_o}\varphi)\sin(\frac{\pi n}{L}z), \tag{6.39}$$

where the parameters of $I_\nu(k\rho)$ are determined entirely by the conditions on the functions of φ and *z*. The orthogonality relations that are needed to obtain the coefficients are obvious. While we can evaluate the sine functions, we'll let the mathematicians deal with the fractional-order Bessel function.

A considerably more exotic expansion is that for the potential given on the side $\varphi = \varphi_o$. Here, we can retain $Z(z) = \sin(k_{zn}z)$ as above, but[DA1] $Q(\varphi)$ is now $Q(\varphi) = \sinh(\alpha(\varphi_o - \varphi))$, where α remains to be determined. Making appropriate changes in the above solution for $\phi_{\nu n}(\vec{r})$, the result is

$$\phi_{\nu n}(\vec{r}) = J_{i\alpha}(\frac{\pi n}{L}\rho)\sinh(\alpha\varphi)\sin(\frac{\pi n}{L}z). \tag{6.40}$$

The quantization of the sine function sets the scaling factor for ρ in the argument of $J_{i\alpha}(x)$ (yes, the order of the Bessel function is imaginary – which may mean that we must use $I_{i\alpha}(x)$ instead of $J_{i\alpha}(x)$. The remaining quantization condition is that $i\alpha$ can only take on values for which the appropriate Bessel function vanishes when $\rho = a$. The statement of the problem is therefore complete. How one actually solves this is another problem that we leave to the next generation.

F. Two-dimensional configurations.

The two-dimensional limit of Eq. (6.1) is realized by assuming that the configuration of interest is long enough so end effects can be ignored. By Eq. (6.16), end effects decrease exponentially away from the end caps, so dropping the term $\partial^2/\partial z^2$ several radii *a* from either end is justified. With a vanishingly small *z* dependence, Eq. (6.1) reduces to

$$\nabla^2\phi = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]\phi = 0. \tag{6.41}$$

This is also obtained by taking the limit $k \to 0$ of Eq. (6.6). Equations (6.8) show that in this limit the Bessel functions reduce to monomials ρ^ν and $\rho^{-\nu}$. These are eigenfunctions of the Eq. (6.41) operator, with eigenvalue l^2.

Specifically, let

$$\phi(\rho,\varphi) = A_{\alpha\beta}\rho^\alpha e^{\beta\varphi}, \tag{6.42a}$$

and require $\nabla^2\phi = 0$. Then

$$\alpha^2 + \beta^2 = 0\,. \tag{6.42b}$$

Equation (6.42b) can be satisfied if $\alpha = \pm i\beta$ or $\beta = \pm i\alpha$, where α and β are real. Traditionally $\beta = \pm\alpha$, but both are used below. If the circular cross section of the configuration is unbroken, then $\alpha = \nu$ is an integer, so

$$\phi(\rho,\varphi) = \sum_{\nu=-\infty}^{\infty} (A_\nu \rho^\nu + B_\nu \rho^{-\nu}) e^{i\nu\varphi}\,. \tag{6.43}$$

Terms are often combined as the linear combinations $\cos\nu\varphi$ and $\sin\nu\varphi$ to avoid complex quantities, with the sum index then going from zero to infinity.

As an example, consider a cylinder of circular cross section with the perimeter at a constant potential $\phi = V$. Writing the interior solution as

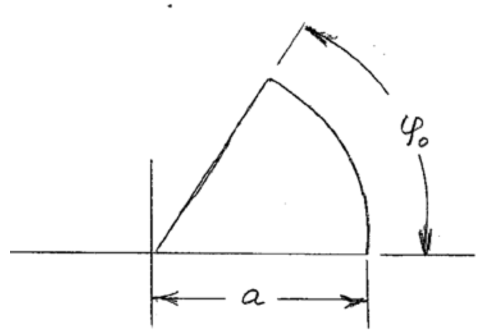
$$\phi(\rho,\varphi) = \sum_{\nu=0}^{\infty} A_\nu \rho^\nu \cos\nu\varphi\,,$$

the orthogonality condition applied to $\phi(a,\varphi) = V$ leads directly to $\nu = 0$, $A_0 = V$, and

$$\phi(\rho,\varphi) = V\,.$$

The example is trivial, but it illustrates the procedure to be followed.

A much more interesting case is shown in the figure at the right. This can be interpreted either as a long groove of opening angle φ_o cut to the center of a long conductor of radius *a*, or a fabricated shell that has this cross section. Assuming that $\phi(a,\varphi) = V(\varphi)$ is specified on the arc and the sides are at zero potential, the expansion for the interior is



$$\phi(\rho,\varphi) = \sum_{\nu=1}^{\infty} A_\nu \rho^{\alpha_\nu} \sin(\alpha_\nu \varphi)\,, \tag{6.44}$$

where $\alpha_\nu = \nu\pi/\varphi_o$. Equation (6.40) simultaneously permits a Fourier expansion of $V(\varphi)$ at $\rho = a$, sets $\phi = 0$ on both sidewalls, and is regular as $\rho \to 0$. Orthogonality here takes the form

$$\int_0^{\varphi_o} d\varphi\, \sin(\alpha_{\nu'}\varphi)\sin(a_\nu \varphi) = \frac{\varphi_o}{2}\delta_{\nu'\nu}\,. \tag{6.45}$$

The coefficients themselves are therefore

$$A_\nu = \frac{2}{\varphi_o a^{\alpha_\nu}} \int_0^{\varphi_o} d\varphi\, V(\varphi)\sin(\alpha_\nu \varphi)\,. \tag{6.46}$$

If $V(\varphi)$ is provided the integral can be evaluated and the result substituted in Eq. (6.40) to obtain $\phi(\rho,\varphi)$ everywhere in the groove.

The result is the two-dimensional version of the physics of lightning rods. Let $V(\varphi) = V$. Then A_ν can be evaluated explicitly, and

$$\phi(\rho,\varphi) = \frac{4V}{\pi}\sum_{\substack{\nu=1 \\ odd\ \nu}}^{\infty}\frac{1}{\nu}\left(\frac{\rho}{a}\right)^{\alpha_\nu}\sin(\alpha_\nu\varphi)\,. \tag{6.47}$$

The interior field is

$$\vec{E}(\rho,\varphi) = -\nabla\phi = -\frac{4V}{\varphi_o a}\sum_{\substack{\nu=1 \\ odd\ \nu}}^{\infty}\left(\frac{\rho}{a}\right)^{\alpha_\nu - 1}\left[\hat{\rho}\sin\alpha_\nu + \hat{\varphi}\cos\alpha_\nu\right]. \tag{6.48}$$

Now $\alpha_\nu = \nu\pi/\varphi_o$. If $\varphi_o < \pi$, all terms reduce to zero for $\rho \to 0$. However, if $\varphi_o > \pi$, then the wedge is convex and the $\nu = 0$ term diverges. By concentrating the electric field at a sharp point (or line in the present example), they can become high enough so breakdown occurs in air. This is why lightning rods work.

An even more interesting case is that where $\phi(\rho,\varphi_o) = V(\rho)$ is specified on the sidewall $\varphi = \varphi_o$. With the Fourier-expansion task falling on ρ , the appropriate generic term in Eq. (6.39) is

$$\phi_\nu(\rho,\varphi) = A_\nu \rho^{i\alpha_\nu} e^{-\alpha_\nu\varphi}\,.$$

Monomials raised to imaginary powers are not commonly encountered, but can be treated by scaling the ρ dependence as

$$\rho^{i\alpha} = (e^{\ln\rho})^{i\alpha} = e^{i\alpha\ln\rho}\,. \tag{6.49}$$

This expression diverges as $\rho \to 0$, because the length scale is set by the sidewall separation, which becomes arbitrarily small as $\rho \to 0$. As noted in a previous section, in this case discrete series fail. In principle, sums can be replaced by integrals, but convergence remains to be demonstrated.

However, an approximate fix is obtained by defining a lower cutoff ρ_o to exclude the singularity at $\rho = 0$. With $\phi = 0$ at $\rho = \rho_o$ and $\rho = a$, the α_ν are quantized as

$$\alpha_\nu = \frac{\nu\pi}{\ln(a/\rho_o)}\,. \tag{6.50}$$

The eigenfunction of the Laplacian are therefore

$$\phi(\rho,\varphi) = \sum_{\nu=1}^{\infty} A_\nu \sin\left(\alpha_\nu \ln(\frac{\rho}{\rho_o})\right)\sinh(\alpha_\nu\varphi)\,. \tag{6.51}$$

Direct substitution verifies that this series meets at least part of the requirements: it satisfies Laplace's Equation, and vanishes at $\rho = \rho_o$ and $\rho = a$.

What about orthogonality? Let

$$\theta = \pi \frac{\ln(\rho/\rho_o)}{\ln(a/\rho_o)}. \tag{6.52}$$

Substituting Eq. (6.48) in the standard orthogonality relation for $\sin\theta$ yields

$$\int_{\rho=\rho_o}^{\rho=a} d\left(\pi \frac{\ln(\rho/\rho_o)}{\ln(a/\rho_o)}\right) \sin\left(\nu'\pi \frac{\ln(\rho/\rho_o)}{\ln(a/\rho_o)}\right) \sin\left(\nu\pi \frac{\ln(\rho/\rho_o)}{\ln(a/\rho_o)}\right) \tag{6.53a}$$

$$= \int_{\rho=\rho_o}^{\rho=a} d\left(\pi \frac{\ln(\rho/\rho_o)}{\ln(a/\rho_o)}\right) \sin\left(\alpha_{\nu'} \ln\frac{\rho}{\rho_o}\right) \sin\left(\alpha_\nu \ln\frac{\rho}{\rho_o}\right) \tag{6.53b}$$

$$= \frac{\pi}{\ln(a/\rho_o)} \int_{\rho_o}^{a} \frac{d\rho}{\rho} \sin\left(\alpha_{\nu'} \ln\frac{\rho}{\rho_o}\right) \sin\left(\alpha_\nu \ln\frac{\rho}{\rho_o}\right) \tag{6.53c}$$

$$= \frac{\pi}{2}\delta_{\nu'\nu}. \tag{6.53d}$$

Therefore

$$\int_{\rho_o}^{a} \frac{d\rho}{\rho} \sin\left(\alpha_{\nu'} \ln(\frac{\rho}{\rho_o})\right) \sin\left(\alpha_\nu \ln(\frac{\rho}{\rho_o})\right) = \frac{1}{2}\ln(\frac{a}{\rho_o})\ \delta_{\nu'\nu}. \tag{6.54}$$

Applying this to Eq. (6.46) with $\varphi = \varphi_o$ yields

$$A_\nu = \frac{2}{\ln(a/\rho_o)\ \sinh(\alpha_\nu \varphi_o)} \int_{\rho_o}^{a} \frac{d\rho}{\rho} V(\rho) \sin(\alpha_\nu \ln(\frac{\rho}{\rho_o}). \tag{6.55}$$

If $V(\rho)$ specified, A_ν can be evaluated and the results substituted into Eq. (6.46) to give ϕ anywhere within the groove.

As an example, let $V(\rho) = V_o$ be a constant. Then the above integral reduces to

$$A_\nu = \frac{4V_o}{\ln(a/\rho_o)\sinh(\alpha_\nu \varphi_o)}, \tag{6.56}$$

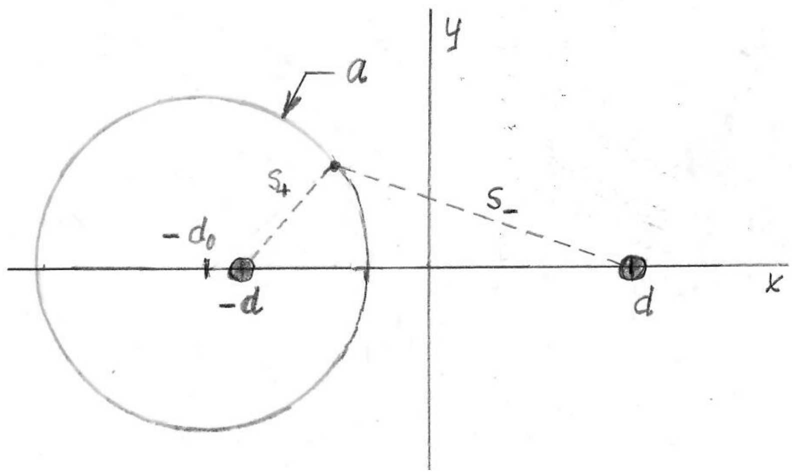
Therefore

$$\phi(\rho,\varphi) = \frac{4V_o}{\ln(a/\rho_o)} \sum_{\nu=1}^{\infty} \sin\left(\alpha_\nu \ln(\frac{\rho}{\rho_o})\right) \frac{\sinh(\alpha_\nu \varphi)}{\sinh(\alpha_\nu \varphi_o)}. \tag{6.57}$$

This may not be the most familiar mathematical expression in the world, but we obtained it by following the procedures developed in Ch. 05.

If the length of the wedge is finite, expansions over the top arc or the ends pr, ovide no essential difficulties. Standard functions are involved, although the range of φ is now limited so $\nu \to \beta$, which must be adjusted accordingly. The Bessel functions are now of fractional order, but their properties are well understood. However, for a sidewall expansion $\nu \to i\nu$, in which case we are dealing with Bessel functions of imaginary order. As noted above, these functions have not made it into textbooks or references, since they have only recently been explored in any detail.

G. Image-charge problem in 2D (2dimage.docx) (25 Jan 2020)



Unlike its 1D planar and 3D spherical-shell counterparts, the image-charge (image-line) problem in two dimensions involves some moderately intense mathematics. As this is a math course as well as a physics course, we discuss this here. It is made more complicated because of logarithmic divergences at both small and large radii.

Beginning with Gauss' Law in two dimensions, we consider a wire of radius a_o carrying a surface charge density σ. At this stage a_o is "small", the exact value does not matter since it does not appear in the result. Obviously in the limit $a_o \to 0$ the wire collapses into a line. Then

$$\rho(\vec{r}) = \sigma\delta(\rho - a_o). \tag{6.58}$$

Applying Gauss' Theorem to Gauss' Law and using a volume in the shape of a circular cylinder of radius ρ and length Δz centered on the wire,

$$\int_V d^3r \nabla \cdot \vec{E} = \int_S d^2r \hat{\rho} \cdot \vec{E} = 2\pi\rho E_\rho \Delta z$$

$$= 4\pi\sigma \int_V d^3r \delta(\rho - a_o) = 4\pi\sigma a_o (2\pi\Delta z), \tag{6.59}$$

or

$$\vec{E} = E_\rho \hat{\rho} = 4\pi\sigma \frac{\rho}{a_o} \hat{\rho}. \tag{6.60}$$

Then since $\vec{E} = -\nabla\phi$, we integrate once more along a radius, we find that

$$\phi(\rho) = \phi_a + 4\pi\sigma a \ln\left(\frac{a_o}{\rho}\right). \tag{6.61}$$

Makes sense: if $\sigma > 0$ and $\phi(a) > 0$ then as ρ increases $\phi(\rho)$ becomes smaller.

We now consider two wires, one with surface charge density $\sigma > 0$ centered at $(x, y) = (-d, 0)$ and the other with surface charge density $(-\sigma)$ centered at $(d, 0)$. Because $\rho^2 = x^2 + y^2$, the potential $\phi(\rho, \varphi)$ at the point (x, y) is

$$\phi(\rho,\varphi) = \phi_o + 4\pi\sigma a \ln\left(\frac{a}{\sqrt{(x+d)^2 + y^2}}\right) - 4\pi\sigma a \ln\left(\frac{a}{\sqrt{(x-d)^2 + y^2}}\right) \tag{6.62a}$$

$$= \phi_o + 4\pi\sigma a \ln\left(\sqrt{\frac{(x-d)^2 + y^2}{(x+d)^2 + y^2}}\right) \tag{6.62b}$$

$$= \phi_o + 2\pi\sigma a \ln\left(\frac{(x-d)^2 + y^2}{(x+d)^2 + y^2}\right). \tag{6.62c}$$

Cross-check: if $x < 0$ then the numerator exceeds the denominator and the ln term is greater than zero.

Equation (6.62c) shows that if the ratio of the distances between the observation point and the two wires is constant, then the result is an equipotential line. This favorable situation occurs under restrictive conditions: (1) the product $|\sigma a|$ is the same for both wires and (2) the σa on the two wires have opposite signs. To investigate further, let this ratio be η, or explicitly

$$\frac{(x-d)^2 + y^2}{(x+d)^2 + y^2} = \eta^2 . \tag{6.63}$$

Combining like terms yields

$$(\eta^2 - 1)x^2 + 2xd(\eta^2 + 1) + (\eta^2 - 1)y^2 + (\eta^2 - 1)d^2 = 0 . \tag{6.64}$$

To simplify the expression, define

$$\beta = \frac{\eta^2 + 1}{\eta^2 - 1}. \tag{6.65}$$

Then

$$x^2 + 2\beta xd + y^2 + d^2 = 0 . \tag{6.66}$$

With the positively charged wire on the side $x < 0$, the ratio $\eta > 1$. Then for this to work for $x < 0$, it's clear that $\beta > 0$, which means in turn that $\eta > 1$, consistent with our above argument.

Equation (6.66) defines a circle with center at $(-d_0, 0)$, where d_0 remains to be determined. To do this efficiently define a new variable $x' = x + d_0$, so $x' = 0$ when $x = -d_0$. Then with $x = x' - d_0$ we obtain

$$0 = (x' - d_0)^2 + 2\beta(x' - d_0)d + y^2 + d^2$$

$$= (x')^2 - 2x'd_0 + d_0^2 + 2\beta x'd - 2\beta d_0 d + y^2 + d^2 . \tag{6.67}$$

The terms linear in x' vanish if

$$d_0 = \beta d . \tag{6.68}$$

Hence the offset of the isopotential circle from the wire is

$$\Delta x = | d_o - d | = (\beta - 1)d . \tag{6.69}$$

Substituting Eq. (6.68) in Eq. (6.67) gives

$$\begin{aligned} (x')^2 + y^2 &= -d_0^2 + 2\beta d_0 d - d^2 \\ &= -\beta^2 d^2 + 2\beta^2 d^2 - d^2 \\ &= (\beta^2 - 1)d^2 . \end{aligned} \tag{6.70}$$

Returning to the original coordinate system, this is a circle of radius

$$a = \left(\sqrt{\beta^2 - 1}\right) d , \tag{6.71}$$

centered at $d_0 = \beta d$, where d and d_o are both negative. The distance between the circle and $x = 0$ is

$$\Delta x = \left(\beta - \sqrt{\beta^2 - 1}\right) d . \tag{6.72}$$

This approaches d as $\beta \to 1$ and zero as $\beta \to \infty$.

Returning to Eq. (6.62c), the potential of the circle is

$$\phi = \phi_o + 2\pi\sigma a \ln\left(\eta^2\right) \tag{6.73a}$$

$$= \phi_o + 4\pi\sigma a \ln \eta . \tag{6.73b}$$

Again, this is consistent: since $\eta > 1$ on the left side of $x = 0$, the potential gets more and more positive as $\eta \to \infty$. Well, actually η is limited to $(2d + a)/a = \frac{2d}{a} + 1$ to 1 if we're to stay on the lh side of $x = 0$. Since nearly everything is written in terms of β and d, we return to Eqs. (6.69) and (6.71) to write

$$\beta = \frac{1 + \xi^2}{1 - \xi^2} , \tag{6.74a}$$

$$d = \Delta x \frac{1 - \xi^2}{2\xi^2} , \tag{6.74b}$$

then convert the above equations. After some algebra we find

$$d_o = \Delta x \frac{1 + \xi^2}{2\xi^2} ; \tag{6.75a}$$

$$\eta = \frac{1}{\xi} = \frac{a}{\Delta x};\tag{6.75b}$$

where as a cross-check η is calculated from both ratios

$$\eta = \frac{d + (d_o + a)}{d - (d_o - a)}; \qquad +(6.76a)$$

$$= \frac{d_o + a + d}{d_o + a - d}.\tag{6.76b}$$

The first and second ratios are recognized as the ratios of the short and long distances along the *x* axis from the intersections of the equipotential circles and the x axis. These are equal, as they must be, and hence provide a cross-check on the results. It is noted that

$$\eta = \frac{a}{\Delta x} = \frac{s_-}{s_+},\tag{6.77}$$

where s_- and s_+ are the distances from any point on the equipotential circle to the negative and positive wires. This is an unexpected symmetry in the configuration.

To obtain the Green function we fix ϕ_o so that $\phi(\vec{r}) = 0$ on the cylinder.