Recall: If f=E-IRM, where ECRM is open, then f is differentiabled XEI iff there exist a linear may L: IRM-IRM such that

that is,

f(x+h) = f(x)+L(h)+ o(||h||).

If such an L exist, it is called the derivative of f at x and written Ofks).

Definition: If X and Y are vector spaces over IR, then we write L(x,Y) for the Space of all linear maps from X into Y. It is well known that L(x,Y) is also a vector space over IR.

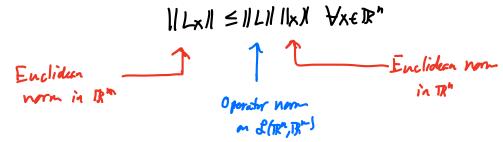
Definition: The operator norm on & (IR", IR") is defined by

 $||L|| = \max \{||L_x|| : x \in \mathbb{R}^n, ||x|| = 1\}.$

Note that ||L| is well defined since \(\text{X} \in |\ ||x||=| \) is compact and \(f: \text{R}^n \in |\ \text{R} \) defined by \(f(x) = || ||x|| \) is continuous. It is straightforward to prove that \(|| \cdot| || \) defines a norm on \(\text{Z} \in \text{R}^n \cdot, \text{R}^n \).

Theorem:

1. If Led (R", R"), the



2. If Led/right, the

1111 = min 80 20: 11Lx11= B 11x11 YXETR".

3. If Le &(R",R") and Me&(R",R"), the

| LM | = | LIIIMI.

Proof: 1. By definition.

=) || Lx|| \(\text{|L|| ||x|| } \(\text{V} \text{ER}^n \) (since this inequality obviously helds for x=0).

2. Write S= FBZO: 11 Lx 11 = Bllx11 +xeR". Then, by #1, 11L11 ES.

Suppose BES. The

This proves #2.

3. We have

11 L Mx 11 = 11 L (Mx) 11 = 11 LI 1 IMX 1 1 XXETR!

By #2, this implies that

| LM | = | LI | | | M | . /

Theorem: Let SZ = FAES(IRM, IRM) [A is invertible]. Then SZ is open, in fact,

Proof: Recall that BE $d(\mathbb{R}^n) = d(\mathbb{R}^n, \mathbb{R}^n)$ belongs to Ω iff B is noneingular (Bx=0 \Rightarrow x=0). Let AE Ω and suppose

Then if xEIR" and x +0, we have

$$B_{X} = A_{X} + (B-A)_{X} \implies ||B_{X}|| \ge ||A_{X}|| - ||(B-A)_{X}||$$

$$\ge ||A_{X}|| - ||B-A|| ||A_{X}||$$

$$\ge ||A_{X}|| - \frac{||A_{X}||}{||A-1||}$$

$$= ||A_{X}|| - \frac{||A-A_{X}||}{||A-1||} \ge ||A_{X}|| - \frac{||A-A_{X}||}{||A-A_{X}||} = 0.$$

Thur B is nonsingular and hence BESZ:/

Corollary: $f: SZ \to SZ$ defined by $f(A) = A^{-1}$ is continuous and invertible, with $f^{-1} = f$. Proof: Since $(A^{-1})^{-1} = A$, it is obvious that f is invertible and $f^{-1} = f$.

Now suppose AG St and

$$||B-A|| \leq \frac{1}{2||A^{-1}||}$$

Then BESE and

$$B^{-1}-A^{-1}=B^{-1}AA^{-1}-B^{1}BA^{-1}=B^{-1}(A-B)A^{-1}$$

$$\Rightarrow$$
 $\|B^{-1}\| \leq \|A^{-1}\| + \frac{\|B^{-1}\|}{2}$

$$\Rightarrow \frac{||B^{-1}||}{2} \leq ||A^{-1}||$$

Now let E>0 be given and define

$$S = m_{34} \left\{ \frac{1}{2||A^{4}||^{1}}, \frac{\varepsilon}{2||A^{-4}||^{2}} \right\}$$

The

$$||B-A|| < S \implies ||B^{-1}-A^{-1}|| \le ||B^{-1}|| ||A-B|| ||A^{-1}||$$

$$\le 2||A^{-1}||^{2}||A-B||$$

$$\le 2||A^{-1}||^{2} \cdot \frac{\varepsilon}{2||A^{-1}||^{2}} = \varepsilon.$$

Thus f is continuous at A.

Theorem (the chain rule): Suppose ECR" and FCR" are open,

f:E-IR", with R(f)CF, and g:F-91R*. If f is differentiable at

XEE and g is differentiable at fk)EF, then h = gof is differentiable at

X, and

$$Dh(x) = Dg(f(x))Of(x).$$

Proof: We have

Nau,

 $\|D_g(f_{(k)})_o(\|p\|)\| \le \|D_g(f_{(k)})\|\|o(\|p\|)\| = o(\|p\|)\|$ (Since $\|D_g(f_{(k)})\|\|i\|$) a constant in \mathbb{R}

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$$\Rightarrow \frac{o(||\text{lipf}(x)p+o(||p|))|)}{(||\text{lipf}(x)||+1)||p||} \rightarrow 0 \text{ as } p \rightarrow 0$$

Thus

Which implies that

as desired.

Note that Dg(f(x)) Df(x) is the product (composition) of two linear maps, and this product is not commutative.