Math 672 Lecture 2

Many problems in linear algebra are either defined on a subspace, or a subspace is involved in their solution.

(I won't keep repeating this.)

Definition Let V be a vector space over a field F.
A subset S of V is called a subspace of V iff S is a vector space (under the same operations defined on V).

Theorem: A subset S of V is a subspace of V iff the following three conditions are satisfied:

- $0 \in S$
- $u, v \in S \Rightarrow u+v \in S$ (S is closed under addition)
- $u \in S, \alpha \in F \Rightarrow \alpha u \in S$ (S is closed under scalar multiplication)

Proof (sketch) If S is a subspace of V, then it is obvious that these conditions hold. Conversely, if S is a subset of V that satisfies the above properties, then it automatically satisfies the following properties of a vector space (*inherited from V):

- $u+v=v+u \quad \forall u,v\in S$ *
- $(u+v)+w=u+(v+w) \quad \forall u,v,w\in S$ *
- S has an additive identity (assumed)
- $1v=v \quad \forall v\in S$ *
- $\alpha(u+v)=\alpha u+\alpha v \quad \forall u,v\in S \ \forall \alpha\in S$ *
- $(\alpha+\beta)u=\alpha u+\beta u \quad \forall u\in S \ \forall \alpha,\beta\in S$ *
- $\alpha(\beta v)=(\alpha\beta)v \quad \forall v\in V \ \forall \alpha,\beta\in F$ *

The only property that isn't obvious is the existence of additive inverses in S. But we have shown that

$$-v=-1\cdot v \quad \forall v\in S$$

and

$$v\in S \Rightarrow -1\cdot v\in S \quad \text{(by assumption)}.$$

Thus S satisfies all of the properties of a vector space. //

Note that V always has at least two subspaces, namely $\{0\}$ (called the <u>trivial</u> vector space) and V itself.

Examples

1. $S=\{x\in\mathbb{R}^3 \mid a_1x_1+a_2x_2+a_3x_3=0\}$, where a_1,a_2,a_3 are given constants in $\mathbb{R}$, is a subspace of $\mathbb{R}^3$.

Proof: We must verify the three properties of a subspace

- First, $a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 = 0$, so $0 = (0,0,0) \in S$.

- Next, if $x, y \in S$, then

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \text{ and } a_1 y_1 + a_2 y_2 + a_3 y_3 = 0$$

$$\Rightarrow a_1(x_1+y_1) + a_2(x_2+y_2) + a_3(x_3+y_3)$$

$$= a_1 x_1 + a_1 y_1 + a_2 x_2 + a_2 y_2 + a_3 x_3 + a_3 y_3$$

$$= (a_1 x_1 + a_2 x_2 + a_3 x_3) + (a_1 y_1 + a_2 y_2 + a_3 y_3) = 0 + 0$$

$$\Rightarrow x + y = (x_1+y_1, x_2+y_2, x_3+y_3) \in S.$$

- Similarly, if $x \in S$, then

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0.$$

Thus, if $\alpha \in \mathbb{R}$, then

$$a_1(\alpha x_1) + a_2(\alpha x_2) + a_3(\alpha x_3)$$

$$= \alpha a_1 x_1 + \alpha a_2 x_2 + \alpha a_3 x_3$$

$$= \alpha (a_1 x_1 + a_2 x_2 + a_3 x_3)$$

$$= \alpha \cdot 0 = 0$$

$$\Rightarrow \alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \in S.$$

Thus S is a subspace of $\mathbb{R}^3$.

2. $V_0 = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$ is a subspace of $C^2[0,1]$.

3. $V_1 = \{u \in C^2[0,1] : u(0) = u(1) = 1\}$ is a <u>subset</u> of $C^2[0,1]$ but not a <u>subspace</u>.

<u>Proof:</u>

$0 \in C^2[0,1]$ is the zero function, and $u(x) \equiv 0$ does not satisfy $u(0) = u(1) = 1$. Thus $0 \notin V_1$ and hence V_1 is not a subspace of $C^2[0,1]$. //

(Usually, if S is not a subspace of V, this fact is most easily proven by showing that $0 \notin S$.)

<u>Definition</u>: If U_1 and U_2 are any subsets of V, we define

$$U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1 \text{ and } u_2 \in U_2\}.$$

Similarly, if $U_1, U_2, \ldots, U_m \subseteq V$, then

$$U_1 + U_2 + \cdots + U_m = \{u_1 + u_2 + \cdots + u_m \mid u_i \in U_i, i = 1, 2, \ldots, m\}.$$

Given $v \in V$ and $U \subseteq V$, we will write

$$v + U = \{v + u \mid u \in U\} \quad (= \{v\} + U).$$

Theorem: If $U_1, U_2, \ldots, U_m$ are subspaces of V, then $U_1 + U_2 + \cdots + U_m$ is also a subspace of V.

Proof: It is straightforward (though tedius) to prove that $U_1 + U_2 + \cdots + U_m$ satisfies the three properties of a subspace:

- Since $0 \in U_i$ for each $i = 1, 2, \ldots, m$ (because U_i is a subspace), it follows that

$$0 + 0 + \cdots + 0 \in U_1 + U_2 + \cdots + U_m$$

$$\Rightarrow 0 \in U_1 + U_2 + \cdots + U_m.$$

- Suppose $u, v \in U_1 + U_2 + \cdots + U_m$. Then, by definition, there exist $u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m$ such that

$$u = u_1 + u_2 + \cdots + u_m$$

and $v_1 \in U_1, v_2 \in U_2, \ldots, v_m \in U_m$ such that

$$v = v_1 + v_2 + \cdots + v_m.$$

But, since each U_i is a subspace, it follows that

$$u_i + v_i \in U_i \quad \forall\, i = 1, 2, \ldots, m.$$

Thus

$$(u_1 + v_1) + (u_2 + v_2) + \cdots + (u_m + v_m) \in U_1 + U_2 + \cdots + U_m$$

$$\Rightarrow (u_1 + u_2 + \cdots + u_m) + (v_1 + v_2 + \cdots + v_m) \in U_1 + U_2 + \cdots + U_m$$

(where we have used commutativity and associativity repeatedly)

$$\Rightarrow u+v \in U_1+U_2+\cdots+U_m.$$

Therefore $U_1+U_2+\cdots+U_m$ is closed under addition.

- Suppose that $u \in U_1+U_2+\cdots+U_m$ and $\alpha \in F$.

  Then there exist $u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m$ such that

  $$u = u_1+u_2+\cdots+u_m.$$

  But then $\alpha u_i \in U_i$ for each $i=1,2,\ldots,m$ (since each U_i is closed under scalar multiplication) and hence

  $$(\alpha u_1)+(\alpha u_2)+\cdots+(\alpha u_m) \in U_1+U_2+\cdots+U_m$$

  $$\Rightarrow \alpha(u_1+u_2+\cdots+u_m) \in U_1+U_2+\cdots+U_m$$

  $$\Rightarrow \alpha u \in U_1+U_2+\cdots+U_m.$$

  Thus $U_1+U_2+\cdots+U_m$ is closed under scalar multiplication, and the proof is complete. //

We are usually interested in sums of subspaces when we can represent V as a sum: $V = U_1+U_2+\cdots+U_m$. Then we can often reduce a problem posed on V to m problems posed on the smaller spaces $U_1, U_2, \ldots, U_m$. But this is usually tenable only if there is a certain uniqueness in the representation.

<u>Definition</u>: Let $U_1, U_2, \ldots, U_m$ be subspaces of V. We say that $U_1 + U_2 + \cdots + U_m$ is a <u>direct sum</u> iff each $u \in U_1 + U_2 + \cdots + U_m$ can be written uniquely as $u = u_1 + u_2 + \cdots + u_m$, where $u_i \in U_i$ for $i = 1, 2, \ldots, m$. In this case, we write $U_1 + U_2 + \cdots + U_m$ as $U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

<u>Examples</u>

1. Let $\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ and define $S = \{\alpha x^n \mid \alpha \in \mathbb{R}\}$. Then
$$\mathcal{P}_n = \mathcal{P}_{n-1} \oplus S$$

2. Let $U = \{\alpha x^{n-1} + \beta x^n \mid \alpha, \beta \in \mathbb{R}\}$. Then
$$\mathcal{P}_n = \mathcal{P}_{n-1} + U,$$
but $\mathcal{P}_{n-1} + U$ is not a direct sum. To see this last point, consider
$$x^{n-1} + x^n \in \mathcal{P}_n.$$
We have

$$x^{n-1} + x^n = 2x^{n-1} + (-x^{n-1} + x^n)$$

$$\begin{pmatrix} x^{n-1} \in \mathcal{P}_{n-1} \\ x^n \in U \end{pmatrix} \qquad \begin{pmatrix} 2x^{n-1} \in \mathcal{P}_{n-1} \\ -x^{n-1} + x^n \in U \end{pmatrix}$$

Thus the uniqueness required by the definition of direct sum fails.

Theorem: Let $U_1, U_2, \ldots, U_m$ be subspaces of V. Then $U_1 + U_2 + \cdots + U_m$ is a direct sum iff

$$(*) \quad u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m \text{ and } u_1 + u_2 + \cdots + u_m = 0 \implies u_1 = u_2 = \cdots = u_m = 0.$$

Proof: If $U_1 + U_2 + \cdots + U_m$ is a direct sum, then there is a unique way to write 0 as the sum of elements of $U_1, U_2, \ldots, U_m$, and hence $(*)$ holds.

Conversely, suppose $(*)$ holds, and let $u \in U_1 + U_2 + \cdots + U_m$. We must show that there is a unique way to write $u = u_1 + u_2 + \cdots + u_m$, where $u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m$.

So suppose

$$u = u_1 + u_2 + \cdots + u_m, \quad u_i \in U_i \text{ for } i = 1, 2, \ldots, m,$$

$$u = v_1 + v_2 + \cdots + v_m, \quad v_i \in U_i \text{ for } i = 1, 2, \ldots, m.$$

It follows that

$$u_1 + u_2 + \cdots + u_m = v_1 + v_2 + \cdots + v_m$$

$\Rightarrow (u_1 - v_1) + (u_2 - v_2) + \cdots + (u_m - v_m) = 0$ (by repeated use of commutativity and associativity of addition; recall that $u_i - v_i = u_i + (-v_i)$)

$\Rightarrow u_1 - v_1 = 0, u_2 - v_2 = 0, \ldots, u_m - v_m = 0$ (by $(*)$, since $u_i - v_i \in U_i$ for each i)

$\Rightarrow u_1 = v_1, u_2 = v_2, \ldots, u_m = v_m.$

This shows that $U_1 + U_2 + \cdots + U_m$ is a direct sum. //

Theorem: Let U_1, U_2 be subspaces of V. Then $U_1 + U_2$ is a direct sum iff $U_1 \cap U_2 = \{0\}$.

Proof: Suppose $U_1 + U_2$ is a direct sum and $u \in U_1 \cap U_2$. Then $u \in U_1$ and $u \in U_2$, which implies that $-u \in U_2$ and hence that

$$0 = u + (-u) \in U_1 + U_2.$$

But, by the previous theorem, this implies that $u=0$ and $-u=0$.

Therefore,

$$u \in U_1 \cap U_2 \Rightarrow u = 0,$$

that is, $U_1 \cap U_2 = \{0\}$.

Conversely, suppose that U_1, U_2 are subspaces of V and $U_1 \cap U_2 = \{0\}$. Suppose $u \in U_1 + U_2$ and

$$u = u_1 + u_2, \quad u_1 \in U_1, u_2 \in U_2,$$

$$u = v_1 + v_2, \quad v_1 \in U_1, v_2 \in V_2.$$

Then

$$u_1 + u_2 = v_1 + v_2$$

$$\Rightarrow u_1 - v_1 = v_2 - u_2$$

$$\Rightarrow u_1 - v_1 \in U_1 \text{ (because } U_1 \text{ is a subspace)}$$

$$\text{and } u_2 - v_2 \in U_2 \text{ (because } U_2 \text{ is a subspace)}$$

$$\Rightarrow u_1 - v_1 \in U_1 \cap U_2 \text{ and } u_2 - v_2 \in U_1 \cap U_2$$

$$\Rightarrow u_1 - v_1 = 0 \text{ and } u_2 - v_2 = 0 \quad (\text{since } U_1 \cap U_2 = \{0\})$$

$$\Rightarrow u_1 = v_1 \text{ and } u_2 = v_2.$$

This shows that $U_1 + U_2$ is a direct sum. //