Tier 1 - Analysis

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Contents

1. Overview

There are three different stages of Tier 1 analysis prelims:
Stage 1: 2017-2023: Contents only including MATH600,
Stage 2: 2011-2016: Contents including both MATH600 and MATH602,
Stage 3: 1996-2006: Contents only including MATH600.
Also, 2006W is a good practice for some big problems but 2018S is super hard.
Please see here for the links to each problem.

2. Prelim Problems

2.1. **Highly Repeated Problems.** These problems are asked more than 3 times.

2.1.1. *Represent Limits as Riemann Integrals.*

Problem 1: Represent Limits as Riemann Integrals

Find the following limits:

(1) (23W-4(b)) $\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{k^4+n^4}$

(2) (22S-4(a)) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)$

(3) (21W-4(a)) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right)$

(4) (20S-5(a)) $\lim_{n\to\infty} \frac{1}{n}\ln\frac{(n+1)(n+2)\cdots(n+n)}{n^n}$

(5) (20W-5(ii)) Let $p>0$ be fixed. $\lim_{n\to\infty} \frac{1}{n^{p+1}}\sum_{k=1}^{n} k^p$

(6) (19S-5(b)) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$

(7) (18W-5(a)) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n}$

(8) (17W-4) $\lim_{n\to\infty} \frac{1^2\sqrt[n]{e}+2^2\sqrt[n]{e^2}+\cdots+n^2\sqrt[n]{e^n}}{n^3}$

(9) (06W-4(a)) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{k}{n}\right)$

(10) (96S-5(a)) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{\pi k}{n}\right)$

In 23W-4(a) and 17W-4(a), they give a crucial hint to do these:
Prove the following theorem:

Theorem 1. *If $f:[0,1]\to\mathbb{R}$ is a continuous function, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx.$$

Solution: Before we prove the Theorem, we state (and prove if you have time) the characterization theorem: (for the proofs, see here.)

Theorem 2. *(Riemann Integral Characterization Theorem) If f is bounded on $[a,b]$. The Following Are Equivalent:*

(0) *f is Riemann integrable on $[a,b]$.*

(1) *(Darboux)*
Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. And let

$$U(P,f) = \sum_{i=1}^{n} \sup f|_{[x_{i-1},x_i]} \Delta x_i, \;\; L(P,f) = \sum_{i=1}^{n} \inf f|_{[x_{i-1},x_i]} \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$. Let

$$\overline{\int_a^b} f(x)\ dx = \inf_P U(P,f), \quad \underline{\int_a^b} f(x)\ dx = \sup_P L(P,f).$$

Then

$$\overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx.$$

In the case, we call

$$\int_a^b f(x)\ dx = \overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx.$$

(2) *For any $\varepsilon > 0$, there exists a partition P such that*

$$U(P,f) - L(P,f) < \varepsilon.$$

(3) *There exists a sequence of partition $\{P_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

In the case, we call

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

(4) *(Riemann)*
There exists $I \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(x_i^*)\Delta x_i - I \right| < \varepsilon \text{ when } \|P\| = \max_{i=1,\cdots,n} \Delta x_i < \delta$$

where $x_i^* \in [x_{i-1}, x_i]$, $i = 1, \cdots, n$ be the intermediate points. In the case, we call

$$\int_a^b f(x)\ dx = I.$$

Now We to prove the Theorem:

Theorem 3. *If $f : [0,1] \to \mathbb{R}$ is a continuous function, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\ dx.$$

Proof. We will use three ways to prove the Theorem.
$\boxed{\text{Way I: Using (1)}}$ (Similar to proving continuous functions are integrable.)
Let $P_n = \left\{ \frac{k}{n} \mid k = 0, \cdots, n \right\}$. Since f is continuous on a compact set $[0, 1]$, f is uniformly continuous on $[0, 1]$. For $\varepsilon > 0$ be given. Hence there exists $\delta > 0$ such that

$$|f(x) - (f)| < \varepsilon \text{ when } |x - y| < \delta.$$

Also, by the Extreme Value Theorem, we have

$$\sup f|_{[x_{i-1}, x_i]} = f(s_i), \ \inf f|_{[x_{i-1}, x_i]} = f(t_i)$$

for some $s_i, t_i \in [x_{i-1}, x_i]$. Hence by the Archimedean property, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Hence when $n >\geq N$, we have $\|P_n\| = \frac{1}{n} \leq \frac{1}{N} < \delta$ and in this case,

$$\begin{aligned} U(P_n, f) - L(P_n, f) &= \sum_{i=1}^{n} \left(\sup f|_{[x_{i-1}, x_i]} - \inf f|_{[x_{i-1}, x_i]} \right) \Delta_{x_i} \\ &= \sum_{i=1}^{n} \left(f(s_i) - f(t_i) \right) (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{n} \left(f(s_i) - f(t_i) \right) \\ &< \frac{1}{n} \sum_{i=1}^{n} \varepsilon = \frac{1}{n} \cdot n\varepsilon = \varepsilon. \end{aligned}$$

(Remark: We are actually proving continuous function is Riemann integrable if we have (2))
Also, note that

$$\begin{aligned} & f(t_k) \leq f\left(\frac{k}{n}\right) = f(x_k) \leq f(s_k) \\ \Longrightarrow & \sum_{k=1}^{n} \frac{1}{n} f(t_i) = L(P_n, f) \leq \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq U(P_n, f) = \sum_{k=1}^{n} \frac{1}{n} f(s_i). \end{aligned}$$

Hence

$$\begin{aligned} & U(P_n, f) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) < U(P_n, f) - L(P_n, f) < \varepsilon, \\ & \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - L(P_n, f) < U(P_n, f) - L(P_n, f) < \varepsilon \\ \Longrightarrow & -\varepsilon < L(P_n, f) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right). \end{aligned}$$

Finally, since f is Riemann integrable, we have

$$\begin{aligned} & L(P_n, f) \leq \underline{\int_0^1} f(x)\ dx = \int_0^1 f(x)\ dx \leq \overline{\int_0^1} f(x)\ dx \leq U(P_n, f) \\ \Longrightarrow & -\varepsilon < L(P_n, f) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \int_0^1 f(x)\ dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq U(P_n, f) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) < \varepsilon \\ \Longrightarrow & \left| \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n} - \int_0^1 f(x)\ dx \right| < \varepsilon \Longrightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\ dx \end{aligned}$$

$\boxed{\text{Way II: Using (1)}}$ (Using integral properties directly)

Let $P_n = \left\{ \frac{k}{n} \mid k = 0, \cdots, n \right\}$ and $x_k^* = \frac{k}{n}$. Since f is continuous on $[0,1]$, f is Riemann integrable on $[0,1]$ and hence f is integrable on $[x_{i-1}, x_i]$, $i = 1, \cdots, n$. (I don't think you need to prove this, the for a proof, see Lemma.). Also, since f is continuous on a compact set $[0,1]$, f is uniformly continuous on $[0,1]$. For $\varepsilon > 0$ be given. Hence there exists $\delta > 0$ such that

$$|f(x) - (f)| < \varepsilon \text{ when } |x - y| < \delta.$$

By the Archimedean property, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. So when $n \geq N$, for any $x \in [x_{i-1}, x_i]$, $i = 1, \cdots, n$, we have

$$|x_i - x| \leq \frac{1}{n} \leq \frac{1}{N} < \delta \Longrightarrow |f(x_i) - f(x)| < \varepsilon.$$

Hence we have

$$\int_{x_{i-1}}^{x_i} |f(x_i) - f(x)| \; dx < \int_{x_{i-1}}^{x_i} \varepsilon = \frac{\varepsilon}{n}.$$

Hence we have,

$$\begin{aligned}
\left| \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n} - \int_0^1 f(x) \; dx \right| &= \left| \sum_{k=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x_i) \; dx - \int_{x_{i-1}}^{x_i} f(x) \; dx \right) \right| \\
&= \left| \sum_{k=1}^{n} \int_{x_{i-1}}^{x_i} (f(x_i) - f(x)) \; dx \right| \\
&\leq \sum_{k=1}^{n} \int_{x_{i-1}}^{x_i} |f(x_i) - f(x)| \; dx < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x) \; dx.$$

$\boxed{\text{Way III: Using (4)}}$ (It's almost immediate.)

Let $P_n = \left\{ \frac{k}{n} \mid k = 0, \cdots, n \right\}$ and $x_k^* = \frac{k}{n}$. Clearly we have $\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{1}{n} = 0$. Since f is continuous, f is integrable on $[0,1]$ (at this point, I don't think you need to prove this). Hence for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_{i=k}^{n} f(x_k^*) \Delta x_k - \int_0^1 f(x) \; dx \right| = \left| \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n} - \int_0^1 f(x) \; dx \right| < \varepsilon \text{ when } \|P\| < \delta.$$

Recall that $\lim_{n\to\infty} \|P\| = 0$, that is, for each $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\|P_n\| < \delta$ when $n \geq N$ that is, when $n \geq N$, we have

$$\left| \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n} - \int_0^1 f(x) \; dx \right| < \varepsilon$$

that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x) \; dx.$$

$\square$

Now we can solve those (easy) limit using the theorem:

(1) Note that $n^2\sum_{k=1}^{n}\frac{k}{k^4+n^4}=\frac{1}{n}\sum_{k=1}^{n}\frac{kn^3}{k^4+n^4}=\frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{\left(\frac{k}{n}\right)^4+1}$.

Hence let $f(x)=\frac{x}{x^4+1}$ and use the theorem. We have

$$\begin{aligned}\lim_{n\to\infty} n^2\sum_{k=1}^{n}\frac{k}{k^4+n^4} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,dx=\int_0^1\frac{x}{x^4+1}\,dx\\ (u=x^2) &= \frac{1}{2}\int_0^1\frac{2x}{x^4+1}\,dx=\frac{1}{2}\int_0^1\frac{1}{u^4+1}\,dx\\ &= \frac{1}{2}\tan^{-1}(u)\Big|_0^1=\frac{\pi}{8}.\end{aligned}$$

(2) There should be a typo. The sum shouldn't be in the limit.

By the Theorem, let $f(x)=\sqrt{x}$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\sqrt{\frac{k}{n}}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\\ &= \int_0^1 f(x)\;dx=\int_0^1\sqrt{x}\;dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1=\frac{2}{3}\end{aligned}$$

(3) Let $f(x)=\sin x$, then by the Theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{k}{n}\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\\ &= \int_0^1 f(x)\;dx=\int_0^1\sin x\;dx=-\cos x\Big|_0^1=-\cos 1+1=1-\cos 1\end{aligned}$$

(4) Note that

$$\begin{aligned}\frac{1}{n}\ln\frac{(n+1)(n+2)\cdots(n+n)}{n^n} &= \frac{1}{n}\ln\left(\frac{n+1}{n}\frac{n+2}{n}\cdots\frac{n+n}{n}\right)\\ &= \frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)\\ &= \frac{1}{n}\left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right).\end{aligned}$$

Let $f(x)=\ln(1+x)$, by the theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln\frac{(n+1)(n+2)\cdots(n+n)}{n^n} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\\ &= \int_0^1 f(x)\;dx=\int_0^1\ln(1+x)\;dx\overset{u=1+x}{=}\int_1^2\ln u\;du\\ (u_2=\ln u,\;dv_2=1du) &= \ln u\cdot u\Big|_1^2-\int_1^2 u\cdot\frac{1}{u}\;du=2\ln 2-1.\end{aligned}$$

(5) Note that

$$\frac{1}{n^{p+1}}\sum_{k=1}^{n}k^p=\frac{1}{n}\frac{1}{n^p}\sum_{k=1}^{n}k^p=\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p.$$

Let $f(x) = x^p$, by the theorem, we have

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\, dx = \int_0^1 x^p \ dx = \frac{1}{p+1} x^{p+1}\Big|_0^1 = \frac{1}{p+1}.$$

Since $p > 0$, we don't need to worry about the improper integral.

(6) Note that

$$\sum_{k=1}^{n} \frac{k}{n^2+k^2} = \sum_{k=1}^{n} \frac{k}{n^2\left(1+\frac{k^2}{n^2}\right)} = \sum_{k=1}^{n} \frac{\frac{k}{n}}{n\left(1+\frac{k^2}{n^2}\right)} = \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}.$$

Let $f(x) = \dfrac{x}{1+x^2}$, by the theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \int_0^1 f(x)dx = \int_0^1 \frac{x}{1+x^2}\ dx \\
(u = 1+x^2) \quad &= \int_{1+0^2}^{1+1^2} \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int_1^2 \frac{1}{u}\ du = \frac{1}{2}\left(\ln u\Big|_1^2\right) = \frac{\ln 2}{2}.
\end{aligned}$$

(7) Note that

$$\sum_{k=1}^{n} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+\frac{k}{n}} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}.$$

Let $f(x) = \dfrac{x}{1+x}$, by the theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \int_0^1 f(x)dx = \int_0^1 \frac{1}{1+x}\ dx \\
(u = 1+x) \quad &= \int_{1+0}^{1+1} \frac{1}{u}\ du = \int_1^2 \frac{1}{u}\ du = \ln u\Big|_1^2 = \ln 2.
\end{aligned}$$

(8) Note that

$$\frac{1^2\sqrt[n]{e} + 2^2\sqrt[n]{e^2}\cdots + n^2\sqrt[n]{e^n}}{n^3} = \frac{1}{n^3}\sum_{k=1}^{n} k^2\sqrt[n]{e^k} = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 e^{\frac{k}{n}}.$$

Let $f(x) = x^2e^x$, by the Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^2\sqrt[n]{e} + 2^2\sqrt[n]{e^2}\cdots + n^2\sqrt[n]{e^n}}{n^3} &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\, dx = \int_0^1 x^2e^x\ dx \\
(u_1 = x^2,\ dv_1 = e^x dx) &= x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\ dx = e - 2\left(\int_0^1 xe^x\ dx\right) \\
(u_2 = x,\ dv_2 = e^x dx) &= e - 2\left(xe^x\Big|_0^1 - \int_0^1 e^x\ dx\right) \\
&= e - 2(e+e-1) = e-2
\end{aligned}$$

(9) Note that

$$\frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\sin\left(\frac{k}{n}\right).$$

Let $f(x) = x\sin x$, by the Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{k}{n}\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\sin\left(\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \int_0^1 f(x)dx = \int_0^1 x\sin x\ dx \\
(u = x,\ dv = \sin x\ dx) \quad &= (x\cdot(-\cos x))\Big|_0^1 - \int_0^1 (-\cos x)\ dx = (-x\cos x + \sin x)\Big|_0^1 \\
&= \sin(1) - \cos(1).
\end{aligned}$$

(10) Note that

$$\frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{\pi k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\sin\left(\pi\frac{k}{n}\right).$$

Let $f(x) = x\sin(\pi x)$, by the Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k\sin\left(\frac{\pi k}{n}\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\sin\left(\pi\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \int_0^1 f(x)dx = \int_0^1 x\sin(\pi x)\ dx \\
(u = x,\ dv = \sin(\pi x)\ dx) \quad &= \left(x\cdot\left(-\frac{1}{\pi}\cos(\pi x)\right)\right)\Bigg|_0^1 - \int_0^1 \left(-\frac{1}{\pi}\cos(\pi x)\right)\ dx \\
&= \left(-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right)\Bigg|_0^1 = \frac{1}{\pi}.
\end{aligned}$$

$\square$

2.1.2. *Riemann Integrability.*

Problem 2: Riemann Integrability

Test if the following functions are Riemann integrable or not, if it is, find the value of the integral.

(1) (24S2-4) Let $f : [0.1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, & \text{if } x = \dfrac{1}{n} \text{ for some } n \in \mathbb{Z}^+, \\ 0, & \text{otherwise.} \end{cases}$

(2) (24S1-4(c)) $f(x) = \begin{cases} x\sin(x) & \text{if } x \in \mathbb{Q} \cap \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \\ \\ 0 & \text{if } x \notin \mathbb{Q} \cap \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]. \end{cases}$

(3) (22S-4(b)) $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Is f Riemann integrable on $[-1, 1]$?

(4) (22W-4=21W-4(b)) $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0, 1] \end{cases}$

(5) (21S-3=19S-5(a)) Let $f : [0, 1] \to \mathbb{R}$ be defined as $f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$.

(6) (20S-5(b)) $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [-1, 1] \\ -x, & \text{if } x \notin \mathbb{Q} \cap [-1, 1] \end{cases}$

(7) (19W-6) $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ x, & \text{if } x \notin \mathbb{Q} \cap [0, 1] \end{cases}$

(8) (05W-3(a)) $x_0 \in (0, 1)$, $f(x) = \begin{cases} 0, & \text{if } x \neq x_0 \\ 1, & \text{if } x = x_0 \end{cases}$

(9) (05-3(b)) $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0, 1] \end{cases}$

Solution: We prove the most famous one first.

(8) This is not Riemann integrable (since it's discontinuous on $[0, 1]$).
Clearly for any partition $P = \{0 = x_0 < \cdots < x_n = 1\}$, we have

$$\inf f|_{[x_{i-1}, x_i]} = 0, \ \inf f|_{[x_{i-1}, x_i]} = 1, \ i = 1, \cdots, n.$$

Hence

$$U(P, f) = \sum_{i=1}^{n} 1(x_i - x_{i-1}) = x_n - x_0 = 1, \ L(P, f) = 0.$$

And thus

$$\overline{\int_0^1} f(x)\, dx = 1 \neq 0 = \underline{\int_0^1} f(x)\, dx.$$

Hence f is not integrable.

(1)

(2) This is not Riemann integrable (since it's discontinuous besides $x = 0$).
Note that $x \sin x \geq 0$ when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (it's easy you can prove that if you want to).
So by $\mathbb{Q}^C$ is dense in $\mathbb{R}$, we have for any partition $P = \{0 = x_0 < \cdots < x_n = 1\}$, we have

$$\inf f|_{[x_{i-1},x_i]} = 0 \Longrightarrow L(P, f) = 0 \Longrightarrow \underline{\int}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx = 0.$$

Note that (Idea: I only want to do the positive part)

$$(x \sin x)' = \sin x + x \cos x > 0,\ x \in \left(0, \frac{\pi}{2}\right]$$

since $x, \sin x$ and $\cos x$ are all positive when $x \in \left(0, \frac{\pi}{2}\right]$ and $f(0) = 0$. Let x_k such that $x_{k-1} \geq 0$ and $x_{k-2} < 0$,that is, $[x_{k-1}, x_k]$ is the first interval in $\left[0, \frac{\pi}{2}\right]$. Hence

$$\sup f|_{[x_{i-1},x_i]} = x_i \sin x_i,\ i = k, \cdots, n \Longrightarrow U(P, f) \geq \sum_{i=k}^{n} x_i \sin x_i (x_i - x_{i-1}).$$

Let $p \geq k$ be the index such that $\frac{\pi}{6} \in (x_{p-1}, x_p]$. Then we have

$$\begin{aligned} U(P, f) &\geq \sum_{i=k}^{n} x_i \sin x_i (x_i - x_{i-1}) \geq \sum_{i=p}^{n} x_i \sin x_i (x_i - x_{i-1}) \\ &\geq \sum_{i=p}^{n} \frac{\pi}{6} \sin \frac{\pi}{6} (x_i - x_{i-1}) = \frac{\pi}{12} \sum_{i=p}^{n} (x_i - x_{i-1}) = \frac{\pi}{12}(x_n - x_{p-1}) \\ &> \frac{\pi}{12}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{36}. \end{aligned}$$

Here we use that $x \sin x$ is increasing on $\left[0, \frac{\pi}{2}\right]$ and $x_{p_1} < \frac{\pi}{6}$. Hence

$$U(P, f) > \frac{\pi}{36} \Longrightarrow \overline{\int}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx \geq \frac{\pi}{36}.$$

Hence f is not Riemann integrable.

Remark

For the problems including rational and irrational numbers, they follow the same logic.

(3) This is Riemann integrable (since it's only discontinuous at points of measure 0).
It suffices to find a partition such that for any $\varepsilon > 0$, we have $U(P, f) - L(P, f) < \varepsilon$.
Since this function is almost zero. So the choice is relatively easy.
For the given $\varepsilon > 0$, choose

$$P = \left\{x_0 = -1,\ x_1 = -\frac{\varepsilon}{4}, x_2 = \frac{\varepsilon}{4},\ x_3 = 1\right\}.$$

Then we have

$$\begin{aligned} U(P, f) &= 1 \cdot (x_1 - x_0) + 1 \cdot (x_2 - x_1) + 1 \cdot (x_3 - x_2) = 2 \\ L(P, f) &= 1 \cdot (x_1 - x_0) + 0 \cdot (x_2 - x_1) + 1 \cdot (x_3 - x_2) \\ &= 1 \cdot \left(1 - \frac{\varepsilon}{4}\right) + 1 \cdot \left(1 - \frac{\varepsilon}{4}\right) = 2 - \frac{\varepsilon}{2} \end{aligned}$$

Thus

$$U(P, f) - L(P, f) = \frac{\varepsilon}{2} < \varepsilon.$$

Hence f is Riemann integrable. See here for generalization.

Since f is integrable, $\int_0^1 f(x)\,dx = \overline{\int_{-1}^{1}} f(x)\,dx = \inf_P U(P,f) = 2$. Here we used that for any partition P, $U(P,f) = 2$.

(4) This is not Riemann integrable (since it's discontinuous besides $x = 0$).
Note that $x \geq 0$ when $x \in [0,1]$. So by $\mathbb{Q}^C$ is dense in $\mathbb{R}$, we have for any partition $P = \{0 = x_0 < \cdots < x_n = 1\}$, we have

$$\inf f|_{[x_{i-1},x_i]} = 0 \Longrightarrow L(P,f) = 0 \Longrightarrow \underline{\int_0^1} f(x)\,dx = 0.$$

For the upper integral, since x is increasing, we have

$$\sup f|_{[x_{i-1},x_i]} = x_i \Longrightarrow U(P,f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}).$$

Let p be the index such that $\frac{1}{2} \in (x_{p-1}, x_p]$. Then we have

$$U(P,f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=p}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=p}^{n} \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{2}(x_n - x_{p-1})$$
$$> \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

Here we use that x is increasing on $[0,1]$ and $x_{p-1} < \frac{1}{2}$. Hence

$$U(P,f) > \frac{1}{4} \Longrightarrow \overline{\int_0^1} f(x)\,dx \geq \frac{1}{4}.$$

Hence f is not Riemann integrable.

(5) This is not Riemann integrable (since it's discontinuous besides $x = 0$).
Note that $x^2 \geq 0$ when $x \in [0,1]$. So by $\mathbb{Q}^C$ is dense in $\mathbb{R}$, we have for any partition $P = \{0 = x_0 < \cdots < x_n = 1\}$, we have

$$\inf f|_{[x_{i-1},x_i]} = 0 \Longrightarrow L(P,f) = 0 \Longrightarrow \underline{\int_0^1} f(x)\,dx = 0.$$

For the upper integral, since x^2 is increasing on $[0,1]$, we have

$$\sup f|_{[x_{i-1},x_i]} = x_i^2 \Longrightarrow U(P,f) = \sum_{i=1}^{n} x_i^2(x_i - x_{i-1}).$$

Let p be the index such that $\frac{1}{2} \in (x_{p-1}, x_p]$. Then we have

$$U(P,f) = \sum_{i=1}^{n} x_i^2(x_i - x_{i-1}) \geq \sum_{i=p}^{n} x_i^2(x_i - x_{i-1}) \geq \sum_{i=p}^{n} \left(\frac{1}{2}\right)^2 (x_i - x_{i-1}) = \frac{1}{4}(x_n - x_{p-1})$$
$$> \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}.$$

Here we use that x^2 is increasing on $[0,1]$ and $x_{p-1} < \frac{1}{2}$. Hence

$$U(P,f) > \frac{1}{8} \Longrightarrow \overline{\int_0^1} f(x)\,dx \geq \frac{1}{8}.$$

Hence f is not Riemann integrable.

(6) This is not Riemann integrable (since it's discontinuous besides $x = 0$).
We need to analyze both $L(P, f)$ and $U(P, f)$ in this case. It's doable but complicated since x and $-x$ behave differently around $x = 0$. We prove a small lemma here:

Lemma 1. If f is Riemann integrable on $[a, b]$, then for an interval $I = [c, d] \subset [a, b]$, f is Riemann integrable on I.

Proof. Since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Consider a partition

$$P' = \{c\} \cup (P \cap [c, d]) \cup \{d\}.$$

Recall (it's direct from definition) that if for any partition $P = P_1 \cup P_2,\ P_1 \cap P_2 = \emptyset$, we have

$$U(P, f) = U(P_1, f) + U(P_2, f),\ L(P, f) = L(P_1, f) + L(P_2, f).$$

Hence we have

$$\begin{aligned}
&U(P', f) + U((P \cup \{c, d\}) \setminus P', f) = U(P \cup \{c, d\}, f) \leq U(P, f),\\
&L(P', f) + L((P \cup \{c, d\}) \setminus P', f) = L(P \cup \{c, d\}, f) \geq L(P, f)\\
\implies &0 \leq (U(P', f) - L(P', f)) + (U((P \cup \{c, d\}) \setminus P', f) - L((P \cup \{c, d\}) \setminus P', f))\\
&\quad < U(P, f) - L(P, f) < \varepsilon\\
\implies &U(P', f) - L(P', f) < \varepsilon
\end{aligned}$$

since $U((P \cup \{c, d\}) \setminus P', f) - L((P \cup \{c, d\}) \setminus P', f) \geq 0$.

Remark

There are several different easier proofs of this:

(i) Note that $\displaystyle\int_c^d f(x)\ dx = \int_a^b f(x)\chi_{[c,d]}(x)\ dx$. And $\chi_{[c,d]}$ is only discontinuous at 2 points hence it's Riemann integrable. Use the fact that if $f(x),\ g(x)$ are Riemann integrable on $[a, b]$, then $f(x)g(x)$ is integrable then we finish the proof.

(ii) Recall that $f(x)$ is Riemann integrable on $[a, b]$ if and only if set discontinuous points of $f(x)$ on $[a, b]$ is of measure 0. And a subset of measure 0 set is (clearly) measure 0. Hence we finish the proof.

However, these proofs use fancier technique than what we'll use to actually solving the problem. □

With the Lemma, it suffices to prove that there is a subinterval such that $f(x)$ is not Riemann integrable. We choose that interval to be $[0, 1]$. Consider partition $P = \{0 = x_0 < \cdots < x_n = 1\}$. Let p be the index such that $\frac{1}{2} \in (x_{p-1}, x_p]$. Since $\mathbb{Q}$ and $\mathbb{Q}^C$ are dense in $\mathbb{R}$, we have

$$\sup f|_{[x_{i-1},x_i]} = x_i \Longrightarrow U(P,f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}),$$

$$\inf f|_{[x_{i-1},x_i]} = -x_i \Longrightarrow L(P,f) = -\sum_{i=1}^{n} x_i(x_i - x_{i-1}) = -U(P,f).$$

For the upper integral, since x is increasing on $[0,1]$, we have

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=p}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=p}^{n} \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{2}(x_n - x_{p-1}) \\ &> \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

Here we use that x is increasing on $[0,1]$ and $x_{p-1} < \dfrac{1}{2}$. Hence

$$U(P,f) > \frac{1}{4} \Longrightarrow \overline{\int_0^1} f(x)\ dx \geq \frac{1}{4},\ L(P,f) < -\frac{1}{4} \Longrightarrow \overline{\int_0^1} f(x)\ dx \leq -\frac{1}{4}.$$

Hence f is not Riemann integrable on $[0,1]$ and thus not Riemann integable on $[-1,1]$ by the Lemma.

(7) This is not Riemann integrable (since it's discontinuous besides $x = 1$).
Note that $0 \leq x \leq 1$ when $x \in [0,1]$. So by $\mathbb{Q}$ is dense in $\mathbb{R}$, we have for any partition $P = \{0 = x_0 < \cdots < x_n = 1\}$, we have

$$\sup f|_{[x_{i-1},x_i]} = 1 \Longrightarrow U(P,f) = \sum_{i=1}^{n}(x_i - x_{i-1}) = 1 \Longrightarrow \underline{\int_0^1} f(x)\ dx = 1.$$

For the lower integral, since x is increasing on $[0,1]$, we have

$$\inf f|_{[x_{i-1},x_i]} = x_{i-1} \Longrightarrow L(P,f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}).$$

Let p be the index such that $\dfrac{1}{2} \in (x_{p-1}, x_p]$. Then we have

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \leq \sum_{i=1}^{p} x_{i-1}(x_i - x_{i-1}) + \sum_{i=p+1}^{n} 1(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{p} \frac{1}{2}(x_i - x_{i-1}) + (1 - x_{p-1}) = \frac{1}{2}(x_p - x_0) + (1 - x_p) \\ &< \frac{1}{2}\left(\frac{1}{2} - 0\right) + \left(1 - \frac{1}{2}\right) = \frac{3}{4}. \end{aligned}$$

Here we use that x is increasing on $[0,1]$ and $x_p > \dfrac{1}{2}$. Hence

$$L(P,f) < \frac{3}{4} \Longrightarrow \underline{\int_0^1} f(x)\ dx \leq \frac{3}{4}.$$

Hence f is not Riemann integrable.

(8) This is just (2) with the point is different. Hence it's Riemenn integrable.

Since f is integrable, $\displaystyle\int_0^1 f(x)\ dx = \underline{\int_0^1} f(x)\ dx = \sup_P L(P,f) = 0$. Here we used that for any partition P, $L(P,f)=0$.

$\square$

2.1.3. *Continuous Functions on Compact Sets.*

Problem 3: Continuous Functions on Compact Sets

Let (X,d_X) and (Y,d_Y) be metric spaces, and suppose $f: X \to Y$ is continuous on X, and X is compact.

(a) (24S1-1(b),13W-8=12W-1=97W-1(a))
Prove that f is uniformly continuous on (X,d_X).

(b) (20W-1(a)=03S-1(b)=96W-1(d))
Prove that $f(X)$ is compact.

(c) (20W-1(b)=11W-2=03W-2(a))
Prove if f is bijective (i.e. one-to-one and onto) then f^{-1} is continuous as well.

Solution:

(a) Given $\varepsilon > 0,\ x \in X$, since f is continuous, there exists $\delta_x > 0$ such that

$$d(f(x),f(y)) < \frac{\varepsilon}{2} \Longleftrightarrow f(y) \in B_{\frac{\varepsilon}{2}}(f(x)) \text{ when } d(y,x) < \delta \Longleftrightarrow y \in B_{\delta_x}(x).$$

Let $\{B_{\frac{\delta_x}{2}}(x) \mid x \in X\}$ be an open covering of X. Since X is compact, we have $n \in \mathbb{N}$ such that

$$X \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i).$$

Now, we choose

$$\delta = \min_{i=1,\cdots,n}\left\{\frac{\delta_{x_i}}{2}\right\},$$

then if $x,y \in X$ and $d(x,y) < \delta$. Then since $X \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$, there exists some i such that $x \in B_{\frac{\delta_{x_i}}{2}}(x_i)$. And also, we have

$$d(x,y) < \delta < \frac{\delta_{x_i}}{2} \Longrightarrow d(y,x_i) < d(y,x) + d(x,x_i) < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}.$$

Since f is continuous at x_i and $x,y \in B_{\delta_{x_i}}(x_i)$, we have $f(x),f(y) \in B_{\frac{\varepsilon}{2}}(f(x_i))$ and hence

$$d(f(x),f(y)) < d(f(x),f(x_i)) + d(f(x_i),f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we finished the proof.

(b) $\boxed{\text{Way I: Open cover}}$
Consider $\{G_\alpha\}_\alpha$ be an open cover of $f(X)$, that is,

$$X \subseteq \bigcup_{\alpha\in I} G_\alpha \Longrightarrow f(X) \subseteq \bigcup_{\alpha\in I} f^{-1}(G_\alpha)$$

Since f is continuous, $f^{-1}(G_\alpha)$ is open for all $\alpha \in I$. Hence $\left\{f^{-1}(G_\alpha)\right\}_\alpha$ is an open cover for X. Now, since X is compact, there exists $\alpha_1, \cdots, \alpha_n \in I$ such that

$$\begin{aligned} X \quad &\subseteq f^{-1}(G_{\alpha_1}) \cup \cdots \cup f^{-1}(\alpha_n) \\ \implies f(X) &\subseteq f\left(f^{-1}(G_{\alpha_1}) \cup \cdots \cup f^{-1}(\alpha_n)\right) = f(f^{-1}(G_{\alpha_1})) \cup \cdots \cup f(f^{-1}(G_{\alpha_n})) \\ &\subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}. \end{aligned}$$

Here we use that $f(A \cup B) = f(A) \cup f(B)$ and $f\left(f^{-1}(B)\right) \subseteq B$. Hence $\{G_\alpha\}_\alpha$ has a finite subcover. Since the choice of $\{G_\alpha\}_\alpha$ is arbitrary, $f(X)$ is compact.

$\boxed{\text{Way II: Sequentially Compact}}$

Use the fact that X is compact if and only if X is sequentially compact. (See here for one direction.) Let $\{y_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $f(X) \implies y_n = f(x_n)$ for some $\{x_n\}_{n\in\mathbb{N}}$ in X. Since X is compact, X is sequentially compact, there exists a converging subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$ for some $x \in X$. Since f is continuous, $f(x_{n_k}) \to f(x) \in f(X)$.

Hence $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ is a converging subsequence of $\{f(x_n)\}_{n\in\mathbb{N}} = \{y_n\}$. Therefore $f(X)$ is sequentially compact and thus $f(X)$ is compact.

(c) We first prove: (To be honest, I don't think we need to prove this.)

Claim 1. If f is bijective, then

$$(f^{-1})^{-1}(A) = f(A)$$

for all $A \subseteq X$ and f^{-1} denotes the inverse function and $()^{-1}$ denotes the preimage.

Proof of claim. If $y \in (f^{-1})^{-1}(A)$, since f is bijective, f^{-1} exists and $\exists!\ x \in A$ such that $f^{-1}(y) = x \implies f(x) = y \in f(A) \implies (f^{-1})^{-1}(A) \subseteq f(A)$. Similarly, if $y \in f(A)$, then $\exists!\ x \in A$ such that $f(x) = y \implies f^{-1}(y) = x \in A \implies y \in (f^{-1})^{-1}(A) \implies f(A) \subseteq (f^{-1})^{-1}(A)$. Hence $(f^{-1})^{-1}(A) = f(A)$.

Let $A \subseteq X$ be a closed set. Since X is compact, A is also compact. By (b), we have $f(A)$ is also compact and therefore $f(A)$ is closed. Hence $(f^{-1})^{-1}(A) = f(A)$ is closed. By the open/closed characterization of continuous function, f^{-1} is continuous.

(We've used the open characterization in proving (b) and I don't think we need to prove this. For the closed set characterization, if you want to prove that, see here.)

$\square$

2.1.4. *Set Distance Function.*

Problem 4: Set Distance Function

Let A be a non-empty subset of a metric space (X, d) and x an element of X. Define the distance from x to A as

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

(a) (19S-4=13S-2(a)=11S-1(b)=03W-1(b))
Prove that the function $f_A : X \to \mathbb{R}$, defined by $f_A(x) := d(x, A)$, for all $x \in X$ satisfies

$$|f_A(x) - f_A(y)| \leq d(x, y) \text{ for all } x, y \in X,$$

and that f_A is uniformly continuous on X.

(b) (15W-2(b)=13S-2(b))
Prove that $\overline{A} = \{x \mid x \in X \text{ and } f_A(x) = 0\}$.

(c) (03W-1(b))
If A is closed, prove that $f_A(x) = 0$ if and only if $x \in A$.

(d) (15W-2(c))
Suppose A and B are non-empty, disjoint, closed subsets of X. Use the function $g := f_A - f_B$ to prove that there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

In 24S1-2 and 97S-1(b), we're asked to define the distance between two sets:

(e) (24S1-2)
Let (X, d) be a compact metric space and A and B be non-empty closed subsets of X. Let

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Show that $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$.

(f) (24W-2=97S-1(b))
Suppose A and B are disjoint, nonempty subsets of a metric space (X, d). Assume A is compact and B is closed. Give meaning to $d(A, B)$ the distance between A and B and prove that $d(A, B) > 0$.

Solution:

(a) Note that for any $z \in A$, by the inf definition, for any $x, y \in A$, we have

$$f_A(x) = d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, z) \leq d(x, y) + d(y, z).$$

By inf definition again, for a fixed $\varepsilon > 0$, we can choose $z \in A$ such that

$$d(y, z) \leq \inf_{a \in A}(y, a) + \varepsilon.$$

Hence

$$f_A(x) \leq d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + f_A(x, y) + \varepsilon.$$

Since ε is arbitrary, let $\varepsilon \to 0$, we have

$$f_A(x) \leq d(x, y) + f_A(y) \Longrightarrow f_A(x) - f_A(y) \leq d(x, y).$$

By switching the role of x, y, we have similarly

$$f_A(y) - f_A(x) \leq d(x, y).$$

Combining the results, we have

$$|f_A(x) - f_A(y)| \leq d(x, y) \text{ for any } x, y \in A.$$

Furthermore, for any $\varepsilon > 0$, choose $\delta = \varepsilon$, then we have

$$d(x, y) < \delta \Longleftarrow |f_A(x) - f_A(y)| \leq d(x, y) < \delta = \varepsilon.$$

Since δ doesn't depend on x, y, f_A is uniformly continuous on X.

(b) Let $x \in \overline{A} = A \cup A'$. If $x \in A$, clearly we have$f_A(x) = 0$. If $x \in A'$, that is, $\forall\, \varepsilon > 0,\ (B(x, \varepsilon) \cap A) \setminus \{x\} \neq \emptyset \Longleftrightarrow \{y \in A \mid 0 < d(x, y) < \varepsilon\} \neq \emptyset$. Let $\{y_n\}_{n \in \mathbb{N}}$ such that $y_n \in \left\{y \in A \mid d(x, y_n) < \dfrac{1}{n}\right\}$, $n \in \mathbb{N}$. (Or you can use the sequence definition of limit point here.) Then we have

$$0 \leq f_A(x) = \inf_{a \in A} d(x, a) \leq \inf_{n \in \mathbb{N}} d(x, y_n) \leq \frac{1}{n} \to 0$$

as $n \to \infty$. Here we used that $\left\{y \in A \mid d(x, y_n) < \dfrac{1}{n}\right\} \subseteq A$. Hence $f_A(x) = 0$. Combining the results, we have

$$\overline{A} \subseteq \{x \mid x \in X \text{ and } f_A(x) = 0\}.$$

$\boxed{\text{Another Way:}}$

Note that $\{x \mid x \in X \text{ and } f_A(x) = 0\} = f_A^{-1}(\{0\})$. Since $\{0\}$ is closed and we proved that $f_A(x)$ is continuous, we have $\{x \mid x \in X \text{ and } f_A(x) = 0\}$ is closed. Also, it's clear that if $x \in A \Longrightarrow f_A(x) = 0$ hence $A \subseteq \{x \mid x \in X \text{ and } f_A(x) = 0\}$. Recall that the closure is the smallest closed set containing itself (we might need to prove this if you have time. So I don't recommend using this.) We have

$$\overline{A} \subseteq \{x \mid x \in X \text{ and } f_A(x) = 0\}.$$

On the other hand, let $f_A(x) = 0$. If $x \in A \subseteq \overline{A}$, we have $x \subseteq \overline{A}$. If $x \notin A$, by the inf definition, we have for any $\varepsilon > 0$, there exists $y \in A$ such that

$$0 < d(x, y) < \inf_{a \in A} d(x, a) + \varepsilon = \varepsilon$$

where $x \neq y$ clearly. Hence by the previous discussion, we have $x \in \overline{A}$. Combining the results, we have

$$\overline{A} = \{x \mid x \in X \text{ and } f_A(x) = 0\}.$$

Note that for $\inf A = a$, we used both there exists $b \in A$ such that $b < a + \varepsilon$ and $b \leq a + \varepsilon$. It doesn't matter if we use either just like we can use $\dfrac{\varepsilon}{2}$, 2ε for ϵ-δ-like arguments.

(c) This is just (b) with $\overline{A} = A$ if A is closed.

(d) Note A, B: closed and disjoint, we have $\overline{A} = A,\ \overline{B} = B$ and $A \cap B = A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Also, note that clearly $f_A,\ f_B > 0$ by definition. By the previous problems, we have

$$x \in A \Longrightarrow f_A(x) = 0,\ f_B(x) > 0 \Longrightarrow g(x) < 0,$$
$$x \in B \Longrightarrow f_A(x) > 0,\ f_B(x) = 0 \Longrightarrow g(x) > 0.$$

Also, it's clear that g is also continuous. Now let

$$U = \{x \mid g(x) > 0\},\ V = \{x \mid g(x) < 0\}.$$

Clearly $U \cap V = \emptyset$ and by previous discussion, $A \subseteq U,\ B \subseteq V$. Furthermore,

$$U = g^{-1}((0, \infty)),\ V = g^{-1}((-\infty, 0))$$

are both open since g is continuous and $(-\infty, 0),\ (0, \infty)$ are open. Hence U, V are our desired open sets.

(e) If $A \cap B \neq \emptyset$, choose $a = b \in A \cap B$ then $d(a, b) = 0 \Longrightarrow d(A, B) = 0$.

If $d(A, B) = 0$, then we can choose some $\{a_n\}_{n\in\mathbb{N}},\ \{b_n\}_{n\in\mathbb{N}}$ where $a_n \in A,\ b_n inB,\ n \in \mathbb{N}$ such that

$$d(x_n, y_n) < \frac{1}{n}.$$

Since A is closed in a compact set, A is compact and hence A is sequentially compact. (I believe you don't have to prove this here.) Hence $a_{n_m} \to a$ for some $a \in A$ where $\{a_{n_m}\}_{m\in\mathbb{N}}$ is a subsequence of $\{a_n\}_{n\in\mathbb{N}}$. Now consider a fixed $\varepsilon > 0$. Since $a_{n_m} \to a$ Hence there exists $M \in \mathbb{N}$ such that when $m > M$,

$$d(a_{n_m}, a) < \frac{\varepsilon}{2}.$$

Now choose $k \in \mathbb{N}$ such that $\dfrac{1}{n_k} < \dfrac{\varepsilon}{2}$ by the Archimedean property. Choose $N = \max\{M, k\}$ then when $m > N$, we have

$$d(a, b_{n_m}) \leq d(a, a_{n_m}) + d(a_{n_m}, b_{n_m}) < \frac{\varepsilon}{2} + \frac{1}{n_m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $a = b_{n_m}$ for some m, then $a \in A \cap B$. If $a \neq b_{n_m}$ for all $m > N$, then for each $\varepsilon > 0$, there exists some $b_{n_m} \in B$ lies in $B(a, \varepsilon)$. Hence a is a limit point of B. Since B is closed, we have $a \in \overline{B} = B \Longrightarrow a \in A \cap B$. Combining the results, we have $A \cap B \neq \emptyset$.

(f) This is just a rephrase of (e).
We suppose the contrary that $d(A, B) = 0$. By the exact same argument, we will have

$$d(a, b_{n_m}) < \varepsilon.$$

Note that in this case $A \cap B = \emptyset$. Hence $a \neq b_{n_m}$ and hence a is a limit point of B. And we have a contradiction that $A \cap B$ is not disjoint by the same argument.

□

2.1.5. *One-sided Derivative.*

Problem 5: One-sided Derivative

(1) (24W-3)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f'(x)$ exists for all $x \neq a$. Prove that, if $f'(x) \to A$ as $x \to a$, then f is differentiable at $x = a$ and $f'(a) = A$.

(2) (21W-2(b)=19W-3=04S-2(b)=03S-2(b)=00W-2(b)=97W-2(b))
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point, differentiable at every point except perhaps at 0, and $\lim_{x\to 0} f'(x) = L$.
Prove that f is differentiable at $x = 0$ and $f'(0) = L$.

(3) (11W-3) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $\lim_{x\to 0^+} f'(x) = 3$. Show that f has a one sided derivative at $x = 0$ and $f'(0) = 3$.

Reference: Baby Rudin, Chapter 5, Exercise 5,9.

Solution:

(1) $\boxed{\text{Way I: Limit}}$

Let $h > a$. Then f is continuous on $[a, h]$ and differentiable on (a, h). Then by the Mean Value Theorem, there exists $c_h \in (a, h)$ such that

$$\frac{f(h) - f(a)}{h - a} = f'(c_h).$$

Note that $c_h \to a^+$ as $h \to a^+$. Then we have

$$\lim_{x\to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h\to a^+} \frac{f(h) - f(a)}{h - a} = \lim_{h\to a^+} f'(c_h) = \lim_{x\to a^+} f'(x) = \lim_{x\to a} f'(x) = A.$$

Similarly, we can consider the interval $[k, a]$ then we can have

$$\lim_{x\to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a^-} f'(x) = \lim_{x\to a} f'(x) = A.$$

Hence f is differentiable at $x = a$ and $f'(a) = A$.

$\boxed{\text{Way II: Sequence}}$

Note that $f'(x) \to A$ as $x \to a$ so we can consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \neq a$, $n \in \mathbb{N}$ and $x_n \to a$ then we also have $f(x_n) \to A$.

Note that by construction, $x_n \neq a$, $n \in \mathbb{N}$, for each $n \in \mathbb{N}$, by the Mean Value Theorem, there exists some $c_n \in \mathbb{R}$ between a and x_n such that

$$\frac{f(x_n) - f(a)}{x_n - a} = f'(c_n).$$

Also, since $x_n \to a$, let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ when $n \geq N$. Since c_n is between a and x_n, we have

$$|c_n - a| \leq |x_n - a| < \varepsilon.$$

So we have $c_n \to a$ thus $f'(c_n) \to A$. Hence

$$\lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = \lim_{n\to\infty} f'(c_n) = A.$$

Since we don't sepcify the direction of x_n, we have f is differentiable at $x = a$ and $f'(a) = A$.

$\boxed{\text{Way III: L'Hôpital's rule}}$
(If you proved the L'Hôpital's rule which I omit here.)

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \overset{L'H}{=} \lim_{x\to a}\frac{f'(x)}{1} = \lim_{x\to a} f'(x) = A.$$

Hence f is differentiable at $x=a$ and $f'(a)=A$.

(2) This is just (1) with $a=0$ and $L=A$.

(3) This is just a special case of (1). For Way I, we just need to consider $[a,h]$. For Way II, we restrict the sequence to be bigger than 0.

□

2.1.6. *Zero Integral.*

Problem 6: Zero Integral

Suppose $f:[a,b]\to\mathbb{R}$ is continuous.

(a) (24S1-4(a), 23S-2(a),14S-7,04S-3(b),03S-3(b),96W-3(b))
If $f(x)\ge 0$ for all $x\in[a,b]$ and $\displaystyle\int_a^b f(x)\,dx=0$, show that $f=0$ everywhere on $[a,b]$.

(b) (23S-2(b),06W-4(b),96S-4(d)) If $\displaystyle\int_a^b x^k f(x)\,dx=0$ for every non-negative integer k, show $f=0$ everywhere on $[a,b]$.

In 96S-4, they gave useful steps to do the above problems:

(b-01) (96S-4(a))
Let $f:[0,1]\to\mathbb{R}$ be continuous and $\displaystyle\int_0^1 f^2(x)\,dx=0$ Prove that $f(x)=0$ for all $x\in[0,1]$.

(b-02) (96S-4(b)=03S-4(b))
State the Weierstrass Approximation Theorem.

(b-03) (96S-4(c))
Suppose $f_n:[0,1]\to\mathbb{R}$ is sequence of continuous functions converging uniformly to f. Prove that f_nf converges uniformly to f^2.

Reference: Rudin, Theorem 7.26, Exercise 7. 20.

Solution: We solve (a), (b-01)-(b-03) then (b).

(a) Suppose the contrary that $f(x)\neq 0$ for some $x\in[a,b]$. Say $f(c)\neq 0$.
Since $f\ge 0$, we have $f(c)>0$. Since f is continuous, choose $\varepsilon=\dfrac{f(c)}{2}>0$, then $\exists\,\delta>0$ such that when $|x-c|<\delta$, we have

$$|f(x)-f(c)|<\varepsilon=\frac{f(c)}{2}\Longrightarrow -\frac{f(c)}{2}<f(x)-f(c)<\frac{f(c)}{2}$$
$$\Longrightarrow 0<\frac{f(c)}{2}<f(x)<\frac{3}{2}f(c).$$

That is, $f(x) > \dfrac{f(c)}{2} > 0 \;\forall\; x \in (c-\delta, c+\delta)$. But since f is continuous, f is Riemann integrable. Therefore we can choose a partition

$$P = \{x_0 = a, x_1 = \max\{a, c-\delta\}, x_2 = \max\{b, c+\delta\}, x_3 = b\}.$$

Then $\displaystyle\int_a^b f(x)\ dx \geq \frac{f(c)}{2}|x_1 - x_2| > 0$ contradicting to the assumption $\displaystyle\int_a^b f(x) = 0$. Hence $f(x) = 0 \;\forall\; x \in [a,b]$.

(b-01) This is just (a) with $g(x) = f^2(x) \geq 0$.

(b-02)

Theorem 4 (Weierstrass Approximation Theorem)**.** *If f is a continuous real/complex function on $[a,b]$, there exists a sequence of complex/real polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that P_n converges to f uniformly on $[a,b]$.*

(b-03) Since f is continuous on a compact set $[0,1]$, f is bounded on $[0,1]$. Say $|f(x)| < M$. Since $f_n \to f$ uniformly, we have $\forall\; \varepsilon > 0,\ \exists\; N \in \mathbb{N}$ then $\forall\; n > N,\ \forall\; x \in [0,1]$, we have

$$|f(x) - f_n(x)| < \frac{\varepsilon}{M}.$$

Thus

$$|f^2(x) - f(x)f_n(x)| = |f(x)||f(x) - f_n(x)| < M\frac{\varepsilon}{M} = \varepsilon.$$

Hence $f_n f \to f^2$ uniformly.

(b) (Here I still write the full solution, but this is just (b-02),(b-03) then (b-01).)
Since f is continuous on a compact set $[a,b]$, f is bounded on $[a,b]$. Say $|f(x)| < M$. Then by the Weierstrass Approximation Theorem, there exists polynomials $\{P_n\}$ such that $\forall\; \varepsilon > 0,\ \exists\; N \in \mathbb{N}$ then $\forall\; n > N,\ \forall\; x \in [a,b]$, we have

$$|f(x) - P_n(x)| < \frac{\varepsilon}{M}.$$

Thus

$$|f^2(x) - f(x)P_n(x)| = |f(x)||f(x) - P_n(x)| < M\frac{\varepsilon}{M} = \varepsilon.$$

Hence $fP_n \to f^2$ uniformly. Note that

$$\int_a^b f(x)\, x^k\, dx = 0 \;\forall\; k \in \mathbb{N} \Longrightarrow \int_a^b f(x)P_n(x)\ dx = 0.$$

Hence we have

$$\int_a^b f^2(x)\ dx = \int_a^b \lim_{n\to\infty} f(x)P_n(x)\ dx = \lim_{n\to\infty}\int_a^b f(x)P_n(x)\ dx = 0.$$

Since f is continuous, by (a), we have $f^2(x) = 0$. Furthermore, using the same estimate in (a) that if $f(c) > 0$ for some $c \in [a,b]$, then $f(x) > \dfrac{f(c)}{2} > 0 \;\forall\; x \in (c-\delta, c+\delta)$ for some $\delta > 0$, which contradicts $f^2(x) = 0$. Hence $f = 0$.

$\square$

2.1.7. *Test of Uniform Convergence.*

Problem 7: Test of Uniform Convergence

(a) (24S2-1)
Let $\varepsilon > 0$ be given and define $f_n : [\varepsilon, \infty) \to \mathbb{R}$ by $f_n(x) = \dfrac{nx}{1+nx^2}$ for each $n \in \mathbb{Z}$. Prove that $\{f_n\}$ converges uniformly to a function $f : [\varepsilon, \infty) \to \mathbb{R}$. What is f?

(b) (23S-5)
Define
$$f_n(x) = \frac{n}{x^2} e^{-\frac{n}{x}}, \qquad x > 0,$$
Show that $f_n \to 0$ uniformly on $(0, \infty)$.

(c) (22S-3)
For $n \geq 1$, let $f_n : [0, \infty) \to \mathbb{R}$ be defined by:
$$f_n(x) = \frac{nx}{1+nx}.$$
(i) Prove that $f_n(x)$ converges to a function $f(x)$ for all $x \in [0, \infty)$ as $n \to \infty$.
(ii) Does $f_n \to f$ uniformly on $[0,1]$? Justify your answer.
(iii) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify your answer.

(d) (21W-3(b)=17W-2=11W-4)
Consider the sequence of functions
$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2} \qquad x \in [0,1] \qquad n \geq 1.$$
(i) Prove that $\{f_n\}$ converges uniformly on $[\delta, 1]$ for every $\delta > 0$ but does not converge uniformly on $[0,1]$.
(ii) Suppose g is a continuous function on $[0,1]$ with $g(0) = 0$. Prove that $\{gf_n\}$ converges uniformly on $[0,1]$.

(e) (14W-8(a))
Set
$$f_n(x) = \begin{cases} \dfrac{n}{x^3} e^{-\frac{n}{2x^2}} & x > 0 \\ 0 & x = 0 \end{cases}.$$
Does $\{f_n(x)\}$ converge uniformly to 0 on $[0, \infty)$?

(f) (04S-4)
Let $f_n(x) = \dfrac{x}{n} e^{-\frac{x}{n}}$.
(i) Does f_n converge uniformly on $[0, \infty]$? Please justify your claim.
(ii) Does f_n converge uniformly on $[0, 100]$? Please justify your claim.

(g) (96W-4(a))
For $n = 1, 2, \cdots$ Define $g_n(x) = \dfrac{e^{-nx}}{n}$. Prove that g_n converges uniformly to 0 on $[0, \infty)$.

Solution:

(a) Note that for any $x \in (0, \infty)$, we have
$$\lim_{n\to\infty} \frac{n}{x^2} e^{-\frac{n}{x}} = \frac{1}{x^2} \lim_{n\to\infty} n e^{-\frac{n}{x}} = \frac{1}{x^2} \lim_{n\to\infty} \frac{n}{e^{\frac{n}{x}}} \overset{L'H}{=} \frac{1}{x^2} \lim_{n\to\infty} \frac{1}{\frac{1}{x} e^{\frac{n}{x}}} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{e^{\frac{n}{x}}} = 0.$$

Hence $f_n \to 0$ pointwise on $(0, \infty)$. Note that

$$f_n(x) = \frac{n}{x^2} e^{-\frac{n}{x}} = \frac{1}{n} \frac{n^2}{x^2} e^{-\frac{n}{x}}.$$

Note that we know that $e^t = 1 + t + \dfrac{t^2}{2} + \cdots$, $t > 0$ hence

$$e^t \geq \frac{t^2}{2} \Longrightarrow e^{-t} \leq \frac{2}{t^2} \Longrightarrow t^2 e^{-t} \leq 2.$$

Let $t = \dfrac{n}{x}$, since $x \in (0, \infty)$, $n \in \mathbb{N}$, we have $t > 0$ and

$$|f_n(x)| = \left| \frac{1}{n} t^2 e^{-t} \right| \leq \frac{2}{n}.$$

Therefore, for any $\varepsilon > 0$, choosing $N \in \mathbb{N}$, $N > \dfrac{2}{\varepsilon}$, then for $n \geq N$, for all $x \in (0, \infty)$, we have

$$|f_n(x) - 0| < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Hence $f_n \to 0$ uniformly on $(0, \infty)$.

(b) (i)

$$\lim_{n\to\infty} \frac{nx}{1+nx} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x} = \frac{x}{x} = 1 := f(x).$$

(ii) Note that

$$|f_n(x) - f(x)| = \left| \frac{-1}{1+nx} \right| = \frac{1}{1+nx}.$$

If $x = \dfrac{1}{n}$, then we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2}.$$

Hence the convergence is not uniform.

(iii) Note that

$$\left(\frac{1}{1+nx} \right)' = -\frac{n}{(1+nx)^2} < 0.$$

So we have

$$|f_n(x) - f(x)| = \frac{1}{1+nx} < f_n(1) = \frac{1}{1+n}.$$

Let $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N+1 > \dfrac{1}{\varepsilon}$, then when $n \geq N$, for all $x \in [1, \infty)$, we have

$$|f_n(x) - f(x)| < \frac{1}{1+n} \leq \frac{1}{1+N} < \varepsilon.$$

Hence $f_n \to f = 1$ uniformly on $[1, \infty)$.

(c) (i) Clearly $\lim\limits_{n\to\infty} f_n(x) = 0 := f(x)$. Note that

$$f_n'(x) = \frac{2x(1-nx)}{(x^2 + (1-nx)^2} \Longrightarrow f_n'(x) = 0 \Longleftrightarrow x = \frac{1}{n}.$$

Also, when $x \in [\delta, 1]$, $f_n'(x) > 0$ when $x < \frac{1}{n}$, $f_n'(x) < 0$ when $x > \frac{1}{n}$. In this case, we can choose $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < \delta \implies n_1 > \frac{1}{\delta}$. So we have when $n \geq n_1$, $\forall\ x \in [\delta, 1]$, we have

$$|f_n(x) - f(x)| = |f_n(x)| = |f_n(\delta)| = \left|\frac{\delta^2}{\delta^2 + (1 - n\delta)^2}\right| = \frac{\delta^2}{\delta^2 + (1 - n\delta)^2}.$$

Hence for any $\varepsilon > 0$, we choose (of course I calculate this first)

$$n \geq N = \frac{1}{\delta} + \sqrt{\left|\frac{1}{\varepsilon} - 1\right|} = \frac{1}{\delta}\left(1 + \sqrt{\left|\frac{\delta^2}{\varepsilon} - \delta^2\right|}\right) > n_1$$

$$\implies -(n\delta - 1)^2 = -(1 - n\delta)^2 < \frac{\delta^2}{\varepsilon} - \delta^2 < (n\delta - 1)^2 = (1 - n\delta)^2$$

$$\implies \frac{\delta^2}{\delta^2 + (1 - n\delta)^2} = f_n(\delta) = |f_n(x) - f(x)| < \varepsilon.$$

$\{f_n\}$ converges uniformly on $[\delta, 1]$ for every $\delta > 0$.

On the other hand, if we take $x = \frac{1}{n}$, then $f_n\left(\frac{1}{n}\right) = 1$ for all $n \in \mathbb{N}$. Hence

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = 1.$$

So $\{f_n\}$ does not converge uniformly on $[0, 1]$.

Remark This question would be easier if we use this (and we will use this afterward):

Theorem 5 (Baby Rudin, Theorem 7.9)**.** *If $f_n \to f$ pointwise on E, then $f_n \to f$ uniformly on E if and only if*

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

Proof. (Though it's almost trivial but there's no proof in the baby Rudin. You might need to prove this.)

If $\sup_{x \in E} |f_n(x) - f(x)| \to 0$, then there exists $N \in \mathbb{N}$ such that $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$ when $n \geq N$. Then clearly when $n \geq N$, for all $x \in E$, we also have

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

If $f_n \to f$ uniformly, then there exists $N \in \mathbb{N}$ such that for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$. We clearly have

$$\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon \implies \sup_{x \in E} |f_n(x) - f(x)| \to 0.$$

□

With this theorem, in the first part, we can directly see that $f_n(\delta) \to 0$ without writing out the choice of N. And the second part, we clearly have

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = 1 \nrightarrow 0.$$

(ii) First of all, we clearly have $gf_n \to 0$ pointwise as well. Let $\varepsilon > 0$ be given. Since g is continuous and $g(0) = 0$, there exists $0 < \delta < 1$ such that

$$|g(x)| < \frac{\varepsilon}{2} \text{ when } |x| < \delta.$$

Also since g is continuous on a compact set $[0,1]$, g is bounded, say $|g(x)| < M$, $\forall\, x \in [0,1]$. Furthermore, by (i), we have $f_n \to 0$ uniformly on $[\delta, 1]$. By the above Theorem, $\sup\limits_{x\in[\delta,1]} |f_n| \to 0$. Hence there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in[\delta,1]} |f_n| < \frac{\varepsilon}{2M} \text{ when } n \geq N.$$

So we have when $n \geq N$, for all $x \in [0,1]$

$$\begin{aligned}|(gf_n)(x) - 0| &\leq \sup_{x\in[0,1]} |gf_n| \leq \sup_{x\in[0,\delta)} |gf_n| + \sup_{x\in[\delta,1]} |gf_n| \\ &\leq 1 \cdot \sup_{x\in[0,\delta)} |g| + M \sup_{x\in[\delta,1]} |f_n| \\ &\leq 1 \cdot \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{2M} < \varepsilon.\end{aligned}$$

Here we use that $|f_n| \leq 1$ we proved in (i). Hence $\{gf_n\}$ converges uniformly on $[0,1]$.

(If you don't want to use Theorem, for $x \in [\delta, 1]$, you can still write out f_n and choose n such that $f_n < \dfrac{\varepsilon}{2M}$ just like the first method in (i).)

(d) Note that

$$f_n'(x) = \frac{n(n-3x^2)}{x^6} e^{-\frac{n}{2x^2}} \implies f_n'(x) = 0 \iff x = \pm\sqrt{\frac{n}{3}}.$$

Hence

$$\sup_{x\in[0,\infty)} |f_n(x) - 0| = \sup_{x\in[0,\infty)} \left| f_n\left(\sqrt{\frac{n}{3}}\right)\right| = \sqrt{\frac{3}{n}} e^{-\frac{2}{3}} \to 0$$

when $n \to \infty$. Hence $\{f_n(x)\}$ converge uniformly to 0 on $[0,\infty)$.

(e) (i) Note that $f_n \to 0$ pointwise clearly. Also

$$f_n'(x) = \frac{e^{-\frac{x}{n}}(n-x)}{n^2} \implies f_n'(x) = 0 \iff n = x.$$

Also, $f_n'(x) > 0$ when $x < n$, $f_n'(x) < 0$ when $x > n$. Hence

$$\sup_{x\in[0,\infty)} |f_n(x) - 0| = \sup_{x\in[0,\infty)} |f_n(n)| = \frac{1}{e} \not\to 0.$$

Hence $f_n(x) \to 0$ not uniformly. (Or use $f_n(n) = \dfrac{1}{e}$ $\forall\, n \in \mathbb{N}$ as we did in (a).)

(ii) When $n > 100$, we have

$$\sup_{x\in[0,100]} |f_n(x) - 0| = f_n(100) = \frac{100}{n} e^{-\frac{100}{n}} \to 0 \text{ when } n \to \infty.$$

Hence f_n converge uniformly on $[0,100]$.

(f) Note that $g_n'(x) = -e^{-nx} < 0$. Hence $g_n(x) < g_n(0)$ for all $x \in [0, \infty)$

$$|g_n(x) - 0| = \left|\frac{e^{-nx}}{n}\right| < \frac{1}{n}.$$

Then for any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$, then when $n \geq N$, we have for all $x \in [0, \infty)$,

$$|g_n(x) - 0| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence g_n converges uniformly to 0 on $[0, \infty)$.

□

2.1.8. *Uniformly Convergence of Series Types.*

Problem 8: Uniformly Convergence of Series Types

Find the following series are uniformly convergent on the indicated intervals:

(1-1) (24W-5(i)=19S-6) $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1+n^6x^2}$ on $\mathbb{R}$

(2-1) (23W-3(a)) $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{\sqrt{n}(1+(nx)^2)}$ on $\mathbb{R}$

(3) (22W-2) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{9n^4+n^6x^4}$ on $\mathbb{R}$

(4-1) (20S-2(a),11S-2(b)=03W-3(b)) $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{n(1+nx^2)}$ on $\mathbb{R}$

(5) (05W-4(a)) $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1+n^4x^2}$ on $[a, \infty),\ a > 0$

Also, for 24W-5(ii), 23W-3(b) and 20S-2(b), we have to evaluate the integral:

(1-2) Express $\displaystyle\int_0^1 \left(\sum_{n=1}^{\infty} \frac{nx}{1+n^6x^2}\right) dx$ as an infinite series, fully justifying each step in the calculation.

(2-2) Prove that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{2x}{\sqrt{n}(1+(nx)^2)}\right) = \sum_{n=1}^{\infty} \frac{\ln(1+n^2)}{n^2\sqrt{n}}.$$

(4-2) Prove that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{2x}{n(1+nx^2)}\right) = \sum_{n=1}^{\infty} \frac{\ln(1+n)}{n^2}.$$

Solution: For the integral part, we need that if f converges uniformly on $\mathbb{R}$, we have for any $a, b \in \mathbb{R}$,

$$\int_a^b f(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.$$

(1-1) Let $f_n(x) = \dfrac{1+n^6x^2}{2}$, $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. When $x = 0$, $f_n(x) = f(x) = 0\ \forall\ n \in \mathbb{N}$, hence $f_n(x)$ converge to $f(x)$ uniformly. Note that since $n^6x^2 > 0$ for all $x \in \mathbb{R},\ x \neq 0$

hence by the AM-GM Inequality, we have

$$\frac{1+n^6x^2}{2} \geq \sqrt{n^6x^2} = |n^3x| \Longrightarrow 1+n^6x^2 \geq 2|n^3x^2|.$$

Hence we have

$$\left|\frac{nx}{1+n^6x^2}\right| \leq \left|\frac{nx}{2|n^3x|}\right| = \frac{n|x|}{2n^3|x|} = \frac{1}{2n^2}.$$

And by p-series test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Hence by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1+n^6x^2}$ converges uniformly when $x \neq 0$. Combining the results, $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1+n^6x^2}$ is uniformly convergent on $\mathbb{R}$.

(1-2) By the theorem, we have

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{\infty} \frac{nx}{1+n^6x^2}\right) dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{nx}{1+n^6x^2}\, dx \\
(u = 1+n^6x^2 \Longrightarrow du = 2n^6xdx) &= \sum_{n=1}^{\infty} \frac{1}{2n^5} \int_1^{1+n^6} \frac{1}{u}\, du \\
&= \sum_{n=1}^{\infty} \frac{1}{2n^5} \left(\ln u \Big|_1^{1+n^6}\right) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\ln(1+n^6)}{n^5}.
\end{aligned}$$

(2-1) Let $f_n(x) = \dfrac{2x}{\sqrt{n}(1+(nx)^2)}$, $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. When $x = 0$, $f_n(x) = 0\ \forall\ n \in \mathbb{N}$. When $x \neq 0$, note that $1+(nx)^2 \geq 2\sqrt{n^2x^2} = n|x|$. Hence we have

$$\left|\frac{2x}{\sqrt{n}(1+(nx)^2)}\right| \leq \left|\frac{2x}{2\sqrt{n}n|x|}\right| = \left|\frac{1}{n^{\frac{3}{2}}}\right| \left|\frac{x}{|x|}\right| \leq \frac{1}{n^{\frac{3}{2}}}.$$

And by p-series test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$. Hence by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{\sqrt{n}(1+(nx)^2)}$ converges uniformly when $x \neq 0$. Combining the results, $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{\sqrt{n}(1+(nx)^2)}$ is uniformly convergent on $\mathbb{R}$.

(2-2) By the theorem, we have

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{\infty} \frac{2x}{\sqrt{n}(1+(nx)^2)}\right) dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{2x}{\sqrt{n}(1+(nx)^2)}\, dx \\
(u = 1+n^2x^2 \Longrightarrow du = n^2 2xdx) &= \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} \int_1^{1+n^2} \frac{1}{u}\, du \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} \left(\ln u \Big|_1^{1+n^2}\right) \\
&= \sum_{n=1}^{\infty} \frac{\ln(1+n^2)}{n^2\sqrt{n}}.
\end{aligned}$$

(3) Let $f_n(x) = \dfrac{x^2}{9n^4 + n^6x^4}$, $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. When $x = 0$, $f_n(x) = f(x) = 0 \; \forall \; n \in \mathbb{N}$, hence $f_n(x)$ converge to $f(x)$ uniformly. Note that since $n^6x^4 > 0$ for all $x \in \mathbb{R}$, $x \neq 0$ hence by the AM-GM Inequality, we have

$$\left|\frac{x^2}{9n^4 + n^6x^4}\right| \leq \left|\frac{x^2}{3n^5x^2}\right| = \frac{1}{3n^5}.$$

And by p-series test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^5} < \infty$. Hence by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{9n^4 + n^6x^4}$ converges uniformly when $x \neq 0$. Combining the results, $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{9n^4 + n^6x^4}$ is uniformly convergent on $\mathbb{R}$.

(4-1) Let $f_n(x) = \dfrac{2x}{n(1 + nx^2)}$, $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. When $x = 0$, $f_n(x) = f(x) = 0 \; \forall \; n \in \mathbb{N}$, hence $f_n(x)$ converge to $f(x)$ uniformly. Note that since $nx^2 > 0$ for all $x \in \mathbb{R}$, $x \neq 0$ hence by the AM-GM Inequality, we have

$$\left|\frac{2x}{n(1 + nx^2)}\right| \leq \left|\frac{2x}{n(\sqrt{n}|x|)}\right| = \frac{1}{n^{\frac{3}{2}}}.$$

And by p-series test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$. Hence by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{n(1 + nx^2)}$ converges uniformly when $x \neq 0$. Combining the results, $\displaystyle\sum_{n=1}^{\infty} \frac{2x}{n(1 + nx^2)}$ is uniformly convergent on $\mathbb{R}$.

(4-2) By the theorem, we have

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{\infty} \frac{2x}{n(1 + nx^2)}\right) dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{2x}{n(1 + nx^2)} \, dx \\
(u = 1 + nx^2 \implies du = n2xdx) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_1^{1+n^2} \frac{1}{u} \, du \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\ln u \Big|_1^{1+n}\right) = \sum_{n=1}^{\infty} \frac{\ln(1 + n)}{n^2}.
\end{aligned}$$

(5) Let $f_n(x) = \dfrac{nx}{1 + n^4x^2}$, $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. Note that the AM-GM inequality doesn't work here since we will arrive at $\dfrac{1}{n}$. But this is actually somehow easier. Note that

$$\left|\frac{nx}{1 + n^4x^2}\right| = \frac{n|x|}{1 + n^4x^2} \leq \frac{n|x|}{n^4x^2} = \frac{1}{n^3|x|} \leq \frac{1}{n^3a} \; \forall \; x \in [a, \infty).$$

And by p-series test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$. Hence by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1 + n^4x^2}$ converges uniformly on $x \in [a, \infty)$.

$\square$

Problem 9: Compactness in Real Numbers

(a) (12S-1=11W-1)
Prove one of the following equivalent statements:
  (i) (04S-1(b)=03S-1(a)=97S-1(c)=96S-1(b))
  Every closed interval $[a, b]$ is a compact subset of $\mathbb{R}$.
  (ii) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

(b) (i) (97S-1(a))
  State the definition of compactness of a metric space in terms of sequences.
  (ii) (96W-1(a))
  Suppose E is a subset of a metric space (X, d). When is E said to be compact?
  (iii) (96W-1(b))
  Verify that the interval $(0, 1]$ is not a compact subset of $\mathcal{R}$.
  (iv) (96W-1(c))
  Let
$$E = \{0\} \cup \left\{\frac{1}{n} \ : \ n = 1, 2, 3, \cdots\right\}.$$
  Prove directly that E is a compact subset of $\mathcal{R}$.

Solution:

(a) Note that in both questions, we will use the Nested Interval Theorem. And the Nested Interval Theorem is also a characterization of completeness of real numbers, I don't think we need to prove this. (See here for the proof.)
  (i) Suppose the contrary that $[a, b]$ is not compact. Then there exists a open covering $\{G_\alpha\}_{\alpha\in I}$ which doesn't have a finite covering. Now let $I_1 = [a, b] := [a_1, b_1]$. Choose $x_1 \in S \cap I_1$. Now, spilt S into two closed intervals. Since $\{G_\alpha\}_{\alpha\in I}$ has no finite covering, at least one of the intervals has no finite covering, say that interval is
$$I_2 = [a_2, b_2], \ |I_2| = |b_2 - a_2| = \frac{b_1 - a_1}{2}.$$
  Choose $x_2 \in S\cap$. Continuing the halving process on the intervals, choosing intervals containing infinite elements, we can have
$$I_k = [a_k, b_k], \ |I_k| = |b_k - a_k| = \frac{b_{k-1} - a_{k-1}}{2} = \cdots = \frac{b_1 - a_1}{2^{k-1}}.$$
  And we choose $x_k \in S \cap I_k$. Note that we have $I_{k+1} \subseteq I_k$, and $|I_k| \to 0$ as $k \to \infty$. Then by the Nested Interval Theorem, we have
$$\exists \ x \in \mathbb{R} \text{ such that } \bigcap_{k\in\mathbb{N}} I_k = \{x\}.$$
  Note that for any clearly we have $x \in I_1 = [a, b]$. Then there exists $\alpha \in I$ such that $x \in G_\alpha$. Also, since G_α is open, there exists $r > 0$ such that $N_r(x) \subseteq G_\alpha$. Now we choose $n \in \mathbb{N}$ such that $\frac{b-a}{2^{n-1}} < r$, then we have
$$I_n \subseteq N_r(x) \subseteq G_\alpha$$

which is a contradiction to the construction of I_n with no finite subcover. Hence $\{G-\alpha\}_{\alpha\in I}$ has a finite subcover. Since the choice of $\{G_\alpha\}_{\alpha\in I}$ is arbitrary, we have $[a,b]$ is compact.

(ii) Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Consider the set $S=\{x_n \mid n\in\mathbb{N}\}$. If S is finite, then there exists $x\in x_n$ repeats infinitely. Just choose $x_{n_k}=x\ \forall\ k\in\mathbb{N}$ then $\{x_{n_k}\}_{k\in\mathbb{N}}$ is our desired subsequence.

If S is infinite. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, S is bounded, say $S\subseteq[a_1,b_1]:=I_1,\ a_1,b_1\in\mathbb{R}$. Choose $x_{n_1}\in S\cap I_1$. Now, spilt S into two closed intervals. Since S is infinite, at least one of the intervals contain infinite elements, choose one such interval to be

$$I_2=[a_2,b_2],\ |I_2|=|b_2-a_2|=\frac{b_1-a_1}{2}.$$

Choose $x_{n_2}\in S\cap I_2,\ n_2>n_1,\ n_2\in\mathbb{N}$. Since $S\cap I_2$ is infinite, such n_2 exists. (If such n_2 doesn't exists, then $n_1>n_i,\ i\in\mathbb{N}$ where $n_i\in S\cap I_2$, contradiction.) Continuing the halving process on the intervals, choosing intervals containing infinite elements, we can have

$$I_k=[a_k,b_k],\ |I_k|=|b_k-a_k|=\frac{b_{k-1}-a_{k-1}}{2}=\cdots=\frac{b_1-a_1}{2^{k-1}}.$$

And we choose $x_{n_k}\in S\cap I_k,\ n_k>n_{k-1},\ n_k\in\mathbb{N}$. Note that we have $I_{k+1}\subseteq I_k$, and $|I_k|\to 0$ as $k\to\infty$. Then by the Nested Interval Theorem, we have

$$\exists\ x\in\mathbb{R}\text{ such that }\bigcap_{k\in\mathbb{N}} I_k=\{x\}.$$

Note that for any $k\in\mathbb{N},\ x_k\in I_k,\ x\in I_k$, then we have

$$|x_{n_k}-x|<|I_k|=\frac{b_1-a_1}{2^{k-1}}.$$

(To be honest, I think you can just write it goes to zero and conclude the result here.) Now for any $\varepsilon>0$, choose $N\in\mathbb{N}$ such that $N>\ln_2\left(\frac{b_1-a_1}{\varepsilon}\right)+1$, then when $k\geq N$

$$|x_{n_k}-x|=\frac{b_1-a_1}{2^{k-1}}<\frac{b_1-a_1}{2^{N-1}}<\varepsilon.$$

Hence $x_{n_k}\to x$ when $k\to\infty$. Since the choice of $\{x_n\}_{n\in\mathbb{N}}$ is arbitrary. We have every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

(b) (i) (X,d) : metric space, $K\subseteq X$. K is sequentially compact if every sequence in K has a subsequence converging to a point in K.

(ii) E is compact if any open cover $\{G_\alpha\}_{\alpha\in I}$ of E has a finite subcover.

(iii) Consider $A_n=\left\{\left(\frac{1}{n},2\right)\right\}$. Clearly, $(0,1]\subseteq\bigcup_{n\in\mathbb{N}}A_n$ hence $\{A_n\}_{n\in\mathbb{N}}$ is an open cover for $(0,1]$. However, for any $k\in\mathbb{N}$,

$$A_1\cup\cdots A_k=\left(\frac{1}{k},2\right)$$

And when $m>k$, we have $\frac{1}{m}<\frac{1}{k}\implies\frac{1}{m}\notin A_1\cup\cdots A_k\implies(0,1]\nsubseteq A_1\cup\cdots A_k$. Hence $\{A_n\}_{n\in bN}$ doesn't have a finte subcover.

(iv) Let $\{G_\alpha\}_{\alpha\in I}$ be an open covering for E. Then for any $n \in \mathbb{N}$, there exists $\alpha_n \in I$ such that $\frac{1}{n} \in G_{\alpha_n}$. Also, there exists $\alpha_0 \in I$ such that $0 \in G_{\alpha_0}$. Since G_{α_0} is open, there exists $r > 0$ such that $N_r(0) \subseteq G_{\alpha_0}$. Now we can choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{r} \Longrightarrow \frac{1}{N} < r \Longrightarrow \frac{1}{N} \in N_r(0) \Longrightarrow \frac{1}{n} \geq \frac{1}{N} \in N_r(0) \subseteq G_{\alpha_0} \text{ when } n \geq N.$$

Hecne

$$E \subseteq G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_{N-1}}$$

which is a finite subcover. Since the choice of $\{G_\alpha\}_{\alpha\in I}$ is arbitrary, we have E is compact.

□

2.1.10. *Uniform Continuity and One Sided Limit.*

Problem 10: Uniform Continuity and One Sided Limit

(1) (24W-4=22W-7=18W-3=96S-2(b))
Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous.
  (a) (and 22S-2(b))
  Prove that f is bounded.
  (b) Prove that $\lim_{x\to 0^+} f(x)$ exists.

(2) (04S-2(a)=96W-2(c))
Suppose E is a bounded subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. Prove that $f(E)$ is a bounded subset of $\mathbb{R}$.

(3) (20W-2)
Suppose $f : (0, 1] \to \mathbb{R}$ is continuous.
  (i) Prove that if f is uniformly continuous then $\lim_{x\to 0^+} f(x)$ exists.
  (ii) Prove that if $\lim_{x\to 0^+} f(x) = 3$ then f is uniformly continuous.

(4) (05W-2(a))
Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x)$ exists and is a finite number. Prove that f is uniformly continuous on $[0, \infty)$.

Solution:

(1) We prove the case $f : (a, b) \to \mathbb{R}$. And this is $a = 0, b = 1$.
Let $\varepsilon = 1$, since f is uniformly continuous, $\exists\ \delta > 0$ such that $\forall\ x, y \in (a, b)$, $|f(x) - f(y)| < 1$ when $|x - y| < \delta$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$, then consider the equidistant divided (or any length smaller than δ) intervals

$$I_k = \left[a + (k-1)\cdot\frac{b-a}{n}, a + k\cdot\frac{b-a}{n}\right],\ k = 1, \cdots, N.$$

Then for each interval, we have $|x - y| < \frac{1}{N} < \delta$ for all $x, y \in I_k$.
Hence when $|x - x_k| < \delta$, for some $x_k \in I_k$, we have $|f(x) - f(x_k)| < 1$, so

$$|f(x)| < |f(x) - f(x_k)| + |f(x_k)| < 1 + |f(x_k)|.$$

Repeating this in all I_k, let

$$M = 1 + \max_{k=1,\cdots,N} |f(x_k)|.$$

Then for all $x \in (a,b)$, then $x \in I_k$ for some k. Hence we have

$$|f(x)| < 1 + |f(x_k)| < M.$$

(2) Idea: In above case, for (a,b), we actually consider $[a,b]$ and the intervals we chose are actually covering of $[a,b]$. So we can use the compactness in $\mathbb{R}$ to do this.

Note that for any set, $\overline{E}$ is closed (if you have time, you can prove this but I don't think it's needed here.) And E is closed, we claim that $\overline{E}$ is also bounded.

Claim 2. If E is bounded, then $\overline{E}$ is bounded.

Proof of claim. There are several ways to do this.
$\boxed{\text{Way I: Use equivalence definition}}$ (I suggest use this and don't write too much on this claim since it's not the point of the this problem.)
Recall that

$$\overline{E} = \bigcap \{E \subseteq F \mid F: \text{ closed in } X\}.$$

Here $X = \mathbb{R}$ and this means that the closure is the smallest closed set that contains E. Since E is bounded, E is contained in a closed ball. And by this definition of the closure, $\overline{E}$ is also contained in that closed ball and hence $\overline{E}$ is also bounded.

$\boxed{\text{Way II: Use limit point definition}}$ (If you wish to use this.)

$\boxed{\text{Way III: Proof by contrapositive}}$ (Prove that if E is not bounded then $\overline{E}$ is not bounded)

By the claim and the Heine-Borel Theorem, we have $\overline{E}$ is compact. Let $\varepsilon = 1$, since f is uniformly continuous, $\exists\ \delta > 0$ such that $\forall\ x, y \in (a,b)$, $|f(x) - f(y)| < 1$ when $|x - y| < \delta$. Now, for that δ, consider the open covering $\{B_\delta(x) \mid x \in E\}$ be an open covering for $\overline{E}$ since if $x \in E'$, $\exists\ \{x_n \in E\}_{n\in\mathbb{N}}$ such that $x_n \to x$, that is, for any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ when $n > N$. Just choose $\varepsilon = \delta$ then we have $x \in B_\delta(x_n)$ for all $n > N$. Since $\overline{E}$ is compact, there exists finite subcovering $\overline{E} \subseteq \bigcup_{i=1}^{n} B_\delta(x_i)$ for some $x_i \in E$, $i = 1, \cdots, n$. Now, for any $x \in B_\delta(x_k)$, we have $|x - x_k| < \delta \Longrightarrow |f(x) - f(x_k)| < 1$. So as in (a), we have

$$|f(x)| < |f(x) - f(x_k)| + |f(x_k)| < 1 + |f(x_k)|.$$

Repeating this in all x_i, $i = 1, \cdots, n$, let

$$M = 1 + \max_{k=1,\cdots,N} |f(x_k)|.$$

Then for all $x \in E$, then $x \in B_\delta(x_k)$ for some k. Hence we have

$$|f(x)| < 1 + |f(x_k)| < M.$$

(3) (i) We prove that: If $f:(a,b)\to\mathbb{R}$ is uniformly continuous on (a,b) for some $a,b\in\mathbb{R}$, then $\lim_{x\to a^+} f(x)$, $\lim_{x\to b^-} f(x)$ both exist and are finite. This is just the case $a=0$. We only prove one side, the other side is identical. (the question also only need one side)

Note that it suffices to prove that any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n\to a^+$ and if $f(x_n)\to L$ is independent of the choice of $\{x_n\}_{n\in N}$, then $\lim_{x\to a^+} f(x)$ exists.
Let $\{x_n\}_{n\in\mathbb{N}}$ such that $a\le x_n\le b$ for all $n\in\mathbb{N}$ and $x_n\to a$. We claim that $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since f is uniformly continuous, for any $\varepsilon>0$, we have $|f(x)-f(y)|<\varepsilon$ when $|x-y|<\delta$. Note that $x_n\to a$, then there exists $N\in\mathbb{N}$ such $|x_n-a|<\frac{\delta}{2}$ when $n\ge N$. Hence if $n,m\ge N$, $|x_n-x_m|\le|x_n-a|+|x_m-a|<\delta$. Thus by the uniform continuity, we have

$$|f(x_n)-f(x_m)|<\varepsilon.$$

Therefore, consider $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in (a,b) and $x_n\to a$, $y_n\to a$. By the previous claim, we have $\{f(x_n)\}_{n\in\mathbb{N}}$, $\{f(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequence, and (since $\mathbb{R}$ is complete) $f(x_n)\to L_1$, $f(y_n)\to L_2$ for some $L_1,\ L_2\in\mathbb{R}$.
It suffices to show that $L_1=L_2$. Suppose the contrary, $L_1\neq L_2$. Choose $\varepsilon=\frac{|L_1-L_2|}{3}$. Then there exists $N_1\in\mathbb{N}$ such that $|f(x_n)-L_1|<\varepsilon$ when $n\ge N_1$ and there exsits $N_2\in\mathbb{N}$ such that $|f(y_n)-L_2|<\varepsilon$ when $n\ge N_2$. Then for $n\ge\max\{N_1,N_2\}$, we have

$$\begin{aligned}|L_1-L_2|&=|L_1-f(x_n)+f(x_n)-f(y_n)+f(y_n)-L_2|\\&\le|f(x_n)-L_1|+|f(x_n)-f(y_n)|+|f(y_n)-L_2|\end{aligned}$$

Hence

$$|f(x_n)-f(y_n)|\ge|L_1-L_2|-|f(x_n)-L_1|-|f(y_n)-L_2|>\frac{|L_1-L_2|}{3}=\varepsilon.$$

However since $x_n\to a,\ y_n\to a, \exists\ N_3,N_4\in\mathbb{N}$ such that $|x_n-a|<\frac{\delta}{2}$, $|y_n-a|<\frac{\delta}{2}\implies|x_n-y_n|=|(x_n-a)+(a-y_n)|\le|x_n-a|+|y_n-a|<\delta$ when $n>\max\{N_3,N_4\}$, this contradicts to the assumption that f is uniformly continuous on $\mathbb{R}$.

(ii) Since $\lim_{x\to 0^+} f(x)=3$, we can define $\tilde{f}(x)=\begin{cases}f(x), & x\in(0,1],\\ 3, & x=0\end{cases}$. Then $\tilde{f}$ is continuous on $[0,1]$. Since $[0,1]$ is compact then $\tilde{f}$ is uniformly continuous on $[0,1]$, that is, $\forall\ \varepsilon>0$, $\exists\ \delta>0$ such that $\forall\ x,y\in[0,1]$, $|f(x)-f(y)|<\varepsilon$ when $|x-y|<\delta$. Note that $(0,1]\subseteq[0,1]$ and $f=\tilde{f}\Big|_{(0,1]}$, hence just replace $x,y\in[0,1]$ to $x,y\in(0,1]\subseteq[0,1]$ we can see that f is uniformly continuous.

(4) This is slightly different that (3)(ii) and it's more delicate. We prove the whole real line version: If $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $\lim_{x\to-\infty} f(x)$, $\lim_{x\to\infty} f(x)$ both exist and are finite, then f is uniformly continuous.

Let $\lim_{x\to\infty} f(x)=M$, $\lim_{x\to-\infty} f(x)=m$. We have for any $\varepsilon>0$, there exists A such that

$$|f(x)-M|<\frac{\varepsilon}{2},\ |f(x)-m|<\frac{\varepsilon}{2}\text{ when } x>A,\ x<-A\text{ respectively.}$$

Hence when $x, y \in (-\infty, -A)$ or (A, ∞), we have

$$|f(x) - f(y)| \leq |f(x) - M(m)| + |f(y) - M(m)| < \varepsilon \text{ when } x > A(x < -A).$$

Also, f is uniformly continuous on $[-A-1, A+1]$ obviously. (We don't want to discuss the behavior of the endpoint)

On $[-A-1, A+1]$, for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ when $|x-y| < \delta$. Let's consider $0 < \delta < 1$. Again, for any $x, y \in \mathbb{R}$, if $x, y \in [-A-1, A+1]$ then we're done. Suppose $x \in (-\infty, -A-1)$, note that if $|x-y| < \delta < 1 \Longrightarrow y \in (-\infty, -A)$. Hence by $\lim\limits_{x \to -\infty} f(x) = m$, we have

$$|f(x) - f(y)| \leq |f(x) - m| + |f(y) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Also, if $x \in (A+1, \infty)$, since $|x-y| < \delta < 1 \Longrightarrow y \in (A, \infty)$.
Hence by $\lim\limits_{x \to -\infty} f(x) = M$, we have

$$|f(x) - f(y)| \leq |f(x) - M| + |f(y) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, combining the result, we have f is uniformly continuous on $\mathbb{R}$.

$\square$

2.1.11. *Sequence of Functions with Convergent Sequence.*

Problem 11: Sequence of Functions with Convergent Sequence

(a) (21S-4(a)=03W-3(a)=00W-2(a))
Suppose (X, d) is a metric space and $f_n : X \to \mathbb{R}$ a sequence of continuous real valued functions on X which converges uniformly to a function $f : X \to \mathbb{R}$. Prove that f is continuous.

(b) (06S-4(a)=03S-4(a),06W-2(b)=96S-3(b))
Suppose E is a subset of $\mathbb{R}$ and $f_n : E \to \mathbb{R}$ is a sequence of *continuous* functions converging *uniformly* to $f : E \to \mathbb{R}$. Let x_n be a sequence in E converging to a point x in E. Prove that the sequence $\{f_n(x_n)\}$ converges to $f(x)$, that is

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Reference: Baby Rudin, Chapter 7, Theorem 7.12, Exercise 9.

Solution:

(a) Let $\epsilon > 0,\ a \in X$ be arbitrary chosen. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $\forall\ x \in X$,

$$d_{\mathbb{R}}(f_n(x), f(x)) < \frac{\varepsilon}{3}\ \forall x \in X \text{ when } n \geq \mathbb{N}.$$

Here $d_{\mathbb{R}}(x, y)$ is a metric in $\mathbb{R}$. (All metric in $\mathbb{R}$ is equivalent, so you can write $|x - y|$ if you like.) Also, since f_n are continuous, f_n are continuous at $x = a$. So there exists $\delta > 0$ such that

$$d_{\mathbb{R}}(f_n(x), f_n(a)) < \frac{\varepsilon}{3} \text{ when } d(x, a) < \delta.$$

Hence when $n \geq N,\ d(x, a) < \delta$, we have

$$\begin{aligned} d_{\mathbb{R}}(f(x), f(a)) &\leq d_{\mathbb{R}}(f(x), f_n(x)) + d_{\mathbb{R}}(f_n(x), f_n(a)) + d_{\mathbb{R}}(f_n(a), f(a)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0,\ a \in X$ are arbitrary chosen, f is continuous on X.

(b) Let $\varepsilon > 0$ be fixed. By (a) (Note that (a) is not a part in the question in the actual prelims, you will have to prove (a) in proving this.), f is continuous, there exists $\delta > 0$ such that

$$d_{\mathbb{R}}(f(y), f(x)) < \frac{\varepsilon}{2} \text{ when } d(y, x) < \delta.$$

Again, since $f_n \to f$ uniformly, there exists $N_1 \in \mathbb{N}$ such that $\forall\ y \in X$,

$$d_{\mathbb{R}}(f_n(y), f(y)) < \frac{\varepsilon}{2} \text{ when } n \geq \mathbb{N}.$$

Also, since $x_n \to x$, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \text{ when } n \geq N_2.$$

Choose $N = \max\{N_1, N_2\}$. Then when $n \geq N$, we have

$$d_{\mathbb{R}}(f_n(x_n), f(x)) \leq d_{\mathbb{R}}(f_n(x_n), f(x_n)) + d_{\mathbb{R}}(f(x_n), f(x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f_n(x_n) \to f(x)$, that is, $\lim_{n\to\infty} f_n(x_n) = f(x)$.

□

2.1.12. *Discrete Topology.*

Problem 12: Discrete Topology

Let X be a set equipped with the discrete metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(1) (14S-9(a))
Prove that d is a metric.
(2) (22S-1(a)∼14S-9(b)(c))
Prove that every subset of X is both open and closed.
(3) (24S2-5(iv),23W-1(b))
Prove that every infinite subset of $Y \subseteq X$ is closed and bounded, but is not compact.
(4) (22S-1(b)=16W-2=15S-3∼14S-9(d))
Prove that every subset of X is compact if and only if it is finite.

Solution:

(1) Standard check:
- $d(x,y) > 0$ if $x \neq y$, $d(x,y) = 0$ if $x = y$: Clear by definition.
- $d(x,y) = d(y,x)$: Clear by definition.
- $d(x,z) \leq d(x,y) + d(y,z)$:
  Case 1: $x = y = z$: $d(x,z) = 0 \leq 0 = 0 + 0 = d(x,y) + d(y,z)$.
  Case 2: $x = y, y \neq z$: $d(x,z) = 1 \leq 1 = 0 + 1 = d(x,y) + d(y,z)$.
  Case 3: $x \neq y,\ y \neq z$: $d(x,z) = 1 \leq 2 = 1 + 1 = d(x,y) + d(y,z)$.

(2) If $0 < r < 1$, then $\forall\, x \in Y$, we have $N_r(x) = \{x\} \subseteq Y$. Hence Y is open. On the other hand, for any $y \in X \setminus Y$, $N_r(y) = \{y\} \subseteq X \setminus Y$. Hence $X \setminus Y$ is also open and Y is closed.

(3) Note that if $r > 1$, we have $N_r(x) = X\ \forall\, x \in Y$ hence $Y \subseteq X = N_r(x)$ and therefore Y is bounded. For the non-compactness, we only have to find a open covering which doesn't have a finite subcover. Take $\{\{x\} \mid x \in Y \subseteq X\}$ is clearly a open cover since we've proved that every subset is open. And it clearly doesn't have finite subcover since Y is infinite. (Or see (4) and we can prove this using proof by contrapositive.)

(4) If Y is compact, we can choose $\{N_r(x)\}_{x \in Y}$ where $0 < r < 1$ be an open cover for Y. Since Y is compact, there exists $x_1, \cdots, x_n \in Y$ such that $Y \subseteq N_r(x_1) \cup \cdots \cup N_r(x_n)$ for some $n \in \mathbb{N}$. Recall that $N_r(x_i) = \{x_i\}$ as discussed before. We have

$$Y = \{x_1, \cdots, x_n\}$$

which is finite.

On the other hand, if Y is finite, say $Y = \{y_1, \cdots, y_n\}$ for some $n \in \mathbb{N}$. Choose $\{G_\alpha\}_{\alpha \in I}$ be an open cover for Y for some index set I. Then there exists $\alpha_1, \cdots, \alpha_n \in I$

such that $y_1 \in G_{\alpha_1}, y_2 \in G_{\alpha_2}, \cdots, y_n \in G_{\alpha_n}$. Hence

$$Y \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

which prove that Y is compact. (Note that this is regardless of the choice of metric d.)

□

2.1.13. *Compactness Implies Sequentially Compactness.*

Problem 13: Compact Implies Sequentially Compact

Suppose (X, d) is a compact metric space.

(a) (24S1-1(a),23S-1(a))
State the open cover definition of the compactness of a metric space.

(b) (23S-1(b)=19S-3=11S-1(a)=05W-1(a))
Starting with the above definition, prove that every sequence in X has a convergent subsequence.

Solution:

(a) (X, d) is compact if and only if for any open covering of X, there exists a finite subcovering that also covers X.

(b) This is compactness implies Sequentially compactness.
We look at the set $E = \{p_n \in X \mid n \in \mathbb{N}\}$. Note that if E is finite then any convergent sequence will have infinite repeating terms of the converging term. For example, $\{p_1, \cdots, p_n, p, p, \cdots\}$ which converges to p and we can take $\{p, p, p, \cdots\}$ as the convergent subsequence.

Now suppose E is infinite. Suppose the contrary that that E doesn't have convergent subsequence. Then every $x \in X$ is not a limit point of E, so there exists r_x such that $N_{r_x}(x)$ contains at most finite points in E. Let $G = \{N_{r_x}(x) \mid x \in X\}$ be an open covering of X. Since X is compact, there exists $N_{r_1}(x_1), \cdots, N_{r_n}(x_n)$ such that $X \subseteq N_{r_1}(x_1) \cup \cdots \cup N_{r_n}(x_n)$. But every $N_{r_i})(x_i)$ only contains finite point of E hence $N_{r_1}(x_1) \cup \cdots \cup N_{r_n}(x_n)$ is also finite and therefore E is finite, which is a contradiction. Hence every sequence in X has a convergent subsequence.

□

2.1.14. *Antiderivatives.*

Problem 14: Antiderivatives

Suppose $f : [a, b] \to \mathbb{R}$ is Riemann Integrable and let

$$F(x) = \int_a^x f(t)\ dt, \qquad x \in [a, b].$$

(a) (24W-6(i)=03-3(a)=97S-3(a)=97W-3(c))
Prove that $F(x)$ is continuous on $[a, b]$.

(b) (24W-6(ii)=03-3(a)=22W-3=97S-3(b))
Prove that if f is continuous at a point p then F is differentiable on p.

(c) (24W-6(ii)=97S-3(b))
Compute $F'(p)$.

Solution:

(a) Note that f is Riemann integrable, then f is bounded, say $|f| \leq M$. Let $p, q \in [a, b]$ and $p > q$, then we have

$$\begin{aligned} |F(q) - F(p)| &= \left| \int_a^q f(t)\ dt - \int_a^q f(t)\ dt \right| = \left| \int_p^a f(t)\ dt + \int_a^q f(t)\ dt \right| \\ &= \left| \int_p^q f(t)\ dt \right| \leq M|p - q|. \end{aligned}$$

Hence for any $\epsilon > 0,\ p, q \in [a, b]$, choose $\delta = \dfrac{\epsilon}{M}$ then we have when $|p - q| < \dfrac{\epsilon}{M} = \delta$, we have

$$|F(q) - F(q)| \leq \varepsilon.$$

Hence $F(x)$ is (uniformly) continuous on $[a, b]$.

(b) Choose $\delta_1 > 0$ and $h > 0$ such that that $\{p + h\} \subset (a, b)$. Consider

$$F(p + h) - F(p) = \int_p^{p+h} f(t)\ dt.$$

Then we have

$$\int_p^{p+h} f(t)\ dt = \int_p^a f(t)\ dt + \int_a^{p+h} f(t)\ dt = \int_a^{p+h} f(t)\ dt - \int_a^p f(t)\ dt$$

which is continuous on $[p, p + h]$ by the previous problem. Note that f is Riemann integrable, then f is bounded, say $m_2 \leq f \leq M_2$. Also, by the Extreme Value Theorem, since $[p, p + h]$ is compact, we have $f(x_m) = m_2,\ f(x_M) = M_2$. Hence we have

$$\begin{aligned} mh \leq F(p + h) - F(p) &= \int_p^{p+h} f(t)\ dt \leq Mh \\ \implies m = f(x_m) \leq &\frac{F(p + h) - F(p)}{h} \leq f(x_M) = M. \end{aligned}$$

Hence by the Intermediate Value Theorem, there exists x^* such that

$$f(x^*) = \frac{F(p + h) - F(p)}{h}$$

where $|x^* - p| < h < \delta_1$.

Also, since f is continuous at p, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(p) - f(y)| < \epsilon$ when $|p - y| < \delta_2$. Hence if we choose

$$\delta = \min\{\delta_1, \delta_2\} \Longrightarrow |h| < \delta,$$

then we have

$$\left|\frac{F(p+h) - F(p)}{h} - f(p)\right| = |f(x^*) - f(x)| < \varepsilon.$$

Hence

$$\lim_{h\to 0} \frac{F(p+h) - F(p)}{h} = f(p).$$

So F is differentiable at p and $F'(p) = f(p)$.

(c) We've done this in (b).

$\square$

2.1.15. *Basic Topology.*

Problem 15: Basic Topology

Consider the following subset in $\mathbb{R}^2$:

(19W-1) $E_1 = \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\big\} \cup \big\{(x, 0) \in \mathbb{R}^2 : x \in [1, 2)\big\} \cup \{(0, 2)\}$,

(17W-5) $E_2 = \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\big\} \cup \big\{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1,\ x \neq 1\big\}$,

(15W-1) $E_3 = \big\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\big\} \cup \left\{\left(\frac{2n}{n+1}, 0\right) \in \mathbb{R}^2 \mid n = 1, 2, \cdots\right\}$,

(14W-2) $E_4 = \{(0, 1) \times (0, 1)\} \cup \big\{(x, 0) \in \mathbb{R}^2 : x \in (1, 2)\big\} \cup \{(3, 0)\}$.

For $i = 1, 2, 3, 4$,

(a) Determine the following sets:
$E_i^0,\ E_i^{'},\ \overline{E_i},\ \partial E_i$ and the set of all isolated points of E_i

(b) Determine whether the set E_i is open, closed, bounded, compact, or connected.

Solution:

(a) We will use $E^\circ = \{x \in E : \exists\, r > 0$ such that $N_r(x) \subseteq E\}$ and $E^{'} = \{x \in X : \forall\, r > 0,\ (N_r(x) \cap E) \setminus \{x\} \neq \emptyset\}$. Draw them out and everything will be clear. Note that the open balls are in $\mathbb{R}^2$ and be careful about the intersection of open balls and 1 dimensional intervals.

For set E_1:
$E_1^\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$E_1^{'} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, 0) : x \in [1, 2]\}$
$\overline{E_1} = E_1 \cup E_1^{'} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, 0) : x \in [1, 2]\} \cup \{(0, 2)\}$
$\partial E_1 = \overline{E_1} \setminus E_1^\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, 0) : x \in [1, 2]\} \cup \{(0, 2)\}$
Isolated points of E_1 : $E_1 \setminus E_1^{'} = \{(0, 2)\}$

For set E_2:
$E_2^\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$E_2^{'} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1,\ x \neq 1\}$

$\overline{E_2} = E_2 \cup E_2^{'} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1,\ x \ne 1\}$
$\partial E_2 = \overline{E_2} \setminus E_2^{\circ} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1,\ x \ne 1\}$
Isolated points of E_2 : $E_2 \setminus E_2^{'} = \emptyset$.

For set E_3:
$E_3^{\circ} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$E_3^{'} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \cup \{(2, 0)\}$ since $\lim\limits_{n\to\infty} \dfrac{2n}{n+1} = 2$.
$\overline{E_3} = E_3 \cup E_3^{'} = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} \cup \{(2, 0)\} \cup \left\{ \left(\dfrac{2n}{n+1}, 0\right) \in \mathbb{R}^2 \mid n = 1, 2, \cdots \right\}$
$\partial E_3 = \overline{E_3} \setminus E_3^{\circ} = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \cup \{(2, 0)\} \cup \left\{ \left(\dfrac{2n}{n+1}, 0\right) \in \mathbb{R}^2 \mid n = 1, 2, \cdots \right\}$
Isolated points of E_3 : $E_3 \setminus E_3^{'} = \left\{ \left(\dfrac{2n}{n+1}, 0\right) \in \mathbb{R}^2 \mid n = 1, 2, \cdots \right\}$

For set E_4:
Note that $\{(0, 1) \times (0, 1)\} = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1),\ y \in (0, 1)\}$.
$E_4^{\circ} = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1),\ y \in (0, 1)\}$
$E_4^{'} = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1],\ y \in [0, 1]\} \cup \{(x, 0) \in \mathbb{R}^2 : x \in [1, 2]\}$
$\overline{E_4} = E_4 \cup E_4^{'} = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1],\ y \in [0, 1]\} \cup \{(x, 0) \in \mathbb{R}^2 : x \in [1, 2]\} \cup \{(3, 0)\}$
$\partial E_4 = \overline{E_4} \setminus E_4^{\circ} = \{(x, 0) : x \in (0, 1)\} \cup \{(x, 1) : x \in (0, 1)\} \cup \{(0, y) : y \in (0, 1)\} \cup \{(1, y) : y \in (0, 1)\} \cup \{(x, 0) \in \mathbb{R}^2 : x \in [1, 2]\} \cup \{(3, 0)\}$
Isolated points of E_4 : $E_4 \setminus E_4^{'} = \{(3, 0)\}$.

(b) For openness, we will use that E is open if and only if $E = E^{\circ}$. For closedness, we will use that E is closed if and only if $E = \overline{E}$. (These are almost directly from definitions but you are welcome to prove them.) For compactness, we will use the Heine-Borel Theorem that E is compact if and only if E is closed and bounded. (This is not that straight forward, see here for the proof.) For connectedness, we will use: A subset E of a metric space (X, d) is called disconnected if there exists $A \subseteq X,\ B \subseteq X$ such that
(i) $E = A \cup B$,
(ii) $A \ne \emptyset,\ B \ne \emptyset$,
(iii) $A \cap \overline{B},\ \overline{A} \cap B = \emptyset$.
If E is not disconnected, then we say E is connected.

For set E_1: $E_1^{\circ} \ne E_1,\ \overline{E_1} \ne E_1$ E_1 is not open nor closed. And $E_1 \subseteq N_3(0) = \{x, y \in \mathbb{R}^2 : x^2 + y^2 < 3^2\}$ so E_1 is bounded. And by the Heine-Borel Theorem, E_1 is not compact. Now we choose $A_1 = \{(0, 2)\},\ B_1 = E_1 \setminus A_1$, then clearly $E_1 = A_1 \cup B_1$ and $\overline{A_1} = A_1,\ \overline{B_1} = \{x, y \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \cup \{(x, 0) :\in [1, 2)\}$. So $A_1 \cap \overline{B_1} = \overline{A_1} \cap B_1 = \emptyset$. Hence E_1 is not connected.

For set E_2: $E_2^{\circ} \ne E_2,\ \overline{E_2} \ne E_2$ E_2 is not open nor closed. And $E_2 \subseteq N_4(0) = \{x, y \in \mathbb{R}^2 : x^2 + y^2 < 4^2\}$ so E_1 is bounded. And by the Heine-Borel Theorem, E_2 is not compact. Now we choose $A_2 = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\},\ B_2 = E_2 \setminus A_2 = \left\{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1,\ x \ne 1\right\}$, then clearly $E_2 = A_2 \cup B_2$ and $\overline{A_2} =$

$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, $\overline{B_2} = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1\}$. So $A_1 \cap \overline{B_1} = \overline{A_1} \cap B_1 = \emptyset$. (See the point $(1,0)$ in each case.) Hence E_2 is not connected.

For set E_3: $E_3^\circ \ne E_3$, $\overline{E_3} \ne E_3$ E_3 is not open nor closed. And $E_3 \subseteq N_3(0) = \{x, y \in \mathbb{R}^2 : x^2 + y^2 < 3^2\}$ so E_3 is bounded. And by the Heine-Borel Theorem, E_3 is not compact. Now we choose $A_3 = \{x, y \in \mathbb{R}^2 : x^2 + y^2 < 1\} \cup \{(1,0)\}$, $B_3 = E_3 \setminus A_3 = \left\{\dfrac{2n}{n+1} : n \ge 2\right\}$, then clearly $E_1 = A_1 \cup B_1$ and $\overline{B_3} = B_3 \cup \{(2,0)\}$, $\overline{A_3} = \{x, y \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. So $A_3 \cap \overline{B_3} = \overline{A_3} \cap B_3 = \emptyset$. Hence E_3 is not connected.

For set E_4: $E_4^\circ \ne E_4$, $\overline{E_4} \ne E_4$ E_4 is not open nor closed. And $E_4 \subseteq N_4(0) = \{x, y \in \mathbb{R}^2 : x^2 + y^2 < 4^2\}$ so E_4 is bounded. And by the Heine-Borel Theorem, E_4 is not compact. Now we choose $A_4 = \{(3,0)\}$, $B_4 = E_4 \setminus A_4$, then clearly $E_4 = A_4 \cup B_4$ and $\overline{A_4} = A_4$, $\overline{B_4} = \{x, y \in \mathbb{R}^2 : x \in [0,1],\ y \in [0,1]\} \cup \{(x,0) :\in [1,2]\}$. So $A_4 \cap \overline{B_4} = \overline{A_4} \cap B_4 = \emptyset$. Hence E_4 is not connected.

$\square$

2.1.16. *Closed Graph Theorem.*

Problem 16: Closed Graph Theorem: 24W-1=21S-1=13S-3=06W-1

Consider two metric space (X, d_X) and (Y, d_Y). The Cartesian product $Z = X \times Y$ becomes a metric space if equipped with the metric

$$d_Z((x,y),(u,v)) = d_X(x,u) + d_Y(y,v).$$

(1) Prove that a sequence $\{(x_n, y_n)\}$ converges to (x,y) in Z if and only if $x_n \to x$ in X and $y_n \to y$ in Y.
(2) Consider the function $f : (X, d_X) \to (Y, d_Y)$. The graph of f is a subset of Z defined by
$$G = \{(x,y) \mid x \in X,\ y \in Y,\ y = f(x)\}.$$
Show that if f is continuous then G is a closed set in Z.
(3) Show that if Y is a compact metric space and G is a closed set in Z then f is continuous.
(4) Show by an example that the result in (3) does not hold true if Y is not compact.

Solution:

(1) If $x_n \to x$ in X and $y_n \to y$ in Y, then for a given $\varepsilon > 0$,

$$\exists\ N_1 \in \mathbb{N} \text{ such that } d_X(x_n, x) < \frac{\varepsilon}{2} \text{ when } n \ge N_1,$$
$$\exists\ N_2 \in \mathbb{N} \text{ such that } d_Y(y_n, y) < \frac{\varepsilon}{2} \text{ when } n \ge N_2.$$

Choose $N = \max\{N_1, N_2\}$ then when $n \ge N$. Then for $n \ge N$, we have

$$d_Z((x_n, y_n),(x,y)) = d_X(x_n, x) + d_Y(y_n, y) < \varepsilon \Longrightarrow (x_n, y_n) \to (x,y) \text{ in } Z.$$

On the other hand, if $(x_n, y_n) \to (x, y)$ in Z, for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that when $n \geq N$, we have

$$d_Z((x_n, y_n), (x, y)) = d_X(x_n, x) + d_Y(y_n, y) < \varepsilon$$
$$\implies \begin{cases} d_X(x_n, x) < \varepsilon \\ d_Y(y_n, y) < \varepsilon \end{cases} \implies x_n \to x \text{ in } X,\ y_n \to y \text{ in } Y$$

since $d_X,\ d_Y$ are non-negative.

(2) Let $(x_n, y_n) \in G$ be an arbitrary sequence in G such that $(x_n, y_n) \to (x, y) \in X \times Y$ for some $x \in X,\ y \in Y$. From the previous question, we have $x_n \to x$ in X and $y_n \to y$ in Y. Note that we have $y_n = f(x_n)$ and $x_n \to x$ in X. Since f is continuous, we have

$$y_n = f(x_n) \to f(x).$$

So we have $y_n \to y$ and $y_n \to f(x)$. Since the limit is unique (you probably don't need to prove that but it's easy anyway), we have $y = f(x)$ so $(x, y) = (x, f(x)) \in G$. Since the sequence is arbitrary chosen, G is closed.

(3) Suppose the contrary that f is not continuous, that is, there exists $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \to x$ but $f(x_n) \not\to f(x)$. Hence there exists $\varepsilon > 0$ such that for infinitely many n,

$$d_Y(f(x_n), f(x)) \geq \varepsilon,$$

thus we can find a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ for $\{x_n\}_{n\in\mathbb{N}}$ such that $d_Y(f(x_n), f(x)) \geq \varepsilon$. Since Y is compact, Y is sequentially compact (I'm pretty sure you don't need to prove this here. See the proof here.) Hence there exists a convergent subsequence $\{f(x_{n_{k_\ell}})\}_{\ell\in\mathbb{N}}$ of $\{f(x_{n_k})\}_{k\in\mathbb{N}}$ converging to some $y \in Y \setminus N_\varepsilon(f(x))$ (since every term in $f(x_{n_k})$ has distance to $f(x)$ bigger than ε). Then consider the sequence in G:

$$\{(x_{n_{k_\ell}}, f(x_{n_{k_\ell}}))\}_{\ell\in\mathbb{N}}$$

with $x_{n_{k_\ell}} \to x$ in X and $f(x_{n_{k_\ell}}) \to y$ in Y. By the first problem, we have

$$(x_{n_{k_\ell}}, f(x_{n_{k_\ell}})) \to (x, y) \text{ in } Z.$$

Since G is closed, we have $(x, y) \in G$ and $y = f(x)$. However, we have

$$d_Y(f(x_{n_k}), f(x)) = d_Y(f(x_{n_k}, y) \geq \varepsilon \implies f(x_{n_{k_\ell}}) \not\to y$$

which is a contradiction. Hence f is continuous.

(The proof seems complicated (since we need two layer of subsequence). But the idea is first choose the "bad points" and form a subsequence. Then use the sequenitally compact to have a "convergent" (second layer) subsequence for the bad point subsequence. And the closedness/convergence will fail since the it's from the bad points.)

(4) A counter-example is $f : [0, \infty) \to [0, \infty)$ defined as

$$f(x) = \begin{cases} \dfrac{1}{x} & , x > 0 \\ 0 & , x = 0 \end{cases}$$

Clearly $X = Y = [0, \infty)$ is not compact since it's not bounded. Also, $\lim_{x \to 0} f(x) \neq 0 = f(0)$, so f is not continuous. We have

$$G = \{(0,0)\} \cap \left\{ \left(x, \frac{1}{x} \right) \middle| x > 0 \right\}.$$

It's clear that $\{(0,0)\}$ is closed. For the second set, let $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n > 0$ and $x_n \to x \neq 0$. Then

$$f(x_n) = \frac{1}{x_n} \to \frac{1}{x} = f(x).$$

Here we used that $x_n \to x \neq 0$, then $\frac{1}{x_n} \to \frac{1}{x}$ when $n \to \infty$. Hence $(x_n, \frac{1}{x_n}) \to (x, \frac{1}{x}) = (x, f(x)) \in \left\{ \left(x, \frac{1}{x} \right) \middle| x > 0 \right\}$. So $\left\{ \left(x, \frac{1}{x} \right) \middle| x > 0 \right\}$ is also closed. Hence G is closed but f is not continuous.

$\square$

3. Remaining Problems Sorted by Subjects

3.1. **Set Theory.**

3.1.1. *19W-4,16S-1.*

Problem 17: 19W-4,16S-1

For any two nonempty subsets A and B of $\mathbb{R}$ define

$$A + B := \{a + b \ : \ a \in A, \ \text{and } b \in B\}.$$

(a) (19W-4(a))
If A and B are bounded below, prove that $A + B$ is bounded below as well, and $\inf(A + B) = \inf A + \inf B$.

(b) (16S-1)
Prove that $\sup(A + B) = \sup A + \sup B$.

(c) (19W-4(b))
If A is bounded above, prove that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ convergent to $\sup A$.

Solution:

(a) Note that since A and B are bounded below, so $\inf A$, $\inf B$ and $\inf(A + B)$ are finite. First note that $a \geq \inf A$, $b \geq \inf B$ $\forall$ $a \in A$, $b \in B \Longrightarrow a + b \geq \inf A + \inf B$. Hence $\inf(A + B) \geq \inf A + \inf B$.

On the other hand, let $\varepsilon > 0$ be given, by the definition of inf, we have

$$\begin{aligned} &\exists \ a \in A \text{ such that } a \leq \inf A + \frac{\varepsilon}{2}, \\ &\exists \ b \in B \text{ such that } b \leq \inf B + \frac{\varepsilon}{2}, \\ \Longrightarrow \ &\inf(A + B) \leq a + b \leq \inf A + \inf B + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\inf(A + B) \leq \inf A + \inf(B)$. Combining the result, we have

$$\inf(A + B) = \inf A + \inf B.$$

$\boxed{\text{Another Way for } \inf(A + B) \leq \inf A + \inf(B)}$ (If you're not comfortable in using $\leq$ in stead of $<$ in inf definition but it doesn't make any difference there.)
Let $a \in A$, $b \in B$, we have

$$\begin{aligned} a = (a + b) - b \geq \inf(A + B) - b &\Longrightarrow \inf(A + B) - b \leq \inf A, \\ \Longrightarrow b &\geq \inf(A + B) - \inf A \\ \Longrightarrow \ \inf B &\geq \inf(A + B) - \inf A \\ \Longrightarrow \ \inf A + \inf B &\geq \inf(A + B). \end{aligned}$$

where we use that $\inf(A + B) - b$ and $\inf(A + B) - \inf B$ are lower bounds for A and B as well.

(b) Note that if either A or B is not bounded above, then $\sup(A+B)=\infty$ and one of the $\sup A,\ \sup B$ is infinite and we have nothing to prove. Now we consider A, B are bounded above, so $\sup A,\ \sup B$ and $\sup(A+B)$ are finite. First note that $a \le \sup A,\ b \le \sup B\ \forall\ a\in A,\ b\in B \Longrightarrow a+b \le \sup A + \sup B$. Hence $\sup(A+B) \le \sup A + \sup B$.

On the other hand, let $\varepsilon > 0$ be given, by the definition of sup, we have

$$\begin{aligned} &\exists\ a\in A \text{ such that } a \ge \sup A - \frac{\varepsilon}{2},\\ &\exists\ b\in B \text{ such that } b \ge \sup B - \frac{\varepsilon}{2},\\ \Longrightarrow\ &\sup(A+B) \ge a+b \ge \sup A + \sup B - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\sup(A+B) \ge \sup A + \sup(B)$. Combining the result, we have

$$\sup(A+B) = \sup A + \sup B.$$

$\boxed{\text{Another Way for } \sup(A+B) \ge \sup A + \sup(B)}$

Let $a\in A,\ b\in B$, we have

$$\begin{aligned} &a = (a+b)-b \le \sup(A+B) - b \Longrightarrow \sup A \le \sup(A+B) - b,\\ \Longrightarrow\ &b \le \sup(A+B) - \sup A\\ \Longrightarrow\ &\sup B \le \sup(A+B) - \sup A\\ \Longrightarrow\ &\sup A + \sup B \le \sup(A+B). \end{aligned}$$

where we use that $\sup(A+B)-b$ and $\sup(A+B)-\sup B$ are upper bounds for A and B as well.

(c) By the definition of sup, for $n\in\mathbb{N}$, there exists

$$a_n \in A \text{ such that } a_n > \sup A - \frac{1}{n} \Longrightarrow |a_n - \sup A| < \frac{1}{n}.$$

Hence for any $\varepsilon > 0$, choose $N\in\mathbb{N}$ such that $N > \frac{1}{\varepsilon}$, then when $n \ge N$, we have

$$|a_n - \sup A| < \frac{1}{n} \le \frac{1}{N} < \varepsilon.$$

Therefore $a_n \to \sup A$.

$\square$

3.1.2. *18S-1,04S-1(a).*

Problem 18: 18S-1,04S-1(a)

(a) (18S-1)
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for all n. Show that

$$\bigcap_{n=1}^{\infty}[a_n, b_n] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right].$$

(b) (04S-1(a))
Suppose $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$ for all n. Show that $\bigcap_{n=1}^{\infty} I_n$ not empty.

Solution: There are related to the nested interval theorem and it is one of the theorem which characterize the completeness of real numbers. Here we use: If $\{a_n\}_{n\in\mathbb{N}}$ is monotone and $|a_n| \leq M,\ \forall\ n \in \mathbb{N}$ then $\{a_n\}_{n\in\mathbb{N}}$ converges.

(a) Since $a_n \leq a_{n+1} < b_{n+1} \leq b_n$, we have $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. Also, we have $a_n \leq b_1,\ b_n \geq a_1$. Hence $\lim\limits_{n\to\infty} a_n,\ \lim\limits_{n\to\infty} b_n$ exists, say $\lim\limits_{n\to\infty} a_n = a,\ \lim\limits_{n\to\infty} b_n = b$.
Suppose that $a > b$, then there exists $N_1 \in \mathbb{N}$ such that when $n \geq N_1$,

$$|a_n - a| < \frac{a-b}{2} \Longrightarrow \frac{a+b}{2} < a_n < \frac{3a-b}{2}.$$

Also, there exists $N_2 \in \mathbb{N}$ such that when $n \geq N_2$,

$$|b_n - b| < \frac{a-b}{2} \Longrightarrow \frac{3b-a}{2} < b_n < \frac{a+b}{2}.$$

Hence when $n \geq \max\{N_1, N_2\}$, we have

$$b_n < \frac{a+b}{2} < a_n \Longrightarrow b_n < a_n$$

which is a contradiction. Hence we have $a \leq b \Longrightarrow a_n \leq a \leq b \leq b_n\ \forall\ n \in \mathbb{N}$. Hence we have

$$[a, b] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right] \subseteq \bigcap_{n=1}^{\infty}[a_n, b_n].$$

Now let $c \in \bigcap\limits_{n=1}^{\infty}[a_n, b_n]$. If $c < a$, then there exists $N_3 \in \mathbb{N}$ such that when $n \geq N_3$,

$$|a_n - a| < a - c \Longrightarrow c < a_n < 2a - c \Longrightarrow c < a_n.$$

Hence we have when $n \geq N_3$, $c \notin [a_n, b_n] \Longrightarrow c \notin \bigcap\limits_{n=1}^{\infty}[a_n, b_n]$. Again, if $c > b$, then there exists $N_4 \in \mathbb{N}$ such that when $n \geq N_4$,

$$|b_n - b| < c - b \Longrightarrow 2b - c < b_n < c \Longrightarrow b_n < c.$$

Hence we have when $n \geq N_4$, $c \notin [a_n, b_n] \Longrightarrow c \notin \bigcap_{n=1}^{\infty}$. Therefore, we have

$$a_n \leq a \leq c \leq b \leq b_n \Longrightarrow c \in [a, b] \Longrightarrow \bigcap_{n=1}^{\infty} [a_n, b_n] \subseteq [a, b] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right].$$

Hence combining the results, we finally have

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right].$$

(b) This is just (a).

$\square$

3.2. **Topology.**

3.2.1. *24S2-7.*

Problem 19: 24S2-7

Let (X, d) be a metric space.

(a) Let $S \subset X$. Define "S is closed."

(b) Let $S_1, S_2, \cdots, S_n$ be closed subsets of X. Prove that $\bigcup_{j=1}^{n} S_j$ is closed.

(c) Find a metric space (X, d) and sets $S_1, S_2, S_3, \cdots$ such that $\bigcup_{j=1}^{\infty} S_j$ fails to be closed.

Solution:

(a)

(b)

(c)

$\square$

3.2.2. *24S2-5(i)(ii),23W-1(a),16S-2.*

Problem 20: 24S2-5(i)(ii),23W-1(a),16S-2

Let (X, d) be a metric space and let Y be a compact subset of X.

(a) (24S2-5(ii), 23W-1(a)=16S-2(a))
Prove that Y is closed and bounded.

(b) (24S2-5(i),16S-2(b))
Give an example to show that the converse of (a) does not hold, i.e., give an example of a metric space X and a closed and bounded subset Y of X such that Y is not compact.

Solution:

(a) First, we prove compactness implies closedness.
Let $p \in X \setminus Y$ be arbitrary chosen and $\varepsilon_x = \dfrac{d(x,p)}{2} > 0$. Consider $\{N_{\varepsilon_x}(x)\}_{x\in X}$, then it's a open covering for X. And since Y is compact, there exists $p_1, \cdots, p_n \in Y$ such that $Y \subseteq N_{\varepsilon_{p_1}}(p_1) \cup \cdots \cup N_{\varepsilon_{p_n}}(p_n)$ for some $n \in \mathbb{N}$. Let $\varepsilon = \min\{d(p,p_1), \cdots, d(p,p_n)\} > 0$.

Claim 3. $N_{\frac{\varepsilon}{2}}(p) \subseteq X \setminus Y$

Proof of claim. Suppose that $\exists\, z \in N_{\frac{\varepsilon}{2}}(p) \cap X$. So there exists $i \in \{1, 2, \cdots, n\}$ such that $z \in N_{\varepsilon_{p_i}}(p_i)$. Then

$$d(p,p_i) \leq d(p,z) + d(z,p_i) < \frac{\varepsilon}{2} + \varepsilon_{p_i} \leq \frac{d(p,p_i)}{2} + \frac{d(p_i,p)}{2} = d(p,p_i)$$

which is a contradiction.

By the claim, we have $X \setminus Y$ is open hence Y is closed.

Next, we prove compactness implies boundedness.

Let $p \in Y$ be arbitrary chosed, then $\{N_n(p)\}_{n\in\mathbb{N}}$ is an open cover for Y. And since Y is compact, there exists $n_1, n_2, \cdots, n_m \in \mathbb{N}$ such that $Y \subseteq N_{n_1}(p) \cup \cdots \cup N_{n_m}(p)$. Now, we choose $r = \max\{n_1, \cdots, n_m\}$ then $Y \subseteq N_r(p)$ which proved that Y is bounded.

(b) A counter-example would be any infinite set with the discrete metric. See the proof in Discrete Topology.

$\square$

3.2.3. *22W-1,05W-1(b).*

Problem 21: 22W-1,05W-1(b): Sequentially Compact and Totally Bounded

(a) (22W-1)
Suppose (X, d) is a complete metric space and, for every $\varepsilon > 0$, there is a positive integer k and $x_1, \cdots, x_k$ in X such that $X = \bigcup_{i=1}^{k} B_\varepsilon(x_i)$. Prove that every sequence in X has a convergent subsequence. Here $B_r(a)$ denotes the a centered open ball of radius r.

(b) (05W-1(b))
Suppose every sequence in X has a convergent subsequence (but X is not given to be compact). Show that, for every $r > 0$, X is contained in the union of a finite number of balls of radius r.

Solution:

(a) This is complete and totally bounded implies Sequentially compact.

Let $\{x_n\}_{n\in\mathbb{N}}$ be sequence. Let $\varepsilon_1 = 1$, then $X \subset B_1(p_1) \cup \cdots \cup B_1(p_n),\ p_i \in X,\ i = 1, \cdots, n$. Then $\exists\ N_1 \in \{1, \cdots, n\}$ such that $B_1(p_{N_1})$ contains infinite many terms of $\{x_n\}_{n\in\mathbb{N}}$. Let $\{x_n^{(1)}\}_{n\in\mathbb{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ in $B_1(p_{N_1})$.

Now repeating the process for $\varepsilon_k = \frac{1}{k},\ k \in \mathbb{N}$. We will have subsequences $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $d(x_n^{(k)}, x^{(k)m}) < \frac{1}{k} \to 0$. Now we consider the sequence $\{x_1^{(1)}, x_2^{(2)}, \cdots, x_n^{(n)}, \cdots\} = \{x_n^{(n)}\}_{n\in\mathbb{N}}$. Then we have $\{x_n^{(n)}\}_{n\in\mathbb{N}}$ is clearly a Cauchy sequence. Finally, since X is complete, $\{x_n^{(n)}\}_{n\in\mathbb{N}}$ is convergent in X hence it's a convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Hence X is sequentially compact, that is, every sequence in X has a convergent subsequence.

(b) This is sequentially compact implies totally bounded.

If X is sequentially compact, take a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$. Now suppose that X is not totally bounded. There $\exists\ \varepsilon > 0$ such that X cannot be covered by finite number of balls of radius ε. Now take $x_1 \in X \Longrightarrow X \setminus B_\varepsilon(x_1) \neq \emptyset$, and

$$x_2 \in X \setminus B_\varepsilon(x_1) \Longrightarrow d(x_1, x_2) \geq \varepsilon$$
$$\vdots$$
$$x_n \in X \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_{n-1}))$$

Then we have $d(x_n, x_m) \geq \varepsilon$, hence there's no Cauchy subsequence and thus no convergent subsequence, contradicting to the sequentially compactness.

$\square$

3.2.4. *20S-1.*

Problem 22: 20S-1

Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $X = X_1 \times X_2$ and $d : X \times X \to [0, +\infty)$ be given by

$$d((x_1, x_2), (x_1^{'}, x_2^{'})) = \sqrt{d_1(x_1, x_1^{'})} + \sqrt{d_1(x_2, x_2^{'})}.$$

(a) Prove that d is a metric on X.

(b) Show that a sequence $\{(x_1^{(n)}, x_2^{(n)})\}_{n=1}^{\infty} \subseteq X$ converges to the point (x_1, x_2) in the metric space (X, d) if and only if $\{x_1^{(n)}\}_{n=1}^{\infty}$ converges to x_1 in the metric space (X_1, d_1) and $\{x_2^{(n)}\}_{n=1}^{\infty}$ converges to x_2 in the metric space (X_2, d_2).

(c) Suppose that (X_1, d_1) and (X_2, d_2) are compact. Show that (X, d) is compact.

Solution:

(a) Let $x = (x_1, x_2),\ y = (y_1, y_2)$ and $z = (z_1, z_2)$. We check that

- $d(x, y) > 0$ if $x \neq y$, $d(x, y) = 0$ if $x = y$: If $x = y \implies x_1 = x_2,\ y_1 = y_2 \implies d_1(x_1, y_1) = d_2(y_1, y_2) = 0$ since d_1 and d_2 are metric. Hence $d(x, y) = 0$. Similarly, if $x \neq y$, then $d_1(x_1, y_1) > 0,\ d_2(y_1, y_2) > 0 \implies d(x, y) > 0$.
- $d(x, y) = d(y, x)$: Clear by definition and d_1 and d_2 are metric.
- $d(x, z) \leq d(x, y) + d(y, z)$: Note that if $a, b \geq 0$, then

$$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} \geq a + b = (\sqrt{a+b})^2 \implies \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

Hence we have

$$\begin{aligned} d(x, z) &= \sqrt{d_1(x_1, z_1)} && + \sqrt{d_2(x_2, z_2)} \\ &\leq \sqrt{d_1(x_1, y_1) + d_1(y_1, z_1)} && + \sqrt{d_2(x_2, y_2) + d_2(y_2, z_2)} \\ &\leq \left(\sqrt{d_1(x_1, y_1)} + \sqrt{d_1(y_1, z_1)}\right) && + \left(\sqrt{d_2(x_2, y_2)} + \sqrt{d_2(y_2, z_2)}\right) \\ &= \left(\sqrt{d_1(x_1, y_1)} + \sqrt{d_2(x_2, y_2)}\right) && + \left(\sqrt{d_1(y_1, z_1)} + \sqrt{d_2(y_2, z_2)}\right) \\ &= d(x, y) && + d(y, z) \end{aligned}$$

(b) If $x_n^{(1)} \to x_1$ in (X_1, d_1) and $x_n^{(2)} \to x_2$ in (X_2, d_2), then for a given $\varepsilon > 0$,

$$\exists\ N_1 \in \mathbb{N} \text{ such that } d_1\left(x_1^{(n)}, x_1\right) < \frac{\varepsilon^2}{4} \text{ when } n \geq N_1,$$
$$\exists\ N_2 \in \mathbb{N} \text{ such that } d_2\left(x_2^{(n)}, x_2\right) < \frac{\varepsilon^2}{4} \text{ when } n \geq N_2.$$

Choose $N = \max\{N_1, N_2\}$ then when $n \geq N$. Then for $n \geq N$, we have

$$d\left(\left(x_1^{(n)}, x_2^{(n)}\right), (x_1, x_2)\right) = \sqrt{d_1\left(x_1^{(n)}, x_1\right)} + \sqrt{d_2\left(x_2^{(n)}, x_2\right)} < \sqrt{\frac{\varepsilon^2}{4}} + \sqrt{\frac{\varepsilon^2}{4}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\left\{\left(x_1^{(n)}, x_2^{(n)}\right)\right\}_{n=1}^{\infty}$ converges (x_1, x_2) in (X, d).

On the other hand, if $\left(x_1^{(n)}, x_2^{(n)}\right) \to (x_1, x_2)$ in (X, d), for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that when $n \geq N$,

$$d\left(\left(x_1^{(n)}, x_2^{(n)}\right), (x_1, x_2)\right) = \sqrt{d_1\left(x_1^{(n)}, x_1\right)} + \sqrt{d_2\left(x_2^{(n)}, x_1\right)} < \sqrt{\varepsilon}$$

$$\Longrightarrow \begin{cases} \sqrt{d_1\left(x_1^{(n)}, x_1\right)} < \sqrt{\varepsilon} \\ \sqrt{d_2\left(x_2^{(n)}, x_2\right)} < \sqrt{\varepsilon} \end{cases} \Longrightarrow \begin{cases} d_1\left(x_1^{(n)}, x_1\right) < \varepsilon \\ d_2\left(x_2^{(n)}, x_2\right) < \varepsilon \end{cases}$$

$$\Longrightarrow x_1^{(n)} \to x_1 \text{ in } (X_1, d_1),\ x_2^{(n)} \to x_2 \text{ in } (X_2, d_2)$$

since $d_1,\ d_2$ are non-negative.

(c) We will prove the sequentially compact. Since (X_1, d_1) is compact, it's sequentially compact thus for any sequence $\left\{x_1^{(n)}\right\}_{n\in\mathbb{N}}$ in X_1 has a convergent subsequence $\left\{x_1^{(n_k)}\right\}_{k\in\mathbb{N}}$, say $x_1^{(n_k)} \to x_1$ when $k \to \infty$. Similarly, for any sequence $\left\{x_2^{(n)}\right\}$ in X_2 has a convergent subsequence $\left\{x_2^{(n_k)}\right\}_{k\in\mathbb{N}}$, say $x_2^{(n_k)} \to x_2$ when $k \to \infty$. Note that $\left\{\left(x_1^{(n_k)}, x_2^{(n_k)}\right)\right\}_{k\in\mathbb{N}}$ is a subsequence of $\left\{\left(x_1^{(n)}, x_2^{(n)}\right)\right\}_{n\in\mathbb{N}}$. By (b), we have

$$\left\{\left(x_1^{(n_k)}, x_2^{(n_k)}\right)\right\}_{k\in\mathbb{N}} \to (x_1, x_2) \in X.$$

Since the choice of $\left\{x_1^{(n)}\right\}_{n\in\mathbb{N}}$ and $\left\{x_2^{(n)}\right\}_{n\in\mathbb{N}}$ is arbitrary. Hence (X, d) is sequentially compact and thus (X, d) is compact.

□

3.2.5. *18S-3.*

Problem 23: 18S-3

Let (X, d) be a complete metric space and let

$$d_c(x, y) := \min\{1, d(x, y)\} \qquad x, y \in X.$$

Show that (X, d_c) is also a complete metric space.

Solution: Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d_c). For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $d_c(x_n, x_m) < \varepsilon$ when $n, m \geq N_1$. Note that if $\varepsilon > 1$, we always have $d_c(x_n, y) \leq 1 < \varepsilon$ for any $y \in X$ so we have nothing to say. Now we consider $0 < \varepsilon \leq 1$ be given, we have

$$d_c(x_n, x_m) = \min\{1, d(x_n, x_m)\} < \varepsilon \Longleftrightarrow d_c(x_n, x_m) = d(x_n, x_m).$$

Note $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (X, d_c) and (X, d) is complete. Hence $\exists\ N_2 \in \mathbb{N}$ such that

$$d_c(x_n, x_m) = d(x_n, x_m) < \varepsilon,\ \forall\ n, m \geq N \Longrightarrow \exists\ x \in X \text{ such that } \lim_{n\to\infty} x_n = x,$$

that is, $\exists\ N_3 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ when $n \geq N_3$. Now for any $1 > \varepsilon > 0$, when $n \geq N_3$, we also have

$$d_c(x_n, x) = d(x_n, x) < \varepsilon \Longrightarrow \lim_{n\to\infty} x_n = x.$$

Thus (X, d_c) is complete. □

3.2.6. *18W-2=06S-1(b)=00W-1(b): Lebesgue Number.*

Problem 24: 18W-2=06S-1(b)=00W-1(b)

Suppose (X, d) is a metric space in which every sequence has a convergent subsequence. Given an open cover $\{U_i\}_{i\in I}$ of X, prove there is a $\rho > 0$ such that every open ball of radius ρ is contained in one of the U_i.

Solution: This is called the Lebesgue number, which gives one way to prove that sequentially compactness implies compactness.

□

3.2.7. *14W-6.*

Problem 25: 14W-6

Suppose that $S \subset \mathbb{R}^2$. A point $x \in \mathbb{R}^2$ is said to be a *condensation point* of S if every neighborhood $N(x)$ has the property that $N(x) \cap S$ is not countable. Show that if S is not countable then there must exist a point $z \in S$ such that z is a condensation point of S.

Solution:

□

3.2.8. *13S-1(a)(b).*

Problem 26: 13S-1(a)(b)

(a) Define a complete metric space and prove that every *compact* metric space K is complete.
(b) Let $C[0,1]$ denote the metric space of continuous real-valued functions defined in $[0,1]$ endowed with the distance

$$d(f,g) = \sup_{t\in[0,1]} (|f(t) - g(t)|), \qquad f, g \in C[0,1].$$

Prove that $C[0,1]$ is complete.

Solution:

(a)
(b)

□

3.2.9. *06S-1(a)=96S-1(a).*

Problem 27: 06S-1(a)=96S-1(a)

Let E be a subset of X and let E' denote the set of limit points of E. Prove that E' is closed.

Solution:

□

3.2.10. *06S-5(a)=00W-5(b),14W-1: Contraction Mappings.*

Problem 28: 06S-5(a)=00W-5(b),14W-1: Contraction Mappings

Suppose (E, d) is a complete metric space and $f : E \to E$ is a contraction mapping i.e. there is a real number k, $0 \le k < 1$, such that $d(f(x), f(y)) \le kd(x, y)$ for all x, y in E.

(a) (14W-1(a))
Prove that f is uniformly continuous on E.

(b) (06S-5(a)=00W-5(b)=14W-1(b))
Prove that f has a unique fixed point i.e. a unique point $p \in E$ such that $f(p) = p$.

Solution:

(a) Fix $\epsilon > 0$ and let $\delta = \epsilon/k$. Then for every $x, y \in E$ satisfying $d(x, y) < \delta$, we have

$$d(f(x), f(y)) \le k \cdot d(x, y) < k \cdot \epsilon/k = \epsilon.$$

(b) First we show existence. Take $x_0 \in E$ and consider the sequence defined by $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$. Note that $d(x_2, x_1) = d(f(x_1), f(x_0)) \le kd(x_1, x_0)$ and also

$$d(x_3, x_2) = d(f(x_2), f(x_1)) \le kd(x_2, x_1) \le k^2 d(x_1, x_0)$$

and by induction we have

$$d(x_n, x_{n-1}) \le k^{n-1} d(x_1, x_0)$$

for $n \ge 2$. We use this inequality to show that $\{x_n\}$ is Cauchy. Given $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that

$$k^N < \frac{\epsilon(1-k)}{d(x_1, x_0)}$$

which implies

$$\frac{k^n d(x_1, x_0)}{1-k} < \epsilon$$

when $n \ge N$. Now when $m \ge n \ge N$ we have

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\le k^{m-1} d(x_1, x_0) + k^{m-2} d(x_1, x_0) + \cdots + k^n d(x_1, x_0) \\ &= k^n d(x_1, x_0) \left(\frac{1 - k^{m-n}}{1-k} \right) \le k^n d(x_1, x_0) \left(\frac{1}{1-k} \right) = \frac{k^n d(x_1, x_0)}{1-k} < \epsilon. \end{aligned}$$

Therefore $\{x_n\}$ is Cauchy in a complete metric space, so it must converge to a limit x. Furthermore, since f is continuous, x must satisfy

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x)$$

and therefore x is a fixed point of f.

For uniqueness, if there exist two distinct fixed points x and y then $d(x, y) > 0$. Therefore

$$d(x, y) = d(f(x), f(y)) \le k \cdot d(x, y)$$

and since $d(x, y) > 0$ this implies $k \ge 1$ which is a contradiction.

□

3.2.11. *03W-1(a).*

Problem 29: 03W-1(a)

Prove that any interval is a connected subset of $\mathbb{R}$.

Solution: Recall the definition of (dis)connectedness: A subset E of a metric space (X, d) is called disconnected if there exists $A \subseteq X,\ B \subseteq X$ such that

(i) $E = A \cup B$,
(ii) $A \neq \emptyset,\ B \neq \emptyset$,
(iii) $A \cap \overline{B},\ \overline{A} \cap B = \emptyset$.

If E is not disconnected, then we say E is connected.

Suppose the contrary that the interval is disconnected, then there exists $A, B \neq \emptyset$ such that $A \cup B = (x, y)$ and $A \cap \overline{B},\ \overline{A} \cap B = \emptyset$. Let $a \in A$ and $b \in B$ and $a < b$. Let $S = [a, b]$ and

$$S_1 = A \cap S,\ S_2 = B \cap S.$$

Clearly $S_1,\ S_2$ are non empty and

$$S = S_1 \cup S_2.$$

Note that

$$\overline{S_2} = \overline{B \cap S} \subseteq \overline{B} \cap \overline{S} = \overline{B} \cap S.$$

Hence

$$S_1 \cap \overline{S_2} = (A \cap S) \cap \overline{B \cap S} \subseteq (A \cap S) \cap (\overline{B} \cap S) = A \cap \overline{B} \cap S = \emptyset \Longrightarrow S_1 \cap \overline{S_2} = \emptyset.$$

And similarly, $\overline{S_1} \cap S_2 = \emptyset$. So S is disconnected.

Now we let $s = \sup S_1 \Longrightarrow s \in \overline{S_1},\ s \notin S_2$ since $\overline{S_1} \cap S_2 = \emptyset$. Note that $s \in S_1$ since if $S \notin S_1$ then $s \notin S_1,\ s \notin S_2$ but $s \in S = S_1 \cup S_2$ which is a contradiction. Hence we have $s \in S_1$ and thus $s \notin \overline{S_2}$ since $S_1 \cap \overline{S_2} = \emptyset$. So s is not a limit point of S_2, that is, there exists $r > 0$ such that

$$(B_r(s) \setminus \{s\}) \cap S_2 = \emptyset \Longrightarrow B_r(c) \subseteq S_2^C.$$

Hence we have there exists $c \notin S_2$ such that (here we use the usual Euclidean metric)

$$s < c < b.$$

However, since $c > s = \sup S_1$, $c \notin S_1$ as well which is a contradiction since $c \in S = S_1 \cap S_2$.

Note that we don't need to assume open or closed interval here, we only use the fact (in fact you can prove this): A set E is an interval if $x < z < y$ and $x \in E,\ y \in E$ then $z \in E$. And we choose our a, b using this.

□

3.2.12. *00W-1(a).*

Problem 30: 00W-1(a)

Suppose (X, d) is a compact metric space. Prove that every infinite subset of X has a limit point.

Solution:

$\square$

3.3. Sequence and Series.

3.3.1. *20S-4,17W-6,96S-3(a).*

Problem 31: 20S-4,17W-6,96S-3(a)

In (a), (b) and (c) below, we assume that $x_n > 0,\ n \in \mathbb{N}$.

(a) (20S-4(a)=17W-6(a)=96S-3(a))

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Prove that $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+2}}$ is also convergent.

(b) (20S-4(b),17W-6(b))

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Prove that $\sum_{n=1}^{\infty} (-1)^n x_n^{\alpha}$ is absolutely convergent for any $\alpha \geq 1$.

(c) (20S-4(c))

Suppose that the series $\sum_{n=1}^{\infty} x_n$ diverges. Prove that the series $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ diverges

Reference: Baby Rudin, Chapter 3, Exercise 11.

Solution:

(a) Since $x_n > 0\ \forall\ n \in \mathbb{N}$, we have

$$\sqrt{x_n x_{n+1}} \leq \frac{x_n + x_{n+1}}{2}.$$

Hence for every $m, k \in \mathbb{N},\ k > m$,

$$\sum_{n=m}^{k} \sqrt{x_n x_{n+2}} \leq \sum_{n=m}^{k} \frac{x_n + x_{n+1}}{2} = \frac{1}{2}\left(\sum_{n=m}^{k} x_n + \sum_{n=m}^{k} x_{n+1}\right) < \frac{1}{2}\left(\sum_{n=m}^{k} x_n + \sum_{n=m}^{k} x_n\right) = \sum_{n=m}^{k} x_n.$$

Since $\sum_{n=1}^{\infty} x_n$ converges, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{n=m}^{k} x_n\right| < \varepsilon.$$

Hence we have

$$\left|\sum_{n=m}^{k} \sqrt{x_n x_{n+2}}\right| < \left|\sum_{n=m}^{k} x_n\right| < \varepsilon.$$

So by the Cauchy criterion, $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+2}}$ is also convergent.

(b) It suffices to check if $\sum_{n=1}^{\infty} |(-1)^n x_n^\alpha| = \sum_{n=1} x_n^\alpha$ (since $x_n > 0$) is convergent or not. Note that since $\sum_{n=1}^{\infty} x_n$ converges, there exists $N \in \mathbb{N}$ such that

$$|x_n| = x_n < 1 \text{ when } n \geq N.$$

Hence we also have

$$x_n^\alpha < x_n < 1$$

since $\alpha \geq 1$ (you might prove this if you have a lot of time). Then by the Comparison Test, we have

$$\sum_{n=N}^{\infty} x_n^\alpha \text{ converges } \Longrightarrow \sum_{n=1}^{\infty} x_n^\alpha \text{ converges}$$

since $\sum_{n=1}^{N} x_n^\alpha$ is finite.

(c) $\boxed{\text{Way I: Direct Proof 1}}$ (This is a little bit overkill.)
Note that if $|x_n| \leq M$ for all $n \in \mathbb{N}$, then

$$\frac{x_n}{1+x_n} \geq \frac{x_n}{1+M}.$$

Since $\sum_{n=1}^{\infty} x_n$ diverges, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ diverges. On the other hand, if $\{x_n\}_{n\in\mathbb{N}}$ is not bounded above. We have the following claim:

Claim 4. There exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \to \infty$.

Proof of claim. We can choose $n_k \in \mathbb{N}$ such that $x_{n_k} > k$ since if $\{x_n\}$ is not bounded above, there exists $n_k \in \mathbb{N}$ such that $x_{n_k} > \max\{x_1, \cdots, x_k, k\} \geq k$ and since $x_{n_k} > x_i$, $i = 1, \cdots, k$, we also have $n_k > k$. Then we can choose $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > k+1$ and $n_{k+1} > n_k$ since again, $\{x_n\}$ is not bounded above, there exists $n_{k+1} \in \mathbb{N}$ such that $x_{n_k} > \max\{x_1, \cdots, x_{n_k}, k+1\} \geq k+1$ and $x_{n_{k+1}} > x_i$, $i = 1, \cdots, n_k$, we have $n_{k+1} > n_k$. Finally, by this construction, we have $x_{n_k} \to \infty$.

By the above Claim, we have

$$\lim_{k\to\infty} \frac{x_{n_k}}{1+x_{n_k}} = \lim_{k\to\infty} \frac{1}{\frac{1}{x_{n_k}}+1} = 1 \neq 0.$$

And this subsequence $\sum_{k=1}^{\infty} \frac{x_{n_k}}{1+x_{n_k}}$ diverges. Finally, we prove the following claim: (I think you might skip that but for safety I prove it here.)

Claim 5. A sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to a if and only if every subsequence converges to a.

Proof of claim. Note that a sequence is a subseqeunce itself, so the only if part is trivial. On the other hand, if $a_n \to a$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ when $n \geq N$. Note that there exists $K \in \mathbb{N}$ such that $n_k \geq N$ when $k \geq K$ since n_i, $i \in \mathbb{N}$ is strictly increasing indices. Hence we also have $d(a_{n_k}, a) < \varepsilon$ when $k \geq K$.

By the claim above, if a sequence has a divergent subsequence, then then sequence is divergent. Finally, combining the results, we have $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ diverges.

$\boxed{\text{Way II: Proof by Contradiction}}$

Suppose the contrary that $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges, then

$$\lim_{n\to\infty} \frac{x_n}{1+x_n} = \lim_{n\to\infty} \left(1 - \frac{1}{1+x_n}\right) = 0 \Longrightarrow \lim_{n\to\infty} x_n = 0.$$

Hence there exists $N \in \mathbb{N}$ such that

$$|x_n| = x_n \leq 1 \Longrightarrow \frac{x_n}{2} = \frac{x_n}{1+1} \leq \frac{x_n}{1+x_n} \text{ when } n \geq N.$$

By the Comparison Test, we have $\displaystyle\sum_{n=1}^{\infty} x_n$ also converges, which is a contradiction. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ diverges.

$\square$

3.3.2. *20W-3.*

Problem 32: 20W-3

Determine for which values of the complex number z the given series is convergent. State any convergence test that you use.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} z^n$.

Reference: Baby Rudin, Chapter 3, Exercise 9.

Solution:

(i) We use the ratio test:

Theorem 6 (Ratio Test)**.** *Let* $\displaystyle\sum_{n\in\mathbb{N}} a_n$ *be a complex series, then let* $R = \limsup_n \left|\frac{a_{n+1}}{a_n}\right|$ *and* $r = \liminf_n \left|\frac{a_{n+1}}{a_n}\right|$. *Then*

(a) *If* $R < 1$, *the series is absolutely convergent.*

(b) *If* $r > 1$, *the series diverges.*

(c) *If* $r \leq 1 \leq R$, *the test is inconclusive.*

Let $a_n = \frac{2^n}{n!}z^n$. We compute

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n}|z| = \lim_{n\to\infty}\frac{2}{n+1}|z| = 0.$$

Thus for all $z \in \mathbb{C}$,

$$\limsup_n\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

Hence the radius of convergence is $R = \infty$. Hence the series converges for all $z \in \mathbb{C}$.

(ii) We use the root test:

Theorem 7 (Root Test)**.** *Let* $\sum_{n\in\mathbb{N}} a_n$ *be a complex series, then let* $R = \limsup_n |a_n|^{\frac{1}{n}}$. *Then*

(a) *If* $R < 1$, *the series is absolutely convergent.*
(b) *If* $R > 1$, *the series diverges.*
(c) *If* $R = 1$, *the test is inconclusive.*

Let $a_n = \sqrt{n}z^n$. We compute

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}|\sqrt{n}z^n|^{\frac{1}{n}} = \lim_{n\to\infty}\sqrt{n}^{\frac{1}{n}}|z| = 1.$$

So $R < 1$ if and only if $|z| < 1$. For $|z| = 1$, note that

$$|\sqrt{n}z^n| = \sqrt{n} \to \infty \text{ when } n \to \infty.$$

Hence the series doesn't converge. Combining the result, the series converge if $|z| < 1$, $z \in \mathbb{C}$.

Note that in both question, we calculate the limit first, since if the limit exists, limsup equals to the value of limit. □

3.3.3. *18W-5(b).*

Problem 33: 18W-5(b)

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent positive term series and z_n is a sequence of complex numbers such that

$$|z_{n+1} - z_n| \leq a_n, \ n = 1, 2, \cdots.$$

Prove that z_n is convergent.

Solution:

□

3.3.4. *15W-7.3).*

Problem 34: 15W-7.3)

Justify or give a counterexample for the following statements:

3) If $a_n > 0$ and $\sum a_n$ is a convergent series, then $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{a_n}}{n^{2/3}}$ is absolutely convergent.

Solution:

3) True.
Note that since $a_n > 0\ \forall\ n \in \mathbb{N} \implies \left|\frac{(-1)^n\sqrt{a_n}}{n^{2/3}}\right| = \frac{\sqrt{a_n}}{n^{2/3}}$. By the AM-GM inequality,

$$\frac{\sqrt{a_n}}{n^{2/3}} = \sqrt{\frac{a_n}{n^{\frac{4}{3}}}} \leq \frac{1}{2}\left(a_n + \frac{1}{n^{\frac{4}{3}}}\right).$$

Since $\displaystyle\sum_{n=1}^{\infty} a_n$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ converges by assumption and p-series test, by the comparison test, we have

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\sqrt{a_n}}{n^{2/3}}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{n^{2/3}}$$

converges and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{a_n}}{n^{2/3}}$ is absolutely convergent.

□

3.3.5. *14S-5.*

Problem 35: 14S-5

Let $\{x_n\}$ be a sequence in the complete metric space (X, d) and for every $\varepsilon > 0$ there exists a convergent sequence $\{y_n\}$ such that $\sup_n d(x_n, y_n) < \varepsilon$. Prove that then $\{x_n\}$ converges.

Solution: Let $\{y_n\}_{n\in\mathbb{N}}$ such that $\sup_n d(x_n, y_n) < \frac{\varepsilon}{3}$. Since $\{y_n\}_{n\in\mathbb{N}}$ is convergent $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy, so we have there exists $N \in \mathbb{N}$ such that

$$d(y_n, y_m) < \frac{\varepsilon}{3} \text{ when } n, m \geq N.$$

Hence when $n, m \geq N$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, y_n) + d(y_n, x_m) + d(y_m, x_m) \leq \sup_n d(x_n, y_n) + d(y_n, x_m) + \sup_m d(x_m, y_m) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, $\{x_n\}_{n\in\mathbb{N}}$ converges.

□

3.3.6. *14W-7(a).*

Problem 36: 14W-7(a)

(a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} \frac{(-1)^n a_n^2}{1+a_n^2}$ converges absolutely.

(b) Prove that

$$\sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{2n+1}-\frac{x^{n+1}}{2n+2}\right)$$

converges pointwise on $[0,1]$, but the sequence of partial sums does not converge uniformly on $[0,1]$.

Solution:

(a) Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, $\lim_{n\to\infty} |a_n| = 0 \implies \exists\ N \in \mathbb{N}$ such that $|a_n| < 1 \implies |a_n|^2 = a_n^2 < |a_n|$. Hence we have, when $n \geq N$

$$\left|\frac{(-1)^n a_n^2}{1+a_n^2}\right| = \frac{a_n^2}{1+a_n^2} < a_n^2 < |a_n|.$$

Hence by the Comparison Test,

$$\sum_{n=N}^{\infty} \frac{(-1)^n a_n^2}{1+a_n^2} \text{ converges absolutely } \implies \sum_{n=1}^{\infty} \frac{(-1)^n a_n^2}{1+a_n^2} \text{ converges absolutely .}$$

$\square$

3.4. **Continuity and Uniform Continuity.**

3.4.1. *Intermediate Value Theorem: 23S-6(a),97W-1(c),96S-2(a).*

Problem 37: Intermediate Value Theorem

(a) (23S-6(a))
Suppose $f : [0,1] \to \mathbb{R}$ is continuous with $f(0) = f(1)$. For each positive integer n, show there are $s_n, t_n \in [0,1]$ with $f(s_n) = f(t_n)$ and $|s_n - t_n| = \dfrac{1}{n}$.

(b) (97W-1(c))
Suppose $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) = 0 = f(b)$ and there is an $x_0 \in (a,b)$ with $f(x_0) > 0$. Prove that there is an open interval (c,d) on which f is positive and $f(c) = 0 = f(d)$.

(c) (96S-2(a))
Let $f : [a,b] \to \mathbb{R}$ be continuous with $f(a) < 0$ and $f(b) > 0$. Let $S = \{x \in [a,b] : f(x) < 0\}$. If c is the supremum of S prove that $f(c) = 0$.

Solution: These are all applications of the Intermediate Value Theorem (you might need to see that quickly). And in 9S-2(a), we're asked to prove the Intermediate Value Theorem in a slightly different form. We prove that first.

(c) Note that S is bounded by $x = b$ hence $\sup S$ exists by the completeness of real numbers. Suppose the contrary that $f(c) \neq 0$. Since f is continuous on $[a,b]$, f is continuous at $x = c$, hence there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{|f(c)|}{2} \text{ when } |x - c| < \delta.$$

<u>Case 1</u>: $f(c) > 0$.
If $f(c) > 0$, when $|x - c| < \delta$, we have

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2} \Longrightarrow f(x) > \frac{f(c)}{2} > 0.$$

Hence there exists $d \in (c - \delta, c) \Longrightarrow d < c$ such that $f(d) > 0$ which make $d < c$ is another upper bound for S, contradicting to c is the least upper bound of S.
<u>Case 2</u>: $f(c) < 0$.
If $f(c) < 0$, when $|x - c| < \delta$, we have

$$\frac{f(c)}{2} < f(x) - f(c) < -\frac{f(c)}{2} \Longrightarrow f(x) < \frac{f(c)}{2} < 0.$$

Hence there exists $e \in (c, c + \delta) \Longrightarrow e > c$ such that $f(e) < 0 \Longrightarrow e \in S$ which contradicts to c as an upper bound. Combining the results, we have $f(c) = 0$.

(a) Let $g(x) = f\left(x + \frac{1}{n}\right) - f(x),\ x \in \left[0, 1 - \frac{1}{n}\right]$. Clearly g is continuous since f is continuous and we have

$$\begin{aligned} g(0) &= f\left(\frac{1}{n}\right) - f(0) \\ g\left(\frac{1}{n}\right) &= f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) \\ g\left(\frac{2}{n}\right) &= f\left(\frac{3}{n}\right) - f\left(\frac{2}{n}\right) \\ &\vdots \qquad\qquad \vdots \\ g\left(1 - \frac{1}{n}\right) &= f(1) - f\left(1 - \frac{1}{n}\right) \\ \implies g(0) + \cdots + g\left(1 - \frac{1}{n}\right) &= f(1) - f(0) = 0 \end{aligned}$$

Note that if there exist $a \in \{1, 2, \cdots, n-1\}$ such that $g\left(\frac{a}{n}\right) = 0 = f\left(\frac{a}{n} + \frac{1}{n}\right) - f\left(\frac{a}{n}\right)$ then we can just choose $s_n = \frac{a}{n},\ t_n = s_n + \frac{1}{n}$ (or reverse the order) then we're done. Otherwise, if all terms are not 0, then there exists $b, c \in \{1, 2, \cdots, n-1\}$ such that $g\left(\frac{b}{n}\right) > 0,\ g\left(\frac{c}{n}\right) < 0$ then since g is continuous, by the Intermediate Value Theorem, there exists ξ between $\frac{b}{n}, \frac{c}{n}$ such that $g(\xi) = f\left(\xi + \frac{1}{n}\right) - f(\xi) = 0$. Then we can choose $s_n = \xi,\ t_n = s_n + \frac{1}{n}$ or vice versa.

(b)

□

3.4.2. *24S2-6,23W-2.*

Problem 38: 24S2-6,23W-2

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a function. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For each of the following statements, either prove it or provide a counterexample.

(a) (24S2-6(a)) Let $U \subset X$ and $V \subset Y$ be given. Define the sets $f(U)$ and $f^{-1}(V)$.

(b) (24S2-6(b)) Let $x_0 \in X$. Give the original definition (the $\varepsilon - \delta$ definition) of continuity of f at x_0.

(c) For each of the following statements, prove the statement if it is a theorem and provide a counterexample if it is not a theorem:

  (i) (24S2-6(c)(i),23W-2(a)) Let $f : X \to Y$ be continuous. Then $f(U)$ is open in Y for every open subset of U of X.

  (ii) (24S2-6(c)(ii)) Let $f : X \to Y$ be continuous. Then $f^{-1}(V)$ is open in X for every open subset V of Y.

  (iii) (23W-2(b)) The inverse image $f^{-1}(F)$ of a closed set $F \subseteq Y$ is always closed.

Solution:

(a)
$$f(U) = \{f(x) \in Y \mid x \in U\}, \; f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

(b) f is continuous at $x = x_0$ if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ when $d_X(x, x_0) < \delta$.

(c) (i) False.

A counter-example would be $f : \mathbb{R} \to \mathbb{R}, \; f(x) = 1$ which is clearly continuous. Let $U = (-1, 1)$ then $f(U) = \{1\}$ which is closed.

(ii) True.

Let $p \in f^{-1}(V)$ be arbitrary chosen then $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $N_\varepsilon(f(p)) \subseteq V$. Also, since f is continuous, for that ε, there exists $\delta > 0$ such that $x \in N_\delta(p) \implies f(x) \in N_\varepsilon(f(p))$. Hence $N_\delta(p) \subseteq f^{-1}(V)$. Since $p \in f^{-1}(V)$ is arbitrary chosen, $f^{-1}(V)$ is open.

(iii) True.

Since F is closed, $Y \setminus F$ is open and hence $f^{-1}(\mathbb{R} \setminus F)$ is open by the (ii). Note that
$$f^{-1}(Y \setminus F) = f^{-1}(f^{-1}(Y)) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$
(you can prove that (preimage preserves almost everything) if you have some extra time. I didn't and everything was fine.) Hence $X \setminus f^{-1}(F)$ is closed and therefore $f^{-1}(F)$ is closed.

$\square$

3.4.3. *22S-2.*

Problem 39: 22S-2

Recall that a function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ whenever $d_X(p, q) < \delta$. Suppose $f : E \to \mathbb{R}$ is uniformly continuous on a bounded subset E of R.

(a) Show that f can be extended to a continuous functions $\tilde{f} : \overline{E} \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in E$.

(b) Prove that f is bounded on E.

Solution: This is one characterization of uniformly continuous function.

(a) Let $a \in \overline{A}$, then there exists $\{a_n\}_{n\in\mathbb{N}},\ a_n \in E$ such that $a_n \to a$. Since $\{a_n\}_{n\in\mathbb{N}}$ is convergent, $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy. Now we prove the following claim.

Claim 6. If $f : X \to Y$ is uniformly continuous where (X, d_X), (Y, d_Y) are metric spaces. Then for any Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$, $\{f(x_n)\}_{n\in\mathbb{N}}$ is also Cauchy.

Proof of claim. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon \text{ when } d_X(x, y) < \delta.$$

Also, since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, then there exists $N \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \delta\ \forall\ n, m \geq N.$$

Hence when $n, m \geq N_1$, we have

$$d_Y(f(x_n), f(x_m)) < \varepsilon.$$

By the Claim, we have $f(a_n)$ is Cauchy. And since $\mathbb{R}$ is complete, $\lim\limits_{n\to\infty} f(a_n)$ exists. Now we define

$$\tilde{f}(a) = \lim_{n\to\infty} f(a_n),\ a \in \overline{E}.$$

Note that if $a \in E$, then clearly we can choose $a_n = a\ \forall\ n \in \mathbb{N}$ then

$$\tilde{f}(a) = \lim_{n\infty} f(a) = f(a) \Longrightarrow \tilde{f}(x) = f(x)\ \forall\ x \in E.$$

Now we prove that $\tilde{f}(x)$ is well-defined (since the input right now is a sequence in the right hand side). Let $\{x_n\}_{n\in\mathbb{N}},\ \{y_n\}_{n\in\mathbb{N}}$ such that $x_n \to a,\ y_n \to a$ then we have $x_n - y_n \to 0 \in \mathbb{R}$. Let $f(x_n) \to a_1,\ f(y_n) \to a_2$. Since f is uniformly continuous, for any $\varepsilon > 0,\ \exists\ \delta > 0$ such that

$$|f(x_n) - f(y_n)| < \varepsilon \text{ when } d(x_n, y_n) < \delta.$$

Also, since $x_n - y_n \to 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, y_n) < \delta$ when $n \geq N$. Hence when $n \geq \mathbb{N}$, we have $|f(x_n) - f(y_n)| < \varepsilon \Longrightarrow f(x_n) - f(y_n) \to 0$. But we also have $f(x_n) - f(y_n) \to a_1 - a_2$, so we have $a_1 - a_2 = 0 \Longrightarrow a_1 = a_2$ since the limit is unique.

Also, we have $\tilde{f}(x)$ is unique since if there's another $h(x)$ satisfies the extension such that $h(a) = \lim\limits_{n\to\infty} f(x_n) = \tilde{f}(a)$

Note that $\tilde{f}$ is clearly continuous on $\overline{E}$ by the sequential definition of continuity by construction. ($f : X \to Y$ is continuous at $x \in X$ if and only if for any sequence

$\{x_n\}_{n\to\infty}$ such that $x_n \to x$, then $f(x_n) \to f(x)$.)

(Extra, we don't need this)
Now we prove $\tilde{f}(x)$ is actually uniformly continuous. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{3} \text{ when } d(x,y) < \delta.$$

Now let $a, b \in \overline{E}$ and $\{a_n\}_{n\in\mathbb{N}},\ \{b_n\}_{n\to\mathbb{N}}$ such that $a_n \to a,\ b_n \to b$ and $d(a,b) < \frac{\delta}{3}$. Let $\varepsilon > 0$ be given. Since $a_n \to a,\ b_n \to b$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$d(a_n, a) < \frac{\delta}{3},\ d(b_n, b) < \frac{\delta}{3}$$

when $n \geq N_1, N_2$ respectively (actually they can combined as one). Hence we have

$$d(a_n, b_n) \leq d(a_n, a) + d(a, b) + d(b_n, b) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Note that $f(a_n) \to \tilde{f}(a),\ f(b_n) \to \tilde{f}(b)$ by the construction of $\tilde{f}$, there exists $N_3, N_4 \in \mathbb{N}$ such that

$$|\tilde{f}(a) - f(a_n)| < \frac{\varepsilon}{3},\ |\tilde{f}(b) - f(b_n)| < \frac{\varepsilon}{3}$$

when $n \geq N_3, N_4$ respectively (actually they can combined as one). Combining all the results, now we have

$$d(a,b) < \frac{\delta}{3} < \delta$$

and

$$\begin{aligned} |\tilde{f}(b) - \tilde{f}(a)| &\leq |\tilde{f}(b) - f(b_n)| + |f(b_n) - f(a_n)| + |f(a_n) - \tilde{f}(a)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $\tilde{f}(x)$ is uniformly continuous on $\overline{E}$.

(b) See (ii) in here.

□

3.4.4. *21W-1,06S-2.*

Problem 40: 21W-1,06S-2

(a) Define a compact metric space. You can skip the definition of a metric space.
(b) Suppose (X, d) is a compact metric space and $f : X \to \mathbb{R}$ is continuous. Prove the f is bounded above and its supremum is attained at some point in X.

As an application (or you can prove those in elementary ways), 2006S-2 asked:
Suppose $f : [a,b] \to \mathbb{R}$ is continuous and let

$$m = \inf_{x\in[a,b]} f(x), \qquad M = \sup_{x\in[a,b]} f(x).$$

(c) Prove that m, M are finite and that the range of f is $[m, M]$.
(d) Suppose $m = a$ and $M = b$. Prove that there is an $x \in [a,b]$ so that $f(x) = x$.

Solution:

(a) (X,d) is compact if and only if for any open covering of X, there exists a finite subcovering that also covers X.

(b) First, we claim that:

Claim 7. $f(X)$ is compact.

Proof of claim. See (b) in here.

By the Claim, we have $f(X) \subseteq \mathbb{R}$ is compact. By the Heine-Borel Theorem, we have $f(X)$ is closed and bounded. Let $M = \sup_{x\in X} f(x),\ m = \inf_{x\in X} f(x)$, then $m \le f(x) \le M$. Now we prove another Claim: (It might be okay if you directly claim this since this is rather straightforward.)

Claim 8. If $A \subseteq \mathbb{R}$ is closed, then $\inf A, \sup A \in A$.

Proof of claim. We prove $\sup A \in A$ and $\inf A \in A$ is similar.

Let $\sup A = \alpha$. Suppose the contrary that $\alpha \in A^C$. Since A is closed, A^C is open. Hence there exists $r > 0$ such that $N_r(\alpha) = (\alpha - r, \alpha + r) \subseteq A^C \implies \alpha - r$ is another upper bound, which is a contradiction to $\alpha = \sup A$.

By the Claim above, $M, m \in f(X)$, that is, there exists $x_m, x_M \in X$ such that

$$f(x_m) = \inf_{x\in X} f(x),\ f(x_M) = \sup_{x\in X} f(x).$$

(c) We need both the Extreme Value Theorem and the Intermediate Value Theorem here.

Claim 9. $|m|, |M| < \infty$ and there exists $c, d \in [a,b]$ such that $f(c) = m,\ f(d) = M$.

Proof of claim. $\boxed{\text{Way I:}}$ Just copy (b). (I recommend this)

$\boxed{\text{Way II:}}$ ($[a,b]$ is easier to manipulate but this is nasty.)
Note that $\inf_{x\in[a,b]} f(x) = \sup_{x\in[a,b]} (-f(x))$. So it suffices to prove that f is bounded above.

Suppose the contrary that f is not bounded above. Then for all $n \in \mathbb{N}$, there exists $\{x_n\}_{n\in\mathbb{N}},\ x_n \in [a,b],\ n \in \mathbb{N}$ such that

$$f(x_n) \ge n.$$

Then by the Bolzano-Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \to x_0 \in \mathbb{R}$ when $k \to \infty$. Since $[a,b]$ is closed, we have $x_0 \in [a,b]$ (x_0 is a "limit" point). However, since f is continuous on $[a,b]$, f is continuous at x_0. Hence we have

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) \ge \lim_{k\to\infty} n_k = \infty.$$

which is a contradiction to $f(x) \in \mathbb{R}$ for all $x \in [a,b]$. Hence $M = \sup_{x\in[a,b]} f(x) < \infty$. Similarly, $m = \inf_{x\in[a,b]} f(x) < \infty$.

Next, since $M = \sup_{x\in[a,b]} f(x)$, for all $n \in \mathbb{N}$, we have $M - \dfrac{1}{n}$ is not an upper bound of $f([a,b])$. Hence there exists $\{y_n\}_{n\in\mathbb{N}},\ y_n \in [a,b], n \in \mathbb{N}$ such that

$$f(y_n) > M - \frac{1}{n}.$$

Again, by the Bolzano-Weierstrass Theorem, there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that $y_{n_K} \to y_0 \in \mathbb{R}$. And similarly, $y_0 \in [a,b]$ and f is continuous at y_0 and thus

$$f(y_0) = \lim_{k\to\infty} f(y_{n_k}) \geq \lim_{k\to\infty}\left(M - \frac{1}{y_{n_k}}\right) = M.$$

But we also have $f(y_0) \leq M$. Hence $f(y_0) = M$. Let $d = y_0$ and similarly, there exists $c \in [a,b]$ such that $f(d) = m$.

We also need the Intermediate Value Theorem here. (I think you need to prove this here. But at least cite the theorem).

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$*

Proof. See (c) for the proof. Note that you prove that if $f : X \to Y$ is continuous and $E \subseteq X$ is connected, then $f(E)$ is connected. Then using the fact that connected sets in $\mathbb{R}$ are intervals to prove the Intermediate Value Theorem. But this is rather complicated.

□

Claim 10. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then for any c such that $f(a) < d < f(b)$, there exists $c \in (a,b)$ such that $f(c) = d$.

Proof of claim. Let $g(x) = f(x) - d$, then $g(a) < 0,\ g(b) > 0$. By the Intermediate Value Theorem, there exists $c \in (a,b)$ such that $g(c) = f(c) - d = 0 \implies f(c) = d$.

From the first Claim, we we have $f([a,b]) \subseteq [m,M]$ and $f(c) = m,\ f(d) = M$. Now suppose $c < d$ ($c > d$ case is identical), we have $[c,d] \subseteq [a,b]$. From the second Claim, for any $f(c) = m < y < M = f(d)$, there exists $x \in (c,d) \subseteq [a,b]$ such that $f(x) = y$. Hence $[m,M] \subseteq f([a,b])$. Combining the results, we have

$$f([a,b]) = [m,M].$$

(d) Let $g : [a,b] \to \mathbb{R}$ be

$$g(x) = x - f(x).$$

Clearly $g(a) \geq 0,\ g(b) \leq 0$. If $g(a) = 0$ or $g(b) = 0$ then we're done and in those cases, $x = a$ and $x = b$. Otherwise, $g(a) > 0\ g(b) < 0$. so by the Intermediate Value Theorem, there exists $c \in (a,b)$ such that $g(c) = f(c) - c = 0 \implies f(c) = c$. Combining the results, we have that there exists $x \in [a,b]$ such that

$$g(x) = 0 \implies f(x) = x.$$

□

3.4.5. *Test of Uniform Continuity: 20S-3(b),97W-1(b),96W-2(a)(b).*

Problem 41: Test of Uniform Continuity

(a) (96W-2(a))
Suppose E is a subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. When is f said to be uniformly continuous?

(b) (20S-3(b),97W-1(b),96W-2(b))
Let

$$(20S-3(b))\ f:\mathbb{R}\to\mathbb{R},\ f(x)=x^2,$$
$$(97W-1(b))\ g:\mathbb{R}\setminus\{0\}\to\mathbb{R},\ g(x)=\frac{1}{x}$$
$$(96W-2(b))\ h:\mathbb{R}\to\mathbb{R},\ h(x)=\sin x.$$

Prove that the function f, g are not uniformly continuous on $\mathbb{R}$ and h is uniformly continuous as a map from $\mathbb{R}$ to $\mathbb{R}$.

(c) (24S2-5(iii))
Let (X, d_X) and (Y, d_Y) be metric spaces. For each of the following statements, prove the statement if it is a theorem and provide a counterexample if it is not a theorem: If $f : X \to Y$ is a uniformly continuous function, then X is compact.

Solution:

(a) f is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, we have

$$|f(x) - f(y)| < \varepsilon \text{ when } |x - y| < \delta.$$

(b) To prove a function is not uniformly continuous, we can either use proof by contradiction or find an $\varepsilon > 0$ such that for all $\delta > 0$, we have $|f(x) - f(y)| \geq \varepsilon$ for some $x, y \in E$ such that $|x - y| < \delta$.

(i) Let $\varepsilon = 2$. Then $\delta > 0$ be fixed. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Then for $x = N + \frac{1}{N},\ y = N \Longrightarrow |x - y| = \frac{1}{N} < \delta$. But

$$|f(x) - f(y)| = 2 + \frac{1}{N^2} > 2 = \varepsilon.$$

Since the choice of $\delta > 0$ is arbitrary, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

(ii) Suppose the contrary that g is uniformly continuous. Let $\varepsilon = 1$, then there exists $\delta > 0$ such that $|g(x) - g(y)| < 1$ when $|x - y| < \delta$. Now by the Archimedean property we can choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Then for $x = \frac{1}{N},\ y = \frac{1}{2N} \Longrightarrow |y - x| = \frac{1}{2N} < \frac{1}{N} < \delta$. So we have

$$|g(x) - g(y)| = N < \varepsilon = 1$$

which is a contradiction to $N \in \mathbb{N}$. Hence $g(x) = \frac{1}{x}$ is not uniformly continuous on $\mathbb{R} \setminus \{0\}$.

(iii) Note that $h(x) = \sin x$ is clearly continuous and differentiable on $\mathbb{R}$. Hence for any $x, y \in \mathbb{R}$, by the Mean Value Theorem, there exists $c \in (x, y)$ such that

$$h'(c) = \cos(c) = \frac{h(y) - h(x)}{y - x} \Longrightarrow |h(x) - h(y)| = |\cos(c)(x - y)| \leq |x - y|.$$

Hence for any $\varepsilon > 0$, choosing $\delta = \varepsilon$, we have when $|x - y| < \delta \Longrightarrow |h(x) - h(y)| < |x - y| < \delta = \varepsilon$. Hence $h(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

If you don't like the Mean Value Theorem (since it needs "continuity" first). We can use the trigonometric identity that

$$\begin{aligned} |\sin x - \sin y| &= \left|2 \sin\left(\frac{x - y}{2}\right) \cos\left(\frac{x + y}{2}\right)\right| \leq 2 \left|\sin\left(\frac{x - y}{2}\right)\right| \\ &\leq 2 \left|\frac{x - y}{2}\right| = |x - y|. \end{aligned}$$

And the rest follows the Mean Value Theorem one. Note that we use that $|\sin x| \leq |x|$. This is clear either we use the high school geometric definition of $\sin x$ (this appears in proving $\lim_{x \to 0} \frac{\sin x}{x} = 1$) or use Taylor series as the definition (like in complex analysis).

(c) Clearly it's false. Take h in (b) then h is uniformly continuous but $\mathbb{R}$ is clearly not compact (since it's not bounded and thus doesn't satisfy the Heine-Borel Theorem).

$\square$

3.4.6. *06W-2(a).*

Problem 42: 06W-2(a)

Let the real valued function $f : [0, 1] \to \mathbb{R}$ have the following two properties:

(i) If $[a, b] \subset [0, 1]$ then $f([a, b])$ contains the interval with end points $f(a)$ and $f(b)$ (i.e. has the intermediate value property).

(ii) For each point $c \in \mathbb{R}$, $f^{-1}(c)$ (the pre-image) is closed.

Prove that f is continuous.

Solution:

$\square$

3.4.7. *03W-2(b).*

Problem 43: 03W-2(b)

Suppose $f(x)$ and $g(x)$ are two real valued, continuous functions on $\mathbb{R}$ so that $(f(x))^2 = (g(x))^2$. If $f(x) \neq 0$ for all x then show that either $f(x) = g(x)$ for all x, or $f(x) = -g(x)$ for all x.

Solution:

$\square$

3.5. **Differentiation.**

3.5.1. *24S1-6.*

Problem 44: 24S1-6

Consider $f(x) = \begin{Bmatrix} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{Bmatrix}$. Show that the n-th derivative $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Solution: First, for $x \neq 0$, we need the following claim:

Claim 11. For $x \neq 0$, we have

$$f^{(n)}(x) = p_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where $p_n(x)$ is a polynomial.

Proof. We prove this by induction.
It's clear that $n = 1 \implies f'(x) = 2x^{-3}e^{-\frac{1}{x^2}}$ and the claim is true. Suppose the claim is true for $n = k$, then for $n = k+1$, we have

$$\begin{aligned} f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' &= \left(p_k\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}\right)' \\ &= \left(p_k^{'}\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2}\right) e^{-\frac{1}{x^2}} + p_k\left(\frac{1}{x}\right)\left(e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}\right) \\ &= \left(\frac{2}{x^3}p_k\left(\frac{1}{x}\right) - \frac{1}{x^2}p_k^{'}\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x^2}} \end{aligned}$$

which is also true since $p_k^{'}$ is clearly also a polynomial. ☐

We prove the original problem by induction.
For $n = 1$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h} \overset{k=\frac{1}{h}}{=} \lim_{k\to\infty} \frac{k}{e^{-k}} \overset{L'H}{=} \frac{1}{2ke^{k^2}} = 0.$$

Suppose the statement is true for $n = k$, then for $n = k+1$, we have

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} \\ (\text{Claim.}) &= \lim_{h\to 0} \frac{p_k\left(\frac{1}{h}\right) e^{-\frac{1}{h^2}} - 0}{h} \overset{m=\frac{1}{h}}{=} \lim_{m\to\infty} \frac{kp_k(m)}{e^{m^2}} \overset{L'H}{=} \lim_{m\to\infty} \frac{p_k(m) + mp_k^{'}(m)}{2me^{m^2}} \\ &\overset{L'H}{=} (\deg p_k(m) \text{ times}) = \lim_{m\to\infty} \frac{c}{q(m)e^{m^2}} = 0 \end{aligned}$$

where c is some constant and $q(m)$ is a polynomial (you can prove that using induction again but it doesn't matter here). Hence by the induction hypothesis, we finish the proof.

☐

3.5.2. *23S-6(b)=18W-4.*

Problem 45: 23S-6(b)=18W-4

Suppose $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable with $f'(a) < 0$ and $f'(b) > 0$. Show there is a $c \in (a, b)$ with $f'(x) = 0$. Please do not use an equivalent theorem to prove this.

Solution: This is the simplified version of the Intermediate Value Theorem for Derivatives (or called Darboux's Theorem).
Consider the restriction $f : [a, b] \to \mathbb{R}$, since $[a, b]$ is compact, the minimum exists, say $f(c)$ is the minimum. We claim that the minimum can't be achieved at $x = a, b$.
$\boxed{\text{Way I: Direct Proof}}$
Note that f is differentiable on $[a, b] \subset \mathbb{R}$, we have

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) \iff \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

That is, for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $\left| \dfrac{f(x) - f(x)}{x - a} - f'(a) \right| < \varepsilon$ when $|x - a| < \delta$. Hence

$$\frac{f(x) - f(a)}{x - a} < f'(a) + \varepsilon.$$

Now, we take $\varepsilon = \dfrac{|f'(a)|}{2}$, then

$$\frac{f(x) - f(a)}{x - a} < f'(a) + \frac{|f'(a)|}{2} = f'(a) - \frac{f'(a)}{2} = \frac{f'(a)}{2}.$$

Hence we have

$$\begin{aligned} &\frac{f(x) - f(a)}{x - a} < \frac{f'(a)}{2} \\ \implies &f(x) - f(a) = \frac{f'(a)}{2}(x - a) < 0 \\ \implies &f(x) < f(a) \end{aligned}$$

when $x - a > 0$, that is, $0 < x - a < \delta \implies a < x < a + \delta$. Hence $f(a)$ is not the local minimum. And we can take the negative sign to show that $f(b)$ is also not the local minimum.
$\boxed{\text{Way II: Proof by Contradiction}}$ See the proof in the whole Theorem and take $k = 0$.

Hence $a < c < b$ and $f(c)$ is the minimum of f. Hence by the Fermat's Theorem, we have $f'(c) = 0$.

Here we prove the full theorem.

Theorem 9 (Darboux's)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Let $k \in \mathbb{R}$ such that $f'(a) < k < f'(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = k$.*

Proof. Let $g(x) = f(x) - kx$, then g is clearly continuous and differentiable and since $[a, b]$ is compact, the minimum exists. We claim that the minimum of g can't be achieved at $x = a, b$.
$\boxed{\text{Way I: Direct Proof}}$
Note that g is also differentiable and $g'(x) = f'(x) - k$ hence $g'(a) < 0$, $g'(b) > 0$. Applying

the exact same process above we can prove that the minimum can't be achieved at $x = a, b$.

$\boxed{\text{Way II: Proof by Contradiction}}$

Suppose the contrary that $g(a)$ is the local minimum, that is, $g(a) \le g(x)\ \forall\ x \in (a, b]$. Hence for all $x \in (a, b]$, we have

$$f(a) - ka \le f(x) - kx \Longrightarrow k \le \frac{f(x) - f(a)}{x - a} \Longrightarrow f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \ge k$$

which is a contradiction. Similarly, if $g(b)$ is the local minimum, we have $g(b) \le g(x)\ \forall\ x \in (a, b]$. Hence for all $x \in (a, b]$, we have

$$f(b) - kb \le f(x) - kx \Longrightarrow k \le \frac{f(b) - f(x)}{b - x} \Longrightarrow f'(b) = \lim_{x \to b^-} \frac{f(b) - f(x)}{b - x} \le k$$

which is also a contradiction.

Finally, there exists $c \in (a, b)$ such that $g(c)$ is the minimum and by the Fermat's Theorem, we have

$$g'(c) = f'(c) - k = 0 \Longrightarrow f'(c) = k.$$

□

□

3.5.3. *22W-8.*

Problem 46: 22W-8

Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $|f(x)| \le 1,\ |f''(x)| \le 1$ for all $x \in \mathbb{R}$. Prove there is an $M > 0$ such that $|f'(x)| \le M$ for all $x \in \mathbb{R}$.

Reference: Baby Rudin, Chapter 5, Exercise 15.

Solution: Let $h > 0$. Since f is twice differentiable, by the Taylor expansion, we have there exsits $\xi \in (x, 2h)$ such that

$$f(x + 2h) = f(x) + \frac{f'(x)}{1!}(2h) + \frac{f''(\xi)}{2!}(2h^2) = f(x) + 2hf'(x) + 2h^2 f''(\xi).$$

Hence we have

$$\begin{aligned}
f'(x) &= \frac{f(x+2h) - f(x)}{2h} - hf''(\xi) \\
\Longrightarrow |f'(x)| &\le \frac{|f(x+2h) - f(x)|}{2h} + h|f''(\xi)| \le \frac{|f(x+2h)| + |f(x)|}{2h} + h|f''(\xi)| \\
&\le \frac{2}{2h} + h = \frac{1}{h} + h \\
\Longrightarrow |f'(x)| &\le 2.
\end{aligned}$$

Here since $h > 0$ we used the AM-GM Inequality to have

$$\frac{1}{h} + h \ge 2\sqrt{\frac{1}{h}h} = 2.$$

□

3.5.4. *21S-2.*

Problem 47: 21S-2

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Reference: Baby Rudin, Chapter 5, Exercise 1.

Solution: Note that $|f(x) - f(y)| \leq (x-y)^2 = |x-y|^2 \implies \left|\dfrac{f(x)-f(y)}{x-y}\right| \leq |x-y|$. Hence for x be given, then for any $y \in \mathbb{R}$, we have

$$0 \leq \implies \left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|.$$

Then by the Squeeze Theorem, we have

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| = \lim_{y\to x}|x-y| = 0.$$

Hence $f'(x) = 0$. Since the choice of x is arbitrary, $f'(x) = 0 \forall\ x \in \mathbb{R}$ and thus f is a constant.

□

3.5.5. *20S-3(a).*

Problem 48: 20S-3(a)

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 1.2 & \text{if } x = 0, \end{cases}$ cannot be the derivative of a differentiable function $F : \mathbb{R} \to \mathbb{R}$.

Solution: We use the theorem:

Theorem 10 (Darboux's)**.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Let $k \in \mathbb{R}$ such that $f'(a) < k < f'(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = k$.*

Proof. See here. □

Suppose the that $F'(x) = f(x)$. Consider $F'\left(\dfrac{2}{\pi}\right) = f\left(\dfrac{2}{\pi}\right) = 1$ and $F'(0) = f(0) = 1.2$. Then by the Darboux's Theorem, there exists $c in \left(0, \dfrac{2}{\pi}\right)$ such that $F'(c) = f(c) = 1.1$. But learly $\left|\sin\left(\dfrac{1}{x}\right)\right| \leq 1$ for all $x \in \left(0, \dfrac{2}{\pi}\right)$ which is a contradiction. Hence f cannot be the derivative of a differentiable function $F : \mathbb{R} \to \mathbb{R}$.

□

3.5.6. *19S-7.*

Problem 49: 19S-7

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$ and $f'(x) > f(x)$ for all $x \geq 0$.

(a) Prove that $f(x) > 0$ on $(0, a]$ for some $a > 0$.

(b) Prove that $f(x) > 0$ for all $x > 0$.

Solution:

(a) Note that $f'(0) = \lim_{x\to 0} \dfrac{f(x)}{x} > f(0) = 0$. Take $\varepsilon = \dfrac{f'(0)}{2}$, then there exists $\delta > 0$ such that

$$\left|\frac{f(x)}{x} - f'(0)\right| < \frac{f'(0)}{2} \text{ when } 0 < |x - 0| < \delta.$$

Choose $0 < a < \delta$, then when $x \in (0, a] \implies |x| < \delta$, we have

$$-\frac{f'(0)}{2} < \frac{f(x)}{x} - f'(0) < \frac{f'(0)}{2} \implies \frac{f(x)}{x} > \frac{f'(0)}{2} \implies f(x) > \frac{f'(0)}{2}x > 0.$$

(b) Suppose the contrary that there exists $b \in \mathbb{R}^+$ such that $f(b) \leq 0$. Then by (a), we can choose $a < b$ such that $f(x) > 0$ on $(0, a]$. Then by the Intermediate Theorem, there exists $c \in (a, b]$ such that $f(c) = 0$. Let c^* be the first point such that $f(c^*) = 0$ then we have $f(x) > 0$ on $(0, c^*)$. So $f(0) = f(c^*) = 0$, by the Rolle's Theorem, there exists $d \in (0, c^*)$ such that $f'(d) = 0$ which is a contradiction to $f'(d) > f(d) > 0$ since $f(x) > 0$ on $(0, c^*)$ and $d \in (0, c^*)$. Hence $f(x) > 0$ for all $x > 0$.

$\boxed{\text{Another Way}}$ (This is a special case of the Grönwall's Inequality)
Let $g(x) = e^{-x}f(x)$, then we have

$$g'(x) = -e^{-x}f(x) + e^{-x}f'(x) = e^{-x}(f'(x) - f(x)) > 0.$$

Hence $g'(x) > 0$ for all $x > 0$ and thus $g(x) > g(0)$ for all $x > 0$ (you may need to prove this calculus result if you have time). Note that $g(0) = e^{-0}f(0) = 0$ and

$$g(x) = e^{-x}f(x) > g(0) = 0 \implies f(x) > 0$$

since $e^{-x} > 0$ for all $x > 0$.

$\square$

3.5.7. *19W-2,15W-3.*

Problem 50: 19W-2,15W-3

Suppose $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0.2019x + \varepsilon \cos x$, where $\varepsilon \in \mathbb{R}$ is given.

(a) Prove that there exists $\varepsilon > 0$ such that the function f is injective.

(b) Prove that for any $\varepsilon \in \mathbb{R}$, the function f is surjective.

Reference: Baby Rudin, Chapter 5, Exercise 3.

Solution:

(a) Consider

$$f'(x) = 0.2019 - \varepsilon \sin x.$$

Choose $\varepsilon > 0$ such that $\varepsilon \sin x < 0.2019$, then we have $f'(x) > 0$. We prove the following claims:

Claim 12. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then if $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Proof. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$$

since $f'(x) > 0$ and $x_2 > x_1$. □

Note that x and $\cos x$ are differentiable so $f(x)$ is clearly differentiable. And it's clear that strictly increasing function is injective since $x_1 < x_2 \implies x_1 \neq x_2 \implies f(x_1) < f(x_2) \implies f(x_1) \neq f(x_2)$. Hence f is injective.

(b) Note that x and $\cos x$ are clearly continuous on $\mathbb{R}$ so f is also continuous on $\mathbb{R}$. Also, note that x is not bounded above or below and $-\varepsilon \leq |\varepsilon \cos x| \leq \varepsilon$ which is bounded. Therefore f is not bounded above or below in $\mathbb{R}$. Now let $d \in \mathbb{R}$ be arbitrary chosen, since f is not bounded below, there exists $a \in \mathbb{R}$ such that $f(a) < d$. Similarly, since f is not bounded above, there exists $b \in \mathbb{R}$ such that $d < f(b)$. Hence we have $f(a) < d < f(b)$. And by the Intermediate Value Theorem, there exists $e \in \mathbb{R}$ such that $f(e) = d$. Since $d \in \mathbb{R}$ is arbitrary chosen, f is surjective.

□

3.5.8. *15W-7.2).*

Problem 51: 15W-7.2)

Justify or give a counterexample for the following statements:

2) The derivative of a differentiable function on $\mathbb{R}$ cannot be onto $(0, 1) \cup (3, 4)$.

Solution:

2)

□

3.5.9. *11S-2(a).*

Problem 52: 11S-2(a)

(a) Suppose $f : (1, \infty) \to \mathbb{R}$ is a bounded differentiable function on $(1, \infty)$ and $\lim_{x\to\infty} f'(x) = b$. Prove that $b = 0$.

Solution:

(a)

$\square$

3.5.10. *97W-2(c).*

Problem 53: 97W-2(c)

(c) Suppose $f : (0, 1] \to \mathbb{R}$ is differentiable and the derivative is bounded. Prove that the sequence $\left\{ f\left(\frac{1}{n}\right) \right\}_{n=1}^{\infty}$ is convergent.

Solution:

(c) Say $|f'(x)| \leq M$, the consider $n, m \in \mathbb{N}, m > n$, then since f is differentiable on $(0, 1]$ hence f is continuous on $(0, 1]$. So clearly f is continuous on $\left[\frac{1}{n}, \frac{1}{m}\right]$ and f is differentiable on $\left(\frac{1}{n}, \frac{1}{m}\right)$, so by the Mean Value Theorem, we have

$$\begin{aligned}
\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right) \right| &= \left| f'(c)\left(\frac{1}{n} - \frac{1}{m}\right) \right| = |f'(c)| \left| \frac{1}{n} - \frac{1}{m} \right| \\
&\leq M \left| \frac{1}{n} - \frac{1}{m} \right| = M \left| \frac{m-n}{mn} \right| = \frac{M(m-n)}{mn} \\
&< \frac{Mm}{mn} = \frac{M}{n}.
\end{aligned}$$

Hence for any $\varepsilon > 0$, choose $N > \frac{M}{\varepsilon}$ by the Archimedean property, then when $n \geq N$, we have $\frac{1}{m} < \frac{1}{n} < \frac{1}{N} < \frac{\varepsilon}{M}$ and

$$\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right) \right| < \frac{M}{n} < M\frac{\varepsilon}{M} = \varepsilon.$$

Hence $\left\{ f\left(\frac{1}{n}\right) \right\}_{n=1}^{\infty}$ is a Cauchy sequence and hence it's convergent. (If you have time you can prove any Cauchy sequence converges.)

$\square$

3.5.11. *Will be updated.*

19W-2=15W-3

17W-7

14S-6

12S-8
06S-3
06W-3
05W-2(b)
03S-2(a)
97W-2(a):thm

3.6. **Riemann Integral.**

3.6.1. *24S-4(a)(b).*

Problem 54: 24S-4(b)

(b) If $f : [0,1] \to \mathbb{R}$ be a continuous function. True or false:

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} f(x)dx = \int_0^1 f(x)dx$$

Justify your answer.

Solution:

(b) This is true.

Note that f is continuous on a compact set $[0,1]$, so f is bounded, say $|f(x)| \le M \;\forall\, x \in [0,1]$ and clearly f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$. Then

$$\begin{aligned}\left|\int_0^1 f(x)\,dx - \int_{\varepsilon}^{1-\varepsilon} f(x)\,dx\right| &= \left|\int_0^{\varepsilon} f(x)\,dx + \int_{1-\varepsilon}^{1} f(x)\,dx\right| \\ &\le \left|\int_0^{\varepsilon} f(x)\,dx\right| + \left|\int_{1-\varepsilon}^{1} f(x)\,dx\right| \\ &\le M\cdot\varepsilon + M\cdot\varepsilon = 2M\varepsilon. \\ &\to 0 \text{ when } \varepsilon \to 0.\end{aligned}$$

Hence when $\varepsilon \to 0$,

$$\int_{\varepsilon}^{1-\varepsilon} f(x)\,dx \to \int_0^1 f(x)\,dx \iff \lim_{\epsilon \to 0}\int_{\epsilon}^{1-\epsilon} f(x)dx = \int_0^1 f(x)dx$$

□

3.6.2. *20W-5(i).*

Problem 55: 20W-5(i)

(i) Suppose that $g : [0,1] \to \mathbb{R}$ is bounded and g^4 is Riemann integrable. Does it follow that g is Riemann integrable? Justify your claim by either proving the statement, or by providing a counter example and proving that g is not Riemann integrable for that example.

Solution:

(i) False.

A counter-example would be

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{Q}^C \end{cases}.$$

Then $g^4(x) = 1$ and it's clearly integrable however, since g is discontinuous at every point of $[0,1]$ and clearly is not measure 0.

□

3.6.3. *18S-4.*

Problem 56: 18S-4

Let f and g be Riemann integrable in the interval (a,b) and assume that

$$\int_a^b f(x)^2 \mathrm{d}x = \int_a^b g(x)^2 \mathrm{d}x = 1.$$

Show that

$$\int_a^b f(x)g(x)\mathrm{d}x \leq 1.$$

Use the above inequality to prove that if f and g are Riemann integrable, then

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leq \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b g(x)^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Reference: See Baby Rudin, Chapter 6, Exercise 10 for the general Hölder's inequality.

Solution: This is just the Cauchy-Schwarz inequality. Just copy your favorite proof here.

Let $h(x) = f(x) - g(x)$, then clearly h is Riemann integrable and

$$h(x)^2 \geq 0 \Longrightarrow \int_a^b h(x)^2 \, dx \geq 0.$$

So we have

$$\begin{aligned} \int_a^b h(x)^2 \, dx &= \int_a^b \left(f(x)^2 - 2f(x)g(x) + g(x)^2\right) dx \\ &= \int_a^b f(x)^2 \, dx - 2\int_a^b f(x)g(x) \, dx + \int_a^b g(x)^2 \, dx \geq 0. \end{aligned}$$

Hence

$$\int_a^b f(x)g(x) \, dx \leq \frac{1}{2}\left(\int_a^b f(x)^2 \, dx + \int_a^b g(x)^2 \, dx\right) = 1.$$

Now we let

$$F(x) = \frac{f(x)}{\left(\int_a^b f(x)^2 \, dx\right)^{\frac{1}{2}}}, \; G(x) = \frac{g(x)}{\left(\int_a^b g(x)^2 \, dx\right)^{\frac{1}{2}}}.$$

Then clearly

$$\int_a^b F(x)^2 \mathrm{d}x = \int_a^b G(x)^2 \mathrm{d}x = \int_a^b |F(x)|^2 \mathrm{d}x = \int_a^b |G(x)|^2 \mathrm{d}x = 1.$$

Then by the previous result, we have

$$\left|\int_a^b F(x)G(x) \, dx\right| \leq \int_a^b |F(x)||G(x)| \, dx \leq 1$$

Rearranging the inequality, we have

$$\left|\int_a^b \frac{f(x)}{\left(\int_a^b f(x)^2 \, dx\right)^{\frac{1}{2}}} \frac{g(x)}{\left(\int_a^b g(x)^2 \, dx\right)^{\frac{1}{2}}} \, dx\right| \leq 1$$

$$\Longrightarrow \left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leq \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b g(x)^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

$\square$

3.6.4. *18W-6=12W-6=97S-2.*

Problem 57: 18W-6=12W-6=97S-2

Given $f : [a, b] \to \mathbb{R}$ is continuous and non-negative and M is the maximum of f on $[a, b]$, prove that

(a) (97S-2(a))
Given $\varepsilon > 0$ prove that there is a closed interval on which f exceeds $M - \varepsilon$.

(b) (18W-6=12W-6=97S-2(b))
Prove that

$$\lim_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} = M.$$

Solution:

(a) Suppose $M > 0$ (if $M < 0$, just replace everything with $-M$). Since f is continuous on a compact set $[a, b]$. Then by the Extreeme Value Theorem, there exists $x_0 \in [a, b]$ such that $f(x_0) = M$. Then there exists $\delta > 0$ such that when $|x - x_0| \leq \delta \implies x \in [x_0 - \delta, x_0 + \delta]$,

$$|f(x_0) - f(x)| = f(x_0) - f(x) < \varepsilon \Longrightarrow f(x) > f(x_0) - \varepsilon = M - \varepsilon.$$

(b) Since $|f(x)| \leq M$ for all $x \in [a, b]$, we have

$$\left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} \leq \left(\int_a^b M^n \, dx \right)^{\frac{1}{n}} = M(b - a)^{\frac{1}{n}}.$$

Hence

$$\limsup_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} \leq \limsup_{n\to\infty} M(b - a)^{\frac{1}{n}} = \lim_{n\to\infty} M(b - a)^{\frac{1}{n}} M.$$

Here we used that $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$ for all $c > 1$. (I don't think we need to prove it here.) Note that we don't know if the limit exists or not. But $\limsup$ always exists.

Also, by the previous part, we have that when $x \in [x_0 - \delta, x_0 + \delta]$, $f(x) > M - \varepsilon$. Therefore

$$\left(\int_a^b (f(x))^n \right)^{\frac{1}{n}} dx \geq \left(\int_{x_0-\delta}^{x_0+\delta} (f(x))^n \, dx \right)^{\frac{1}{n}} > \left(\int_{x_0-\delta}^{x_0+\delta} (M - \varepsilon)^n \, dx \right)^{\frac{1}{n}} = (2\delta)^{\frac{1}{n}} (M - \varepsilon).$$

Hence

$$\liminf_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} \geq \liminf_{n\to\infty} (2\delta)^{\frac{1}{n}} (M - \varepsilon) = \lim_{n\to\infty} (2\delta)^{\frac{1}{n}} (M - \varepsilon) = M - \varepsilon.$$

Hence

$$M - \varepsilon \leq \liminf_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} \leq M.$$

Since $\varepsilon > 0$ is arbitrary chosen, we have

$$\liminf_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} = M \Longrightarrow \lim_{n\to\infty} \left(\int_a^b (f(x))^n \, dx \right)^{\frac{1}{n}} = M.$$

$\square$

3.6.5. *15S-2.*

Problem 58: 15S-2

If $f : \mathbb{R} \to \mathbb{R}$ is convex then show that

$$\int_a^b f(x)\ dx \leq \frac{b-a}{2}(f(a)+f(b)), \qquad \forall\ a < b.$$

Solution: Recall that a function $f : X \to Y$ is convex if and only if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall\ x, y \in X,\ \ \lambda \in (0,1).$$

Let $a, b \in \mathbb{R},\ \ a < b$ be given and consider for all $x \in (a,b)$, we can write

$$x = a + \lambda(b-a) = \lambda b + (1-\lambda)a,\ \ \lambda \in (0,1).$$

Also, we have

$$\lambda = \frac{x-a}{b-a}.$$

Since f is convex on (a,b), we have

$$\begin{aligned} f(x) = f(\lambda a + (1-\lambda)b) &\leq \lambda f(a) + (1-\lambda)f(b) \\ &= \frac{x-a}{b-a}f(a) + \left(1 - \frac{x-a}{b-a}\right)f(b) \\ &= ... = \frac{f(b)-f(a)}{b-a}(x-a) + f(a). \end{aligned}$$

(Note that the latter formula is just the secant line passing through $(a, f(a)), (b, f(b))$. And the definition of convexity is basically "function curve always lies below the secant line". We're just doing the reverse.) Hence we have

$$\begin{aligned} f(x) &\leq \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \\ \implies \int_a^b f(x)\ dx &\leq \int_a^b \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)\ dx \\ &= \left(\frac{f(b)-f(a)}{b-a}\right)\int_a^b (x-a)\ dx + f(a)\int_a^b 1\ dx \\ &= \left(\frac{f(b)-f(a)}{b-a}\right)\left[\frac{1}{2}(x-a)^2\right]_a^b + f(a)[x]_a^b \\ &= \frac{b-a}{2}(f(b)-f(a)) + (b-a)f(a) \\ &= \frac{b-a}{2}(f(a)+f(b)). \end{aligned}$$

Since the choice of a, b is arbitrary, we have

$$\int_a^b f(x)\ dx \leq \frac{b-a}{2}(f(a)+f(b)), \qquad \forall\ a < b.$$

$\square$

3.6.6. *97W-3(a)(b),96W-3(a).*

Problem 59: 97W-3(a)(b),96W-3(a)

(a) State a characterization for Riemann Integrability of f.
(b) If f is discontinous at only one point in $[a, b]$, prove that f is Riemann Integrable.

Solution:

(a) Though we don't need proofs, I will still provide them here. (I will write an extra handout for this.)

Theorem 11. *(Riemann Integral Characterization Theorem) If f is bounded on $[a, b]$. The Following Are Equivalent:*

(0) *f is Riemann integrable on $[a, b]$.*

(1) *(Darboux)*
Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. And let

$$U(P, f) = \sum_{i=1}^{n} \sup f|_{[x_{i-1},x_i]}\Delta x_i, \; L(P, f) = \sum_{i=1}^{n} \inf f|_{[x_{i-1},x_i]}\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$. Let

$$\overline{\int_a^b} f(x)\ dx = \inf_P U(P, f), \quad \underline{\int_a^b} f(x)\ dx = \sup_P L(P, f).$$

Then

$$\overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx.$$

In the case, we call

$$\int_a^b f(x)\ dx = \overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx.$$

(2) *For any $\varepsilon > 0$, there exists a partition P such that*

$$U(P, f) - L(P, f) < \varepsilon.$$

(3) *There exists a sequence of partition $\{P_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0.$$

In the case, we call

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

(4) *(Riemann)*
There exists $I \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\sum_{i=1}^{n} f(x_i^*)\Delta x_i - I\right| < \varepsilon \text{ when } \|P\| = \max_{i=1,\cdots,n} \Delta x_i < \delta$$

where $x_i^* \in [x_{i-1}, x_i]$, $i = 1, \cdots, n$ be the intermediate points. In the case, we call

$$\int_a^b f(x)\ dx = I.$$

Proof. We start with (0)$\Longleftrightarrow$ (1) as the definition.

(1) $\Longrightarrow$ (2) By definition of sup and inf, for any $\varepsilon > 0$, there exists P_1 such that

$$U(P, f) \le \overline{\int_a^b} f(x)\ dx + \frac{\varepsilon}{2} = \int_a^b f(x)\ dx + \frac{\varepsilon}{2}$$

and there exists P_2 such that

$$U(P, f) \ge \underline{\int_a^b} f(x)\ dx - \frac{\varepsilon}{2} = \int_a^b f(x)\ dx - \frac{\varepsilon}{2}.$$

Then let $P = P_1 \cup P_2$ then we have

$$\begin{aligned} U(P, f) &\le U(P_1, f) \le \int_a^b f(x)\ dx + \frac{\varepsilon}{2} \\ L(P, f) &\ge L(P_2, f) \le \int_a^b f(x)\ dx - \frac{\varepsilon}{2} \\ \Longrightarrow U(P, f) &- L(P, f) < \varepsilon. \end{aligned}$$

Note that we used the fact that

$$U(P_2, f) \le U(P_1, f),\ L(P_2, f) \ge L(P_1, f)$$

if $P_2 \subset P_1$ (P_2 is finer than P_1).

(2) $\Longrightarrow$ (1) Note that for any partition P, by definition, we always have

$$L(P, f) \le \underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx \le U(P, f).$$

Then we have

$$\overline{\int_a^b} f(x)\ dx - \underline{\int_a^b} f(x)\ dx \le U(P, f) - \underline{\int_a^b} f(x)\ dx \le U(P, f) - L(P, f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx.$$

(1) $\Longrightarrow$ (3) If f is integrable, we have $\int_a^b f(x)\ dx = \overline{\int_a^b} f(x)\ dx = \underline{\int_a^b} f(x)\ dx$.Let $n \in \mathbb{N}$. By the definition of inf and sup, there exist partition P_1 and P_2 such that

$$\begin{aligned} L(P_1, f) &> \int_a^b f(x)\ dx - \frac{1}{2n}, \\ U(P_1, f) &< \int_a^b f(x)\ dx + \frac{1}{2n}. \end{aligned}$$

Let $P_n = P_1 \cup P_2$, since P_n is finer than both P_1 and P_2, we have

$$\begin{aligned} L(P_n, f) \ge L(P_1, f) &> \int_a^b f(x)\ dx - \frac{1}{2n}, \\ U(P_n, f) \le U(P_n, f) &< \int_a^b f(x)\ dx + \frac{1}{2n}. \end{aligned}$$

Hence

$$0 \le U(P_n, f) - L(P_n, f) < \left(\int_a^b f(x)\ dx + \frac{1}{2n}\right) - \left(\int_a^b f(x)\ dx - \frac{1}{2n}\right)$$
$$= \frac{1}{n} \to 0 \text{ when } n \to \infty.$$

(2) $\Longrightarrow$ (3) If f is integrable, for every $n \in \mathbb{N}$, there exists a partition P_n such that

$$0 \le U(P_n, f) - L(P_n, f) < \frac{1}{n} \to 0 \text{ when } n \to \infty.$$

See the remark below for more details.

(3) $\Longrightarrow$ (1) Suppose the contrary that $\inf_P U(P, f) = \overline{\int_a^b} f(x)\ dx \ne \underline{\int_a^b} f(x) \ = \sup_P L(P, f)$. The by the definition of inf and sup, we have that for every partition, we have

$$0 < \overline{\int_a^b} f(x)\ dx - \underline{\int_a^b} f(x) \le U(P, f) - L(P, f)$$

which is a contradiction to $U(P_n, f) - L(P_n, f) \to 0$.

(3) $\Longrightarrow$ (2) Since $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$U(P_n, f) - L(P_n, f) < \varepsilon.$$

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) Since we need a number I in (4), I assume (1) $\Longrightarrow$ (2) here.
Since $I = \int_a^b f(x)\ dx = \overline{\int_a^b} f(x)\ dx = \inf_P U(P, f) = \sup_P L(P, f) = \underline{\int_a^b} f(x)\ dx$, for any given $\varepsilon > 0$, there exists a partition P_0 such that

$$U(P_0, f) - L(P_0, f) < \frac{\varepsilon}{2} \Longrightarrow I - \frac{\varepsilon}{2} < L(P_0, f)$$

since $I = \inf_P U(P, f) \le U(P_0, f)$. Now we let

$$P_0 = \{x_0^0 = a < x_1^0 < \cdots < x_n^0 = b\}.$$

If we let $\delta_0 = \min_{1\le i\le n} (x_i - x_{i-1})$ and construct another partition P such that

$$\|P\| < \delta_0.$$

Let

$$P = \{x_0 = a < x_1 < \cdots < x_m = b\}$$

(clearly $m > n$ by construction, and at each interval of P, $[x_{j-1}, x_j]$, at most one x_i^0 is contained in that interval $i = 0, \cdots, n,\ j = 0, \cdots, m$)

Now we let

$$[x_{j_k-1}, x_{j_k}) \text{ contains } x_k^0,\ k = 0, \cdots, n.$$

(We left the right end to be open so the choice of j_k is unique.)
(<u>Goal</u> We want $L(P, f) > I - \frac{\varepsilon}{2} \Longrightarrow$ Riemann sum $> L(P, f) > I - \frac{\varepsilon}{2}$ then we're almost done (upper sum is symmetric.) But now we only know $L(P_0 \cup P, f) \ge L(P_0, f)$. And that P will be our desired partition.)

Consider

$$
\begin{aligned}
&L(P_0 \cup P, f) - L(P, f) \\
=&\sum_{k=1}^{n} \left(\left(\inf f|_{[x_{j_k-1}, x_k^0]}(x_k^0 - x_{j_k-1}) + \inf f|_{[x_k^0, x_{j_k}]}(x_{j_k} - x_k^0) \right) - \inf f|_{[x_{j_k-1}, x_{j_k}]}(x_{j_k} - x_{j_k-1}) \right) \\
=&\sum_{k=1}^{n} \left(\left(\inf f|_{[x_{j_k-1}, x_k^0]} - \inf f|_{[x_{j_k-1}, x_{j_k}]} \right) (x_k^0 - x_{j_k-1}) + \left(\inf f|_{[x_k^0, x_{j_k}]} - \inf f|_{[x_{j_k-1}, x_{j_k}]} \right) (x_{j_k} - x_k^0) \right) \\
\leq&(M-m) \sum_{k=1}^{n} (x_{j_k} - x_{j_k-1}) \leq (M-m) \cdot n \cdot \|P_1\|.
\end{aligned}
$$

Here we use that the contribution of the difference only happens on $[x_{j_k-1}, x_{j_k}]$ since other intervals are the same in $P_0 \cup P$ and P. Also, since f is bounded, we assume that $m \leq f(x) \leq M$ and hence inf difference will smaller than $M - m$. Hence we have

$$0 \leq L(P_0 \cup P, f) - L(P, f) \leq n(M-m)\|P\|$$

and

$$L(P, f) \geq L(P_0 \cup P, f) - n(M-m)\|P\| > I - \frac{\varepsilon}{2} - n(M-m)\|P\|$$

since $L(P_0 \cup P, f) \geq L(P_0, f) > I - \frac{\varepsilon}{2}$. Recall that $\|P\| < \delta_0$. So if we choose

$$\delta = \min \left\{ \delta_0, \frac{\varepsilon}{2n(M-m)} \right\},$$

then if $\|P\| < \delta$, we have

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i \geq L(P_1, f) \geq I - \frac{\varepsilon}{2} - n(M-m)\|P_1\| > I - \varepsilon$$

where $x_i^* \in [x_{i-1}, x_i],\ i = 1, \cdots, n$.

Similarly, we can do the exact same thing for $U(P, f)$ from the same construction of partition P from the same P_0. And we will have

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i \leq U(P, f) < I + \varepsilon.$$

Here the choice of δ is exactly the same since the difference of sup is also bounded by $M - m$. Combining the results, we have

$$\left| \sum_{i=1}^{n} f(x_i^*)\Delta x_i - I \right| < \varepsilon.$$

Remark This proof looks difficult but the idea is quite straight-forward. (It looks bad since writing out all the points and things is messy.)

Idea is from (2), we create a special partition, and we need to construct a "finer" partition to be our desired partition. But to do the estimate, we can only do upper sum and lower sum (and the Riemann sum will always lie between them). To simplify the estimate, we construct the partition with only one point inside, and this do the trick since the contributions are all from the same interval. See the handout for more details and more characterization.

(4) $\implies$ (2) For a fixed $\varepsilon > 0$. Choose $s_i, t_i \in [x_{i-1}, x_i],\ i = 1, \cdots n$ such that

$$f(s_i) \geq \sup f|_{[x_{i-1},x_i]} - \frac{\varepsilon}{2(b-a)} \text{ and } f(t_i) \leq \inf f|_{[x_{i-1},x_i]} + \frac{\varepsilon}{2(b-a)}.$$

Also, by assumption, we have there exists P and $\delta > 0$ such that $\|P\| < \delta$ and

$$\left|\sum_{i=1}^{n} f(s_i)\Delta x_i - I\right| < \frac{\varepsilon}{4}, \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - I\right| < \frac{\varepsilon}{4}.$$

Then we have

$$\begin{aligned} U(P, f) &= \sum_{i=1}^{n} \sup f|_{[x_{i-1},x_i]}\Delta x_i \leq \sum_{i=1}^{n} \left(f(t_i) + \frac{\varepsilon}{4(b-a)}\right)\Delta x_i \\ &= \sum_{i=1}^{n} f(s_i)\Delta x_i + \frac{\varepsilon}{4(b-a)} \cdot (b-a) = \sum_{i=1}^{n} f(s_i)\Delta x_i + \frac{\varepsilon}{4} \end{aligned}$$

where $\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a$. And similarly,

$$L(P, f) \geq \sum_{i=1}^{n} f(t_i)\Delta x_i - \frac{\varepsilon}{4}.$$

Hence we have

$$\begin{aligned} U(P, f) - L(P, f) &< \sum_{i=1}^{n} f(s_i)\Delta x_i - \sum_{i=1}^{n} f(t_i)\Delta x_i + \frac{\varepsilon}{2} \\ &\leq \left|\sum_{i=1}^{n} f(s_i)\Delta x_i - \sum_{i=1}^{n} f(t_i)\Delta x_i\right| + \frac{\varepsilon}{2} \\ &= \left|\left(\sum_{i=1}^{n} f(s_i)\Delta x_i - I\right) - \left(\sum_{i=1}^{n} f(t_i)\Delta x_i - I\right)\right| + \frac{\varepsilon}{2} \\ &\leq \left|\sum_{i=1}^{n} f(s_i)\Delta x_i - I\right| + \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - I\right| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Remark

(i) We can see that (2) is the most useful one. And we usually don't use (4).

(ii) In (3), we can acutally write

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f).$$

But in general, for sequences $\{a_n\}_{n\in\mathbb{N}},\ \{b_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}(a_n - b_n) = 0 \not\Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

We can just take $a_n = b_n$ with no limit as an easy counter-example.

So we actually need to prove that $\lim_{n\infty} L(P_n, f)$ and $\lim_{n\infty} U(P_n, f)$ exist and equal.

If $U(P_n, f) - L(P_n, f) \to 0$ (so f is integrable), then

$$0 \le U(P_n, f) - \int_a^b f(x)\, dx \le U(P_n, f) - L(P_n, f),$$

$$0 \le \int_a^b f(x)\, dx - L(P_n, f) \le U(P_n, f) - L(P_n, f).$$

Hence by the squeeze lemma, we have

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \int_a^b f(x)\, dx.$$

$\square$

(b) We prove a more general result: If f is discontinuous at countably/finitely many points, then f is Riemann integrable.

Proof. We will prove the characterization (2).
Let $|f| < M$ (Note that f should be bounded. If f is not bounded, f is not Riemann integrable. See the handout for the proof) and $\varepsilon > 0$ be given and f is discontinuous at the points (besides endpoints)

$$d_1 < d_2 < \cdots < d_n < \cdots .$$

We choose $l_n, u_n,\ n \in \mathbb{N}$ such that for each n,

$$|u_n - l_n| < \frac{\varepsilon}{4M \cdot 2^n} \text{ and } d_n \in (u_n, l_n).$$

If d_i is an endpoint, we just choose one of the l_i, u_i. Then we have (the estimate of intervals including the discontinuities)

$$\sum_{n=1}^{\infty} \left(\sup f|_{[l_n,u_n]} - \inf f|_{[l_n,u_n]}\right)(u_n - l_n) \le \sum_{n=1}^{\infty}(2M)(u_n - l_n) \le \sum_{n=1}^{\infty} 2M \cdot \frac{\varepsilon}{4M \cdot 2^n} = \frac{\varepsilon}{2}$$

where we use that $\sup f|_{[l_n,u_n]} - \inf f|_{[l_n,u_n]} \le \left|\sup f|_{[l_n,u_n]}\right| + \left|\inf f|_{[l_n,u_n]}\right| \le 2M$. Now consider

$$I = [a, b] \setminus \left(\bigcup_{n\in\mathbb{N}} (l_n, u_n) \cup E\right).$$

where E is the union of intervals containing endpoints. It's clear that I is a union of closed intervals and on each interval f is continuous and hence integrable. Write

$$I = \bigcup_{j\in\mathbb{N}} I_j$$

hence for each interval I_j, there exists partition P_j such that

$$U(P_j, f|_{I_j}) - L(P_j, f|_{I_j}) < \frac{\varepsilon}{2 \cdot 2^j}.$$

Now we let

$$P = \bigcup_{j\in\mathbb{N}} P_J \cup \{l_n, u_n \mid i \in \mathbb{N}\}.$$

Hence we have

$$\begin{aligned}
U(P,f)-L(P,f) &= \sum_{j=1}^{\infty}\big(U(P_j,f|_{I_j})-L(P_j,f|_{I_j})\big)+\sum_{n=1}^{\infty}\big(\sup f|_{[l_n,u_n]}-\inf f|_{[l_n,u_n]}\big)(u_n-l_n)\\
&< \sum_{j=1}^{\infty}\frac{\varepsilon}{2\cdot 2^j}+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}\sum_{j=1}^{\infty}\frac{1}{2^j}+\frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}$$

Hence f is integrable by the characterization (2). □

□

3.6.7. *Will be updated.*

20W-5(a),05W-3(c)
18S-4
13S-4
13W-1
12S-6

04S-3(a)
00W-3
97W-3(a)(b),96W-3(a)
96S-5(b):FTC

3.7. **Sequence of Functions and Power Series.**

3.7.1. *24S2-2.*

Problem 60: 24S2-2

(a) Derive the Taylor series in powers of x for the function $f(x) = \dfrac{1}{1-x}$.

(b) Prove directly that the series from part (a) converges uniformly to f on any interval $[-r, r]$, where $r \in (0, 1)$.

(c) Explain why the result in part (b) implies that the series converges pointwise to f on the interval $(-1, 1)$.

(d) Prove that the series diverges at the endpoints, $x = \pm 1$.

Solution:

(a)

(b)

(c)

(d)

$\square$

3.7.2. *24S1-3.*

Problem 61: 24S1-3

Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n + n}{n^2}$$

(a) Converges uniformly in $[0, 1]$.

(b) Does not converge absolutely at any $x \in [0, 1]$.

Reference: Baby Rudin, Chapter 7, Exercise 6.

Solution:

(a) Let $S_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n + n}{n^2}$. Note that when $x \in [0, 1] \implies |x| \le 1$. Consider the partial sum for any $k > m,\ k, m \in \mathbb{N}$, we have

$$\begin{aligned}
|S_k(x) - S_m(x)| = \left| \sum_{n=m}^{k} (-1)^n \frac{x^n + n}{n^2} \right| &\le \left| \sum_{n=m}^{k} (-1)^n \frac{x^n}{n^2} \right| + \left| \sum_{n=m}^{k} (-1)^n \frac{1}{n} \right| \\
&\le 1^2 \sum_{n=m}^{k} \frac{1}{n^2} + \left| \sum_{n=m}^{k} (-1)^n \frac{1}{n} \right|.
\end{aligned}$$

Note that both $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ both converge by the p-series test (or integral test if you like to use that) and alternating series test. Hence

$$\exists\ N_1 \in \mathbb{N} \text{ such that } \left| \sum_{n=m}^{k} \frac{1}{n^2} \right| < \frac{\varepsilon}{2},$$

$$\exists\ N_2 \in \mathbb{N} \text{ such that } \left| \sum_{n=m}^{k} (-1)^n \frac{1}{n} \right| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$, then when $m, k \geq N$, we have

$$|S_k(x) - S_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence by the Cauchy criterion, $S_n(x)$ converges uniformly.

(b) Note that

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^n + n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{x^n + n}{n^2} = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n}.$$

And clearly $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the series diverge and hence doesn't converge absolutely.

$\square$

3.7.3. *24S1-5.*

Problem 62: 24S1-5

Recall that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions defined on the same metric space (K, d) is said to be equicontinuous at a point $x_0 \in K$, if for all $\epsilon > 0$, there exists a $\delta > 0$ so that $d(f_n(x_0), f_n(x)) < \epsilon$ for all $n \in \mathbb{N}$ whenever $d(x_0, x) < \delta$.
Let $\{f_n\}$ be equicontinuous at every $x \in K$, and pointwise convergent on a compact set K. Prove that $\{f_n\}$ is uniformly continuous on K.

Reference: Baby Rudin, Chapter 7, Exercise 16.

Solution: I think this one has typos. Since equicontinuity clearly implies continuity and continuous function on compact set is uniformly continuous. (It would be weird if this is actually asking continuous function on compact set is uniformly continuous.) There are several different results under the same setting:

(1) If $f_n \to f$ pointwise, then f is continuous (so it's uniformly continuous).
(2) If $f_n \to f$, then $f_n \to f$ uniformly. (I believe it's this since it's in the Rudin Exercise.)

We will proof both. Note that we're given pointwise equicontinuous. (Be careful! It's different from the definition in the Rudin! The δ here may depend on $x_0 \in K$. So we need to prove we can imply the rudin's version that ε only depends on ε.) So we claim:

Claim 13. If $f_n : X \to Y$ and X is compact and $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous on K, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$

$$d_Y(f_n(x), f_n(y)) < \varepsilon \text{ when } d_X(x, y) < \delta.$$

Proof of claim. This is exactly how we prove continuous functions are uniformly continuous on compact sets. (I think you need to prove it here since they give you different definition.)

Given $\varepsilon > 0$, $x \in K$, since f_n is equicontinuous, there exists $\delta_x > 0$ such that for all $n \in \mathbb{N}$

$$d_Y(f_n(x), f_n(y)) < \frac{\varepsilon}{2} \iff f_n(y) \in B_{\frac{\varepsilon}{2}}(f_n(x)) \text{ when } d_X(y, x) < \delta \iff y \in B_{\delta_x}(x).$$

Let $\{B_{\frac{\delta_x}{2}}(x) \mid x \in X\}$ be an open covering of X. Since X is compact, we have $n \in \mathbb{N}$ such that

$$X \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i).$$

Now, we choose

$$\delta = \min_{i=1,\cdots,n} \left\{ \frac{\delta_{x_i}}{2} \right\},$$

then if $x, y \in X$ and $d_X(x, y) < \delta$. Then since $X \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$, there exists some i such that $x \in B_{\frac{\delta_{x_i}}{2}}(x_i)$. And also, we have

$$d(x, y) < \delta < \frac{\delta_{x_i}}{2} \implies d(y, x_i) < d(y, x) + d(x, x_i) < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}.$$

Since f_n is equicontinuous at x_i and $x, y \in B_{\delta_{x_i}}(x_i)$, we have $f_n(x), f_n(y) \in B_{\frac{\varepsilon}{2}}(f_n(x_i))$ and hence for all $n \in \mathbb{N}$

$$d(f_n(x), f_n(y)) < d(f_n(x), f_n(x_i)) + d(f_n(x_i), f_n(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we finished the proof.

Now we prove the two possible problems.

(1) Let $x \in K,\ \varepsilon > 0$ be arbitrarily chosen. Since $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous, by the above Claim, there exists $\delta > 0$ such that for all $x, y \in K$,when $d(x, y) < \delta$, we have

$$d(f_n(y), f_n(x)) < \varepsilon.$$

Now let $x, y \in K$ with $d(x, y) < \delta$. Since $f_n \to f$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$d(f(x), f_n(x)) < \frac{\varepsilon}{3} \text{ when } n \geq N_1 \text{ and } d(f(y), f_n(y)) < \frac{\varepsilon}{3} \text{ when } n \geq N_2.$$

Hence when $n \geq \max\{N_1, N_2\}$, we have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence f is uniformly continuous.

(2) Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous, by the above Claim, there exists $\delta > 0$ such that for all $x, y \in K$,when $d(x, y) < \delta$, we have

$$d(f_n(y), f_n(x)) < \frac{\varepsilon}{3}.$$

Note that $\{B_\delta(x) \mid x \in K\}$ is an open covering for K. Since K is compact, there exists $x_1, \cdots, x_k,\ k \in \mathbb{N}$ such that

$$K \subseteq B_\delta(x_1) \cup \cdots B_\delta(x_k).$$

Since $f_n \to f$ pointwise on each $x_i,\ i = 1, \cdots, k$, there exists $N_i \in \mathbb{N}$ such that when $m, n \geq N_i$,

$$d(f_n(x_i), f_m(x_i)) < \frac{\varepsilon}{3}.$$

So if $n \geq \max\{N_1, \cdots, N_k\}$, when $d(x, x_i) < \delta$, we have

$$d(f_n(x_i), f_m(x_i)) < \frac{\varepsilon}{3}\ \forall\ i = 1, \cdots, k.$$

Now, for any given $x \in K \subseteq \bigcup_{i=1}^{k} B_\delta(x_i) \Longrightarrow x \in B_\delta(x_i)$ for some $i \in \{1, \cdots, k\}$. Hence $d(x, x_i) < \delta$ and hence

$$d(f_n(x), f_n(x_i)) < \frac{\varepsilon}{3}.$$

Finally, for $n \geq \max\{N_1, \cdots, N_k\}$, we have

$$\begin{aligned} d(f_n(x), f_m(x)) &\leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f_m(x_i)) + d(f_,(x_i), f_m(x)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence by the Cauchy criterion, $f_n \to f$ uniformly.

$\square$

3.7.4. *24W-7(i).*

Problem 63: 24W-7(i)

Prove or give a counterexample. You are allowed to quote known theorems in a proof.

(i) Suppose that $f_n : [a,b] \to \mathbb{R}$ is continuous for all $n = 1, 2, \cdots$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$, where $f : [a,b] \to \mathbb{R}$ is another continuous function. Then $f_n \to f$ uniformly on $[a,b]$.

Solution:

(i) This is false.

There are plenty of counter-examples ((c) in here is a good one). Another counter-example is: Construct a sequence of function $f_n : [0,1] \to \mathbb{R}$ where f_n represents an isosceles triangle with a vertex at the origin of base $\dfrac{1}{n}$ and height $2n$, that is

$$f_n(x) = \begin{cases} 4n^2 x & , 0 \le x \le \dfrac{1}{2n}. \\ -4n^2\left(x - \dfrac{1}{n}\right) & , \dfrac{1}{2n} \le x \le \dfrac{1}{n}. \\ 0 & , \dfrac{1}{n} \le x \le 1. \end{cases}$$

Clearly $f_n \to f = 0$ pointwise since when $n > \dfrac{1}{x} \implies x > \dfrac{1}{n}$ then $|f_n(x) - f(x)| = 0 < \varepsilon$ and $f = 0$ is clearly continuous. However, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 2n \not\to 0$. Hence $f_n \to f$ not uniformly. (This Theorem is really useful.)

(Note that the idea is just contructing a function with "spikes".)

$\square$

3.7.5. *21S-4.*

Problem 64: 21S-4

(a) Prove that a uniform limit of continuous functions is continuous.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}.$$

Prove that f is differentiable on $\mathbb{R}$ and that f' is continuous on $\mathbb{R}$.

Solution:

(a) See (a) here.

(b) Recall that (Baby Rudin, Theorem, 7.17) if $f_n : [a,b] \to \mathbb{R}$ is differentiable and $\{f_n(x_0)\}$ converges for some point $x_0 \in [a,b]$ and $\{f_n'\}$ converges uniformly on $[a,b]$ then $f_n \to f$ uniformly and

$$f'(x) = \lim_{n\to\infty} f_n'(x), \ \forall\, x \in [a,b].$$

Hence we have if $\sum_{n=1} f_n(x)$ converges for some $x_0 \in [a,b]$ and $\sum_{n=1} f_n'(x)$ converges uniformly on $[a,b]$, then

$$\sum_{n=1} f_n'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right).$$

Now we let

$$f_n(x) = \frac{\sin(nx)}{n^3} \implies f_n'(x) = \frac{\cos(nx)}{n^2}.$$

Note that

$$|f_n'(x)| = \left|\frac{\cos(nx)}{n^2}\right| \leq \frac{1}{n^2}$$

and clearly $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges hence $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly by the Weierstrass M-test. Hence we have f is differentiable and since $f_n'(x)$ is continuous, $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ is continuous.

□

3.7.6. *21W-3(a).*

Problem 65: 21W-3(a)

(a) Suppose $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly to a function $f : [0,1] \to \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{1} f_n(x)\ dx = \int_0^1 f(x)\ dx.$$

Solution:

(a) We first claim that

Claim 14. Suppose $f_n : E \to \mathbb{R}$. If $f_n \to f$ uniformly and f_n is bounded, then f is bounded and f_n is uniformly bounded.

Proof of claim. Since f_n is bounded, for all $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f_n| \leq M_n$. Also, since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that when $n \geq N$, for all $x \in E$, $|f_n(x) - f(x)| < 1$. Hence we have

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N.$$

Hence f is bounded. Furthermore, let

$$M = \max\{M_1, \cdots, M_{N-1}, 1 + M_N\}.$$

Then when $n \geq N$, we have

$$|f_n(x)| = |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + 1 + M_n < M + 2.$$

Hence f_n is unformly bounded.

Now, let $\varepsilon > 0$ be given. since f_n is continuous on a compact set $[0,1]$, f_n is uniformly continuous and thus f_n is bounded on $[0,1]$. By the above Claim, $|f_n| \le M$ for some $M > 0$. Let $N_1 \in \mathbb{N}$ such that

$$\frac{M}{n} < \frac{\varepsilon}{2}.$$

Also, since $f_n \to f$ uniformly, there exists $N_2 \in \mathbb{N}$ such that when $n \ge N_2$, for all $x \in [0,1]$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then when $n \ge N$, we have

$$\begin{aligned}
\left| \int_{\frac{1}{n}}^1 f_n(x)\ dx - \int_0^1 f(x)\ dx \right| &\le \left| \int_{\frac{1}{n}}^1 f_n(x)\ dx - \int_0^1 f_n(x)\ dx \right| + \left| \int_0^1 f_n(x)\ dx - \int_0^1 f(x)\ dx \right| \\
&= \left| \int_0^{\frac{1}{n}} f_n(x)\ dx dx \right| + \left| \int_0^1 (f_n(x) - f(x))\ dx \right| \\
&\le \int_0^{\frac{1}{n}} |f_n(x)|\ dx + \int_0^1 |f_n(x) - f(x)|\ dx \\
&< \frac{M}{n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen,

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f_n(x)\ dx = \int_0^1 f(x)\ dx.$$

□

3.7.7. *20W-4,19W-5.*

Problem 66: 20W-4,19W-5

Let $f_n : [0,1] \to [0,1]$, $n \ge 1$, be differentiable functions on $[0,1]$ satisfying

$$\begin{aligned}
f_n(0) &= 1 && \forall\ n \in \mathbb{N}, \\
f_n'(x) &\le 0 && \forall\ n \in \mathbb{N},\ \forall\ x \in [0,1], \\
\lim_{n\to\infty} f_n(x) &= 0 && \forall\ x \in (0,1].
\end{aligned}$$

(i) (20W-4(i)=19W-5(a))
Show that $\{f_n\}$ is uniformly convergent in $[\delta, 1]$ for all $\delta > 0$.

(ii) (20W-4(ii)=19W-5(b))
Show that $\{f_n\}$ is not uniformly convergent in $[0,1]$.

(iii) (19W-5(c))
Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = 0.$$

Solution:

(i) Since $f_n'(x) \le 0 \ \forall\ x \in [0,1],\ n \in \mathbb{N}$, we take $f(x) = 0$. We have for any $\delta > 0$,

$$\sup_{x\in[\delta,1]} |f_n(x) - f(x)| = \sup_{x\in[\delta,1]} |f_n(x)| = |f_n(\delta)| \to 0$$

when $n \to \infty$ since $\lim_{n\to\infty} f_n(x) = 0\ \forall\ x \in (0,1]$. By the Theorem, we have $\{f_n\}$ is uniformly convergent in $[\delta, 1]$ for all $\delta > 0$.

(ii) Note that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |f_n(x)| = |f_n(0)| = 1 \nrightarrow 0.$$

By the Theorem, we have $\{f_n\}$ is is not uniformly convergent in $[0,1]$.

(iii) Let $0 < \delta < 1$ be given. By (i), we have $f_n \to 0$ uniformly on $[\delta, 1]$. We have

$$\lim_{n\to\infty} \int_\delta^1 f_n(x)\ dx = \int_\delta^1 \lim_{n\to\infty} f_n(x)\ dx = \int_\delta^1 0\ dx = 0.$$

Also, note that $f_n(0) = 1$ for all $n \in \mathbb{N}$ and f_n are decreasing, we have $|f_n(x)| = f_n(x) \le 1$ for all $x \in [0,1]$ since $f_n : [0,1] \to [0,1]$. So we have

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 f_n(x)\ dx &= \lim_{n\to\infty} \int_0^\delta f_n(x)\ dx + \lim_{n\to\infty} \int_\delta^1 f_n(x)\ dx = \lim_{n\to\infty} \int_0^\delta f_n(x)\ dx \\ &\le \lim_{n\to\infty} \delta \cdot 1 = \delta.\end{aligned}$$

Since $\delta > 0$ is arbitrary chosen, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = 0.$$

□

3.7.8. *16S-6.*

Problem 67: 16S-6

Let $S_n(x) := \sum_{k=1}^{n} \frac{e^{ikx}}{k^3}$.

(a) Prove that for all real x, $s_n(x)$ converges to a differentiable function $s(x)$ on $\mathbb{R}$.

(b) Prove that the infinite series

$$\sum_{k=1}^{\infty} \frac{e^{ikx}}{k^3}$$

converges for all real x whenever it is not an integer multiple of 2π.

Solution:

(a)

(b)

□

3.7.9. *15W-4.*

Problem 68: 15W-4

Consider the function $f : [0,1) \to \mathbb{R}$, given by

$$f(x) = \sum_{n=0}^{\infty} (n+1)x^n, \quad \text{for all } x \in [0,1).$$

a) Prove that the power series that defines f converges uniformly on any interval $[0,r] \subset [0,1)$, and justify that f is continuous on $[0,1)$.
b) Justify that for any $x \in [0,1)$,

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \int_0^x (n+1)t^n\, dt, \quad \text{and prove that } f(x) = \frac{1}{(1-x)^2}.$$

Solution:

a) (This is essentially proving Baby Rudin, Theorem 8.1.)
Consider

$$\lim_{n\to\infty} \sqrt[n]{|n+1|} = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n+1)} = e^{\lim_{n\to\infty} \frac{\ln(n+1)}{n}} \overset{L'H}{=} e^{\lim_{n\to\infty} \frac{1}{n+1}} = 1.$$

Hence the radius of convergence

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|n+1|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n+1}} = 1.$$

Let $0 < r < 1$ be given and choose $s \in \mathbb{R}$ such that $r < s < R = 1$. Hence we have

$$\frac{1}{s} > \frac{1}{R} = 1 = \limsup_{n\to\infty} \sqrt[n]{n+1}.$$

Then there exists $N \in \mathbb{N}$ such that $\dfrac{1}{s} > \sqrt[n]{n+1}$ when $n \geq N_1$. Then we have

$$\left|\sum_{k=n}^{\infty}(k+1)x^k\right| \leq \sum_{k=n}^{\infty} |k+1|\,|x|^k = \sum_{k=n}^{\infty} \left(\sqrt[k]{k+1}\right)^k |x|^k < \sum_{k=n}^{\infty} \frac{1}{s^k} r^k = \frac{\left(\frac{r}{s}\right)^n}{1-\frac{r}{s}}.$$

Here $\left|\dfrac{r}{s}\right| < 1$, so the geometric series converges. Also, choose

$$N_2 = \left\lfloor \frac{\ln\left(\varepsilon\left(1-\frac{r}{s}\right)\right)}{\ln\left(\frac{r}{s}\right)} \right\rfloor + 1.$$

Then when $n \geq \max\{N_1, N_2\}$, we have (believe my calculation)

$$\left|\sum_{k=n}^{\infty}(k+1)x^k\right| < \varepsilon.$$

Hence $\sum_{n=0}^{\infty}(n+1)x^n$ converges uniformly on $[0,r]$. (And since r is arbitrary chosen, $\sum_{n=0}^{\infty}(n+1)x^n$ converges absolutely on $[0,1)$.

Finally, since $\sum_{n=0}^{\infty}(n+1)x^n$ converges to $f(x)$ uniformly, by the term by term differentiation theorem, we know that $f'(x)$ exists (I don't think you need to prove this. This is even longer). Hence $f(x)$ is continuous.

b) Note that

$$\int_0^x (n+1)t^n\ dt = \left((n+1)\frac{t^{n+1}}{n+1}\right)\Big|_0^x = x^{n+1}.$$

Consider

$$\sum_{n=0}^{\infty}\int_0^x (n+1)t^n\ dt = \sum_{n=0}^{\infty} x^{n+1},$$

then we have

$$\frac{1}{R} = \limsup_{n\to\infty} 1 = 1$$

which is the same radius of convergence as $\sum_{n=0}^{\infty}(n+1)x^n$. Hence when $x \in [0,1)$, we have term by term differentiation that

$$g(x) = \sum_{n=0}^{\infty} x^{n+1} \Longrightarrow g'(x) = \sum_{n=0}^{\infty}(n+1)x^n = f(x),\ \forall\ x \in [0,1).$$

Then by the Fundamental Theorem of Calculus, for $x \in [0,1)$, we have

$$\int_0^x f(t)\ dt = g(x) - g(0) = \sum_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x} \Longrightarrow f(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}.$$

□

3.7.10. *15W-5.*

Problem 69: 15W-5

Use Weierstrass' approximation theorem to prove that there exists a sequence of polynomials with real coefficients $\{P_n\}_{n=1}^{\infty}$ such that $P_n(0) = 0$, for all $n \in \mathbb{N}$ and $P_n(x)$ converges uniformly to $|\sin x|$ on $[-\pi, \pi]$.

Solution:

□

3.7.11. *14W-7(b).*

Problem 70: 14W-7(b)

(b) Prove that

$$\sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}\right)$$

converges pointwise on $[0,1]$, but the sequence of partial sums does not converge uniformly on $[0,1]$.

Solution:

(b) Note that $\sum_{n=1}^{\infty} x^n$ converges when $x \in [0,1)$. Then when $x \in [0,1)$,

$$\sum_{n=0}^{m} \frac{x^{2n+1}}{2n+1} \le \sum_{n=0} x^{2m+1}, \ \sum_{n=0}^{m} \frac{x^{n+1}}{2n+2} \le \sum_{n=0}^{m+1} x^n.$$

And they all converge on $x \in [0,1)$. When $x = 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.\end{aligned}$$

Clearly, $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n}$ is decreasing. So by the alternating series test, the series converges. Combining the results,

$$\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right)$$

converges pointwise on $[0,1]$.

For the uniform convergence part, let

$$S_N(x) = \sum_{n=0}^{N} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right), \ S(x) = \sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right).$$

If $S_N \to S$ uniformly, then S is continuous. (I don't think you need to prove it here.) Now, consider when $x \in (0,1]$. Observe that

$$\frac{x^{2n+1}}{2n+1} = \int_0^x u^{2n}\, du, \ \frac{x^{n+1}}{2n+2} = \int_0^x \frac{u^n}{2}\, du.$$

Also, we have

$$\sum_{n=0}^{\infty} u^{2n} = \frac{1}{1-u^2}, \ \sum_{n=0}^{\infty} \frac{u^n}{2} = \frac{1}{2(1-u)}$$

both converge uniformly on $[0,x]$. Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right) &= \sum_{n=0}^{\infty} \int_0^x \left(u^{2n} - \frac{u^n}{2} \right) du \\ &= \int_0^x \left(\sum_{n=0}^{\infty} u^{2n} - \sum_{n=0}^{\infty} \frac{u^n}{2} \right) du \\ &= \int_0^x \frac{1}{1-u^2}\, du - \int_0^x \frac{1}{2(1-u)}\, du = \ldots \\ &= \frac{1}{2}(\log(1+x) - \log(1-x)) - \frac{1}{2}\log(1+x) \\ &= \frac{1}{2}\log(1+x).\end{aligned}$$

And when $x = 1$, $S(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. Recall that since (I think you can directly cite this series.)

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \text{ converges on } [0,1)$$

$$\implies \int_0^x \frac{1}{1+u}\, du = \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

Hence by the Abel Limit Theorem (Baby Rudin, Theorem 8.2), we have

$$\lim_{x\to 1} \log(1+x) = \log 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

Finally, we have

$$S(x) = \begin{cases} \frac{1}{2}\log(1+x), & x \in [0,1) \\ \log 2, & x = 1 \end{cases}$$

which is not continuous at $x = 1$. Hence $S_N \to S$ not uniformly.

□

3.7.12. *13S-1(c).*

Problem 71: 13S-1(c)

Solution:

□

3.7.13. *13W-7.*

Problem 72: 13W-7

Solution:

□

3.7.14. *05W-4(b).*

Problem 73: 05W-4(b)

(b) Suppose $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions which converges uniformly to a function $f : [a,b] \to \mathbb{R}$. Show either by constructing an example or by an existence argument that f need not be differentiable on $[a,b]$.

Solution:

(b) A counter-example would be:

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Clearly $\lim_{n\to\infty} f_n(x) = \sqrt{x^2} = |x|$. pointwise and $f_n(x)$ are all differentiable but $f(x)$ is not differentiable. It suffices to prove the convergence is uniform. Note that

$$\left(\sqrt{x^2}+\sqrt{\frac{1}{n}}\right)^2 - \left(x^2+\frac{1}{n}\right) = x^2 + 2\sqrt{\frac{x^2}{n}} + \frac{1}{n} - x^2 - \frac{1}{n} = 2\sqrt{\frac{x^2}{n}} \geq 0$$

Hence

$$f_n(x) = \sqrt{x^2+\frac{1}{n}} \leq \sqrt{x^2} + \sqrt{\frac{1}{n}}.$$

So we have

$$|f_n(x) - f(x)| = \left|\sqrt{x^2+\frac{1}{n}} - \sqrt{x^2}\right| \leq \sqrt{\frac{1}{n}}.$$

For any $\varepsilon > 0$, choose $N > \dfrac{1}{\varepsilon^2}$, then when $n \geq N$, for any $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| \leq \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \varepsilon.$$

Hence $f_n \to f$ uniformly.

□

3.7.15. *97W-4.*

Problem 74: 97W-4

Solution:

□

3.7.16. *96W-4(b).*

Problem 75: 96W-4(b)

(d) (96W-4(b))
Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions which converge uniformly on $[a, b]$. Define

$$F_n(x) = \int_a^x f_n(t)\ dt, \qquad \forall\ x \in [a, b],\ n = 1, 2, \cdots .$$

Prove that F_n converges uniformly on $[a, b]$.

Solution: Since f_n converges to a function uniformly, by the Cauchy criterion, let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that when $m, n \geq N$, for all $x \in [a, b]$, we have

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{b-a}.$$

Now consider

$$\begin{aligned}
|F_n(x) - F_m(x)| &= \left| \int_a^x f_n(t)\ dt - \int_a^x f_m(t)\ dt \right| = \left| \int_a^x (f_n(t) - f_m(t))\ dt \right| \\
&\leq \int_a^x |f_n(t) - f_m(t)|\ dt \leq \int_a^x \frac{\varepsilon}{b-a}\ dt = \left(\frac{\varepsilon t}{b-a} \right) \Big|_a^x \\
&= \frac{x-a}{b-a}\varepsilon \leq \epsilon
\end{aligned}$$

for all $x \in [a, b]$. Hence by the Cauchy Criterion again, F_n converges uniformly on $[a, b]$.

(The good thing about the Cauchy criterion is that we don't need to know what function f_n converges to.)

$\square$

3.7.17. *Will be updated.*

13S-1(c)

13W-7

3.8. **Multivariable Calculus.**

3.8.1. *24S2-8.*

Problem 76: 24S2-8

Let $v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the (time-dependent) vector field defined by

$$v(x,t) = (x_1 e^t + x_2, x_2 e^t - x_1),$$

let C denote the unit circle, and define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \int_C \vec{v} \cdot \vec{n}\, ds,$$

where $\vec{n} = \vec{n}(x)$ denotes the outward pointing unit normal vector at $x \in C$. Calculate $\phi'(t)$ and justify your work.

Solution:

$\square$

3.8.2. *24W-7(ii).*

Problem 77: 24W-7(ii)

Prove or give a counterexample. You are allowed to quote known theorems in a proof.

(ii) Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = (x^2 y, -xy^2)$ and let Ω be any bounded domain (connected open set) in $\mathbb{R}^2$ such that $\partial\Omega$ is smooth. Then $\displaystyle\int_{\partial\Omega} F \cdot n = 0$, where $\partial\Omega$ is the boundary of Ω and $n = n(x,y)$ denotes the outward pointing unit normal to $\partial\Omega$ at $(x,y) \in \partial\Omega$.

Solution:

(ii) Let $\Omega = S \setminus C$ where $C = \{(x,y) \mid x^2 + y^2 = 1\}$ where S be a simply connected domain containing the unit circle with smooth domain. Then Ω is not simply connected (since not all interior region of closed curves is contained in Ω). Since S is simply connected, we can use the Divergence Theorem (or Green's Theorem) that

$$\begin{aligned}
&\int_{\partial S} F \cdot ndS = \int_{\partial\Omega \cup \partial C} F \cdot ndS = \iint_R \mathrm{div} F\, dA = 0 \\
\implies &\int_{\partial\Omega} F \cdot ndS = -\int_{\partial C} F \cdot ndS \\
&\qquad = -\int_{\partial C} (x^2 y, -xy^2) \cdot (x,y) dS \\
&\qquad = -\int_0^{2\pi} (\cos^3\theta \sin\theta - \cos\theta \sin^3\theta)\, d\theta
\end{aligned}$$

$\square$

3.8.3. *23S-3.*

Problem 78: 23S-3

Let $\mathbf{F} = x\mathbf{i}$, D the region above the cone $z = \sqrt{x^2+y^2}$ but below $z = 1$, S boundary of D, and $\mathbf{n}$ the outward pointing unit normal to S

(a) Compute $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ directly (without using the divergence theorem).

(b) Compute $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ directly using the divergence theorem. You can do this part without any integration.

Solution:

(a) Recall that (Cal 3) the general parametrization of a surface S

$$(u,v)\in R,\ r(u,v) = (x(u,v), y(u,v), z(u,z)),\ dS = |r_u\times r_v|dA,\ \mathbf{n} = \frac{r_u\times r_v}{|r_u\times r_v|}$$

and therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\frac{r_u\times r_v}{|r_u\times r_v|}|r_u\times r_v|\,dA = \iint_A \mathbf{F}\cdot(r_u\times r_v)\,dA.$$

In this case, we take $r(x,y) = (x, y, \sqrt{x^2+y^2})$, $(x,y)\in R$ where we need $z = \sqrt{x^2+y^2}\le 1 \implies x^2+y^2\le 1$ hence we take $R = \{(x,y)\mid x^2+y^2\le 1\}$. So we calculate

$$\begin{aligned}
r_x\times r_y &= \left(1,0,\frac{\partial}{\partial x}(\sqrt{x^2+y^2})\right)\times\left(0,1,\frac{\partial}{\partial y}(\sqrt{x^2+y^2})\right)\\
&= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}}\\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}}\end{vmatrix} = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)
\end{aligned}$$

Note that since the z component of the $r_x\times r_y$ is positive, meaning that the normal we found is upward, which is inward normal to the surface (it's easily checked by the graph, or you can use different parametrization to get outward normal directly), so we need to consider normal to be $-r_x\times r_y$ as the problem asked. Hence (we use the polar coordinate $x = r\cos\theta$, $y = r\sin\theta$, $dA = rdrd\theta$ here)

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R (x,0,0)\cdot\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)\,dA\\
&= \iint_R \frac{x^2}{\sqrt{x^2+y^2}}\,dA\\
&= \int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\cancelto{r}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}}\,rdrd\theta = -\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\,drd\theta\\
&= \int_0^{2\pi}\left(\frac{1}{3}\cos^2\theta\right)\,d\theta = \cdots\\
&= \frac{\pi}{3}
\end{aligned}$$

(b) As (a), add a surface $B = \{(x,y,z) \mid x^2+y^2 \le 1,\ z = 1\}$ then $S \cup B$ is clearly closed and piecewise smooth and $\mathbf{F}$ is C^2. Let E be the region enclosed by $S \cup B$, by the Divergence Theorem, we have

$$\begin{aligned}
\iint_{S\cup B} \mathbf{F}\cdot\mathbf{n}\, dS &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 1\ dV (= \operatorname{Vol}(E)) \\
&= \iint_R \int_{\sqrt{x^2+y^2}}^1 1\ dz\ dA = \int_0^{2\pi}\int_0^1\int_r^1 r dz dr d\theta \\
&= \int_0^{2\pi}\int_0^1 (1-r)\ r dr d\theta = \cdots \\
&= \frac{\pi}{3}
\end{aligned}$$

Therefore

$$\frac{\pi}{3} = \iint_{S\cup B} \mathbf{F}\cdot\mathbf{n}\ dS = \iint_S \mathbf{F}\cdot\mathbf{n}\ dS + \iint_B \mathbf{F}\cdot\mathbf{n}\ dS.$$

Hence

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\ dS &= \frac{\pi}{3} - \iint_B \mathbf{F}\cdot\mathbf{n}\ dS \\
&= \frac{\pi}{3} - \iint_B (x,0,0)\cdot(0,0,1)\ dS \\
&= \frac{\pi}{3}
\end{aligned}$$

which is the same answer as we did in (a).
(Due to my poor calculation skill, they might be both wrong. Please double check this. And the problem say we may do this without integration, so it would probably ve fine to just write $\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \operatorname{Vol}(E)$.)

□

3.8.4. *23W-5.*

Problem 79: 23W-5

(a) Find the maximum of the function $f(x_1,\cdots,x_n) = (x_1x_2\cdots x_n)^2$ under the constraint

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

(b) Use part (a) to deduce the following inequality: for any $a_1, a_2, \cdots, a_n \ge 0$,

$$(a_1a_2\cdots a_n)^{\frac{1}{n}} \le \frac{a_1+\cdots+a_n}{n}.$$

Solution:

(a) When we see extreme values under constraints, we think of the Lagrange multipliers. Let $g(x_1,\cdots,x_n) = x_1^2+\cdots+x_n^2 - 1$, We have

$$\nabla f = \left(\frac{2x_1^2\cdots x_n^2}{x_1},\cdots,\frac{2x_1^2\cdots x_n^2}{x_n}\right),\quad \nabla g = (2x_1,\cdots,2x_n).$$

By the Theory of the Lagrange multipliers, if f has extreme values at points satisfying the constraint $g = 1$, then $\nabla f = \lambda g$, that is

$$\begin{cases} \dfrac{2x_1^2 \cdots x_n^2}{x_1} = 2\lambda x_1 & \Longrightarrow \ x_1^2 \cdots x_n^2 = \lambda x_1^2, \\ \qquad \vdots & \qquad\qquad \vdots \\ \dfrac{2x_1^2 \cdots x_n^2}{x_n} = 2\lambda x_n & \Longrightarrow \ x_1^2 \cdots x_n^2 = \lambda x_n^2. \end{cases}$$

If one of the $x_1, \cdots, x_n$ is 0 then it achieves its minimum 0. Otherwise, we have $x_1^2 = x_2^2 = \cdots = x_n^2 \Longrightarrow x_i = \pm\dfrac{1}{\sqrt{n}}$, $i = 1, \cdots, n$ by $g(x_1, \cdots, x_n) = 1$ hence the maximum of f is $\left(\dfrac{1}{n}\right)^n$.

(b) Note that if $a_1 = \cdots = a_n = 0$, the result is obviously true. We only prove the case that not all a_i, $i = 1, \cdots, n$ are 0. Since $a_i > 0$, let $S = a_1 + \cdots + a_n$, $x_i^2 = \dfrac{a_i}{S}$, $i = 1, \cdots, n$. Then

$$x_1^2 + \cdots + x_n^2 = \frac{a_1 + \cdots + a_n}{S} = 1.$$

Hence by (a), we have

$$\begin{aligned} & (x_1^2)(x_2^2)\cdots(x_n^2) \le \left(\frac{1}{n}\right)^n \\ \Longleftrightarrow\ & \frac{a_1}{S}\frac{a_2}{S}\cdots\frac{a_n}{S} = \frac{a_1 a_2 \cdots a_n}{S^n} \le \left(\frac{1}{n}\right)^n \\ \Longleftrightarrow\ & a_1 a_2 \cdots a_n \le \left(\frac{S}{n}\right)^n = \left(\frac{a_1 + \cdots + a_n}{n}\right)^n \\ \Longleftrightarrow\ & (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \le \frac{a_1 + \cdots + a_n}{n} \end{aligned}$$

□

3.8.5. *23W-6.*

Problem 80: 23W-6

Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = 0$, with the normal vector chosen such that the normal at $(x, y, z) = (0, 0, 1)$ is $(0, 0, 1)$. Compute the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \left(x + e^{z^3}, x, y\cos(z^2)\right)$.

Solution: Note that $\mathbf{F}$ is clearly C^2 hence (you can write that if you have time)

$$\text{div curl } \mathbf{F} = 0$$

where div $\mathbf{F} = \nabla \cdot \mathbf{F}$, curl $\mathbf{F} = \nabla \times \mathbf{F}$. Let B be the interior region where S and $z = 1$ intersect, that is $B = \{(x, y, z) \mid z = 1,\ x^2 + y^2 \le 1\}$. Clearly $S \cup B$ is piecewise smooth, then by the Divergence Theorem, we have

$$\iint_{S \cup B} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iiint_V \text{div curl } \mathbf{F}\, dV = 0$$

where V is the solid enclosed by S and $z = 0$. Hence

$$\begin{aligned}
0 = \iint_{S \cup B} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS + \iint_B (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \\
\implies \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS &= -\iint_B (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\end{aligned}$$

And the curl and outer normal at B are

$$\nabla \times \mathbf{F}\Big|_B = \text{curl } \mathbf{F}\Big|_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & x & y \end{vmatrix} = (1, 0, 1), \ \mathbf{n}\Big|_B = (0, 0, -1).$$

Therefore

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS &= -\iint_B (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \\
&= -\iint_B (1, 0, 1) \cdot (0, 0, -1) \, dS = \iint_B 1 \, dS = \text{Area}(B) \\
&= \pi.
\end{aligned}$$

□

3.8.6. *22S-6.*

Problem 81: 22S-6

Consider the following cube in $\mathbb{R}^3$:

$$V = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Let S denote the boundary of the cube and let $\mathbf{F}(x, y, z)$ be the vector field given by

$$\mathbf{F}(x, y, z) = (x^2 sin(\pi y) z e^z, z, xy^2).$$

Compute the surface integral of the second kind (flux integral)

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{n}$ denotes the normal vector to S pointing outwards of V.

Solution: It's clear that $\mathbf{F}$ is C^1, S is piecewise smooth and V is simply connected. Hence by the Divergence Theorem, we have

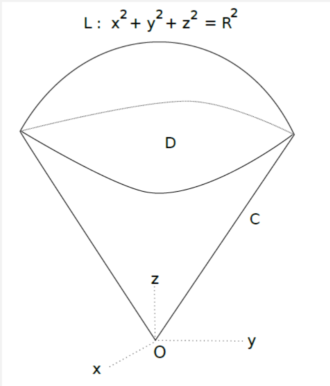
$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\ dS &= \iiint_V \text{div } \mathbf{F}\ dV \\
&= \int_0^1\int_0^1\int_0^1 (2x\sin(\pi y)ze^z + 0 + 0)\ dxdydz \\
&= \int_0^1\int_0^1 \left(\left[x^2\sin(\pi y)ze^z\right]_0^1\right)\ dydz = \int_0^1\int_0^1 \sin(\pi y)ze^z\ dydz \\
&= \int_0^1 \left[-\frac{1}{\pi}\cos(\pi y)ze^z\right]_0^1\ dz = \frac{2}{\pi}\int_0^1 ze^z\ dz \\
&= \frac{2}{\pi}\left([ze^z]_0^1 - \int_0^1 e^z\ dx\right) \\
&= \frac{2}{\pi}.
\end{aligned}$$

$\square$

3.8.7. *22W-6,21W-5(b).*

Problem 82: 22W-6,21W-5(b)

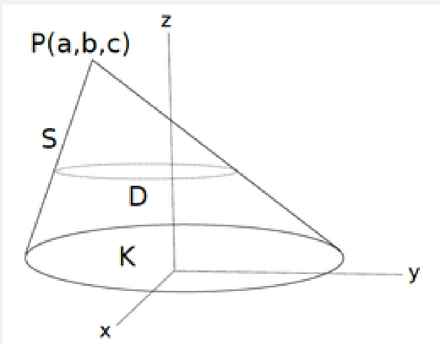
(a) (22W-6)
Let D be the conical solid shown with the boundary of D consisting of the surfaces L and C where L lies on the sphere $x^2 + y^2 + z^2 = R^2$ and C is formed by joining the edge of L to the origin.



Find a relation between the volume of D, the surface area of L, and the surface area of C.

(b) (21W-5(b))
Suppose K is a region in the x, y plane and $P(a, b, c)$ a point above K. Let S be the 'conical surface' formed by drawing lines from P to the boundary of K. If D is the 'conical solid' enclosed by S and K, prove that

$$\text{Volume}(D) = \frac{c}{3}\text{Area}(K).$$



Solution:

(a) It's clear that D is simply connected and ∂D is piecewise smooth. To get the volume, we consider using the Divergence Theorem from a C^1 vector field $\mathbf{F}$ with div $\mathbf{F} = c$ where c is a constant. Note that $\mathbf{n}$ on L is (x, y, z), if we need to find the surface area of L, we need $\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot (x, y, z) = d$ where d is another constant. Finally, note that for any point (x, y, z) on C, we have the connecting vector is (x, y, z). Hence if we choose $\mathbf{F} \parallel (x, y, z) \implies \mathbf{F} \cdot \mathbf{n} = 0$.

Now we choose

$$\mathbf{F}(x, y, z) = (x, y, z) \implies \text{div } \mathbf{F} = 3, \; \mathbf{F} \cdot (x, y, z) = x^2 + y^2 + z^2 = R^2.$$

which is clearly C^1. It's clear that the D is simply connected and its boundary is piecewise smooth. Hence by the Divergence Theorem, we have

$$\begin{aligned}\iiint_E \operatorname{div} \mathbf{F}\, dV &= \iint_E 3\, dV = 3\mathrm{Vol}(E) = \iint_C \mathbf{F}\cdot\mathbf{n}\, dS + \iint_L \mathbf{F}\cdot\mathbf{n}\, dS \\ &= \iint_L \mathbf{F}\cdot\mathbf{n}\, dS + 0 \\ &= \iint_L (x,y,z)\cdot(x,y,z)\, dS = \iint_C R^2\, dS = R^2\mathrm{Area}(L).\end{aligned}$$

Hence we have

$$\mathrm{Vol}(D) = \frac{R^2\mathrm{Area}(L)}{3}.$$

(b) It's clear that D is simply connected and ∂D is piecewise smooth. To get the volume, we consider using the Divergence Theorem from a C^1 vector field $\mathbf{F}$ with div $\mathbf{F} = c$ where c is a constant. Note that $\mathbf{n}$ on K is $(0,0,-1)$, if we need to find the surface area of K, we need $\mathbf{F}(x,y,0)\cdot(0,0,-1) = d$ where d is another constant. Finally, note that for any point (x,y,z) on ∂K (K is in x-y plane, here $z = 0$), we have the connecting vector is $(x-a, y-b, -c)$. Hence if we choose $\mathbf{F} \parallel (x-a, y-b, z-c) \implies \mathbf{F}\cdot\mathbf{n} = 0$.

Now we choose

$$\mathbf{F}(x,y,z) = (x-a, y-b, z-c) \implies \operatorname{div} \mathbf{F} = 3,\ \mathbf{F}(x,y,0)\cdot(0,0,-1) = c.$$

which is clearly C^1. It's clear that the D is simply connected and its boundary is piecewise smooth. Hence by the Divergence Theorem, we have

$$\begin{aligned}\iiint_D \operatorname{div} \mathbf{F}\, dV &= \iint_K 3\, dV = 3\mathrm{Volume}(D) = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS + \iint_K \mathbf{F}\cdot\mathbf{n}\, dS \\ &= 0 + \iint_K c\, dS = c\mathrm{Area}(K).\end{aligned}$$

Hence we have

$$\mathrm{Volume}(D) = \frac{c}{3}\mathrm{Area}(K).$$

□

3.8.8. *21S-6.*

Problem 83: 21S-6

Consider the vector field

$$\mathbf{F}(x,y,z) = (2x^2y, xz^2, 4yz).$$

Compute the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

where S is the surface of the rectangular box bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 2,\ z = 0,\ z = 3$, and where the normal vector $\mathbf{n}$ points towards the exterior of the box.

Solution: It's clear that S is piecewise smooth and the region enclosed by S, say E, is simply connected and $\mathbf{F}$ is C^1. Hence by the Divergence Theorem, we have

$$\begin{aligned}
\iint_S \mathbf{F}) \cdot \mathbf{n}\, dS &= \iiint_E \text{div } \mathbf{F}\, dV \\
&= \int_0^3 \int_0^2 \int_0^1 (4xy + 0 + 4y)\, dzdydx = \int_0^3 \int_0^2 \int_0^1 4y(x+1)\, dxdydz \\
&= \int_0^3 \int_0^2 4y \left[\frac{(x+1)^2}{2}\right]_0^1 dxdydz = \int_0^3 \int_0^2 6y\, dydz \\
&= \int_0^3 6\left[\frac{y^2}{2}\right]_0^2 dz = \int_0^3 12\, dz = 12(3-0) \\
&= 36
\end{aligned}$$

□

3.8.9. *20S-6.*

Problem 84: 20S-6

Consider the surface

$$S : \{(x, y, z) : x^2 + y^2 + z^2 = 1, \quad z \geq 0\},$$

with the normal vector chosen so that the normal at $(0, 0, 1)$ is $(0, 0, 1)$. For the vector field

$$\mathbf{F}(x, y, z) := \left(2x + e^{yz}, zx^2, z - (x^2 + y^2)\right),$$

compute the surface integral of the second kind (flux integral)

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution: We add a surface $B = \{(x, y, z) \mid x^2 + y^2 \leq 1,\ z = 0\}$. Let E be the region enclosed by $S \cup B$. Then E is closed, $S \cup B$ is piecewise smooth and $\mathbf{F}$ is C^2. Let E be the region enclosed by $S \cup B$. By the Divergence Theorem, we have

$$\begin{aligned}
\iint_{S\cup B} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \text{div } \mathbf{F}\, dV &= \iiint_E (2 + 0 + 1)\, dV (= 3\text{Vol}(E)) \\
&= 3\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi\, d\rho d\theta d\phi = \cdots \\
&= 2\pi.
\end{aligned}$$

Therefore

$$2\pi = \iint_{S\cup B} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_B \mathbf{F} \cdot \mathbf{n}\, dS.$$

Hence

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= 2\pi - \iint_B \mathbf{F} \cdot \mathbf{n}\, dS \\
&= 2\pi - \iint_B (2x + 1, 0, -1) \cdot (0, 0, -1)\, dS \\
&= 2\pi - \iint_B 1\, dS = 2\pi - \text{Area}(B) = 2\pi - \pi = \pi.
\end{aligned}$$

$\square$

3.8.10. *20W-7.*

Problem 85: 20W-7

Let V be a solid in 3-space bounded by an orientable closed surface S, let $\mathbf{n}$ be the unit outer normal to S. Let $\mathbf{F}$ be a continuously differentiable vector field on V.

(i) Under the above mentioned setting, state the Divergence Theorem.

(ii) Let S be the surface of a unit cube, $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$, and let $\mathbf{n}$ be the unit outer normal to S. If $\mathbf{F} = (x^2, y^2, z^2)$, use the Divergence Theorem to evaluate the surface integral of $\iint_S \mathbf{F} \cdot \mathbf{n} dS$. Verify the result by evaluating the surface integral directly.

Solution:

(i) (I write the whole version here.) Let E be a simply connected region with $\partial E = S$ be piecewise smooth surface oriented by its unit outer normal. If $\mathbf{F}$ is a C^1 vector field defined near E, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\ dV.$$

(ii) It's clear that S is piecewise smooth, V is simply connected and $\mathbf{F}$ is C^1. Hence by the Divergence Theorem, we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dS &= \iiint_V \operatorname{div} \mathbf{F}\ dV = \iiint_V (2x + 2y + 2z)\ dV \\
&= 2 \int_0^1 \int_0^1 \int_0^1 (x + y + z)\ dxdydz \\
&= 6 \int_0^1 \int_0^1 \int_0^1 x\ dxdydz = 6 \cdot (1 - 0) \cdot (1 - 0) \cdot \left[\frac{x^2}{2}\right]_0^1 \\
&= 3.
\end{aligned}$$

On the other hand, direct computation gives (This is tedious. I will just write the calculations.)

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dS &= \sum_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot \mathbf{n}\ dS \\
&= \iint_{S_1} (1, y^2, z^2) \cdot (1, 0, 0)\ dS + \iint_{S_2} (x^2, 1, z^2) \cdot (0, 1, 0)\ dS + \\
&\quad \iint_{S_3} (0, y^2, z^2) \cdot (-1, 0, 0)\ dS + \iint_{S_4} (x^2, 0, z^2) \cdot (0, -1, 0)\ dS + \\
&\quad \iint_{S_5} (x^2, y^2, 1) \cdot (0, 0, 1)\ dS + \iint_{S_6} (0, y^2, z^2) \cdot (0, 0, -1)\ dS \\
&= \text{Area}(S_1) + \text{Area}(S_2) + 0 + 0 + \text{Area}(S_5) + 0 \\
&= 3.
\end{aligned}$$

$\square$

3.8.11. *19S-1.*

Problem 86: 19S-1

Let D be the three dimensional region which is below $z = x^2 + y^2$, above the x, y plane and inside the cylinder $x^2 + y^2 = 1$.

(a) Compute $\iiint_D z\, dV$ using triple integration. Drawing a sketch of D will help.

(ii) Compute $\iiint_D z\, dV$ by relating it to a surface integral and computing the surface integral.

Solution:

(a) We use the cylindrical coordinate that

$$\begin{aligned}\iiint_D z\, dV &= \int_0^{2\pi}\int_0^1\int_0^{r^2} z\, rdzdrd\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{r^5}{2}\, drd\theta = \cdots = \frac{\pi}{6}.\end{aligned}$$

(b) Let

$$\mathbf{F}(x, y, z) = \left(0, 0, \frac{z^2}{2}\right)$$

which is clearly C^1. Now, we let $S = \partial D$, then S is piecewise smooth and D is simply connected. Then by the Divergence Theorem, we have

$$\begin{aligned}&\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_D z\, dV \\ \implies &\iiint_D z\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.\end{aligned}$$

Note that $S = S_1 \cup S_2 \cup S_3$ where $S_1 = \{(x, y, z) \mid x^2 + y^2 = 1,\ 0 \le z \le 1\}$, $S_2 = \{(x, y, 0) \mid x^2 + y^2 = 1\}$, $S_3 = \{(x, y, z) \mid z = x^2 + y^2\}$. And we have

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_{S_1} \left(0, 0, \frac{z^2}{2}\right)\cdot(x, y, 0)\, dS = 0, \\ \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_{S_2} \left(0, 0, \frac{z^2}{2}\right)\cdot(0, 0, -1)\, dS = 0\end{aligned}$$

since $\mathbf{F} = (0, 0, 0)$ on $z = 0$. For S_3, we parametrize S_3 as

$$r(r, \theta) = (r\cos\theta, r\sin\theta, r^2),\ r \in [0, 1],\ \theta \in [0, 2\pi].$$

Hence (careful about orientation, this is actually "inward")

$$\begin{aligned}r_r \times r_\theta &= (\cos\theta, \sin\theta, 2r) \times (-r\sin\theta,\ r\cos\theta, 0) \\ &= \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r).\end{aligned}$$

So we have

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^1 \left(0, 0, \frac{r^4}{2}\right) \cdot (-2r^2 \cos\theta, -2r^2 \sin\theta, r)\, dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \frac{r^5}{2}\, dr d\theta = 2\pi \left[\frac{r^6}{12}\right]_0^1 = \frac{\pi}{6}.$$

□

3.8.12. *19W-8.*

Problem 87: 19W-8

Consider the surface $S := \{(x, y, z) : x^2 + y^2 = 4, \quad -1 \leq z \leq 1\}$, with the normal vector chosen so that the normal at $(0, 2, 0)$ is $(0, 1, 0)$. For the vector field

$$\mathbf{F}(x, y, z) := (x + z \sin y, ze^x + 2y, \cos z),$$

compute the surface integral of the second kind (flux integral)

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution: We add two surfaces $B_1 = \{(x, y, z) \mid x^2+y^2 \leq 4,\ z = -1\}$, $B_2 = \{(x, y, z) \mid x^2+y^2 \leq 4,\ z = 1\}$ Let E be the region enclosed by $S \cup B_1 \cup B_2$. Then the E is clearly closed, $S \cup B_1 \cup B_2$ is piecewise smooth and $\mathbf{F}$ is C^2. By the Divergence Theorem, we have

$$\iint_{S\cup B_1 \cup B_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E (1 + 2 - \sin z)\, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_{-1}^1 (3 - \sin z)\, r dz dr d\theta = \int_0^{2\pi} [3rz + r\cos z]_{-1}^1\, dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 6r\, dr d\theta = 2\pi[3r^2]_0^2$$

$$= 24\pi.$$

Therefore

$$= \iint_{S\cup B} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_B \mathbf{F} \cdot \mathbf{n}\, dS.$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = A - \iint_{B_1} \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{B_2} \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= 24\pi - \iint_{B_1} (x - \sin y, -e^x + 2y, \cos(-1)) \cdot (0, 0, -1)\, dS$$

$$- \iint_{B_2} (x + \sin y, e^x + 2y, \cos(1)) \cdot (0, 0, 1)\, dS$$

$$= 24\pi + \cos(-1)\mathrm{Area}(B_1) - \cos(1)\mathrm{Area}(B_2) = A + 4\pi\cos(-1) - 4\pi\cos(1)$$

$$= 24\pi.$$

□

3.8.13. *18S-5.*

Problem 88: 18S-5

Let $f : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|},$$

and let $\Omega \in \mathbb{R}^3$ be a bounded open set whose boundary is the union of orientable piecewise smooth surfaces, so that the divergence theorem can be applied in Ω. Show that if $\mathbf{0} \notin \overline{\Omega}$, then

$$\int_{\partial\Omega} \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{S}(\mathbf{x}) = 0,$$

whereas, if $\mathbf{0} \in \Omega$, then

$$\int_{\partial\Omega} \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{S}(\mathbf{x}) = 1.$$

Solution: Note that

$$\mathbf{x} = (x, y, z) \Longrightarrow |\mathbf{x}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

Note that

$$\frac{\partial f}{\partial x} = \frac{1}{4\pi} \frac{-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}} \cdot 2x}{(x^2 + y^2 + z^2)} = \frac{1}{4\pi} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

Hence

$$\nabla f(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right).$$

Also we have

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} &= \frac{-1\left((x^2 + y^2 + z^2)^{\frac{3}{2}}\right) + x\left(\frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}} \cdot 2x\right)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-(x^2 + y^2 + z^2)^{\frac{3}{2}} + 3x^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-(x^2 + y^2 + z^2) + 3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{-(x^2 + y^2 + z^2) + 3y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \quad \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{-(x^2 + y^2 + z^2) + 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Hence

$$\operatorname{div}(\nabla f) = \frac{-3(x^2 + y^2 + z^2) + 3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0$$

Now, it's obvious that we can use Divergence Theorem on $\overline{\Omega}$, which is closed and bounded and vector field ∇f. If $\mathbf{0} = (0, 0, 0) \notin \overline{\Omega}$, then by the Divergence Theorem, we have

$$\int_{\partial\Omega} \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{S}(\mathbf{x}) = \int_{\Omega} \operatorname{div}(\nabla f(\mathbf{x}))\ dV = 0.$$

If $\mathbf{0} = (0, 0, 0) \in \overline{\Omega}$, we remove a small ball $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = R^2\}$, R is sufficiently small. Let D be the space region between S and Ω, then div $\nabla f = 0$ on D. And we can use

Divergence Theorem on D. Consider the orientation of D that the outward normal towards D on S is inward. So the unit outer normal on S with respect to D is $\mathbf{n} = \dfrac{-(x,y,z)}{R}$. Hence

$$\left(\iint_{\partial\Omega} + \iint_S\right) \nabla f(\mathbf{x}) \cdot \mathbf{n}\, dS = \iiint_{D^*} \operatorname{div}(\nabla f(\mathbf{x}))\, dV = 0.$$

Hence

$$\begin{aligned}
\iint_{\partial\Omega} \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{S}(\mathbf{x}) &= -\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_S \nabla f(\mathbf{x}) \cdot \frac{-(x,y,z)}{R}\, dS \\
&= -\iint_S \frac{1}{4\pi}\left(\frac{-x}{R^3}, \frac{-y}{R^3}, \frac{-z}{R^3}\right) \cdot \frac{-(x,y,z)}{R}\, dS \\
&= \frac{-1}{4\pi R^2} \iint_S 1\, dS = \cdots = \frac{-1}{4\pi R^2}(4\pi R^2) \\
&= -1.
\end{aligned}$$

(Of course you can calculate the area of the sphere but I don't think we need to.)
This is a famous example. **The answer should be** -1 not 1. There must be a typo.

□

3.8.14. *15S-1.*

Problem 89: 15S-1

Let S be the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, with the normal vector pointing downwards. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

where

$$\mathbf{F}(x,y,z) = \left(yz + e^{x^3}, z^{10} - \cos y, x - y + \frac{1}{1+z^4}\right).$$

Solution: Note that $\mathbf{F}$ is clearly C^2 hence (you can write that if you have time)

$$\text{div curl } \mathbf{F} = 0$$

where div $\mathbf{F} = \nabla \cdot \mathbf{F}$, curl $\mathbf{F} = \nabla \times \mathbf{F}$. Let B be the interior region where S and $z = 0$ intersect, that is $B = \{(x,y,z) \mid z = 0,\ x^2 + y^2 \leq 1\}$. Clearly $S \cup B$ is piecewise smooth, then by the Divergence Theorem, we have

$$\iint_{S\cup B} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iiint_V \text{div curl } \mathbf{F}\, dV = 0$$

where V is the solid enclosed by S and $z = 1$. Hence

$$\begin{aligned}
0 = \iint_{S\cup B} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} + \iint_B (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
\implies \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= -\iint_B (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\end{aligned}$$

And the curl and outer normal at B are

$$\nabla \times \mathbf{F}\Big|_B = \text{curl } \mathbf{F}\Big|_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + e^{x^3} & 1 - \cos y & x - y + \dfrac{1}{2} \end{vmatrix} = (-1,-1,-1), \ \mathbf{n}\Big|_B = (0,0,1).$$

Therefore

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= -\iint_B (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -\iint_B (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \\ &= -\iint_B (-1,-1,-1) \cdot (0,0,1)\, dS = \iint_B -1\, dS = -\text{Area}(B) \\ &= -\pi. \end{aligned}$$

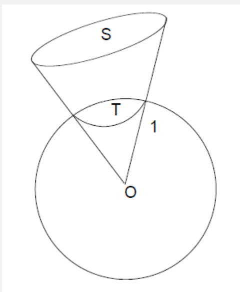
□

3.8.15. *12S-4.*

Problem 90: 12S-4

Let S be a connected surface in $\mathbb{R}^3$ so that every line through the origin intersects S at most once and each point of S is more than one unit away from the origin. Let T be the subset of the unit sphere consisting of points on the unit sphere where the lines joining the origin to a point of S intersect the unit sphere - so T is the radial shadow of S (see the sketch). If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{n}$ denotes the unit normal to S (pointing away from the origin) show that

$$Area(T) = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}|^3}\, dS.$$



Solution: $Area(T)$ is called solid angles if you're interested.

We let

$$\mathbf{F}(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}^3|} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

which is clearly C^1. Now we consider E be the region enclosed by S, T and the connected lines between them, say it's L. It's clear that the E is simply connected and its boundary is piecewise smooth. Note that

$$\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{1\left((x^2+y^2+z^2)^{\frac{3}{2}}\right) - x\left(\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x\right)}{(x^2+y^2+z^2)^3}$$
$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$
$$= \frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

And similar calculation holds for other components, so we have

$$\text{div } \mathbf{F} = \frac{3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0.$$

Hence by the Divergence Theorem, we have

$$0 = \iiint_E \text{div } \mathbf{F}\, dV = \left(\iint_S + \iint_T + \iint_L\right) \mathbf{F}\cdot\mathbf{n}\, dS.$$

Note that for any point (x, y, z) on S, we have the connecting line is (x, y, z). Hence $\mathbf{F} \parallel (x, y, z) \Longrightarrow \mathbf{F}\cdot\mathbf{n} = 0$. On T, we have $\mathbf{n} = -(x, y, z)$. Hence

$$\iint_T \mathbf{F}\cdot\mathbf{n}\, dS = \iint_T \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot -(x,y,z)\, dS = -\iint_T \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dS$$
$$= -\iint_T \frac{1}{1^{\frac{3}{2}}}\, dS = -\iint_T 1\, dS = -Area(T).$$

Hence we have

$$0 = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS + \iint_T \mathbf{F}\cdot\mathbf{n}\, dS + \iint_L \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS - Area(T) + 0$$
$$\Longrightarrow Area(T) = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \frac{\mathbf{r}}{|\mathbf{r}^3|}\cdot\mathbf{n}\, dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{r}|^3}\, dS.$$

□

3.8.16. *Will be updated.*
21S-6,20S-6,20W-7,19W-8,18W-1,17W-3,16W-1,15S-1
22W-6,21W-5(b),13W-4,12S-4,11S-5,06S-6(b),03W-6(b),96S-7,96W-6

18S-5
16S-7: Stokes
14S-2
14W-5(c)

11W-8
06S-6(a)
06W-6
05W-6(a)

19S-1,04S-6,03S-6 Find F to use divergence thm

05W-6(b),03W-6(a): surface area

00W-6

97S-4

97W-5: Green's

3.9. **Multivariable Analysis.**

Most of the problems in this section are directly application of the Implicit Function Theorem and the Inverse Function Theorem:

Theorem 12 (Implicit Function Theorem)**.** *Let $f : E \to \mathbb{R}^n$ where $E \subseteq \mathbb{R}^{n+m}$ such that $f(a,b) = 0$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ for some $(a,b) \in E$. Let $A = f'(a,b)$, $A(h,k) = A_x h + A_y k$, $A_x h = A(h,0)$, $A_y = A(0,k)$ for any $(h,k) \in E$. Then if A_x is invertible then there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ such that $(a,b) \in U$, $b \in W$ and for every $y \in W$, there exists a unique x such that $(x,y) \in U$, $f(x,y) = 0$. Furthermore, let such $x = g(y)$, then $g : W \to \mathbb{R}^n$ is C^1 satisfying*

$$g(b) = a,\ f(g(y),y) = 0 \text{ and } g'(b) = -A_x^{-1}A_y.$$

Theorem 13 (Inverse Function Theorem)**.** *Let $f : E \to \mathbb{R}^n$ where $E \subseteq \mathbb{R}^n$ be C^1. If $f'(a)$ is invertible and $f(a) = b$ for some $a \in E$, $b \in \mathbb{R}^n$ then there exist open sets $U, V \in \mathbb{R}^n$ such that $a \in U$, $b \in V$ and f is injective on U and $f(U) = V$ so the inverse of f exists. Furthermore, if g is the inverse of f. we have $g(f(x)) = x$, $\forall\ x \in U$, and g is C^1.*

3.9.1. *24S2-4.*

Problem 91: 24S2-4

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x) = (x_1^3 + x_2^3 + x_2, x_1 + e^{x_2}$, where $x = (x1, x2)$. Prove that there exist open neighborhoods U of $(0,0)$ and V of $(0,1)$ such that $G : U \to V$, defined by $G(x) = F(x)$ for all $x \in U$, is invertible. Compute $(G^{-1})'(0,1)$, the Jacobian of G^{-1} evaluated at $(0,1)$.

Solution:

□

3.9.2. *24S1-7.*

Problem 92: 24S1-7

Consider the system of equation

$$e^x + 2e^{2y} + 3e^{3u} + 4e^{4v} + u = 10$$
$$e^x + 3e^{3y} + 6e^{6u} + 9e^{9v} + v = 19.$$

Noting that $u = 0, v = 0, x = 0, y = 0$ is a solution of this system, what can you say about solving this system for x, y in terms of u, v? Please be precise and also verify the hypothesis of any theorem you use.

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 2,\ F(x, y, u, v) = (f_1, f_2) = (e^x + 2e^{2y} + 3e^{3u} + 4e^{4v} + u - 10, e^x + 3e^{3y} + 6e^{6u} + 9e^{9v} + v - 19)$ Note that $F(0, 0, 0, 0) = 0$. We have

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} e^x & 4e^{2y} & 1 + 9e^{3u} & 16e^{4v} \\ e^x & 9e^{3y} & 36e^{6u} & 1 + 81e^{9v} \end{pmatrix}.$$

Hence for $a = (0, 0), b = (0, 0)$, we have

$$A(0, 0, 0, 0) = \begin{pmatrix} 1 & 4 & 10 & 16 \\ 1 & 9 & 36 & 82 \end{pmatrix},\ A_x(0, 0) = \begin{pmatrix} 1 & 4 \\ 1 & 9 \end{pmatrix},\ A_y(1, 1) = \begin{pmatrix} 10 & 16 \\ 36 & 82 \end{pmatrix}.$$

Since $\det A_x = 5 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) = f(u, v)$.
Remark I follow the Rudin's definition, in this question it's a little bit confusing. Be sure to understand what a, b, x, y, h, k mean, especially x, y, they are NOT the usual x and y. □

3.9.3. *24W-7(iii).*

Problem 93: 24W-7(iii)

Prove or give a counterexample. You are allowed to quote known theorems in a proof.

(iii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $f'(0) = 2$. Then there exists $\epsilon > 0$, an open interval (a, b), and $g : (a, b) \to (-\epsilon, \epsilon)$ such that

$$g(f(x)) = x\ \forall\ x \in (-\epsilon, \epsilon),$$
$$f(g(x)) = x\ \forall\ x \in (a, b).$$

Solution:

(iii) This is true. And this is the 1 dimensional Inverse Function Theorem. (If you don't have time, at least cite the multivariable version of the Inverse Function Theorem.)

We outline the proof of the Inverse Functon Theorem here and prove the 1 dimensional version here:

(1) There exists $f|_U : U \to V$ such that $f|_U$ is a bijection.

Proof. ince $f \in C^1$, $f'(x)$ is continuous at $x = 0$. Hence there exists $\varepsilon > 0$ (the problem need ε...) such that when $|x - 0| = |x| < \varepsilon \implies x \in (-\varepsilon, \varepsilon)$

$$|f'(x) - f'(0)| < 2 \implies 0 < f'(x) < 4.$$

Hence f is strictly increasing on $U = (-\varepsilon, \varepsilon)$ and $f|_U$ is one-to-one on U. Choose $V = f(U)$, then $f|_U$ is a bijection. □

(2) $V = f(U)$ is open.

Proof. In higher dimension, we can use the contraction mapping to prove this. In one dimension, this is not a problem.
Let $a = f(-\varepsilon),\ b = f(\varepsilon)$. Then by the Intermediate Value Theorem, for any $c \in (a, b),\ \exists\ d \in V$ such that $f(d) = c \implies (a, b) \subseteq V$. Also, since f is strictly increasing in U, $a < e < b$ for all $e \in U \implies V \subseteq (a, b)$. Hence $V = (a, b)$ and it's clearly open. □

(3) $\exists\, g : V \to U$ such that $g = f^{-1}$.

Proof. This is immediate from $f|_U : U \to V$ is a bijection. □

(4) $g \in C^1$: We don't need it here. In one dimension, we can just use the limit and directly caculate that limit, it would be

$$g'(x) = \frac{1}{f'(g(x))}, \ \forall\, x \in I.$$

And it's clearly C^1 since f is C^1 thus f' is continuous.

Hence results follow.

□

3.9.4. *24W-8=19S-2,22W-5.*

Problem 94: 24W-8=19S-2,22W-5

The system of equations

$$e^x + e^{2y} + e^{3u} + e^{4v} = 4 \text{ and } e^x + e^y + e^u + e^v = 4$$

has the solution $(x, y, u, v) = (0, 0, 0, 0)$.

(1) (24W-8=19S-2)
Prove that, for every (x, y) near $(0, 0)$, the above system has a unique solution $(x, y, u(x, y), v(x, y))$ with $(u((x, y), v(x, y))$ near $(0, 0)$, and compute $\dfrac{\partial u}{\partial y}(0, 0)$.

(2) (22W-5)
What can you say about solving this system for x, y in terms of u, v? Please be precise and also verify the hypothesis of any theorem you use.

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 2,\ F(x, y, u, v) = (f_1, f_2) = (e^x + e^{2y} + e^{3u} + e^{4v} - 4, e^x + e^y + e^u + e^v - 4)$ Note that $F(0, 0, 0, 0) = 0$. We have

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} e^x & 2e^{2y} & 3e^{3u} & 4e^{4v} \\ e^x & e^y & e^u & e^v \end{pmatrix}.$$

Hence for $a = (0, 0), b = (0, 0)$, we have

$$A(0, 0, 0, 0) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix},\ A_x(0, 0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix},\ A_y(1, 1) = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}.$$

(1) (This is to solve u, v in terms of x, y.)
Since $\det A_y(0, 0) = -1 \neq 0$, $A_y(0, 0)$ is invertible. Hence by the Implicit Function Theorem, there exists a unique C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(u, v) = f(x, y) = (u(x, y), v(x, y))$. Furthermore, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(0,0)} = -A_y^{-1} A_x = -\frac{1}{-1} \begin{pmatrix} 1 & -4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}.$$

Therefore, $\dfrac{\partial u}{\partial y}(0, 0) = u_y|_{(0,0)} = 2$.

(2) (This is to solve x, y in terms of u, v.)
Since $\det A_x(0,0) = -1 \neq 0$, $A_x(0,0)$ is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) = g(u, v)$.

$\square$

3.9.5. *23S-4.*

Problem 95: 23S-4

Consider the function $u(x,y) = x^3 + x + y^2,\ v(x,y) = x^2 + y^5$; note that if $(x,y) = (1,1)$ then $(u,v) = (3,2)$. What does the Inverse Function Theorem say about expressing x, y in terms of u, v for (u,v) near $(3,2)$? Also, compute the partial derivative y_u when $(u,v) = (3,2)$ and $(x,y) = (1,1)$.

Solution: We need to use the Inverse Function Theorem here (or you can think of $n = m = 2$ in the Implicit Function Theorem here, but the problem stated that we need to use the Inverse Function Theorem here.)

Let $f(x,y) = (u(x,y), v(x,y)) = (x^3 + x + y^2, x^2 + y^5),\ a = (1,1),\ b = (3,2)$ then

$$u_x = 3x^2 + 1,\ u_y = 2y,\ v_x = 2x,\ v_y = 5y^4$$

clearly they are all of C^1 near $(x,y) = (1,1)$ so f is also C^1 near $(x,y) = (1,1)$. Now we need to check that $f'(x,y)$ is invertible, that is, $\det f'(x,y) \neq 0$ at $(x,y) = (1,1)$. We calculate that

$$\det f'(x,y)\Big|_{(1,1)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3x^2+1 & 2y \\ 2x & 5y^4 \end{vmatrix}_{(1,1)} = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 16 \neq 0.$$

Hence by the Inverse Function Theorem we have that (x,y) can be expressed as a differentiable function of (u,v) near $(x,y) = (1,1)$. Furthermore, we have

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}_{(1,1)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1}_{(1,1)} = \frac{1}{\det f'(x,y)} \begin{pmatrix} 5y^4 & -2y \\ -2x & 3x^2+1 \end{pmatrix}_{(1,1)} = \frac{1}{16} \begin{pmatrix} 5 & -2 \\ -2 & 4 \end{pmatrix}$$

Therefore, $y_u = \dfrac{-2}{16} = -\dfrac{1}{8}$.

□

3.9.6. *23W-7.*

Problem 96: 23W-7

Justify using an appropriate theorem that the equation

$$x^3 + y^3 + z^3 - xyz = 8$$

defines a differentiable function $z = f(x,y)$ in a neighborhood of $(1,1)$, such that $f(1,1) = 2$, and compute

$$\frac{\partial f}{\partial x}(1,1) \text{ and } \frac{\partial f}{\partial y}(1,1).$$

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 1,\ m = 2,\ F(z,x,y) = x^3 + y^3 + z^3 - xyz - 8$. Note that $F(2,1,1) = 0$. We have

$$A = \begin{pmatrix} F_z & F_x & F_y \end{pmatrix} = \begin{pmatrix} 3z^2 - xy & 3x^2 - yz & 3y^2 - xz \end{pmatrix}.$$

Hence for $a = (2), b = (1,1)$, we have

$$A(2,1,1) = \begin{pmatrix} 11 & 1 & 1 \end{pmatrix},\ A_x(2) = \begin{pmatrix} 11 \end{pmatrix},\ A_y(1,1) = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Clearly A_x is invertible and $A_x^{-1} = \begin{pmatrix} \frac{1}{11} \end{pmatrix}$. Hence by the Implicit Function Theorem, there exists a C^1 function f such that $z = f(x,y)$ (it's the g in the Theorem description) and

$$f'(1,1) = \begin{pmatrix} \frac{\partial f}{\partial x}(1,1) & \frac{\partial f}{\partial y}(1,1) \end{pmatrix} = -A_x^{-1}A_y\Big|_{(1,1)} = -\begin{pmatrix} \frac{1}{11} \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{11} & -\frac{1}{11} \end{pmatrix}.$$

Hence

$$\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1) = -\frac{1}{11}.$$

□

3.9.7. *22S-5.*

Problem 97: 22S-5

Consider the function $F : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x,y,z,u,v) = \begin{pmatrix} xy + xuv + uv^2 - 3 \\ u^2v + xuv - 2 \end{pmatrix}.$$

Prove that u and v can be expressed as functions $u = u(x,y,z)$ and $v = v(x,y,z)$ of x,y,z near the point $(x,y,z,u,v) = (1,1,1,1,1)$. Clearly state which result(s) you are using.

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 3,\ F(x,y,z,u,v) = F(u,v,x,y,z) = (f_1, f_2) = (xy + xuv + uv^2 - 3, u^2v + xuv - 2)$. Note that $F(1,1,1,1,1) = 0$. We have

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} xv + v^2 & xu + 2u & y + uv & x & 0 \\ 2uv + xv & u^2 + xu & uv & 0 & 0 \end{pmatrix}.$$

Hence for $a = (1,1), b = (1,1,1)$, we have

$$A(1,1,1,1,1) = \begin{pmatrix} 2 & 2 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{pmatrix},\ A_x(1,1) = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix},\ A_y(1,1) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $\det A_x = -5 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $(u,v) = f(x,y,z)$.

□

3.9.8. *21S-5.*

Problem 98: 21S-5

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous second-order partial derivatives and suppose the first derivatives of f vanish at a. Let

$$\Delta = \det \begin{pmatrix} D_{xx}f(a) & D_{xy}f(a) \\ D_{xy}f(a) & D_{yy}f(a) \end{pmatrix} = D_{xx}f(a)D_{yy}f(a) - (D_{xy}f(a))^2.$$

Assume that $\Delta > 0$ and $D_{xx}f(a) > 0$. Prove that f has a local minimum at a.

Solution: (I slightly change the notations into the ones I'm more familiar with.)
Let $a = (a_1, a_2) \in \mathbb{R}^2$. Consider $g(t) = f(a_1 + th_1, a_2 + th_2)$ where $0 \le t \le 1$ and $h_1, h_2 \in \mathbb{R}$. Then g is differentiable since $f \in C^2$ and

$$g(0) = f(a_1, a_2), \;\; g(1) = f(a_1 + h_1, a_2 + h_2).$$

Consider the 2^{nd} order Taylor's expansion, we have

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(\theta) \text{ where } 0 \le \theta \le 1\text{: remainder.}$$

And

$$g'(0) = \frac{\partial}{\partial t}\left[f(a_1 + th_1, a_2 + th_2)\right]_{t=0} = f_x x_t + f_y y_t = h_1 f_x + h_2 f_y = 0$$

since $f_x = f_y = 0$ at a. Also

$$\begin{aligned} g''(0) &= \frac{\partial}{\partial t}(f_x x_t + f_y y_t) \\ \text{(chain rule)} &= h_1(f_{xx}x_t + f_{xy}y_t) + h_2(f_{yx}x_t + f_{yy}y_t) \\ &= h_1(h_1 f_{xx} + h_2 f_{xy}) + h_2(h_1 f_{yx} + h_2 f_{yy}) \\ (f \in C^2 \implies f_{xy} = f_{yx}) &= h_1^2 f_{xx} + 2h_1h_2 f_{xy} + h_2^2 f_{yy} \\ &= \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) &= g(1) - g(0) = g'(0) + \frac{1}{2}g''(\theta) = \frac{1}{2}g''(\theta) \\ &= \frac{1}{2}\begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}_{(a_1+\theta h_1, a_2+\theta h_2)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} h_1^2 f_x x + 2h_1h_2 f_{xy} + h_2^2 f_y y &= f_{xx}\left[h_1^2 + \frac{2f_{xy}}{f_{xx}}h_1h_2\right] + f_{yy}h_2^2 \\ &= f_{xx}\left(h_1 + \frac{f_{xy}}{f_{xx}}h_1h_2\right)^2 + \frac{f_{xx}f_{yy} - (f_{xy})^2}{f_{xx}}h_2^2. \end{aligned}$$

Hence

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) = f_{xx}\left(h_1 + \frac{f_{xy}}{f_{xx}}h_1h_2\right)^2 + \frac{f_{xx}f_{yy} - (f_{xy})^2}{f_{xx}}h_2^2$$

which is a quadratic equation in h_1. Note that $f \in C^2 \Longrightarrow f_{xx}$ is continuous. Hence if $f_{xx}(a_1, a_2) > 0 \Longrightarrow f_{xx}(a_1 + \theta h_1, a_2 + \theta h_2) > 0$ for h_1, h_2 sufficiently small (at this stage I don't think you need to prove this). Also,

$$\Delta > 0 \Longrightarrow [f_{xx} f_{yy} - (f_{xy})^2]_{a_1+\theta h_1, a_2+\theta h_2} > 0.$$

Hence

$$\left[f_{xx} \left(h_1 + \frac{f_{xy}}{f_{xx}} h_1 h_2 \right)^2 + \frac{f_{xx} f_{yy} - (f_{xy})^2}{f_{xx}} h_2^2 \right]_{a_1+\theta h_1, a_2+\theta h_2} > 0,$$

that is

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) > 0 \Longrightarrow f(a_1, a_2) = f(a) \text{ is a local minimum.}$$

(This looks tedious but it's just calculations and a small quadratic equation. You might need to get familiar with general higher dimensional case with the Hessian matrix.)

□

3.9.9. *21W-5(a).*

Problem 99: 21W-5(a)

(a) For (x, y) near $(0, 0)$ and (u, v) near $(1, 1)$, the differentiable functions $u(x, y), v(x, y)$ are the unique solution of the system of equations

$$x^2 u^2 + (y - 2)uv^2 = -2$$

$$ye^u + xe^v + uv = 1.$$

Compute u_x, v_x when $(x, y) = (0, 0)$.

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 2,\ F(u, v, x, y) = (f_1, f_2)$, where

$$f_1 = x^2 u^2 + (y - 2)uv^2 + 2,\ f_2 = ye^u + xe^v + uv - 1.$$

Note that $F(1, 1, 0, 0) = 0$. We have

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x^2 u + (y-2)v^2 & 2(y-2)uv & 2xu^2 & uv^2 \\ ye^u + v & xe^v + u & e^v & e^u \end{pmatrix}.$$

Hence for $a = (1, 1), b = (0, 0)$, we have

$$A(1, 1, 0, 0) = \begin{pmatrix} -2 & -4 & 0 & 1 \\ 1 & 1 & e & e \end{pmatrix},\ A_x(1, 1) = \begin{pmatrix} -2 & -4 \\ 1 & 1 \end{pmatrix},\ A_y(0, 0) = \begin{pmatrix} 0 & 1 \\ e & e \end{pmatrix}.$$

Since $\det A_x = 2 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(u, v) = f(x, y)$. Furthermore, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(0,0)} = -A_x^{-1} A_y = -\frac{1}{2} \begin{pmatrix} 1 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ e & e \end{pmatrix} = \begin{pmatrix} -2e & -\dfrac{1}{2} - 2e \\ e & \dfrac{1}{2} + e \end{pmatrix}.$$

Therefore, $u_x(0, 0) = -2e,\ v_x(0, 0) = e$.

□

3.9.10. *20S-7.*

Problem 100: 20S-7

Let $u(x,y), v(x,y)$ be the unique simultaneous solution of the equations

$$\begin{cases} xu^3 + (y+1)uv = 6 \\ yu^2 + v^2 + xy = 9, \end{cases}$$

for (x,y) near $(0,0)$ and (u,v) near $(2,3)$. Compute u_x, u_y, v_x and v_y at the point $(x,y) = (0,0)$. Clearly state every theorem that you use.

Solution: We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 2,\ F(u,v,x,y) = (f_1, f_2)$, where

$$f_1 = xu^3 + (y+1)uv - 6,\ f_2 = yu^2 + v^2 + xy - 9.$$

Note that $F(2,3,0,0) = 0$. We have

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3xu^2 + (y+1)v & (y+1)u & u^3 & uv \\ 2yu & 2v & y & u^2 + x \end{pmatrix}.$$

Hence for $a = (2,3), b = (0,0)$, we have

$$A(2,3,0,0) = \begin{pmatrix} 3 & 2 & 8 & 6 \\ 0 & 6 & 0 & 4 \end{pmatrix},\ A_x(2,3) = \begin{pmatrix} 3 & 2 \\ 0 & 6 \end{pmatrix},\ A_y(0,0) = \begin{pmatrix} 8 & 6 \\ 0 & 4 \end{pmatrix}.$$

Since $\det A_x = 18 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(u,v) = f(x,y)$. Furthermore, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(0,0)} = -A_x^{-1}A_y = -\frac{1}{18}\begin{pmatrix} 6 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 8 & 6 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -\dfrac{8}{3} & -\dfrac{14}{9} \\ 0 & -\dfrac{2}{3} \end{pmatrix}.$$

Therefore, $u_x(0,0) = -\dfrac{8}{3},\ u_y(0,0) = -\dfrac{14}{9},\ v_x(0,0) = 0,\ v_y(0,0) = -\dfrac{2}{3}$.

$\square$

3.9.11. *20W-6.*

Problem 101: 20W-6

Consider the equations

$$\begin{cases} x^2 + 2\exp(y) + z = 4 \\ 2x^3 + 4y^2 + z = 2 \end{cases}$$

(i) Show that there exists a neighborhood of $(1, 0, 1)$ on which (y, z) can be expressed as a differentiable function of x.

(ii) Compute $\dfrac{\partial y}{\partial x}$ and $\dfrac{\partial z}{\partial x}$ at $(1, 0, 1)$.

Solution:

(i) We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 1,\ F(y, z, x) = (f_1, f_2)$, where

$$f_1 = x^2 + 2\exp(y) + z - 4,\ f_2 = 2x^3 + 4y^2 + z - 2.$$

Note that $F(0, 1, 1) = 0$. We have

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} & \dfrac{\partial f_1}{\partial x} \\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} & \dfrac{\partial f_2}{\partial x} \end{pmatrix} = \begin{pmatrix} 2e^y & 1 & 2x \\ 8y & 1 & 6x^2 \end{pmatrix}.$$

Hence for $a = (0, 1), b = 1$, we have

$$A(0, 1, 1) = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 6 \end{pmatrix},\ A_x(0, 1) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},\ A_y(1) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

Since $\det A_x = 2 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(y, z) = f(x)$.

(ii) Furthermore, we have

$$\begin{pmatrix} y_x \\ z_x \end{pmatrix}_{(0,1)} = -A_x^{-1} A_y = -\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}.$$

Therefore, $\dfrac{\partial y}{\partial x}(0, 1) = y_x(0, 1) = 2,\ \dfrac{\partial z}{\partial x}(0, 1) = z_x(0, 1) = -6.$

□

3.9.12. *19W-7.*

Problem 102: 19W-7

Consider the equations

$$\begin{cases} x^2 + 2y^2 + u^2 + v = 6 \\ 2x^3 + 4y^2 + u + v^2 = 9 \end{cases}$$

(i) Show that there exists a neighborhood of $(1, -1, -1, 2)$ on which (u, v) can be expressed as a differentiable function of (x, y).

(ii) Compute $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ at $(1, -1, -1, 2)$.

Solution:

(i) We need to use the Implicit Function Theorem here:

In this case, take $n = 2,\ m = 2,\ F(u, v, x, y) = (f_1, f_2)$, where

$$f_1 = x^2 + 2y^2 + u^2 + v - 6,\ f_2 = 2x^3 + 4y^2 + u + v^2 - 9.$$

Note that $F(-1, 2, 1, -1) = 0$. We have

$$A = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2u & 1 & 2x & 4y \\ 1 & 2v & 6x^2 & 8y \end{pmatrix}.$$

Hence for $a = (-1, 2), b = (1, -1)$, we have

$$A(-1, 2, 1, -1) = \begin{pmatrix} -2 & 1 & 2 & -4 \\ 1 & 4 & 6 & -8 \end{pmatrix},\ A_x(-1, 2) = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix},\ A_y(1, -1) = \begin{pmatrix} 2 & -4 \\ 6 & -8 \end{pmatrix}.$$

Since $\det A_x = -9 \neq 0$, A_x is invertible. Hence by the Implicit Function Theorem, there exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(u, v) = f(x, y)$.

(ii) Furthermore, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(1,-1)} = -A_x^{-1} A_y = -\frac{1}{-9} \begin{pmatrix} 4 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 6 & -8 \end{pmatrix} = \begin{pmatrix} \dfrac{2}{9} & -\dfrac{8}{9} \\ -\dfrac{14}{9} & \dfrac{20}{9} \end{pmatrix}.$$

Therefore, $\dfrac{\partial u}{\partial x}(1, -1) = u_x(0, 1) = \dfrac{2}{9},\ \dfrac{\partial v}{\partial x}(1, -1) = v_x(1, -1) = -\dfrac{14}{9}.$

□

3.9.13. *18S-2.*

Problem 103: 18S-2

Let $\Omega \subset \mathbb{R}^n$ be a convex open set and, for each pair of points $\mathbf{x}, \mathbf{y} \in \Omega$, let $[\mathbf{x}, \mathbf{y}] = \{(1-t)\mathbf{x} + t\mathbf{y} : t \in [0,1]\}$ be the straight segment connecting the points. Show that if $\mathbf{f} : \Omega \to \mathbb{R}^n$ is continuous with continuous first derivatives, then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \sqrt{n}|\mathbf{x} - \mathbf{y}| \max_{\mathbf{z} \in [\mathbf{x},\mathbf{y}]} \|D\mathbf{f}(\mathbf{z})\|,$$

where

$$\|D\mathbf{f}(\mathbf{z})\|^2 = \sum_{i,j=1}^{n} \left| \frac{\partial f_i}{\partial x_j}(\mathbf{z}) \right|^2,$$

$f_1, \cdots, f_n$ being the components of $\mathbf{f}$.

Solution: This is the Mean-Value Theorem for vector valued functions.

Theorem 14. *Let $\boldsymbol{f} : S \to \mathbb{R}^m$ be continuous with continuous first derivative where $S \subseteq \mathbb{R}^n$ be an open convex set. Then for each pair of points $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, let $[\boldsymbol{x}, \boldsymbol{y}] = \{(1-t)\boldsymbol{x} + t\boldsymbol{y} : t \in [0,1]\}$ be the straight segment connecting the points. Then there exists for any $\boldsymbol{a} \in \mathbb{R}^m$, there exist $\boldsymbol{c} \in [\boldsymbol{x}, \boldsymbol{y}]$ such that*

$$\boldsymbol{a} \cdot (\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})) = \boldsymbol{a} \cdot (\boldsymbol{f}'(\boldsymbol{c})(\boldsymbol{y} - \boldsymbol{x}))\,.$$

Here $\boldsymbol{f}'(\boldsymbol{c})(\boldsymbol{y} - \boldsymbol{x})$ is the (total) derivative at $\mathbf{c}$ evaluated at point $\boldsymbol{y} - \boldsymbol{x}$.

Proof. Let $\gamma(t) = (1-t)\mathbf{x} + t\mathbf{y} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ and hence $[\mathbf{x}, \mathbf{y}] = \{\gamma(t) : t \in [0,1]\}$. Now consider

$$g(t) = \mathbf{a} \cdot \mathbf{f}(\gamma(t)).$$

Note that since $f \in C^1$, by the chain rule, we have

$$g'(t) = \mathbf{a} \cdot \mathbf{f}'(\gamma(t))(\mathbf{y} - \mathbf{x}).$$

Hence g is differentiable. Note that $g(0) = \mathbf{a} \cdot \mathbf{f}(\mathbf{x})$, $g(0) = \mathbf{a} \cdot \mathbf{f}(\mathbf{y})$. Hence by the (1 dimensional) Mean Value Theorem, there exists s such that

$$g(1) - g(0) = g(0) = \mathbf{a} \cdot \mathbf{f}(\mathbf{y}) - g(0) = \mathbf{a} \cdot \mathbf{f}(\mathrm{x}) = g'(s)(1-0) = \mathbf{a} \cdot \mathbf{f}'(\gamma(s))(\mathbf{y} - \mathbf{x}).$$

Let $\mathrm{c} = \gamma(s) = (1-s)\mathbf{x} + s\mathbf{y}$, then since S is convex, $z \in [\mathbf{x}, \mathbf{y}]$ and we have

$$\mathbf{a} \cdot (\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) = \mathbf{a} \cdot (\mathbf{f}'(\mathbf{c})(\mathbf{y} - \mathbf{x}))\,.$$

□

Note that $\mathbf{f} = (f_1, \cdots, f_n)$ and recall that for any vector $\mathbf{u}, \mathbf{v} \in \Omega$,

$$\mathbf{f}'(\mathbf{u})(\mathbf{v}) = (\nabla f_1(\mathbf{u}) \cdot \mathbf{v}, \cdots, \nabla f_n(\mathbf{u}) \cdot \mathbf{v}) = \sum_{i=1}^{n} (\nabla f_i(\mathbf{u}) \cdot \mathbf{v}) e_i$$

where

$$\nabla f_i(\mathbf{u}) = \left(\frac{\partial f_i}{\partial x_1}(\mathbf{u}), \cdots, \frac{\partial f_i}{\partial x_n}(\mathbf{u}) \right)$$

and e_i be the i-th component of the standard basis of $\mathbb{R}^n$. Now, we use the above Theorem, let

$$\mathbf{a} = \frac{\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\|} \implies \|a\| = 1$$

then

$$|\mathbf{a}\cdot(\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x}))\,| = \frac{\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\|^2}{\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\|} = \|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| = |\mathbf{a}\cdot(\mathbf{f}'(\mathbf{c})(\mathbf{y}-\mathbf{x}))\,|$$
$$\implies \|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\| \le \|a\|\|\mathbf{f}'(\mathbf{c})(\mathbf{y}-\mathbf{x})\| = \|\mathbf{f}'(\mathbf{c})(\mathbf{y}-\mathbf{x})\|$$

by the Cauchy-Schwarz Inequality and $\|a\| = 1$. Also, by the Cauchy-Schwarz Inequality again, we have

$$\begin{aligned}\|\mathbf{f}'(\mathbf{c})(\mathbf{y}-\mathbf{x})\| &= \left\|\sum_{i=1}^{n}(\nabla f_i(\mathbf{c})\cdot(\mathbf{y}-\mathbf{x}))e_i\right\| \le \sum_{i=1}^{n}\|(\nabla f_i(\mathbf{c})\cdot(\mathbf{y}-\mathbf{x}))e_i\| \\ &= \sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\cdot(\mathbf{y}-\mathbf{x})\|\|e_i\| = \sum_{i=1}^{n}|\nabla f_i(\mathbf{c})\cdot(\mathbf{y}-\mathbf{x})| \\ &\le \sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\|\|\mathbf{y}-\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}\|\sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\|.\end{aligned}$$

Finally, note that

$$\begin{aligned}\|D\mathbf{f}(\mathbf{c})\|^2 &= \sum_{i=1}^{n}\left(\left|\frac{\partial f_i}{\partial x_1}(\mathbf{c})\right|^2 + \cdots + \left|\frac{\partial f_i}{\partial x_n}(\mathbf{c})\right|^2\right) = \sum_{i=1}^{n}\left(\left(\frac{\partial f_i}{\partial x_1}(\mathbf{c})\right)^2 + \cdots + \left(\frac{\partial f_i}{\partial x_n}(\mathbf{c})\right)^2\right) \\ &= \sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\|^2.\end{aligned}$$

Hence by the Cauchy-Schwarz Inequality once again, we have

$$\begin{aligned}\left(\sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\|\right)^2 &= |(\|\nabla f_i(\mathbf{c})\|,\cdots,\|\nabla f_n(\mathbf{c})\|)\cdot(1,\cdots,1)|^2 \\ &\le \left(\sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\|^2\right)\left(\sum_{i=1}^{n}1\right) = n\|D\mathbf{f}(\mathbf{c})\|^2\end{aligned}$$

and consequently we have

$$\sum_{i=1}^{n}\|\nabla f_i(\mathbf{c})\| \le \sqrt{n}\|D\mathbf{f}(\mathbf{c})\| \le \sqrt{n}\max_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\|D\mathbf{f}(\mathbf{z})\|$$

since $\mathbf{c}\in[\mathbf{x},\mathbf{y}]$ in the Theorem. By choosing $\|\mathbf{u}-\mathbf{v}\| = |\mathbf{u},\mathbf{v}|$ be the Euclidean norm for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, we have

$$|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \le \sqrt{n}|\mathbf{x}-\mathbf{y}|\max_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\|D\mathbf{f}(\mathbf{z})\|.$$

Remark In the hint, they suggest we prove $f : S \to \mathbb{R}$ version: (just take $m = 1,\ a = 1$)

$$f(\mathbf{y}) - f(\mathbf{x}) = \nabla f(\mathbf{c})\cdot(\mathbf{y}-\mathbf{x}).$$

The proof is identical to the Theorem we proved and they key the proof of the above Theorem is also to reduce to the 1 dimensional Mean Value Theorem, so we do an extra inner product. □

3.9.14. *Will be updated.*
22S-5: implicit
22W-5,21W-5(a),20S-7,20W-6,19S-2,19W-7,06S-5(b),06W-5,04S-5(b),03W-5(b),97W-6(c): implicit sys of equation type

18S-2: derivative

14W-5(a),05W-5,03W-5(a),00W-5(a),96S-6(a),96W-5(b)(c): Inverse

04S-5(a)
03S-5

00W-4
97W-6(a)(b),96S-6(b),96W-5(a): differentiable

3.10. **Miscellaneous.**

3.10.1. *16W-3=15S-4.*

Problem 104: 16W-3=15S-4

For any $n \geq 2$ let $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, define the set

$$X = \{f : (0,1) \to \mathbb{R}; f \text{ continuous at all } x \in (0,1)\},$$

and let $|f|_n := \max_{x \in I_n} |f(x)|$.

(a) Show that for every $f, g \in X$, the quantity

$$d(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{|f-g|_n}{1 + |f-g|_n}$$

is finite and defines a metric on X.

(b) Suppose $f \in X$ and f_k is a sequence in X. Show that $f_k \to f$ in the metric on X if and only if $f_k \to f$ uniformly on I_n for every n.

(c) Show that (X, d) is complete.

Solution:

(a)
(b)
(c)

□

15S-2
03W-4: The ONLY Riemann Stieltjes (which I need to review that shit

3.11. **Empty.**

3.11.1. *Empty.*

Problem 105: Empty

Solution:

□

4. Link to Each Problem

The following are the links to each problem:
(* means it hasn't finished and numbers without links are left to be updated.)

4.1. **24S2.** 1*, 2*, 3*, 4*, 5(i)(ii)(iii)(iv), 6, 7*, 8*.

4.2. **24S1.** 1.(a)(b), 2, 3, 4.(a)(b)(c), 5, 6, 7.

4.3. **24W.** 1, 2, 3, 4, 5, 6, 7.(i)(ii)* (iii), 8

4.4. **23S.** 1, 2, 3 4 5 6.(a) (b)

4.5. **23W.** 1.(a)(b), 2, 3, 4, 5, 6, 7,

4.6. **22S.** 1, 2, 3, 4.(a)(b), 5, 6

4.7. **22W.** 1, 2, 3, 4, 5, 6, 7, 8

4.8. **21S.** 1, 2, 3, 4, 5, 6

4.9. **21W.** 1, 2, 3.(a)(b) 4.(a)(b) 5.(a)(b)

4.10. **20S.** 1, 2, 3.(a)(b), 4, 5.(a)(b), 6, 7

4.11. **20W.** 1, 2, 3, 4, 5.(i)(ii), 6, 7

4.12. **19S.** 1, 2, 3, 4, 5.(a)(b), 6, 7

4.13. **19W.** 1, 2, 3, 4, 5, 6, 7, 8

4.14. **18S.** 1, 2, 3, 4, 5

4.15. **18W.** 1, 2*, 3, 4, 5.(a)(b)*, 6

4.16. **17S.** No file

4.17. **17W.** 1, 2, 3, 4, 5, 6, 7

4.18. **16S.** 1, 2, (3,4,5: Tier 2), 6, 7

4.19. **16W.** 1, 2, 3, (4,5,6,7: Tier 2)

4.20. **15S.** 1, 2, 3, 4, (5,6,7,8: Tier 2)

4.21. **15W.** 1, 2, 3, 4, 5, (6,7: Tier 2) 7. 2)*3)(1),4),5): Tier 2)

4.22. **14S.** (1: Tier 2) 2, 3, 4, 5, 6, 7, 8, 9

4.23. **14W.** 1, 2.(a)(b)((c): Tier 2), (3,4: Tier 2), 5, 6*, 7.(a)(b)

4.24. **13S.** 1.(a)(b)*(c), 2, 3, 4, (5,6,7: Tier 2)

4.25. **13W.** 1, (2,3: Tier 2), 4, (5,6: Tier 2), 7, 8

4.26. **12S.** 1, (2,3: Tier 2), 4, (5: Tier 2), 6, (7: Tier 2), 8

4.27. **12W.** 1, 2, (3,4,5: Tier 2), 6, 7, (8,9: Tier 2)

4.28. **11S.** 1.(a)(b), 2.(a)*(b), (3,4: Tier 2), 5

4.29. **11W.** 1, 2, 3, 4, (5,6,7: Tier 2), 8

4.30. **07-10.** No files

4.31. **06S.** 1, 2*, 3, 4.(a)(b) 5.(a)(b), 6

4.32. **06W.** 1.(a)*(b)* 2.(a)*(b), 3, 4.(a)(b), 5, 6

4.33. **05S.** No file

4.34. **05W.** 1.(a)*(b)* 2.(a)(b), 3, 4.(a)(b), 5, 6

4.35. **04S.** 1.(a)*(b) 2.(a)(b), 3.(a)(b), 4, 5, 6

4.36. **04W.** No file

4.37. **03S.** 1.(a)(b)* 2.(a)(b), 3.(a)(b), 4.(a)(b) 5, 6

4.38. **03W.** 1.(a)(b) 2.(a)*(b)* 3.(a) (b), 4, 5, 6

4.39. **01-02.** No files

4.40. **00S.** No file

4.41. **00W.** 1.(a)*(b) 2.(a)(b) 3, 4, 5, 6

4.42. **98-99.** No files

4.43. **97S.** 1.(a)(b)(c) 2, 3, 4

4.44. **97W.** 1.(a)*(b)(c)*, 2.(a)(b)(c), 3.(a)(b)*(c), 4, 5, 6

4.45. **96S.** 1.(a)*(b), 2.(a)*(b), 3.(a)(b), 4, 5.(a)(b) 6, 7

4.46. **96W.** 1.(a)(b)(c)*(d)*, 2.(a)(b)(c), 3.(b)*(b), 4.(a)(b), 5.(a)(b) 6