Chapter 7. Boundary-condition problems in two and three dimensions: Spherical coordinates (14 Oct 2020).

A. Perspective.

Configurations defined by (spherical coordinate) = constant that can be solved with elementary functions or series consisting of elementary functions are essentially limited to empty space or spherical shells. Both of these have analytic Green functions, which are already familiar through previous treatments or homework assignments. Fortunately, spherical shapes are more common than shapes like (for example) ice-cream cones, so it is no real sacrifice to skip more exotic mathematics such as Legendre functions of the first kind, which in more familiar terms would involve spherical harmonics with complex l and m, and monomials raised to complex powers.

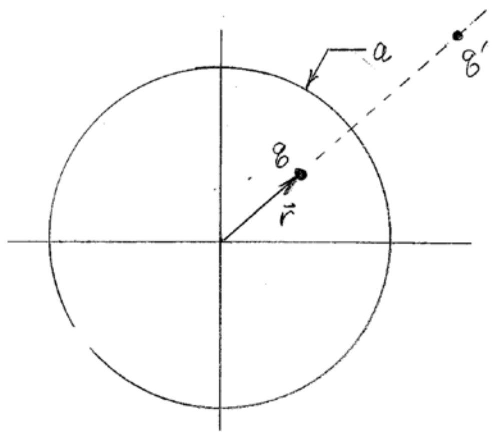
As with the previous two chapters, this chapter is mainly dedicated to series-expansion solutions of Laplace's Equation for the potential and the Green-function equation. We have already encountered analytic Green functions for empty space and spherical shells, so why consider series expansions? For four reasons. First, this is another area where we can apply basic methods, so we gain more experience by generating these expansions. Second, multipole expansions of the empty-space Green function underly the treatment of permittivity, permeability, and radiation, among other physical phenomena, and the expansion in spherical coordinates is far more efficient than a direct series expansion of the analytic expression. Third, if a phenomenon can be described accurately enough by only one or two leading terms in an expansion, much better physical insight can often be obtained from the expansion than from the analytic expression itself. Finally, the graduate-level E&M course is traditionally the one where graduate students receive instruction in higher math.

Following up on the last point, discussions of the associated Legendre polynomials and spherical harmonics constitute a large fraction of Jackson's Chs. 2 and 3. While we go beyond standard treatments, explaining how Legendre functions reduce to associated Legendre polynomials and finally Legendre polynomials, we adopt the more common approach of concentrating on empty space and spherical shells for specific applications. Examples follow, and the chapter ends with a treatment of multipole moments and the connection to spherical Bessel functions.

B. Image-charge solution of the spherical shell.

When discussing configurations involving Cartesian coordinates, we noted that if an image-charge solution exists for a configuration, start with that, because the Green function of the configuration is analytic. This makes more sense in Cartesian than in spherical coordinates, where most of our problems are more conveniently solved by expansions. However, we start by following the path taken with Cartesian coordinates, deriving the analytic Green function for a grounded spherical shell of radius *a* centered on the origin. This Green function has already appeared in several homework problems, so it should be familiar.

The configuration is shown in the diagram. A grounded spherical shell of radius *a* encloses a point charge *q* at $\vec{r}_q = (x_q, y_q, z_q) = (r_q, \theta_q, \varphi_q)$. The objective is to find the potential everywhere within the shell. In principle, the Green function follows immediately by setting $q = 1$ and replacing $\vec{r}_q$ with $\vec{r}\,'$. Jackson covers this in his Sec. 2.1, although his treatment is more accurately described as confirming the validity of the solution rather than obtaining the solution itself. The derivation provides us insight as to how mathematicians would approach this problem, so it is worth considerint in detail.



Drawing from our experience with the grounded plane, which can be considered the $a \to \infty$ limit of the shell, we discard the shell, extend the volume of interest to all space, place a second charge q' at a location $\vec{r}_{q'}$, then seek to determine whether a q' and $\vec{r}_{q'}$ exist such that the sum of the potentials from *q* and q' satisfies $\phi(a, \theta, \varphi) = 0$ for any θ and φ. In analytic form the potential is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_q|} + \frac{q'}{|\vec{r} - \vec{r}_{q'}|} \tag{7.1a}$$

$$= \frac{q}{\sqrt{r^2 + r_q^2 - 2rr_q \hat{r} \cdot \hat{r}_q}} + \frac{q'}{\sqrt{r^2 + r_{q'}^2 - 2rr_{q'} \hat{r} \cdot \hat{r}_{q'}}}. \tag{7.1b}$$

With the observer at $\vec{r} = a\hat{r}$, the equation to solve is therefore

$$0 = \frac{q}{\sqrt{a^2 + r_q^2 - 2ar_q \cos\gamma_q}} + \frac{q'}{\sqrt{a^2 + r_{q'}^2 - 2ar_{q'} \cos\gamma_{q'}}}, \tag{7.2}$$

where γ_q and $\gamma_{q'}$ are the angles between $\vec{r}$ and $\vec{r}_q$, and between $\vec{r}$ and $\vec{r}_{q'}$, respectively. With only the variables q', $r_{q'}$, and $\gamma_{q'}$ at our disposal, a general solution is impossible unless $\cos\gamma_q = \cos\gamma_{q'}$. Thus the two charges must lie on the same radius line, as shown in the figure. We can anticipate this from the image-charge solution for the conducting plane, which puts both charges on a line normal to the plane, noting that the shell solution must reduce to the planar result in the limit that a approaches infinity.

This leaves q' and its location $r_{q'}$ along the radius line to be determined. Looking again at Eq. (7.2), we identify one solution, specifically $q' = -q$ and $r_{q'} = r_q$. But we have already seen this before: this is just the trivial solution requiring ϕ to be zero outside the conductor. Thus as with the conducting plane, we look for a second solution that places $r_{q'} > a$, i.e., outside our volume of interest.

Maybe we can find a solution that makes the two radicals identical, for example one that effectively turns a^2 into r_q^2 and vice versa under the radical. This requires factoring out a^2 and replacing it with r_q^2, leaving a^2 behind. We see that this is possible if we set

$$r_{q'} = a^2 \big/ r_q \,. \tag{7.3}$$

Making this substitution we find

$$0 = \frac{q}{\sqrt{a^2 + r_q^2 - 2ar_q \cos\gamma_q}} + \frac{q'}{\sqrt{a^2 + \dfrac{a^4}{r_q^2} - 2a\dfrac{a^2}{r_q}\cos\gamma_q}} \tag{7.4a}$$

$$= \frac{q}{\sqrt{a^2 + r_q^2 - 2ar_q \cos\gamma_q}} + \frac{q'}{\dfrac{a}{r_q}\sqrt{r_q^2 + a^2 - 2r_q a \cos\gamma_q}} \,. \tag{7.4b}$$

If we set

$$q' = -q\frac{a}{r_q} \,, \tag{7.5}$$

we have succeeded – we have the nontrivial solution.

In principle Eq. (7.4b) can be converted to the Green function $G(\vec{r}, \vec{r}\,')$ by making the substitutions $q \to 1$, $\vec{r}_q \to \vec{r}$, and $\vec{r}_{q'} \to \vec{r}\,'$. However, $\vec{r}_{q'}$ does not appear. Moreover, in Eq. (7.4b) *a* plays the role of both $r_{q'}$ and the radius of the shell. To fix this we first note that Eq. (7.3) turns Eq. (7.5) into

$$q' = -q\frac{r_{q'}}{a} = -q\frac{r'}{a} \,. \tag{7.6}$$

Given this, we identify *a* in both radicals as r'. With these substitutions

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{a}{r'\left|\vec{r} - \dfrac{a^2}{(r')^2}\vec{r}\,'\right|} \tag{7.9}$$

and the problem is solved.

This is an equation that we have seen several times before. It is easily shown that $G(\vec{r}, \vec{r}\,') = G(\vec{r}\,', \vec{r})$. We also note that both $\vec{r}$ and $\vec{r}\,'$ are in the volume of interest, hence r and

r' are either *both* greater or *both* less than a. This places the second singularity outside the volume of interest, another requirement of a Green function. Finally, in specific applications involving a point charge q it is also straightforward to show that the integral over the inside of the shell of the surface charge density σ induced by q is equal to $-q$, which also follows because $\phi(\vec{r}) = 0$ outside the grounded shell.

The above derivation is specific to three dimensions. As seen in Ch. 6, while an image-charge relation can also be obtained for a cylinder of circular cross section, the math leading up to it is significantly different.

C. Laplacian in spherical coordinates: eigenfunctions and orthogonality relations.

We now consider series expansions. In spherical coordinates the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{7.10}$$

The appearance of a common factor r^{-2} in all terms suggests that for $\nabla^2\phi = 0$ the dependence of the radial eigenfunction on r might be much simpler than that of the polar eigenfunction on θ. This is the reverse of the situation for its cylindrical equivalent, where the angular and axial eigenfunctions are simple and the radial one is not. Although the eigenfunctions of $\partial^2/\partial\varphi^2$ are the usual $Q(\varphi) = e^{i\nu\varphi}$ or $e^{\pm\nu\varphi}$, for the configurations that we consider here only $e^{im\varphi}$, where m is a positive or negative integer, is used here. In addition, $e^{im\varphi}$ is usually absorbed with the eigenfunction $P(\theta)$ of the θ operator to form the spherical harmonic $Y_{lm}(\theta,\varphi)$.

We now develop the details. Using separation of variables, write

$$\nabla^2\phi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] R(r)P(\theta)Q(\varphi) = 0. \tag{7.11}$$

Recalling our experience with cylindrical coordinates in the two-dimensional limit, assume the trial function $R(r) = r^l$. Applying the radial operator we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) r^l = \frac{1}{r^2}\frac{d}{dr}\left(l r^{l+1}\right) = \frac{l(l+1)}{r^2} r^l.$$

It is easily verified that the same coefficient is obtained if $R(r) = r^{-l-1}$. The fact that there is a factor $1/r^2$ left over in both cases means that neither r^l nor r^{-l-1} are truly eigenfunctions of the r operator. However, they are particularly useful because this leftover factor of r^{-2} is just what we need to make r^{-2} a common factor in all 3 terms. It can therefore be eliminated. With $Q(\varphi) = e^{im\varphi}$ and $R(r)$ either r^l or r^{-l-1}, Eq. (7.11) reduces to

$$\nabla^2\phi = \left[l(l+1) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right]\frac{1}{r^2} R(r)P(\theta)e^{\pm im\varphi} = 0. \tag{7.12}$$

We have therefore

$$\left[l(l+1) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] P_l^m(\theta) = 0\,, \tag{7.13}$$

where the $P_l^m(\theta)$, the eigenfunctions of the θ operator, are the *associated Legendre polynomials* with eigenvalues $\left(-l(l+1)\right)$ for all allowed values of *m*. The eigenvalue is negative, showing that the solutions $P(\theta)$ are oscillatory, have orthogonality relations, and therefore can be used for expansions. When combined with the eigenfunctions $e^{im\varphi}$ of the φ operator and properly scaled, the results are the spherical harmonics $Y_{lm}(\theta,\varphi)$. Specifically,

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}\,, \tag{7.17}$$

where $l = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots, \pm l$.

To place the above in its most general context, write $\cos\theta = z$, in which case $\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{dz}$. Making this substitution and doing some algebra, Eq. (7.13) can be written as

$$\left((1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + l(l+1) - \frac{m^2}{1-z^2}\right) w_l^m(z) = 0\,. \tag{7.15}$$

Equation (7.15) defines the *Legendre functions* $\omega_l^m(z)$, where the *degree* is *l* and the *order* is m. Both degree and order are arbitrary complex constants.

The Legendre functions are well known. They are related to the Gauss hypergeometric function $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ according to

$$w_l^m(z) = \frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{m/2} {}_2F_1(-l, l+1; 1-m; \frac{1-z}{2})\,. \tag{7.16}$$

Unless you work with these functions routinely Eq. (7.16) probably doesn't mean much, but as physicists we can still extract some information from this expression by noting that the structure of the prefactor suggests (correctly) that the $\omega_l^m(z)$ can be separated into two categories depending on whether $|z|$ is lesser or greater than 1. If $|z| \leq 1$, then the $w_l^m(z) = P_l^m(z)$ are *Legendre functions of the first kind*, which are regular at $z = 0$. If $|z| > 1$, then the $w_l^m(z) = Q_l^m(z)$ are *Legendre functions of the second kind*, which diverge as $z \to 0$. In either case *l* and/or *m* can be complex. From our experience with cylindrical coordinates, complex *l* and *m* open the possibility of analyzing more complicated surfaces, for example expanding $\phi(\vec{r})$ along the radial coordinate of a cone with a spherical cap However, we do not pursue these complications here.

Focusing our attention on functions appropriate to spherical shells and point charges in empty space, we now restrict the Legendre functions of the first kind even further, to where *l* and *m* are integers, specifically $l \geq 0$, and $|m| \leq l$. These $w_l^m(z) = P_l^m(\cos\theta)$ are the *associated Legendre polynomials of the first kind*. The explicit use of $z = \cos\theta$ drives home two points: first, that the condition $|z| \leq 1$ is satisfied automatically, and second, that *z* is real. Both are necessary additional conditions for the existence of these functions. Jackson goes into considerable detail on their properties, although the only essential aspect for our development is one more reduction: to the *Legendre polynomials* $P_l(\cos\theta)$ for azimuthally symmetric configurations ($m = 0$). These are discussed in more detail below.

Rather than use them directly, it is more efficient to combine the $P_l^m(\cos\theta)$ with the complex exponentials $e^{im\varphi}$ and define a new set of functions, the *spherical harmonics* $Y_{lm}(\theta,\varphi)$:

Consistent with their negative eigenvalues $-l(l+1)$, these form a complete orthonormal set according to

$$\int_{-1}^{1} d(\cos\theta) \int_{0}^{2\pi} d\varphi \, Y^*_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}, \tag{7.18}$$

and obey the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi) = \delta(\cos\theta' - \cos\theta)\delta(\varphi - \varphi'). \tag{7.19}$$

The first few spherical harmonics are

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}; \tag{7.20a}$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta; \quad Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi}; \tag{7.20b,c}$$

$$Y_{20}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\left(3\cos^2\theta - 1\right); \quad Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\varphi}; \tag{7.20d,e}$$

$$Y_{2,\pm2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{\pm i2\varphi}. \tag{7.20f}$$

If $m = 0$, the φ dependences vanish. Since azimuthal symmetry is a common situation, it makes sense to define a new set of functions, the *Legendre polynomials*, according to

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{7.21}$$

The first few of these, typically written as $P_l(x)$, are

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{1}{2}(3x^2 - 1); \quad P_3(x) = \frac{5}{2}(x^3 - x). \tag{7.22a-d}$$

These are normalized to $P(1) = 1$ and obey the orthogonality relation

$$\int_{-1}^{1} dx P_{l'}(x) P_l(x) = \frac{2}{2l+1} \delta_{l'l} \ . \tag{7.23}$$

They have the same eigenvalue $(-l(l+1))$ as the $Y_{lm}(\theta, \varphi)$. Equation (7.23) shows that the $P_l(x)$ are orthogonal, but not orthonormal.

Other relations that are useful in azimuthally symmetric expansions are

$$P_l(0) = 0 \, , \tag{7.24a}$$

if l is odd; and

$$P_l(0) = \frac{(-1)^{l/2}}{2^l} \binom{l}{l/2} = \frac{(-1)^{l/2}}{2^l} \frac{l!}{\big((l-l/2)!\big)\big((l/2)!\big)} = \frac{(-1)^{l/2}}{2^l} \frac{l!}{\big((l/2)!\big)^2} \, , \tag{7.24b}$$

if l is even. Also, for $l \geq 1$

$$P_l(x) = \frac{1}{2l+1} \left(\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} \right) . \tag{7.25}$$

Equations (7.24) and (7.25) are useful in evaluating integrals over Legendre polynomials, and will be important in problem assignments.

A relevant relation connecting Legendre polynomials and spherical harmonics is the *addition theorem*:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \, , \tag{7.26a}$$

where

$$\cos\gamma = \hat{r} \cdot \hat{r}' = \sin\theta \, \sin\theta' \, \cos(\varphi - \varphi') + \cos\theta \, \cos\theta' \, , \tag{7.26b}$$

where γ is the angle between $\vec{r}$ and $\vec{r}\,'$. This follows by assuming that $\hat{r}' = \hat{z}$, in which case the configuration is trivially azimuthally symmetric and the angle between $\hat{r}$ and $\hat{r}'$ is the polar angle θ. This result is then projected into an arbitrary coordinate system, thereby yielding Eqs. (7.26).

The Legendre functions of the first kind are connected to the Bessel function of the previous chapter by

$$\lim_{\nu \to \infty} \left[\nu^{\mu} P_{\nu}^{-\mu}\big(\cos(x/\nu)\big) \right] = J_{\mu}(x) \, , \ \ x > 0 \, . \tag{7.27}$$

D. Series expansion of $\phi(\vec{r})$; the spherical shell.

The standard example for spherical coordinates is the potential given on a spherical shell of radius *a*. We start with the direct expansion of $\phi(\vec{r})$, which is the easiest if no charge density is present (four other approaches are given in Sec. F.) For the example, we assume that the shell is at a constant potential *V*. To capitalize directly on orthogonality, we write $V = \sqrt{4\pi}\, V\, Y_{00}(\theta,\varphi)$.

For $r < a$ the appropriate expansion is

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_{lm}(\theta,\varphi). \tag{7.28}$$

The coefficients are obtained through orthogonality:

$$\int_\Omega d(\cos\theta')d\varphi' V\, Y^*_{l'm'}(\theta',\varphi') = \int_\Omega d(\cos\theta')d\varphi' V\, (\sqrt{4\pi} Y_{00}) Y^*_{l'm'}(\theta',\varphi')$$

$$= \sqrt{4\pi}\, V \delta_{0l'}\delta_{0m'}$$

$$= \int_\Omega d(\cos\theta')d\varphi' \left(Y^*_{l'm'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_{lm}(\theta,\varphi) \right) = A_{l'm'}. \tag{7.30}$$

Thus

$$\phi(\vec{r}) = A_{00} r^0 Y_{00}(\theta,\varphi) = \sqrt{4\pi} V \frac{1}{\sqrt{4\pi}}$$

$$= V\,. \tag{7.31}$$

For the exterior solution, the only difference is that $r^l \to r^{-l-1}$. In this case $A_{00} = \sqrt{4\pi} V a$, and the result is

$$\phi(\vec{r}) = V \frac{a}{r}. \tag{7.32}$$

Because the source potential is azimuthally symmetric, the calculation can also be done with Legendre polynomials. In this case the factors $\sqrt{4\pi}$ do not appear. The results are the same.

E. Green function of empty space; potential of a ring of charge.

We next consider the Green function of empty space. Although Eq. (7.9) gives a closed-form analytic expression for $G(\vec{r},\vec{r}\,')$, expansions form the basis for the treatment of radiation, among other topics, since in the far-field region only the leading terms are required. Because we have not developed a set of orthogonal functions involving *r* (although we could), our only option is a double-summation expansion, where the delta function in the radial direction is generated by the second derivative of a continuous function with discontinuous slope. Accordingly, write

$$G_D(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi). \tag{7.33}$$

The A_{lm} are determined as usual by substituting Eq. (7.27) in the defining equation, applying orthogonality, then integrating over the singularity. Proceeding:

$$\nabla^2 G(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{l(l+1)}{r^2}\right)\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi)\,. \tag{7.34a}$$

$$= -4\pi\delta(r-r')\frac{1}{r}\delta(\cos\theta-\cos\theta')\frac{1}{r}\delta(\varphi-\varphi')\,. \tag{7.34b}$$

Here, we have replaced the angular part of the operator with its eigenvalue, and written the delta function in spherical coordinates. With the eigenvalue of r^l and r^{-l-1} both equal to $l(l+1)$, the expression is obviously zero, consistent with the delta function, except at the singularity, where it is ill-defined. As usual, we resolve this indeterminacy by integrating the expression over a vanishingly small region of *r* that contains r', thereby casting evaluation into well-defined regions.

We next capitalize on the orthogonality. Multiplying both sides by $Y_{l'm'}^*(\theta,\varphi)$, integrating over the solid angle, and using orthogonality gives

$$A_{l'm'}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{l'(l'+1)}{r^2}\right)\frac{r_<^{l'}}{r_>^{l'+1}} = -4\pi\delta(r-r')\frac{1}{r^2}Y_{l'm'}^*(\theta',\varphi')\,. \tag{7.35}$$

Now integrating both sides according to $\int_{r'-\delta}^{r'+\delta} r^2 dr$ and taking the limit $\delta \to 0$,

$$A_{l'm'}\left(r^2\frac{d}{dr} - l'(l'+1)\right)\frac{r_<^{l'}}{r_>^{l'+1}}\Bigg|_{r'-\delta}^{r'+\delta} = A_{l'm'}r'^2\left(-(l'+1)\frac{r'^{l'}}{r'^{l'+2}} - l'\frac{r'^{l'-1}}{r'^{l'+1}}\right). \tag{7.36a}$$

$$= -A_{l'm'}(2l'+1) = -4\pi\, Y_{lm}^*(\theta',\varphi')\,. \tag{7.36b,c}$$

The contribution from $l(l+1)/r^2$ vanishes in this limit. Therefore

$$A_{lm} = \frac{4\pi}{2l+1}Y_{lm}^*(\theta',\varphi')\,, \tag{7.37}$$

and

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\,. \tag{7.38}$$

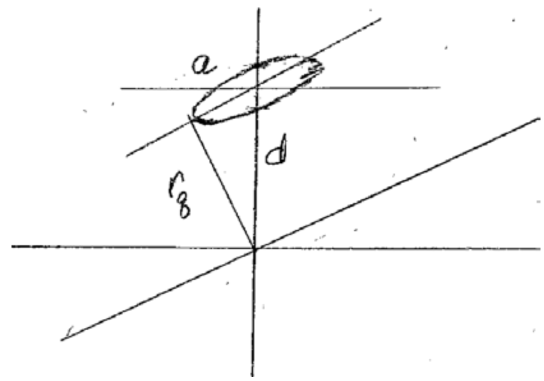
This form is particularly convenient because each term factors into a part belonging to the source and a part belonging to the observer. Jackson arrives at the same result by generalizing simpler expressions, but Eq. (7.38) is obtained naturally if one starts from first principles. This expression becomes the potential of a point charge when multiplied by *q* and with $\vec{r}\,'$ replaced with $\vec{r}_q$. If the configuration is azimuthally symmetric, then $m=0$ and Eq. (7.38) reduces to

$$G_D(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta') P_l(\cos\theta)\,. \tag{7.39}$$

Because calculations should be checked by reducing them to known results, we consider a point charge *q* located at the origin. Then $\rho(\vec{r}\,') = q\delta(\vec{r}\,')$. With $r' = 0$, only the $l = 0$ term survives. The result, obtained with either expression, is

$$\phi(\vec{r}) = 4\pi \frac{q}{r} Y_{00}^*(\theta,\varphi) Y_{00}(\theta,\varphi) = 4\pi \frac{q}{r}\frac{1}{4\pi} = \frac{q}{r}\,. \tag{7.40}$$

Another instructional example is to evaluate the potential of a ring of charge of radius *a* centered on the *z* axis, located at a distance *d* above the $z = 0$ plane and parallel to it. If the ring carries a total charge *Q*, then



$$\rho(\vec{r}\,') = \frac{Q}{2\pi a}\delta(r' - r_q)\frac{1}{r'}\delta(\theta' - \theta_o) \tag{7.41a}$$

where

$$r_q = \sqrt{a^2 + d^2}\;;\;\; \cos\theta_o = d/r_q\,. \tag{7.41b,c}$$

As a standard precaution, we verify that the volume integral of $\rho(\vec{r}\,')$ is indeed *Q*:

$$\int_V d^3r'\,\rho(\vec{r}\,') = \int_0^\infty r'^2\,dr' \int_{-1}^{1} d(\cos\theta') \int_0^{2\pi} d\varphi' \frac{Q}{2\pi a}\delta(r' - r_q)\frac{1}{r'}\delta(\theta' - \theta_o)\,. \tag{7.42}$$

Before the θ' integral can be evaluated, the differential must be brought into agreement with the argument of $\delta(\theta' - \theta_o)$. This is done by expanding the differential as $d(\cos\theta') = -\sin\theta'\,d\theta'$. In so doing the integration limits also change from (-1) and 1 to π and 0, which when reversed eliminates the minus sign of the differential. We obtain

$$\int_V d^3r'\,\rho(\vec{r}\,') = \int_0^\infty r'^2\,dr' \int_{-\pi}^{\pi} \sin\theta'\,d\theta' \int_0^{2\pi} d\varphi' \frac{Q}{2\pi a}\delta(r' - r_q)\frac{1}{r'}\delta(\theta' - \theta_o)\,. \tag{7.43}$$

Now the integration over $\delta(\theta' - \theta_o)$ can be performed. The result is

$$\int_V d^3r'\,\rho(\vec{r}\,') = \frac{Q}{2\pi a} r_q^2 \sin\theta_o \frac{1}{r_q} 2\pi = Q\,, \tag{7.44}$$

since $\sin\theta_o = a/r_q$. While it might appear that $\rho(\vec{r}) = (Q/2\pi a)\,\delta(\cos\theta' - \cos\theta_o)$ is equally valid, this version scales the effective charge density with θ' and hence gives the wrong result, as is easily verified.

Before considering the expansion versions of $G_D(\vec{r},\vec{r}\,')$, we investigate the analytic representation given by the bulk term in the standard Green-function expression for $\phi(\vec{r})$. Inserting the front end of Eq. (7.9) into this expression gives

$$\phi(\vec{r}) = \frac{Q}{2\pi a}\int_0^\infty r'\,dr'\int_{-\pi}^{\pi}\sin\theta'\,d\theta'\int_0^{2\pi}d\varphi'\frac{\delta(r'-r_q)\delta(\theta'-\theta_o)}{\sqrt{r^2+r'^2-2rr'\hat{r}\cdot\hat{r}'}} \tag{7.45a}$$

$$= \frac{Q}{2\pi}\int_0^{2\pi}d\varphi'\frac{1}{\sqrt{r^2+r_q^2-2rr_q(\sin\theta\sin\theta_o\cos(\varphi-\varphi')+\cos\theta\cos\theta_o)}}\,. \tag{7.45b}$$

This is an elliptic integral $K(z)$ of the first kind, although additional work is necessary to cast it into standard form.

Nevertheless, it is possible to show that the above expression is independent of the azimuth angle φ of the observer, which we expect because the symmetry is determined by the source, not the observer. Equation (7.45b) is of the form

$$I = \int_0^{2\pi}d\varphi'\;f(\cos(\varphi-\varphi')) = \int_0^{2\pi}d\varphi'\;f(\cos(\varphi'-\varphi))\,. \tag{7.46}$$

Changing variables leads to

$$I = \int_{-\varphi}^{2\pi-\varphi}d\varphi'\cos\varphi' = \int_0^{2\pi}d\varphi'\cos\varphi'\,, \tag{7.47}$$

which follows because $\cos\varphi$ is periodic in 2π. The point is therefore demonstrated. Equation (7.45b) is therefore

$$\phi(\vec{r}) = \frac{Q}{2\pi}\int_0^{2\pi}d\varphi'\frac{1}{\sqrt{r^2+r_q^2-2rr_q(\sin\theta\sin\theta_o\cos\varphi'+\cos\theta\cos\theta_o)}}\,. \tag{7.48}$$

Since a φ' dependence remains, azimuthal symmetry does *not* mean that the φ' integral is independent of φ'.

As a cross-check on the result, $\phi(\vec{r}) \to Q/r$ as $r \to \infty$. If the observer is on the *z* axis, then $\sin\theta = 0$ and

$$\phi(z) = \frac{Q}{\sqrt{z^2+r_q^2-2zr_q\cos\theta_o)}}\,. \tag{7.49}$$

We now consider the expansion of $1/|\vec{r}-\vec{r}'|$. Capitalizing on azimuthal symmetry we write

$$\phi(\vec{r}) = \sum_{l=0}^{\infty}\int_0^\infty r'^2\,dr'\int_0^\pi\sin\theta'\,d\theta'\int_0^{2\pi}d\varphi'\frac{Q}{2\pi a}\delta(r'-r_q)\frac{1}{r}\delta(\theta'-\theta_o)\left(\frac{r_<^l}{r_>^{l+1}}\right)P_l(\cos\theta')P_l(\cos\theta)$$

$$= Q\sum_l^\infty\left(\frac{r_<^l}{r_>^{l+1}}\sin\theta_o\,P_l(\cos\theta_o)\right)P_l(\cos\theta)\,, \tag{7.50}$$

where θ_o is that used in Eqs. (7.41). The θ' and φ' integrations are trivial, and independent of the greater than/less than decision involving r_q and r. We emphasize that this is between r and r_q, *not* between r and a or d. Again, as a cross-check on the result, $\phi(\vec{r}) \to Q/r$ as $r \to \infty$.

It is worth repeating that the system is rotationally symmetric thanks to the source, not the observer. The observer sees the azimuthal symmetry of the source configuration at any location.

F. Potential $V(\theta,\varphi)$ specified on a spherical shell: 5 approaches.

There are at least 5 way to address this calculation, two using direct expansions and three using different types of Green functions. Because we are emphasizing methods as well as results, we consider all five, since this provides an excellent opportunity to compare them. The section ends by applying the results to a spherical shell of radius a where the potential of the top half is $\phi(a,\theta,\varphi) = V$ and the bottom half zero. This section differs from the earlier example of constant V in that any function $V(\theta,\varphi)$ is allowed.

(1) Direct expansion of $\phi(\vec{r})$. Consider first the external solution. Write

$$\phi_{ext}(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi) \tag{7.51}$$

and use orthogonality where $\phi(a,\theta,\varphi) = V(\theta,\varphi)$. Then

$$\phi_{ext}(\vec{r}) == \sum_{l=0}^{\infty}\sum_{m=-l}^{l} V_{lm}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\varphi)\,, \tag{7.52a}$$

(7.52b)

where

$$V_{lm} = \int_{-1}^{1} d(\cos\theta')\int_{0}^{2\pi} d\varphi' V(\theta',\varphi')Y_{lm}^{*}(\theta',\varphi')\,. \tag{7.52c}$$

For the interior solution replace a/r with r/a and reduce the power of l by 1:

$$\phi_{\text{int}}(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} V_{lm}\left(\frac{r}{a}\right)^{l} Y_{lm}(\theta,\varphi)\,. \tag{7.53}$$

V_{lm} is not affected. The specific case of $V(\theta,\varphi) = V$ is done in Sec. D.

(2) If $V(\theta,\varphi) = V(\theta)$ is azimuthally symmetric,

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} A_l \frac{1}{r^{l+1}} P_l(\cos\theta)\,. \tag{7.54}$$

Orthogonality gives

$$\phi_{ext}(\vec{r}) = \sum_{l=0}^{\infty} V_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)\,, \tag{7.55}$$

where

$$V_l = \frac{2l+1}{2}\int_{-1}^{1} d(\cos\theta')V(\theta')P_l(\cos\theta')\,. \tag{7.56}$$

The interior solution is the same with a/r replaced with r/a and the power of l reduced by 1.

The remaining three approaches all use Green functions, calculating $\boldsymbol{\phi(\vec{r})}$ via

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int_S d^2r'\phi(\vec{r}\,')\hat{n}\cdot\nabla_{\vec{r}\,'}G_D(\vec{r},\vec{r}\,')\,. \tag{7.57}$$

In evaluating $\hat{n}\cdot\nabla_{\vec{r}\,'}G_D(\vec{r},\vec{r}\,')$, a major potential source of error is the sign, because $\hat{n}=\hat{r}$ for the interior solution and $\hat{n}=-\hat{r}$ for the exterior solution. The sign can be incorporated automatically by writing $\hat{n}=\hat{r}\,\mathrm{sgn}(a-r)$, for example

$$\hat{n}\cdot\nabla_{\vec{r}\,'}G_D(\vec{r},\vec{r}\,') = \mathrm{sgn}(a-r)\frac{\partial G}{\partial r'}\,, \tag{7.58}$$

but the writeups are less intimidating if the two results are listed as separate solutions.

(3) The first Green-function example is based on the analytic expression Eq. (7.9). The relevant derivative is

$$\frac{\partial G}{\partial r'} = -\frac{2r'-2r\cos\gamma}{2[r^2+r'^2-2rr'\cos\gamma]^{3/2}} + \frac{2(r^2/a^2)r'-2r\cos\gamma}{2[\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma]^{3/2}} \tag{7.59a}$$

$$= -\frac{a-r\cos\gamma}{[r^2+a^2-2ra\cos\gamma]^{3/2}} + \frac{2(r^2/a)-2r\cos\gamma}{2[r^2+a^2-2ra\cos\gamma]^{3/2}} \tag{7.59b}$$

$$= \frac{r^2-a^2}{a[r^2+a^2-2ra\cos\gamma]^{3/2}}\,, \tag{7.59c}$$

where

$$\cos\gamma = \hat{r}\cdot\hat{r} = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'\,. \tag{7.59d}$$

Combining this with the sgn function yields

$$\hat{n}\cdot\nabla\frac{\partial G}{\partial r'} = -\frac{|\,r^2-a^2\,|}{a[r^2+a^2-2ra\cos\gamma]^{3/2}}\,. \tag{7.60}$$

Then

$$\phi(\vec{r}) = \frac{a\,|\,r^2-a^2\,|}{4\pi}\int_{-1}^{1} d\left(\cos\theta'\right)\int_0^{2\pi} d\varphi'\frac{\phi(\theta',\varphi')}{[r^2+a^2-2ra\cos\gamma]^{3/2}}\,. \tag{7.61}$$

This result is valid for r either greater or less than a.

(4) Now consider expansions. We start with the Green functions of the shell, which differ according to whether the volume of interest is inside or outside. The spherical-harmonic expansion of the first term of $G_D(\vec{r},\vec{r}\,')$ of Eq. (7.9) is the same for both, and is given by Eq. (7.38). The expansions for the image-charge terms are not. We consider the inside solution first, where $r, r' < a$. Doing the math, the image term is

$$G_{D,\mathrm{int}}(\vec{r},\vec{r}\,') = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{1}{a}\left(\frac{rr'}{a^2}\right)^{l} Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\,. \tag{7.62}$$

The complete Green function for $r, r' < a$ is

$$G_{D,\mathrm{int}}(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{rr'}{a^2}\right)^{l}\right)Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\,. \tag{7.63a}$$

For $r, r' > a$

$$G_{D,ext}(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right)Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\,. \tag{7.63b}$$

Both expressions satisfy the general conditions that the functions are unchanged if $\vec{r}$ and $\vec{r}\,'$ are interchanged, and in both terms r appear to the correct power of l. To obtain Eqs. (7.63), the scaling factor (a/r') in Eq. (6.63a) remains a scaling factor: it cannot be brought into the denominator.

For spherically symmetric configurations the above reduce to their Legendre-polynomial equivalents:

$$G_{D,\mathrm{int}}(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{rr'}{a^2}\right)^{l}\right)P_l(\cos\theta')P_l(\cos\theta) \qquad \text{for } r<a\,; \tag{7.64a}$$

$$G_{D,ext}(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right)P_l(\cos\theta')P_l(\cos\theta) \qquad \text{for } r>a\,; \tag{7.64b}$$

The next step is to evaluate the derivatives. For the external Green function $\hat{n} = -\hat{r}$ and we have

$$-\hat{r}\cdot\nabla_{\vec{r}\,'}G_{D,ext} = -\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{(r')^l}{r^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right)_{r'=a} Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{7.65a}$$

$$= -\frac{4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1} Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\,. \tag{7.65}$$

Substituting this result into the surface integral yields Eq. (7.x), which it must.

If the configuration is azimuthally symmetric, the same calculation yields

$$-\hat{r}\cdot\nabla_{\vec{r}'}G_{D,ext}=-\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\left.\left(\frac{(r')^{l}}{r^{l+1}}-\frac{1}{a}\left(\frac{a^{2}}{rr'}\right)^{l+1}\right)\right|_{r'=a}P_{l}(\cos\theta')P_{l}(\cos\theta) \tag{7.66a}$$

$$=-\sum_{l=0}^{\infty}\left(\frac{2l+1}{a^{2}}\left(\frac{a}{r}\right)^{l+1}\right)P_{l}(\cos\theta')P_{l}(\cos\theta). \tag{7.66b}$$

Substituting this result into the surface integral gives

$$\phi_{ext}(\vec{r})=\sum_{l=0}^{\infty}\left(\frac{2l+1}{2}\left(\frac{a}{r}\right)^{l+1}\left(\int_{-1}^{1}d(\cos\theta')V(\theta')P_{l}(\cos\theta')\right)P_{l}(\cos\theta)\right), \tag{7.67}$$

which is what we had before.

Repeating the calculation for the interior solution yields the same result with the factor (a/r) inverted and the power of l reduced by one. The sign remains the same thanks to the fact that here $\hat{n}=\hat{r}$. The results are

$$\hat{r}\cdot\nabla_{\vec{r}'}G_{D,\mathrm{int}}=\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left.\left(\frac{r^{l}}{(r')^{l+1}}-\frac{1}{a}\left(\frac{rr'}{a^{2}}\right)^{l}\right)\right|_{r'=a}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{7.68a}$$

$$=-\frac{4\pi}{a^{2}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r}{a}\right)^{l}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi). \tag{7.68b}$$

(look to cite the earlier equation.) For azimuthally symmetric potentials $V(\theta)$,

$$\hat{r}\cdot\nabla_{\vec{r}'}G_{D,\mathrm{int}}=\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\left.\left(\frac{r^{l}}{(r')^{l+1}}-\frac{1}{a}\left(\frac{rr'}{a^{2}}\right)^{l}\right)\right|_{r'=a}P_{l}(\cos\theta')P_{l}(\cos\theta) \tag{7.69a}$$

$$=-\sum_{l=0}^{\infty}\left(\frac{2l+1}{a^{2}}\left(\frac{r}{a}\right)^{l}\right)P_{l}(\cos\theta')P_{l}(\cos\theta). \tag{7.69b}$$

Substituting this result into the surface integral gives

$$\phi_{\mathrm{int}}(\vec{r})=\sum_{l=0}^{\infty}\left(\frac{2l+1}{2}\left(\frac{r}{a}\right)^{l}\left(\int_{-1}^{1}d(\cos\theta')V(\theta')P_{l}(\cos\theta')\right)P_{l}(\cos\theta)\right). \tag{7.70}$$

As an application, let $V(\theta,\varphi)=V$ for the top hemisphere $0\leq\theta\leq\pi/2$ and zero for the bottom hemisphere. Considering the external solution, the expression for ϕ is

$$\phi_{ext}(\vec{r})=\sum_{l=0}^{\infty}\left\{V\frac{(2l+1)}{2}\int_{0}^{1}P_{l}(\cos\theta)d\theta\right\}\left(\frac{a}{r}\right)^{l+1}P_{l}(\cos\theta), \tag{7.71}$$

where the term in large braces is the coefficient A_l. Equation (7.25) allows the integral to be evaluated for $l \geq 1$:

$$\int_0^1 dx P_l(x) = \frac{1}{2l+1}\left(P_{l+1}(x) - P_{l-1}(x)\right)_0^1 = \frac{1}{2l+1}\left(P_{l-1}(0) - P_{l+1}(0)\right). \tag{7.72}$$

The contribution from the upper limit vanishes since $P_l(1) = 1$ for all l. Now using Eqs. (7.24), we obtain

$$A_l = V\frac{(-1)^{(l-1)/2}((l-1)!)a^{l+1}}{2^{(l-1)}\left(((l-1)/2)!\right)^2}\frac{2l+1}{l+1}. \tag{7.73}$$

The term $l = 0$ must be treated as a special case. From the original integral and using $P_0(\cos\theta) = 1$, it is evident that A_0 is just the average potential, or in this case $A_0 = V/2$.

The integral cannot be done in general, but if the observer is on the positive *z* axis, then $\theta = 0$ and $\cos\gamma = \cos\theta'$. Now if V is either a constant or zero, the integrals can be done in closed form. The integral over φ' simply delivers a factor of 2π. Taking the more general route of supposing that $\phi(\theta) = V$ for $\cos\theta_0 \leq \cos\theta \leq \cos\theta_1$ then:

$$\phi(z) = \frac{a\,|z^2 - a^2|\,V}{4\pi} 2\pi \int_{\cos\theta_0}^{\cos\theta_1} d\left(\cos\theta'\right)\frac{1}{[z^2 + a^2 - 2za\cos\theta']^{3/2}} \tag{7.74a}$$

$$= \frac{a\,|z^2 - a^2|\,V}{2}\int_{-\cos\theta_0}^{-\cos\theta_1} d\left(-\cos\theta'\right)\frac{1}{[z^2 + a^2 + 2za\cos\theta']^{3/2}} \tag{7.74b}$$

$$= \frac{a\,|z^2 - a^2|\,V}{4za}\int_{-2xa\cos\theta_1}^{-2xa\cos\theta_0} d\left(2za\cos\theta'\right)\frac{1}{[z^2 + a^2 + 2za\cos\theta']^{3/2}} \tag{7.74c}$$

$$= \frac{|z^2 - a^2|\,V}{4z}\left[-2\frac{1}{\sqrt{z^2 + a^2 + x}}\right]_{x=-2za\cos\theta_1}^{x=-2za\cos\theta_0} \tag{7.74d}$$

$$= \frac{|z^2 - a^2|\,V}{2z}\left[\frac{1}{\sqrt{z^2 + a^2 - 2za\cos\theta_1}} - \frac{1}{\sqrt{z^2 + a^2 - 2za\cos\theta_0}}\right] \tag{7.74e}$$

and we have a solution. For our example with $\phi = V$ for $0 \leq \theta \leq \pi/2$, Eq. (7.74e) reduces to

$$\phi(z) = \frac{|z^2 - a^2|\,V}{2z}\left[\frac{1}{|z - a|} - \frac{1}{\sqrt{z^2 + a^2}}\right] \tag{7.75}$$

A short calculation shows that this reduces to $\phi(a) = V$, as it must.

What about the situation if the entire shell is at a potential *V*? In this case $\cos\theta_0 = -1$ and $\cos(\theta_1) = 1$. Since the configuration is isotropic, any axis *z* is equivalent, so in the above we can replace *z* with *r*. Continuing:

$$\phi(r) = \frac{a\,|r^2-a^2|\,V}{2ra}\left[\frac{1}{\sqrt{r^2+a^2-2ra}} - \frac{1}{\sqrt{r^2+a^2+2ra}}\right] \tag{7.76a}$$

$$= V\frac{|r^2-a^2|}{2r}\left[\frac{1}{|r-a|} - \frac{1}{|r+a|}\right] \tag{7.76b}$$

If $r < a$ then this reduces to

$$\phi(\vec{r}) = V\frac{|r^2-a^2|}{2r}\left[\frac{1}{a-r} - \frac{1}{r+a}\right] = V\frac{a^2-r^2}{2r}\left[\frac{r+a-a+r}{a^2-r^2}\right] \tag{7.77}$$

$$= V\,.$$

If $r > a$ this reduces to

$$\phi(\vec{r}) = V\frac{a}{r}\,. \tag{7.78}$$

Thus everything reduces as it should. Math is nothing if not consistent.

G. Multipole moments.

Jackson discusses multipoles in his Sec. 4.1. Multipole expansions provide an increasingly detailed overview of a localized charge distribution $\rho(\vec{r}\,')$ for an observer at $\vec{r}$, where $|\vec{r}| >> |\vec{r}\,'|$. They come in two varieties depending on whether spherical or Cartesian coordinates are used. The spherical-coordinate multipole moments q_{lm} are more efficient, since their number for a given *l* is equal to the number of *m*, or $(2l+1)$. The Cartesian moments are the same as the spherical-coordinate multipole moments for $l = 0$ and 1, but rapidly get out of hand for higher numbers. For example, there are 9 instead of 5 Cartesian moments for $l = 2$. We return to this below.

In the meantime, starting with the spherical-coordinate version, we suppose a local charge distribution $\rho(\vec{r})$, then substitute it in the volume term of the general expression for $\phi(\vec{r})$ using the Green-function expansion Eq. (7.38). The result is

$$\phi(\vec{r}) = \int_V d^3r'\,\rho(\vec{r}\,')\left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{(r')^l}{r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right). \tag{7.79}$$

Factoring out the $\vec{r}\,'$ dependence, write

$$q_{lm} = \int_V d^3r'\,r'^l\,\rho(\vec{r}\,')Y_{lm}^*(\theta',\varphi')\,, \tag{7.80}$$

then rewrite Eq. (7.79) as

$$\phi(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{q_{lm}}{r^{l+1}}\,Y_{lm}(\theta,\phi)\,. \tag{7.81}$$

In Eq. (7.79) it is understood that $|\vec{r}\,'| << |\vec{r}|$, where $\vec{r}\,'$ describes the largest extent of ρ. If the charge distribution is spherically symmetric, then the above expressions can be simplified using

$$\sqrt{\frac{4\pi}{2l+1}}Y_{l0}(\theta,\varphi) = P_l(\cos\theta)\,. \tag{7.82}$$

A list of spherical harmonics, Legendre polynomials, and relations involving spherical harmonics and associated Legendre polynomials is given in Appendix 1.

Because the multipole contributions to $\phi(\vec{r})$ are proportional to r^{-l-1} if the distance between source and observer is reasonable, only a few terms may be necessary for an adequate description of $\phi(\vec{r})$. The dipole is particularly important because, as noted above, it provides the basis for understanding the macroscopic dielectric function and permeability of materials on the atomic scale. In particular, since magnetic monopoles have yet to be found, the dipole is the first nonvanishing contributor to magnetics. Finally, unless a process is forbidden, the dipole term is usually all is needed to calculate the physical response of a system to any external field.

The Cartesian multipoles are defined by the expansion

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}\,'|} &= \left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right)^{-1/2} \\ &= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2r^5}\left(3(\vec{r}\cdot\vec{r}\,')^2 - r^2 r'^2\right) + \ldots\end{aligned} \tag{7.83}$$

where the leading part of the last term is from a second-order expansion of the second term in braces in Eq. (7.83), and the trailing part from a first -order expansion. The monopole, dipole and quadrupole moments are obtained by integrating $\rho(\vec{r}\,')$ over $\vec{r}\,'$. This operation leads to the definitions

$$q = \int_V d^3r'\,\rho(\vec{r}\,')\,; \tag{7.84a}$$

$$\vec{p} = \int_V d^3r'\,\vec{r}\,'\rho(\vec{r}\,')\,; \tag{7.84b}$$

and

$$Q_{ij} = \int_V d^3r'\,\rho(\vec{r}\,')\left(3r_i' r_j' - r'^2\,\delta_{ij}\right). \tag{7.84c}$$

Then

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{r_i r_j}{r^5} + \ldots \tag{7.85}$$

for $i, j = 1,\ 2,\ 3$.

To within scaling factors, the spherical and Cartesian monopole and dipole moments map directly onto each other for $l=0$ and $l=1$. However, for $l=2$ there are nine Q_{ij} and only five q_{2m}. As Jackson notes, this discrepancy occurs because the Q_{ij} are reducible. Three are immediately eliminated as independent quantities because $Q_{ij} = Q_{ji}$. Another elimination follows because $Q_{11} + Q_{22} + Q_{33} = 0$, as easily verified from Eq. (7.84c). Therefore, the nine Q_{ij} reduce to five independent quantities, so the spherical and Cartesion sets remain consistent. However, it can be appreciated that with 27 Cartesian 19ctupole moments vs. seven q_{3m}, the Cartesian expansion rapidly gets out of hand.

H. Connection to spherical Bessel functions.

Monomial eigenfunctions were also obtained with cylindrical coordinates, which we found were limits of Bessel functions of argument $(k\rho)$ as $k \to 0$. The same is true in three dimensions, , but the connection is not as direct. To show this, we must introduce an extra dimension (time) via the homogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\psi(\vec{r},t) = 0\,, \tag{7.86}$$

then assume a time dependence $e^{-i\omega t}$. Substituting $\psi(\vec{r},t) = \psi(\vec{r})e^{-i\omega t}$ yields the Helmholtz Equation

$$(\nabla^2 + k^2)\psi(\vec{r}) = 0\,. \tag{7.87}$$

Solutions of the Helmholtz Equation are well known in electromagnetic and quantum-mechanical scattering theory, and consist of the spherical Bessel functions $j_l(kr)$ and $y_l(kr)$.

The procedure is as follows. Inserting the eigenvalues for the angular operator gives the starting equation

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) - \frac{l(l+1)}{r^2} + k^2 \right) R(r)Y_{lm}(\theta,\varphi) = 0\,. \tag{7.88}$$

One immediate difference relative to the cylindrical equivalent is that the eigenvalue appearing in Eq. (7.88) is *l*, not *m*. Thus when Eq. (7.88) is reduced to Bessel's Equation, the order of the Bessel function will be determined by *l*. We can meet the requirement that the argument of the Bessel function is dimensionless by multiplying Eq. (7.88) by r^2 and defining the dimensionless variable *z* as $z = kr$. With this change Eq. (7.88) becomes

$$\left(\frac{\partial}{\partial z}\left(z^2 \frac{\partial}{\partial z} \right) - l(l+1) + z^2 \right) R(z) = 0\,. \tag{7.89}$$

It is conventional to evaluate the first of the derivatives in the left-hand term, in which case Eq. (7.89) becomes

$$\left(z^2 \frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} - l(l+1) + z^2 \right) R(z) = 0\,. \tag{7.90}$$

If we compare this to the second form of Bessel's Equation given in Ch. 6, we see that it has some possibilities. The first derivative in Eq. (7.90) is multiplied by 2 instead of 1, and ν^2 is replaced by $l(l+1)$. But suppose we define a new function $R'(z)$ according to

$$R(z) = \frac{R'(z)}{\sqrt{z}}\,. \tag{7.91}$$

Substituting Eq. (7.91) into the individual terms in Eq. (7.90) gives

$$0=\left(z^{3/2}\frac{d^2R'}{dz^2}-\sqrt{z}\frac{dR'}{dz}+\frac{3}{4}\frac{R'}{\sqrt{z}}\right)+\left(2\sqrt{z}\frac{dR'}{dz}-\frac{R'}{\sqrt{z}}\right)-l(l+1)\frac{R'}{\sqrt{z}}+z^{3/2}R'. \tag{7.92}$$

Combining terms and multiplying the above expression by $\sqrt{z}$ yields

$$0=z^2\frac{d^2R'}{dz^2}+z\frac{dR'}{dz}-\frac{1}{4}R'-l(l+1)R'+z^2R'. \tag{7.93}$$

This is starting to look promising, but we can do better. Let's define

$$l'=l-1/2. \tag{7.94}$$

Making this substitution, Eq. (7.93) becomes

$$0=\left(z^2\frac{d^2}{dz^2}+z\frac{dR'}{dz}-l'^2+z^2\right)R' \tag{7.95}$$

and the parallel with Bessel's Equation is exact. Recalling that l is an integer, the l' are half-integers.

The eigenfunctions of the radial operator are the *spherical Bessel functions*. Those regular at the origin are

$$j_n(z)=\sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z), \tag{7.96}$$

where n is defined according to

$$l'=n+1/2=l-1/2. \tag{7.97}$$

These can be expressed in terms of elementary functions. For example

$$j_o(z)=\frac{\sin z}{z}; \tag{7.98a}$$

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z}; \tag{7.98b}$$

etc. Those for the functions that diverge at $z=0$ are

$$y_o(z)=-\frac{\cos z}{z}; \tag{7.99a}$$

$$y_1(z)=-\frac{\cos z}{z^2}-\frac{\sin z}{z}; \tag{7.99b}$$

etc.

To connect to our electrostatic results, we write $z=kr$, and for the functions regular at the origin take the limit

$$f_n(r) = \lim_{k\to 0}\left(\frac{1}{k^n} j_n(kr)\right) = \frac{r^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)} . \tag{7.100}$$

The corresponding result for the functions $y_n(kr)$ that diverge at the origin is

$$g_n(r) = \lim_{k\to 0}\left(k^{n+1} y_n(kr)\right) = -\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}{r^{n+1}} . \tag{7.101}$$

Thus the connection to electrostatics is obtained. We will see more of spherical Bessel functions when we discuss radiation, scattering, and diffraction.