Math 672 Lecture 26

Theorem: Let V be a complex inner product space and let $T \in \mathcal{L}(V)$. Then

$$\langle T(v), v \rangle_V \in \mathbb{R} \quad \forall v \in V$$

iff T is self-adjoint.

Proof: If T is self-adjoint, then, for any $v \in V$,

$$\langle T(v), v \rangle_V = \langle v, T(v) \rangle_V \quad \text{(because } T \text{ is self-adjoint)}$$
$$= \overline{\langle T(v), v \rangle_V} \quad \text{(by properties of } \langle \cdot, \cdot \rangle\text{)}.$$

Thus $\langle T(v), v \rangle_V \in \mathbb{R}$.

Conversely, suppose $\langle T(v), v \rangle_V \in \mathbb{R}$ for all $v \in V$. Then, for $v \in V$,

$$\langle T(v), v \rangle_V = \langle v, T^*(v) \rangle_V \Rightarrow \langle T^*(v), v \rangle_V = \langle T(v), v \rangle_V$$

(since $\langle v, T^*(v) \rangle_V \in \mathbb{R}$), and therefore

$$\langle (T^* - T)(v), v \rangle_V = 0 \quad \forall v \in V.$$

By the above result, this implies that $T^* - T = 0$, that is, $T^* = T$. //

Theorem: Let V be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), let $T \in \mathcal{L}(V)$ be a self-adjoint operator, and let $v_1, v_2 \in V$ be eigenvectors of T corresponding to distinct eigenvalues λ_1, λ_2. Then v_1 and v_2 are orthogonal.

Proof: We have

$$\lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle T v_1, v_2 \rangle = \langle v_1, T v_2 \rangle \quad (\text{since } T \text{ is self-adjoint})$$

$$= \langle v_1, \lambda_2 v_2 \rangle$$

$$= \lambda_2 \langle v_1, v_2 \rangle \quad (\text{since } \lambda_2 \in \mathbb{R})$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$$

$$\Rightarrow \langle v_1, v_2 \rangle = 0 \quad (\text{since } \lambda_1 - \lambda_2 \neq 0). //$$

Note that self-adjoint operators have special properties w.r.t. eigenvalues and eigenvectors: All eigenvalues are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal. Another class of operators with special properties is the following:

Definition: Let V be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{L}(V)$. We say that T is <u>normal</u> iff $T^*T = TT^*$.

(Every self-adjoint operator is normal, but not vice versa.)

Recall: If V is a finite-dimensional vector space over F and $T \in \mathcal{L}(V)$, then T is diagonalizable iff there exists a basis $\{v_1, \dots, v_n\}$ of V consisting of eigenvectors of T, say

$$T(v_j) = \lambda_j v_j, \quad j = 1, 2, \dots, n.$$

(Then the matrix of T w.r.t. $\{v_1, \dots, v_n\}$ is $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$.)

Note that

$$T\left(\sum_{j=1}^{n} \alpha_j v_j\right) = \sum_{j=1}^{n} \lambda_j \alpha_j v_j \quad \forall \sum_{j=1}^{n} \alpha_j v_j \in V.$$

Consider the case that V is a complex inner product space and there exists an orthonormal basis $\{v_1, \dots, v_n\}$ of V consisting of eigenvectors of T. Then

$$v \in V \Rightarrow v = \sum_{j=1}^{n} \langle v, v_j \rangle v_j$$

$$\Rightarrow T(v) = \sum_{j=1}^{n} \lambda_j \langle v, v_j \rangle v_j \quad \forall v \in V.$$

What's the adjoint of T?

$$\langle T(v), u \rangle = \left\langle \sum_{j=1}^{n} \lambda_j \langle v, v_j \rangle v_j, u \right\rangle$$

$$= \sum_{j=1}^{n} \lambda_j \langle v, v_j \rangle \langle v_j, u \rangle$$

$$= \left\langle v, \sum_{j=1}^{n} \overline{\lambda_j} \overline{\langle v_j, u \rangle} v_j \right\rangle$$

$$= \langle v, \sum_{j=1}^{n} \bar{\lambda}_j \langle u, v_j \rangle v_j \rangle.$$

Thus

$$T^*(u) = \sum_{j=1}^{n} \bar{\lambda}_j \langle u, v_j \rangle v_j \quad \forall\, u \in V.$$

<u>If every λ_j is real</u>, then $\bar{\lambda}_j = \lambda_j$ $\forall j$ and $T^* = T$, that is, T is self-adjoint.

<u>If not all λ_j's are real</u>, then we have $T^*T = TT^*$ (since $\bar{\lambda}_j \lambda_j = \lambda_j \bar{\lambda}_j$) and thus T is normal. Here is the verification that $T^*T = TT^*$:

$$T^*(T(v)) = \sum_{j=1}^{n} \bar{\lambda}_j \langle T(v), v_j \rangle v_j$$

$$= \sum_{j=1}^{n} \bar{\lambda}_j \langle \sum_{i=1}^{n} \lambda_i \langle v, v_i \rangle v_i, v_j \rangle v_j$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \bar{\lambda}_j \lambda_i \langle v, v_i \rangle \langle v_i, v_j \rangle v_j$$

$$= \sum_{j=1}^{n} \bar{\lambda}_j \lambda_j \langle v, v_j \rangle v_j \quad (\text{since } \langle v_i, v_j \rangle = \delta_{ij}),$$

$$T(T^*(v)) = \text{---} = \sum_{j=1}^{n} \lambda_j \bar{\lambda}_j \langle v, v_j \rangle v_j = T^*(T(v)).$$

Thus, if T can diagonalized by an orthonormal basis of eigenvectors, then T is normal (and self-adjoint in the special case that all eigenvalues are real). We want to prove the converses:

- If T is normal, then T can be diagonalized by an orthonormal basis of eigenvectors.
- If T is self-adjoint, then T can be diagonalized by an orthonormal basis of eigenvectors, and the eigenvalues are real.

We need the following properties of normal operators.

<u>Lemma</u>: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$). Then

$$\|T(v)\| = \|T^*(v)\| \quad \forall v \in V$$

iff T is normal.

$$(T^*T - TT^*)^* =$$

<u>Proof</u>: We have

$$T \text{ is normal} \iff T^*T - TT^* = 0$$

(Note that $T^*T - TT^*$ is self-adjoint)

$$\iff \langle (T^*T - TT^*)(v), v \rangle = 0 \quad \forall v \in V$$

$$\iff \langle T^*T(v), v \rangle = \langle TT^*(v), v \rangle \quad \forall v \in V$$

$$\iff \langle T(v), T(v) \rangle = \langle T^*(v), T^*(v) \rangle \quad \forall v \in V$$

$\iff \|T(v)\|^2 = \|T^*(v)\|^2 \quad \forall v \in V.$ //

Lemma: Let V be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{L}(V)$ be normal. If λ, v is an eigenpair of T, then $\bar{\lambda}, v$ is an eigenpair of T^* (λ, v in the real case).

Proof: Recall that $(\lambda I)^* = \bar{\lambda} I$ and hence

$$(T - \lambda I)^* = T^* - \bar{\lambda} I.$$

If λ, v is an eigenpair of T, then

$$\begin{aligned}(T-\lambda I)v = 0 &\Rightarrow \|(T-\lambda I)^* v\| = 0 \\ &\Rightarrow \|(T^* - \bar{\lambda} I)v\| = 0 \\ &\Rightarrow (T^* - \bar{\lambda} I)v = 0,\end{aligned}$$

and hence $\bar{\lambda}, v$ is an eigenpair of T^*. //

Theorem: Let V be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), let $T \in \mathcal{L}(V)$ be normal, and let v_1, v_2 be eigenvectors of T corresponding to distinct eigenvalues λ_1, λ_2, respectively, of T. Then v_1 and v_2 are orthogonal.

Proof: By assumption, $T(v_1) = \lambda_1 v_1$ and $T(v_2) = \lambda_2 v_2$ ($v_1 \neq 0, v_2 \neq 0$), and $T^*(v_2) = \bar{\lambda}_2 v_2$ by the preceding result. We thus have

($T^*(v_2) = \lambda_2 v_2$ in the real case)

$$\lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle T(v_1), v_2 \rangle$$

$$= \langle v_1, T^*(v_2) \rangle$$

$$= \langle v_1, \bar{\lambda}_2 v_2 \rangle \qquad (= \langle v_1, \lambda_2 v_1 \rangle)$$

$$= \lambda_2 \langle v_1, v_2 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$$

$\Rightarrow \langle v_1, v_2 \rangle = 0$ (since $\lambda_1 \neq \lambda_2$ by assumption). //