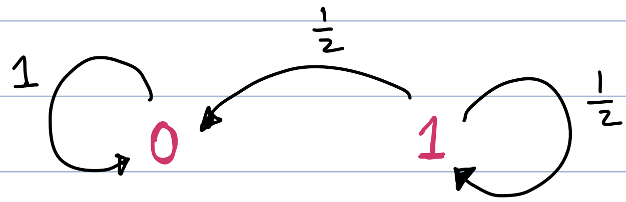
② $Q = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$ is <u>not</u> regular

since $p_{01}^{(n)} = 0 \quad \forall n$.



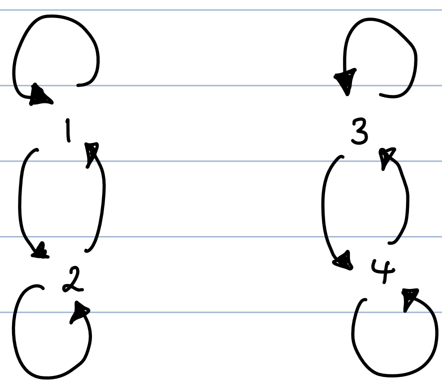
Some chains satisfy the following weaker condition.

<u>Definition</u> : X is irreducible if $\forall i, j \in S$,

$\exists\, n_0 < \infty$ s.t $P_{ij}^{(n_0)} > 0$.

Ex : ①

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{array}$$



Q is <u>not</u> irreducible (Q is reducible).

② $$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is irreducible but not regular.

$$Q^2 = Q^4 = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q^3 = Q^5 = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore, $P_{ii}^{(n)} > 0 \iff n$ is even.

We say Q is periodic with period 2.

First Passage Time

Now we are interested in determining how long it takes for X to travel from one state to another.

Definition: Let X be a MC with $X_0 = i$.

(a) When $i \neq k$, the first passage time from i to k is defined to be

$$T_{ik} = \min\{ n > 0 : X_n = k \mid X_0 = i \}$$

The mean first passage time is

$$\mu_{ik} = E[T_{ik}].$$

(b) When $i = k$, the recurrence time of i is defined to be

$$T_i = \min\{ n > 0 : X_n = i \mid X_0 = i \}$$

The mean recurrence time is

$$\mu_i = E(T_i)$$

Ex:

$$Q = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$X_0 = 1.$

i) Find μ_{12}

ii) Find μ_1.

i)
$$P(T_{12} = r) = \frac{2}{3}\left(\frac{1}{3}\right)^{r-1} \; ; \; r \geq 1$$

$$\mu_{12} = E[T_{12}] = \frac{3}{2}$$

ii)
$$P(T_1 = r) = \begin{cases} \frac{1}{3} & ; r = 1 \\ \frac{2}{3} \cdot \left(\frac{3}{4}\right)^{r-2} \cdot \frac{1}{4} & ; r > 1 \end{cases}$$

$$\mu_1 = \frac{1}{3} + \sum_{r=2}^{\infty} r \cdot \frac{1}{6} \cdot \left(\frac{3}{4}\right)^{r-2} = \boxed{\frac{11}{3}}$$

Stationary Distributions

How does a MC $\{X_n\}$ behave in the long run?

Definition: The vector $\pi = (\pi_j)_{j \in S}$ is called a stationary distribution if

i) $\pi_j \geq 0 \quad \forall j \in S$

ii) $\sum_{j=1} \pi_j = 1$

iii) $\pi = \pi P$ or $\pi_j = \sum_{i \in S} \pi_i P_{ij}, \forall j$.

Note that,

$$\pi P = \pi$$

$$\pi P^2 = \pi P = \pi$$

$$\vdots$$

$$\pi P^n = \pi, \quad \forall n \geq 0.$$

We say $\{X_n\}$ is stationary because the distribution of X_n does not change a time passes.

If $\pi_i = P(X_0 = i)$, $i \in S$, then π is the distribution of X_0.

Then the distribution $\pi^{(n)}$ of X_n is

$$\pi^{(n)} = \pi \cdot Q^{(n)} = \pi .$$

Ex: Find the stationary distribution of X with a trasition matrix

(a) $$Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

Solution : $$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

$$0.8\pi_1 + 0.3\pi_2 = \pi_1 \implies \pi_2 = \frac{2}{3}\pi_1$$

Since $$\pi_1 + \pi_2 = 1$$

$$\pi_1 + \frac{2}{3}\pi_1 = 1 \implies \pi_1 = \frac{3}{5} = 0.6$$

$$\pi_2 = \frac{2}{5} = 0.4$$

In the long run, the MC will be in state 1 60% of the time.

MATH 630

11/18 Problems (Markov Chains)

Name:

Let $\{X_n\}$ be a i.i.d. Let $S_n = \sum_{i=1}^{n} X_i$ with $S_0 = 0$, and $Z_n = \sum_{j=0}^{n} S_j$. Which of the following are Markov chains?

(1) S_n

$S_{n+1} = S_n + X_{n+1}$. Since X_{n+1} is independent of S_n, S_n is a MC.

(2) Z_n

$$Z_{n+1} = Z_n + S_{n+1}$$
$$= Z_n + S_n + X_{n+1}$$
$$= Z_n + Z_n - Z_{n-1} + X_{n+1}$$
$$= 2Z_n - Z_{n-1} + X_{n+1} \Rightarrow Z_{n+1} \text{ depends on } Z_n \text{ \& } Z_{n-1} \Rightarrow \text{Not a MC.}$$

(3) (S_n, Z_n)

$$S_{n+1} = S_n + X_{n+1}$$
$$Z_{n+1} = Z_n + S_{n+1} = Z_n + S_n + X_{n+1}$$

Since X_{n+1} is independent of X_i for $i \leq n$, (S_{n+1}, Z_{n+1}) only depends on S_n & Z_n

$\Rightarrow (S_n, Z_n)$ is a MC

(4) Let $\{X_n\}$ and $\{Y_n\}$ be independent Markov chains. Prove that the MC

$$Z_n = (X_n, Y_n)$$

is regular.

This will be discussed on 11/20.

(5) X has the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(a) Find μ_1

(b) Find μ_2

(c) Find the stationary distribution of $\boldsymbol{\pi}$.

(a) $$P[T_1 = r] = \begin{cases} \frac{1}{2} & ; r = 1 \\ \frac{1}{2}\left(\frac{3}{4}\right)^{r-2}\frac{1}{4} & ; r \geq 2 \end{cases}$$

$$\mu_1 = E[T_1] = \frac{1}{2} + \sum_{r=2}^{\infty} r \cdot \frac{1}{2}\left(\frac{3}{4}\right)^{r-2} \cdot \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{8}\left(\frac{3}{4}\right)^{-2} \cdot \sum_{r=2}^{\infty} r \cdot \left(\frac{3}{4}\right)^{r}$$

$$= \frac{1}{2} + \frac{1}{8} \cdot \frac{16}{9} \cdot \left[\frac{\frac{3}{4}}{\left(1 - \frac{3}{4}\right)^2} - \frac{3}{4}\right]$$

$$= \frac{1}{2} + \frac{2}{9} \cdot \left[12 - \frac{3}{4}\right] = \frac{1}{2} + \frac{2}{9} \cdot \frac{45}{4} = \frac{1}{2} + \frac{5}{2} = \boxed{3}$$

(b) Similar to (a)

(c) $$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 &= \pi \\ \pi_1 + \pi_2 &= 1 \end{aligned}\right\} \quad \begin{aligned} \pi_1 &= \frac{1}{3} \\ \pi_2 &= \frac{2}{3} \end{aligned}$$