Math 600 Lecture 29

Theorem: Suppose $f:[a,b]\to\mathbb{R}$ is bounded and has only finitely many points of discontinuity in $[a,b]$. Then f is Riemann integrable on $[a,b]$.

Proof: Let the points of discontinuity of f be $t_1,\dots,t_k$ and suppose $|f(x)|\le M$ $\forall x\in[a,b]$. Let $\varepsilon>0$ be given. Wlog, assume that

$$\min\{|t_i-t_j| : 1\le i<j\le k\} > \frac{\varepsilon}{8kM}$$

(if this does not hold, we can always replace ε with a smaller number).

For $j=1,2,\dots,k$, define

$$u_j=\max\left\{t_j-\frac{\varepsilon}{4kM},a\right\},\quad v_j=\min\left\{t_j+\frac{\varepsilon}{4kM},b\right\}.$$

Then each t_j belongs to exactly one of $[u_1,v_1],[u_2,v_2],\dots,[u_k,v_k]$, and these intervals are disjoint.

Consider $S=[a,b]-\bigcup_{j=1}^{k}(u_j,v_j)$. Since S is compact, f is uniformly continuous on S, so there exists $\delta>0$ such that

$$x,y\in S,\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2(b-a)}.$$

Define a partition $P=\{x_0,x_1,\dots,x_n\}$ on $[a,b]$ to satisfy

$$u_j\in P \text{ and } v_j\in P\ \ \forall j=1,2,\dots,k,$$

$$(u_j,v_j)\cap P=\emptyset\ \ \forall j=1,2,\dots,k,$$

$$\forall i=1,2,\dots,n,\ (x_{i-1}\ne u_j\ \forall j=1,\dots,k)\Rightarrow \Delta x_j<\delta.$$

Now consider

$$U(P,f)-L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i .$$

Write $\{1,2,\ldots,n\} = J_1 \cup J_2$, where

$$i \in J_1 \Rightarrow [x_{i-1},x_i] \neq [u_j,v_j] \ \forall j=1,\ldots,k,$$

$$i \in J_2 \Rightarrow [x_{i-1},x_i] = [u_j,v_j] \text{ for some } j=1,\ldots,k.$$

Then

$$U(P,f)-L(P,f) = \sum_{i\in J_1} (M_i-m_i)\Delta x_i + \sum_{i\in J_2} (M_i-m_i)\Delta x_i .$$

For $i \in J_1$,

$$\Delta x_i < \delta \Rightarrow M_i - m_i < \frac{\varepsilon}{2(b-a)}$$

and thus

$$\sum_{i\in J_1} (M_i-m_i)\Delta x_i < \frac{\varepsilon}{2(b-a)} \sum_{i\in J_1} \Delta x_i \leq \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

For $i \in J_2$, we can only say that $M_i - m_i \leq 2M$. Thus

$$\sum_{i\in J_2} (M_i-m_i)\Delta x_i \leq 2M \sum_{i\in J_2} \Delta x_i < 2M \cdot k \cdot \frac{\varepsilon}{4kM} = \frac{\varepsilon}{2}$$

Thus $U(P,f)-L(P,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which shows that f is Riemann integrable on $[a,b]$. //

We now have three sufficient conditions for $f:[a,b]\to\mathbb{R}$ to be Riemann integrable on $[a,b]$:

1. f is continuous

2. f is monotonic

3. f is bounded and has only finitely many discontinuities on $[a,b]$.

Properties of the Riemann integral

Theorem: Let $[a,b] \subset \mathbb{R}$ be given $(b>a)$ and let V be the set of all Riemann integrable functions on $[a,b]$. Then V is closed under addition and scalar multiplication, that is,

$$f,g \in V \Rightarrow f+g \in V,$$

$$f \in V, \alpha \in \mathbb{R} \Rightarrow \alpha f \in V.$$

Moreover,

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \forall f,g \in V,$$

$$\int_a^b \alpha f = \alpha \int_a^b f \quad \forall f \in V \; \forall \alpha \in \mathbb{R}.$$

Proof: Suppose $f,g \in V$ and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 on $[a,b]$ such that

$$U(P_1,f) - L(P_1,f) < \frac{\varepsilon}{2},$$

$$U(P_2,g) - L(P_2,g) < \frac{\varepsilon}{2}.$$

Define $P' = P_1 \cup P_2$; then we know that

$$U(P',f) - L(P',g) < \frac{\varepsilon}{2},$$

$$U(P',g) - L(P',g) < \frac{\varepsilon}{2}.$$

Also,

$$\sup\{f(x)+g(x) \mid x_{j-1} \le x \le x_j\} \le \sup\{f(x) \mid x_{j-1} \le x \le x_j\} + \sup\{g(x) \mid x_{j-1} \le x \le x_j\} \quad \forall j$$

$$\Rightarrow \quad U(P', f+g) \le U(P', f) + U(P', g)$$

and

$$\inf\{f(x)+g(x) \mid x_{j-1} \le x \le x_j\} \ge \inf\{f(x) \mid x_{j-1} \le x \le x_j\} + \inf\{g(x) \mid x_{j-1} \le x \le x_j\}$$

$$\Rightarrow \quad L(P', f+g) \ge L(P', f) + U(P', g).$$

Therefore,

$$\begin{aligned} U(P', f+g) - L(P', f+g) &\le U(P', f) + U(P', g) - (L(P', f) + L(P', g)) \\ &= U(P', f) - L(P', f) + U(P', g) - L(P', g) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $f+g$ is Riemann integrable. Moreover,

$$L(P', f) + L(P', g) \le L(P', f+g) \le U(P', f+g) \le U(P', f) + U(P', g),$$

which clearly implies that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

The proof for αf is even simpler. //

<u>Theorem</u>: 1. Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be Riemann integrable. Then

$$f(x)\le g(x)\ \forall x\in[a,b] \Rightarrow \int_a^b f \le \int_a^b g.$$

2. If $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then so is $|f|$, and

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

<u>Proof</u>: 1. Since f is Riemann integrable, we have

$$\int_c^b f(x)dx = \inf\left\{ \sum_{j=1}^{n} \sup\{f(x) \mid x_{j-1}\le x\le x_j\}\Delta x_j \,\middle|\, P=\{x_0,x_1,\dots,x_n\}\in\mathcal{P}\right\}$$

and similarly for $\int_c^b g(x)dx$. We have

$$f(x)\le g(x)\quad \forall x\in[a,b]$$

$$\Rightarrow \forall\, P=\{x_0,\dots,x_n\}\in\mathcal{P},\ \sup\{f(x)\mid x_{j-1}\le x\le x_j\}\le \sup\{g(x)\mid x_{j-1}\le x\le x_j\},\ j=1,\dots,n$$

$$\Rightarrow \forall P\in\mathcal{P},\ U(P,f)\le U(P,g)$$

$$\Rightarrow \inf\{U(P,f)\mid P\in\mathcal{P}\}\le \inf\{U(P,g)\mid P\in\mathcal{P}\}$$

$$\Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx.$$

2. Since f is Riemann integrable on $[a,b]$, for all $\varepsilon>0$, there exists $P\in\mathcal{P}$ such that

$$U(P,f)-L(P,f)<\varepsilon.$$

We must prove that the same is true for $|f|$. For this, it suffices to prove that

$$|U(P,|f|)-L(P,|f|)| \le U(P,f)-L(P,f) \;\; \forall\, P\in\mathcal{P}.$$

Now, for $P=\{x_0,x_1,\dots,x_n\}\in\mathcal{P}$,

$$U(P,f)-L(P,f)=\sum_{j=1}^{n}(M_j-m_j)\Delta x_j,$$

where

$$M_j-m_j=\sup\{f(x)\mid x_{j-1}\le x\le x_j\}-\inf\{f(x)\mid x_{j-1}\le x\le x_j\}$$

$$=\sup\{f(x)-f(y)\mid x_{j-1}\le x,y\le x_j\}$$

and

$$U(P,|f|)-L(P,|f|)=\sum_{j=1}^{n}(M_j'-m_j')\Delta x_j,$$

where

$$M_j'-m_j'=\sup\{|f(x)|:x_{j-1}\le x\le x_j\}-\inf\{|f(x)|:x_{j-1}\le x\le x_j\}$$

$$=\sup\{|f(x)|-|f(y)|:x_{j-1}\le x,y\le x_j\}.$$

But the reverse triangle inequality yields

$$|f(x)|-|f(y)|\le|f(x)-f(y)|$$

$$\Rightarrow M_j'-m_j'\le M_j-m_j$$

$$\Rightarrow U(P,|f|)-L(P,|f|)\le U(P,f)-L(P,f).$$

Thus $|f|$ is Riemann integrable on $[a,b]$. Since

$$-|f|\le f\le|f|,$$

it follows from #1 that

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

$$\Longrightarrow \left| \int_a^b f \right| \le \int_a^b |f|. \quad //$$