Math 600 Lecture 25

Lemma (Rolle's theorem): Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), and satisfy $f(a)=f(b)$. Then there exists $c\in(a,b)$ such that $f'(c)=0$.

Proof: We know that f attains its maximum and minimum on $[a,b]$. Write $M=f(a)=f(b)$.

Case 1: $\min\{f(x)\mid a\le x\le b\}<M$. Then the minimum of f is attained on (a,b); hence f has a local minimizer $c\in(a,b)$, and hence $f'(c)=0$.

Case 2: $\max\{f(x)\mid a\le x\le b\}>M$. Then the maximum of f is attained on (a,b); hence f has a local maximizer $c\in(a,b)$, and hence $f'(c)=0$.

If neither Case 1 nor Case 2 holds, then $f(x)=M\ \forall x\in[a,b]$, in which case $f'(c)=0\ \forall c\in(a,b)$. //

Theorem (the mean value theorem): Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Proof: Define $g:[a,b]\to\mathbb{R}$ by

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a).$$

Then $g(a)=g(b)=0$ and Rolle's theorem applies: There exists $c\in(a,b)$ such that

$$g'(c)=0$$

$$\Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \quad //$$

<u>Theorem</u> (the generalized mean value theorem): Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c\in(a,b)$ such that

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c).$$

<u>Proof:</u> Define $h:[a,b]\to\mathbb{R}$ by $h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$. By the MVT, there exists $c\in(a,b)$ such that

$$h'(c) = \frac{h(b)-h(a)}{b-a}$$

$$\Rightarrow (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = \frac{h(b)-h(a)}{b-a}.$$

But

$$\begin{aligned} h(b)-h(a) &= (f(b)-f(a))g(b) - (g(b)-g(a))f(b) - (f(b)-f(a))g(a) \\ &\qquad + (g(b)-g(a))f(a) \\ &= (f(b)-f(a))(g(b)-g(a)) - (g(b)-g(a))(f(b)-f(a)) \\ &= 0, \end{aligned}$$

and the result follows. //

<u>Theorem</u>: Let $f:I\to\mathbb{R}$, where $I\subset\mathbb{R}$, suppose $(a,b)\subset I$, and let f be differentiable in (a,b). Then

- $f'(x) \geq 0 \ \forall x \in (a,b) \Rightarrow f$ is increasing on (a,b);
- $f'(x) \leq 0 \ \forall x \in (a,b) \Rightarrow f$ is decreasing on (a,b);
- $f'(x) = 0 \ \forall x \in (a,b) \Rightarrow f$ is constant on (a,b).

<u>Proof:</u> Follows immediately from the MVT. //

L'Hôpital's rule

<u>Theorem</u> (version 1): Suppose $f:(a,b)\to\mathbb{R}$, $g:(a,b)\to\mathbb{R}$ are differentiable and satisfy

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$$

If $g'(x) \neq 0$ for all $x\in(a,b)$ and

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

exists (including the case that $\frac{f'(x)}{g'(x)} \to \infty$ or $\frac{f'(x)}{g'(x)} \to -\infty$), then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)},$$

exists and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Moreover, the above holds if "$x\to a^+$" is everywhere replaced by "$x\to b^-$".

<u>Proof:</u> Note that f,g can be extended to be continuous on $[a,b)$ by defining $f(a)=g(a)=0$. By the generalized MVT, for all $x\in(a,b)$,

there exists $c_x \in (a,x)$ such that

$$(f(x)-f(a))g'(c_x) = (g(x)-g(a))f'(c_x)$$

$$\Rightarrow f(x)g'(c_x) = g(x)f'(c_x) \quad (\text{since } f(a)=g(a)=0).$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}.$$

(Note that $g'(x) \neq 0$ for all $x \in (a,b)$ implies that $g(x) \neq 0$ for all $x \in (a,b)$.)

But then, since $c_x \to a^+$ as $x \to a^+$,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

The proof in the case of $x \to b^-$ is similar. //

__Theorem__ (version 2): Suppose $f:(a,b)\to\mathbb{R}$, $g:(a,b)\to\mathbb{R}$ are differentiable and satisfy

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty.$$

If $g'(x) \neq 0$ for all $x \in (a,b)$ and

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

exists (in $\mathbb{R}$), then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)},$$

exists and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Moreover, the above holds if "$x \to a^+$" is everywhere replaced by "$x \to b^-$".

Proof: Let $y \in (a,b)$ be fixed. Then, for all $x \in (a,y)$, there exists $c_x \in (x,y)$ such that

$$(f(x)-f(y))g'(c_x) = (g(x)-g(y))f'(c_x)$$

$$\Rightarrow \frac{f(x)-f(y)}{g(x)} = \frac{g(x)-g(y)}{g(x)} \frac{f'(c_x)}{g'(c_x)}$$ (obviously, $g(x) \to \infty$ as $x \to a^+$ implies that $g(x) \neq 0$ $\forall x$ near a)

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)} - \frac{f'(c_x)}{g'(c_x)} \cdot \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}.$$

Write

$$L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$c \in (a, a+\delta) \Rightarrow \left| \frac{f'(c)}{g'(c)} - L \right| < \frac{\varepsilon}{3},$$

Choose $y \in (a, a+\delta)$. Then, since $c_x \in (x,y) \subset (a,y) \subset (a, a+\delta)$,

$$x \in (a,y) \Rightarrow c_x \in (a, a+\delta) \Rightarrow \left| \frac{f'(c_x)}{g'(c_x)} - L \right| < \frac{\varepsilon}{2}.$$

Since

$$\frac{f'(c_x)}{g'(c_x)} \cdot \frac{g(y)}{g(x)}, \frac{f(y)}{g(x)} \to 0 \text{ as } x \to a^+,$$

there exists $\gamma \in (0, y-a)$ such that

$$x \in (a, a+\gamma) \Rightarrow \left| \frac{f'(c_x)}{g'(c_x)} \cdot \frac{g(y)}{g(x)} \right| < \frac{\varepsilon}{3}, \left| \frac{f(y)}{g(x)} \right| < \frac{\varepsilon}{3}.$$

But then

$$x \in (a, a+\gamma) \Rightarrow \frac{f(x)}{g(x)} - L = \frac{f'(c_x)}{g'(c_x)} - L - \frac{f'(c_x)}{g'(c_x)} \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

$$\Rightarrow \left| \frac{f(x)}{g(x)} - L \right| \le \left| \frac{f'(c_x)}{g'(c_x)} - L \right| + \left| \frac{f'(c_x)}{g'(c_x)} \frac{g(y)}{g(x)} \right| + \left| \frac{f(y)}{g(x)} \right|$$

$$\Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L,$$

as desired

A similar argument works in the case $x \to b^-$. //

The proof of version 2 can be modified to obtain the same result if $x \to -\infty$ (instead of $x \to a^+$) or $x \to \infty$ (instead of $x \to b^-$).