# Math 600 Lecture 26

## Taylor's theorem

Given $f: I \to \mathbb{R}$, $I \subset \mathbb{R}$, we define $f''$ (the second derivative of $f$) by $f'' = (f')'$. Thus

$$f''(t) = \lim_{x \to t} \frac{f'(x) - f'(t)}{x - t} \quad \text{(assuming this limit exists)}$$

and the domain of $f''$ consists of all $t$ for which

- $f$ is differentiable on some open interval containing $t$ (so that $\lim_{x\to t} \frac{f'(x)-f'(t)}{x-t}$ makes sense), and
- $\lim_{x \to t} \frac{f'(x) - f'(t)}{x - t}$ exists.

If we say that $f''$ exists on $(a,b)$ (or that $f$ is twice differentiable on $(a,b)$), this implies that $f$ is differentiable on $(a,b)$ (so that $f'$ is defined on $(a,b)$) and $f'$ is differentiable on $(a,b)$.

Similarly,

$$f''' = (f'')', \; f^{(4)} = (f''')', \; \ldots, \; f^{(n)} = (f^{(n-1)})'.$$

If we say that $f^{(n)}$ exists on $(a,b)$, this implies that $f, f', \ldots, f^{(n-1)}$ all exist and are differentiable on $(a,b)$.

Recall: If $f$ is at least $(n-1)$-times differentiable on an open interval containing $\alpha$, then the Taylor polynomial $p_{n-1}$ is the polynomial

$$p_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k.$$

The significance of $p_{n-1}$ is that it is the unique polynomial of degree at most $n-1$ satisfying

$$p_{n-1}(\alpha) = f(\alpha),\ p_{n-1}'(\alpha) = f'(\alpha), \ldots, p_{n-1}^{(n-1)}(\alpha) = f^{(n-1)}(\alpha).$$

In spite of this, there is no guarantee that $p_{n-1}$ is a good approximation to $f$ for $x \neq \alpha$.

Taylor's theorem provides such a guarantee.

<u>Theorem</u>: Let $f: (a,b) \to \mathbb{R}$ be $n$-times differentiable on $(a,b)$ and assume that $\alpha \in (a,b)$. Then, for all $t \in (a,b)$, there exists $c$ lying between $\alpha$ and $t$ such that

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k + \frac{f^{(n)}(c)}{n!}(t-\alpha)^n.$$

<u>Proof</u>: Write

$$p_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k$$

and let $M \in \mathbb{R}$ satisfy

$$f(t) = p_{n-1}(t) + M(t-\alpha)^n,$$

that is, define

$$M = \frac{f(t) - p_{n-1}(t)}{(t-\alpha)^n}.$$

Then define $g:(a,b)\to\mathbb{R}$ by

$$g(x) = f(x) - p_{n-1}(x) - M(x-\alpha)^n.$$

Note that

$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - p_{n-1}^{(k)}(\alpha) = 0 \quad \forall k=0,1,\dots,n-1$$

(since the kth derivative of $(x-\alpha)^n$ is $n(n-1)\cdots(n-k)(x-\alpha)^{n-k}$, which equals zero at $x=t$ if $k<n$).

Since $g(\alpha) = g(t) = 0$, there exists $c_1$ between $0$ and $\alpha$ such that $g'(c_1) = 0$.

Since $g'(\alpha) = g'(c_1) = 0$, there exists $c_2$ between $0$ and $c_1$ such that $g''(c_1) = 0$.

$\vdots$

Since $g^{(n-1)}(\alpha) = g^{(n-1)}(c_{n-1}) = 0$, there exists $c_n$ between $0$ and $c_{n-1}$ such that $g^{(n)}(c_n) = 0$

But

$$g^{(n)}(x) = f^{(n)}(x) - n!M.$$

Thus

$$g^{(n)}(c_n) = 0 \Rightarrow M = \frac{f^{(n)}(c_t)}{n!},$$

where $c_t = c_n$. //

## Examples

1. $f(x) = e^x, \alpha = 0$. Let $R$ be any positive real number. Note that

$$f^{(n)}(x) = e^x \ \forall x\in\mathbb{R} \ \forall n\in\mathbb{Z}^+.$$

Thus, if $t\in(-R,R)$, then $c_t\in(-R,R)$ and hence

$$|f^{(n)}(c_t)| = e^{c_t} \le e^R$$

and have

$$|f(t) - p_{n-1}(t)| = \frac{|f^{(n)}(c_t)|}{n!}|t|^n \le \frac{e^R|t|^n}{n!}.$$

Since

$$\frac{|t|^n}{n!} \to 0 \text{ as } n \to \infty,$$

we see that

$$|f(t) - p_{n-1}(t)| \to 0 \text{ as } n \to \infty \ \forall t \in (-R, R),$$

that is,

$$p_{n-1}(t) \to f(t) \text{ as } n \to \infty \ \forall t \in (-R, R),$$

that is,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k = e^t \ \forall t \in (-R, R).$$

Since $R$ was arbitrary, we see that, in fact,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k = e^t \ \forall t \in \mathbb{R}.$$

Moreover, $f^{(k)}(0) = e^0 = 1 \ \forall k$, so we obtain

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \ \forall t \in \mathbb{R}.$$

2. $f(x) = \sin(x)$, $\alpha = 0$

$f(x) = \sin(x)$, $f(0) = 0$

$f'(x) = \cos(x)$, $f'(0) = 1$ $\qquad \left(f^{(n)}(0) = 0 \ \forall \text{ even } n\right)$

$f''(x) = -\sin(x)$, $f''(0) = 0$

$f'''(x) = -\cos(x)$, $f'''(0) = -1$

$\vdots$

(The pattern $0, 1, 0, -1, 0, 1, 0, -1, \ldots$ continues.)

We obtain the following Taylor series:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Taylor's theorem is

$$\sin(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \frac{f^{(2n+2)}(c_x)}{(2n+2)!} x^{2n+2}$$

We have

$$|f^{(2n+2)}(c_x)| \leq 1$$

(since the even derivatives are all $\pm \sin$) and hence

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right| \leq \frac{|x|^{2n+2}}{(2n+2)!} \to 0 \text{ as } n \to \infty \quad \forall x \in \mathbb{R}$$

Thus

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}.$$

Similarly

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \forall x \in \mathbb{R}.$$