Math 600 Lecture 5

A real number x is called <u>algebraic</u> if there exists $n \in \mathbb{Z}^+$ and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0.$$

Is the set of algebraic numbers countable or uncountable? Why?

<u>Definition</u>: Let X be a set and suppose $d : X \times X \to \mathbb{R}$ satisfies

- $d(x,y) \geq 0$ $\forall x, y \in X$ and $(d(x,y) = 0$ iff $x = y)$;
- $d(y,x) = d(x,y)$ $\forall x, y \in X$;
- $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x, y, z \in X$ (the <u>triangle inequality</u>).

Then d is called a <u>metric</u> on X and (X, d) is called a <u>metric space</u> (or X is called a metric space under d).

<u>Examples</u>

1. $X = \mathbb{R}$, $d(x,y) = |x-y|$
2. $X = \mathbb{R}^k$, $d(x,y) = \|x-y\|_2$, where $\|x\|_2 = \left[\sum_{j=1}^{k} x_j^2\right]^{1/2}$. (Note that $\|\cdot\|_2$ is called the <u>Euclidean norm</u> on $\mathbb{R}^k$.)
3. $X =$ any set, $d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$

This is called the <u>discrete metric</u>

Note that #1 is a special case of #2. Proving the triangle inequality for the Euclidean norm is a bit tricky. Outline:

- $\forall x \in \mathbb{R}^k$, $\|x\|_2 = [x \cdot x]^{1/2}$, where $x \cdot y = \sum_{j=1}^{k} x_j y_j \quad \forall x, y \in \mathbb{R}^k$

  ($x \cdot y$ is the <u>dot product</u> of x and y).

- The dot product is symmetric and bilinear:

$$x \cdot y = y \cdot x \ \forall x, y \in \mathbb{R}^k \text{ and } (\alpha x + \beta y) \cdot z = \alpha (x \cdot z) + \beta (y \cdot z) \ \forall x, y, z \in \mathbb{R}^k, \ \forall \alpha, \beta \in \mathbb{R}$$

- The dot product satisfies the Cauchy-Schwarz inequality:

$$|x \cdot y| \leq \|x\|_2 \|y\|_2 \quad \forall x, y \in \mathbb{R}^k;$$

  with equality iff $y = \alpha x$ for some $\alpha \in \mathbb{R}$. The proof of this inequality is the tricky part.

- $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2 \quad \forall x, y \in \mathbb{R}^k$ (the triangle inequality for norms)

  <u>Proof:</u> $\|x+y\|_2^2 = (x+y) \cdot (x+y) = x \cdot x + 2 x \cdot y + y \cdot y$

$$= \|x\|_2^2 + 2 x \cdot y + \|y\|_2^2$$

$$\leq \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2 \quad \text{(Cauchy-Schwarz)}$$

$$= (\|x\|_2 + \|y\|_2)^2$$

$$\Rightarrow \|x+y\|_2 \leq \|x\|_2 + \|y\|_2. \ //$$

- We now have

$$d(x,z) = \|x - z\|_2 = \|(x-y) + (y-z)\|_2 \leq \|x-y\|_2 + \|y-z\|_2 = d(x,y) + d(y,z)$$

  as desired.

(The other two properties of a metric are easy to verify for $d(x,y) = \|x-y\|_2$.)

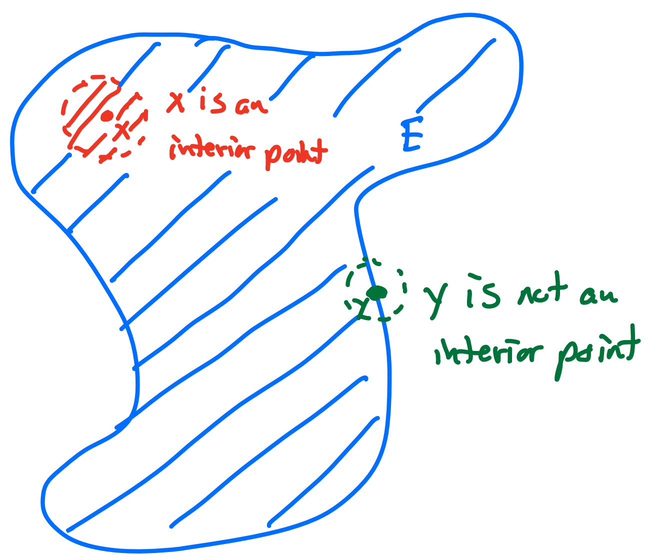
All three properties of a metric are easy to verify for the discrete metric.

<u>Definitions</u>: Let (X,d) be a metric space.

- Given $x \in X$ and $r > 0$, the <u>open ball of radius r centered at x</u> is the set

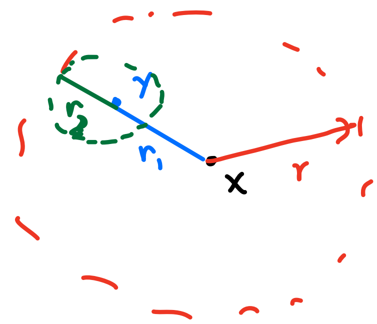
$$B_r(x) = \{y \in X \mid d(y,x) < r\}.$$

- Suppose $E \subset X$. We say that $x \in X$ is an <u>interior point</u> of E iff there exists $r > 0$ such that $B_r(x) \subset E$



- Suppose $E \subset X$. We say that E is <u>open</u> iff every $x \in E$ is an interior point of E (that is, iff for all $x \in E$, there exists $r > 0$ such that $B_r(x) \subset E$).

Lemma: Let (X,d) be a metric space, let $x \in X$, and let $r>0$. Then $B_r(x)$ is an open set.



Proof: Suppose $y \in B_r(x)$, $y \neq x$. Define $r_1 = d(y,x)$ and $r_2 = r - r_1$. We will prove that $B_{r_2}(y) \subset B_r(x)$. Indeed,

$$z \in B_{r_2}(y) \Rightarrow d(z,y) < r_2 \Rightarrow d(z,x) \leq d(z,y) + d(y,x) < r_2 + r_1 = r.$$

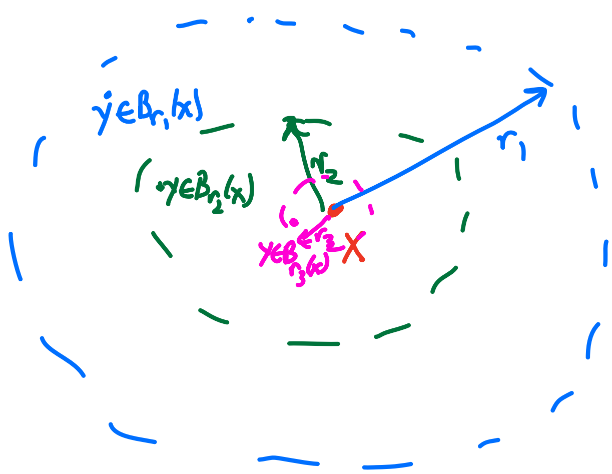
Thus

$$z \in B_{r_2}(y) \Rightarrow z \in B_r(x)$$

and therefore $B_{r_2}(z) \subset B_r(x)$. This proves that $B_r(x)$ is open. //

Definition: Let (X,d) be a metric space.

- Let $E \subset X$. We say that $x \in X$ is a <u>limit point</u> of E iff for all $r>0$, there exists $y \in E \cap B_r(x)$ such that $y \neq x$.



We write E' for the set of all limit points of E and define the <u>closure</u> of E to be the set $\tilde{E} = E \cup E'$.

- Let $E \subset X$. We say that x is an <u>isolated point</u> of E iff $x \in E$ and x is not a limit point of E. (Thus $x \in E$ is a limit point of E iff there exists $r > 0$ such that $E \cap B_r(x) = \{x\}$.)
- Let $E \subset X$. We say that E is <u>closed</u> iff every limit point of E lies in E.

<u>Theorem</u>: Let (X,d) be a metric space, let $E \subset X$, and suppose x is a limit point of E. Then, for all $r > 0$, $B_r(x)$ contains infinitely many points of E.

<u>Proof</u>: It suffices to construct a sequence of distinct points of E that lie in $B_r(x)$. Since x is a limit point of E and $r > 0$, there exists $x_1 \in B_r(x) \cap E$, $x_1 \neq x$. Defining $r_1 = d(x_1, x)$, there exists $x_2 \in B_{r_1}(x) \cap E$, $x_2 \neq x$. By definition, x_2 cannot equal x_1. We continue in this fashion: Given distinct $x_1, \dots, x_j \in B_r(x) \cap E$ with $r_1 > r_2 > \dots > r_j$, where $r_i = d(x_i, x)$, we choose $x_{j+1} \in B_{r_j}(x) \cap E$, $x_{j+1} \neq x$. Such an x_{j+1} must exist because x is a limit point of E. In this way, we construct the desired sequence. //

<u>Corollary</u>: A finite set $E \subset X$ has no limit points.

<u>Definition</u>: Let X be a set and E a subset of X. The <u>complement</u> of E in X is the set

$$X \setminus E = \{x \in X \mid x \notin E\}.$$

If X is understood (especially, if X is a metric space and $E \subset X$), we often write E^c for $X \setminus E$.

<u>Lemma</u>: Let X be a set and $E \subset X$. Then

$$E \cap E^c = \emptyset, \quad X = E \cup E^c, \quad (E^c)^c = E.$$

<u>Theorem</u>: Let (X,d) be a metric space and $E \subset X$. Then E is open iff E^c is closed. (Equivalently, E is closed iff E^c is open.)

<u>Proof</u>: Suppose first that E is open. We must show that E^c contains all of its limit points. Equivalently, we must show that if $x \in E$, then x is not a limit point of E^c (this is equivalent since $X = E \cup E^c$ and $E \cap E^c = \emptyset$). So assume that $x \in E$. Then there exists $r > 0$ such that $B_r(x) \subset E$, which implies that $B_r(x) \cap E^c = \emptyset$. But this implies that x is not a limit point of E^c. Hence E^c must contain all of its limit points, and hence E^c is closed.

Conversely, suppose E^c is closed. Then E^c contains all of its limit points. Let $x \in E$. Then, since $x \notin E^c$, x is not a limit point of E^c, and hence there exists $r > 0$ such that $B_r(x) \cap E^c = \emptyset$. But this implies that $B_r(x) \subset E$. Since $x \in E$ was arbitrary, this proves that E is open. //