Math 600 Lecture 11

Recall: $A, B \subset X$ are <u>separated</u> iff $(A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset)$.

$E \subset X$ is <u>connected</u> iff E cannot be written as $E = A \cup B$, where A and B are separated

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Theorem: A subset E of $\mathbb{R}$ is connected iff it is an interval, that is, iff

$$(*) \qquad (x, y \in E \text{ and } x < z < y) \Rightarrow z \in E.$$

Proof: Suppose first that E is not connected, that is, that there exist nonempty separated sets $A, B \subset \mathbb{R}$ such that $E = A \cup B$. We wish to prove that $(*)$ fails. Choose $x \in A$ and $y \in B$ and assume, without loss of generality, that $x < y$. Define

$$z = \sup \{ A \cap [x,y] \}.$$

Note that $A \cap [x,y]$ is bounded above by y, so z is well defined, and

$z \in \bar{A}$ (if $z \notin A$, then z must be a limit point of A; otherwise, there would be a smaller upper bound). Since A and B are separated, $z \notin B$. In particular, $z < y$.

If $z \notin A$, then $z \notin E = A \cup B$ and $x < z < y$, so $(*)$ fails, as desired.

If $z \in A$, then $z \notin \bar{B}$, so there exists $z_1 \in (z, y)$ such that $z_1 \notin B$. But then

$$z_1 > z \Rightarrow z_1 \notin A \cap [x,y] \Rightarrow z_1 \notin A$$

and we see that

$$x < z_1 < y \text{ and } z \notin E = A \cup B.$$

Thus $(*)$ fails in this case also.

Conversely, suppose that (*) fails. Then there exist $x, y, z \in \mathbb{R}$ such that

$$x, y \in E \text{ and } z \notin E \text{ and } x < z < y.$$

Define

$$A = E \cap (-\infty, z),$$

$$B = E \cap (z, \infty).$$

Then A and B are nonempty ($x \in A, y \in B$), A and B are separated (since $A \subset (-\infty, z)$, $B \subset (z, \infty)$, and $(-\infty, z), (z, \infty)$ are separated), and $E = A \cup B$. Thus E is separated. //

Recall: $E \subset X$ is perfect iff E is closed and every point of E is a limit point of E.

Theorem: If $k \in \mathbb{Z}^+$ and $E \subset \mathbb{R}^k$ is nonempty and perfect, then E is uncountable.

Proof: Since E is nonempty and perfect, E is infinite (a finite set has no limit points). Let us assume, by way of contradiction, that E is countable. Then we can write E as a sequence: $E = \{x_n\}$. Choose $r_1 > 0$ arbitrarily and define $V_1 = B_{r_1}(x_1)$. Choose V_2 to be an open ball with the following properties:

- V_2 is centered at a point lying in E;
- $x_1 \notin \overline{V_2}$;
- $\overline{V_2} \subset V_1$.

This is possible since x_1 is a limit point and hence V_1 contains infinitely many points of E. (Note that x_2 may or may not belong to V_2). Next, choose V_3 to be an open ball centered at a point of E, and such that $x_2 \notin \overline{V_3}$ and $\overline{V_3} \subset V_2$.

Continue in this fashion to construct a sequence $\{V_n\}$ of open balls, each centered at a point of E, such that

$$x_n \notin \overline{V_{n+1}} \text{ and } \overline{V_{n+1}} \subset V_n \quad \forall n \in \mathbb{Z}^+.$$

Define $C_n = \overline{V_n} \cap E$. Then each C_n is compact (since $\overline{V_n}$ is compact by the Heine-Borel theorem and E is closed) and each C_n is nonempty (since the center of V_n lies in E for every $n \in \mathbb{Z}^+$). Hence, by a previous theorem

$$C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset,$$

and obviously $C \subset E$. But, by construction, $x_n \notin C_{n+1}$ for all $n \in \mathbb{Z}^+$, and hence $x_n \notin C$ for all $n \in \mathbb{Z}^+$. This is a contradiction, since $E = \{x_n\}$ and $C \subset E$. This contradiction shows that E cannot be countable. //

Corollary: $\mathbb{R}$ and $\mathbb{R}^k$ are uncountable. Any open interval in $\mathbb{R}$ is uncountable, and any nonempty open set in $\mathbb{R}^k$ is uncountable.

Sequences

Definition: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges (or $\{x_n\}$ is convergent) iff there exists $x \in X$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow d(x_n, x) < \varepsilon.$$

In this case, we say that $\{x_n\}$ converges to x and write $x_n \to x$ or

$$x = \lim_{n \to \infty} x_n.$$

The point x is called the limit of $\{x_n\}$.

If $\{x_n\}$ does not converge, it is said to diverge or to be divergent.

Recall: A sequence $\{x_n\} \subset X$ is actually a function $x : \mathbb{Z}^+ \to X$, where we write x_n instead of $x(n)$. We also use the symbol $\{x_n\} = \{x_n \mid n \in \mathbb{Z}^+\}$ to denote x; that is, we denote x by its range. To say that $\{x_n\}$ is bounded is to say that the set $\{x_n\}$ is bounded: there exists $x \in X$ and $R > 0$ such that $\{x_n\} \subset B_R(x)$, that is, for all $n \in \mathbb{Z}^+$, $d(x_n, x) < R$.

Theorem: Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ converges to $x \in X$ iff, for all $r > 0$, $B_r(x)$ contains all but finitely many terms in $\{x_n\}$.
2. If $\{x_n\}$ converges, its limit is unique.

3. If $\{x_n\}$ converges, then it is bounded.

Also:

4. If $E \subset X$ and $x \in X$ is a limit point of E, then there is a sequence $\{x_n\} \subset E$ such that $x_n \to x$

Proof:

1. Suppose $x_n \to x$. Then, for all $r > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow d(x_n, x) < r,$$

that is, such that

$$\{x_n \mid n \geq N) \subset B_r(x).$$

Thus all but infinity many terms of $\{x_n)$ belong to $B_r(x)$.

Conversely, suppose $\{x_n\} \subset X$, $x \in X$, and, for all $r > 0$, all but finitely many terms m $\{x_n\}$ belong to $B_r(x)$. Let $\varepsilon > 0$ be given. Since all but finitely many terms of $\{x_n\}$ belong to $B_\varepsilon(x)$, there exists $N \in \mathbb{Z}^+$ such that

$$\{x_n \mid n \in N\} \subset B_\varepsilon(x),$$

that is, such that

$$n \geq N \Rightarrow d(x_n, x) < \varepsilon.$$

Thus $x_n \to x$.

2. Suppose $\{x_n\} \subset X$ and $x_n \to x \in X$. We will show that if $x' \in X$ and $x' \neq x$, then $\{x_n\}$ does not converge to x'. Define $r = \frac{1}{2} d(x', x)$. Then all but finitely many terms of the sequence belong to $B_r(x)$. Since

$$B_r(x) \cap B_r(x') = \emptyset,$$

it follows that $B_r(x')$ contains at most finitely many terms of the sequence.

Thus $x_n \not\to x'$.

3. Suppose $\{x_n\} \subset X$ and $x_n \to x \in X$. Then there exists $N \in \mathbb{Z}^+$ such that

$$n \geq N \Rightarrow d(x_n, x) < 1.$$

Define

$$R = \max\{d(x_1,x), d(x_2,x), \ldots, d(x_{N-1},x), 1\}.$$

Then

$$d(x_n, x) \leq R \quad \forall n \in \mathbb{Z}^+,$$

and hence $\{x_n\}$ is bounded.

4. Suppose $E \subset X$ and $x \in X$ is a limit point of x. Then, for each $n \in \mathbb{Z}^+$, there exists a point x_n in

$$B_{1/n}(x) \cap E.$$

It is then easy to show that $x_n \to x$. //