FIM/ECG/MA 528 Midterm #2 (Due: Monday November 7th, 2022)

1. There are 5 Questions
2. Each question worth 20 points
3. Answer all 5 questions correctly for maximum exam score
4. Grading for each question:
   - 0 pts: work is generally incoherent
   - 10 pts: about half the work is correct
   - 15 pts: most of the work is correct but answer is not written out in full
   - 19 pts: the problem is almost entirely correct but there are 1 or more small mistakes
   - 20 pts: a completed answer with no mistakes

5. **You must show all steps to receive full credit.**
6. Simply writing a number that you know is correct is incoherent.

Problem #1: (20 points) A stock index is currently 990, the risk-free rate is 5%, and the dividend yield on the index is 2%. Use a three-step tree to value an 18-month European put option with a strike price of 1,000 when the volatility is 20% per annum.

Solution:

For 18 months with 3 steps, we have $\Delta t = .5$. We use a CRR model with $u = e^{.2\sqrt{.5}}$=1.1519 and $d = e^{-.2\sqrt{.5}}$=.8681. The risk-neutral 'up' probability is

$$p = \frac{e^{(.05-.02).5} - e^{-.2\sqrt{.5}}}{e^{.2\sqrt{.5}} - e^{-.2\sqrt{.5}}} = .5180$$

The possible values of the stock at maturity are

$$S^{u\mathrm{uu}} = 1{,}513.20$$

$$S^{uud} = S^{udu} = S^{duu} = 1{,}140.40$$

$$S^{u\mathrm{dd}} = S^{\mathrm{dd}u} = S^{du\mathrm{d}} = 859.44$$

$$S^{\mathrm{ddd}} = 647.71.$$

The put option pays only when there are 2 or 3 'down' moves, and in which case the payoff is

$$\left(1000 - S^{u\mathrm{dd}}\right)^{+} = \left(1000 - S^{\mathrm{du}d}\right)^{+} = \left(1000 - S^{\mathrm{ddu}}\right)^{+} = 140.56$$

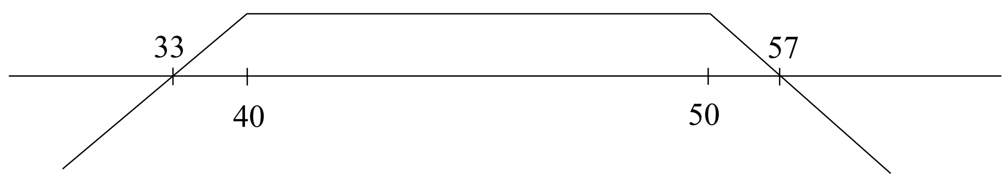
$$\left(1000 - S^{\mathrm{ddd}}\right)^{+} = 352.29.$$

Thus, the European put option value is

$$\boldsymbol{p = e^{-.05\times 3/2}\left((1-p)^3 \times 352.29 + \binom{3}{2}(1-p)^2 p \times 140.56\right) = 83.69.}$$

Problem #2: (20 points) A trader sells a strangle by selling a 6-month European call option with a strike price of \$50 for \$3 and selling a 6-month European put option with a strike price of \$40 for \$4. For what range of prices of the underlying asset in 6 months does the trader make a profit?

Solution: The position earns \$7 initially and has the following payoff diagram:



So the trader ends with a profit if the underlying finishes **between \$33 and \$57.**

Problem #3: (20 points) A stock is trading at $105, you're long a 3-month American call with strike K = $100, and the risk-free rate is 5%. No dividends will be paid in the coming 3 months, and you have a strong feeling that the asset price will go down leaving the option OTM. Compare the following two strategies:

a) Exercise the option and invest the proceeds.
b) Short the asset, hold the option, and invest the proceeds.

Which strategy is better? Show your computations.

Solution:

For strategy a) we exercise to obtain $5, which after 3 months is $\boldsymbol{\$5e^{.05\times.25} = \$5.06}$.

For strategy b) we short the stock to obtain $105, which after 3 months is $\$105\mathrm{e}^{.05\times.25} = \$106.32$. The obligation to repurchase the stock and return to the original owner is at most the strike value of the call option, K=$100. Therefore, the least this strategy will **net is $6.32.**

Clearly **strategy b) is better** than strategy a), as it takes advantage of the short-selling insurance offered by the option. In general, it is never optimal to early exercise an American call on a non-dividend paying stock.

Problem #4: Consider the following model for a non-dividend-paying stock price,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where W(t) is a standard Wiener process. Let $S(0) = 100, \mu = .05$, and $\sigma = .2$.

1. (10 points) What are the 2-sided 95% confidence limits for the stock price at the end of 1 year? That is, find interval $[a, b]$ such that $\mathrm{P}(S(1) > b) = .025$ and $\mathrm{P}(S(1) < a) = .025$.
2. (10 points) Suppose the stock is sold short at time t=0 and the proceeds of the sale invested in a risk-free account at 4% annual (continuously compounded). How much additional cash is needed to cover losses at the end of 1 year with 95% confidence?

Solution:

The solution to this SDE is $\mathrm{S(t)} = \mathrm{S(0)exp}\left(\left(\mu\text{-}\frac{\sigma^2}{2}\right)\mathrm{t} + \sigma \mathrm{W(t)}\right)$. The natural logarithm is normally distributed with mean $\left(\mu\text{-}\frac{\sigma^2}{2}\right)\mathrm{t}$ and standard deviation $\sigma\sqrt{\mathrm{t}}$, which for t=1 are .03 and .2, respectively. For part 1, we seek a and b such that

$$.025 = \mathrm{P}(S(1) > b) = \mathrm{P}\left(\mathrm{Z} > \frac{\log(\mathrm{b}/100)\text{-}.03}{.2}\right) = (\mathrm{Z} > 1.96)$$
$$.025 = \mathrm{P}(S(1) < \mathrm{a}) = \mathrm{P}\left(\mathrm{Z} < \frac{\log(\mathrm{a}/100)\text{-}.03}{.2}\right) = (\mathrm{Z} < \text{-}1.96)$$

where Z is a standard normal. Thus, $\boldsymbol{a} = \mathbf{100}\boldsymbol{exp}(-\mathbf{.2} \times \mathbf{1.96} + \mathbf{.03}) = \mathbf{69.63}$ **and** $\boldsymbol{b} = \mathbf{100}\boldsymbol{exp}(\mathbf{.2} \times \mathbf{1.96} + \mathbf{.03}) = \mathbf{152.50}$**.**

For part 2, the short sale's loss increases as $S(1)$ increases, and so we seek an amount $VaR_{95\%}$ that, when invested initially at the risk-free, will cover time t=1 losses 95% into the tail. $VaR_{95\%}$ is found by solving this equation,

$$\mathrm{P}(S(1)\text{-}100\mathrm{e}^{.04} > VaR_{95\%} \times \mathrm{e}^{.04}) = .05.$$

If we use the SDE solution, we have

$$\mathrm{P}(100\exp(.03 + .2 \times \mathrm{Z})\text{-}100\mathrm{e}^{.04} > VaR_{95\%} \times \mathrm{e}^{.04}) = .05,$$

and for a standard normal we have $\mathrm{P}(\mathrm{Z} > 1.645) = .05$. Therfore,

$$100\exp(.03 + .2 \times 1.645)\text{-}100\mathrm{e}^{.04} = VaR_{95\%} \times \mathrm{e}^{.04},$$

which means

$$\mathbf{VaR_{95\%}} = \mathbf{100} \times \left(\mathbf{e}^{-\mathbf{.01}+\mathbf{.2}\times\mathbf{1.645}} - \mathbf{1}\right) = \mathbf{37.58}.$$

Problem #5: Consider the following model for the short-term interest rate

$$dr(t) = \lambda(\bar{r} - r(t))dt + \sigma dW(t),$$

where W(t) is a standard Wiener process. Short-term loans made at this rate earn $r(t)$, such that an initial principal P_0 at end of 1 year will be

$$\mathrm{P}_1 = \mathrm{P}_0 \mathrm{e}^{\int_0^1 \mathrm{r(t)dt}}.$$

1. (15 points) Show that $\int_0^1 r(t)dt$ is normally distributed.
2. (5 points) What is expected value of P_1 for $\lambda = 5, \bar{r} = .04, \sigma = .2,$ and given $r(0) = .05$?

Solution:

Part 1.) We learned in lecture that the solution to the SDE is

$$\mathrm{r(t)} \ = \ \mathrm{r(0)e^{-\lambda t}} + \bar{r}\left(1\text{-}\mathrm{e^{-\lambda t}}\right) + \sigma \int_0^{\mathrm{t}} \mathrm{e^{-\lambda(t-s)}dW(s)},$$

which is normally distributed because the stochastic integral's integrand is deterministic. For the integrated rate, we have

$$\int_0^1 \mathrm{r(t)dt} = \int_0^1 \left(\mathrm{r(0)e^{-\lambda t}} + \bar{r}\left(1\text{-}\mathrm{e^{-\lambda t}}\right)\right) \mathrm{dt} + \sigma \int_0^1 \int_0^{\mathrm{t}} \mathrm{e^{-\lambda(t-s)}dW(s)\, dt}.$$

The non-stochastic term is

$$\mathrm{m} := \int_0^1 \left(\mathrm{r(0)e^{-\lambda t}} + \bar{r}\left(1\text{-}\mathrm{e^{-\lambda t}}\right)\right) \mathrm{dt} = \bar{r} + \frac{\mathrm{r(0)}\text{-}\bar{r}}{\lambda}\left(1\text{-}\mathrm{e^{-\lambda}}\right).$$

For the stochastic term, we switch the order of integration, and we see

$$\sigma \int_0^1 \int_0^{\mathrm{t}} \mathrm{e^{-\lambda(t-s)}dW(s)\, dt} = \sigma \int_0^1 \int_{\mathrm{s}}^1 \mathrm{e^{-\lambda(t-s)}\, dtdW(s)} = \frac{\sigma}{\lambda} \int_0^1 \left(1\text{-}\mathrm{e^{-\lambda(1-s)}}\right)\mathrm{dW(s)},$$

which is a stochastic integral with deterministic integrand, and therefore is normally distributed. In particular, this stochastic term has mean zero and variance

$$\mathrm{s} := \mathrm{E}\left(\frac{\sigma}{\lambda}\int_0^1 \left(1\text{-}\mathrm{e^{-\lambda(1-s)}}\right)\mathrm{dW(s)}\right)^2 = \left(\frac{\sigma}{\lambda}\right)^2 \int_0^1 \left(1\text{-}\mathrm{e^{-\lambda(1-s)}}\right)^2 \mathrm{ds} = \left(\frac{\sigma}{\lambda}\right)^2 \left(1\text{-}\frac{2\left(1\text{-}\mathrm{e^{-\lambda}}\right)}{\lambda} + \frac{1\text{-}\mathrm{e^{-2\lambda}}}{2\lambda}\right).$$

Finally, we can conclude that $\int_0^1 \mathrm{r(t)dt}$ is normally distributed with mean m and variance s.

Part 2.) Using the normal distribution from Part 1.), we have $\mathrm{E}P_1 = P_0 \mathrm{E} e^{\int_0^1 r(t)dt} = P_0 \mathrm{e}^{\mathrm{m}+.5\times\mathrm{s}}$. If we insert the given numbers, we have

$$m = .04 + \frac{.05\text{-}.04}{5}(1\text{-}e^{-5}) = .0420$$

$$s = \left(\frac{.2}{5}\right)^2 \left(1\text{-}\frac{2(1\text{-}e^{-5})}{5} + \frac{1\text{-}e^{-10}}{10}\right) = .0011.$$

Therefore,

$$\mathbf{EP_1 = P_0 e^{.0408+.5\times.0011} = P_0 \times 1.0435.}$$