Theorem: Let $h(Y)$ be any function of & $E[h(Y)^2] < \infty$.
Then

$$E\left((X - h(Y))^2\right) \geq E\left((X - E(X|Y))^2\right)$$

Further, if $h(Y)$ is any function s.t

$$E\left((X - h(Y))^2\right) = E\left((X - E(X|Y))^2\right), \quad \text{then}$$

$$E\left((h(Y) - E(X|Y))^2\right) = 0$$

Proof: $E[(X - h(Y))^2] = E[(X - E(X|Y) + E(X|Y) - h(Y))^2]$

$$= E\left[(X - E(X|Y))^2 + (E(X|Y) - h(Y))^2\right] + \underbrace{2E\left[(X - E(X|Y))(E(X|Y) - h(Y))\right]}_{\circledast} \tag{1}$$

$$\circledast = E\left[(X - E(X|Y))(E(X|Y) - h(Y))\right]$$

$$= E\left[E\left[(X - E(X|Y))(E(X|Y) - h(Y))\right] \mid Y\right] \quad \left(\text{since } E(X) = E(E(X|Y))\right)$$

$$= E\left[(E(X|Y) - h(Y))\, \underbrace{E\left[(X - E(X|Y)) \mid Y\right]}_{=0}\right] \quad \left(\text{Because } E(g(Y)X|Y) = g(Y)\,E(X|Y)\right)$$

because

$$E\left[X - E(X|Y) \mid Y\right] = E[X|Y] - E\left[E(X|Y)|Y\right]$$
$$= E[X|Y] - E[X|Y]$$
$$= 0$$

Therefore,

$$E\left[(X-h(Y))^2\right] = E\left[(X-E(X|Y))^2 + (E(X|Y)-h(Y))^2\right]$$
$$\geq E\left[(X-E(X|Y))^2\right]$$

Continuous random variables

We say X is continuous if its distribution function $F_X(x)$ is continuous. We say X is absolutely continuous if

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$. f is called the probability density function of X. If F is differentiable at some point x, then

$$f(x) = \frac{d}{dx} F(x)$$

(*) In this course, we will call abs. continuous rvs, continuous rvs.

Remark: $f(x)$ is <u>not</u> a probability.

$$P(x < X \leq x + dx) = F(x + dx) - F(x) \approx f(x)\, dx$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$$

In general,

$$P(X \in A) = \int_{A} f(x)\, dx$$

for any $A \subset \mathbb{R}$ such that $X^{-1}(A) \in \mathcal{F}$.

Theorem: If $F(x)$ is continuous, then
$P(X = x) = 0 \quad \forall x.$

Proof: $$P(X = x) = \lim_{n \to \infty} P\left(x - \frac{1}{n} < X \leq x\right)$$

$$= \lim_{n \to \infty} \left[F(x) - F\left(x - \tfrac{1}{n}\right)\right]$$

$$= F(x) - F(x) \quad (\text{because } F \text{ is cts})$$

$$= 0$$

$\square$

Ex:
$$f(x) = \begin{cases} 2x \ ; & 0 \leq x \leq 1 \\ 0 \ ; & \text{otherwise} \end{cases}$$

If $0 \leq x \leq 1$,
$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{0}^{x} 2u \cdot du = x^2$$

$$F(x) = \begin{cases} 0 \ ; & x < 0 \\ x^2 \ ; & 0 \leq x \leq 1 \\ 1 \ ; & x > 1 \end{cases}$$

Definition: If X is a cts rv, then

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

whenever this integral exists.

Theorem: $$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

for any function g s.t this integral exists.

Theorem: If X has a density function with $f(x) = 0$ when $x < 0$,

$$E(X) = \int_0^{\infty} (1 - F(x))\, dx = \int_0^{\infty} P(X > x)\, dx$$

Proof:

$$\int_0^{\infty} P(X > x)\, dx = \int_0^{\infty} \int_{y=x}^{\infty} f(y)\, dy\, dx$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} f(y)\, dx\, dy = \int_0^{\infty} y\, f(y)\, dy$$

$$= \int_{-\infty}^{\infty} y\, f(y)\, dy = E(X)$$

$\square$

Special continuous functions

① The exponential distribution

X is called an exponential rv if

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

For $x > 0$,

$$F(x) = \int_0^x f(y)\,dy = 1 - e^{-\lambda x}$$

Notation : $X \sim \exp(\lambda)$

Theorem : $E(X) = \frac{1}{\lambda}$

$$V(X) = \frac{1}{\lambda}$$

Ex:

The number of arrivals at a store, for some specific unit of time can be modeled by Poisson (λt). Let T_1 be the time until the first arrival. Then

$$P(T_1 > t) = P(\text{no arrivals in } [0,t]) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$$\Rightarrow P(T_1 \leq t) = 1 - e^{-\lambda t}.$$

$$\Rightarrow T_1 \sim \exp(\lambda).$$

Normal Distribution

X has a normal rv if

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \quad -\infty < x < \infty .$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \, dy$$

Notation : $X \sim N(\mu, \sigma^2)$

When $\mu = 0$, $\sigma = 1$, we say $Z \sim N(0,1)$ is standard normal.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$$

$$Z = \frac{X - \mu}{\sigma}$$

$$P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right)$$

$$= P(X \leq \mu + \sigma z) = F(\mu + \sigma z)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$y = \frac{x - \mu}{\sigma}$$

when $x = \mu + \sigma z$, $y = z$. Therefore,

$$P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy = \Phi(z)$$

Therefore, $\frac{X - \mu}{\sigma} \sim N(0, 1)$.