Math 600 Lecture 28

Recall: A partition P on $[a,b]$ is a set $\{x_0, x_1, \dots, x_n\} \subset [a,b]$ satisfying

$$a = x_0 < x_1 < \dots < x_n = b.$$

Let $f : [a,b] \to \mathbb{R}$ be bounded (I forgot to mention the assumption of boundedness in Lecture 27).

We define the upper and lower (Darboux) sums of f relative to P by

$$U(P,f) = \sum_{j=1}^{n} M_j \Delta x_j, \quad M_j = \sup\{f(x) \mid x_{j-1} \le x \le x_j\}, \quad \Delta x_j = x_j - x_{j-1},$$

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j, \quad m_j = \inf\{f(x) \mid x_{j-1} \le x \le x_j\},$$

respectively.

If P, P' are partitions on $[a,b]$, we say that P' is a <u>refinement</u> of P iff $P \subset P'$.

If P' is a refinement of P, then

$$-\infty < L(P,f) \le L(P',f) \le U(P',f) \le U(P,f) < \infty \quad \text{(since } f \text{ is bounded)}$$

always holds. Hence, we define

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \mathcal{P}\},$$

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \in \mathcal{P}\},$$

where $\mathcal{P}$ is the set of all partitions of $[a,b]$.

We say that f is <u>Riemann integrable</u> on $[a,b]$ iff

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

In this case,

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx \left(= \underline{\int_a^b} f(x)dx\right)$$

is called the <u>Riemann integral</u> of f on $[a,b]$.

Given $[a,b]$, we will always write $\mathbb{P}$ for the set of all partitions on $[a,b]$.

<u>Lemma</u>: Let $f: [a,b] \to \mathbb{R}$ be bounded. Then

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

<u>Proof</u>: Let P_1, P_2 be any partitions of $[a,b]$ and define $P' = P_1 \cup P_2$. Then P' is a refinement of both P_1 and P_2, and hence

$$L(P_1,f) \leq L(P',f) \leq U(P',f) \leq U(P_2,f)$$

$$\Rightarrow L(P_1,f) \leq U(P_2,f) \quad \forall P_1,P_2 \in \mathbb{P}$$

$$\Rightarrow \underline{\int_a^b} f(x)dx = \sup\{L(P_1,f) \mid P_1 \in \mathbb{P}\} \leq U(P_2,f) \quad \forall P_2 \in \mathbb{P}$$

$$\Rightarrow \underline{\int_a^b} f(x)dx \leq \inf\{U(P_2,f) \mid P_2 \in \mathbb{P}\} = \overline{\int_a^b} f(x)dx,$$

as desired. //

Theorem: Let $f:[a,b]\to\mathbb{R}$. Then f is Riemann integrable iff for all $\varepsilon>0$, there exist a partition P of $[a,b]$ such that

$$U(P,f)-L(P,f)<\varepsilon.$$

Proof: Suppose first that f is Riemann integrable on $[a,b]$ and define

$$I=\inf\{U(P,f)\mid P\in\mathcal{P}\}=\sup\{L(P,f)\mid P\in\mathcal{P}\}=\int_a^b f(x)\,dx.$$

Let $\varepsilon>0$ be given. Then there exist $P_1\in\mathcal{P}$ such that

$$I\le U(P_1,f)<I+\frac{\varepsilon}{2}$$

and $P_2\in\mathcal{P}$ such that

$$I-\frac{\varepsilon}{2}<L(P_2,f)\le I.$$

But then, with $P'=P_1\cup P_2$, we have

$$I-\frac{\varepsilon}{2}<L(P_2,f)\le L(P',f)\le U(P',f)\le U(P_1,f)<I+\frac{\varepsilon}{2}$$

$$\Rightarrow\quad U(P',f)-L(P',f)<\left(I+\frac{\varepsilon}{2}\right)-\left(I-\frac{\varepsilon}{2}\right)=\varepsilon.$$

Conversely, suppose there exists $\varepsilon>0$ such that, for all $P\in\mathcal{P}$,

$$U(P,f)-L(P,f)\ge\varepsilon$$

$$\Leftrightarrow\quad U(P,f)\ge L(P,f)+\varepsilon\quad\forall P\in\mathcal{P}.$$

Given any $P_1,P_2\in\mathcal{P}$ and $P'=P_1\cup P_2$, we have

$$U(P_1,f)\ge U(P',f)\ge L(P',f)+\varepsilon\ge L(P_1,f)+\varepsilon$$

and thus

$$U(P_1,f)\ge L(P_2,f)+\varepsilon\quad\forall P_1,P_2\in\mathcal{P}$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx = \inf\{U(P_1,f) \mid P_1 \in \mathcal{P}\} \geq L(P_2,f)+\varepsilon \quad \forall P_2 \in \mathcal{P}$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx \geq \sup\{L(P_2,f) \mid P_2 \in \mathcal{P}\}+\varepsilon = \underline{\int_a^b} f(x)\,dx + \varepsilon$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx.$$

This completes the proof. //

Theorem: Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is Riemann integrable on $[a,b]$.

Proof: Let $\varepsilon>0$ be given. Since f is continuous on the compact interval $[a,b]$, it is uniformly continuous, and hence there exists $\delta>0$ such that

$$x,y\in[a,b] \text{ and } |x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}.$$

Let $P=\{x_0,x_1,\dots,x_n\}$ be any partition of $[a,b]$ with mesh size $(\max\{\Delta x_j \mid j=1,\dots,n\})$ less than δ. Note that

$$\max\{f(x) \mid x_{j-1}\leq x\leq x_j\} - \min\{f(x) \mid x_{j-1}\leq x\leq x_j\} < \frac{\varepsilon}{b-a} \quad \forall j=1,\dots,n$$

$$\Rightarrow M_j - m_j < \frac{\varepsilon}{b-a} \quad \forall j=1,\dots,n$$

$$\Rightarrow U(P,f)-L(P,f) = \sum_{j=1}^{n} M_j\,\Delta x_j - \sum_{j=1}^{n} m_j\,\Delta x_j$$

$$= \sum_{j=1}^{n} (M_j - m_j)\Delta x_j$$

$$< \sum_{j=1}^{n} \frac{\varepsilon}{b-a}\Delta x_j = \frac{\varepsilon}{b-a}\sum_{j=1}^{n}\Delta x_j = \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

Thus, for all $\varepsilon>0$, there exists $P\in\mathcal{P}$ such that

$$U(P,f)-L(P,f)<\varepsilon.$$

Hence f is Riemann integrable on $[a,b]$. //

<u>Theorem</u>: Let $f:[a,b]\to\mathbb{R}$ be monotonic. Then f is Riemann integrable on $[a,b]$.

<u>Proof</u>: We will prove the theorem in the case that f is increasing; the proof in the case that f is decreasing is similar.

Let $\varepsilon>0$ be given and choose $n\in\mathbb{Z}^+$ sufficiently large that

$$\frac{(b-a)(f(b)-f(a))}{n}<\varepsilon$$

(Note that $f(b)-f(a)\geq 0$ since f is increasing. We assume that $f(b)-f(a)>0$, since otherwise f is constant and the result is trivial.) Let P be the uniform partition on $[a,b]$ with n subintervals: $P=\{x_0,x_1,\dots,x_n\}$, where

$$x_j=a+j\Delta x,\ j=0,1,\dots,n,\ \text{where } \Delta x=\frac{b-a}{n}.$$

Note that, since f is increasing, $M_j=f(x_j)$, $m_j=f(x_{j-1})$. Thus

$$U(P,f)-L(P,f)=\sum_{j=1}^{n}M_j\Delta x-\sum_{j=1}^{n}m_j\Delta x \quad (\text{note that } \Delta x_j=\Delta x\ \forall j)$$

$$=\sum_{j=1}^{n}(M_j-m_j)\Delta x$$

$$=\Delta x\sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))$$

$$=\Delta x(f(b)-f(a))=\frac{(b-a)}{n}(f(b)-f(a))<\varepsilon$$

by assumption. This shows that f is Riemann integrable on $[a,b]$. //

Note: $\sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))$ is called a <u>telescoping sum</u>:

$$\sum_{j=1}^{n}(f(x_j)-f(x_{j-1})) = \cancel{f(x_1)}-f(x_0)+\cancel{f(x_2)}-\cancel{f(x_1)}+\cancel{f(x_3)}-\cancel{f(x_2)}+\cdots+f(x_n)-\cancel{f(x_{n-1})}$$

$$= f(x_n)-f(x_0)$$

$$= f(b)-f(a).$$