Math 672 Lecture 20

Theorem: Let V be a complex normed vector space. Then the norm $\|\cdot\|$ of V is defined by an inner product iff the parallelogram law holds:

$$\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2)\quad \forall u,v\in V.$$

Proof: If $\|\cdot\|$ is defined by an inner product $\langle\cdot,\cdot\rangle$, then, for $u,v\in V$,

$$\begin{aligned}\|u+v\|^2+\|u-v\|^2&=\langle u+v,u+v\rangle+\langle u-v,u-v\rangle\\&=\langle u,u\rangle+\cancel{\langle u,v\rangle}+\cancel{\langle v,u\rangle}+\langle v,v\rangle\\&\quad+\langle u,u\rangle-\cancel{\langle u,v\rangle}-\cancel{\langle v,u\rangle}+\langle v,v\rangle\\&=2\|u\|^2+\|v\|^2.\end{aligned}$$

The proof of the converse is even trickier than in the real case, and will be omitted. //

Theorem (the projection theorem): Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. Let S be a subspace of V and let $x\in V$ be given. Then $v\in S$ is a best approximation to x from S (that is, v satisfies $\|v-x\|\le\|u-x\|\ \forall u\in S$) iff

$$\langle v-x,w\rangle=0\quad\forall w\in S.$$

Moreover, there exists a unique best approximation to x from S.

<u>Proof</u>: I will give the proof in the real case; the complex case is similar but a little trickier. Let $v \in V$ be given and note that

$$S = \{v + tw \mid t \in \mathbb{R} \text{ and } w \in S\}.$$

Choose an arbitrary $w \in S, w \neq 0$, and define $\varphi: \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \varphi(t) &= \|x - (v+tw)\|^2 \\ &= \|(x-v) - tw\|^2 \\ &= \langle x-v-tw, x-v-tw\rangle \\ &= \langle x-v, x-v\rangle - 2t\langle x-v, w\rangle + t^2\langle w,w\rangle. \end{aligned}$$

Note that φ is a convex quadratic and $\varphi(0) = \|x-v\|^2$. Thus

$$\begin{aligned} &\|x-v\| \leq \|x-(v+tw)\| \ \forall t \in \mathbb{R} \\ \iff\ &\varphi(0) \leq \varphi(t) \ \forall t \in \mathbb{R} \\ \iff\ &\varphi'(0) = 0 \\ \iff\ &\langle x-v, w\rangle = 0 \quad (\text{since } \varphi'(t) = 2t\langle w,w\rangle - 2\langle x-v,w\rangle). \end{aligned}$$

It then follows that

$$\begin{aligned} &\|x-v\| \leq \|x-u\| \ \forall u \in S \\ \iff\ &\|x-v\| \leq \|x-(v+tw)\| \ \forall t \in \mathbb{R}\ \forall w \in S \\ \iff\ &\langle x-v, w\rangle = 0 \ \forall w \in S. \end{aligned}$$

This proves the first result; we now must prove that there is a unique $v \in S$ that satisfies

$$(*) \quad \langle x-v, w\rangle = 0 \quad \forall w \in S.$$

Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for S. First note that $(*)$ is equivalent to

$$(**) \quad \langle x-v, u_i\rangle = 0 \quad \forall i=1,2,\ldots,n$$

Let $v \in S$ be given by

$$v = \sum_{j=1}^{n} \alpha_j u_j.$$

Then v satisfies $(**)$ iff

$$\langle x - \sum_{j=1}^{n} \alpha_j u_j, u_i\rangle = 0, \quad i=1,2,\ldots,n$$

$$\Leftrightarrow \langle x, u_i\rangle - \sum_{j=1}^{n} \alpha_j \langle u_j, u_i\rangle = 0, \quad i=1,2,\ldots,n$$

$$\Leftrightarrow \sum_{j=1}^{n} \langle u_j, u_i\rangle \alpha_j = \langle x, u_i\rangle, \quad i=1,2,\ldots,n$$

$$\Leftrightarrow G\alpha = b, \quad G \in \mathbb{R}^{n\times n}, \; G_{ij} = \langle u_j, u_i\rangle, \; b \in \mathbb{R}^n, \; b_i = \langle x, u_i\rangle.$$

It remains only to prove that G is invertible, since then there is a unique $v \in S$ satisfying $(*)$. Since G is square (i.e.

defines a linear operator from $\mathbb{R}^n$ into itself), it suffices to prove that G is nonsingular:

$$G\beta = 0 \Rightarrow (G\beta)_i = 0, \quad i = 1, 2, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{n} \beta_i (G\beta)_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij} \beta_j \beta_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} \langle u_j, u_i \rangle \beta_j \beta_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} \beta_j u_j, u_i \right\rangle \beta_i = 0 \quad \text{(by linearity in the first argument)}$$

$$\Rightarrow \left\langle \sum_{j=1}^{n} \beta_j u_j, \sum_{i=1}^{n} \beta_i u_i \right\rangle = 0 \quad \text{(by linearity in the second argument)}$$

$$\Rightarrow \left\| \sum_{j=1}^{n} \beta_j u_j \right\|^2 = 0$$

$$\Rightarrow \sum_{j=1}^{n} \beta_j u_j = 0$$

$$\Rightarrow \beta_1 = \cdots = \beta_n = 0 \quad \text{(since } \{u_1, \ldots, u_n\} \text{ is linearly independent)}$$

$$\Rightarrow \beta = 0.$$

Thus G is nonsingular and the proof is complete. //

Note that the proof of the projection theorem shows how to compute the best approximation (examples in the next lecture).

Orthogonal and orthonormal bases

Theorem: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let $\{u_1, \ldots, u_k\} \subseteq V$ be an orthogonal set with $u_j \neq 0$ for $j = 1, 2, \ldots, k$. Then $\{u_1, \ldots, u_k\}$ is linearly independent.

Proof: If $\alpha_1, \ldots, \alpha_k \in F$ satisfy

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0,$$

then

$$\langle u_i, \alpha_1 u_1 + \cdots + \alpha_k u_k \rangle = \langle u_i, 0 \rangle = 0 \quad \forall i = 1, 2, \ldots, k$$

$$\Rightarrow \alpha_1 \langle u_i, u_1 \rangle + \cdots + \alpha_k \langle u_i, u_k \rangle = 0 \quad \forall i = 1, 2, \ldots, k$$

$$\Rightarrow \alpha_i \langle u_i, u_i \rangle = 0 \quad \forall i = 1, 2, \ldots, k \quad (\text{since } \langle u_i, u_j \rangle = 0 \text{ for } j \neq i)$$

$$\Rightarrow \alpha_i = 0 \quad (\text{since } u_i \neq 0 \Rightarrow \langle u_i, u_i \rangle \neq 0).$$

This completes the proof. //

Definition: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$). We say that $\{u_1, \ldots, u_k\}$ is orthonormal iff

$$\langle u_i, u_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq k.$$

Theorem: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let $\{u_1, \ldots, u_n\}$ be an orthogonal basis for V. Then

$$v = \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \quad \forall v \in V.$$

If $\{u_1, \ldots, u_n\}$ is orthonormal, this simplifies to

$$v = \sum_{j=1}^{n} \langle v, u_j \rangle u_j \quad \forall v \in V.$$

<u>Proof</u>: Let $v \in V$. Since $\{u_1, \ldots, u_n\}$ is a basis for V, we have

$$v = \sum_{j=1}^{n} \alpha_j u_j$$

for some $\alpha_1, \ldots, \alpha_n \in F$. It follows that

$$\begin{aligned}
\langle v, u_i \rangle &= \left\langle \sum_{j=1}^{n} \alpha_j u_j, u_i \right\rangle \\
&= \sum_{j=1}^{n} \alpha_j \langle u_j, u_i \rangle \\
&= \alpha_i \langle u_i, u_i \rangle \quad (\text{since } \langle u_j, u_i \rangle = 0 \text{ for } j \neq i)
\end{aligned}$$

$$\Rightarrow \alpha_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}, \quad i = 1, 2, \ldots, n.$$

If the basis is orthonormal, then $\langle u_i, u_i \rangle = 1$ for all i. //

Given a general (nonorthogonal) basis $\{u_1, \ldots, u_n\}$ for V, we normally have to solve a system of equations to express v as

$$v = \sum_{j=1}^{n} \alpha_j u_j.$$

We have now seen two situations in which it is easy to finds the weights in a linear combination.

- Given a basis $\{u_1, \dots, u_n\}$ for V and its dual basis $\{\varphi_1, \dots, \varphi_n\}$ for V', we have

$$v = \sum_{j=1}^{n} \varphi_j(v) u_j \quad \forall v \in V.$$

(Common examples: V is a space of functions and $\{u_1, \dots, u_n\}$ is a <u>nodal basis</u>, so that $\varphi_j(v) = v(x_j)$, where x_j is the jth interpolation node.)

- The basis $\{u_1, \dots, u_n\}$ is orthogonal or orthonormal.

One last fact about orthonormal sets:

<u>Theorem</u>: Let V be an inner product space over F ($\mathbb{R}$ or $\mathbb{C}$) and let $\{u_1, \dots, u_k\} \subseteq V$ be an orthonormal set. Then

$$v = \sum_{j=1}^{k} \alpha_j u_j \Rightarrow \|v\|^2 = \sum_{j=1}^{k} |\alpha_j|^2.$$

<u>Proof</u>: We have

Note the need for a different "dummy" index

$$\|v\|^2 = \langle v, v \rangle = \left\langle \sum_{j=1}^{k} \alpha_j u_j, \sum_{i=1}^{k} \alpha_i u_i \right\rangle$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \alpha_j \overline{\alpha_i} \langle u_j, u_i \rangle$$

$$= \sum_{j=1}^{k} \alpha_j \overline{\alpha_j} \quad (\text{since } \langle u_j, u_i \rangle = \delta_{ij})$$

$$= \sum_{j=1}^{k} |\alpha_j|^2. \quad //$$