Math 600 Lecture 15

Statement: Let $\{x_n\}$ be a metric space in X. Then $\{x_n\}$ converges to a point in X iff $\{x_n\}$ is Cauchy.

- Name a space in which this statement is true.
- Name a space in which this statement is false.

<u>Definition</u>: Let $\{x_n\}$ be a sequence in $\mathbb{R}$. The <u>limit superior</u> and <u>limit inferior</u> of $\{x_n\}$ are defined by

$$\limsup_{n\to\infty} x_n = \lim_{N\to\infty} \left(\sup\{x_n \mid n \geq N\}\right)$$

and

$$\liminf_{n\to\infty} x_n = \lim_{N\to\infty} \left(\inf\{x_n \mid n \geq N\}\right).$$

Note: The sequence $\{s_N\}$ defined by

$$s_N = \sup\{x_n \mid n \geq N\}$$

is decreasing: $s_{N+1} \leq s_N$ because $\{x_n \mid n \geq N+1\} \subset \{x_n \mid n \geq N\}$.

In general, if U and V are subsets of $\mathbb{R}$ and $U \subset V$, then

$$\sup U \leq \sup V.$$

(Similarly, $\{\inf\{x_n \mid n \geq N\}\}$ is increasing.)

Hence, $\limsup_{n\to\infty} x_n$ is well defined, though it can equal either a real number or $\pm\infty$. (Similarly, $\liminf_{n\to\infty} x_n$ is well defined as a real number or $\pm\infty$.)

<u>Theorem</u>: Let $\{x_n\}$ be a sequence of real numbers.

1. If $\limsup_{n\to\infty} x_n = \infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \infty$.
2. If $\limsup_{n\to\infty} x_n = U \in \mathbb{R}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to U$, and U is the largest subsequential limit of $\{x_n\}$.
3. If $\liminf_{n\to\infty} x_n = -\infty$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to -\infty$.
4. If $\liminf_{n\to\infty} x_n = L \in \mathbb{R}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to L$, and L is the smallest subsequential limit of $\{x_n\}$.

<u>Proof</u>: 1. Suppose $\limsup_{n\to\infty} x_n = \infty$. This implies that

$$\sup\{x_n \mid n \geq N\} = \infty \quad \forall N \in \mathbb{Z}^+$$

(since $\{\sup\{x_n \mid n \geq N\}\}$ is a decreasing sequence). We construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \infty$, as follows: Choose $n_1 \in \mathbb{Z}^+$ such that

$$x_{n_1} > 1.$$

Clearly such an n_1 exists; otherwise, $\{x_n\}$ is bounded above

Next, choose $n_2 > n_1$ such that $x_{n_2} > 2$; if this is no possible, then

$\{x_n \mid n \geq n_1+1\}$ is bounded above.

Continuing, suppose we have $n_1 < n_2 < \cdots < n_k$ so that

$$x_{n_j} > j \ , \ j = 1, 2, \ldots, k.$$

There must exist $n_{k+1} > n_k$ such that $x_{n_{k+1}} > k+1$; otherwise, $\{x_n \mid n \geq n_k + 1\}$

is bounded above.

In this way, we construct a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} > k \quad \forall k;$$

it follow that $x_{n_k} \to \infty$, as desired.

2. Suppose $\limsup\limits_{n\to\infty} x_n = x \in \mathbb{R}$. We will construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \in \left(x - \tfrac{1}{k}, x + \tfrac{1}{k}\right) \quad \forall k \in \mathbb{Z}^+.$$

Note first that $\{\sup\{x_n \mid n \geq N\}\}$ is decreasing; hence

$$\sup\{x_n \mid n \geq N\} \geq x \quad \forall N \in \mathbb{Z}^+.$$

Since

$$\sup\{x_n \mid n \geq N\} \to x \text{ as } N \to \infty,$$

there exists $N' \in \mathbb{Z}^+$ such that

$$N \geq N' \Rightarrow \sup\{x_n \mid n \geq N\} \in [x, x+1)$$

So there exists $x_{n_1} \geq N'$ such that $x_{n_1} \in (x-1, x+1)$ (why not $x_{n_1} \in [x, x+1)$?)

Now suppose we have $n_1 < n_2 < \cdots < n_k$ such that

$$x_{n_j} \in \left(x - \frac{1}{j}, x + \frac{1}{j}\right) \quad \forall j = 1, \ldots, k.$$

Since

$$\sup\{x_n \mid n \geq N\} \to x \text{ as } N \to \infty,$$

there exists $N' \in \mathbb{Z}^+$ such that

$$N > N' \Rightarrow \sup\{x_n \mid n \geq N\} \in \left[x, x + \frac{1}{k+1}\right).$$

Hence there exists $n_{k+1} > \max\{N', n_k\}$ such that

$$x_{n_{k+1}} \in \left(x - \frac{1}{k+1}, x + \frac{1}{k+1}\right)$$

(if not, $\sup\{x_n \mid n > n_k\} \leq x - \frac{1}{k+1}$).

Thus there exists a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right) \quad \forall k \in \mathbb{Z}^+;$$

obviously $x_{n_k} \to x$.

(#3, #4 are proved analogously to #1, #2.) //

Introduction to series

Recall:

$$\forall x \in \mathbb{R}, \; e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Suppose e were rational, say $e = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$. Then

$$\frac{p}{q} = e = \sum_{n=0}^{q} \frac{1}{n!} + \sum_{n=q+1}^{\infty} \frac{1}{n!}$$

$$\Rightarrow \frac{q!\,p}{q} = \sum_{n=0}^{q} \frac{q!}{n!} + \sum_{n=q+1}^{\infty} \frac{q!}{n!}$$

Note that

$$\frac{q!\,p}{q} = p(q-1)! \in \mathbb{Z}^+,$$

$$\sum_{n=0}^{q} \frac{q!}{n!} \in \mathbb{Z}^+ \quad \left(\text{since } \frac{q!}{n!} = q(q-1)\cdots(n+1) \ \forall\, n \le q\right)$$

and, for $n > q$,

$$\frac{q!}{n!} = \frac{1}{(q+1)\cdots n} \le \frac{1}{(q+1)^{n-q}}.$$

Thus

$k = n - q$

$n = q+1 \Rightarrow k = 1$

$$\sum_{n=q+1}^{\infty} \frac{q!}{n!} \le \sum_{n=q+1}^{\infty} \left(\frac{1}{q+1}\right)^{n-q}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{q+1}\right)^{k} \quad (\text{a geometric series})$$

$$= \frac{1}{1 - \frac{1}{q+1}} - 1$$

$$= \frac{q+1}{q} - \frac{q}{q} = \frac{1}{q} < 1 \quad (\text{since obviously } q > 1).$$

But then we have

$$\frac{q!p}{q} = \sum_{n=0}^{q} \frac{q!}{n!} + \sum_{n=q+1}^{\infty} \frac{q!}{n!}$$

$$(\text{integer}) = (\text{integer}) + (\text{some number in } (0,1)),$$

a contradiction. Thus e cannot be rational

The next few lectures present the theory of infinite series like

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

Later in the course, we study power series like

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$