Convergence of Random variables

Let $\{X_n\}$ be a sequence of random variables. In this section, we aim to understand the convergence of X_n. Since $X_n : \Omega \longrightarrow \mathbb{R}$, $\{X_n\}$ is a sequence of functions.

We will discuss four ways of interpreting the statement $X_n \to X$ as $n \to \infty$.

Definition: Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, P)$.

(a) Almost sure convergence: We say $X_n \xrightarrow{a.s.} X$ if

$$P\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = 1.$$

(b) r^{th} mean convergence: We say $X_n \xrightarrow{r} X$, if $E[|X_n|^r] < \infty \quad \forall n \in \mathbb{N}$ &

$$\lim_{n\to\infty} E|X_n - X|^r = 0. \quad (r \geq 1).$$

(c) Convergence in probability: We say $X_n \xrightarrow{P} X$ if $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} P\left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon\right\} = 0.$$

(d) Convergence in distribution: We say $X_n \xrightarrow{D} X$ if

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$$

$\forall x$ s.t. $F_X(x) = P(X \le x)$ is continuous.

Remark: These modes of convergence are NOT equivalent.

The following implications hold in general:

$$\begin{tikzcd} X_n \xrightarrow{a.s} X \arrow[r, Rightarrow] & X_n \xrightarrow{P} X \arrow[r, Rightarrow] & X_n \xrightarrow{D} X \\ X_n \xrightarrow{r} X \arrow[ur, Rightarrow] & & \end{tikzcd}$$

Theorem: If $X_n \xrightarrow{1} X$, then $X_n \xrightarrow{P} X$.

Proof: Let $\varepsilon > 0$. Then, by Markov's inequality,

$$P\{|X_n - X| > \varepsilon\} \le \frac{E|X_n - X|}{\varepsilon} \longrightarrow 0 \text{ as } n \to \infty$$

$\square$

Theorem: If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{D} X$.

Lemma: Let X, Y be random variables. Then $\forall a \in \mathbb{R}$, $\forall \varepsilon > 0$,

$$P(Y \le a) \le P(X \le a + \varepsilon) + P(|X - Y| > \varepsilon).$$

Proof of lemma:

$$P(Y \leq a) = P(Y \leq a, X \leq a+\varepsilon) + P(Y \leq a, X > a+\varepsilon)$$

$$\leq P(X \leq a+\varepsilon) + P(Y - X \leq a - X, a - X < -\varepsilon).$$

Since $Y - X \leq a - X$ & $a - X < -\varepsilon \Rightarrow Y - X < -\varepsilon$,

$$P(Y - X \leq a - X, a - X < -\varepsilon) \leq P(Y - X < -\varepsilon).$$

Therefore,

$$\begin{aligned} P(Y \leq a) &\leq P(X \leq a+\varepsilon) + P(Y - X < -\varepsilon) \\ &\leq P(X \leq a+\varepsilon) + P(Y - X < -\varepsilon) + P(Y - X > \varepsilon) \\ &= P(X \leq a+\varepsilon) + P(|Y - X| > \varepsilon) \end{aligned}$$

$\square$

Proof of theorem:

Let $a \in \mathbb{R}$, $\varepsilon > 0$. Then, by the previous lemma,

$$P(X_n \leq a) \leq P(X \leq a+\varepsilon) + P(|X_n - X| > \varepsilon).$$

Similarly,

$$P(X \leq a-\varepsilon) \leq P(X_n \leq a) + P(|X_n - X| > \varepsilon).$$

Therefore, it follows that

$$P(X \leq a-\varepsilon) - P(|X_n - X| > \varepsilon) \leq P(X_n \leq a) \leq P(X \leq a+\varepsilon) + P(|X_n - X| > \varepsilon)$$

Since $X_n \xrightarrow{P} X$,

$$P(X \leq a-\varepsilon) \leq \lim_{n\to\infty} P(X_n \leq a) \leq P(X \leq a+\varepsilon).$$

or $$F(a-\varepsilon) \leq \lim_{n\to\infty} F_n(a) \leq F(a+\varepsilon).$$

If F is continuous at a, then, as $\varepsilon \to 0$,

$$F(a) \leq \lim_{n\to\infty} F_n(a) \leq F(a)$$

$$\Rightarrow \lim_{n\to\infty} F_n(a) = F(a).$$

$$\Rightarrow X_n \xrightarrow{D} X$$

$\square$

Law of large numbers

Theorem: (WLLN) Let $\{X_n\}$ be a sequence of independent random variables with $E(X_i) = \mu < \infty$ & $V(X_i) = \sigma^2 < \infty$.

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty.$$

Proof: Let $S_n = \sum_{i=1}^{n} X_i$. Then,

$$E(S_n) = n\mu.$$

Let $\varepsilon > 0$.

By Chebyshev's inequality,

$$P\left[\left|\frac{1}{n}S_n - \mu\right| > \varepsilon\right] \leq \frac{E\left[\left(\frac{1}{n}S_n - \mu\right)^2\right]}{\varepsilon^2}$$

$$= \frac{E\left[(S_n - n\mu)^2\right]}{n^2\varepsilon^2} = \frac{V(S_n)}{n^2\varepsilon^2} = \frac{n \cdot V(X_i)}{n^2\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Ex: Let $X_1, X_2, \ldots \sim B(m, p)$ be independent. Then,

$$\frac{X_1 + X_2 + \cdots X_n}{n} \xrightarrow{P} E(X_i) = mp \quad \text{as} \quad n \to \infty.$$

Ex: Let $X_1, X_2, \ldots \sim B(m, p)$ be independent. Discuss the convergence of $\frac{X_1^2 + \cdots + X_n^2}{n}$ as $n \to \infty$.

If $\{X_n\}$ is a sequence such that $E[X_i] = \mu_i$, we say $\{X_n\}$ obeys WLLN if $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \frac{(\mu_1 + \cdots + \mu_n)}{n} \right| \geq \varepsilon \right) = 0$$

Theorem: <u>Strong law of large numbers</u> (SLLN)

Let $X_1, X_2, \ldots$ be i.i.d & $E(X_i) = \mu < \infty$.

Then

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} \mu \quad \text{as } n \to \infty.$$

<u>Kolmogorov's theorem</u>: Let $\{X_n\}$ be i.i.d. Then $\{X_n\}$ obey WLLN if & only if $E|X_n| < \infty$.

<u>Chebyshev's theorem</u>: Let $\{X_n\}$ be a sequence such that X_i & X_j are independent for $i \neq j$. If $\exists M > 0$ s.t $V(X_n) \leq M$ $\forall n$, then $\{X_n\}$ obeys WLLN.

Markov's theorem: Let $\{X_n\}$ be a sequence of rvs such that

$$\lim_{n \to \infty} \frac{1}{n^2} V[X_1 + \dots + X_n] = 0.$$

Then $\{X_n\}$ obeys the WLLN.

Kolmogorov's theorem 2: Let $\{X_n\}$ be independent rvs with $V(X_n) = \sigma_n^2 < \infty$.

If $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty$, then $\{X_n\}$ obeys SLLN.