Math 600 Lecture 21

Recall:

- $f: E \to Y$ ($E \subset X$) is continuous at $p \in E$ iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ (x \in E \text{ and } d_X(x,p) < \delta) \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

(the definition).

- f is continuous at $p \in E$ iff $\lim_{x \to p} f(x) = f(p)$ (equivalent condition).

- f is continuous at $p \in E$ iff

$$\forall \{p_n\} \subset E\ (p_n \to p \Rightarrow f(p_n) \to f(p))$$

(equivalent condition).

We say that f is continuous (or continuous on E) iff f is continuous at every $p \in E$.

Here's another equivalent condition:

<u>Theorem</u>: Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f: X \to Y$. Then f is continuous iff the inverse image of every open set in Y is open in X.

<u>Proof:</u> Suppose first that f is continuous and let $V \subset Y$ be open. We wish to show that $U = f^{-1}(V)$ is open in X. Let $x \in U$; then $f(x) \in V$ and, since V is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset V$. Since f is continuous at x, there exists $\delta > 0$ such that

$$d_X(u,x) < \delta \Rightarrow d_Y(f(u), f(x)) < \varepsilon,$$

that is,

$$u \in B_\delta(x) \Rightarrow f(u) \in B_\varepsilon(f(x)) \subset V.$$

Thus

$$f(B_\delta(x)) \subset V \Rightarrow B_\delta(x) \subset U.$$

This shows that U is open.

Conversely, suppose

$$V \subset Y \text{ is open} \Rightarrow f^{-1}(V) \text{ is open in } X.$$

Let $x \in X$ and let $\varepsilon > 0$ be given. Then $B_\varepsilon(f(x))$ is open in Y and hence $U = f^{-1}(B_\varepsilon(x))$ is open in X. Since $x \in U$, there exists $\delta > 0$ such that $B_\delta(x) \subset U$. But then

$$f(B_\delta(x)) \subset B_\varepsilon(f(x)),$$

that is,

$$d_X(u,x) < \delta \Rightarrow d_Y(f(u), f(x)) < \varepsilon.$$

Thus f is continuous at x. Since x was chosen arbitrarily, this shows that f is continuous on X. //

<u>Lemma</u>: Let X, Y be sets and let $f: X \to Y$. Then, for all $V \subset Y$,

$$f^{-1}(V^c) = f^{-1}(V)^c \quad (\text{i.e. } f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)).$$

Also, if $\{S_a \mid a \in A\}$ is a collection of subsets of Y, then

$$f^{-1}\left(\bigcup_{a \in A} S_a\right) = \bigcup_{a \in A} f^{-1}(S_a) \quad \text{and} \quad f^{-1}\left(\bigcap_{a \in A} S_a\right) = \bigcap_{a \in A} f^{-1}(S_a).$$

Corollary: Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f: X \to Y$. Then f is continuous iff the inverse image of every closed set in Y is closed in X.

Theorem: Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f: X \to Y$ be continuous. If $E \subset X$ is compact, then $f(E)$ is compact.

Proof: Suppose f is continuous, $E \subset X$ is compact, and $\{G_\alpha \mid \alpha \in A\}$ is an open cover of $f(E)$ (thus each G_α is an open subset of Y, and $f(E) \subset \bigcup_{\alpha \in A} G_\alpha$).

But

$$f(E) \subset \bigcup_{\alpha \in A} G_\alpha \Rightarrow E \subset f^{-1}\Big(\bigcup_{\alpha \in A} G_\alpha\Big) = \bigcup_{\alpha \in A} f^{-1}(G_\alpha).$$

By the previous theorem, $f^{-1}(G_\alpha)$ is open for each $\alpha \in A$ and hence $\{f^{-1}(G_a) \mid a \in A\}$ is an open cover of E. Since E is compact, there exist $a_1, a_2, \dots, a_n \in A$ such that

$$E \subset \bigcup_{j=1}^{n} f^{-1}(G_a)$$

$$\Rightarrow f(E) \subset f\Big(\bigcup_{j=1}^{n} f^{-1}(G_{a_j})\Big) = \bigcup_{j=1}^{n} f(f^{-1}(G_{a_j})).$$

Now, $f(f^{-1}(G_a))$ may not equal G_a (why?), but we have

$$f(f^{-1}(G_a)) \subset G_a \quad \forall a \in A.$$

Thus

$$f(E) \subset \bigcup_{j=1}^{n} G_{a_j}.$$

This shows that $f(E)$ is compact. //

Corollary: If $f: X \to Y$ is continuous and $E \subset X$ is compact, then $f(E)$ is closed and bounded.

Corollary: Let (X, d) be a metric space, let $f: X \to \mathbb{R}$ be continuous, and let $E \subset X$ be compact. Then f attains its maximum and minimum on E; that is, there exist $x_1, x_2 \in E$ such that

$$f(x_1) \le f(x) \quad \forall x \in E$$

and

$$f(x_2) \ge f(x) \quad \forall x \in E.$$

Proof: Since $f(E)$ is bounded, $\inf f(E)$ and $\sup f(E)$ are real numbers. If $\sup f(E) \notin f(E)$, then there exists a sequence $\{p_n\} \subset E$ such that

$$f(p_n) \to \sup f(E).$$

But then, since $f(E)$ is closed, $\sup f(E) \in f(E)$. Thus $\sup f(E) \notin f(E)$ is impossible. Hence

$$\sup f(E) \in f(E) \Rightarrow \exists x_2 \in E,\ f(x_2) = \sup f(E)$$

$$\Rightarrow f(x_2) \ge f(x) \quad \forall x \in E.$$

Similarly for $\inf f(E)$. //

<u>Theorem</u>: Suppose $(X,d_X),(Y,d_Y)$ are metric spaces, X is compact, and $f: X \to Y$ is continuous. If f is invertible (i.e., if f is bijective), then f^{-1} is also continuous.

<u>Proof:</u> It suffice to prove that

$$U \subset X \text{ open} \Rightarrow (f^{-1})^{-1}(U) \text{ is open},$$

that is,

$$U \subset X \text{ open} \Rightarrow f(U) \text{ is open}.$$

So let $U \subset X$ be open. Then

$$U^c \text{ is closed} \Rightarrow U^c \text{ is compact}$$ (since a closed subset of a compact set is compact)

$$\begin{aligned} &\Rightarrow f(U^c) \text{ is compact} \\ &\Rightarrow f(U^c) \text{ is closed} \\ &\Rightarrow f(U)^c \text{ is closed} \\ &\Rightarrow f(U) \text{ is open.} \ // \end{aligned}$$

Note that $f(U^c) = f(U)^c$ is valid only because f is bijective (whereas $f^{-1}(V^c) = f^{-1}(V)^c$ is always valid). Also, $(f^{-1})^{-1}(U) = f(U)$ is only true (in fact, it's only meaningful) because f is bijective (invertible).

(Note the contrasts: In general, i.e. if f is not assumed to be bijective,

$f^{-1}(Y) = X$ but $f(X)$ may not equal Y,

$\forall V \subset Y$, $f^{-1}(V) \cap f^{-1}(V^c) = \emptyset$, but there may exist $U \subset X$ such that $f(U) \cap f(U^c) \neq \emptyset$.)