Math 672 Lecture 6

Definition: Let V, W be vector spaces over F and let $T: V \to W$ be given. We say that T is a <u>linear map</u> iff

$$T(u+v) = T(u) + T(v) \quad \forall u, v \in V$$

and

$$T(\alpha u) = \alpha T(u) \quad \forall u \in V \ \forall \alpha \in F.$$

[Synonyms for linear map: linear transformation, linear function, linear operator. However, our author reserves "linear operator" to describe maps of the form $T: V \to V$ (domain and codomain are the same). Most authors do not use "linear function" even though it is perfectly correct.]

Lemma: Let V and W be vector spaces over F and let $T: V \to W$ be given. Then T is linear iff

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad \forall u, v \in V \ \forall \alpha, \beta \in F.$$

Proof: Suppose first that T is linear. Then, for any $u, v \in V$ and

$\alpha, \beta \in F$,

$$T(\alpha u + \beta v) = T(\alpha u) + T(\beta v) \quad \text{(by the first property of linearity)}$$

$$= \alpha T(u) + \beta T(v) \quad \text{(by the second property, applied twice).}$$

Conversely, suppose T satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad \forall u, v \in V \ \forall \alpha, \beta \in F.$$

Then, for $u, v \in V$,

$$T(u+v) = T(1 \cdot u + 1 \cdot v) = 1 \cdot T(u) + 1 \cdot T(v) = T(u) + T(v).$$

For $u \in V, \alpha \in F$, we have

$$T(\alpha u) = T(\alpha u + 0 \cdot u) = \alpha T(u) + 0 T(u) = \alpha T(u).$$

Thus T is linear. //

Note: From now on, if we write "Let $T: V \to W$ be linear," it is assumed that V and W are vector spaces over a common field F.

Lemma: Let $T: V \to W$ be linear. Then

$$T(0) = 0 \quad (\text{i.e. } T(0_V) = 0_W).$$

Proof: We have

$$T(0_V) = T(0_F \cdot 0_V) = 0_F\, T(0_V) \quad (\text{since } T \text{ is linear})$$
$$= 0_W \quad (\text{since } T(0_V) \in W). \,/\!/$$

Note that we can always regard F itself as a 1-dimensional vector space over F (except for notation, we can say that $F = F^1$).

Lemma: Let $T: F \to F$ be a linear map. Then there exists $\alpha \in F$ such that

$$T(x) = \alpha x \quad \forall\, x \in F.$$

Proof: Define $\alpha = T(1)$. Then, for all $x \in F$,

$$T(x) = T(x \cdot 1) = x\, T(1) = x\alpha = \alpha x. \,/\!/$$

Theorem: Let V be a nontrivial finite-dimensional vector space, let $\{v_1, v_2, \dots, v_n\}$ be a basis for V, and let $w_1, w_2, \dots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ such that

$$T(v_j) = w_j \quad \forall\, j = 1, 2, \dots, n.$$

Proof: Each $u \in V$ can be written uniquely as

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n u_n.$$

Thus we can define $T: V \to W$ by

$$T(u) = \alpha w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n, \text{ where } u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

It follows immediately that this map satisfies

$$\begin{aligned} T(v_j) &= T(0 \cdot v_1 + \cdots + 0 v_{j-1} + 1 \cdot v_j + 0 \cdot v_{j+1} + \cdots + 0 v_n) \\ &= 0 \cdot w_1 + \cdots + 0 w_{j-1} + 1 \cdot w_j + 0 w_{j+1} + \cdots + 0 w_n \\ &= w_j \end{aligned}$$

for all $j = 1, 2, \ldots, n$. It is easy, though tedious, to prove that T is linear: If $u, v \in V$ then there exist $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n \in F$ such that

$$u = \gamma_1 v_1 + \cdots + \gamma_n v_n, \quad v = \delta_1 v_1 + \cdots + \delta_n v_n$$

But then

$$u + v = (\gamma_1 + \delta_1) v_1 + \cdots + (\gamma_n + \delta_n) v_n$$

$$\begin{aligned} \Rightarrow T(u+v) &= (\gamma_1 + \delta_1) w_1 + \cdots + (\gamma_n + \delta_n) w_n \\ &= \gamma_1 w_1 + \delta_1 w_1 + \cdots + \gamma_n w_n + \delta_n w_n \\ &= (\gamma_1 w_1 + \cdots + \gamma_n w_n) + (\delta_1 w_1 + \cdots + \delta_n w_n) \\ &= T(u) + T(v). \end{aligned}$$

Similarly, if $u \in V$ and $\alpha \in F$, then there exist $\gamma_1, \ldots, \gamma_n \in F$

such that

$$u = \gamma_1 v_1 + \cdots + \gamma_n v_n$$

$$\Rightarrow \alpha u = \alpha(\gamma_1 v_1 + \cdots + \gamma_n v_n)$$

$$= \alpha(\gamma_1 v_1) + \cdots + \alpha(\gamma_n v_n)$$

$$= (\alpha\gamma_1) v_1 + \cdots + (\alpha\gamma_n) v_n$$

$$\Rightarrow T(\alpha u) = (\alpha\gamma_1) w_1 + \cdots + (\alpha\gamma_n) w_n$$

$$= \alpha(\gamma_1 w_1) + \cdots + \alpha(\gamma_n w_n)$$

$$= \alpha(\gamma_1 w_1 + \cdots + \gamma_n w_n)$$

$$= \alpha T(u).$$

Thus T is linear, and we have proved existence.

Now suppose $T: V \to W$ and $S: V \to W$ are linear maps satisfying

$$T(v_j) = w_j \quad \forall j = 1, 2, \ldots, n$$

and

$$S(v_j) = w_j \quad \forall j = 1, 2, \ldots, n.$$

We must show that $T = S$, that is, that

$$T(u) = S(u) \quad \forall u \in V.$$

So let u be an arbitrary element of V. Then there exist $\alpha_1, \ldots, \alpha_n \in F$ such that $u = \alpha_1 v_1 + \cdots + \alpha_n v_n$. We then have

$$
\begin{aligned}
T(u) &= T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) && \text{(by linearity)} \\
&= \alpha_1 w_1 + \cdots + \alpha_n w_n && (\text{since } T(v_j) = w_j \ \forall j) \\
&= \alpha_1 S(v_1) + \cdots + \alpha_n S(v_n) && (\text{since } S(v_j) = w_j \ \forall j) \\
&= S(\alpha_1 v_1 + \cdots + \alpha_n v_n) && \text{(by linearity)} \\
&= S(u).
\end{aligned}
$$

Thus $S = T$ and hence T is unique. //

Examples of linear maps

1. "The" identity operator: Let V be a vector space over F and define $I : V \to V$ by

$$I(v) = v \quad \forall v \in V.$$

Then I is a linear map, called the identity operator. (I write "the" identity operator because there are actually many identity operators, one for each V.)

2. Differentiation: Define $D : C'[a,b] \to C[a,b]$ by $Df = f'$ $\forall f \in C'[a,b]$.

Then D is linear (this is a theorem from calculus).

3. <u>Integration</u>: We cannot define $Q: C[a,b] \to C^1[a,b]$ by

$$(Qf)(x) = \int f(x)\,dx$$

because f does not have a unique antiderivative (thus Q, given by this formula, is not well defined.)

We can define $Q: C[a,b] \to C^1[a,b]$ by

$$(Qf)(x) = \int_a^x f(t)\,dt.$$

(Question to think about: Are D and Q inverses of each other?)

4. Let $A \in F^{m \times n}$. Recall that matrix-vector multiplication is defined by

$$(Ax)_i = \sum_{j=1}^{n} A_{ij} x_j \quad , \quad i = 1, 2, \ldots, m \qquad (x \in \mathbb{R}^n, Ax \in \mathbb{R}^m)$$

or, equivalently,

$$Ax = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \qquad \text{(a linear combination of } A_1, A_2, \ldots, A_n\text{)},$$

where $A_1, A_2, \ldots, A_n$ are the columns of A. It can be verified (easy but tedious) that

$$A(\alpha x+\beta y)=\alpha Ax+\beta Ay \quad \forall\, x,y\in F^n \ \forall \alpha,\beta\in F.$$

Thus, if we define $T: F^n \to F^m$ by $T(x)=Ax \ \forall x\in F^n$, then T is linear.