Lemma: (a) If $A_1 \subset A_2 \subset \dots \in \mathcal{F}$, then

$$\lim_{n \to \infty} P(A_n) = P\left(\bigcup_n A_n\right)$$

(b) If $B_1 \supset B_2 \supset \dots \in \mathcal{F}$, then

$$\lim_{n \to \infty} P(B_n) = P\left(\bigcap_n B_n\right)$$

Proof: (a) Observe that

$$\bigcup_n A_n = A_1 \cup (A_2 - A_1) \cup \dots$$

is a union of disjoint sets.

$$P\left(\bigcup A_n\right) = P(A_1) + \sum_{n=2}^{\infty} P(A_n - A_{n-1})$$

Since $A_{n-1} \subset A_n$,

$$P(A_n - A_{n-1}) = P(A_n) - \underbrace{P(A_n \cap A_{n-1})}_{P(A_{n-1})}$$

$$= P(A_n) - P(A_{n-1})$$

Therefore,

$$P\left(\bigcup A_n\right) = P(A_1) + \sum_{n=2}^{\infty} \left(P(A_n) - P(A_{n-1})\right)$$

$$P\left(\bigcup A_n\right) = P(A_1) + \lim_{N\to\infty} \sum_{n=2}^{N} \left(P(A_n) - P(A_{n-1})\right)$$

$$= P(A_1) + \lim_{N\to\infty} \left[P(A_N) - P(A_1) \right]$$

$$= \lim_{N\to\infty} P(A_N)$$

$\square$

(b) HW

Proof of proposition :

(ii) Let $x \le y$.

$$B_x = \{\omega : X(\omega) \le x\}$$
$$B_y = \{\omega : X(\omega) \le y\}$$

Notice that $B_x \subset B_y$. Therefore,

$$P(B_x) \le P(B_y)$$
$$F(x) \le F(y).$$

$\square$

(iii) $B_n = \{\omega : X(\omega) \le n\}$

Notice that $B_1 \subset B_2 \subset \cdots$ &

$$\bigcup_{n=1}^{\infty} B_n = \Omega$$

By the previous lemma,

$$1 = P(\Omega) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} F(n)$$

(iv) Let $\{y_n\}$ be a decreasing sequence such that $y_n \to x$.

Let $B_{y_n} := \{\omega \mid X(\omega) \leq y_n\}$

Notice that $B_{y_1} \supset B_{y_2} \supset \dots$ &

$$\bigcap_i B_{y_i} = B_x$$

Therefore,

$$\lim_{n\to\infty} \underbrace{P(B_{y_n})}_{F(y_n)} = P\left(\bigcap B_{y_n}\right) = \underbrace{P(B_x)}_{F(x)}$$

$$\Rightarrow \lim_{n\to\infty} F(y_n) = F(x) \quad \square$$

(v)
(vi) } HW

Theorem: Let F be the distribution function of X. Then

i) $P(X < x) = F(x-)$

ii) $P(X = x) = F(x) - F(x-)$

iii) If $a<b$, $P[\omega : a \leq X(\omega) \leq b] = F(b) - F(a)$

iv) $P(\omega : X(\omega) > x) = 1 - F(x)$

Proof:

(i) Let $B_x = \{\omega : X(\omega) \leq x\}$.

Observe that

$$\{\omega : X(\omega) < x\} = \bigcup_n \{X \leq x - \tfrac{1}{n}\}$$

$$= \bigcup_n B_{x-\frac{1}{n}}$$

Notice that $\{B_{x-\frac{1}{n}}\}$ is an increasing sequence of sets.

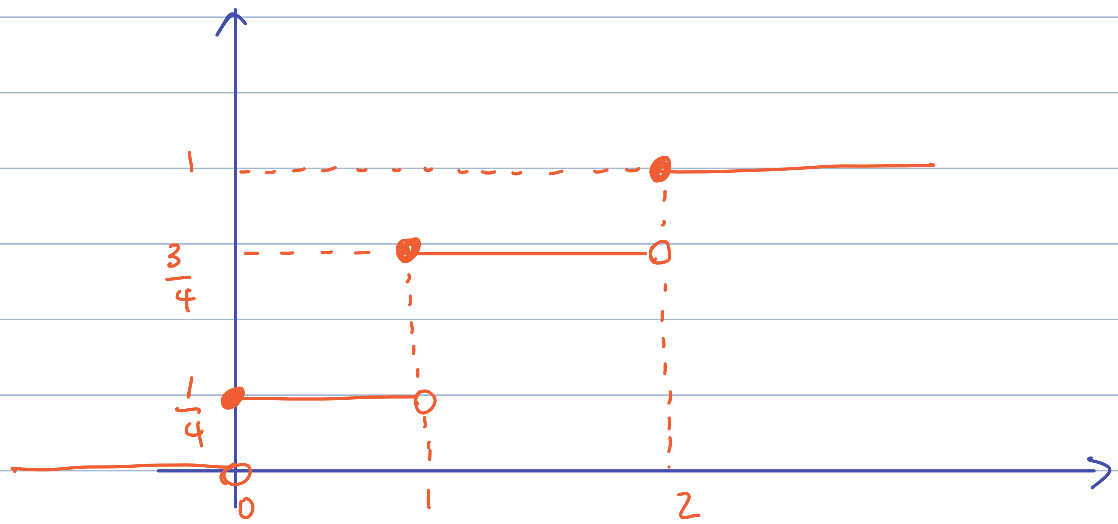
Therefore,

$$P(X<x) = P\left(\bigcup_n B_{x-\frac{1}{n}}\right) = \lim_{n\to\infty} P(B_{x-\frac{1}{n}})$$

$$= \lim_{n\to\infty} F(x - \tfrac{1}{n}) = F(x-)$$

Ex :

$$F_X(x) = \begin{cases} 0 & ; \ x < 0 \\ 1/4 & ; \ 0 \leq x < 1 \\ 3/4 & ; \ 1 \leq x < 2 \\ 1 & ; \ x \geq 2 \end{cases}$$



i) $P(X < 1) = F(1-) = \frac{1}{4}$

ii) $P(X = 2) = F(2) - F(2-)$

$= 1 - \frac{3}{4} = \frac{1}{4}$

Practice Quiz 2

If $P(A|C) \geq P(B|C)$, $P(A|C^c) \geq P(B|C^c)$,

then $P(A) \geq P(B)$

Solution:

$$
\begin{aligned}
P(A) &= \underbrace{P(A|C)}_{\geq P(B|C)} P(C) + \underbrace{P(A|C^c)}_{\geq P(B|C^c)} P(C^c) \\
&\geq P(B|C)\, P(C) + P(B|C^c)\, P(C^c) \\
&= P(B)
\end{aligned}
$$