Expectation of Discrete random variables

Definition: The expectation of a discrete rv X is given by

$$E(X) = \sum_{x : p(x) > 0} x \, p(x)$$

$$= \sum_{x : p(x) > 0} x \cdot P(X = x) .$$

when $\displaystyle\sum_{x : p(x) > 0} |x| \cdot p(x) < \infty$.

Ex: A fair die is rolled twice

$X = \#\ 1s$ observed

$$E(X) = 0 \cdot P(X=0) + 1 \cdot \underbrace{P(X=1)}_{10/36} + 2 \cdot \underbrace{P(X=2)}_{1/36}$$

$$= \frac{1}{3}$$

Ex: Suppose X has a pmf

$$p(n) = \frac{1}{n(n+1)} , \quad n = 1, 2, 3, \ldots$$

$$\sum |x| \cdot p(x) = \sum_{x=1}^{\infty} x \cdot \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \frac{1}{x+1} \text{ diverges .}$$

$\Rightarrow E(X)$ is undefined.

Theorem: Let X be a discrete rv & $g : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_x |g(x)| \, p(x) < \infty .$$

Then $E[g(X)] = \sum g(x) \, p(x)$

Proof: Let $\{y_j\}$ denote all the possible values $g(X)$ can take.

Define $A_j := \{x : g(x) = y_j\}$

$\left(g(A_j) = \{y_j\} \right)$

Let $Y = g(X)$.

$$E[g(X)] = E[Y] = \sum_j y_j \, P(Y = y_j)$$

$$= \sum_j y_j \cdot P(X \in A_j)$$

$$= \sum_j y_j \sum_{x \in A_j} P(X = x)$$

$$= \sum_j \sum_{x \in A_j} g(x) \, P(X = x) \quad \text{since } g(x) = y_j \ \forall x \in A_j .$$

$$= \sum_{x} g(x)\, P(X = x) \quad \text{since} \quad A_i \cap A_j = \phi, \ \forall i, j.$$

Prop:

(I) $E[X+Y] = E[X] + E[Y]$

(II) $E[kX] = kE[X]$, $\forall k \in \mathbb{R}$.

(III) $X \leq Y \Rightarrow E(X) \leq E(Y)$

(IV) $X = k$ (constant), $E(X) = k$.

(V) $E(\mathbb{1}_A) = P(A)$

Proof: (I) Let $\{x_i\}$ be the values X takes

$\{y_j\}$ denote the values Y takes &

$\{z_k\}$ be the values $Z = X+Y$ takes.

$A_k = \{(i,j) : x_i + y_j = z_k\}$

Notice that $P(X+Y = z_k) = \sum_{(i,j) \in A_k} P(X = x_i, Y = y_j)$

$$E(Z) = \sum_{k} z_k\, P(X+Y = z_k)$$

$$= \sum_{k} z_k \sum_{(i,j) \in A_k} P(X = x_i, Y = y_j)$$

$$= \sum_{k} \sum_{A_k} (x_i + y_j)\, P(X = x_i, Y = y_j)$$

$$= \sum_{i} \sum_{j} (x_i + y_j)\, P(X = x_i, Y = y_j) = \sum_{i} \sum_{j} x_i\, P(X = x_i, Y = y_j)$$

$$+ \sum_{i} \sum_{j} y_j\, P(X = x_i, Y = y_j) = \sum_{i} x_i\, P(X = x_i) + \sum_{j} y_j\, P(Y = y_j) = E(X) + E(Y) \ \square$$

Moments

Definition: Let X be a rv with $E(X) < \infty$. The variance of X is defined by

$$V(X) = E(X - E(X))^2$$

Prop: $V(X) = E(X^2) - E(X)^2$

Prop: $V(aX + b) = a^2 V(X)$.

Ex: Prediction:

Let X be a rv. What t best predicts X (i.e. minimizes $E[(X-t)^2]$)?

$$R(t) = E[(X-t)^2]$$

$$= E[X^2 - 2tX + t^2]$$

$$R(t) = E[X^2] - 2t\,E(X) + t^2$$

$$\frac{d}{dt} R(t) = -2E(X) + 2t = 0 \Rightarrow t = E(X).$$

The value that best predicts X is $E(X)$.

Special Discrete Distributions

① Bernoulli Distribution.

We say X is Bernoulli if

$$P(X=1) = p \quad , \quad P(X=0) = 1-p =: q$$

Ex: If A is an event, then 1_A is a Bernoulli RV.

$$E(X) = p$$

$$V(X) = p(1-p)$$

② Binomial RV

Let $X_1, X_2, \dots, X_n$ be independent Bernoulli RVs, Then $X = \sum_{i=1}^{n} X_i$ is called a binomial RV with parameters n, p

$X = \#$ successes of n independent Bernoulli RVs.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad , \quad k = 0, 1, 2, \dots, n$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1 \checkmark$$

Notation $X \sim B(n,p)$

Proposition : $E(X) = p$

$V(X) = np(1-p)$

Ex : $X \sim B(n,p)$

$$E(a^X) = \sum_{k=0}^{n} a^k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (ap)^k (1-p)^{n-k}$$

By the binomial thm

$$= \boxed{(ap + (1-p))^n}$$