8.1: Arc Length

1 Theory

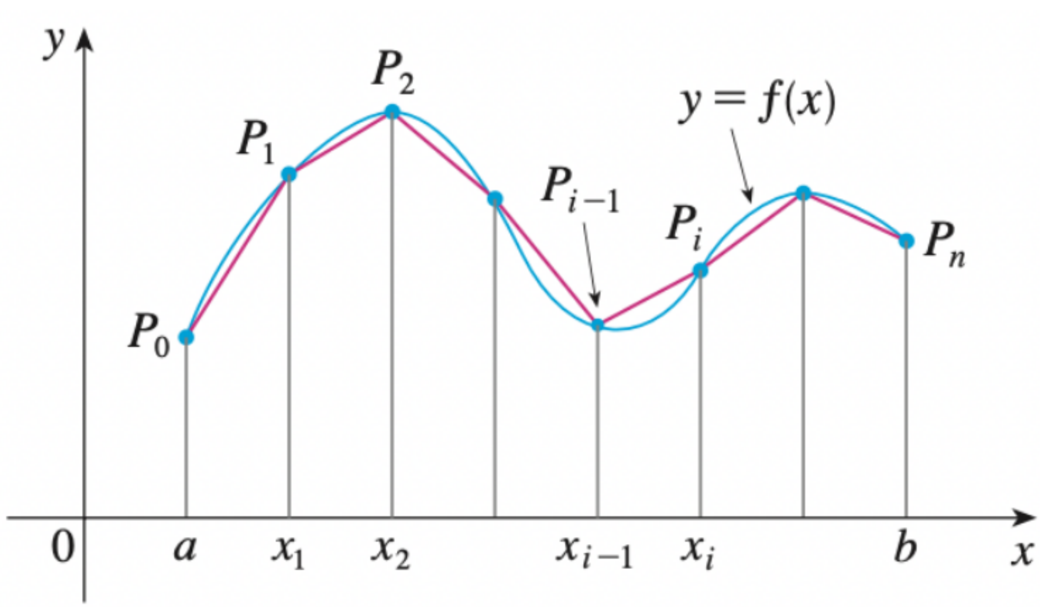
Often we talk about the area under a curve $f(x)$ over an interval $[a, b]$. However, sometimes we are more interested in the *length* of the curve over a given interval instead. The length of a curve over an interval gives us something called *arc length* which serves to be a very important idea when we eventually want to talk about curves that can't be described as functions of x or y (such as the path of a particle, ant, or your average NJ driver). Eventually arclength will be an important part of vector calculus. Before we continue it will be quite useful to recall the MVT:

Mean Value Theorem: Given a function $f(x)$ that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point c between a and b (i.e. $a < c < b$) such that

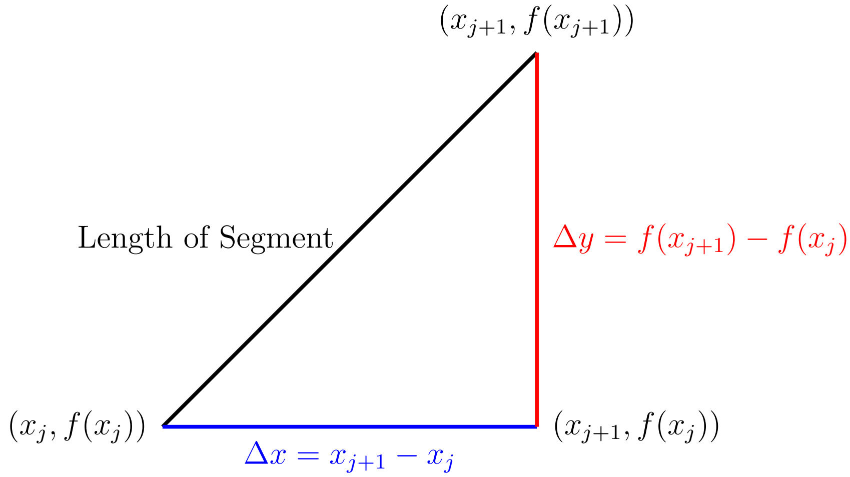
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

The MVT effectively says, that if you can a take derivative of your function, then no matter what the average slope is between two points is, the derivative must equal that value *at some point*. Typically, with the MVT we don't get more info about where the point c is located.

The construction of arclength is not too bad. The idea is, much like a lot of math, to first break the curve into things which are a lot easier to figure out the distance for. In this case, we take the curve and consider measuring straight lines between points instead of the wiggly stuff.



If each sub-interval is some $[x_j, x_{j+1}]$, then from all these straight lines, we can use the Pythagorean theorem to figure out the distance of each piece. For any random straight line you pick we get the following picture



Hence, the length of a given segment is then we can do the following:

$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 \left(1 + \frac{\Delta y^2}{\Delta x^2}\right)} = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = \Delta x \sqrt{1 + \left(\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}\right)^2}$$

By the MVT there has to exist some z_j between x_j and x_{j+1} where

$$\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} = f'(z_j).$$

Plugging this into our equation gives us

$$\Delta x \sqrt{1 + [f'(z_j)]^2}.$$

Then if we add all the different segments together we get our Riemann sum looking expression

$$\sum_{j=1}^{n} \Delta x \sqrt{1 + [f'(z_j)]^2}$$

If we then limit $\Delta x \to 0$ then we get two things.

1. Our Riemann sum becomes an integral (by definition of the integral)
2. Since $\Delta x = x_{j+1} - x_j$ and $x_j < z_j < x_{j+1} = x_j + \Delta x$, as $\Delta x \to 0$ by squeezing we must have $z_j = x_j$.

2 Arc Length

Arc Length: Given a function $f(x)$ that has continuous derivative on $[a, b]$ (i.e. f' is continuous on $[a, b]$). Then the length of the curve $y = f(x)$ on the interval $[a, b]$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly, if we have a function of y, like $g(y) = x$ then its arc length on the interval $[c, d]$ is given by

$$\int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

An important consequence of the arc length formula is the arc length function. It has a lot of usefulness when you want to do calculus on things more extreme than a flat 2D-coordinate system, such as a *surface with curvature.* The **arc length function** is

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

You might commonly see the arc length appear not in the above form but instead in the form of a **differential**

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad \text{or} \qquad ds = \sqrt{(dx)^2 + (dy)^2}$$

We won't talk about them in calc 2, but you might seen these in physics and calc 3.

Example 1. Find the Arc Length of the curve $\underbrace{y^2 = x^3}$ between the points $(1, 1)$ and $(4, 8)$.

This arc length formula requires us to express the curve as a function of one variable.

$$\Rightarrow \quad y^2 = x^3$$

$$y = x^{3/2}$$

$$x = y^{2/3}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\frac{dx}{dy} = \frac{2}{3} y^{-1/3}$$

$$\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx$$

$$\int_1^8 \sqrt{1 + \left(\frac{2}{3}\frac{1}{y^{1/3}}\right)^2}\, dy$$

This integral is harder to do so we'll do the dx

$$\int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left[\left(1 + \frac{9}{4}x\right)^{3/2}\right]_1^4$$

$$= \frac{8}{27}\left[(1+9)^{3/2} - \left(1 + \frac{9}{4}\right)^{3/2}\right]$$

$$S(y) = \int_a^y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dt$$

Example 1.5. 1. Derive an $\underbrace{\text{arclength function}}$ for $x = \ln(\cos(y))$.

$$\frac{dx}{dy} = \frac{1}{\cos(y)} \cdot -\sin(y) = -\tan(y)$$

$$S(y) = \int_a^y \sqrt{1+(-\tan(t))^2}\, dt$$

$$= \int_a^y \sqrt{1+\tan^2(t)}\, dt$$

$$= \int_a^y \sqrt{\sec^2 t}\, dt$$

$$= \int_a^y |\sec t|\, dt = \Big[\ln|\sec t + \tan t|\Big]_a^y$$

for y & a where sec pos.

2. Find the arclength of the curve $\underbrace{x = \ln(\cos(y)) \text{ on } 0 \le y \le \frac{\pi}{3}}$.

Based on above this is just

$$S\left(\tfrac{\pi}{3}\right) - S(0) = \ln\left|\sec\left(\tfrac{\pi}{3}\right) + \tan\left(\tfrac{\pi}{3}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln\left|2 + \frac{1}{\sqrt{3}}\right| - \cancel{\ln|1+0|}^{\,0}$$

$$= \ln\left(2 + \frac{\sqrt{3}}{3}\right)$$

Example 2: Set up a definite integral of the arc length of the parabola $y^2 = x$ from $(0,0)$ and $(1,1)$.

already in a form for "dy" arc length

$2y = \frac{dx}{dy}$

$$\int_0^1 \sqrt{1 + (2y)^2}\, dy$$

$$= \underbrace{\int_0^1 \sqrt{1 + 4y^2}\, dy} \to \text{IBP or Trig-Sub.}$$

Example 3: Set up a definite integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1,1)$ to the point $\left(2, \frac{1}{2}\right)$.

$y = \frac{1}{x}$

$\frac{dy}{dx} = -\frac{1}{x^2} dx$

$$\int_1^2 \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx$$

$$= \int_1^2 \underbrace{\sqrt{1 + \frac{1}{x^4}}}_{\text{trig sub. prob.}}\, dx$$

Example 4: Set up a definite integral for the length of the arc of the function $y = e^x + \frac{1}{4}e^{-x}$ where $0 \leq x \leq 2$.

$$\frac{dy}{dx} = e^x - \frac{1}{4}e^{-x}$$

$$\int_0^2 \sqrt{\underbrace{1 + \left(e^x - \frac{1}{4}e^{-x}\right)^2}}\, dx$$

Simplifying this

$$1+\left(e^{x}-\frac{1}{4}e^{-x}\right)^{2}$$

$$1+(e^{x})^{2}-2\cdot\frac{1}{4}e^{x}e^{-x}+\left(\frac{1}{4}e^{-x}\right)^{2}$$

$$1+(e^{x})^{2}-\frac{1}{2}+\left(\frac{1}{4}e^{-x}\right)^{2}$$

$$(e^{x})^{2}+\frac{1}{2}+\left(\frac{1}{4}e^{-x}\right)^{2}$$

$$(e^{x})^{2}+2\cdot\frac{1}{4}e^{x}e^{-x}+\left(\frac{1}{4}e^{-x}\right)^{2}$$

$$=\left(e^{x}+\frac{1}{4}e^{-x}\right)^{2}$$

Back to the integral

$$\int_0^2\sqrt{1+\left(e^{x}-\frac{1}{4}e^{-x}\right)^{2}}\,dx=\int_0^2\sqrt{\left(e^{x}-\frac{1}{4}e^{-x}\right)^{2}}\,dx$$

$$=\int_0^2 e^{x}-\frac{1}{4}e^{-x}\,dx$$

$$=\left[e^{x}+\frac{1}{4}e^{-x}\right]_0^2$$

$$=e^{2}+\frac{1}{4}\frac{1}{e^{2}}-\left(1+\frac{1}{4}\right)=e^{2}+\frac{1}{4}\cdot\frac{1}{e^{2}}-\frac{5}{4}$$