DIS 2 : 7.4, 7.8, 6.1

Problem 1. Determine the convergence of the following improper integrals. No need to compute the value should they exist.

(a) $\displaystyle\int_0^2 \frac{1}{x-1}\,dx$

(b) $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2}\,dx$

(d) $\displaystyle\int_e^\infty \frac{1}{\ln x}\,dx$

Problem 2.

(a) $\displaystyle\int \frac{d}{(x+a)(x+b)}\,dx$

(b) $\displaystyle\int \frac{-3x^2+10x-36}{(x-3)(x^2+2)}\,dt$

Problem 3.

(a) Find c such that the area above $y = x^2$ and below $y = c$ is equal to the area above $y = c$ and below $y = x^2$ on the interval $[0, 1]$

(b) Let D be the region enclosed by the curves $y = 4 - x^2$, $y = 2x + 1$, and $x = 0$. Find its area.

Problem 4. This problem will detail an application of some of the material we have covered, and is for the more interested student.

The **Laplace Transform** is a function that takes *functions as its input* and spits out *functions as its output* and is used in solving differential equations (and therefore is used for "real world problems"). The nice thing about the Laplace transform is sometimes you will have a system in a "time domain" that is hard to work with, but if we transform it to a system in a "Laplace domain" (or "Frequency domain") it might be easier to work with.

Traditionally, the input function will depend on t (often for time) and the output function function will depend on s (for some kind of frequency). Let $\mathcal{L}$ denote this transform and let F be the result of applying $\mathcal{L}$ to a function f. Formally, it *can* be defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

Note that in the above integral s is treated like a constant since the integral only sees t as a variable. Now, in the full theory, s is allowed to be a complex number, but for us, we will just only think of it as real.

Let's first compute the Laplace transform of 1. That is, compute

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$$

You should get $\frac{1}{s}$. What's interesting we took a constant function (the function that is only ever 1) and found that its Laplace Transform is not only nonconstant, but also singular at a point ($s = 0$). Let's now try to compute the Laplace transform of e^t. That is, compute

$$\mathcal{L}\{e^t\} = \int_0^\infty e^s e^{-st}\,dt$$

You should get $\dfrac{1}{s-1}$.

The above few statements are neat, but where does its power come from you may be asking. Consider the Laplace transform of the 1st derivative that is:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt$$

Use IBP to get an integral inolving $f(t)$. If $F(s)$ is the Laplace transform of $f(t)$ then what you'll find is

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

So now consider the differential equation $f'(t) = f(t)$ with the initial condition $f(0) = 1$. Apply the Laplace transform to this system and solve for $F(s)$.

When you do so, you should get

$$F(s) = \frac{1}{s-1}$$

Does this look like anything above? It should look like one of the above Laplace Transforms you computed. If we now try to get back the original function $f(t)$ we can apply the "Inverse Laplace transform" to both sides

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

What function does this become (based on what we've done in this problem)?

At this point we have solved the initial value differential equation problem or IVP for short. But here's something amazing, once you computed what the Laplace transform of something was, to actually solve the Calculus problem of solving a differential equation all we did was some algebra (solve for $F(s)$).

This is only one *very narrow* application of the Laplace Transform, but it comes up as a decent technique for solving differential equations. Furthermore, since all we did after doing the transform was algebra, we can program this into a computer to solve for us, and now we're *really cooking.*