Section 7.3 Trigonometric Substitution

Background

Last class, we reviewed some important trigonometric identities and used them to solve tricky trigonometric integrals. Today we use those identities to solve integrals involving **square roots and polynomials** via **trigonometric substitution**.

The Set-Up: When Trig Sub Helps

Trig substitution is designed for integrals containing expressions like:

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}, \qquad \text{where } a \text{ is a constant}$$

The goal is to choose a substitution that turns the square root into a simple trig expression using a Pythagorean identity.

Core Identities (Pythagorean)

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

Trig Substitution Table

Expression	**Substitution**	**Identity Used**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Standard Workflow (Checklist)

For each trig substitution integral:

Step 1: **Identify the root form** ($\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$) and pick the matching substitution.

Step 2: **Differentiate** the substitution to rewrite dx in terms of $d\theta$.

Step 3: **Rewrite the entire integrand** in terms of θ (including the square root).

Step 4: **Simplify** using trig identities; integrate with trig techniques.

Step 5: **Draw a reference triangle** to convert back to x (or use inverse trig carefully).

Step 6: **Back-substitute** and include $+C$.

Triangle Back-Sub Templates

Use these to return from θ to x.

If $x = a\sin\theta$

Then $\sin\theta = \dfrac{x}{a}$, so draw a right triangle with:

$$\text{opposite} = x, \quad \text{hypotenuse} = a, \quad \text{adjacent} = \sqrt{a^2 - x^2}.$$

Common outputs: $\cos\theta = \dfrac{\sqrt{a^2 - x^2}}{a}$.

If $x = a\tan\theta$

Then $\tan\theta = \dfrac{x}{a}$, so draw a right triangle with:

$$\text{opposite} = x, \quad \text{adjacent} = a, \quad \text{hypotenuse} = \sqrt{a^2 + x^2}.$$

Common outputs: $\sec\theta = \dfrac{\sqrt{a^2 + x^2}}{a}$.

If $x = a\sec\theta$

Then $\sec\theta = \dfrac{x}{a}$, so $\cos\theta = \dfrac{a}{x}$ and draw a right triangle with:

$$\text{adjacent} = a, \quad \text{hypotenuse} = x, \quad \text{opposite} = \sqrt{x^2 - a^2}.$$

Common outputs: $\tan\theta = \dfrac{\sqrt{x^2 - a^2}}{a}$.

(1) $\int \sqrt{1 - x^2}\, dx$

(2) $\int \sqrt{1+x^2}\,dx$

(3) $\int \frac{1}{x^2\sqrt{x^2+2}}\,dx$

(4) $\displaystyle\int \frac{1}{(4x^2+9)^2}\,dx$

(5) $\int x^3\sqrt{1+4x^2}\,dx$