7.3: Trig Sub

The goal of this lesson is to develop another technique of integration to handle integrals that we currently cannot deal with such as $\int \sqrt{1-x^2}\,dx$ (though this may be doable with a lot of work using integration by parts). This technique will be reliant on the geometry of triangles.

Consider a circle with radius 1 centered at the origin of an (x, y)-coordinate system. The collection of (x, y) points that line on this circle satisfy the equation $x^2+y^2 = 1$. If we consider the angle θ made by a line from the origin to the circle's edge, then we define

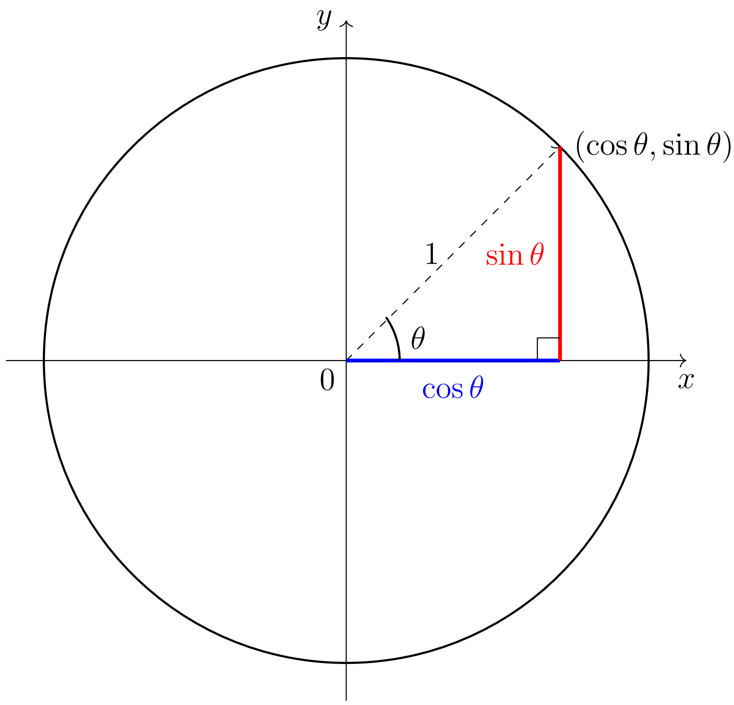
- $\sin(\theta)$ = the y-value of the unit circle at the angle θ
- $\cos(\theta)$ = the x-value of the unit circle at the angle θ
- $\tan(\theta)$ = the slope of the line at the angle θ

By slope formula we get that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

By definition, we have that for any θ,

$$-1 \leq \sin\theta \leq 1 \qquad \text{and} \qquad -1 \leq \cos\theta \leq 1$$



Notationally, $\text{trig}^{\#}(\theta)$ is short hand for $(\text{trig}(\theta))^{\#}$ for any $\# \neq -1$.

Consider a circle of radius r. Then the (x, y) coordinates of this circle satisfy the equation $x^2 + y^2 = r^2$. If we divide the equation by r^2 we get the equation

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

(See picture below // Geometrically we scaled the circle down or up to the unit circle). Then by definition, the $(\frac{x}{r}, \frac{y}{r})$ coordinates of this circle satisfy the expressions

$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \cos\theta = \frac{x}{r}$$

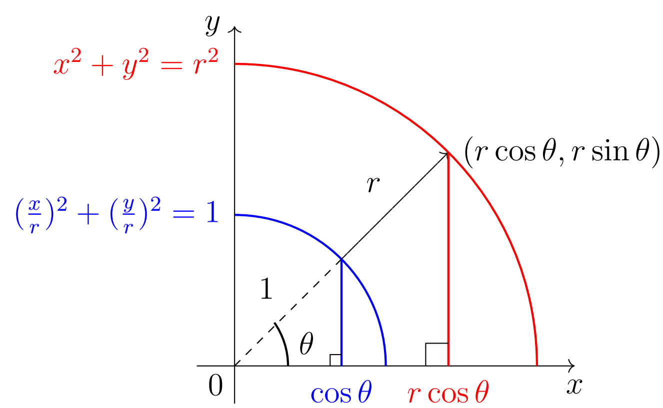
However, if we interpret this information regarding the triangles below

- $\sin\theta$ is equal to the length of the opposite side divided by the radius of the circle (i.e. the hypotenuse of the triangle).
- $\cos\theta$ is equal to the length of the adjacent side divided by the radius of the circle (i.e. the hypotenuse of the triangle).

This gives us

$$\sin\theta = \frac{opp}{hyp} \qquad \text{and} \qquad \cos\theta = \frac{adj}{hyp}$$

From the definition of Tangent we get $\tan\theta = \frac{opp}{adj}$.



Once we have the necessary tools. Consider the integral $\int \sqrt{1-x^2}\,dx$. This can be related to a right-triangle with hypotenuse 1 and one of the legs is has length x. We can then form the substitution $\sin\theta = x$ from soh-cah-toa, and then figure out what our "dx" must be by differentiating $\sin\theta = x$ to get $\cos\theta d\theta = dx$ and so

$$\int \sqrt{1 - x^2}\,dx = \int \sqrt{1 - \sin^2\theta}\,\cos\theta d\theta$$

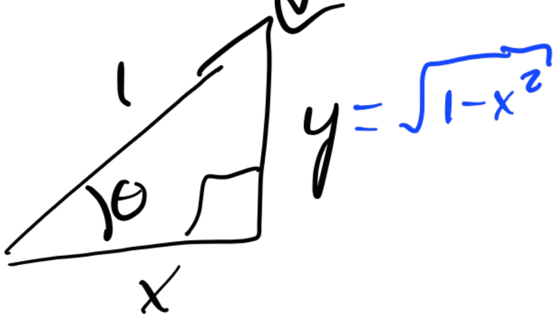
From here the problem reduces to material from 7.2.

Example 1. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

$$a^2 + b^2 = c^2$$

$$1^2 - x^2 = y^2$$

$$1 = x^2 + y^2$$



$\text{trig}(\theta) = \frac{x}{H}$

$\cos(\theta) = x$

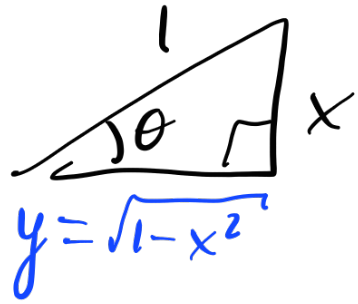
$-\sin\theta\, d\theta = dx$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}}$$

$$= \int \frac{-\sin\theta\, d\theta}{\sin\theta}$$

$$= -\int d\theta = -\theta + C$$

$$= -\arccos x + C$$



$\sin\theta = x$

$\cos\theta\, d\theta = dx$

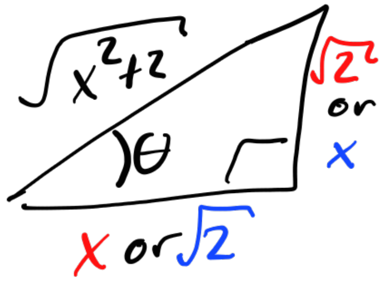
$$\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta = \int d\theta$$

$$= \theta + C$$

$$= \arcsin x + C$$

$a^2+b^2=c^2$

Example 2. $\int \frac{1}{x^2\sqrt{x^2+2}}\,dx$



$x^2+(\sqrt{2})^2=y^2$

The red path says $\quad trig(\theta) = \frac{x}{\#} = \frac{adj}{opp} = \cot(\theta)$ (ew)

The blue path says $\quad trig(\theta) = \frac{x}{\#} = \frac{opp.}{adj} = \tan\theta$

$\Rightarrow \quad \tan\theta = \frac{x}{\sqrt{2}} \Rightarrow \quad x = \sqrt{2}\tan\theta$

$dx = \boxed{\sqrt{2}\sec^2\theta\, d\theta}$

$$\int \frac{dx}{x^2\sqrt{x^2+2}} = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{(\sqrt{2}\tan\theta)^2\sqrt{(\sqrt{2}\tan\theta)^2+2}}$$

$$= \sqrt{2}\int \frac{\sec^2\theta \quad d\theta}{2\tan^2\theta\sqrt{2\tan^2\theta+2}}$$

$\tan^2\theta + 1 = \sec^2\theta$

$\sqrt{2\tan^2\theta+2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta$

$$= \frac{\sqrt{2}}{2}\int \frac{\sec^2\theta}{\tan^2\theta\,\sqrt{2}\sec\theta}\,d\theta = \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \frac{1}{2}\int \frac{\cos^2\theta}{\cos\theta\sin^2\theta}\,d\theta = \frac{1}{2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$\hookrightarrow \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2}u^{-1} + C$

$= -\frac{1}{2}\frac{1}{\sin\theta} + C = -\frac{1}{2}$

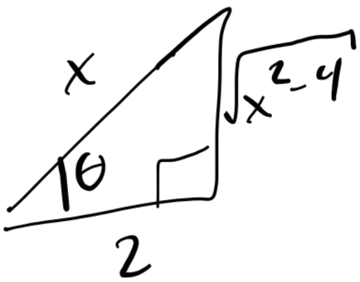
Example 3. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$

$x = 2\sec\theta \to \frac{x}{2} = \sec\theta \to \frac{2}{x} = \cos\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

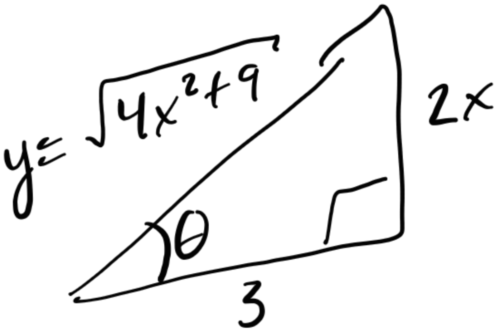


Example 4. $\displaystyle\int \frac{1}{(4x^2+9)^2}\,dx$

$$a^2 + b^2 = c^2$$

$$4x^2 + 9 = y^2$$

$$(2x)^2 + 3^2 = y^2$$



$$\tan\theta = \frac{2x}{3} \Rightarrow \frac{3}{2}\tan\theta = x$$

$$\frac{3}{2}\sec^2\theta\, d\theta = dx$$

$$\int \frac{1}{(4x^2+9)^2}\,dx = \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{\left(\cancel{4}\cdot\frac{9}{\cancel{4}}\tan^2\theta + 9\right)^2} = \frac{3}{2}\int \frac{\sec^2\theta}{(9\sec^2\theta)^2}\,d\theta$$

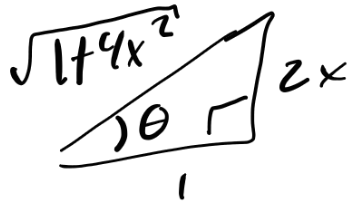
$$= \frac{3}{2}\int \frac{\sec^2\theta}{81\sec^4\theta}\,d\theta = \frac{3}{2\cdot 81}^{\frac{1}{27}}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{54}\int \cos^2\theta\, d\theta = \frac{1}{54}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{108}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

Example 5. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ where $a > 0$ is constant.

Example 6. $\int \frac{x^3}{\sqrt{1+4x^2}}\,dx$



$\frac{1}{2}\tan\theta = x$

$\frac{1}{2}\sec^2\theta\, d\theta = dx$

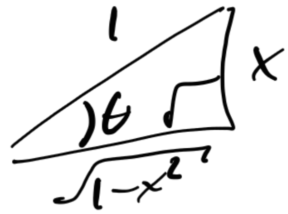
$$\int \frac{\frac{1}{8}\tan^3\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta}{\sec\theta} = \frac{1}{16}\int \tan^3\theta \sec\theta\, d\theta$$

$$= \frac{1}{16}\int \tan\theta\,(\sec^2\theta - 1)\sec\theta\, d\theta = \frac{1}{16}\int u^2 - 1\, du$$

$$= \frac{1}{16}\left[\frac{1}{3}u^3 - u\right] + C = \frac{1}{16}\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C$$

$$= \frac{1}{16}\left[\frac{1}{3}\left(\frac{1}{\sqrt{1+4x^2}}\right)^3 - \frac{1}{\sqrt{1+4x^2}}\right] + C$$

Example 7. $\int \sqrt{1-x^2}\, dx$



$\sin\theta = x$

$\cos\theta\, d\theta = dx$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int 1 + \cos 2\theta\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{2}\left[\arcsin x + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arcsin x + x\cdot\sqrt{1-x^2}\right] + C$$

Example 8. $\displaystyle\int \sqrt{1+x^2}\,dx$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$