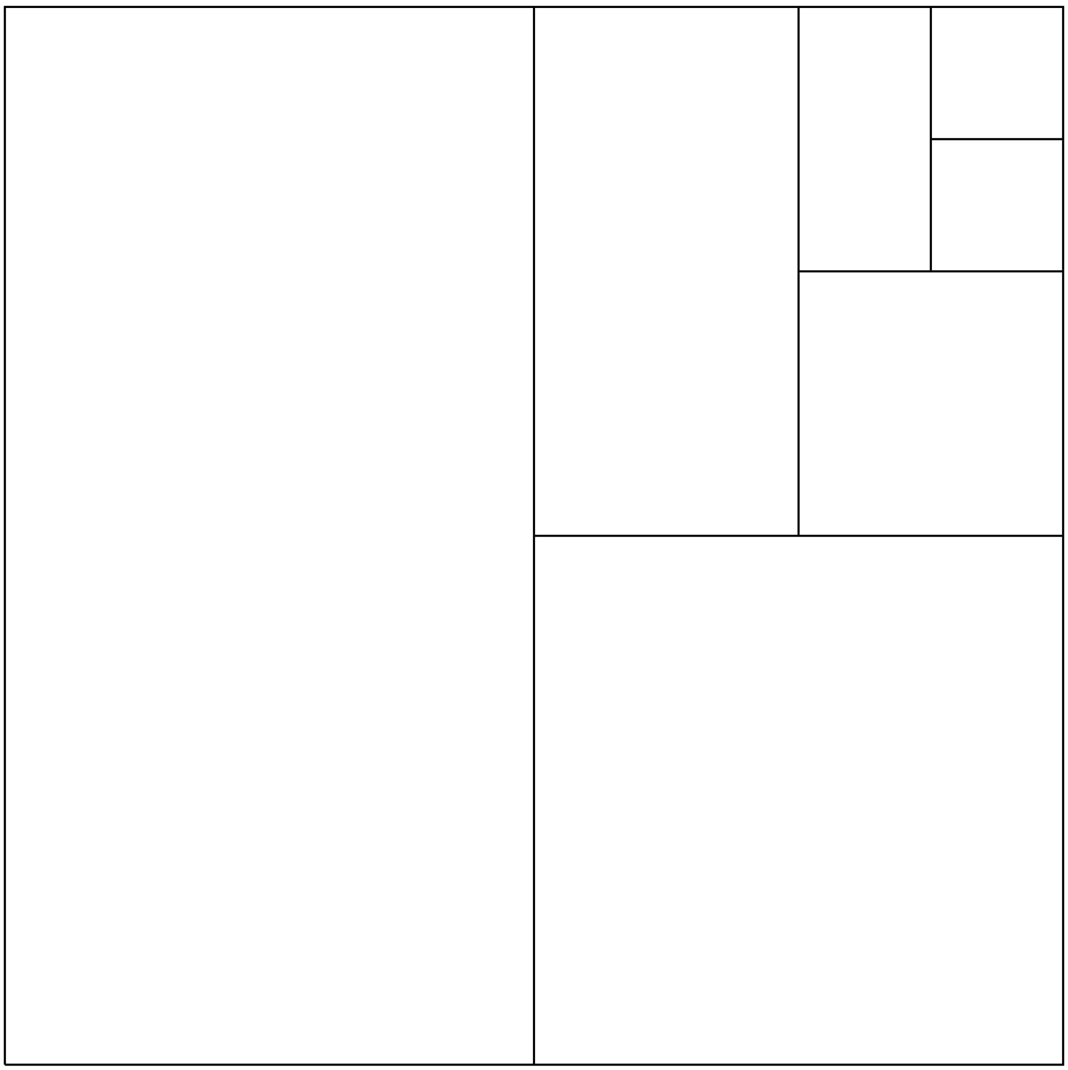
11.2: Series

Consider the following: We have a square with length 1. By the area equation for a square, this means the square has area 1. However, what happens if we cut the square in half? We get two things that each have an area of $\frac{1}{2}$. Maybe it's not surprising we get $\frac{1}{2} + \frac{1}{2} = 1$, but happens if we cut one of these halves in half? $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$. What happens if we do this again, and again, and again, and ...



Wgeb we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an **infinite series** (or simply a **series**) denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

We can also omit the decorations on a series if we are not concerned with the value of the series or if we are talking about it in the *abstract*, $\sum a_n$.

However, to make sense of a series we need to base it in some math. Denote s_n to be the **nth partial sum**,

$$s_n \sum_{j=1}^{n} a_j = a_1 + a_2 + ... + a_n$$

Note that for each n, the n-th partial sum s_n is just a number, and so $\{s_n\}_{n=1}^{\infty}$ forms a *sequence of real numbers.* So, a series $\sum a_n$ is called **convergent** if the sequence of partial sums, s_n, is convergent. That is,

- The series $\sum a_n$ is **convergent** if its sequence of partial sums is convergent. In this case we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

- If the sequence of partial sums is divergent, then we say the series $\sum a_n$ is **divergent.**

The value of the limit $\lim s_n$ is called the **sum** of the series (i.e. $\sum a_n$ is the sum). Much like improper integrals, the symbol $\sum_{n=1}^{\infty} a_n$ only has meaning if the associated limits exist.

Example 1. Consider a series $\sum a_n$ whose partial sums are given by $s_n = \dfrac{2n}{3n+5}$. Determine if the series converges, and find its sum if it does.

Example 2: Evaluate the sum of the series $\sum_{n=1}^{\infty} a_n$, where the partial sum $s_n = 2\left(\frac{7}{11}\right)^n$ is given. Justify your answer.

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (We have already considered the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$ on page 708.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3
$$\boxed{s_n = \frac{a(1 - r^n)}{1 - r}}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n\to\infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leqslant -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n\to\infty} s_n$ does not exist. Therefore the geometric series diverges in those cases. ■

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Example 3: Determine whether the series is convergent or divergent. If the series is convergent, find its sum. If not, write DIVERGENT and explain a valid argument for full credit.

$$\sum_{n=1}^{\infty} \frac{5^{n-1}}{(-6)^n}$$

Example 4: Is the series $\sum_{n=1}^{\infty} 2^{n-1}3^{-n}$ convergent or divergent? If the series is convergent, find its sum. If not, write DIVERGENT and explain a valid argument for full credit.

The General Geometric Series:

Consider the sequence $a_n = ar^{n+k}$. Then by the geometric series test if $|r| < 1$ we have for any nonnegative integer ℓ,

$$\sum_{n=\ell}^{\infty} a_n = \frac{a_\ell}{1-r}$$

That is, **you can find what** goes on top of the fraction above **by *plugging in the starting index* of your series into the sequence**

Theorem:

Suppose the series $\sum a_n$ is convergent. Then we must have $\lim_{n\to\infty} a_n = 0$.

Divergence Test:

If $\lim_{n\to\infty} a_n \neq 0$ or is DNE then the series $\sum a_n$ is divergent.

The **opposite is NOT true.** Consider $\sum \frac{1}{n}$

Example 5. Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is convergent or divergent.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Definition A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

Writing out the first several terms in the sequence of partial sums $\{s_n\}$, we see that

$$\begin{aligned}
s_1 &= a_1 = b_1 - b_2 \\
s_2 &= a_1 + a_2 = (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3 \\
s_3 &= a_1 + a_2 + a_3 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4 \\
&\ \ \vdots
\end{aligned}$$

Example 6. Express the following series as telescoping series and determine their convergence. If they are convergent, find their sum.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Example 7. Determine of the following series converge or diverge. Provide sufficient reasoning, and compute the sums of all convergent series.

1. $\displaystyle\sum_{n=3}^{\infty} e^n \pi^{-n+1}$

2. $\displaystyle\sum_{n=4}^{\infty} \ln\left(\frac{n+2}{n+3}\right)$

3. $\displaystyle\sum_{n=7}^{\infty} \frac{1}{n^2} - \frac{1}{n^2+2n+1}$

4. $\displaystyle\sum_{n=5}^{\infty} a_n$ where the partial sums are given by $s_n = 3$