7.1: Integration by Parts (IBP)

The chain rule let us ask "*is this integral a chain rule*?" and now we will learn Integration by Parts which will let us ask "*is this integral a* ***product rule***?" To see how it comes up, let's look at the product rule and try to integrate it:

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

$$f(x)g(x) = \int \frac{d}{dx}\Big[f(x)g(x)\Big]dx = \int \Big[f'(x)g(x) + f(x)g'(x)\Big]\,dx$$

$$f(x)g(x) = dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

If we move one of the integrals to the other side we get integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

If we relabel $f(x)$ as u and $g(x)$ as v (that is, set $u = f(x)$ and $v = g(x)$), then by the chain rule, $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$ (or more familiarly: "$du = f'(x)\,dx$" and "$dv = g'(x)\,dx$") and we get Integration-By-Parts (sometimes shorten to IBP):

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

aka

$$\int udv = uv - \int vdu$$

or: "ultra*violet - SUPER (voo*du)"

When integrating by parts you will want to choose dv so that you can actually integrate the function. This means, your priority will be setting dv first in many cases. A there are many ways to try and remember the order, my suggestion is:

LIATE which stands for "**L**et's **I**ntegrate **A** **T**errible **E**quation" it's short hand for

- **L**: Logarithms (i.e. $\ln x$, $\log_b(x)$, etc.)
- **I**: Inverse Trig (i.e. $\arctan x$, $\arcsin x$, etc.)
- **A**: Algebraic (i.e. x, x^a, dx, etc.)
- **T**: Trig (i.e. $\sin x$, $\cos x$, etc.)
- **E**: Exponentials (i.e. e^x, 2^x, a^x, etc.)

Things higher on the list are harder to integrate so you most likely set that as u and set dv to be whatever appear below it. Other acronyms that you might've seen or will see are: LIPET, ILATE, ILPTE, etc. When it comes to inverse trig and logs, you can

Example 1. Find $\int x \sin x \, dx = uv - \int v \, du$

L
I
A $\to u = x$
T $\to dv = \sin x \, dx$
E

derivative

$u = x$

$du = dx$

$dv = \sin x \, dx$

$v = -\cos x$

integrate

$= x(-\cos x) - \int (-\cos x) \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

Example 2. Find $\int \ln x \, dx = uv - \int v \, du$

L $\to u = \ln x$
I
A $\to dv = 1 \cdot dx$
T
E

$\int$ $dv = dx$

$v = x$

$\frac{d}{dx}$ $u = \ln x$

$du = \frac{1}{x} dx$

$= x \ln x - \int x \frac{1}{x} \, dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

Example 3. Find $\int e^x \sin x \, dx$ $= e^x \sin x - \boxed{\int e^x \cos x \, dx}$

L
I
A
T $\to u = \sin x \to du = \cos x \, dx$
E $\to dv = e^x dx \to v = e^x$

$\int e^x \cos x \, dx$

$u = \cos x$
$du = -\sin x \, dx$
$dv = e^x dx$
$v = e^x$

$= e^x \cos x - \int -\sin x \, e^x dx$

$$\int e^x \sin x \, dx = e^x \sin x - \left[e^x \cos x + \int \sin x \, e^x dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$+ \int e^x \sin x \, dx$ $\qquad + \int e^x \sin x \, dx$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{e^x}{2} \left[\sin x - \cos x \right] + C$$

Example 4. Find $\int_0^1 t^2 e^t \, dt$

$$\int t^2 e^t \, dt = t^2 e^t - 2\int t e^t \, dt = t^2 e^t - 2te^t + 2\int e^t \, dt$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

$$\int_0^1 t^2 e^t \, dt = \left[t^2 e^t - 2te^t + 2e^t \right]_0^1$$

$$= \left[(1)^2 e^1 - 2(1) e^1 + 2e^1 \right]$$

$$- \left[0 \cdot 1e - 2 \cdot 0 \cdot 1e + 2e^0 \right]$$

$$= \left[e - 2e + e \right] - \left[2 \right] = -2$$