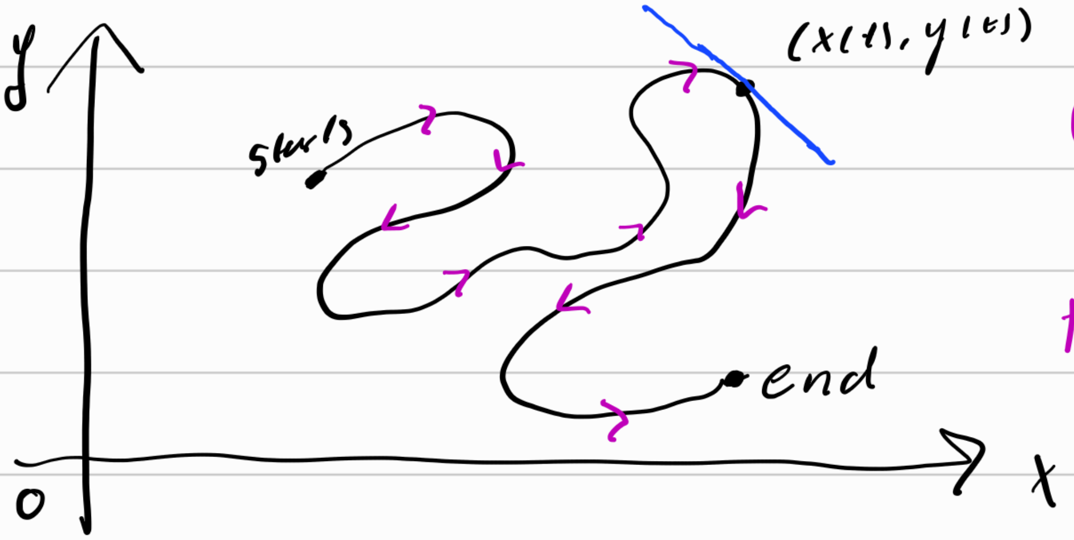
§ 10.2 Calculus w/ Parametric eq.

Consider parametric curve



We can still talk about the slope of tangent lines

To actually talk about the slope of the tangent lines we need vector calc. (i.e. Calc. 3)

But we at least motivate the slope of the tangent using "some special" single variable calc.

For our diravation we assume y is a function of x (ie. $y(x)$)

If x itself is given parametrically by t we'll derivative with respect to t.

$$\frac{d}{dt}\left[y(x(t))\right] = y'(x(t)) \cdot x'(t)$$ newton's notation

rewritting this in Leibniz' notation

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}}$$

useful form of parametric Derivative

this

So the (x, y) derivative (or slope) is given by

$\frac{dy/dt}{dx/dt}$ is useful for more general arguements

$\frac{y'(t)}{x'(t)}$ is more useful for calculation.

The "parametric derivative" is still rise/run b/c

$\frac{y'}{x'}$ → rise / → run

So another view the derivative we just found is:

$$\frac{d}{dx}\left[\ddot\smile \right] = \frac{\frac{d}{dt}\left[\ddot\smile \right]}{dx/dt}$$

To talk the 2nd derivative we'll exploit this

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

$$\rightsquigarrow \frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{y'(t)}{x'(t)}\right]$$

$$= \frac{x'(t)\cdot y''(t) - y'(t)\cdot x''(t)}{[x'(t)]^2}$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)\cdot y''(t) - y'(t)\cdot x''(t)}{[x'(t)]^3}$$

I DON'T Reccomend Committing this

to MEMORY I think it's better to know

$$\frac{d}{dx}\left[\because\right] = \frac{\frac{d}{dt}[\because]}{x'(t)}$$

Going Back to tangent lines:

- If $y'(t)=0$ & $x'(t)\neq 0$, then we have a horiz. tangent
- If $x'(t)=0$ & $y'(t)\neq 0$, then we have a vert. tangent
- If Both $x'(t)$ & $y'(t)$ are zero then we have to do some kind of limit arguement to classify the tangent.

$\boxed{\text{example}}$ The cycloid given parametrically by

$x(\theta)=r(\theta-\sin(\theta))$ $\quad$ $y(\theta)=r(1-\cos(\theta))$

$$\frac{dy}{dx} = \frac{\text{rise}}{\text{run}} = \frac{y'(\theta)}{x'(\theta)} = \frac{r(0+\sin\theta)}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

The potential horiz. tangents are when $\sin\theta = 0$

$\hookrightarrow$ @ $\theta = 0, \pi, 2\pi, 3\pi, \ldots$
$-\pi, -2\pi, \ldots$ $\Big\} \Rightarrow \pi k$ ← integer k.

The potential vert. tangents $\Rightarrow 1-\cos\theta = 0$

$\hookrightarrow$ @ $\theta = 0, 2\pi, 4\pi, \ldots$
$-2\pi, -4\pi, \ldots$ $\Big\}$ $2\pi k$ ← integer k

So $\pi, 3\pi, 5\pi, -\pi, -3\pi, \ldots$ etc. are all horiz. tangents because there is no overlap w/ the potential vertical tangents

When we overlap:

We will only classify $\theta = 0$ fully.

So note "$\frac{\sin(0)}{1-\cos(0)} = \frac{0}{0}$"

⇒ we look

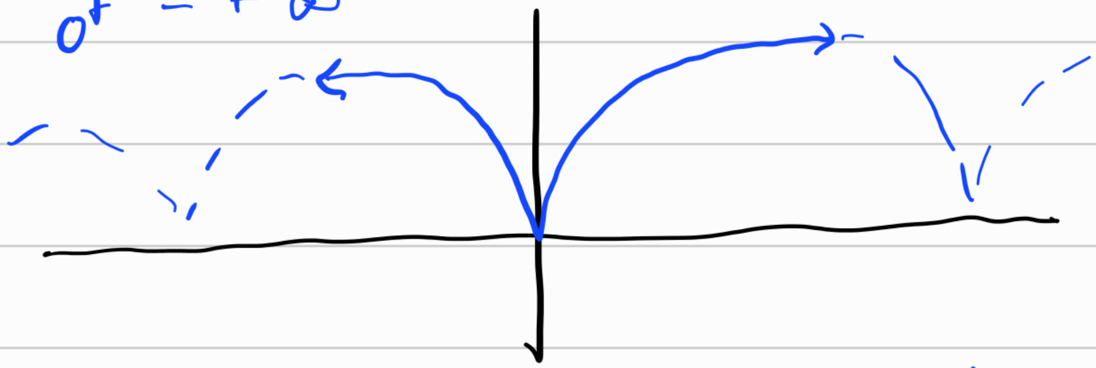
$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \text{"}\frac{0}{0}\text{"}$$

L'Hôpital's rule

$$= \lim_{\theta \to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = \text{"}\frac{1}{0^+}\text{"}$$

A this point we'll have a vert. tangent

"$\frac{1}{0^+} = +\infty$"



↳ Another to see this is we effectively are in the "$x' = 0$ & $y' \neq 0$" case

10.2: Calculus with Parametric Curves

1 Derivatives

Now that we are talking about curves that are described parametrically, that is, $(x(t), y(t))$ is some collection of points in a (x, y)-grid that we can trace out "forwards and backwards in time," it's time to do some calculus with these objects. A lot of it comes as a consequence of the chain rule:

<u>Derivatives of Parametric Curves:</u>
Consider the parametric curve $(x(t), y(t))$. If y is also a differentiable funciton of x (that is, we can write $y(x)$ and take a derivative), then by the chain rule:

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ y'(t) &= y'(x(t))x'(t) \end{aligned}$$

and if $x'(t)$ is nonzero then we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

Geometrically (using calc 3 concepts), the point on the curve $(x(t), y(t))$ a tangential velocity vector given by $(x'(t), y'(t))$. This vector lies in the line tangent to the curve at a given point. i.e. the slope of that vector (a rise over run of sorts) gives the slope of the curve at that point.

Consequently we also see that that if we replace y with $\frac{dy}{dx}$ we get

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{\frac{d}{dt}\left(\frac{dx}{dy}\right)}{\frac{dx}{dt}} \\ &= \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3} \end{aligned}$$

Example 1. Consider the curve give by $x = t^2$ and $y = t^3 - 3t$. We will

(a) Determine where the tangents are vertical and horizontal

(b) Determine where the curve is concave up or down

(c) Show that a parametric curve can have multiple tangent lines at a point (but not for a t value), specifically at $(x, y) = (3, 0)$

(cont. from prev page) $x = t^2$ and $y = t^3 - 3t$

$x(t) = t^2$ & $y(t) = t^3 - 3t$

Recall horiz tangents are $y' = 0$ & $x' \neq 0$
vert. tangents are $x' = 0$ & $y' \neq 0$

$x'(t) = 2t$

$0 = 2t$

$t = 0$

Potential vert. tangent

$y'(t) = 3t^2 - 3$

$0 = 3t^2 - 3$

$t = \pm 1$

Potential horiz. tangent

No OverLap!

So, @ $t = 0$ vert. tangent

@ $t = 1$ & $t = -1$ horiz. tangents.

$\boxed{\text{Any kind of overlap means a limit argument to classify them}}$

Recall $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-3}{2t} = \frac{3}{2}t - \frac{3}{2}\frac{1}{t}$$

$$\frac{d}{dt}\left[\quad\right] = \frac{3}{2} + \frac{3}{2}\frac{1}{t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}+\frac{3}{2}\frac{1}{t^2}}{2t} = \frac{3}{4t} + \frac{3}{4t^3} = \frac{3t^2+3}{4t^3}$$

To determine concavity $\quad \dfrac{d^2y}{dx^2} \geq 0 \quad$ C.U.

$\dfrac{d^2y}{dx^2} < 0 \quad$ C.D.

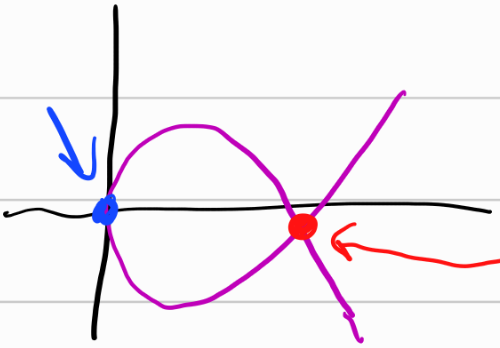
→ Set $\dfrac{d^2y}{dx^2} = 0$ & set up a sign chart

$\dfrac{3t^2+3}{4t^3} = 0 \rightarrow$ no real solutions

So by the below sign chart

C.D. on $(-\infty, 0)$ & C.U. on $(0, \infty)$

t	-1	0	1
$3t^2+3$	$+$	$+$	$+$
$4t^3$	$-$	0	$+$
d^2y/dx^2	$-$	DNE	$+$
Curve	C.D.	pot. inflection pt.	C.U.



2 different tangents depending on what "time" t we arrive @ the point

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$x = t^2, \quad y = t^3 - 3t$

Fun Bonus: Consider $t = \pm\sqrt{3}$

Note $(x(\sqrt{3}), y(\sqrt{3})) = (3, 0)$

& $(x(-\sqrt{3}), y(-\sqrt{3})) = (3, 0)$

$\boxed{\text{Same point @ different times}}$

But tangent @ $t=\sqrt{3}$: $\dfrac{dy}{dx} = \dfrac{3\cdot 3 - 3}{2\cdot\sqrt{3}} = \dfrac{6}{2\sqrt{3}} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

@ $t = -\sqrt{3}$: $\dfrac{dy}{dx} = \dfrac{3\cdot 3 - 3}{-2\sqrt{3}} = \dfrac{6}{-2\sqrt{3}} = -\sqrt{3}$

2 Area lmao no

Don't write any of this down

Area with parametric curves is an incredibly large and complicated topic. Most if not all introductory textbooks don't cover this material well, and we *certainly* don't have the time to also cover this material well. For now, we will only consider the simplest case and note this cane still produce *weird, but* ***valid*** results.

Area with Parametric Curves (Simplified):

Consider the parametric curve $(x(t), y(t))$ and the integral $\int_a^b y\, dx$ (i.e. **using the idea that an integral collects together the width*height of a bunch of rectangles of width** dx **and height** y), then using the substitution $y = y(t)$ and $x = x(t)$ then

$$\int_a^b y\, dx = \int_\alpha^\beta y(t)x'(t)\, dt \qquad \left(\text{or} \quad \int_\beta^\alpha y(t)x'(t)\, dt\right)$$

where $x(\alpha)$ is either a or b and $x(\beta)$ is the remaining one. Similarly, by a similar idea:

$$\int_c^d x\, dy = \int_\gamma^\delta x(t)y'(t)\, dt \qquad \left(\text{or} \quad \int_\delta^\gamma x(t)y'(t)\, dt\right)$$

But why is this simplified? Well, the textbook only covers this case, but it is possible to have negative areas within this framework and it is still valid. The details are quite messy and not suitable for calc 2.

Example 1. Finding the area under the first arch of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$.

Skip

Arc length (Ver 2) with parametric curves.

↳ Is most naturally described w/ parametric equations.

Recall 8.1

the way we approx. length of curves is w/ these triangles

We're gonna take Approx. $\approx \sqrt{\Delta x^2 + \Delta y^2}$

when x & y are given parametrically a a change in x or y is a change coming from the parametric variable

Arc length for a Parametric System:

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

equiv.

$$\int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

3 Arc Length

Arc Length of Parametric Curves:

Consider the parametric curve $(x(t), y(t))$ and recall that (one version of) arc length is given by the integral $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. If we follow the same substitution idea from before we get

$$\text{Arc Length} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Note that this means that **we don't need to write y in terms of x or vise-versa. This version of Arc Length** (and the version used in calc 3 and beyond) **can be used for any parametric description of a curve**. In fact, we could've derived this version of Arc Length back in section 8.1 if we knew about parametric equations, and then we could have ignored the mean value theorem step.

Writing $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ we get that arc length is by definition $\int ds$ (if you go back in the notes, this ds is present for the 8.1 material).

Define the **arc length function** for a parametrized curve as

$$s(t) := \int_\alpha^t ds = \int_\alpha^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note this means $\boxed{s'(t) = \sqrt{x'(t)^2 + y'(t)^2}}$

Example 1. Finding the length of the $\underbrace{\text{first arch}}_{[0,2\pi]}$ of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$.

$x(\theta) = r(\theta - \sin\theta)$ $y(\theta) = r(1 - \cos\theta)$

$x'(\theta) = r(1 - \cos\theta)$ $y'(\theta) = r\sin\theta$

$$\int_0^{2\pi} \sqrt{[r(1-\cos\theta)]^2 + [\ r\sin\theta\ ]^2}\, d\theta$$

$$\int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\; d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2\left(1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}\right)}\; d\theta$$

$$= \int_0^{2\pi} r \sqrt{1 - 2\cos\theta + 1}\; d\theta$$

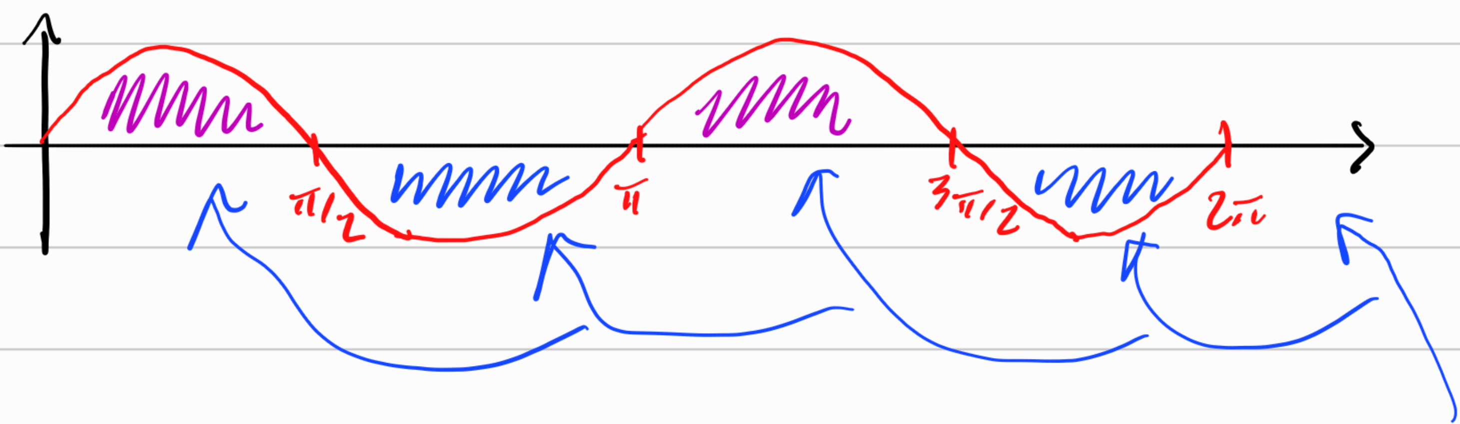
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= r \int_0^{2\pi} \sqrt{\underbrace{2 - 2\cos\theta}_{\rightarrow \text{Half-Angle Identity}}}\; d\theta$$

$$= r \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\; d\theta$$

$$= r \int_0^{2\pi} 2 \cdot \left|\sin\left(\frac{\theta}{2}\right)\right|\; d\theta$$

Because of the absolute values we have break up the integral over the regions where $\sin\left(\frac{\theta}{2}\right)$ is pos. & neg.



$$\int_0^{2\pi} |\sin(\tfrac{\theta}{2})| \, d\theta =$$

$$\int_0^{\pi/2} \sim \; + \int_{\pi/2}^{\pi} \sim \; + \int_{\pi}^{3\pi/2} \sim \; + \int_{3\pi/2}^{2\pi} \sim$$

However, using symmetry of sine each Area in this picture are equal

$$2r \cdot 4 \int_0^{\pi/2} \sin\left(\frac{\theta}{2}\right) d\theta$$

4 Hard Problems

Problem 1. **(Hard)** Find the length of the curve given by $x = t \sin t$ and $y = t \cancel{\sin}\ \cos t$ on $0 \le t \le \frac{\pi}{2}$.

$$x'(t) = \sin t + t\cos t$$

$$y'(t) = \cos t - t\sin t$$

$$\Rightarrow \int_0^{\pi/2} \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\, dt$$

$$\sin^2 + 2t\sin t\cos t + t^2\cos^2 t$$

$$+ \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t$$

$$1 + 0 + t^2$$

$$= \int_0^{\pi/2} \sqrt{1+t^2}\, dt$$

Nasty Trig-sub problem

$\boxed{\text{See last example of 7.3 notes}}$

$$= \frac{1}{2}\left[\sqrt{1+x^2} + \ln\left|\sqrt{1+x^2} + x\right|\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\sqrt{1+4\pi^2} + \ln\left(\sqrt{1+4\pi^2} + 2\pi\right)\right.$$

$$\left. - \left(\sqrt{1} + \cancelto{0}{\ln(\sqrt{1} + 0)}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{1+4\pi^2} + \ln\left(2\pi + \sqrt{1+4\pi^2}\right) - 1\right]$$