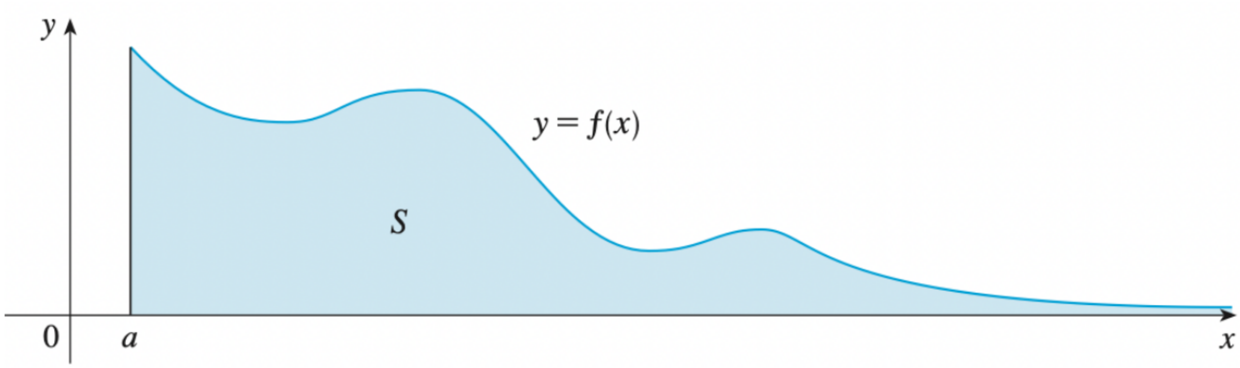
7.8: Improper Integrals

Consider a function $f(x)$. Often we talk about the area under the curve $f(x)$ as an integral over some kind of interval $[a, b]$ where a and b are real numbers. But how to we talk about the area under a curve $f(x)$ if our interval goes on forever?



That is, what happens if we take the interval $[a, b]$ and we send b to ∞? (i.e. $b \to \infty$) This gives us the idea of Improper Integrals. For some reason in the US, we say there are two types of improper integrals, and so, we'll follow the conventions:

In general, when we say "improper integral" we mean that $\int f(x)\, dx$ has something "bad" happening within the domain of integration. These can be thought of coming in two flavours (which can be combined).

Type I Improper Integral: For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the integral $\displaystyle\int_a^t f(x)\, dx$ exists for every number $t \geq a$ then we define

$$\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$$

  provided the **limit** exists.

- If the integral $\displaystyle\int_t^b f(x)\, dx$ exists for every number $t \leq b$ then we define

$$\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx$$

  provided the **limit** exists.

In other words: A **Type I Improper Integral** is an integral where **one or both** bounds is $\pm\infty$.

Some Important teminology:

- If an improper integral gives you a number i.e.

$$\int_a^\infty f(x)\,dx = \# \qquad \text{or} \qquad \int_{-\infty}^b f(x)\,dx = \#$$

  Then we say the Improper Integral is **convergent.**

- If we DON'T get a number (so every other case) we say Improper Integral is **divergent.**

Lastly, if both $\int_{-\infty}^t f(x)\,dx$ and $\int_t^\infty f(x)\,dx$ are convergent then we can write:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^t f(x)\,dx + \int_t^\infty f(x)\,dx$$

Example 1. Determine the convergence of $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$.

Example 2. Determine the convergence of $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ for all values of p.

We can summarise this result as

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for ______________ and **diverges** for ______________

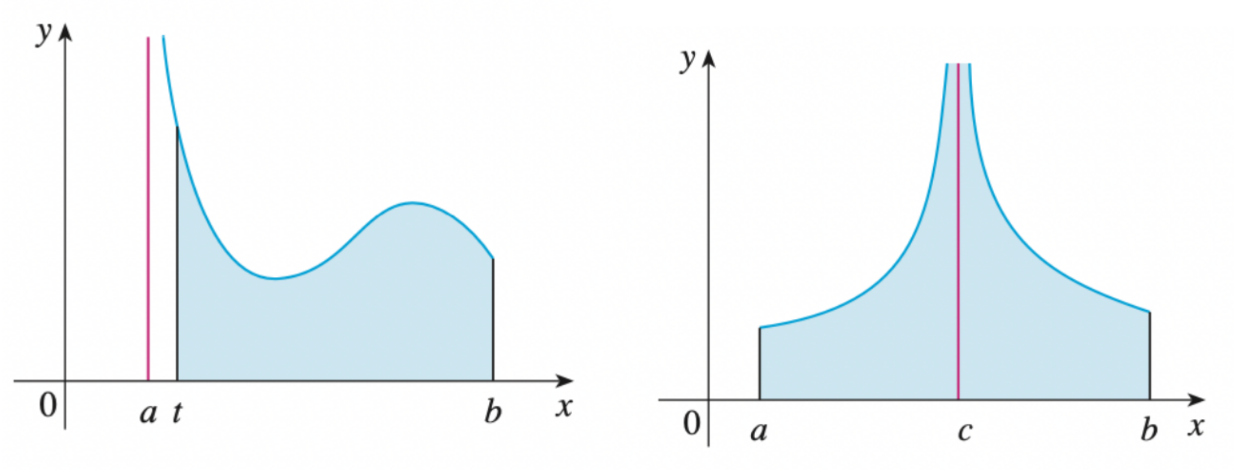
Example 3. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{0} \frac{x}{(x^2+4)^3}\,dx$$

Example 4. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

So now let's look at the area of functions that "blow up:"



Type II Improper Integral: For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the function $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then we define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

  provided the **limit** exists.

- If the function $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then we define

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

  provided the **limit** exists.

Just like Type I Improper Integrals, we use the words **convergent** and **divergent** if the limits exists. Another important Type II Improper Integral is:

- If the function $f(x)$ has a discontinuity at c where $a < c < b$ and the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent, then we define:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

  provided the **limit** exists.

In other words: A **Type II Improper Integral** is an integral where there is a **discontinuity of the function somewhere** in the domain of integration.

Example 5. Determine if the convergence of $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$. Find its value if it converges.

Example 6. Determine if the convergence of $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$. Find its value if it converges.

Sometimes, evaluating an integral is hard, but if all we care about is a question about convergence, then we have a handy theorem that takes care of things for us:

The Comparison Theorem:

Suppose that f and g are continuous functions where $f(x) \geq g(x) \geq 0$ for all $x \geq a$ where a is some constant number.

- **IF** $\displaystyle\int_a^\infty f(x)\,dx$ is **convergent**, **then** this implies $\displaystyle\int_a^\infty g(x)\,dx$ is **convergent**.
- **IF** $\displaystyle\int_a^\infty g(x)\,dx$ is **divergent**, **then** this implies $\displaystyle\int_a^\infty f(x)\,dx$ is **divergent**.

This theorem can be thought of as:

- If the **BIGGER thing** converges then so does the smaller one.
- If the **SMALLER thing** diverges then so does the bigger one.

Example 7. Show that $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.