

## 10.2: Calculus with Parametric Curves

### 1 Derivatives

Now that we are talking about curves that are described parametrically, that is,  $(x(t), y(t))$  is some collection of points in a  $(x, y)$ -grid that we can trace out “forwards and backwards in time,” it’s time to do some calculus with these objects. A lot of it comes as a consequence of the chain rule:

#### Derivatives of Parametric Curves:

Consider the parametric curve  $(x(t), y(t))$ . If  $y$  is also a differentiable function of  $x$  (that is, we can write  $y(x)$  and take a derivative), then by the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

$$y'(t) = \color{red}{y'(x(t))} \color{blue}{x'(t)}$$

and if  $x'(t)$  is nonzero then we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

Geometrically (using calc 3 concepts), the point on the curve  $(x(t), y(t))$  a tangential velocity vector given by  $(x'(t), y'(t))$ . This vector lies in the line tangent to the curve at a given point. i.e. the slope of that vector (a rise over run of sorts) gives the slope of the curve at that point.

Consequently we also see that if we replace  $y$  with  $\frac{dy}{dx}$  we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dx}{dy} \right)}{\frac{dx}{dt}} \\ &= \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3} \end{aligned}$$

**Example 1.** Consider the curve give by  $x = t^2$  and  $y = t^3 - 3t$ . We will

- (a) Determine where the tangents are vertical and horizontal
- (b) Determine where the curve is concave up or down
- (c) Show that a parametric curve can have multiple tangent lines at a point (but not for a  $t$  value), specifically at  $(x, y) = (3, 0)$

(cont. from prev page)  $x = t^2$  and  $y = t^3 - 3t$

## 2 Area

Area with parametric curves is an incredibly large and complicated topic. Most if not all introductory textbooks don't cover this material well, and we *certainly* don't have the time to also cover this material well. For now, we will only consider the simplest case and note this can still produce *weird, but valid* results.

### Area with Parametric Curves (Simplified):

Consider the parametric curve  $(x(t), y(t))$  and the integral  $\int_a^b y \, dx$  (i.e. **using the idea that an integral collects together the width\*height of a bunch of rectangles of width  $dx$  and height  $y$** ), then using the substitution  $y = y(t)$  and  $x = x(t)$  then

$$\int_a^b y \, dx = \int_{\alpha}^{\beta} y(t)x'(t) \, dt \quad (\text{or } \int_{\beta}^{\alpha} y(t)x'(t) \, dt)$$

where  $x(\alpha)$  is either  $a$  or  $b$  and  $x(\beta)$  is the remaining one. Similarly, by a similar idea:

$$\int_c^d x \, dy = \int_{\gamma}^{\delta} x(t)y'(t) \, dt \quad (\text{or } \int_{\delta}^{\gamma} x(t)y'(t) \, dt)$$

But why is this simplified? Well, the textbook only covers this case, but it is possible to have negative areas within this framework and it is still valid. The details are quite messy and not suitable for calc 2.

**Example 1.** Finding the area under the first arch of the cycloid  $x = r(\theta - \sin \theta)$  and  $y = r(1 - \cos \theta)$ .

### 3 Arc Length

#### Arc Length of Parametric Curves:

Consider the parametric curve  $(x(t), y(t))$  and recall that (one version of) arc length is given by the integral  $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ . If we follow the same substitution idea from before we get

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

Note that this means that **we don't need to write  $y$  in terms of  $x$  or vise-versa. This version of Arc Length** (and the version used in calc 3 and beyond) **can be used for any parametric description of a curve**. In fact, we could've derived this version of Arc Length back in section 8.1 if we knew about parametric equations, and then we could have ignored the mean value theorem step.

Writing  $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$  we get that arc length is by definition  $\int ds$  (if you go back in the notes, this  $ds$  is present for the 8.1 material).

Define the **arc length function** for a parametrized curve as

$$s(t) := \int_{\alpha}^t ds = \int_{\alpha}^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Note this means  $s'(t) = \sqrt{x'(t)^2 + y'(t)^2}$

**Example 1.** Finding the length of the first arch of the cycloid  $x = r(\theta - \sin \theta)$  and  $y = r(1 - \cos \theta)$ .

## 4 Hard Problems

**Problem 1. (Hard)** Find the length of the curve given by  $x = t \sin t$  and  $y = t \cos t$  on  $0 \leq t \leq \frac{\pi}{2}$ .