7.3: Trig Sub

The goal of this lesson is to develop another technique of integration to handle integrals that we currently cannot deal with such as $\int \sqrt{1-x^2}\,dx$ (though this may be doable with a lot of work using integration by parts). This technique will be reliant on the geometry of triangles.

Consider a circle with radius 1 centered at the origin of an (x, y)-coordinate system. The collection of (x, y) points that line on this circle satisfy the equation $x^2+y^2 = 1$. If we consider the angle θ made by a line from the origin to the circle's edge, then we define

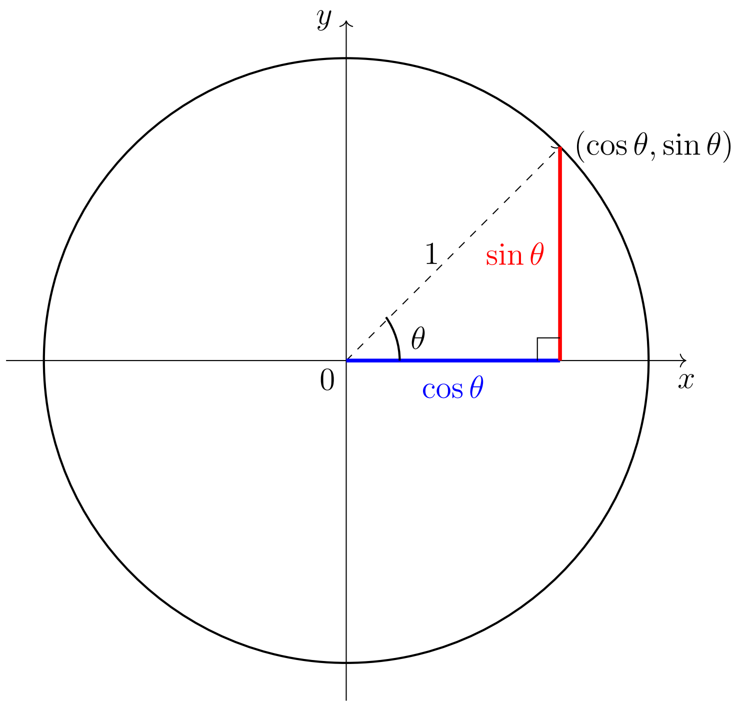
- $\sin(\theta) =$ the y-value of the unit circle at the angle θ
- $\cos(\theta) =$ the x-value of the unit circle at the angle θ
- $\tan(\theta) =$ the slope of the line at the angle θ

By slope formula we get that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

By definition, we have that for any θ,

$$-1 \le \sin\theta \le 1 \qquad \text{and} \qquad -1 \le \cos\theta \le 1$$



Notationally, $\text{trig}^{\#}(\theta)$ is short hand for $(\text{trig}(\theta))^{\#}$ for any $\# \neq -1$.

Consider a circle of radius r. Then the (x, y) coordinates of this circle satisfy the equation $x^2 + y^2 = r^2$. If we divide the equation by r^2 we get the equation

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

(See picture below // Geometrically we scaled the circle down or up to the unit circle). Then by definition, the $(\frac{x}{r}, \frac{y}{r})$ coordinates of this circle satisfy the expressions

$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \cos\theta = \frac{x}{r}$$

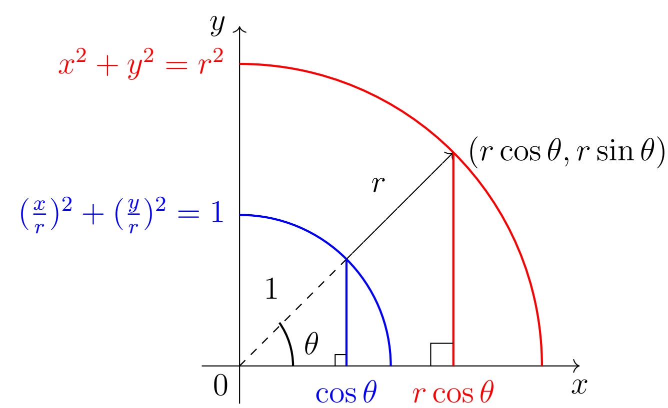
However, if we interpret this information regarding the triangles below

- $\sin\theta$ is equal to the length of the opposite side divided by the radius of the circle (i.e. the hypotenuse of the triangle).
- $\cos\theta$ is equal to the length of the adjacent side divided by the radius of the circle (i.e. the hypotenuse of the triangle).

This gives us

$$\sin\theta = \frac{opp}{hyp} \qquad \text{and} \qquad \cos\theta = \frac{adj}{hyp}$$

From the definition of Tangent we get $\tan\theta = \frac{opp}{adj}$.



Once we have the necessary tools. Consider the integral $\int \sqrt{1-x^2}\,dx$. This can be related to a right-triangle with hypotenuse 1 and one of the legs is has length x. We can then form the substitution $\sin\theta = x$ from soh-cah-toa, and then figure out what our "dx" must be by differentiating $\sin\theta = x$ to get $\cos\theta d\theta = dx$ and so

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\,\cos\theta d\theta$$

From here the problem reduces to material from 7.2.

Example 1. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx$

Example 2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+2}}\,dx$

Example 3. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$

Example 4. $\displaystyle\int \frac{1}{(4x^2+9)^2}\,dx$

Example 5. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ where $a > 0$ is constant.

Example 6. $\displaystyle\int \frac{x^3}{\sqrt{1+4x^2}}\, dx$

Example 7. $\int \sqrt{1-x^2}\, dx$

Example 8. $\displaystyle\int \sqrt{1+x^2}\,dx$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$