11.1: Sequences

Up until this point we have only talked about functions that exist on a continuous collection of numbers, often denoted as $f(x)$ where x can be any real number. However, functions are a lot more general for this and can have more than just a real number as an input. *Restricting our attention to functions evaluated at only the (positive) integers gives us* *sequences.*

> A **sequence** is a list of numbers written in some prescribed order. All of the following are different representations of a sequence:
>
> - $a_1, a_2, a_3, ..., a_n, ...$
>
> - $\{a_n\}_{n=\#}^{\infty}$
>
> - $\{a_n\}$ (when the starting index is not important).
>
> Some examples of a sequence of numbers are:
>
> - $1, 2, 3, 4, 5, 6, 7...$
>
> - $1, -1, 234, -e, 2.3934579, 0, \pi...$
>
> - $\{2n + 1\}$
>
> Sequences can be thought of as function which assigns a number to a particular **(positive) integer** n. That is, given a function $f(x)$ we can make a sequence by declaring its values to be whatever f gives us:
>
> $$a_n = f(n)$$
>
> Depending if you're a mathematician or computer scientist or someone who likes to argue, we call the set of positive integers the **natural numbers** denoted by $\mathbb{N}$. Some people consider 0 to be a natural number, others don't (I don't).

Example 1. Given the following sequences, write out the first few terms

1. $a_n = \frac{1}{2^n}$

2. $\left\{(-1)^n \dfrac{n+1}{3^n}\right\}_{n=0}^{\infty}$

Example 2. Find the general form for a term in the sequence: $\left\{\dfrac{3}{5}, -\dfrac{4}{25}, \dfrac{5}{125}, -\dfrac{6}{625}, \dfrac{7}{3125}, ...\right\}$.

UD ID: _____   Name: _____

11.1 Sequences Continued

> If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} c_n = L$, then $\lim\limits_{n \to \infty} b_n = L$.

Example 1: Determine whether the sequence converges or diverges. If it converges, find the limit. If it diverges, write DIVERGES.

$$\lim_{n \to \infty} \frac{\cos^2(n)}{4^n}$$

> **6 Theorem**   If $\lim\limits_{n \to \infty} |a_n| = 0$, then $\lim\limits_{n \to \infty} a_n = 0$.

Example 2: Determine whether the sequence converges or diverges. If it converges, find the limit. If it diverges, write DIVERGES.
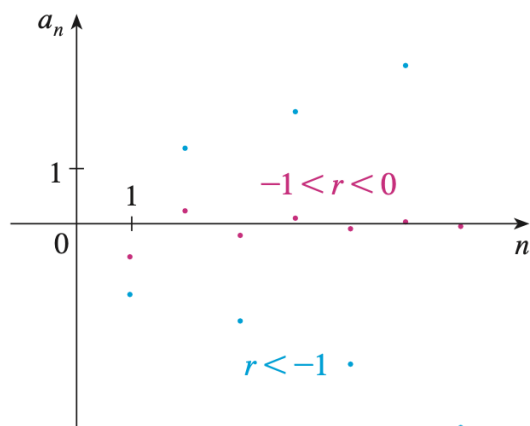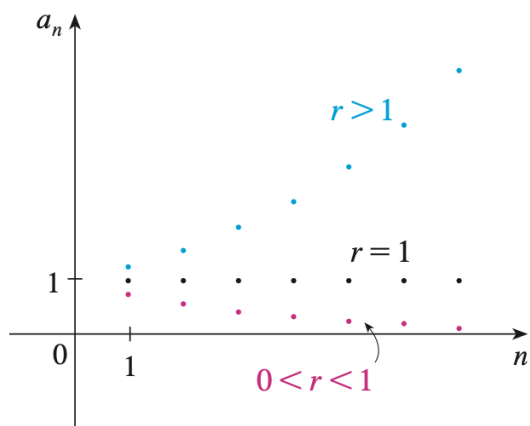
$$\lim_{n \to \infty} \frac{6n^3 - n^2 + 7n}{2n^2 + 1}$$

Example 3: Find $\displaystyle \lim_{n \to \infty} \sin\left(\frac{\pi}{n}\right)$.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example 4

> **10 Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.
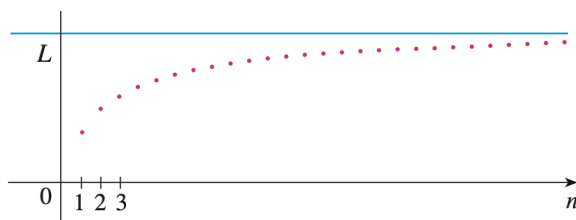
> **11 Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that
>
> $$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$
>
> It is **bounded below** if there is a number m such that
>
> $$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$
>
> If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

> **12 Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

4