7.4: Integration of Rational Functions and Partial Fractions

The goal of this lesson is to develop another technique of integration to handle integrals of **rational functions**.

Recall that a **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 + a_0$$

i.e. $\quad x^2+1 \quad x \quad x-1 \quad 7 \quad 4x^{62}+7x^7+3x-2.$

The highest ordered power is called the **degree** of the polynomial. So, in the above examples they are (in order) 2,1,1,0, and 62. A **rational function** $f(x)$ is a function that is a fraction involving two polynomials $P(x)$ and $Q(x)$ on the top and bottom:

$$f(x) = \frac{P(x)}{Q(x)}.$$

Put another way. A rational function is a fraction that has a polynomial on top and a polynomial on the bottom. Some examples of this would be

$$\frac{x^2+1}{x-1} \qquad \frac{x-1}{x^2+1} \qquad \frac{x-1}{x+1} \qquad x$$

We say that a rational function is **improper** if the degree of the bottom is less than or equal to the degree of the top. We say that a rational function is **proper** if the bottom has a bigger degree than the top. In the above examples, only the red expression is proper.

To integrate **improper rational functions** we first need to do polynomial long division:

Example 1. $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

To integrate a proper rational function, we need to "break up" the integral is a clever way. This is called **partial fraction decomposition**. The idea comes from wanting to combine fractions with different denominators we have to multiply by what's missing:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\frac{d}{d} + \frac{c}{d}\frac{b}{b} = \frac{ad + cb}{bd}$$

Partial fraction decomposition is about reversing this process. That is, we start with the fraction $\frac{E}{bd}$ and we want to find numbers A and C that let's us decompose it into

$$\frac{E}{bd} = \frac{A}{b} + \frac{C}{d}.$$

Example 2. Decompose the rational function into partial fractions: $\dfrac{x+5}{x^2+x-2}$.

As you might expect with rational function integration based off of partial fractions, we'll need to integrate $\dfrac{1}{ax+b}$ a lot, so, in short:

$$\int \frac{dx}{ax+b} =$$

Case I: The denominator $f(x) = \frac{P(x)}{Q(x)}$ is a product of distinct linear factors (ie. a bunch of degree 1 polynomials):

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

i.e. $\quad (x+1)(x-2) \quad x(x-2) \quad 4x(3x+1)(5x-2) \quad$ *etc.*

The partial fraction decomposition (for $Q(x)$) is:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_n}{a_nx + b_n}$$

Example 2.5. Evaluate $\displaystyle\int \frac{x+5}{x^2+x-2}dx$

Case II: The denominator $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor $(a_1x + b_1)$ is repeated r times, that is

$$Q(x) = (a_1x + b_1)^r(a_2x + b_2)\cdots(a_kx + b_k)$$

In this case, there exist constants $A_1, A_2, ...A_k$ to be determined such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r} + \frac{A_{r+1}}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Example 3: Consider $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$.

(a) Write out the form of the partial fraction decomposition of the function $\displaystyle\frac{4x}{(x-1)^2(x+1)}$.

(b) Evaluate the integral.

MATH 242 SPRING 2025
Chap 7 Techniques of Integration

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7.4 Integration of Rational Functions by Partial Fractions: Continued

Case III: The denominator $Q(x)$ contains irreducible factors, none of which is repeated. If $Q(x)$ has the factor (ax^2+bx+c), where $b^2-4ac<0$, then in addition to the partial fractions in Case I and II, the expression $P(x)/Q(x)$ will have a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

For example,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

How to solve:

A given problem $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$.

Step 1: Write out the form of the partial fraction decomposition of the function $\dfrac{P(x)}{Q(x)}$.

Step 2: Evaluate the integral.

Recall:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Example 1: Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

Case IV: The denominator $Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in Case III, we should have partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example 2: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^2}$$

Example 3: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$$\frac{x^4 + x^2 + 1}{(x^2 + 1)(2x^2 + 3)^2}$$