6.1 Area Between Curves

Motivation: Area via Riemann Sums

When finding the area of an awkward-shaped region where functions can be used, we approximate it using Riemann sums:

$$\text{Area of } R \approx \sum_{j=1}^{n} f(x_j)\,\Delta x,$$

where each $f(x_j)$ represents the height of a small rectangle and Δx is its width.

By refining the size of the rectangles, we obtain the integral:

$$\text{Area of } R = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x = \int_a^b f(x)\,dx.$$

Area Between Two Curves (as functions of x)

If $f(x)$ and $g(x)$ are two curves such that $f(x) \geq g(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ on $[a, b]$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

If the "top" and "bottom" functions switch on the interval, a robust formula is

$$A = \int_a^b |f(x) - g(x)|\,dx,$$

and in practice you split the integral at intersection points where the top/bottom relationship changes.

Example 1 Find the area bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \dfrac{\pi}{2}$.

When Curves Are Better Described as Functions of y

Sometimes we cannot conveniently write the boundary curves as functions of x (for instance, when a region has left/right edges that fail the vertical line test). In that case, describe the curves as functions of y:

$$x = f(y), \qquad x = g(y).$$

If a region R is bounded by the curves $x = f(y)$ and $x = g(y)$ such that $f(y) \geq g(y)$ for $y \in [c, d]$, and by the horizontal lines $y = c$ and $y = d$, then the area is

$$A = \int_c^d \big(f(y) - g(y)\big)\, dy$$

(i.e., *right minus left*).

Example 2

Find the area bounded by $y = x - 1$ and $y^2 = 2x + 6$.

Example 3

Find the area bounded by $y = \ln x$ and $y = 1$, in the first quadrant.