§ 10.1 Parametric Equations

A parametric equation is a system of n equations/variables that depend on at Most n-1 variables.

eg 2-variables, 1-Parametric input

$x = f(t)$ ie. The path of a
$y = g(t)$ particle on a 2D grid

eg 3-variables, 1-Parametric input (Calc 3)

$x = f(t)$ ie. The path of a
$y = g(t)$ particle in 3D space
$z = h(t)$ or the path of a 3D in an animation

eg 3-variables, 2-Parametric input (Calc 3)

$x = f(u, v)$ ie. A surface

$$y = g(u,v)$$
$$z = h(u,v)$$

Parametric Equations is how a lot of modern applied math is done.

- As often the parameter t is thought of as "time"

Some special terminology for a parametric system

- If there is only 1 parametric input variable
  ↳ We call it a "(parametric) curve"
- If there is only 2 parametric input variable
  ↳ We call it a "surface"
- If there is n parametric input variable
  ↳ We call it an "n-manifold" or "manifold" or "hyper-surface"

10.1: Parametric Curves

Up until now we have only talked about the situation in which 1 variable is described as a function of another variable $y = f(x)$, $x = g(y)$, $r(\theta)$, etc. However, in a lot of cases, given 2 variables, its quite unlikely we will be able to do this. The simplest example is $x^2 + y^2 = 1$, the unit circle. However, there is a way to describe the equation of a circle using only *1 variable*. This idea gives us parametric equations and paves the way for calc 3 / vector calculus.

Parametric Equations:

The variables x and y are given by a **parametric system of equations** if there is a 3rd variable t such that

$$x = f(t) \qquad \text{and} \qquad y = g(t).$$

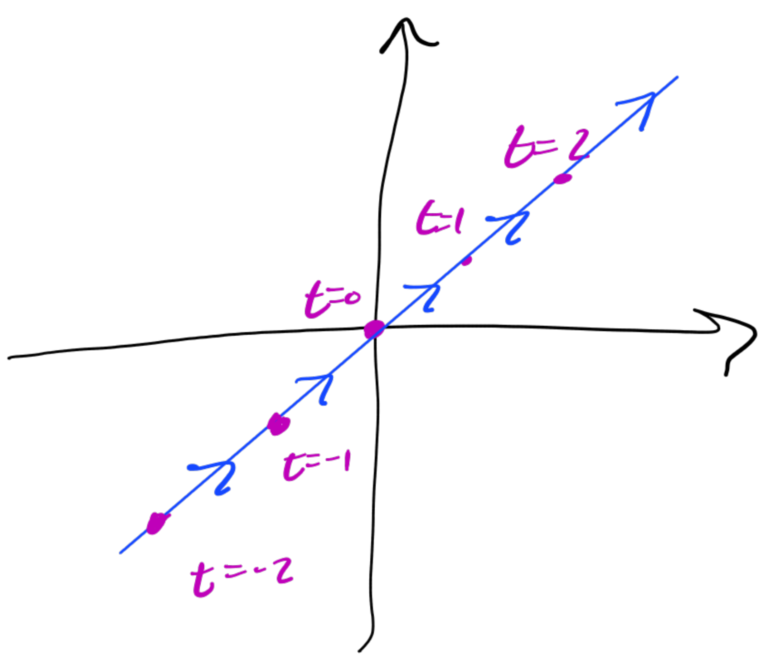
This equivalent to saying:

- A **parametric description of** x **and** y
- A **parametrization of** x **and** y **by** t
- x **and** y **are given parametrically by** t

In some sense, a parametric description of the variables x and y gives a sense of following the "*path over time*" that x and y sketch out.

Example 1. What is the function that the parametric equations $x = t$ and $y = t$ describe?

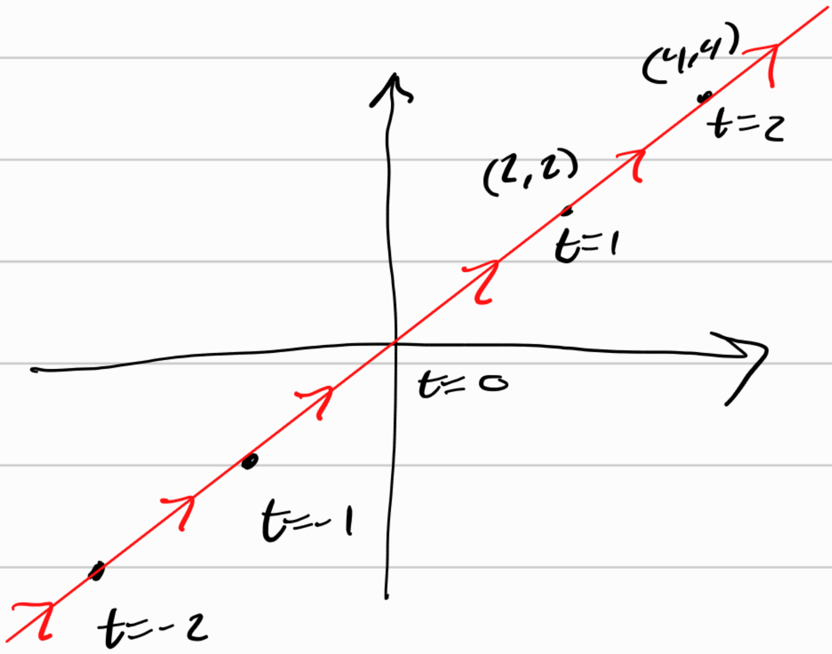
t	x	y
0	0	0
1	1	1
2	2	2
-1	-1	-1
-2	-2	-2



Note Parametric curves have direction

Consider instead of $x = 2t$ & $y = 2t$ then

t	x	y
0	0	0
1	2	2
2	4	4
-1	-2	-2
-2	-4	-4



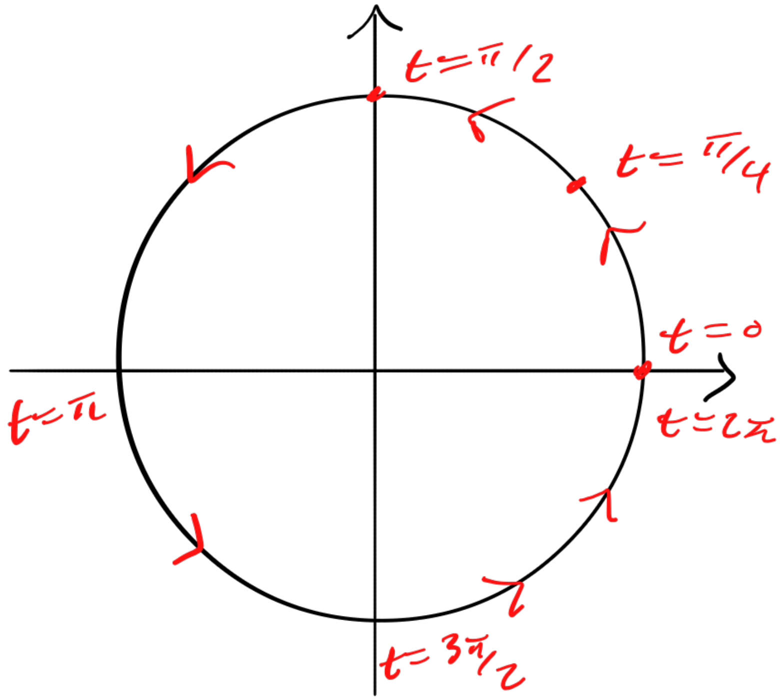
So parametric curves have both a direction & speed associated w/ them.

Example 2. What is the graph of the parametric curve $x = t^2 - 1$ and $y = t - 1$? Can you rewrite this system as 1 equation with only 2 variables instead of 2 equations with 3 variables?

Example 3. This shows us a circle is a 1-D object (a curve)

(a) What is the shape that the parametric equations $x = \cos(t)$ and $y = \sin(t)$ describe on the interval $0 \leq t \leq 2\pi$?

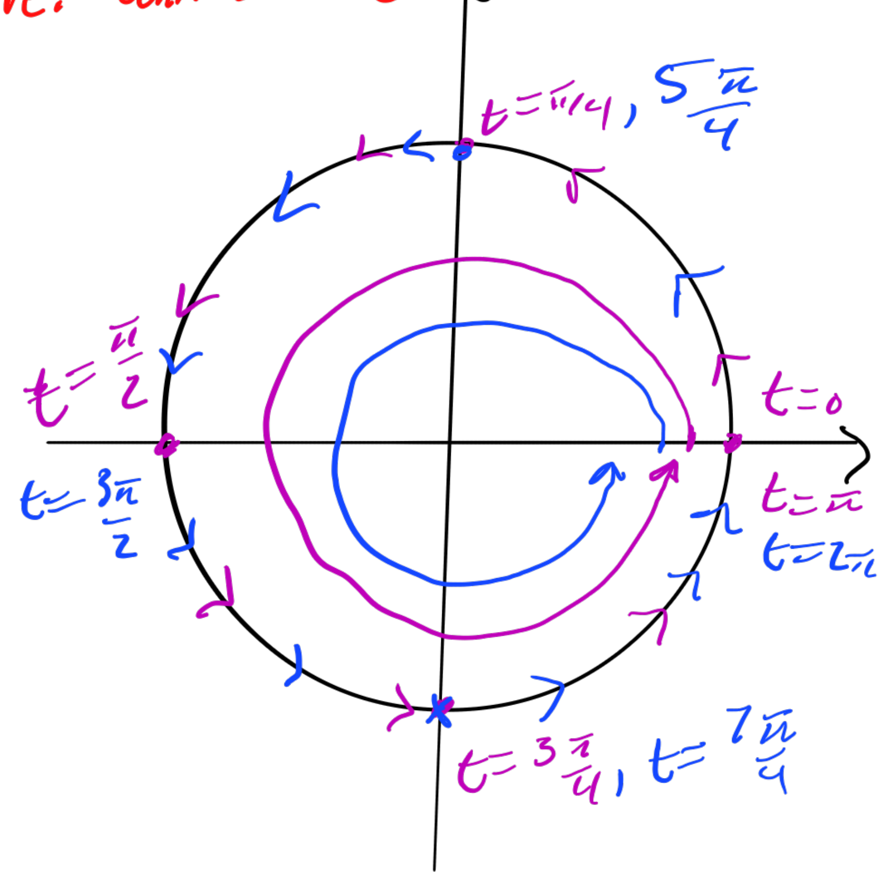
t	x	y
0	1	0
$\frac{\pi}{6}$	$\sqrt{3}/2$	$1/2$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1

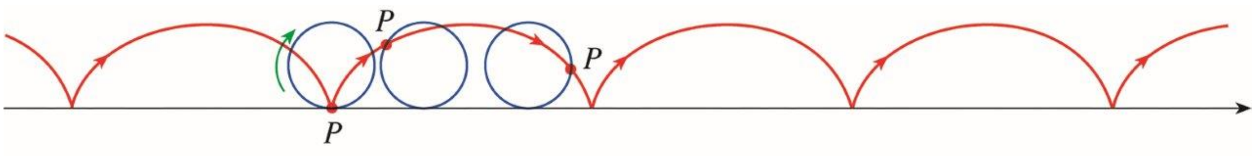


(b) What do the parametric equations $x = \cos(2t)$ and $y = \sin(2t)$ describe on the interval $0 \leq t \leq 2\pi$?

ie. Unit circle @ 2 times speed.

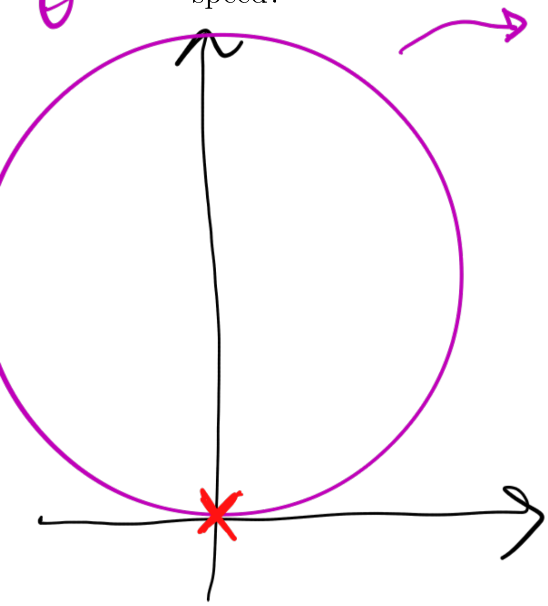
t	$2t$	x	y
0	0	1	0
$\pi/4$	$\pi/2$	0	1
$\pi/2$	π	-1	0
$3\pi/4$	$3\pi/2$	0	-1
π	2π	1	0

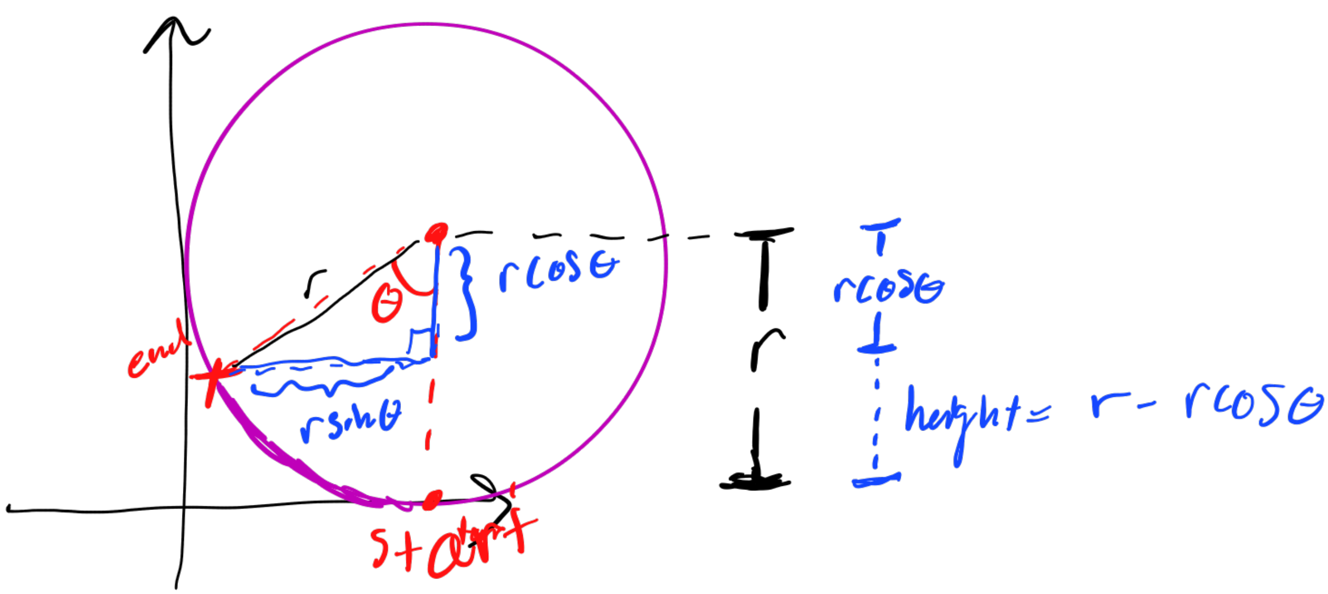


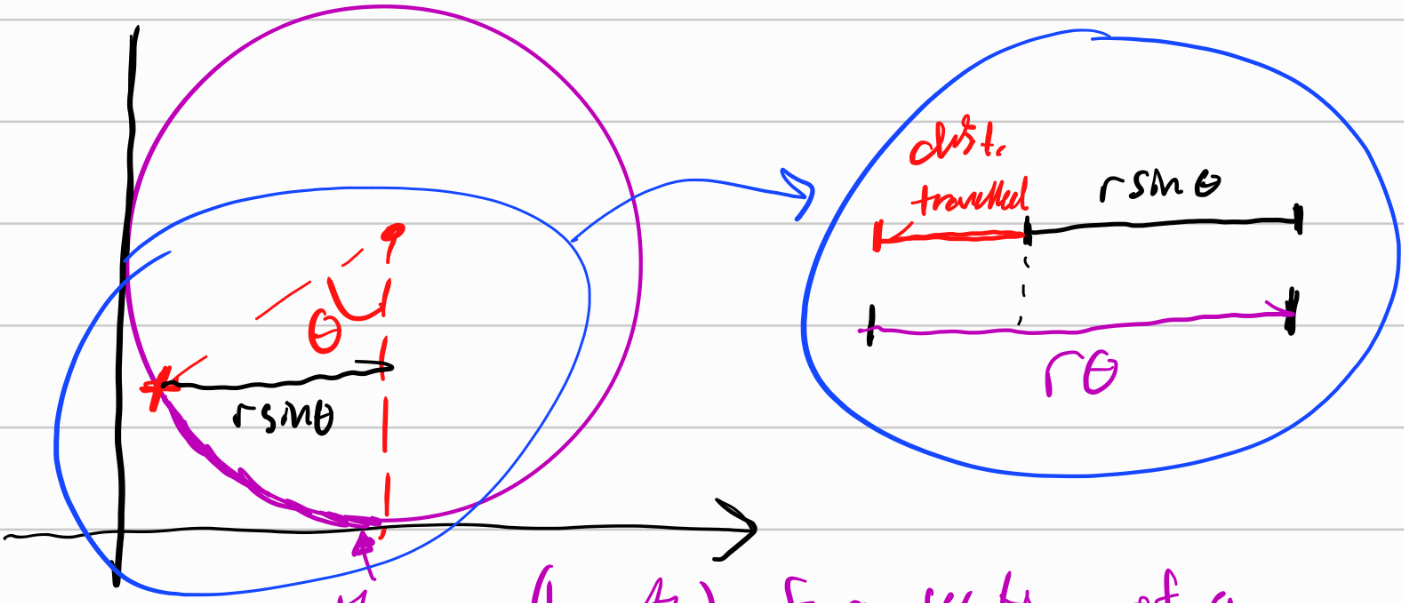


Example 4. Consider a pebble P stuck in a tire. If we assume the pebble starts at the bottom of the tired, how do we model the motion of the pebble if the tired moves at unit speed?

$\theta =$







the arc(length) of a section of a circle is $r\theta$

$$\Rightarrow \quad x = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = r - r\cos\theta = r(1 - \cos\theta)$$

If the wheel has angular velocity ω then $\theta = \omega t$ & so we have the parametric system

$$x = r(\omega t - \sin(\omega t))$$
$$y = r(1 - \cos(\omega t))$$
$$\theta = \omega t$$

If r is not unif. (i.e. depends on θ) then $r = f(\theta)$ & the system expands.