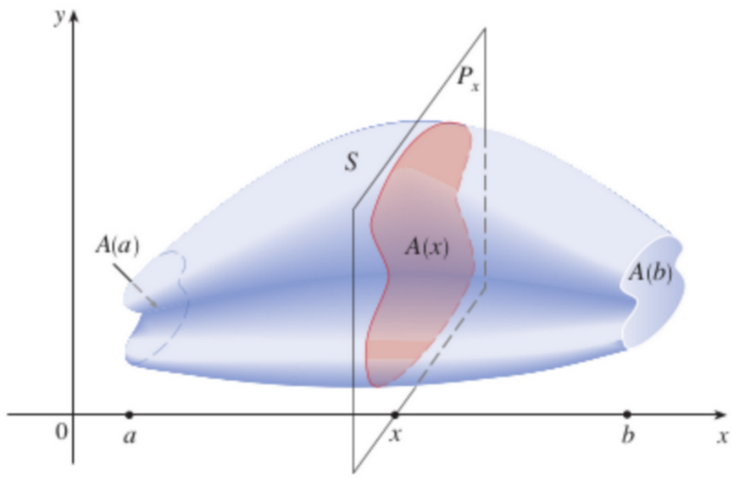
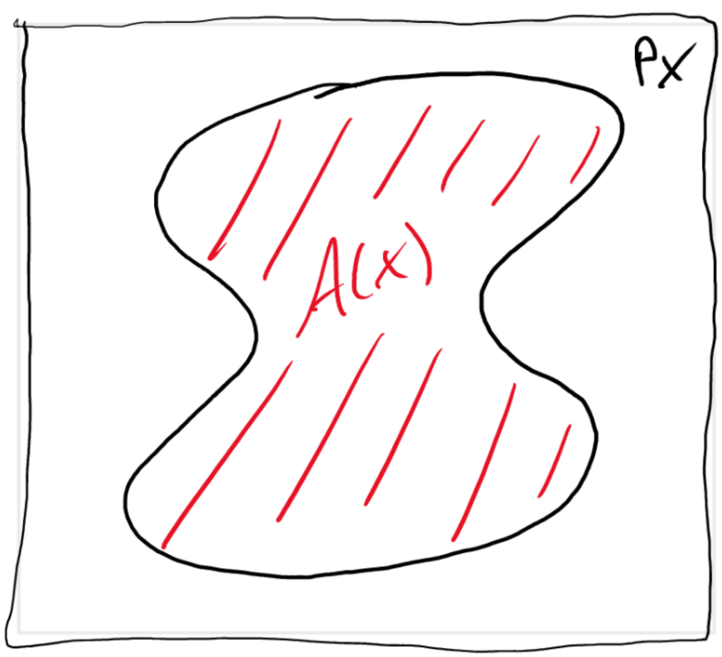
Section 6.2 Volume + Disk/Washer Method

Definition of Volume





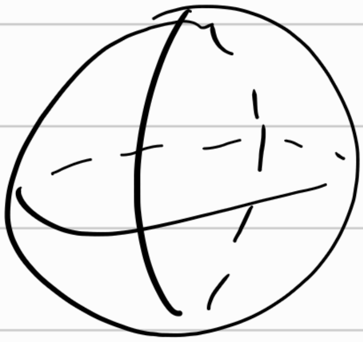
For a given x-value on the the axis A(x) is just a #.

Consider a solid S whose "ends" we can place at $x = a$ and $x = b$. Let P_x be a plane that is perpendicular to the x-axis and goes through a point x(as pictured above). Let $A(x)$ be the cross-sectional area of S that lies on P_x. Then the volume of S is given by

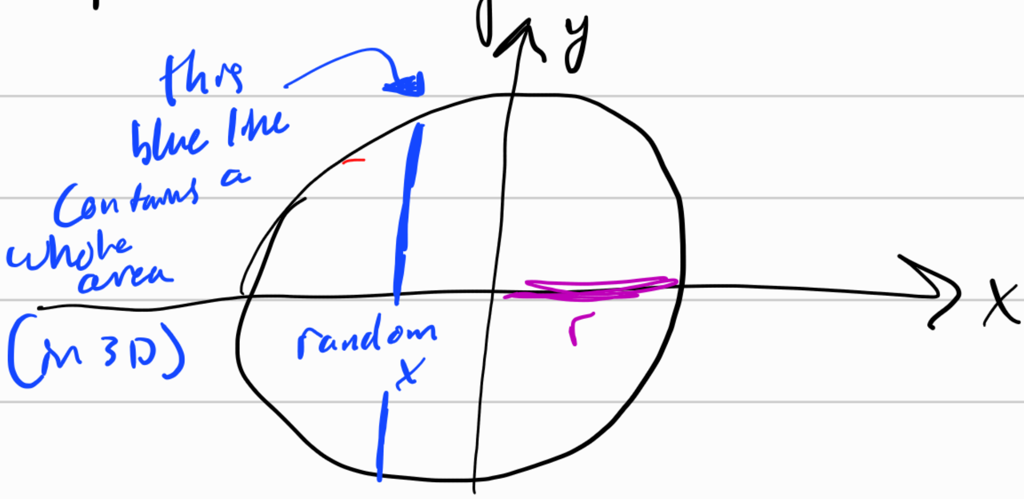
$$\text{Volume of } S \text{ on } [a, b] = \int_a^b A(x)\, dx$$

[Video of Bread Slice]

Let's consider the volume of a sphere



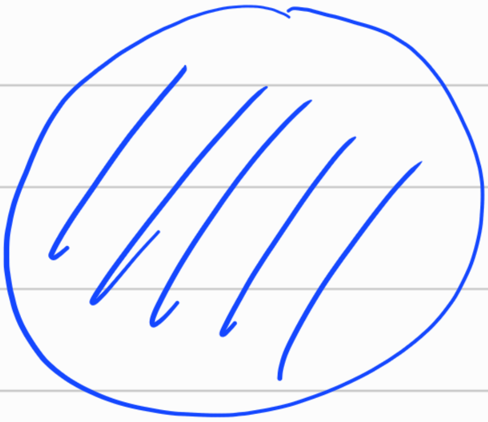
By our definition we should slice the sphere along an axis

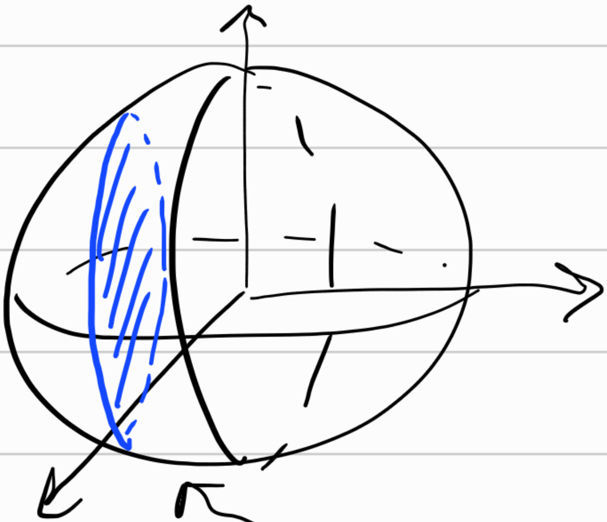


We would then like to find each area associated w/ a slice of the sphere @ an x-value

If we rotate the picture & look straight @ the x-axis

This is the blue x-value slice →

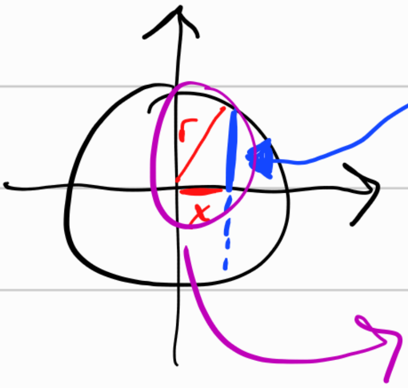


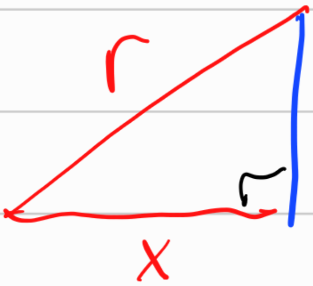


So $A(x) = \pi(\text{radius})^2$

the radius of this is

This dist.





radius $= \sqrt{r^2 - x^2}$

So we get $\int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} r^2 - x^2 \, dx$

As you might be able to tell,

- Finding volumes can be quite complicated.
- As such we will only concern ourselves w/ 1 kind of volume:

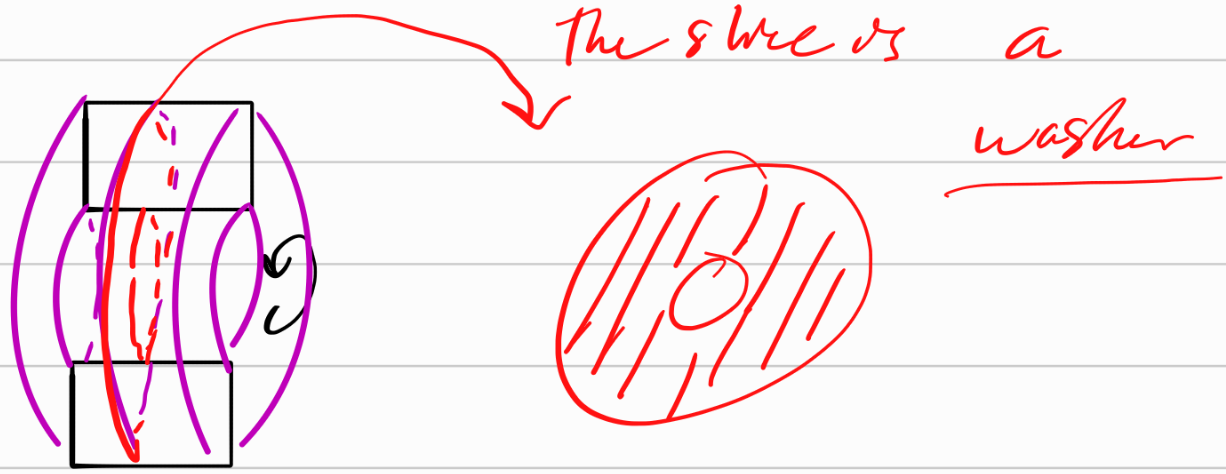
Volumes generated by rotating a Region

There are 2 methods for this

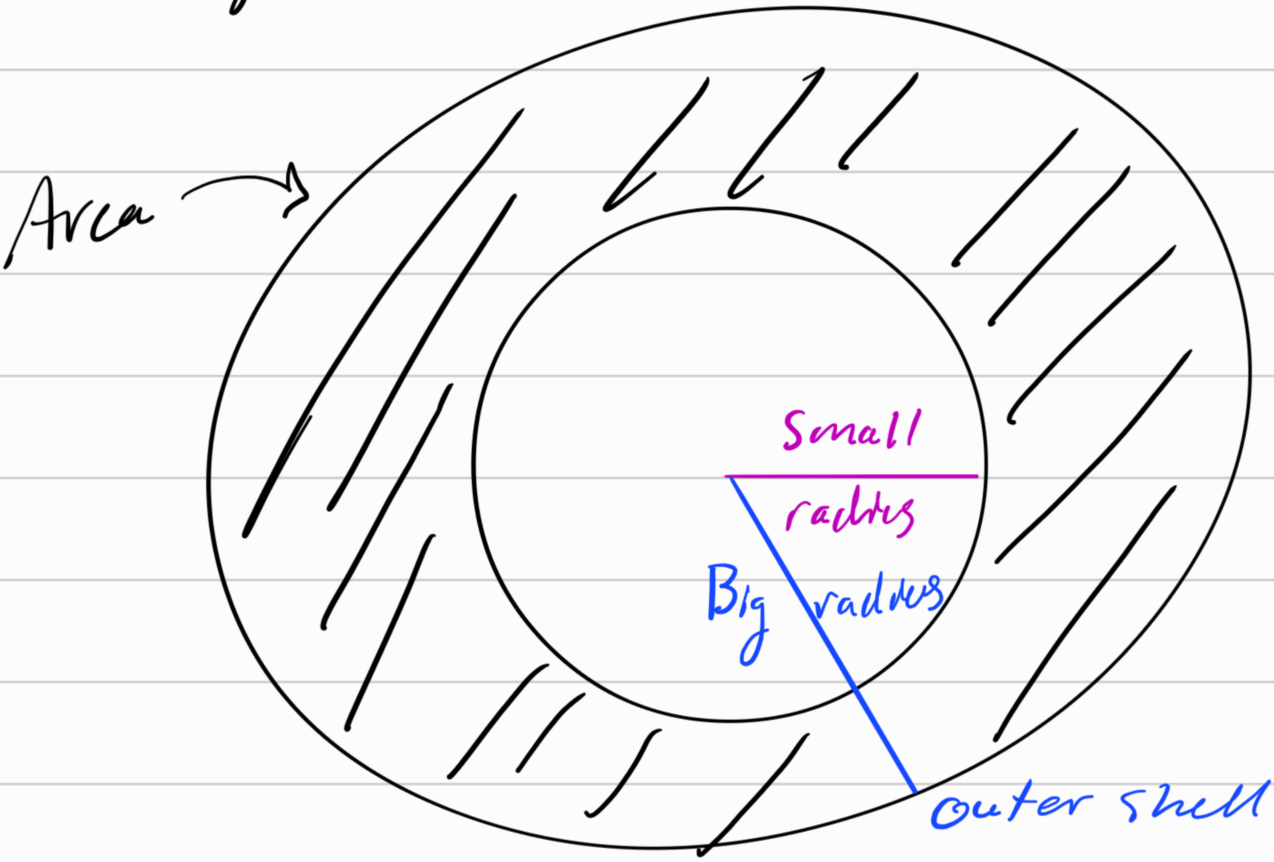
(i) Disk/washer Method (6.2)

(ii) Shell method (6.3)

In the Washer method we generated volumes by taking a region & rotating it perpendicularly to the axis of rotation



& A washer can be thought of as a circle w/ a small circle taken away from it:



$\Rightarrow$ Area Big $-$ Area small

$$\pi(\text{Big Rad})^2 - \pi(\text{Small rad})^2$$

ie. $\pi\left[\left(\begin{matrix}\text{outer}\\ \text{rad}\end{matrix}\right)^2 - \left(\begin{matrix}\text{inner}\\ \text{rad}\end{matrix}\right)^2\right]$

Since this is our "simple" piece

The equation we get from an "exact approx" is

$$\pi \int_{\text{start}}^{\text{end}} \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 d(x \text{ or } y)$$

Outer rad = furthest dist. to axis of rotation

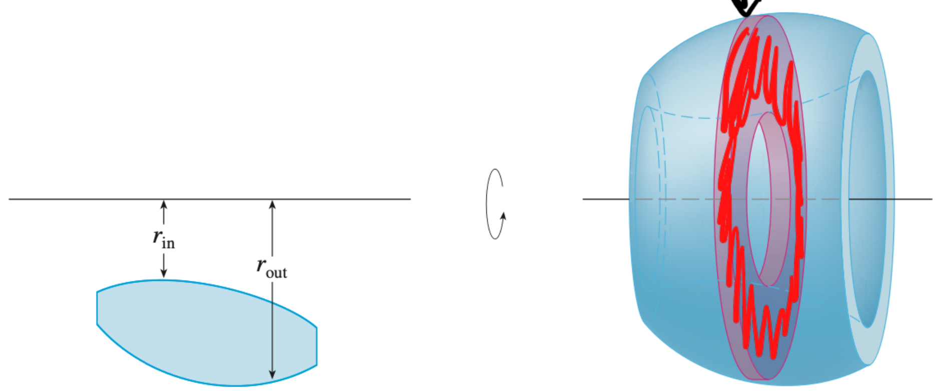
inner rad = closest dist. to axis of rotation.

If the region you want to rotate has the axis of rotation as its boundary/edge then inner rad = 0 always & this becomes "the disk method"

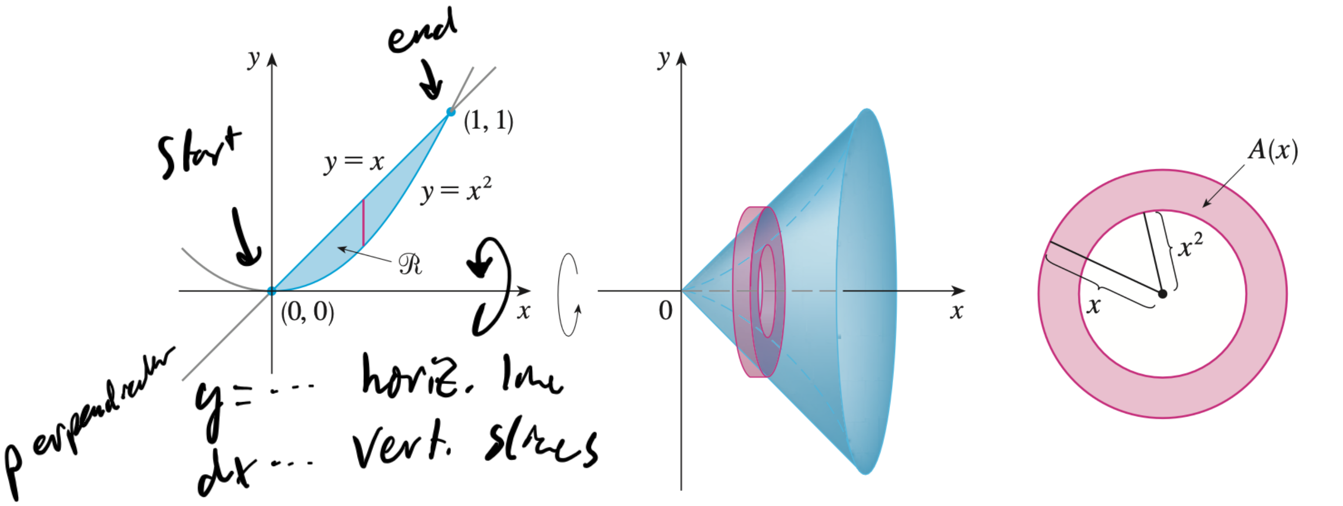
- **Washer Method**: If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

the area slice (red)



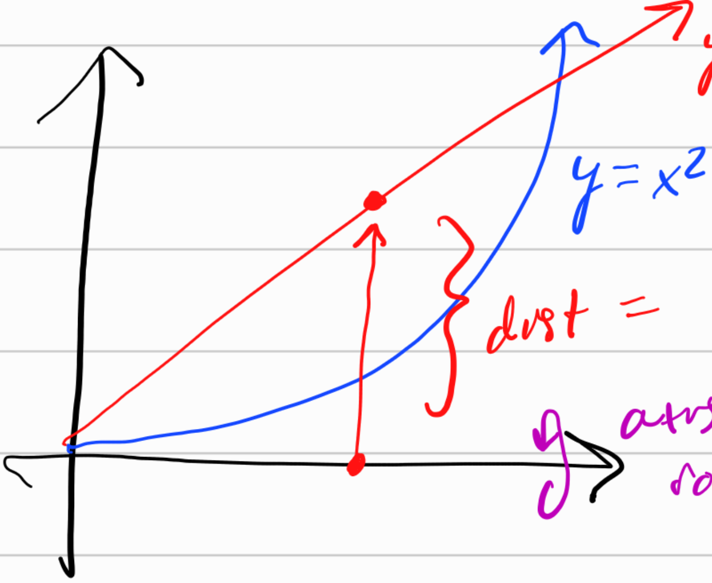
Example 3: The region $\mathfrak{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Set up the volume V of the resulting solid.



(a) (b) (c)

$$\int_0^1 \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx$$

Outer rad = furthest dist.



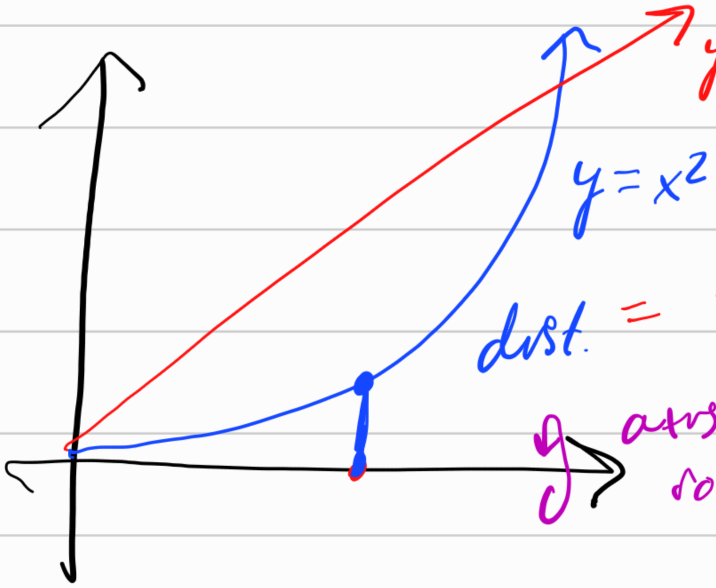
dist = final pos. − initial pos. = top − bot

axis of rotation (located @ $y=0$)

location of AoR

outer rad $= x - 0 = x$

location of outer edge



dist. = final pos. − initial pos. = top − bot

axis of rotation (located @ $y=0$)

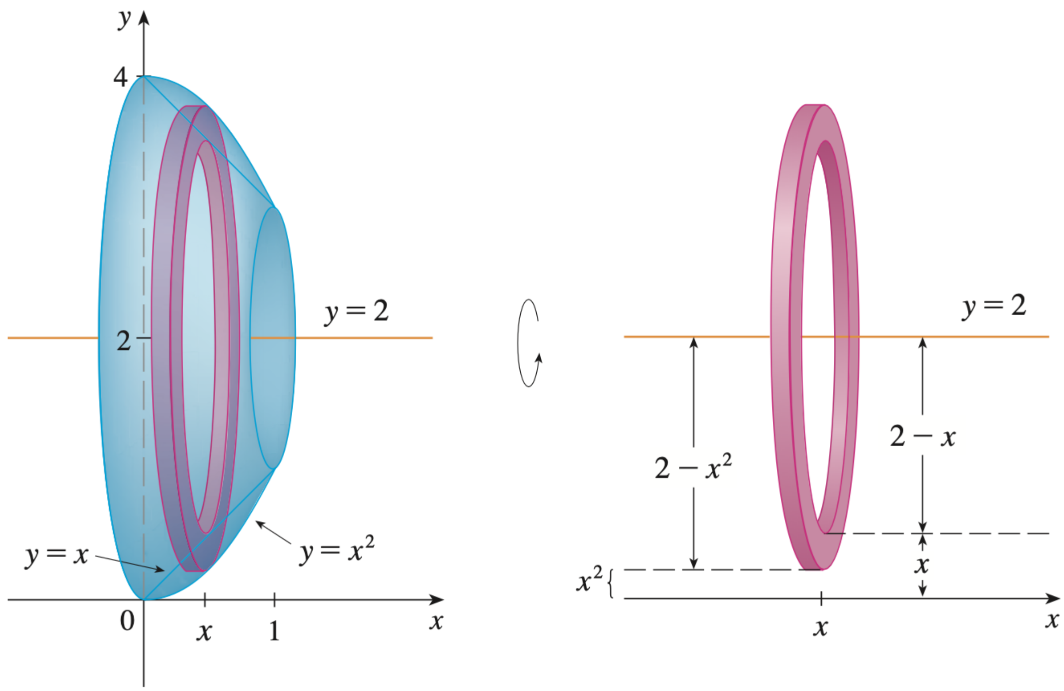
location of AoR

inner rad $= x^2 - 0 = x^2$

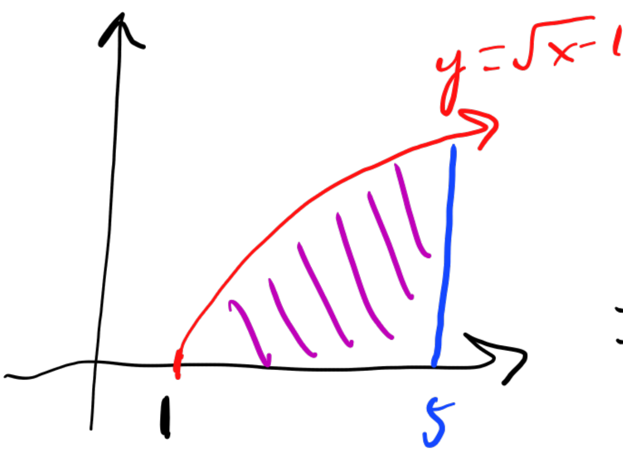
location of inner edge

$$\Rightarrow \int_0^1 \left(\text{outer rad}\right)^2 - \left(\text{inner rad}\right)^2 dx = \int_0^1 (x)^2 - (x^2)^2 dx$$

Example 4: Set up the volume formula V of the solid obtained by rotating the region in Example 3 about the line $y = 2$.



Problem 6.2.13. (a) Find the area bounded between the curves $y = \sqrt{x-1}$, $y = 0$, and $x = 5$.

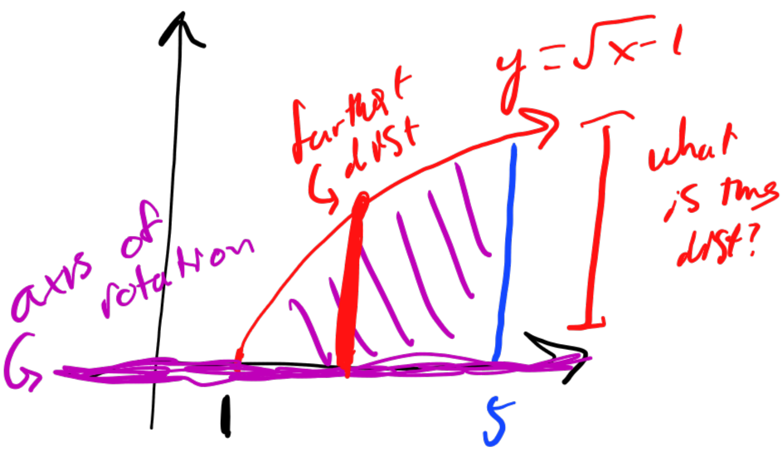


$$\int_1^5 \sqrt{x-1}\,dx$$

$$= \frac{2}{3}\Big[(x-1)^{3/2}\Big]_1^5 = \frac{2}{3}\left((4)^{3/2} - 1\right)$$

$$= \frac{2}{3}(8-1) = \frac{14}{3}$$

(b) Find the volume using (via washer method) generated by the area from part (a) about the y-axis.



$$\pi \int_1^5 \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx$$

$\boxed{\text{note this is zero if the region touches the axis of rotation}}$

inner rad = closest dist. to axis of rotation

$= 0$

outer rad = furthest dist. from axis of rotation

$= \sqrt{x-1} - 0$

$= \sqrt{x-1}$

$$\pi\int_1^5 \left(\sqrt{x-1}\right)^2 - (0)^2\,dx = \pi\int_1^5 x-1\ dx$$

$$= \pi\left[\frac{1}{2}x^2 - x\right]_1^5$$

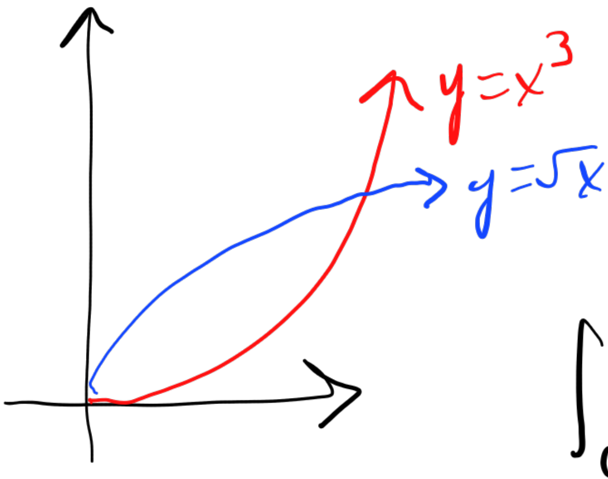
$$= \pi\left(\frac{1}{2}5^2 - 5\right) - \pi\left(\frac{1}{2} - 1\right)$$

$$= 5\pi\left(\frac{5}{2} - \frac{2}{2}\right) - \pi\left(-\frac{1}{2}\right)$$

$$= 5\pi\left(\frac{3}{2}\right) + \frac{\pi}{2} = \frac{15}{2}\pi + \frac{\pi}{2} = \frac{16\pi}{2}$$

$$= 8\pi$$

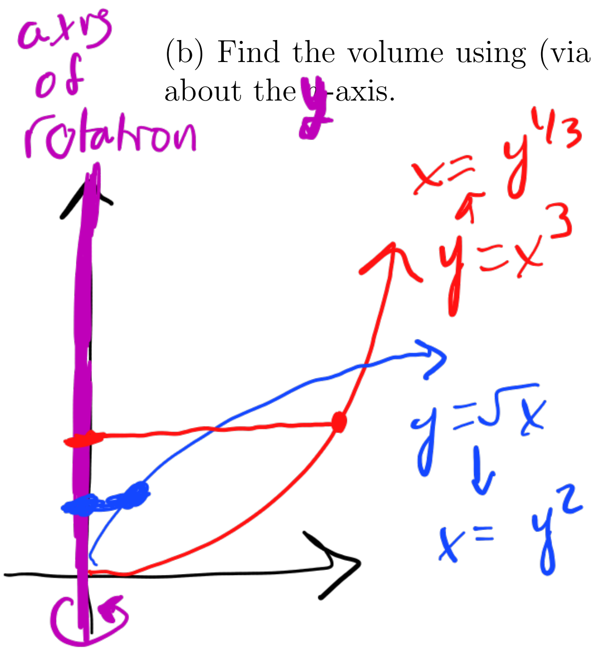
Problem 6.2.19. (a) Find the area bounded between the curves $y = x^3$ and $y = \sqrt{x}$



$$x^3 = \sqrt{x}$$
$$x^6 = x$$
$$\Rightarrow x = 0 \text{ or } 1$$

$$\int_0^1 \sqrt{x} - x^3\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$
$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

(b) Find the volume using (via washer method) generated by rotating the area in part (a) about the y-axis.



$$\pi \int_?^? \begin{pmatrix}\text{Out} \\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner} \\ \text{rad}\end{pmatrix}^2 dy$$

Out. rad = furthest dist.

$= y^{1/3} - 0$

(ie. Right − left)

inner rad = closest dist.

$= y^2 - 0$

(right − left)

$$\pi \int_{?}^{?} (y^{1/3})^2 - (y^2)^2 \, dx$$

To find ? we set these equal

$$y^{1/3} = y^2$$

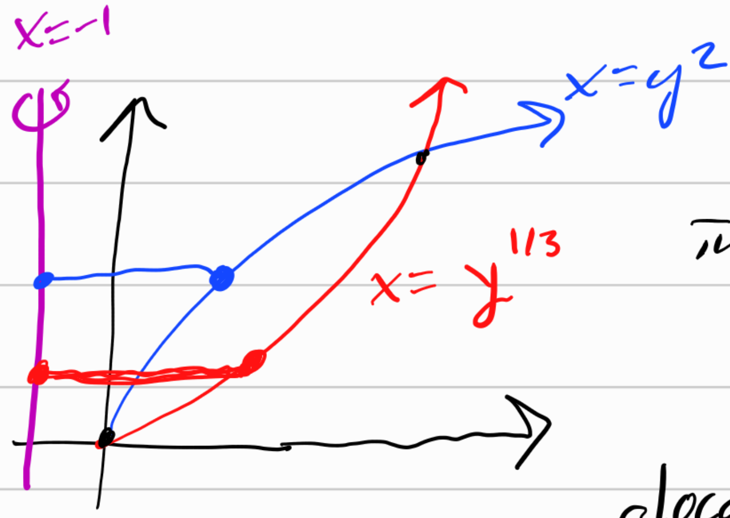
$$\left.\begin{aligned} y &= y^6 \\ 0 &= y^6 - y \\ 0 &= y(y^5 - 1) \end{aligned}\right] \Rightarrow \boxed{\begin{gathered} y = 0 \\ \text{or} \\ y = 1 \end{gathered}}$$

$$\pi \int_0^1 y^{2/3} - y^4 \, dy$$

$$= \pi \left[\frac{3}{5} y^{5/3} - \frac{1}{5} y^5 \right]_0^1$$

$$= \pi \left[\frac{3}{5} - \frac{1}{5} \right] = \frac{2\pi}{5}$$

What if rotated about $x = -1$?



$$\pi\int_0^1 \begin{pmatrix}\text{out}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{in}\\ \text{rad}\end{pmatrix}^2 dy$$

location of the outside edge

out rad = furthest dist = $y^{1/3} - (-1)$

location of axis of rotation

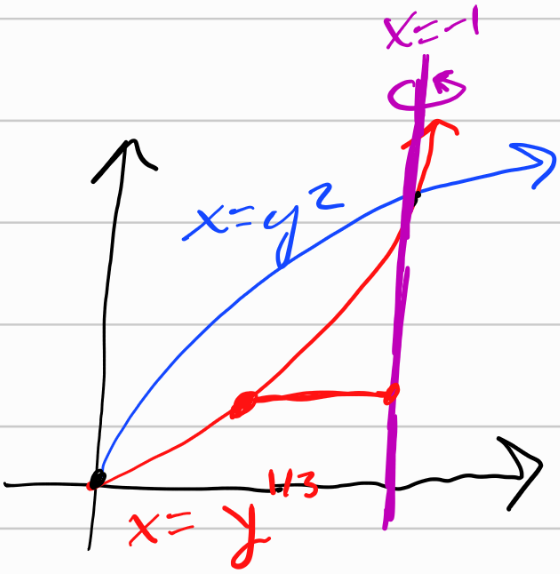
in rad = closest dist. = $y^2 - (-1)$

location of inside edge

=>

$$\pi\int_0^1 (y^{1/3}+1)^2 - (y^2+1)^2 \, dy$$

What if we rotated about $x=1$?



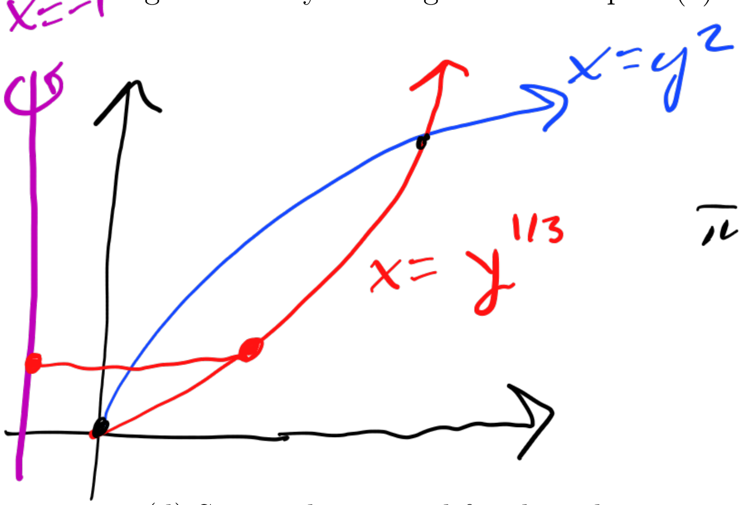
$$\pi \int_0^1 \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dy$$

Inner rad $= 1 - y^{1/3}$

Outer rad $= 1 - y^2$

$$\pi \int_0^1 (1-y^2)^2 - (1-y^{1/3})^2 \, dy$$

Problem 6.2.19(cont.). (c) Set up the integral for the volume using (via washer method) generated by rotating the area in part (a) about the line $x = -1$.



$$\pi \int_0^1 y^{1/3} - (-1)$$

(d) Set up the integral for the volume using (via washer method) generated by rotating the area in part (a) about the line $x = 1$.

(e) Set up the integral for the volume using (via washer method) generated by rotating the area in part (a) about the line $y = -1$.