6.3: U-Substitution (Chain Rule Integration)

The chain rule lets us take derivatives of nested functions. Now we will use it *backwards* to compute integrals. To see where u-substitution comes from, start with the chain rule:

$$\frac{d}{dx}\big[f(g(x))\big] = f'(g(x))\,g'(x).$$

If we integrate both sides with respect to x, we get:

$$\int \frac{d}{dx}\big[f(g(x))\big]\,dx = \int f'(g(x))\,g'(x)\,dx.$$

So,

$$f(g(x)) = \int f'(g(x))\,g'(x)\,dx.$$

This is the basic idea behind **u-substitution**: if you see a function and something proportional to its derivative multiplying it, then it may be the integral of a chain rule derivative.

U-Substitution Setup

Let

$$u = g(x).$$

Then

$$\frac{du}{dx} = g'(x) \qquad \text{or more familiarly:} \qquad du = g'(x)\,dx.$$

So the integral

$$\int f'(g(x))\,g'(x)\,dx$$

becomes

$$\int f'(u)\,du.$$

A practical checklist

When doing u-substitution, your goal is to turn an x-integral into a u-integral:

1. Given $\int f(g(x))dx$, define $u = g(x)$
2. Find $\frac{du}{dx}$ and solve for du
3. Rewrite the integral in terms of u

4. Solve the integral with respect to u
5. Substitute $g(x)$ for u in your answer

When the integral is definite

If you have a **definite integral**, you have two clean options:

- **Method A:** Convert back to x after integrating, then evaluate using the original x-bounds.
- **Method B:** Rewrite the integral in terms of u, convert bounds to u-bounds, evaluate with respect to u.

Method B is often faster and cleaner.

Example 1. Find $\displaystyle\int 6x(3x^2 + 5)^7 \, dx$

Example 2. Find $\displaystyle\int \frac{2x}{x^2+1}\,dx$

Example 3. Find $\displaystyle\int e^{4x-1}\,dx$

Example 4. Find $\displaystyle\int_0^1 (5x+2)^3\,dx$