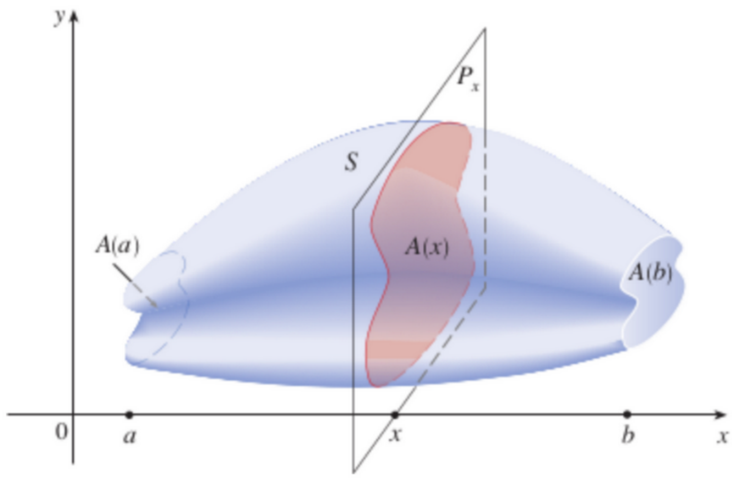
Section 6.2 Volume + Disk/Washer Method

Definition of Volume



Consider a solid S whose "ends" we can place at $x = a$ and $x = b$. Let P_x be a plane that is perpendicular to the x-axis and goes through a point x(as pictured above). Let $A(x)$ be the cross-sectional area of S that lies on P_x. Then the volume of S is given by

$$\text{Volume of } S \text{ on } [a, b] = \int_a^b A(x)\, dx$$

Since Volume is given by

$$\int_a^b A(x \text{ or } y)\, d(x \text{ or } y)$$

where A is some kind of area function (i.e. our slice)

& a & b denote the start & end of our volume.

→ It would be nice to have a few "nice to work w/ areas"

In Calc 2 there is only 1 "nice" volume

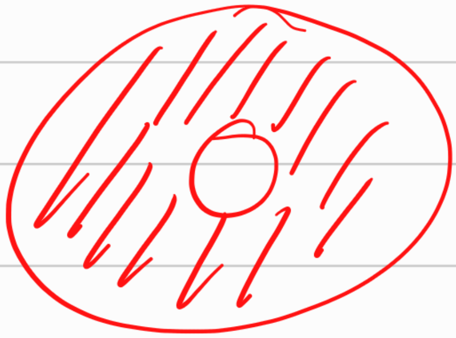
⇒ Volumes generated by rotations.

→ → ↳ Disk/Washer Method (6.2) ⇐ ⇐

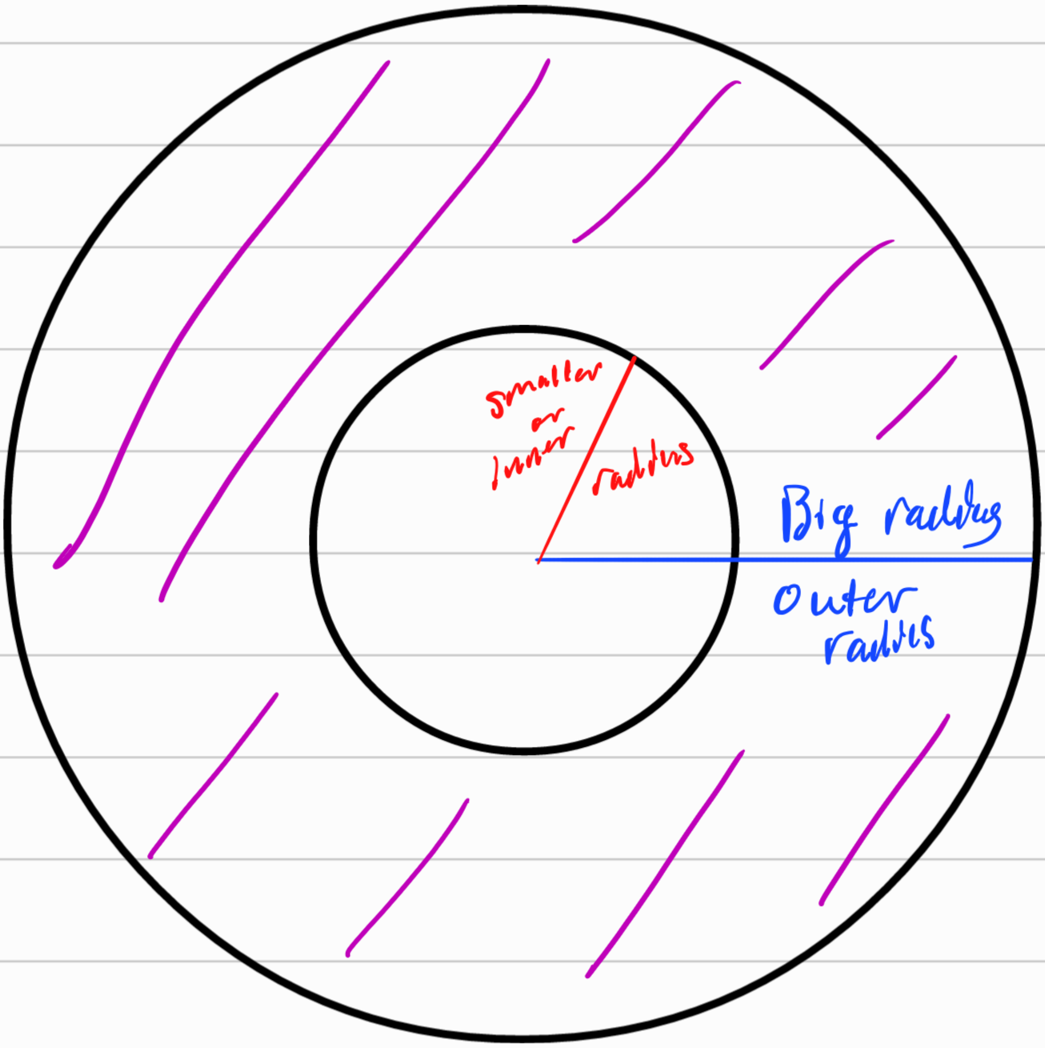
↳ Shell Method (6.3)

The names of these methods come from the shape of the area associated w/ each slice ($A(x)$)

This Slice has a front view



This shape is a washer



Total Area = Area of Big Circle − Area of Small circle

$$= \pi \begin{pmatrix} \text{Big} \\ \text{radius} \end{pmatrix}^2 - \pi \begin{pmatrix} \text{small} \\ \text{radius} \end{pmatrix}^2$$

For each slice we have

$$A(x) \approx \pi \left[\begin{pmatrix} \text{outer} \\ \text{radius} \end{pmatrix}^2 - \begin{pmatrix} \text{inner} \\ \text{radius} \end{pmatrix}^2 \right]$$

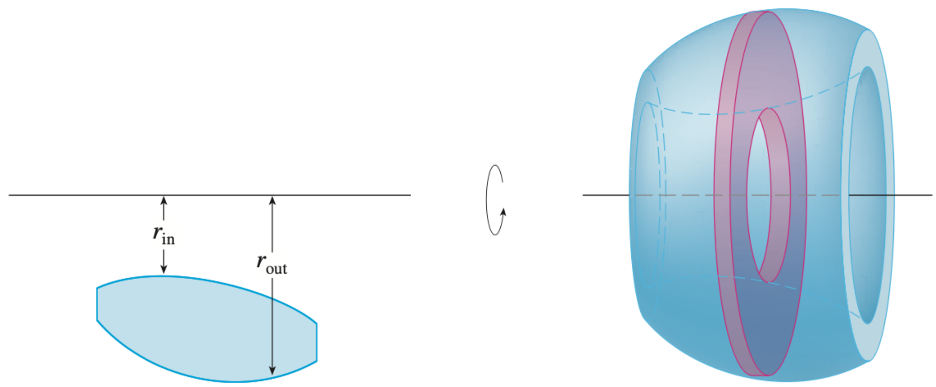
A washer method integral is then

$$\pi \int_a^b \left[\begin{pmatrix} \text{outer} \\ \text{rad} \end{pmatrix}^2 - \begin{pmatrix} \text{inner} \\ \text{radius} \end{pmatrix}^2 \right] d\left(x \text{ or } y \right)$$

$\Rightarrow$ The dx or dy (d-whatever) slices need to be Perpendicular to the axis of rotation (AoR)

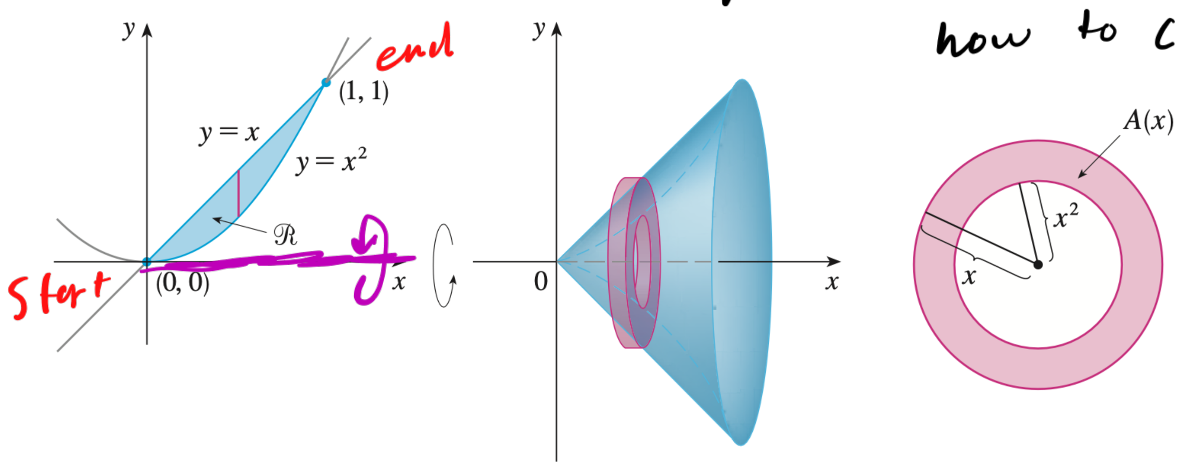
- **Washer Method**: If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$



Example 3: The region $\Re$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Set up the volume V of the resulting solid.

This example is highlighting how to construct this



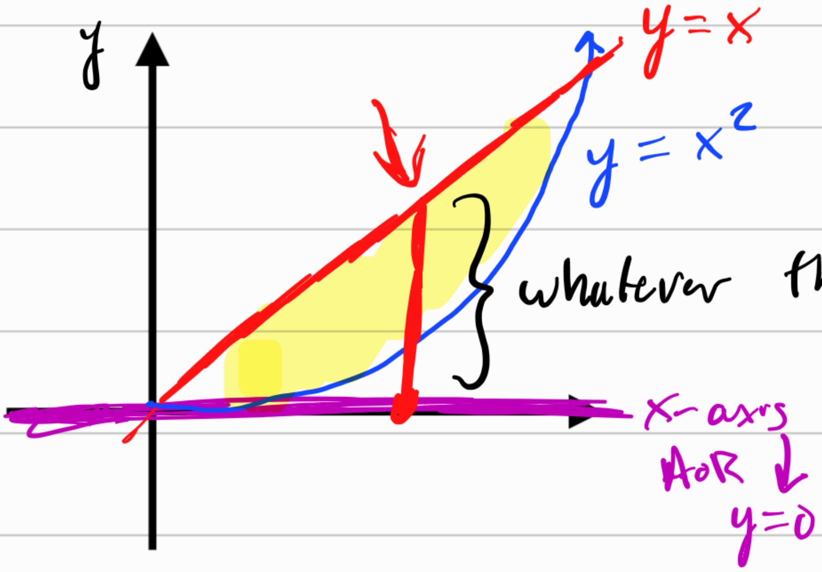
(a) (b) (c) ■

AoR is x-axis => horiz. line => vert. slices (perpendicular)

↳ y = # eq. => opp. slice i.e. dx

The general form is $\pi \int_{\text{start}=0}^{\text{end}=1} \left(\text{out rad}\right)^2 - \left(\text{inner rad}\right)^2 \, dx$

(dist. final pos. − initial pos.)



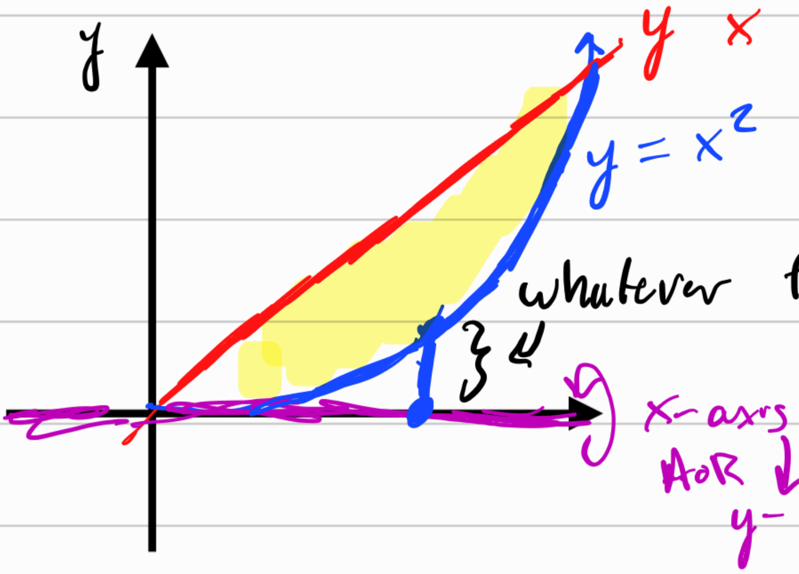
whatever this distance is, is the furthest I could possibly be @ a given x-value

Outer radius = furthest distance from the AoR

$= x - 0$ ← location of the AoR

↑

Is the curve $y=x$ (i.e. the outer edge of the region)



whatever this dist. is

inner radius = closest dist. to AoR $= x^2 - 0$ ← location of AoR

is the inner edge of the region

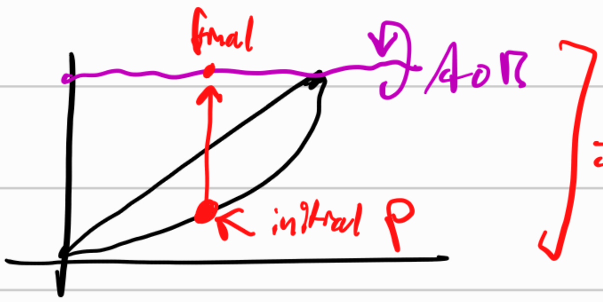
$$\pi \int_0^1 \left(\begin{smallmatrix} \text{out} \\ \text{rad} \end{smallmatrix}\right)^2 - \left(\begin{smallmatrix} \text{inner} \\ \text{rad} \end{smallmatrix}\right)^2 dx$$

$$= \pi \int_0^1 (x)^2 - (x^2)^2 \, dx$$

keep in mind these parantheses & squares are apart of the Washer/Disk method formula.

Q: Do we always subtract the AoR

A: No. We'll in a future example.



$\Rightarrow$ AoR − p

Q: How did we determine dx vs dy?

A: Geometry (or variables)

A washer/disk "slice" must be perpendicular to the AoR

In the prev. example we were rotating

about a horiz. line (x-axis)
So the washer wants a perpendicular (i.e. vert. in this case) slice.

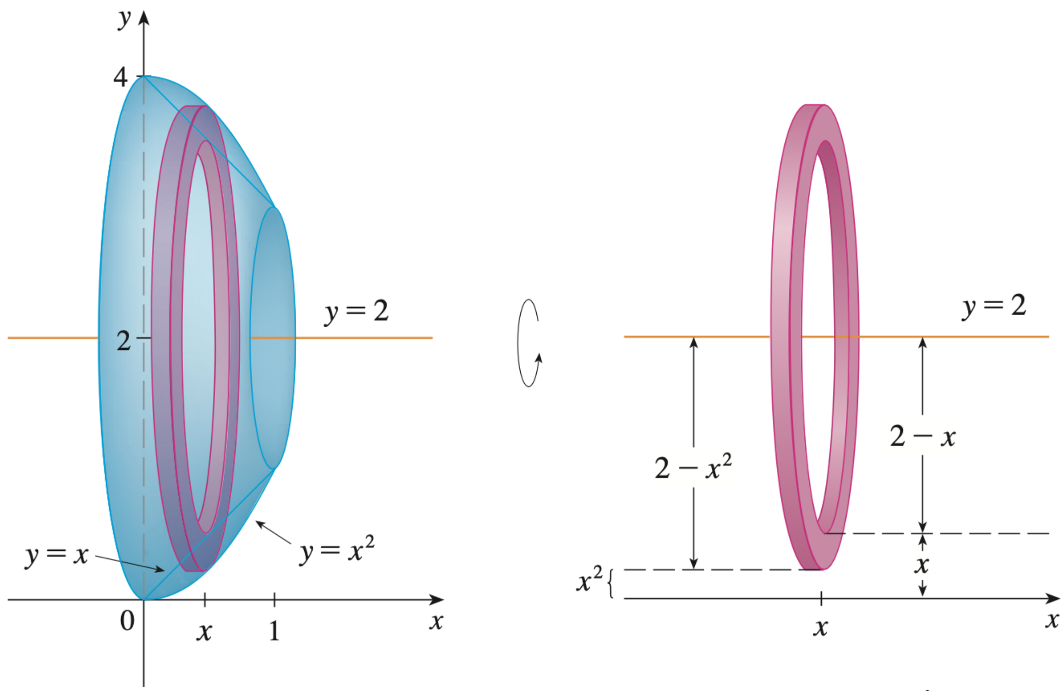
$\boxed{\text{Vert. slice} \Rightarrow dx \text{ integral}}$

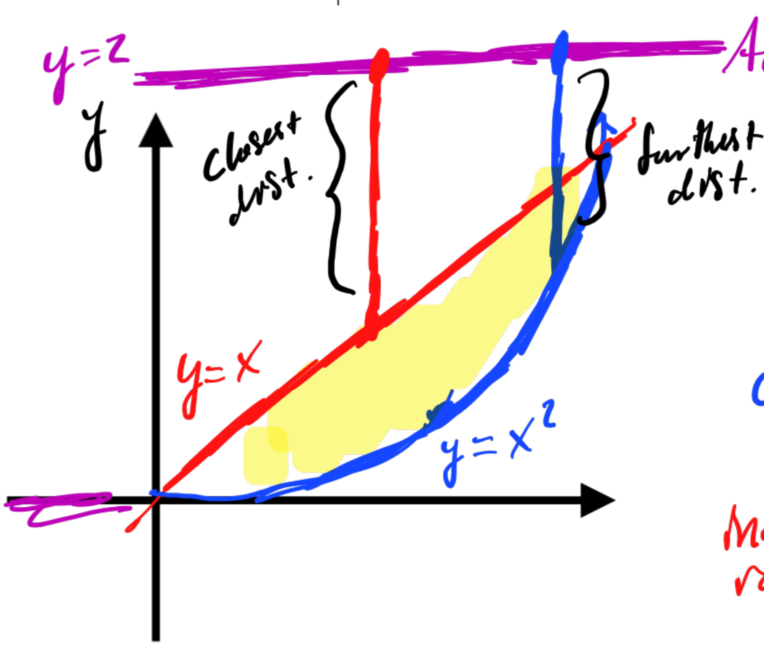
Alternatively
about a $y = \#$ line (x-axis)
So the washer wants the opp. variable slice
So $\boxed{y = \# \text{ line} \Rightarrow dx \text{ integral}}$

Q: If the AoR is the x-axis ($y = \dots$) then dx slice
AoR is ... y-axis ($x = \dots$) then dy slice?

A: ONLY for disk / washer

Example 4: Set up the volume formula V of the solid obtained by rotating the region in Example 3 about the line $y = 2$.





AoR → horiz. line.

⇒ vert. slice ⇒ dx

$$\pi \int_0^1 \left(\text{outer rad}\right)^2 - \left(\text{inner rad}\right)^2 dx$$

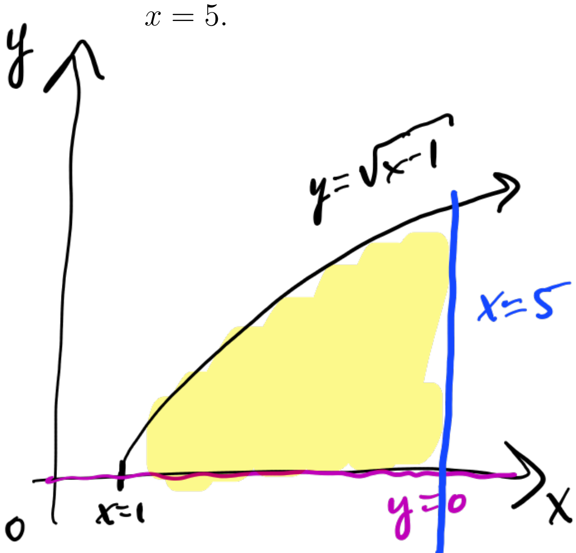
outer rad = top - bot = $2 - x^2$ (outer edge)

AoR

inner rad = top - bot = $2 - x$ (inner edge)

$$\Rightarrow \quad \pi \int_0^1 (2-x^2)^2 - (2-x)^2 \, dx$$

Problem 6.2.13. (a) Find the area bounded between the curves $y = \sqrt{x-1}$, $y = 0$, and $x = 5$.



$$\Rightarrow \int_{\text{Start}}^{\text{end}} \text{top} - \text{bot}\; dx$$

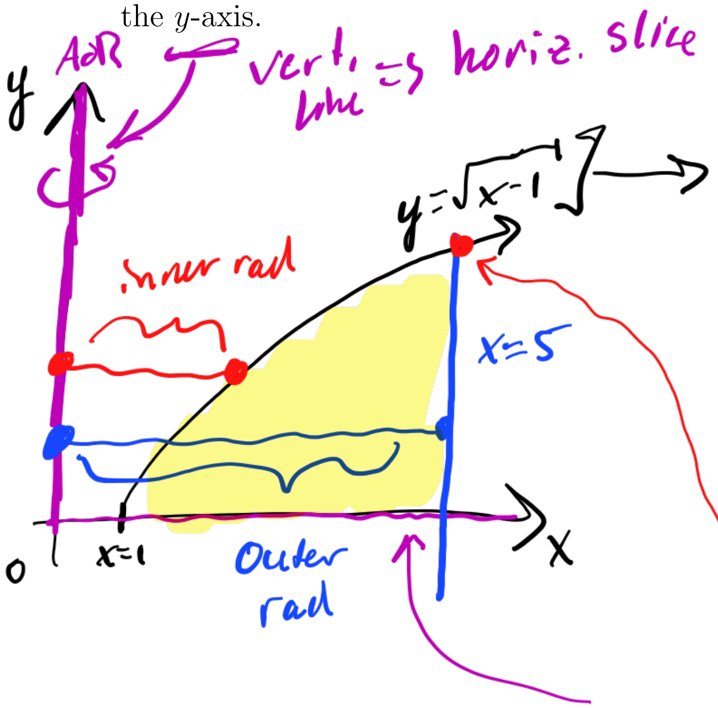
$$= \int_1^5 \sqrt{x-1} - 0\; dx$$

$$= \frac{2}{3}\left[(x-1)^{3/2}\right]_1^5 = \frac{2}{3}\left((4)^{3/2} - 1\right)$$

$$= \frac{2}{3}(8-1) = \frac{14}{3}$$

(b) Find the volume using (via washer method) generated by the area from part (a) about the y-axis.

vert. line ⇒ horiz. slice ⇒ dy ⇒ need x= ⋯ eq.



$$y = \sqrt{x-1}$$

$$y^2 = x-1$$

$$y^2+1 = x$$

AoR

$$\pi \int_?^? (\underbrace{5-0}_{\text{outer edge}})^2 - (\underbrace{y^2+1-0}_{\text{inner edge}})^2\, dy$$

Start is $y=0$ based on picture

end is whatever y-value this is

$(x=5)$ set equal to $(x=y^2+1)$

$$5 = y^2+1$$

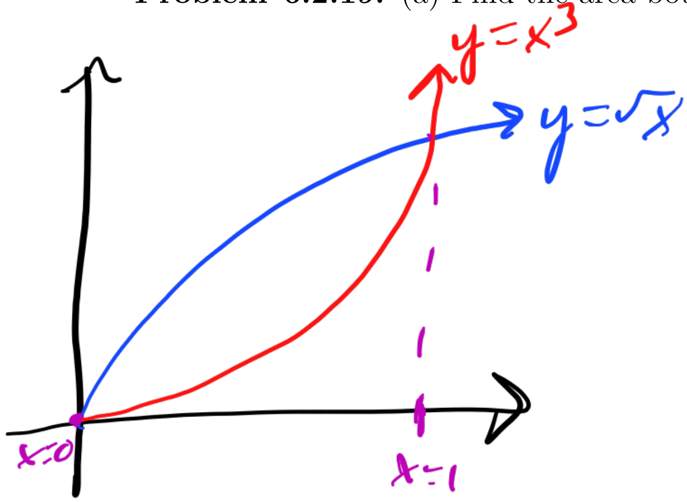
$$4 = y^2$$

$\Rightarrow \quad y = \pm 2$

But based on the picture we should choose $y = 2$

So $\Rightarrow \quad \pi \int_0^2 (5)^2 - (y^2+1)^2 \, dy$

Problem 6.2.19. (a) Find the area bounded between the curves $y = x^3$ and $y = \sqrt{x}$



$$\int_{\text{start}=0}^{\text{end } 1} \overset{\sqrt{x}}{\text{top}} - \overset{x^3}{\text{bot}}\ dx$$

$$\int_{\text{start}}^{\text{end}} \text{right} - \text{left}\ dy$$

$\sqrt{x} = x^3$

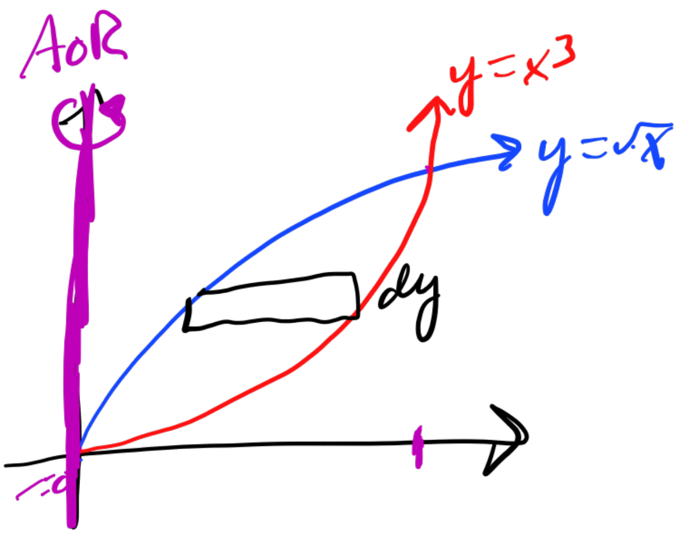
$x = x^6$

$0 = x^6 - x$

$= x(x^5 - 1)$

$x = 0$ & $x = 1$

(b) Find the volume using (via washer method) generated by rotating the area in part (a) about the y-axis.



AoR = y-axis

Vert. line (i.e. x= eq)

So the washer/disk wants perpendicular slice (i.e. opp. var. slice)

$\Rightarrow$ dy (horiz. slice)

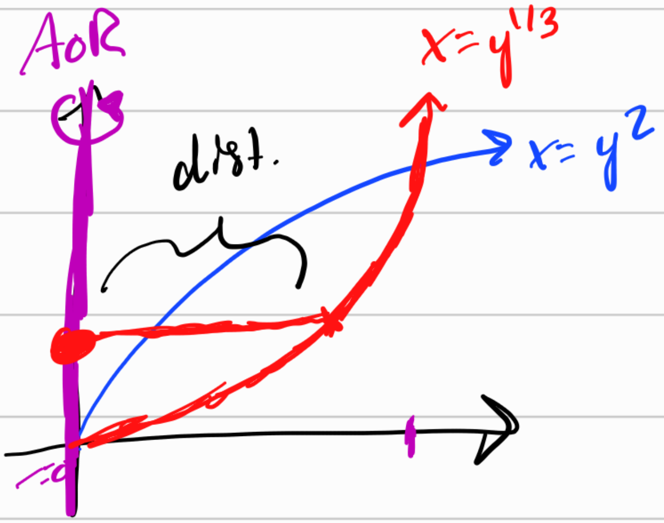
Because we're a dy integral every edge eq. needs to be put into "$x = \ldots$ form"

$y = x^3$ solve for x

$\Rightarrow y^{1/3} = x$

$y = \sqrt{x}$ solve for x

$\Rightarrow y^2 = x$



Washer

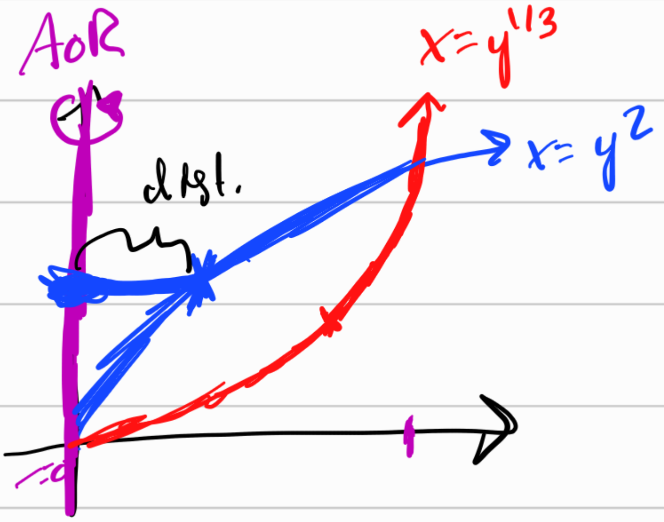
$$\pi\int_{\text{start}}^{\text{end}} \left(\begin{matrix}\text{outer}\\ \text{rad}\end{matrix}\right)^2-\left(\begin{matrix}\text{inner}\\ \text{rad}\end{matrix}\right)^2 dy$$

outer rad = furthest dist. to AoR = for "dy directions"

= right − left

= $y^{1/3}$ − 0

location of outer edge ↗ ↖ location of AoR



Inner rad = closest dist.

= y^2 − 0

location of inner edge ↗ ↖ location of AoR

$$\pi\int_{\text{start}}^{\text{end}} (y^{1/3})^2-(y^2)^2\, dy$$

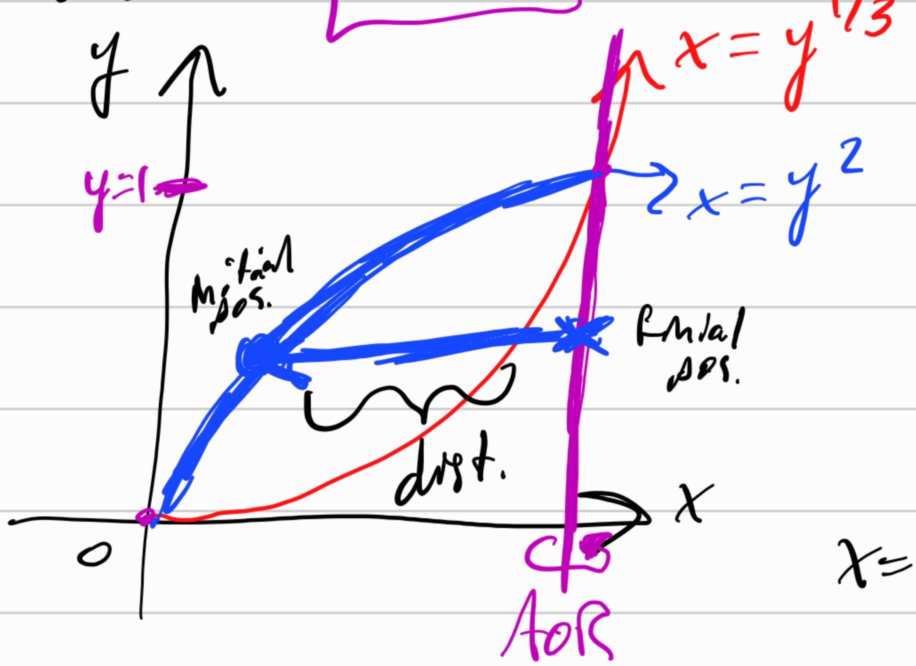
solving for start & end pts => $y^{1/3}=y^2$

$y=y^6$

$y=0$ & $y=1$ $\leftarrow$ $\left[\begin{aligned} 0 &= y^6 - y \\ &= (y^5-1)y \end{aligned}\right.$

$$\pi\int_0^1 (y^{1/3})^2 - (y^2)^2\,dy$$

What if we rotated the same region about $\boxed{x=1}$?



(Exact same Region as before)

$x=1 \Rightarrow$ vert. line

$\Rightarrow$ horiz slice $\Rightarrow$ $\boxed{dy}$

$$\pi\int_0^1 \left(\begin{matrix}\text{outer}\\ \text{rad}\end{matrix}\right)^2 - \left(\begin{matrix}\text{inner}\\ \text{rad}\end{matrix}\right)^2 dy$$

Despite changing the AoR, the region is the same so the bounds are the same.

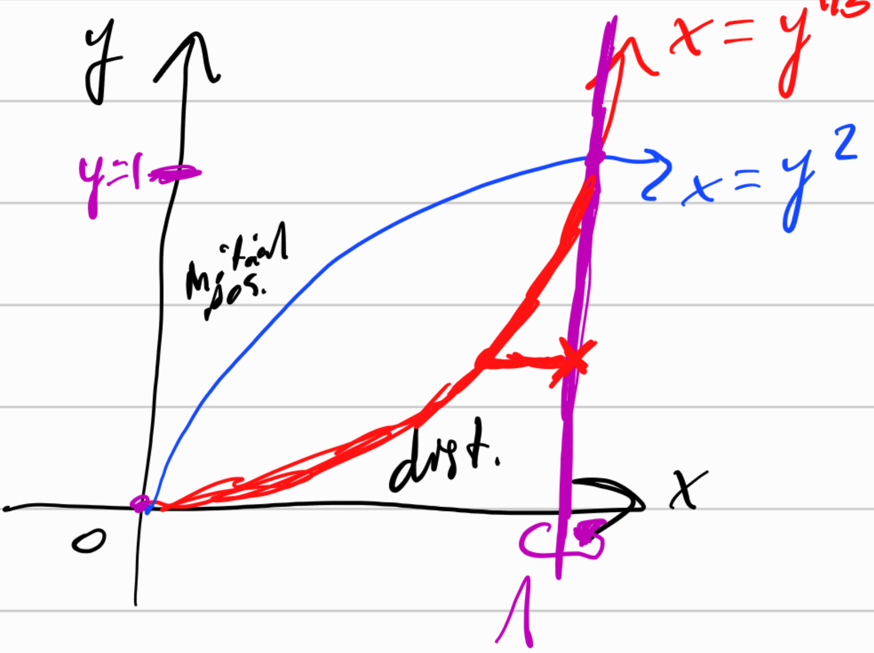
Outer rad = furthest dist. AoR = right - left (for dy)

$= 1 - y^2$

Location of AoR

location of Outer edge

Note for disk/washer / Shell any mention of "Radius" Always involves the AoR



Inner rad = Closest dist.

= right - left (dy)

$= 1 - y^{1/3}$

AoR

Inner edge.

$$\pi \int_0^1 (1-y^2)^2 - (1-y^{1/3})^2 \, dy$$

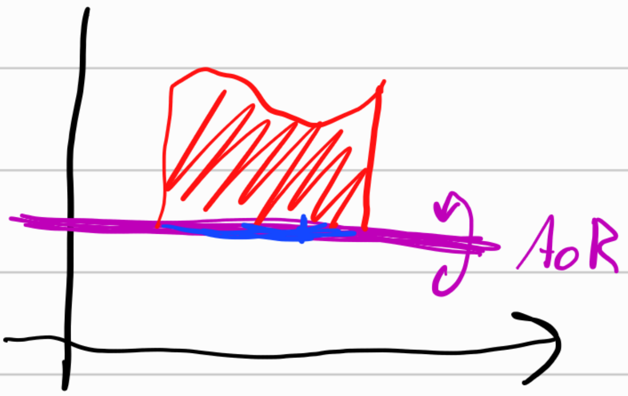
Q: How do we know if the radius is
AoR − edge vs. edge − AoR.

A: look @ the picture & follow

dy: Right − left

dx: top − bot.

The Disk is just Washer, but the AoR is the "inner edge" in this case inner rad = 0.



washer

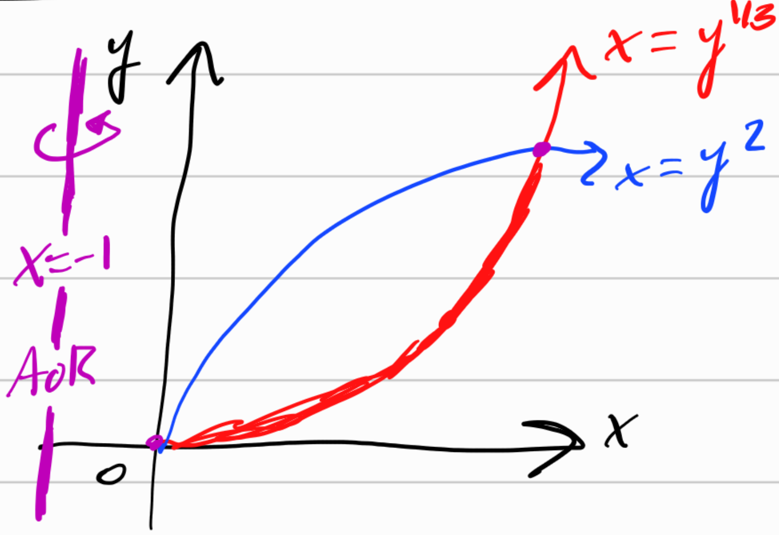
$$\boxed{\pi \int_a^b \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx}$$

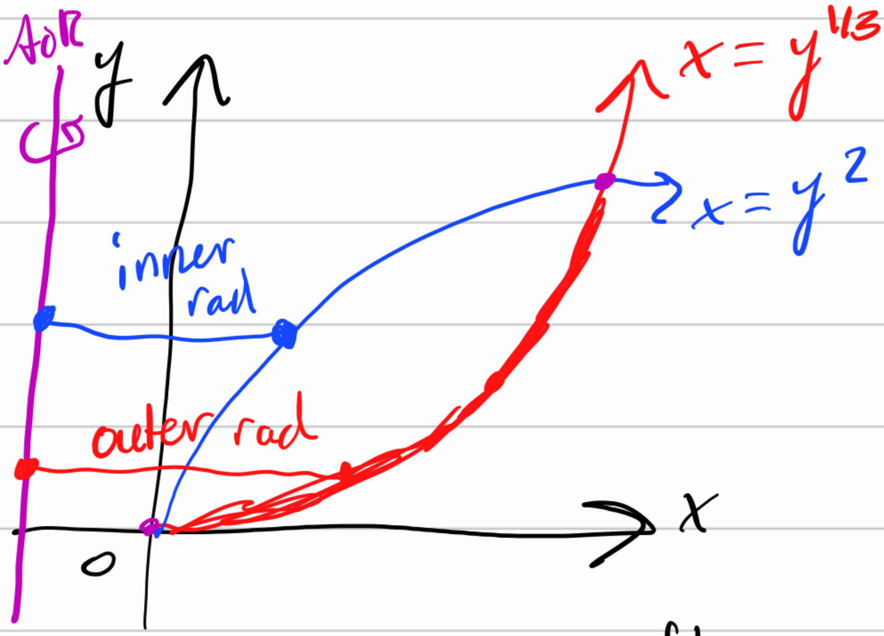
$$\Rightarrow \boxed{\pi \int_a^b \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 dx}$$

Disk method

→ Area eq are circles

What if we rotate the same region as prev. example about @ $x=-1$?



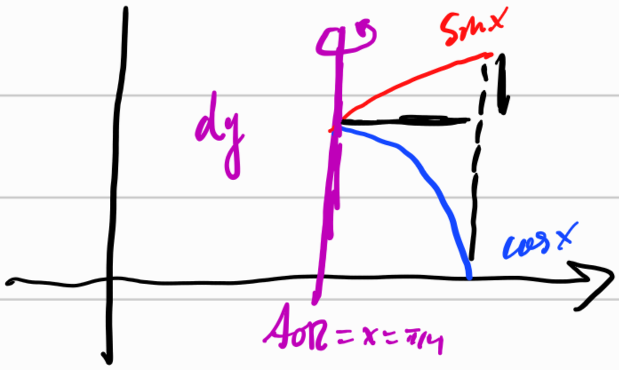


Same region as before so same bounds

$$\Rightarrow \pi \int_0^1 \Big(\underbrace{y^{1/3}}_{\text{outer edge}} - \underbrace{(-1)}_{\text{AoR}}\Big)^2 - \Big(\underbrace{y^2}_{\text{inner edge}} - \underbrace{(-1)}_{\text{AoR}}\Big)^2 \, dy$$

(right - left) (right - left)

Q: what about more complicated setup.



$y = \cos x$, $y = \sin x$
$[\frac{\pi}{4}, \frac{\pi}{2}]$
rotated @ $x = \frac{\pi}{4}$

If only we could still do a dx integral
↳ This gives a reason for the Shell method.