Section 6.2: Volume + Disk / Washer Method

Definition of Volume

Consider a solid S whose "ends" lie at $x = a$ and $x = b$. Let P_x be a plane perpendicular to the x–axis at position x. Let $A(x)$ denote the cross–sectional area of S on P_x.

$$\boxed{\text{Volume of } S = \int_a^b A(x)\,dx}$$

Here:

- $A(x)$ is an area function (area of each slice)
- a and b are the start and end of the solid

Volumes Generated by Rotation

In Calculus II, we focus on **volumes generated by rotation**. Two primary methods will be used:

- Disk / Washer Method (Section 6.2)
- Shell Method (Section 6.3)

The method name comes from the **shape of the cross–sectional slice**.

Disk Method

The disk method applies only when the region **directly touches the axis of rotation**. The radius r is always measured as the **distance from the curve to the axis of rotation**. In this scenario, each cross-section perpendicular to the axis of rotation is a **solid disk**.

Disk Method Formula

$$\boxed{V = \pi \int_a^b r^2\, d(\text{slice variable})}$$

Washer Method

A washer is formed when a region is rotated around an axis and the slice has:

- an **outer radius** r_{out}
- an **inner radius** r_{in}

$$A = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2$$

Washer Method Formula

$$\boxed{V = \pi \int_a^b \left[r_{\text{out}}^2 - r_{\text{in}}^2 \right] \, d(\text{slice variable})}$$

Important rule: The slice variable (dx or dy) must be **along the axis of rotation**.

Choosing dx vs. dy

- Rotating about a **horizontal axis** $\Rightarrow$ vertical slices $\Rightarrow dx$
- Rotating about a **vertical axis** $\Rightarrow$ horizontal slices $\Rightarrow dy$

This rule applies to the **disk and washer methods only**.

Key Reminders

- Radii are always measured **from the axis of rotation**
- radius = (furthest boundary) - (closest boundary)
- A disk is just a washer with inner radius = 0

Example 1

The region enclosed by $y = x$ and $y = x^2$ is rotated about the x–axis.

- Identify the axis of rotation
- Determine whether slices are vertical or horizontal
- Identify r_{out} and r_{in}
- Write the volume integral (do not evaluate)

Example 2

The same region is rotated about the line $y = 2$.

- Sketch the region and axis of rotation
- Identify outer and inner radii as distances to the axis
- Write the washer method integral

Practice Problems

Problem 6.2.13

(a) Find the area bounded by:

$$y = \sqrt{x-1}, \quad y = 0, \quad x = 5$$

(b) Using the washer method, set up (but do not evaluate) the volume generated by rotating the region about the y–axis.

Problem 6.2.19

(a) Find the area bounded by:

$$y = x^3 \quad \text{and} \quad y = \sqrt{x}$$

(b) Using the washer method, set up the volume obtained by rotating the region about the x–axis.