7.3: Trig Sub

The goal of this lesson is to develop another technique of integration to handle integrals that we currently cannot deal with such as $\int \sqrt{1-x^2}\,dx$ (though this may be doable with a lot of work using integration by parts). This technique will be reliant on the geometry of triangles.

Consider a circle with radius 1 centered at the origin of an (x, y)-coordinate system. The collection of (x, y) points that line on this circle satisfy the equation $x^2+y^2 = 1$. If we consider the angle θ made by a line from the origin to the circle's edge, then we define

- $\sin(\theta)$ = the y-value of the unit circle at the angle θ
- $\cos(\theta)$ = the x-value of the unit circle at the angle θ
- $\tan(\theta)$ = the slope of the line at the angle θ

By slope formula we get that

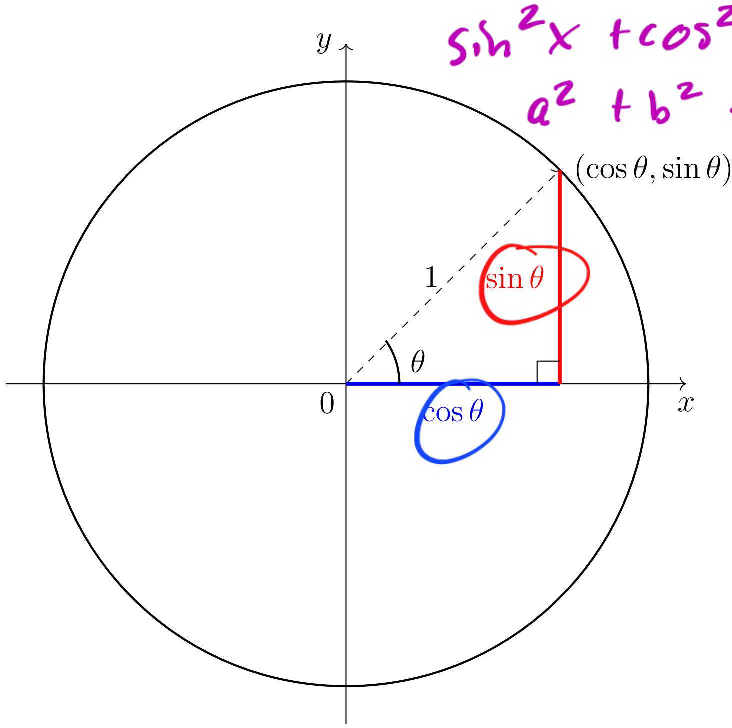
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

By definition, we have that for any θ,

$$-1 \le \sin\theta \le 1 \qquad \text{and} \qquad -1 \le \cos\theta \le 1$$

$\sin^2 x + \cos^2 x = 1$

$a^2 + b^2 = c^2$



Notationally, $\text{trig}^{\#}(\theta)$ is short hand for $(\text{trig}(\theta))^{\#}$ for any $\# \ne -1$.

Consider a circle of radius r. Then the (x, y) coordinates of this circle satisfy the equation $x^2 + y^2 = r^2$. If we divide the equation by r^2 we get the equation

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

(See picture below // Geometrically we scaled the circle down or up to the unit circle). Then by definition, the $(\frac{x}{r}, \frac{y}{r})$ coordinates of this circle satisfy the expressions

$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \cos\theta = \frac{x}{r}$$

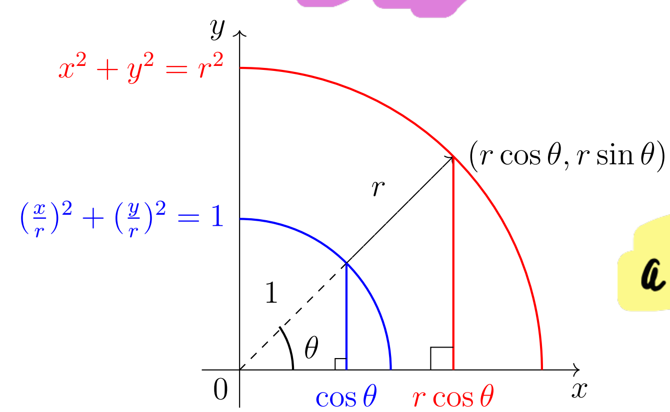
However, if we interpret this information regarding the triangles below

- $\sin\theta$ is equal to the length of the opposite side divided by the radius of the circle (i.e. the hypotenuse of the triangle).
- $\cos\theta$ is equal to the length of the adjacent side divided by the radius of the circle (i.e. the hypotenuse of the triangle).

This gives us

$$\sin\theta = \frac{opp}{hyp} \qquad \text{and} \qquad \cos\theta = \frac{adj}{hyp}$$

From the definition of Tangent we get $\tan\theta = \frac{opp}{adj}$.



$a^2 + b^2 = c^2$

Once we have the necessary tools. Consider the integral $\int \sqrt{1 - x^2}\, dx$. This can be related to a right-triangle with hypotenuse 1 and one of the legs is has length x. We can then form the substitution $\sin\theta = x$ from soh-cah-toa, and then figure out what our "dx" must be by differentiating $\sin\theta = x$ to get $\cos\theta d\theta = dx$ and so

$$\int \sqrt{1 - x^2}\, dx = \int \sqrt{1 - \sin^2\theta}\, \cos\theta d\theta$$

From here the problem reduces to material from 7.2.

"A nice example of trig-sub"

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Integral Techniques

1. (i) "Do it"
2. (ii) u-sub
3. (iii) IBP
4. (iv) ?
5. (v) Trig-sub

The idea of trig sub is to match something in the integral to:

$$\boxed{a^2 + b^2 = c^2}$$

In the integral: $\underbrace{1 - x^2}$ we can just call this a variable y^2

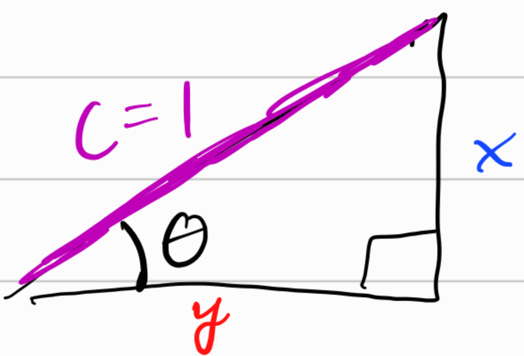
$$1 - x^2 = y^2$$

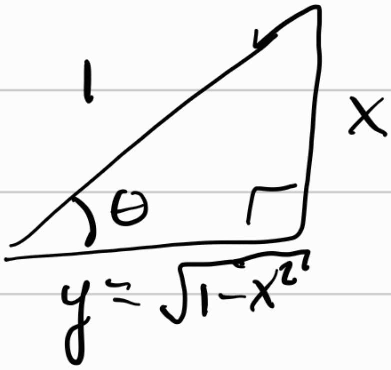
$+x^2 \qquad +x^2$

$$\boxed{1 = x^2 + y^2}$$

c^2

have the same form





$\text{trig}(\theta) = \frac{x}{\#}$

In this case $\frac{\text{opp.}}{\text{hyp}} \Rightarrow \sin\theta$

$\boxed{\sin\theta = \frac{x}{1} = x} \longrightarrow \theta = \arcsin x$

$\cos\theta \, d\theta = dx$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$\sin^2 x + \cos^2 = 1$

$\cos^2 x = 1 - \sin^2 x$

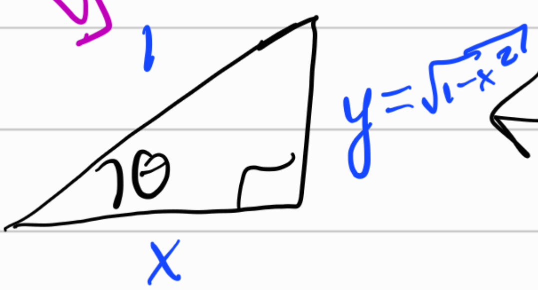
$$= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}} d\theta = \int \frac{\cos\theta}{\cos\theta} d\theta = \int d\theta$$

Started w/ dx integral need to end w/ x function

$= \theta + C = \arcsin x + C$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

This looks like a "trig-sub" integral b/c it almost has $a^2 + b^2 = c^2$. So we can "arbitrarily" set $1 - x^2 = y^2$



$\rightsquigarrow \boxed{1 = x^2 + y^2}$

$$\mathrm{trig}(\theta) = \frac{x}{H} = \frac{\text{adj}}{\text{hyp.}} \Rightarrow \cos\theta$$

$$\rightsquigarrow \quad \frac{x}{1} = \cos\theta \longrightarrow \theta = \arccos x$$

$$dx = -\sin\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-\sin\theta \, d\theta}{\sqrt{1-\cos^2\theta}} = -\int \frac{\sin\theta}{\sin\theta} d\theta = -\int d\theta$$

$$= -\theta + C = -\arccos x + C$$

Q: why does $\theta = \arccos x$?

↳ A: B/C we made the substitution

$\cos\theta = x$

Apply arccos to both sides

$$\arccos(\cos\theta) = \arccos(x)$$

$$\Rightarrow \theta = \arccos x$$

Example 3. $\int \frac{1}{\sqrt{x^2-4}}\,dx$

For trig sub the goals are

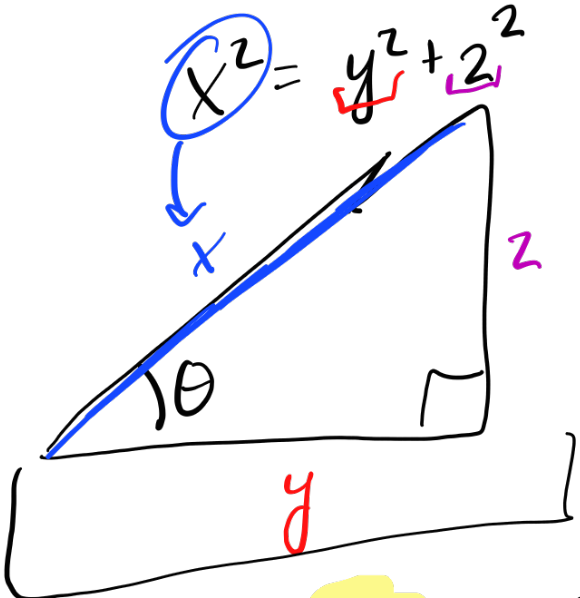
(i) Match something to $a^2+b^2=c^2$

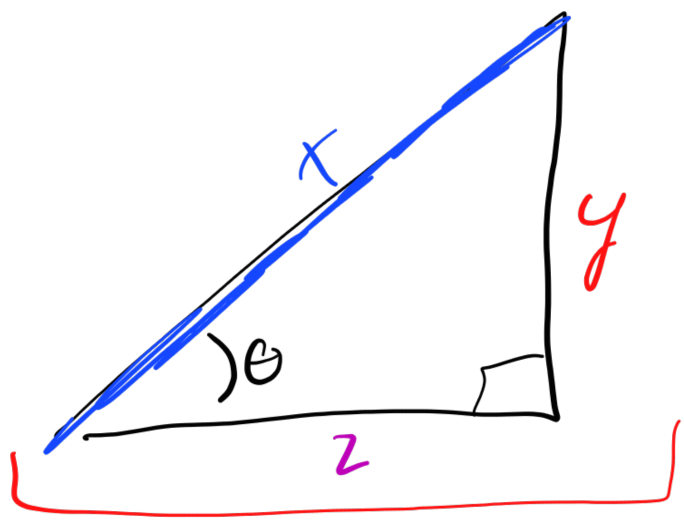
(ii) Figure out $\text{trig}(\theta) = \frac{x}{\#}$

$$x^2 - 4 = y^2$$

$$x^2 = y^2 + 4 = y^2 + (\sqrt{4})^2 = y^2 + 2^2$$

$$x^2 = y^2 + 2^2$$





from Two possible triangle what gives $\text{trig}(\theta) = \frac{x}{\#}$?

Triangle 1: $\frac{x}{\#} = \frac{\text{hyp.}}{\text{opp.}} = \frac{1}{\frac{\text{opp.}}{\text{hyp.}}} = \frac{1}{\sin\theta} = \csc\theta$

Triangle 2: $\frac{x}{\#} = \frac{\text{hyp.}}{\text{adj.}} = \frac{1}{\frac{\text{adj.}}{\text{hyp.}}} = \frac{1}{\cos\theta} = \sec\theta$

using the fraction rule $\frac{1}{\frac{a}{b}} = \frac{b}{a}$

The substitutions that work are

$x = 2\csc\theta$ & $\boxed{x = 2\sec\theta}$

Doing the trig-sub $x = 2\sec\theta$

$\rightarrow dx = 2\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{\sqrt{(2\sec\theta)^2-4}} = 2\int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{4\sec^2\theta-4}}$$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$4\tan^2\theta = 4\sec^2\theta - 4$

$$\rightarrow = 2\int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{4\tan^2\theta}}$$

$$= 2\int \frac{\sec\theta\tan\theta\, d\theta}{2\tan\theta}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

We started w/ dx integral

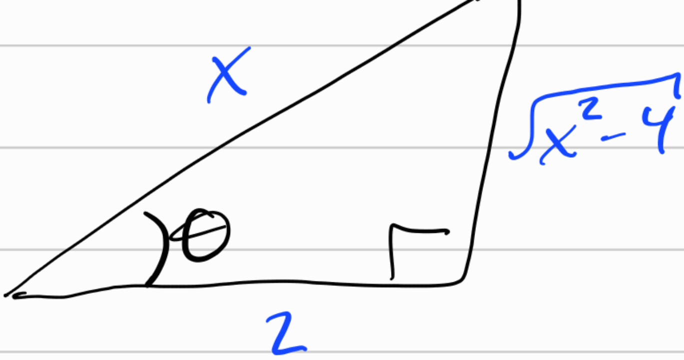
So final Answer should have x

$\hookrightarrow$ we go back to the triangle we made (or make the triangle)

$$\boxed{x = 2\sec\theta}$$

$\Rightarrow \frac{x}{2} = \sec\theta$

$\Rightarrow \frac{2}{x} = \cos\theta$



$$\boxed{\tan\theta = \frac{\text{opp.}}{\text{adj}} = \frac{\sqrt{x^2-4}}{2}}$$

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left| \frac{x}{2} + \frac{\sqrt{x^2-4}}{2} \right| + C$$

Q: What does integral need for trig-sub?

↳ A: You need something "close enough" to $a^2+b^2=c^2$

Q: What does integral need for trig-s u b?

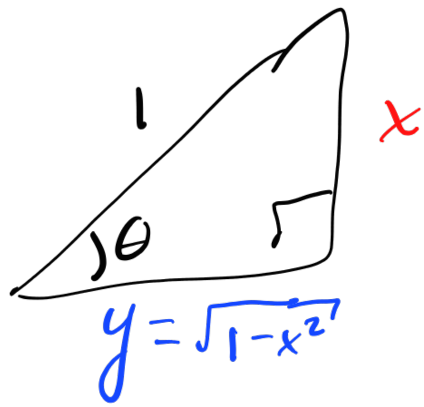
A : You need something "close enough" to $a^2+b^2=c^2$

Example 7. $\int \sqrt{1-x^2}\,dx$

$$1 - x^2 = y^2$$

$$\rightarrow 1 = x^2 + y^2$$

$$\text{trig}(\theta) = \frac{x}{1} = \frac{\text{opp.}}{\text{hyp.}}$$



$$\sin\theta = x$$

$$\cos\theta\,d\theta = dx$$

Q: why didn't I use $\cos\theta = x$

A: Didn't feel it.

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta} \cdot \cos\theta\,d\theta$$

$$= \int \sqrt{\cos^2\theta} \cdot \cos\theta\,d\theta = \int \underbrace{\cos^2\theta\,d\theta}$$

For $\boxed{\text{this kind}}$ of integral we use $\boxed{\text{Half-Angle}}$

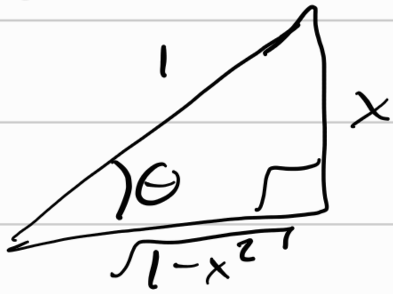
$$\int \sin^{\text{even}} \cdot \cos^{\text{even}}\,d\sim$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\int 1 + \cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\underbrace{\sin(2\theta)}\right] + C$$

$\sin(2\theta) = 2\sin\theta\cos\theta$



$$= \frac{1}{2}\left[\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$

$\boxed{\sin\theta = \frac{x}{1}}$

$\cos\theta = \frac{\sqrt{1-x^2}}{1}$

$$= \frac{1}{2}\left[\arcsin x + x\sqrt{1-x^2}\right] + C$$

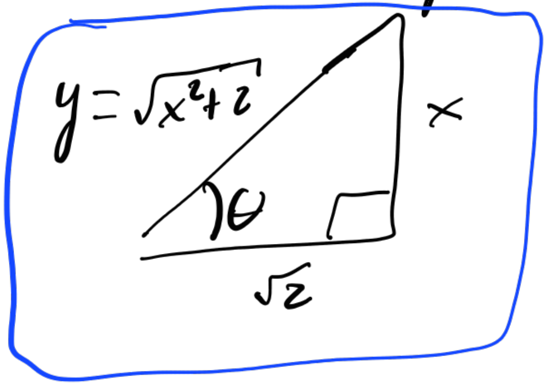
$\theta = \arcsin x$

Example 2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+2}}\,dx$

$$\begin{cases} x^2 + 2 = y^2 \\ x^2 + (\sqrt{2})^2 = y^2 \end{cases}$$

① Determine trig-sub via $\rightarrow$ or $\underbrace{\text{memorization}}$

$$\tan\theta = \frac{x}{\sqrt{2}} \iff$$



$$\sqrt{2}\tan\theta = x$$

$$\sqrt{2}\sec^2\theta\,d\theta = dx$$

$$\int \frac{dx}{x^2\sqrt{x^2+2}} = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{(\sqrt{2}\tan\theta)^2\sqrt{(\sqrt{2}\tan\theta)^2+2}}$$

Plug in trig-sub.

Pull out $\sqrt{2}$ & Simplify

$$= \sqrt{2}\int \frac{\sec^2\theta\,d\theta}{2\tan^2\theta\sqrt{\boxed{2\tan^2\theta+2}}}$$

Pythagorean id: $\tan^2\theta + 1 = \sec^2\theta$

Pull out $\frac{1}{2}$ & Simplying

$$= \frac{\sqrt{2}}{2}\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{2\sec^2\theta}}$$

Pull out $\frac{1}{\sqrt{2}}$ & Simplying

$$= \frac{\sqrt{2}}{2}\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{2}\sec\theta}$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Convert trig into sine & cosine

$$= \frac{1}{2}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

if you don't know where to go

fraction rules

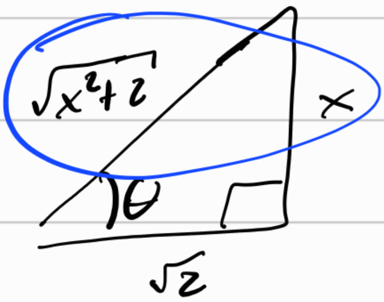
$$= \frac{1}{2}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$\boxed{u=\sin\theta \to du=\cos\theta\,d\theta}$ u-sub

$$= \frac{1}{2}\int\frac{du}{u^2} = -\frac{1}{2}\,\frac{1}{u} + C$$

$$= -\frac{1}{2}\,\frac{1}{\sin\theta} + C = -\frac{1}{2}\csc\theta + C$$

Start w/ x
end w/ x



$$\csc\theta = \frac{\sqrt{x^2+2}}{x}$$

$$= -\frac{1}{2}\,\frac{\sqrt{x^2+2}}{x} + C$$

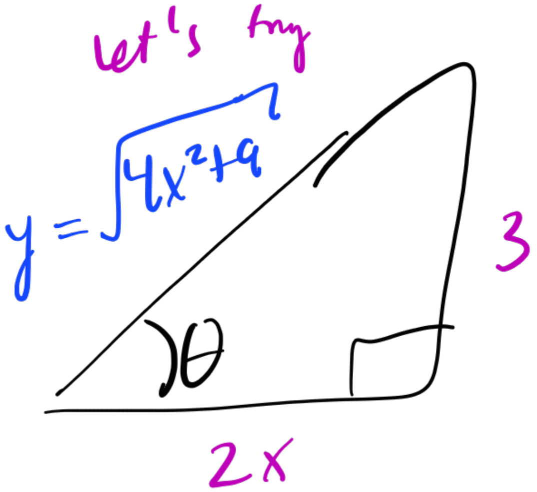
Example 4. $\displaystyle\int \frac{1}{(4x^2+9)^2}\,dx$

$$a^2+b^2=c^2$$

$$4x^2+9=y^2$$

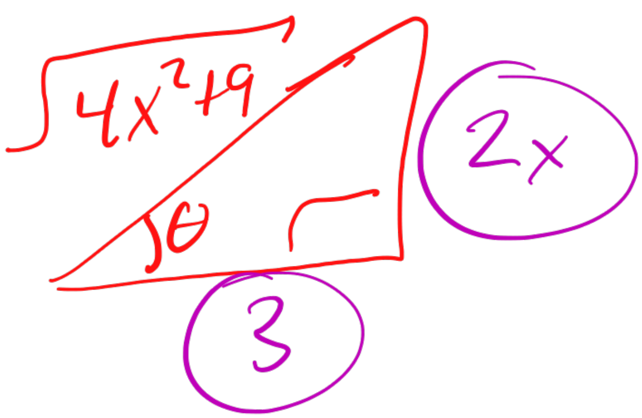
$$2^2\cdot x^2+3^2=y^2$$

$$\underbrace{(2x)^2}+3^2=\underbrace{y^2}$$



$$\text{trig}(\theta)=\frac{x}{\#}?=\frac{\text{adj}}{\text{opp.}}=\cot(\theta)=\frac{2x}{3}\rightarrow \text{ew}$$

Don't like the current trig-sub? Then just swap the legs



$$\tan\theta=\frac{2x}{3}$$

So the trig-sub to use is

$$\frac{3}{2}\tan\theta=x \quad \& \quad \frac{3}{2}\sec^2\theta\, d\theta=dx$$

$$\int \frac{dx}{(4x^2+9)^2} = \int \frac{\frac{3}{2}\sec^2\theta \, d\theta}{\left(4\cdot\left(\frac{3}{2}\tan\theta\right)^2+9\right)^2}$$

Plug in trig-sub.

$$= \frac{3}{2}\int \frac{\sec^2\theta \, d\theta}{(9\tan^2\theta+9)^2}$$

Simplify

$$= \frac{3}{2}\int \frac{\sec^2\theta}{(9\sec^2\theta)^2} d\theta$$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \frac{3}{2}\int \frac{\sec^2\theta}{81\sec^4\theta} d\theta$$

$$= \frac{3}{2\cdot 81}\int \frac{1}{\sec^2\theta} d\theta$$

$\sec\theta = \frac{1}{\cos\theta} \Rightarrow \frac{1}{\sec\theta} = \cos\theta$

$$= \frac{1}{2\cdot 27}\int \cos^2\theta \, d\theta$$

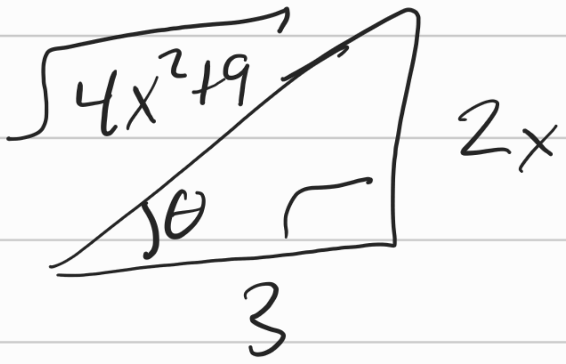
half-angle

$$= \frac{1}{54}\int \frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= \frac{1}{108}\int 1+\cos 2\theta \, d\theta$$

$$= \frac{1}{108}\left[\theta + \frac{1}{2}\underbrace{\sin 2\theta}_{\text{Double angle}}\right] + C$$

$$= \frac{1}{108}\left[\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$



$$\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$$

$$\cos\theta = \frac{3}{\sqrt{4x^2+9}}$$

$$\tan\theta = \frac{2x}{3} \Rightarrow \theta = \arctan\left(\frac{2x}{3}\right)$$

$$= \frac{1}{108}\left[\arctan\left(\frac{2x}{3}\right) + \frac{6x}{4x^2+9}\right] + C$$

This can be done w/ u-sub, IBP, or Trig-Sub.

Example 6. $\boxed{\displaystyle\int \frac{x^3}{\sqrt{1+4x^2}}\,dx}$

$1+4x^2 = y^2$

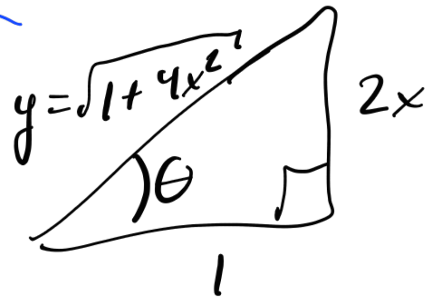
$1+(2x)^2 = y^2$

$\tan\theta = 2x$

$\frac{1}{2}\tan\theta = x$

$\frac{1}{2}\sec^2\theta\,d\theta = dx$

$\Longleftrightarrow$



$$\int \frac{x^3}{\sqrt{1+4x^2}}\,dx = \int \frac{\left(\frac{1}{2}\tan\theta\right)^3 \frac{1}{2}\sec^2\theta\,d\theta}{\sqrt{1+4\left(\frac{1}{2}\tan\theta\right)^2}}$$

Plug in trig sub

$$= \frac{1}{2}\int \frac{\frac{1}{8}\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}}$$

simplify

$\boxed{1+\tan^2\theta = \sec^2\theta}$

$$= \frac{1}{16}\int \frac{\tan^3\theta\sec^2\theta}{\sec\theta}\,d\theta$$

$$= \frac{1}{16}\int \tan^3\theta\sec\theta\,d\theta$$

u-sub w/ $u = \sec\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$\boxed{\text{$u = \tan\theta$ doesn't work}}$

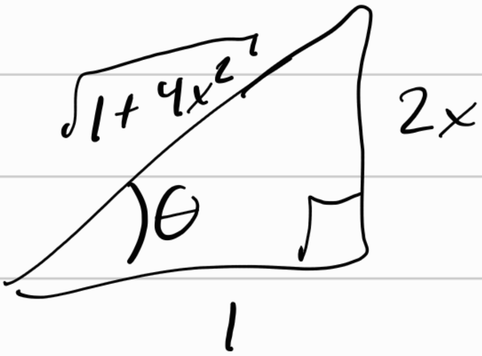
$$= \frac{1}{16}\int \underbrace{\tan^2\theta}_{\sec^2\theta - 1 \,\to\, u^2-1}\ \underbrace{\sec\theta\tan\theta\, d\theta}_{du}$$

$$= \frac{1}{16}\int u^2-1\, du$$

$$= \frac{1}{16}\left[\frac{1}{3}u^3 - u\right] + C$$

$$= \frac{1}{16}\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C$$

Start w/ x



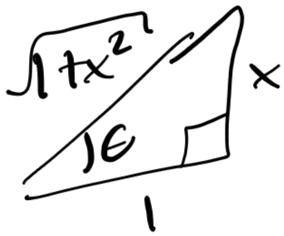
$$\sec\theta = \frac{\text{hyp.}}{\text{adj}} = \frac{\sqrt{1+4x^2}}{1}$$

$$\frac{1}{16}\left[\frac{1}{3}(1+4x^2)^{3/2} - \sqrt{1+4x^2}\right] + C$$

end w/ x

Example 8. $\int \sqrt{1+x^2}\, dx$

$$1+x^2=y^2 \Rightarrow$$

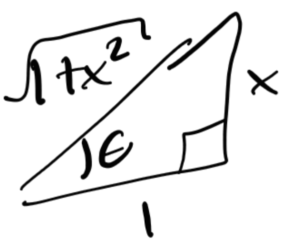


$$\iff \quad \tan\theta = x, \qquad \sec^2\theta\, d\theta = dx$$

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta$$

$$= \int \sqrt{\sec^2\theta}\,\sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$$

See next page



$$= \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

$\tan\theta = x$

$\sec\theta = \dfrac{\sqrt{1+x^2}}{x}$

$$= \frac{1}{2}\left[\sqrt{1+x^2} + \ln\left|\frac{\sqrt{1+x^2}}{x} + x\right|\right] + C$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$\int \sec^3\theta \, d\theta$ is done w/ IBP & has the "repeating trick" like $\int e^x \sin x \, dx$

$u = \sec\theta$ $dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta \, d\theta$ $v = \tan\theta$

$$\int \sec^3\theta \, d\theta = \int \sec\theta \sec^2\theta \, d\theta$$

IBP

$$= \sec\theta\tan\theta - \int \tan\theta \sec\theta \tan\theta \, d\theta$$

simplify

$$= \sec\theta\tan\theta - \int \sec\theta \tan^2\theta \, d\theta$$

trig identity

$$= \sec\theta\tan\theta - \int \sec\theta (\sec^2\theta - 1) \, d\theta$$

distribute

$$= \sec\theta\tan\theta - \underbrace{\int \sec^3\theta - \sec\theta \, d\theta}_{\text{split integral}}$$

This was always here

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \int \sec\theta \, d\theta - \int \sec^3\theta \, d\theta$$

$$+\int \sec^3\theta \, d\theta \qquad\qquad +\int \sec^3\theta \, d\theta$$

$$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$\Rightarrow \int \sec^3\theta \, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C$$