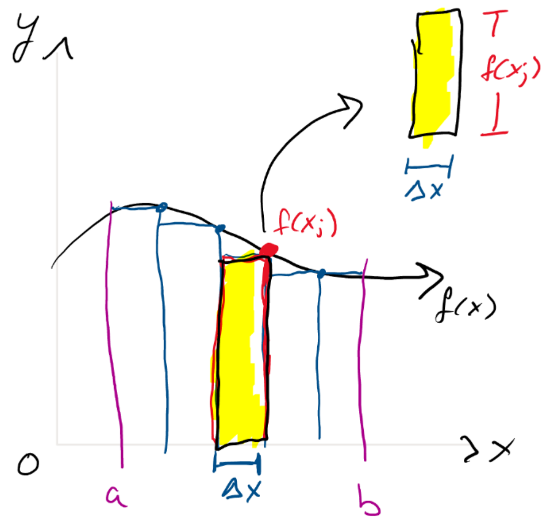
Section 6.1 Area Between Curves

When finding the area of an awkward shaped where functions can be used. We approximate it using Riemann sums. That is



$$\text{Area of } R \approx \sum_{j=1}^{n} f(x_j)\Delta x$$

where each $f(x_j)$ represents a height of our small rectangle and Δx is our width. By refining the size of our rectangles we get the integral:

$$\text{Area of } R = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x = \int_a^b f(x)\, dx$$

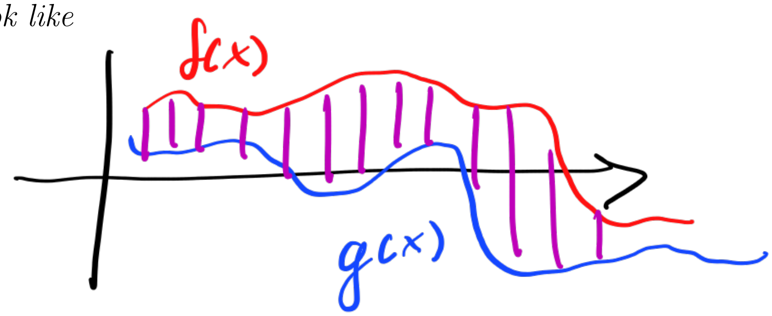
If $f(x)$ and $g(x)$ are two curves such that $f(x) \geq g(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ on the interval $[a, b]$ is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

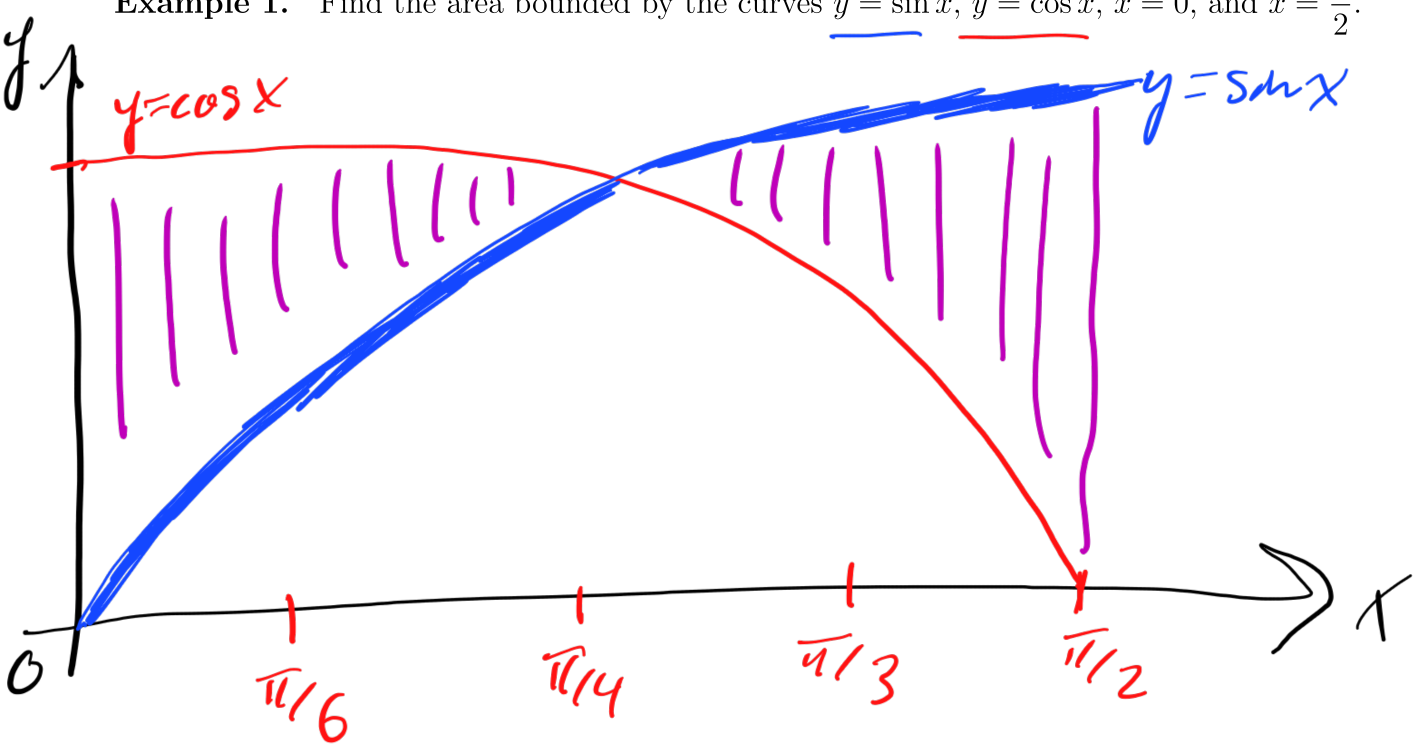
If we were to draw a picture then

- $f(x)$ would be the top function (i.e. closer to $+\infty$)
- $g(x)$ would be the bottom function (i.e. closer to $-\infty$)
- Visually, $f(x)$ is (above / **below**) $g(x)$.

A sketch *could look like*



Example 1. Find the area bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.



According to our idea of area

we should have $\int_0^{\pi/2}$ top - bot dx

but @ $\pi/4$ the top & bot functions swap

So on $[0, \frac{\pi}{4}]$
cos x is on top
sin x is on bottom

$\rightarrow \int_0^{\pi/4} \cos x - \sin x \, dx$

negatives of each other

On $[\frac{\pi}{4}, \frac{\pi}{2}]$
cos x is on bottom
sin x is on top.

$\rightarrow \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$

Note $\cos x - \sin x = -(\sin x - \cos x)$

So we could encapsulate this idea as

$$\int_0^{\pi/2} |\cos x - \sin x| \, dx$$

Since the absolute values would keep track of which function is

on top/bot
(since it makes all neg. values pos.)

$$\int_0^{\pi/4} \cos x - \sin x \, dx = \left[\sin x + \cos x\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) = \sqrt{2} - 1$$

$$\int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx = \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}$$

$$= -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = -1 + \sqrt{2}$$

So Area $= \int_0^{\pi/2} |\cos x - \sin x| \, dx = 2(\sqrt{2} - 1)$

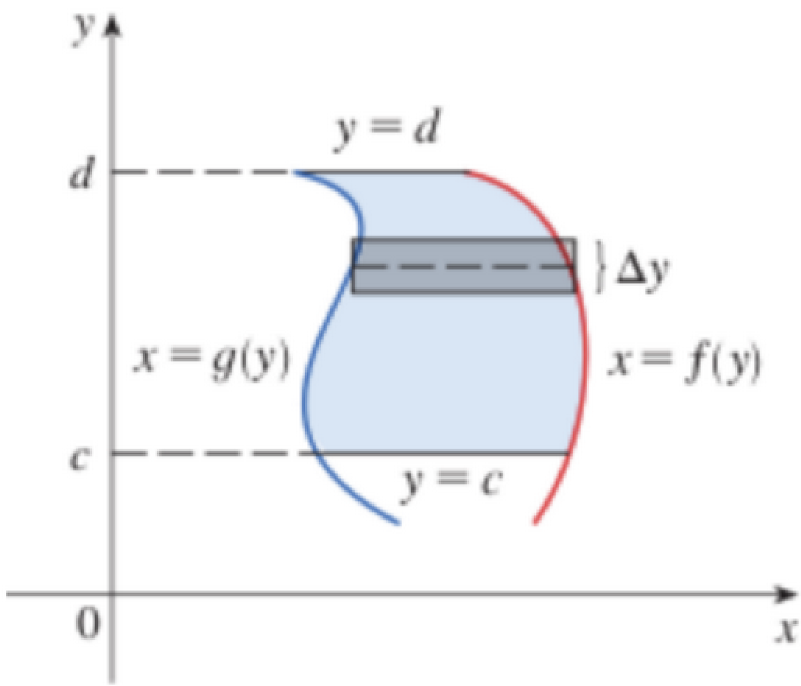
In general, **the Area between two curves** $f(x)$ and $g(x)$ on $[a, b]$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

So, what if the region can't be described as functions of x?

We can try & describe them as functions of y & then do the same math.

Consider a region as below. Here, the left and right edges are NOT functions of x, but instead can be described as functions of y. ie. The curves are described by some function $f(y)$ and some function $g(y)$.



If an area R (as above) is bounded by the curves $x = f(y)$ and $x = g(y)$ curves such that $f(y) \geq g(y)$ and the horizontal lines $y = c$ and $y = d$, then

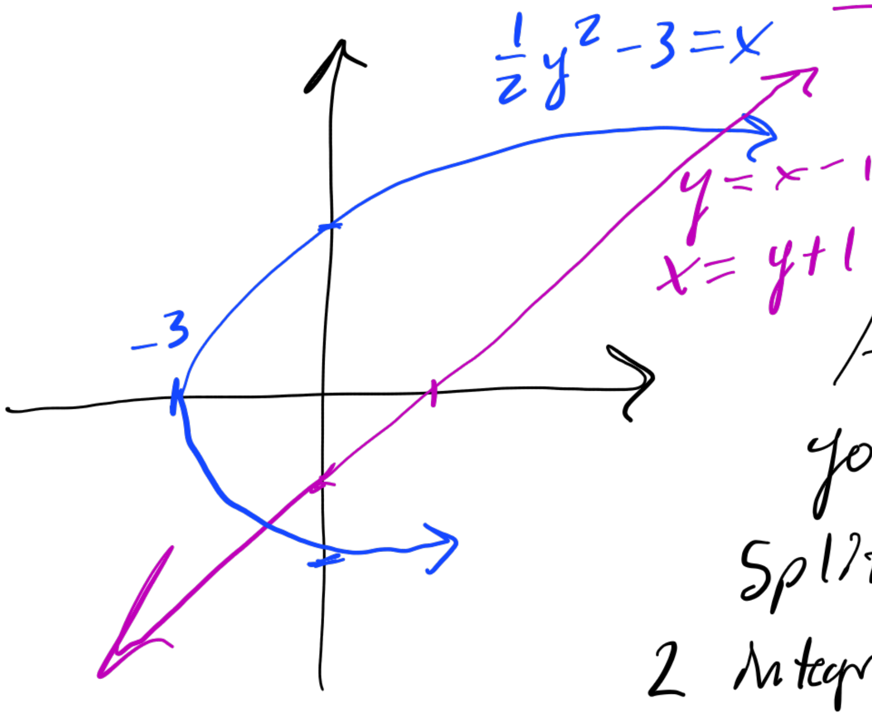
- the **right** function $f(y)$ would be the closer to ($+\infty$ / $-\infty$)
- the **left** function $g(y)$ would be the closer to ($+\infty$ / $-\infty$)
- This means that the **right function** / "more positive" function $f(y)$ plays the role of the (top / bottom)
- This means that the ~~**left**~~ **function** / "more negative" function $f(y)$ plays the role of the (top / bottom)

If an area R (as above) is bounded by the curves $x = f(y)$ and $x = g(y)$ curves such that $f(y) \geq g(y)$ and the horizontal lines $y = c$ and $y = d$, then

Area of A = $\int_c^d f(y) - g(y) \, dy$

ie. right − left

Example 2. Find the area bounded by $y = x - 1$ and $\boxed{y^2 = 2x + 6}$.



As a "dx" integral you would have to split the integral into 2 integrals for the 2 different bottoms. But for a "dy integral" it can be left as just 1 integral.

$$\int_{?}^{?} \text{right} - \text{left} \; dy$$

$$\int (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dx$$

To figure out the bounds we do as we always do. "Set 'them' equal to each other"

$$\frac{1}{2}y^2 - 3 = y+1$$

$$\frac{1}{2}y^2 - y - 4 = 0$$

$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0 \longrightarrow \text{so } y = 4 \text{ \& } y = -2$$

$$\Rightarrow \int_{-2}^{4} y+1 - \left(\frac{1}{2}y^2 - 3\right) dy$$

$$= \int_{-2}^{4} y+1 - \frac{1}{2}y^2 + 3 \, dy$$

$$= \int_{-2}^{4} y - \frac{1}{2}y^2 + 4 \, dy$$

$$= \left[\frac{1}{2}y^2 - \frac{1}{6}y^3 + 4y\right]_{-2}^{4}$$

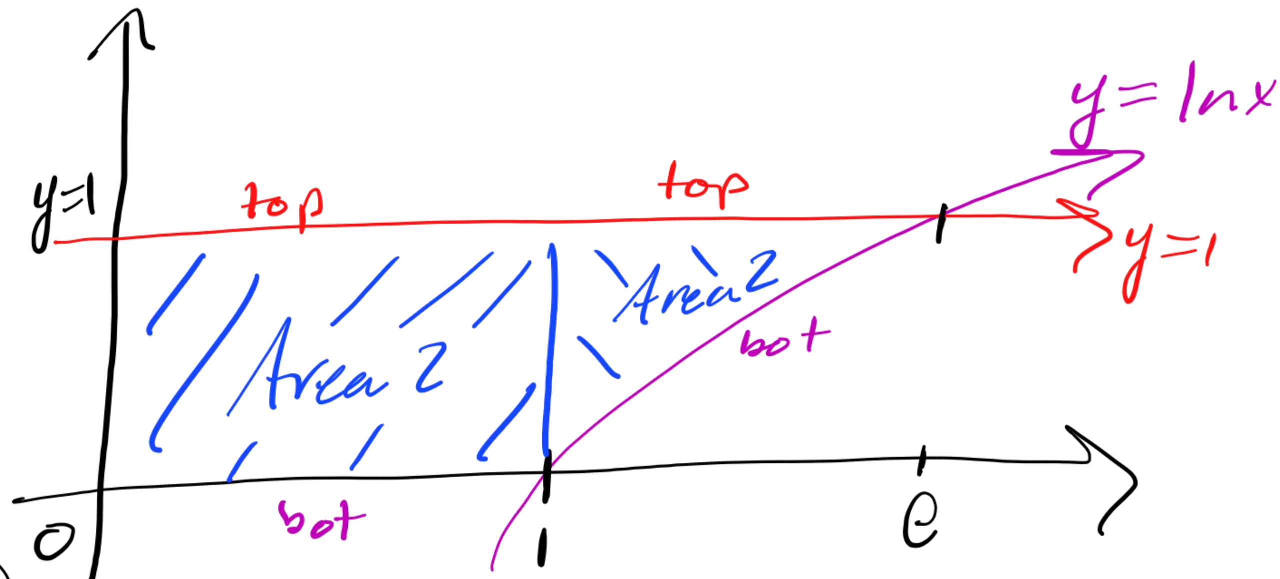
$$= \frac{1}{2}\cdot 4^2 - \frac{1}{6}4^3 + 4\cdot 4 - \left[\frac{1}{2}(-2)^2 - \frac{1}{6}(-2)^3 + 4(-2)\right]$$

$$= 4^2\left(\frac{1}{2} - \frac{1}{6}\cdot 4 + 1\right) - 4\left[\frac{1}{2} + \frac{1}{6}(-2) + (-2)\right]$$

$$= 16\left(\frac{3}{6} - \frac{4}{6} + \frac{6}{6}\right) - 4\left[\frac{3}{6} - \frac{2}{6} - \frac{12}{6}\right]$$

$$= \frac{1}{6}\left[16\cdot 5 - 4\cdot(-11)\right]$$

Example 3. Find the area bounded by $y = \ln x$ and $y = 1$, in the first quadrant.

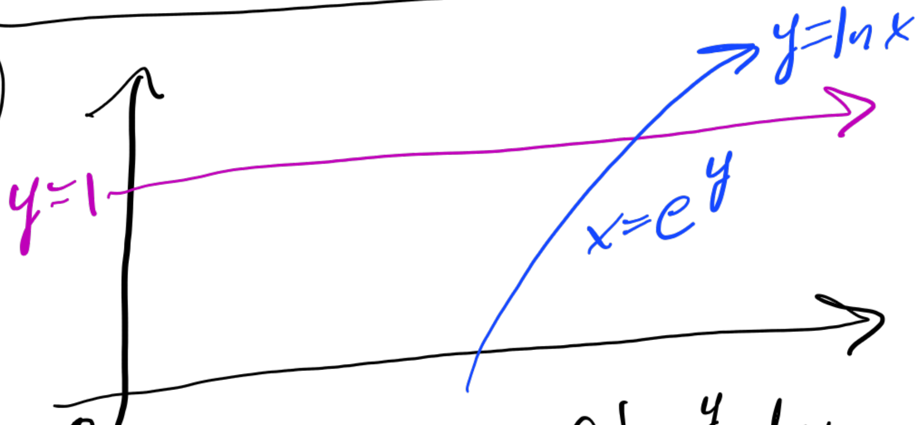


dx

Total area = Area 1 + Area 2

$$= \int_0^1 1 - 0 \, dx + \int_1^e 1 - \ln x \, dx$$

dy



$$\int_0^1 e^y - 0 \, dy = \int_0^1 e^y \, dy$$