7.2: Trigonometric Integrals

The goal is to now build techniques to handle integrals specifically involving trig function. This will give us the tools needed to do 7.3 - trig sub. The idea of 7.2 can be boiled down to 2 things:

- Trig-Identities
- u-sub

We either do one of the above things or both for integrals in this section (and whenever you see trig integrals in general).

Trig Identities

Pythagorean Identities:

I'll give you these

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

Double Angle Identities:

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

Half Angle Identities:

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

Product-to-Sum Formulas

Not for our class, but important for engineers

$$\sin A \cos B = \frac{1}{2}\Big[\sin(A-B) + \sin(A+B)\Big]$$
$$\sin A \sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$
$$\cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

not on an exam

Note in this class we won't make use of the product-to-sum formulas much, but for many who are going into engineering these are useful to have in your back pocket.

example $\int \sin^2 x \cos^3 x \, dx$

7.2 material "just a form of u-sub"

This won't work

Try $u = \underbrace{\cos x}$

$du = \underbrace{-\sin x \, dx}$

$$\int \sin^2 x \cos^3 x \, dx = \int \underbrace{\sin x}_{?} \cdot \underbrace{\cos^3 x}_{u^3} \cdot \underbrace{\sin x \, dx}_{(-du)}$$

This u-subs fail to work b/c we deal w/ sin x

→ In u-sub the star of the show is du

→ In "7.2 Integrals" we have to resolve "du" 1st.

This will work

Try $u = \sin x$

$du = \cos x \, dx$

Example 1. $\int \sin^2 x \cos^3 x \, dx = \int \underbrace{\sin^2 x}_{u^2} \cdot \underbrace{\cos^2 x}_{} \cdot \underbrace{\cos x \, dx}_{du}$

$\cos^2 x + \sin^2 = 1$

$\cos^2 x = 1 - \sin^2 x$

Trig identity

$1 - \sin^2 x$

u-sub

$1 - u^2$

$$= \int u^2 (1 - u^2) \, du = \int u^2 - u^4 \, du$$

$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Example 2. $\int \sin^5 x \cos^2 x \, dx$

If you try $u = \sin x$, $du = \cos x \, dx$ you get $\int u^5 \cos x \, du$

& the $\cos x$ doesn't go away nicely

Try $u = \cos x$, $du = -\sin x \, dx$ then $\int \sin^5 x \cos^2 x \, dx = \int \underbrace{\sin^4 x}_{} \underbrace{\cos^2 x}_{u^2} \underbrace{\sin x \, dx}_{(-du)}$

$$\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2 = (1 - u^2)^2$$

trig id: $\sin^2 x = 1 - \cos^2 x$

u-sub

$$= -\int \overbrace{(1-u^2)^2}^{\text{expand}} u^2\,du = -\int \underbrace{(1-2u^2+u^4)\,u^2}_{\text{distribute}}\,du$$

$$= -\int u^2 - 2u^4 + u^6\,du$$

distribute (now or later)

$$= \int 2u^4 - u^2 - u^6\,du$$

$$= \frac{2}{5}u^5 - \frac{1}{3}u^3 - \frac{1}{7}u^7 + C$$

$$= \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x - \frac{1}{7}\cos^7 x + C$$

Example 3. $\int \tan^6 x \sec^4 x\, dx$

won't work

try $u = \sec x$

$du = \sec x \tan x\, dx$

$$\int \underbrace{\tan^5 x}_{?} \underbrace{\sec^3 x}_{u^3} \cdot \underbrace{\sec x \tan x\, dx}_{du}$$

Didn't work try

$u = \tan x$

$du = \sec^2 x\, dx$

$$\int \underbrace{\tan^6 x}_{u^6} \underbrace{\sec^2 x}_{} \cdot \underbrace{\sec^2 x\, dx}_{du}$$

$\sec^2 x = \tan^2 x + 1$

$\tan^2 x + 1 \xrightarrow{u\text{-sub}} u^2 + 1$

$$\int \tan^6 x \sec^4 x\, dx = \int u^6 (u^2+1)\, du = \int u^8 + u^6\, du$$

$$= \frac{1}{9} u^9 + \frac{1}{7} u^7 + C = \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C$$

Example 4. $\int_0^{\pi/2} (1-\sin x)^2\,dx \xrightarrow[\text{indefinite prob.}]{\text{convert}} \int (1-\sin x)^2\,dx$

$$= \int \underbrace{1 - 2\sin x}_{\text{this is good}} + \underbrace{\sin^2 x}_{\text{this is a pain}}\,dx$$

this is a pain

↳ we can use half-angle

$\Rightarrow \boxed{\sin^2 x = \frac{1-\cos(2x)}{2}}$

$$= \int 1 - 2\sin x + \boxed{\frac{1}{2} - \frac{1}{2}\cos(2x)\,dx}$$

$$= \int \frac{3}{2} - 2\sin x - \frac{1}{2}\cos(2x)\,dx$$

u-sub step that I skipped.

$$= \frac{3}{2}x - 2(-\cos x) - \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + C$$

$$\int_0^{\pi/2} \sin^2 x\,dx = \left[\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}\right) + 2\cancel{\cos}\left(\frac{\pi}{2}\right)^{\nearrow 0} - \frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)$$

$$- \left[\frac{3}{2}\cdot \cancel{0}^{\nearrow 0} + 2\cancel{\cos}(0)^{\nearrow 1} - \frac{1}{4}\cancel{\sin}(2\cdot 0)^{\nearrow 0}\right]$$

$$= \frac{3}{4}\pi - \frac{1}{4}\cancel{\sin}(\pi)^{\nearrow 0}$$

$$-(2)$$

$$= \boxed{\frac{3}{4}\pi - 2}$$

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example 5 $\int \sin x \sec^5 x \, dx$ 2 ways

① Convert to sine & cosine's then u-sub

$$\int \sin x \sec^5 x \, dx = \int \frac{\sin x}{\cos^5 x}\, dx \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}$$

$$= \int -\frac{du}{u^5} = -\left(\frac{1}{-4} u^{-4}\right) + c$$

$$= \frac{1}{4}\frac{1}{\cos^4 x} + c = \frac{1}{4}\sec^4 x + c$$

② Convert $\sin x \cdot \sec x \longrightarrow \tan x$

$$\sin x \cdot \sec x = \frac{\sin x}{\cos x} = \tan x$$

$$\int \sin x \sec^5 x\,dx = \int \tan x \sec^4 x\,dx$$

then do 7.2 tricks

$u = \tan x$

$du = \sec^2 x\,dx$

$$\tan x \rightarrow u, \quad \sec^4 x\,dx = \sec^2 x \cdot \sec^2 x\,dx$$

$$\sec^2 x \rightarrow \tan^2 x + 1 \rightarrow u^2+1, \quad \sec^2 x\,dx \rightarrow du$$

$\Rightarrow$

$$\int u(u^2+1)\,du = \int u^3 + u\,du$$

$$= \frac{1}{4}u^4 + \frac{1}{2}u^2 + C = \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + C$$

Despite looking different Both are correct.
(Plot them on Desmos)

Example 6 $\int \sqrt{1+\cos(2x)}\,dx$

Half-angle: $\cos^2 x = \frac{1+\cos(2x)}{2} \rightarrow 2\cos^2 x = \boxed{1+\cos(2x)}$

$$\int \sqrt{1+\cos(2x)}\,dx = \int \sqrt{2\cos^2 x}\,dx$$

$$= \int \sqrt{2}\cdot\sqrt{\cos^2 x}\,dx = \sqrt{2}\int |\cos x|\,dx$$

Note $\sqrt{x^2} = |x|$

$\boxed{\text{Since } f(x)=\sqrt{x} \text{ is not a true inverse for } x^2}$

To eval. the integral further you need bounds.

Example 5. $\displaystyle\int \sin x \sec^5 x \; dx$

Example 6. $\displaystyle\int \sqrt{1+\cos(2x)} \; dx$

Example 7. $\displaystyle\int \sin(5x)\cos(4x) \; dx$

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Chap 7 Techniques of Integration

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7.2 Trigonometric Integrals

In this section, we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

Example 3: Evaluate $\displaystyle\int \tan^6 x \sec^4 x \, dx$

Done above

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7.2 Trigonometric Integrals Continued

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Example 7: Evaluate $\int \sin(5x)\cos(4x)\,dx$

$$\sin(5x)\cos(4x) = \frac{1}{2}\left[\sin(5x - 4x) + \sin(5x + 4x)\right]$$

$$= \frac{1}{2}\left[\sin(x) + \sin(9x)\right]$$

$$\int \sin(5x)\cos(4x)\,dx = \frac{1}{2}\int \sin x + \sin(9x)\,dx$$

$$= \frac{1}{2}\left[-\cos x - \frac{1}{9}\cos(9x)\right] + C$$