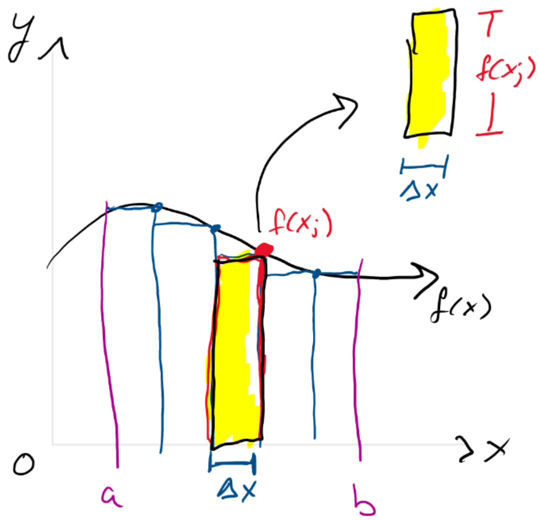
Section 6.1 Area Between Curves

When finding the area of an awkward shaped where functions can be used. We approximate it using Riemann sums. That is



$$\text{Area of } R \approx \sum_{j=1}^{n} f(x_j)\Delta x$$

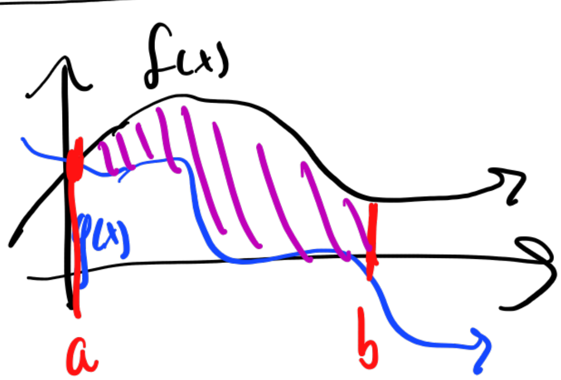
where each $f(x_j)$ represents a height of our small rectangle and Δx is our width. By refining the size of our rectangles we get the integral:

$$\text{Area of } R = \lim_{n\to\infty} \sum_{j=1}^{n} \boxed{f(x_j)\Delta x} = \int_a^b f(x)\, dx$$

simple shapes

approx.

refine until exact



$f(x) \geq g(x)$ on $[a,b]$

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

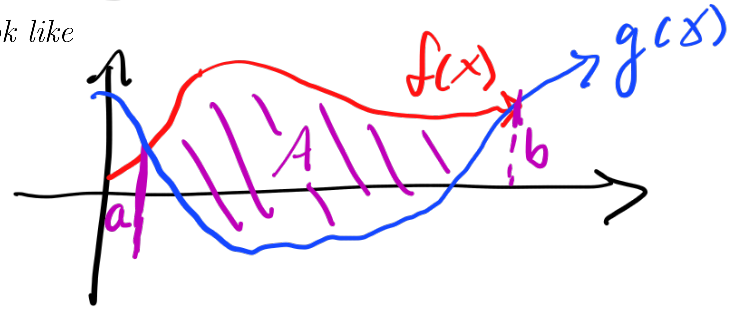
If $f(x)$ and $g(x)$ are two curves such that $f(x) \geq g(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ on the interval $[a, b]$ is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

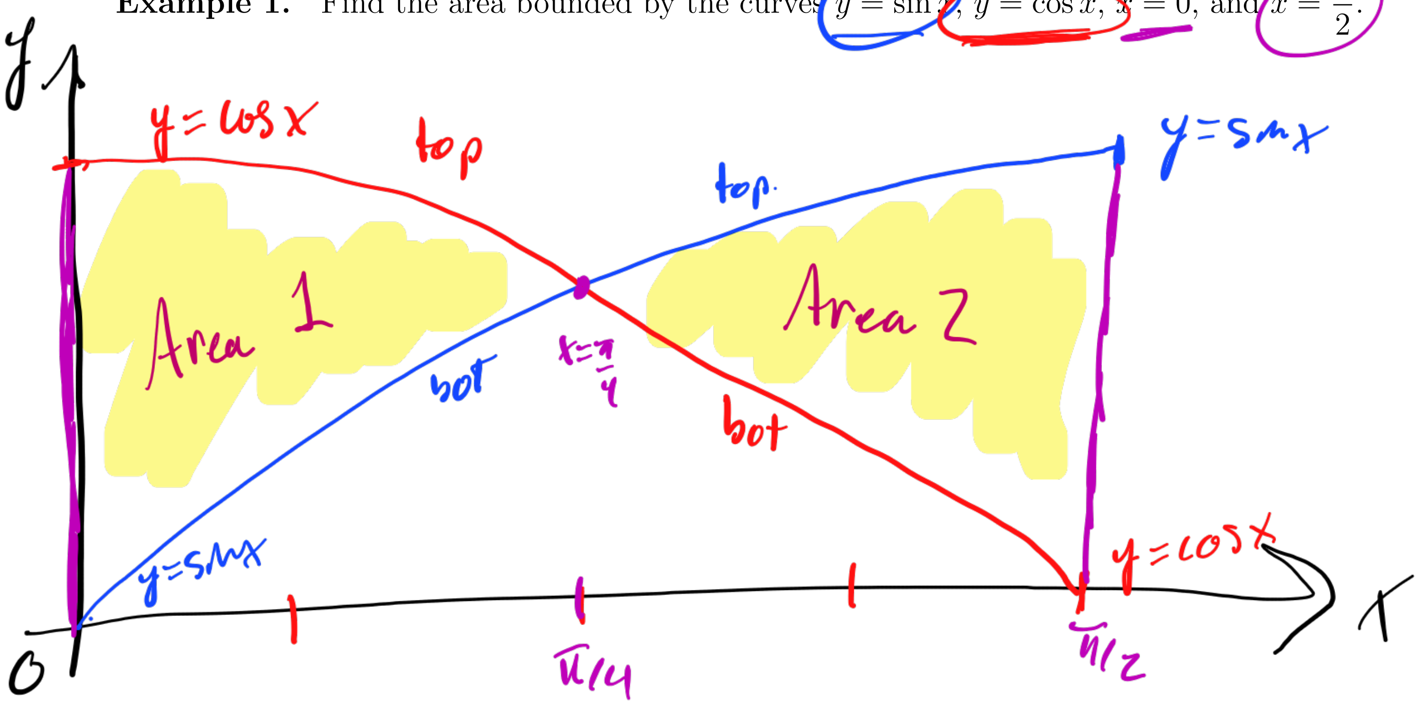
If we were to draw a picture then

- $f(x)$ would be the top function (i.e. closer to $+\infty$)
- $g(x)$ would be the bottom function (i.e. closer to $-\infty$)
- Visually, $f(x)$ is (**above** / **below**) $g(x)$.

A sketch *could look like*



Example 1. Find the area bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.



$$\text{Area 1} = \int_0^{\pi/4} \underbrace{\cos x}_{\text{"top"}} - \underbrace{\sin x}_{\text{"bot"}}\,dx$$

$$\text{Area 2} = \int_{\pi/4}^{\pi/2} \underbrace{\sin x}_{\text{"top"}} - \underbrace{\cos x}_{\text{"bot"}} \, dx$$

Note: $\cos x - \sin x = -(\sin x - \cos x)$

↳ which means we track area using a function that tracks negative signs (i.e. always make a # pos.)

⇒ absolute value

Definition — Area between $f(x)$ & $g(x)$ on $[a,b]$

$$\boxed{\int_a^b \left| f(x) - g(x) \right| dx}$$

note the absolute value bars

Going back to the cosine-sine example

$$\text{Total Area} = \int_0^{\pi/2} \left| \cos x - \sin x \right| dx$$

To actually compute this we would up the integral @ pts. where the top & bottom functions swap.

$$\int_0^{\pi/2} |\cos x - \sin x| \, dx$$

$$= \underbrace{\int_0^{\pi/4} \cos x - \sin x \, dx}_{\text{here } \cos x \text{ is on top \& } \sin x \text{ is on the bot.}} + \underbrace{\int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx}_{\text{here } \sin x \text{ is on top \& } \cos x \text{ is on the bot.}}$$

I will ask for you to compute where the functions meet.

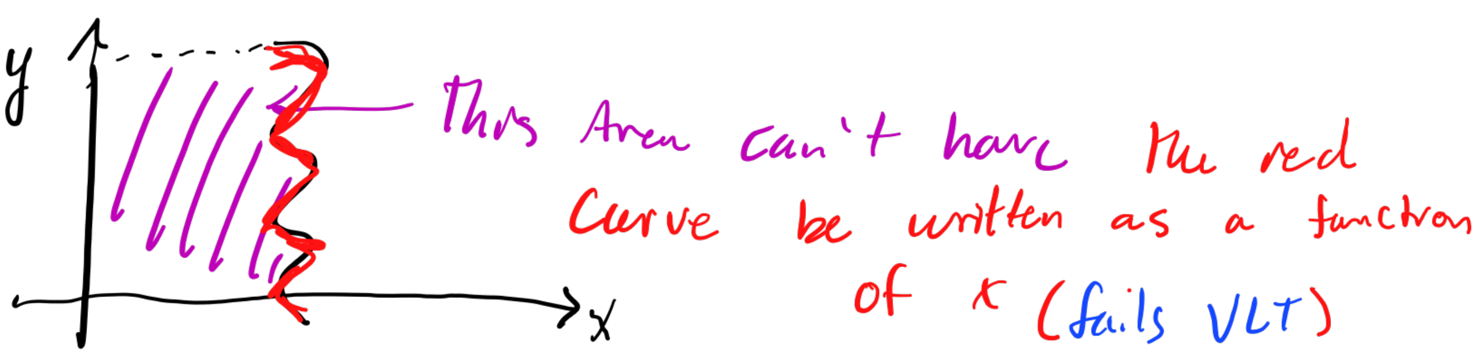
→ So on an exam you should be able to compute this point is $\pi/4$

In general, **the Area between two curves** $f(x)$ and $g(x)$ on $[a,b]$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

A lot of the time we talk about equations that are written as functions of x ie. $y = f(x)$.

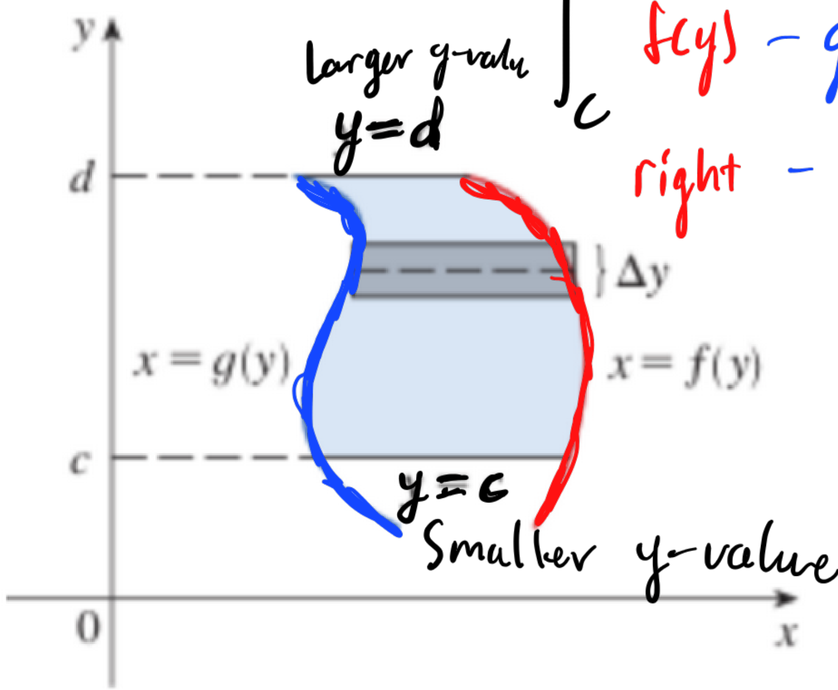
But sometimes we can't always write our equations w/ functions of x. Consider:



Consider a region as below. Here, the left and right edges are NOT functions of x, but instead can be described as functions of y. ie. The curves are described by some function $f(y)$ and some function $g(y)$.

$\int_c^d f(y) - g(y)\, dy$

right − left



If an area R (as above) is bounded by the curves $x = f(y)$ and $x = g(y)$ curves such that $f(y) \geq g(y)$ and the horizontal lines $y = c$ and $y = d$, then

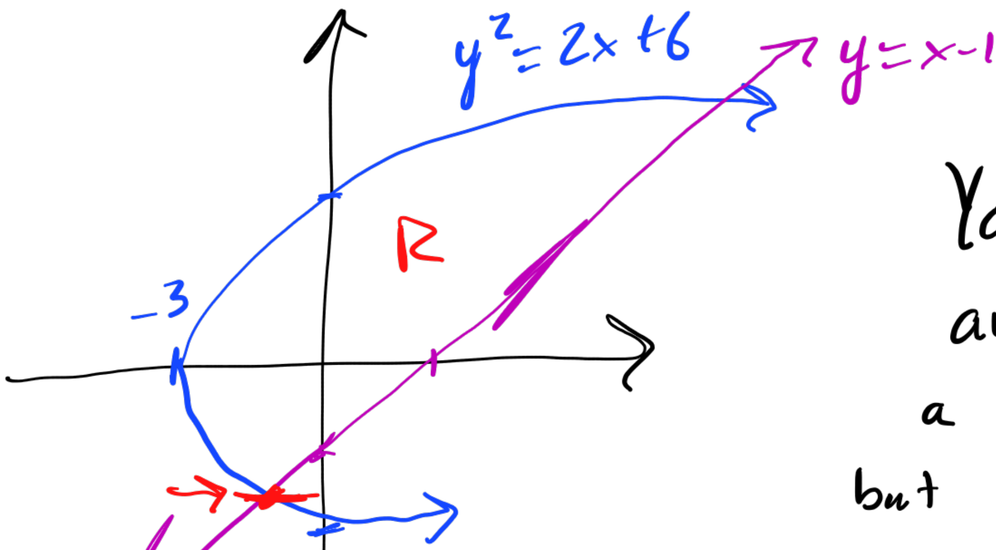
- the **right** function $f(y)$ would be the closer to ($+\infty$ / $-\infty$)
- the **left** function $g(y)$ would be the closer to ($+\infty$ / $-\infty$)
- This means that the **right function** / "more positive" function $f(y)$ plays the role of the (top / bottom)
- This means that the ~~**left**~~ **function** / "more negative" function $f(y)$ plays the role of the (top / bottom)

If an area R (as above) is bounded by the curves $x = f(y)$ and $x = g(y)$ curves such that $f(y) \geq g(y)$ and the horizontal lines $y = c$ and $y = d$, then

Area of A = $\int_c^d f(y) - g(y)\, dy$

ie. right − left

Example 2. Find the area bounded by $y = x - 1$ and $\boxed{y^2 = 2x + 6}$.



The bottom function (for dx integral) changes

You can find area of R using a "dx" integral, but you would need 2 of them

As "dy" integral we only need 1 integral

$$\int_{?}^{?} \text{right} - \text{left} \; dy$$

$y = x - 1 \rightsquigarrow$ convert this into "$x = f(y)$"

$y + 1 = x$

$y^2 = 2x + 6 \rightsquigarrow$ convert this into "$x = g(y)$"

$$\frac{y^2-6}{2} = x$$

$$\int_{?}^{?} (y+1) - \left(\frac{y^2-6}{2}\right) dx$$

$\boxed{\text{To find the bounds we just "set the eq equal to each other \& solve" or look @ the picture}}$

Setting the 2 functions equal & solving:

$$y+1 = \frac{y^2-6}{2}$$

$$2y+2 = y^2-6$$

$$0 = y^2-2y-8$$

$$0 = (y-4)(y+2)$$

$$\Rightarrow \quad y=-2 \quad \& \quad y=4$$

So the integral is

$$\int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy$$

Q: How do we know if a function is more pos. vs. neg.

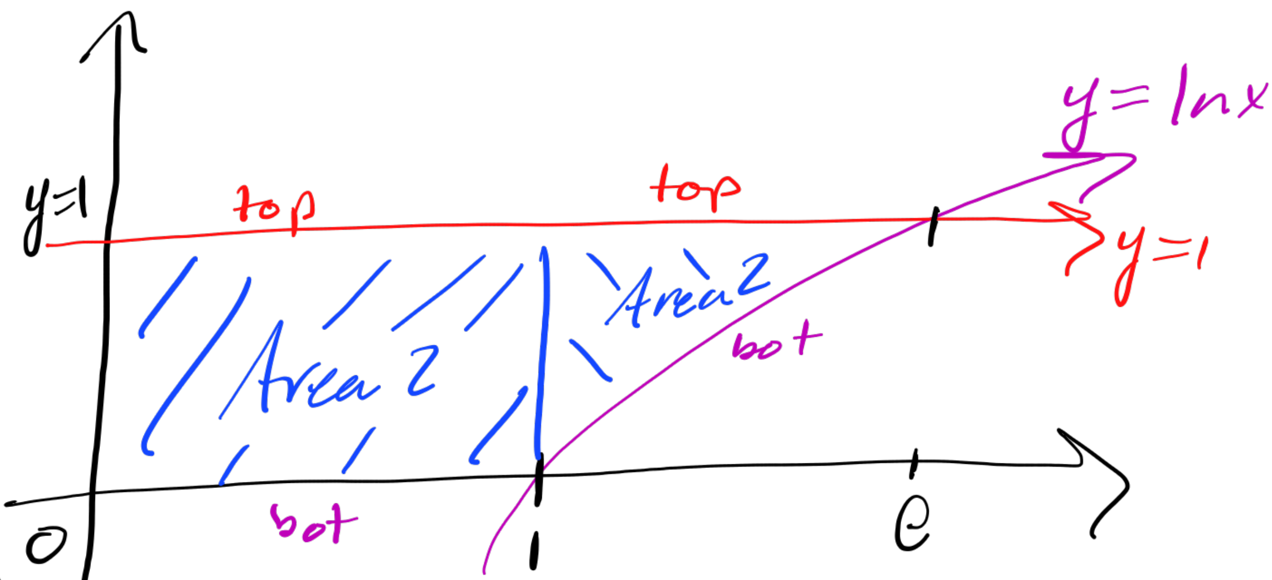
For "dx" integral higher (top) → pos.
lower (bot.) → neg.

For "dy" integral right → pos.
left → neg.

I Reccomend a picture Always for all Chapter 6 problems.

Example 3. Find the area bounded by $y = \ln x$ and $y = 1$, in the first quadrant.

As a dx integral, this would be an annoying problem since @ $x=1$ the bottom function changes which means 2 integrals

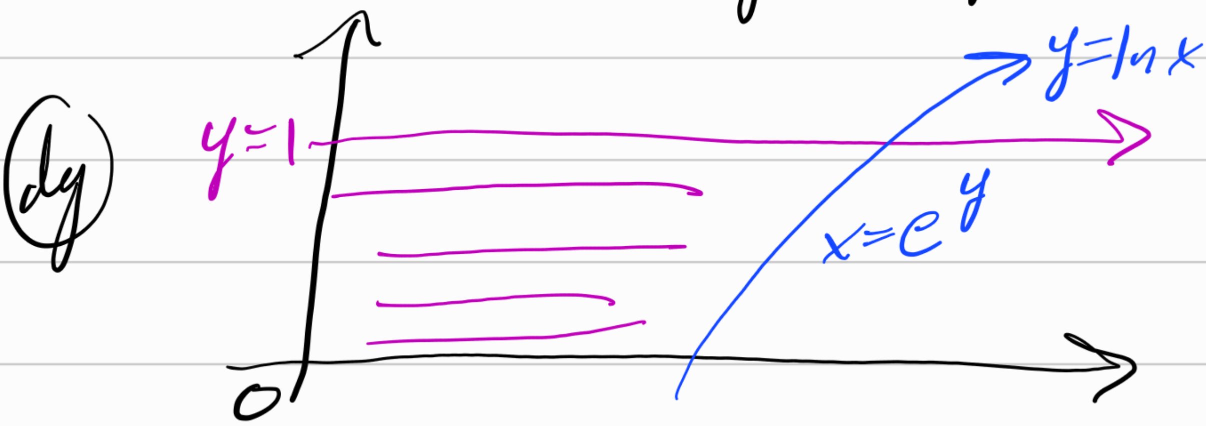


dx

Total area = Area 1 + Area 2

$$= \underbrace{\int_0^1 1 - 0 \, dx}_{\text{This integral is fine}} + \underbrace{\int_1^e 1 - \ln x \, dx}_{\text{This is IBP}}$$

If we do a dy integral



We only have 1 area to worry about

- The right function is $y = \ln x$
  converted $\longrightarrow x = e^y$
- The left function is $x = 0$
  (i.e. the y-axis)

The region starts @ $y=0$ & ends @ $y=1$
(from the picture)

So,

$$\int_0^1 e^y - 0 \, dy = \underbrace{\int_0^1 e^y \, dy}_{\text{Super nice integral.}}$$