§ 10.1 Parametric Equations

A parametric system is a system of n equation (or variables) that depend on AT MOST n-1 variables

eg 2-variable w/ 1-Parametric input

$\boxed{\text{Our Main thing in Calc 2}}$

$$x = f(t) \quad \& \quad y = g(t)$$

ie. The variables x & y are described by the variable t.

Often t is thought as "time"

equiv. we can write $x(t)$ & $y(t)$.

→ example "Path of a particle on 2D grid"

eg 3-variable system w/ 1-Parametric input (calc 3)

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

ie. $x(t)$ $y(t)$ $z(t)$

eg 3-Variable system w/ 2-Parameter input

$$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$$

⇒ This describes a surface in a 3D space.

Keep in Mind for any of these to be a parametric system they all have to be described by a common set of variables.

Some special terminology for this stuff

- If a parametric has exactly 1 parametric input
  ↳ We call a "(parametric) curve"
  ↳ This is how we classify "1-D objects"
- If a paramatric system has exactly 2 inputs
  ↳ call this a "surface"
  ↳ 2-D object
- If a parametric system n inputs
  ↳ call this "n-manifold"
  ↳ n-D object.

10.1: Parametric Curves

Up until now we have only talked about the situation in which 1 variable is described as a function of another variable $y = f(x)$, $x = g(y)$, $r(\theta)$, etc. However, in a lot of cases, given 2 variables, its quite unlikely we will be able to do this. The simplest example is $x^2 + y^2 = 1$, the unit circle. However, there is a way to describe the equation of a circle using only *1 variable*. This idea gives us parametric equations and paves the way for calc 3 / vector calculus.

Parametric Equations:

The variables x and y are given by a **parametric system of equations** if there is a 3rd variable t such that

$$x = f(t) \qquad \text{and} \qquad y = g(t).$$

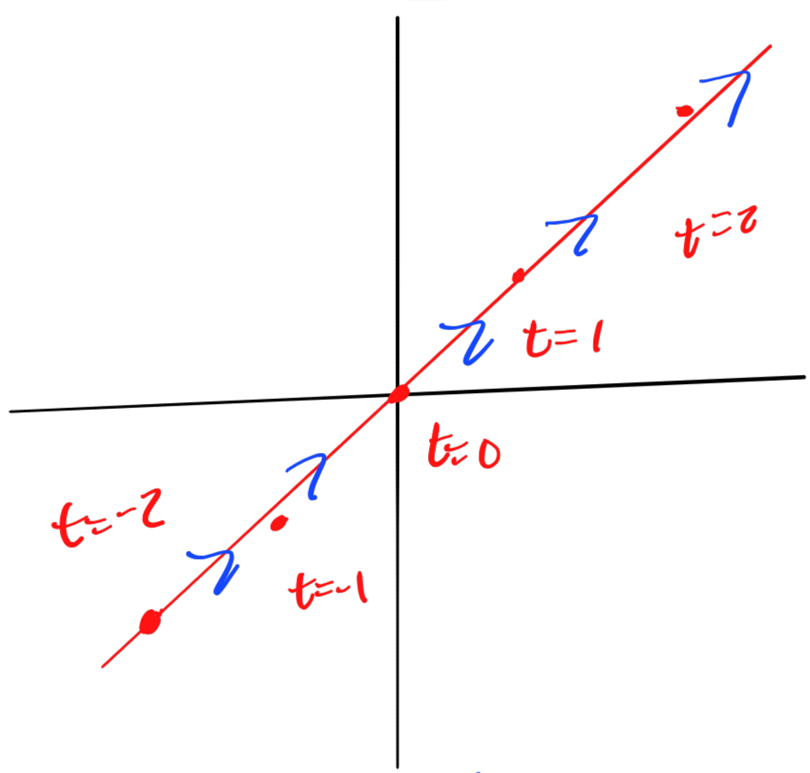
This equivalent to saying:

- A **parametric description of x and y**
- A **parametrization of x and y by t**
- **x and y are given parametrically by t**

In some sense, a parametric description of the variables x and y gives a sense of following the "*path over time*" that x and y sketch out.

Example 1. What is the function that the parametric equations $x = t$ and $y = t$ describe?

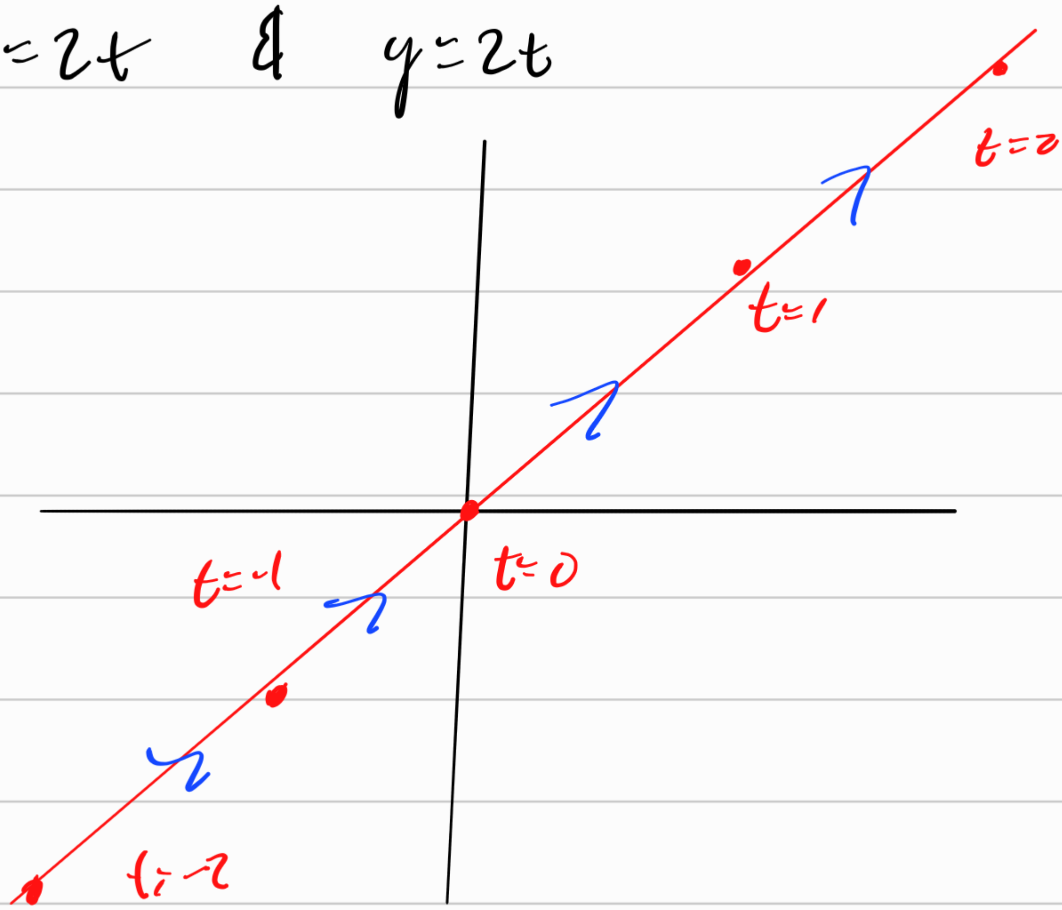
t	x	y
0	0	0
1	1	1
2	2	2
-1	-1	-1
-2	-2	-2



Note Parametric curves have a direction

Consider $x = 2t$ & $y = 2t$

t	x	y
0	0	0
1	2	2
2	4	4
-1	-2	-2
-2	-4	-4



Note this is the same line as before, but "We go through it faster"

Parametric curves have both a "sense of speed & direction" → forms the basis for vector calc. (ie. Calc 3)

Q: How do we reduce Parametric system (in Calc 2) to one (x,y) equation?

A: Either its doable or impossible

Example 2. What is the graph of the parametric curve $x = t^2 - 1$ and $y = t - 1$? Can you rewrite this system as 1 equation with only 2 variables instead of 2 equations with 3 variables?

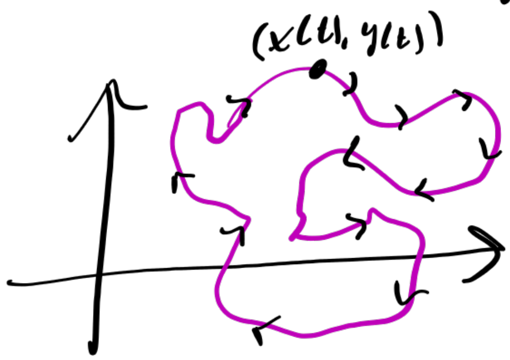
let's take $x = t^2 - 1$ & $y = t - 1$

& reduce it to an (x, y) equation

$\Rightarrow$ we have to solve for t in one of the equations.

NOTE: This may not be possible

Since A key feature about parametric system is that we can described objects which can't be represented by a function.



Absolutely no way to write an (x, y) function, But we can write a parasystem as $(x(t), y(t))$

$x = t^2 - 1$ & $y = t - 1$

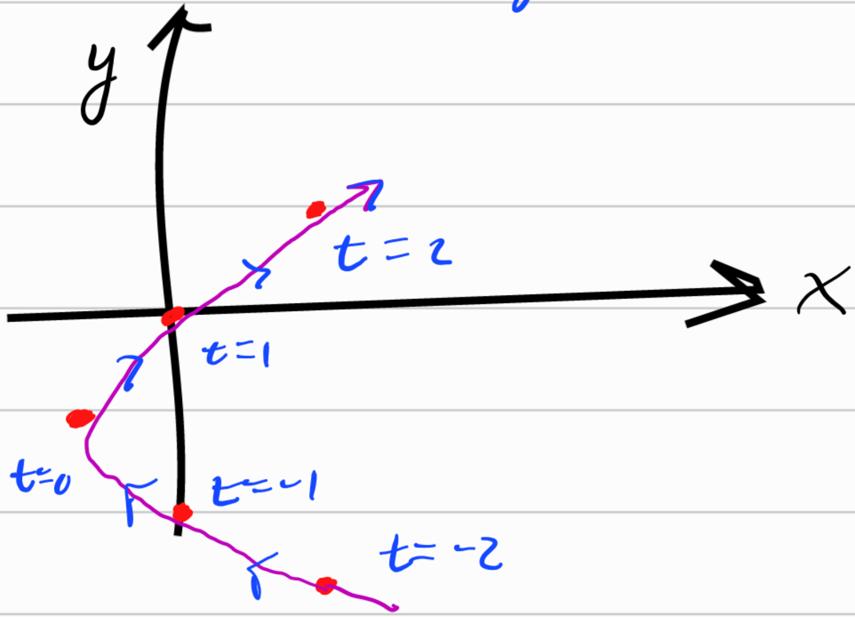
$y + 1 = t$

$x = (y+1)^2 - 1$

Back to the plotting question:

t	x	y
0	-1	-1
1	0	0
2	3	1
-1	3	-2
-2	0	-3

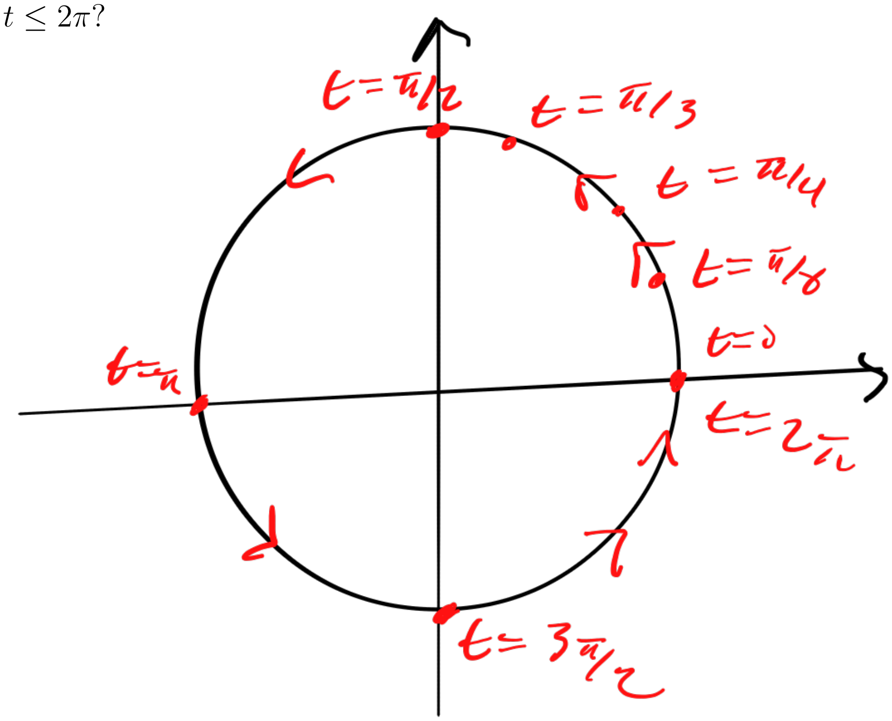
$x = t^2 - 1$ & $y = t - 1$



Example 3.

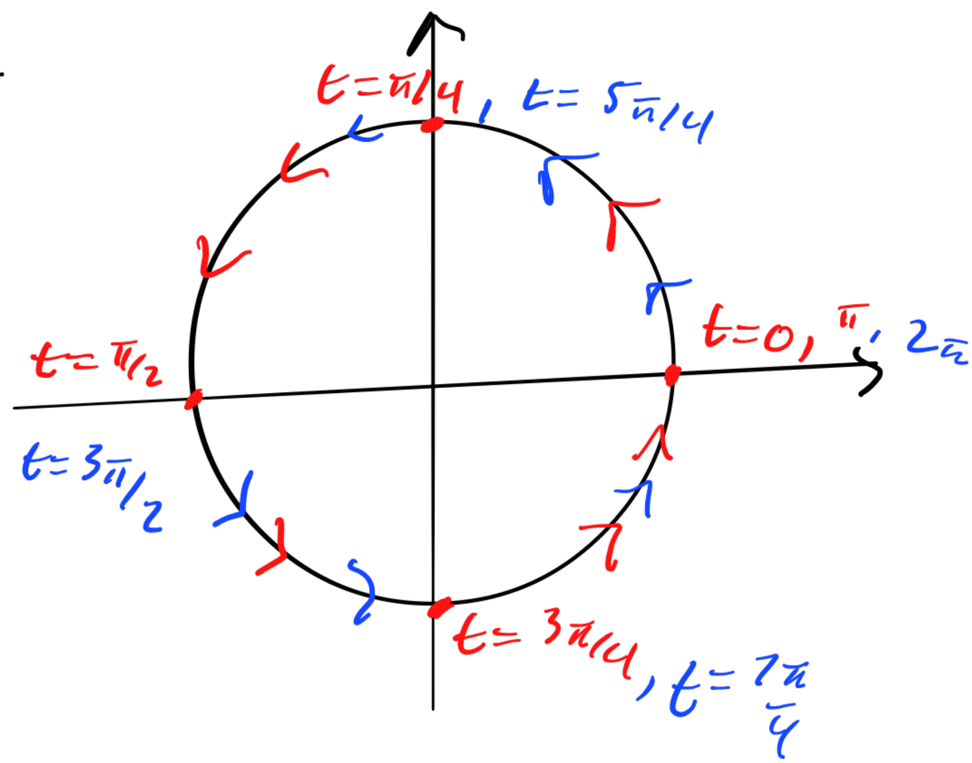
(a) What is the shape that the parametric equations $x = \cos(t)$ and $y = \sin(t)$ describe on the interval $0 \le t \le 2\pi$?

t	x	y
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1



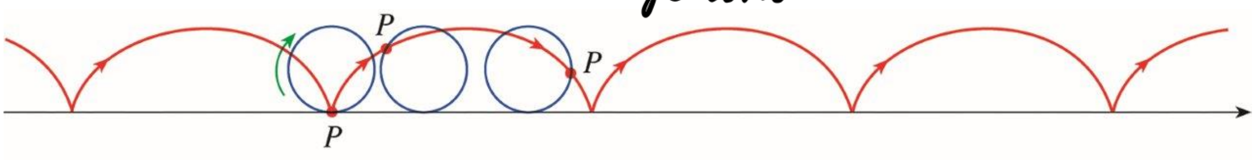
(b) What do the parametric equations $x = \cos(2t)$ and $y = \sin(2t)$ describe on the interval $0 \le t \le 2\pi$?

t	$2t$	x	y
0	0	1	0
$\pi/4$	$\pi/2$	0	1
$\pi/2$	π	-1	0
$3\pi/4$	$3\pi/2$	0	-1
π	2π	1	0

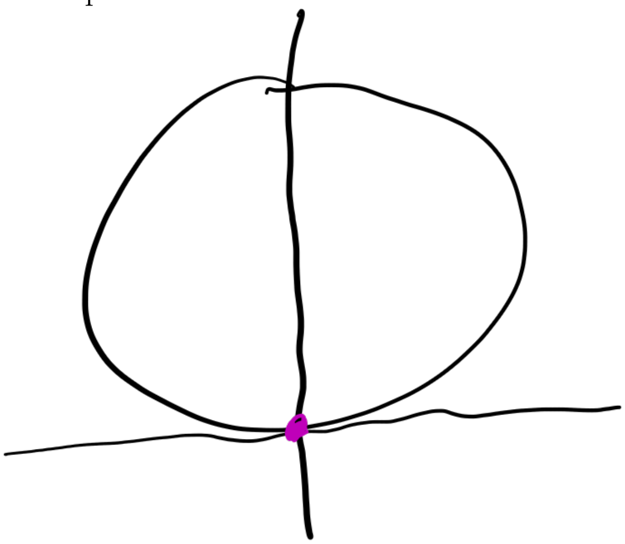


i.e. we go twice around the unit circle

cycloid

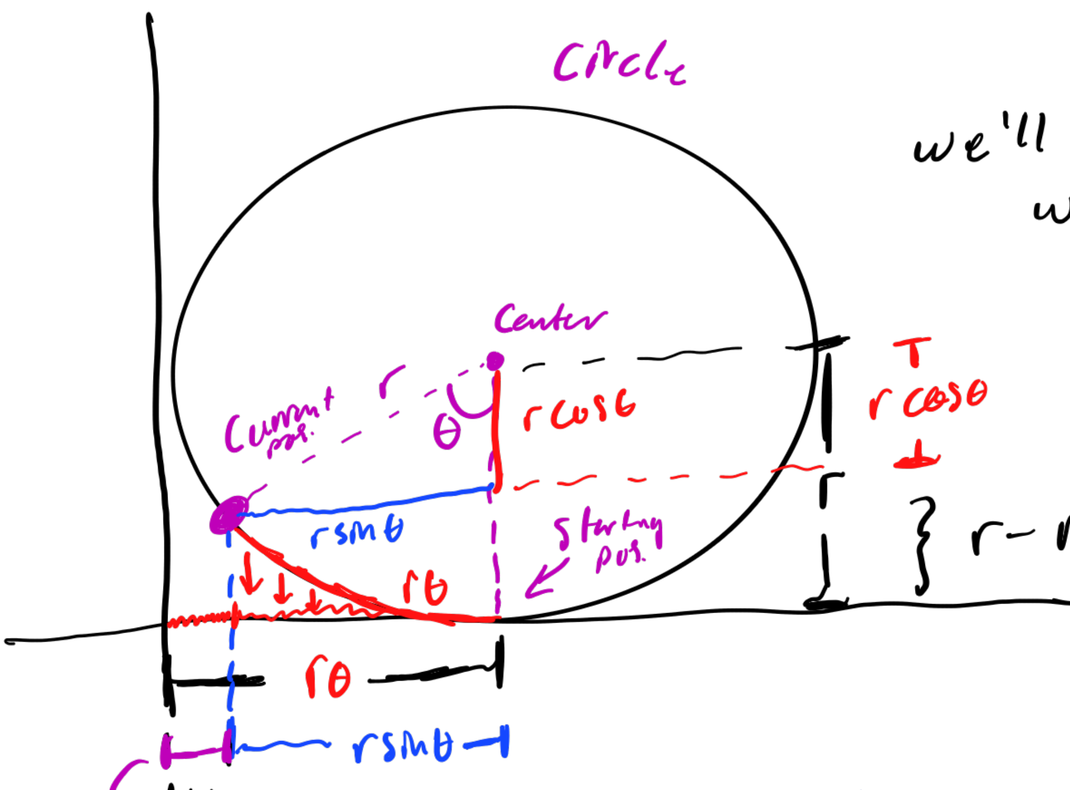


Example 4. Consider a pebble P stuck in a tire. If we assume the pebble starts at the bottom of the tired, how do we model the motion of the pebble if the tired moves at unit speed?



We want track the (x, y) pos. of the pebble as the tire moves

we'll meas. things wrt θ.



$r - r\cos\theta =$ height

dist. travelled (x-dir.) $= r\theta - r\sin\theta$

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

If we were interested in the wheel having an angular velocity ω then $\theta = \omega \cdot t$ where t is time

So the whole system is given parametrically by time

$$x = r(\omega t - \sin(\omega t))$$
$$y = r(1 - \cos(\omega t))$$

We could further complicate the system by changing the shape to say an ellispse or giving the wheel "bumps" or irregularities

↳ This means r becomes a function of θ (& therefore t)