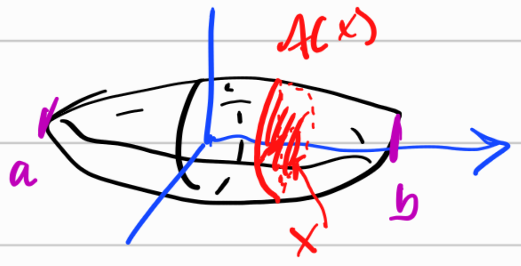
§ 6.3 Shell Method

Recall our definition is that we have some kind of volume "lying" on a coordinate system



We integrate over a Area function which tracks "Area slices"

$$\text{Vol} = \int_a^b A(x)\,dx$$

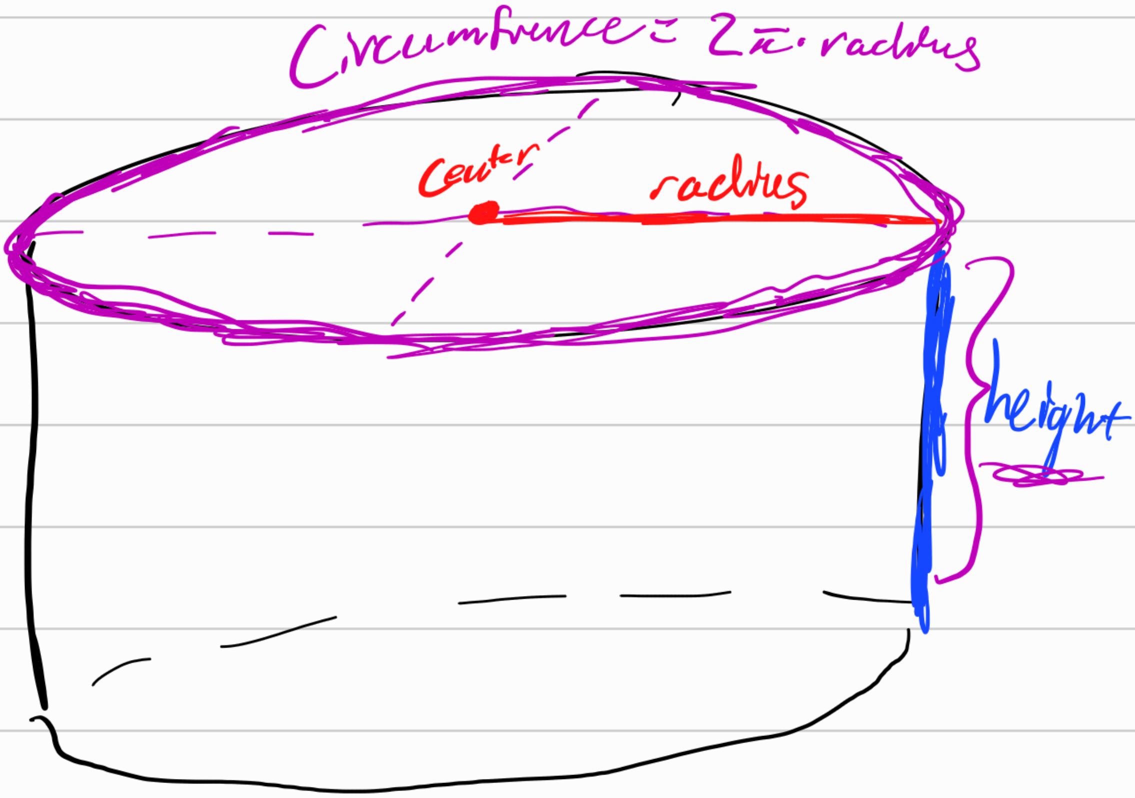
(based on this visual)

If you want this done more generally than w/ Shell or Washer method
→ look @ section 6.2 in textbook.

Yesterday we had areas slices that were washers (or disks).
Today area slices which are the surface area of cylindrical SHELLS

Recall cylinders:

2 important components
- radius
- height



The surface area of the outside SHELL is

$$\underbrace{2\pi \cdot \text{radius} \cdot \text{height}}$$

This will simple components

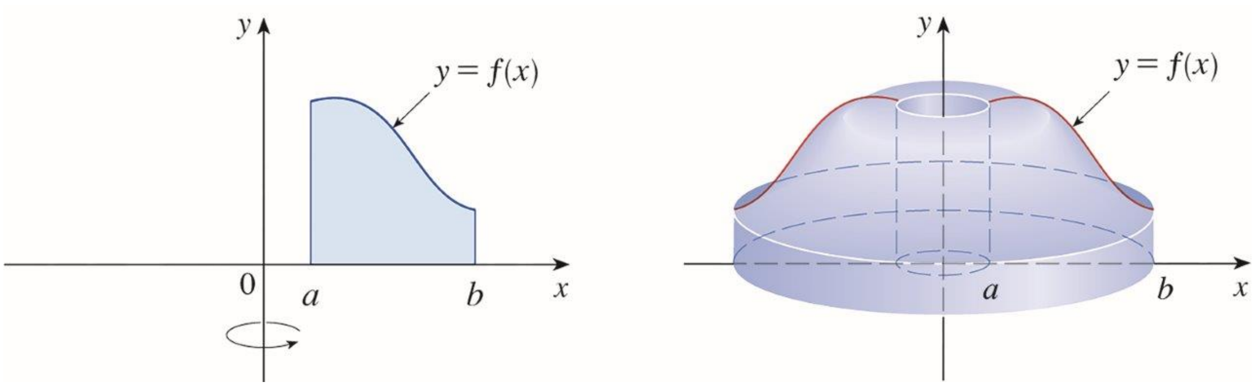
Consider

From rectangle we can associate a "volume" of cylinder surface area × "thickness"

$$2\pi \cdot \text{radius} \cdot \text{height} \cdot dx$$

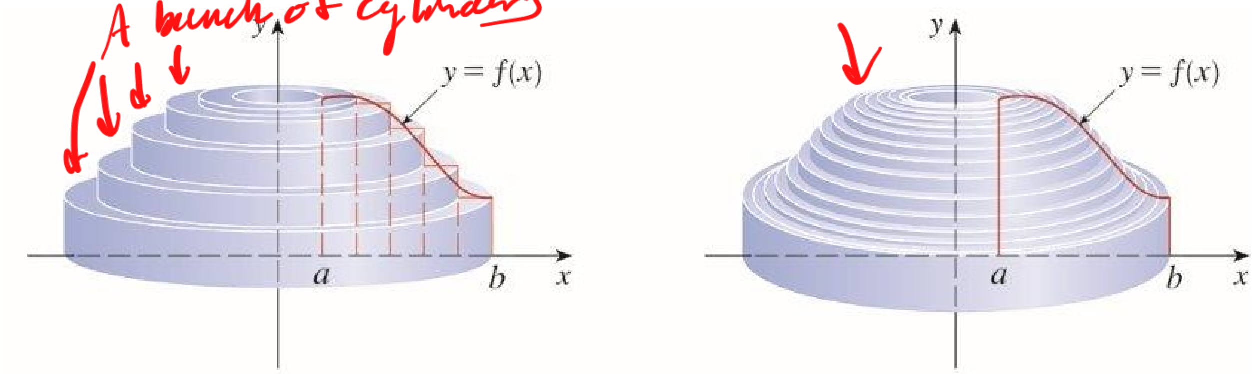
6.3: Shell Method

Consider the following region bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$. We will then rotate the region around the y-axis.

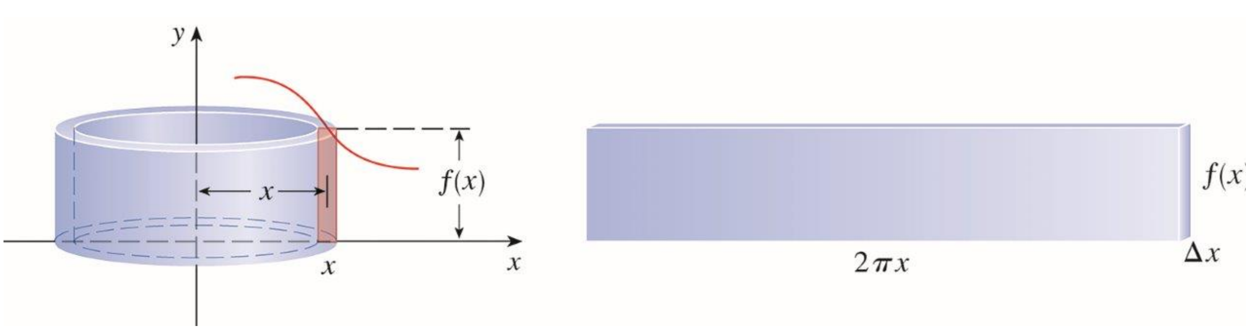


Using the Washer method for this region would be a pain. Instead we can use instead cylindrical shells to find the volume.

A bunch of cylinders



Since each volume chunk is given by a cylinder we can find the "length" of each chunk using the circumference of the associated circle (2π·radius) and the height of each chunk. For this specific example, the radius is x (the distance to the y-axis) and the height is the value of the top function $y = f(x)$ minus the bottom function $y = 0$.



In Shell our d-whatever slices should be Parallel to AoR

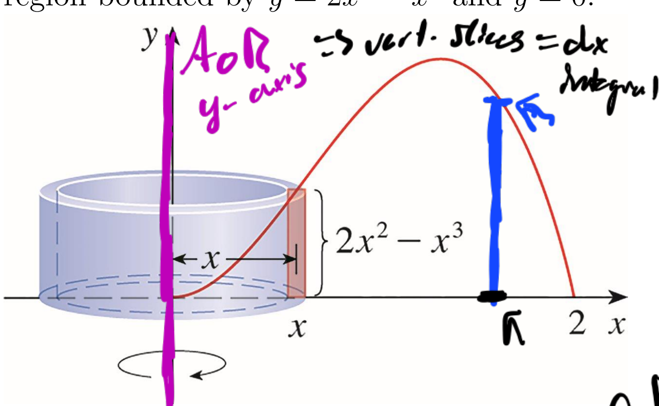
In general the shell method is given by (either dx or dy)

Volume via shell method $= \int_a^b 2\pi\, r(x)\, h(x)\, dx$ (x or y)

"current dist" refers to our dist from the AoR @ a given x or y value

- $r(x)$ is the radius of the shell (i.e. "current" distance to axis of rotation)
- $h(x)$ is the height of the shell (i.e. top-bottom or right-left)

Example 1. Find the volume of the solid obtain by rotating about the y-axis with the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



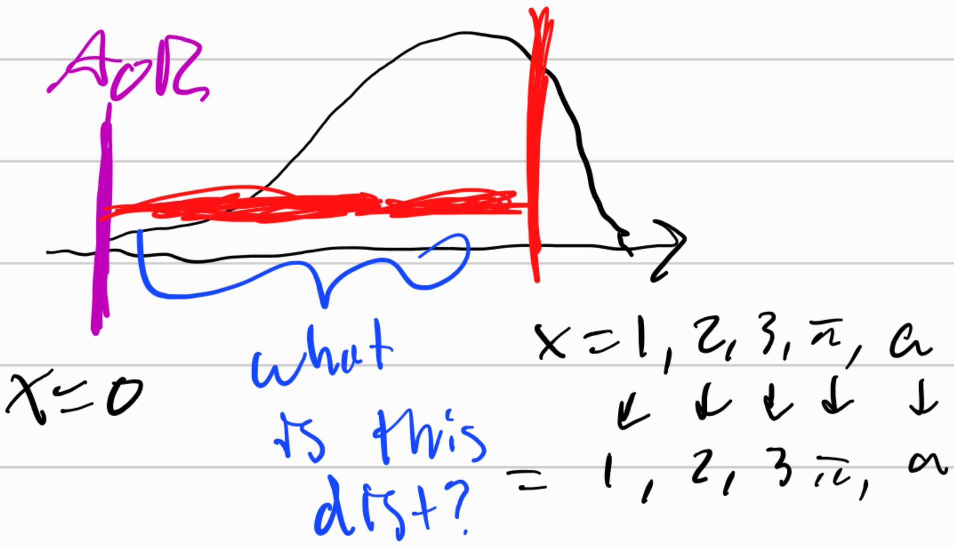
In this I'm highlighting radius & height

General form: $2\pi \int_a^b \text{radius} \cdot \text{height}\; d(x \text{ or } y)$

$$2\pi \int_0^2 \text{rad} \cdot \text{height}\; dx$$

height = $\underbrace{\text{top part of the region} - \text{bot part of the region}}_{dx \text{ integral}} = \underbrace{2x^2 - x^3}_{\text{top}} - \underbrace{0}_{\text{bot}}$

radius = "current" dist. to the AoR



We can denote our current unknown position in an integral by x & so our

current dist to AoR from our current position x $= x - 0 = x$ (variable)

location of AoR

Our marker for current position

(you use y for a dy integral)

$$2\pi \int_0^2 x \, (2x^2 - x^3) \, dx$$

rad height

In a shell method integral the radius

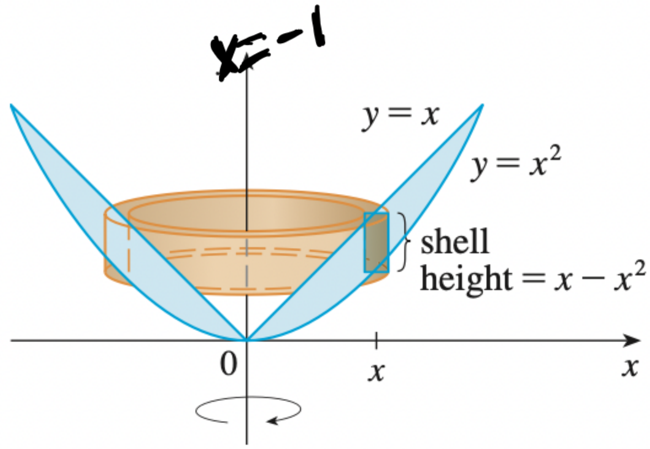
term always features the axis of rotation's location & "our current position" (denoted by x or y)

↳ Another way of saying "current position" is that the variable x or y denotes the position of the slice.

Example 2: Find the volume of the solid obtained by rotating about ~~the y-axis~~ $x=-1$ with the region between $y = x$ and $y = x^2$.

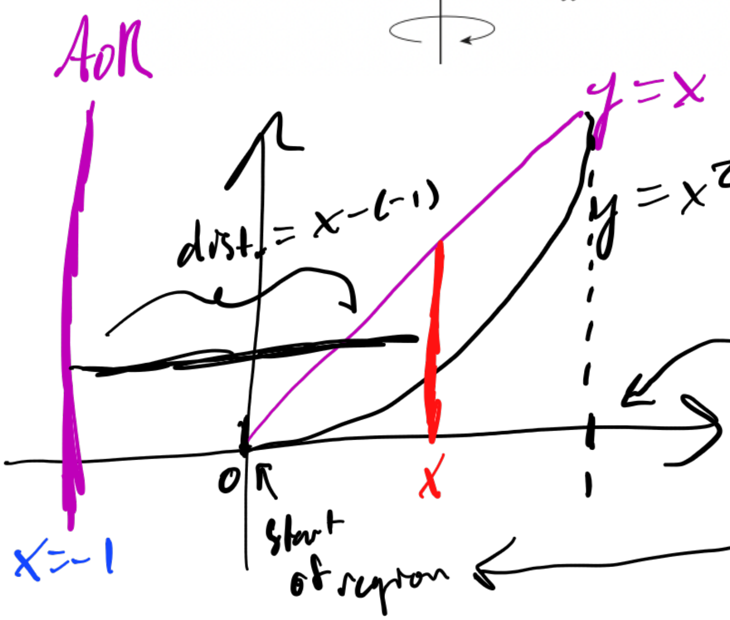
↳ vert. line & shell method (parallel)

↳ vert. slices ⇒ dx



vert. line ⇒ $x = \#$ eq.

shell method (same var. slice) → dx



end of region

$$2\pi \int_0^1 \text{rad} \cdot \text{height} \, dx$$

height = top − bot = $x - x^2$

location AoR

radius = "current dist. to AoR" = $x - (-1)$

"current pos."

Q: In shell method is radius always x or y ± AoR?

A: Basically, yes. Either it's this or that

AoR ± x or y

$$2\pi \int_0^1 (x+1)(x-x^2)\,dx$$

Note In the last 2 examples: We rotated about a vertical "w/ dx integrals" using shell method

If instead we did the washer/disk method we would need to use "dy integral"

But in any case both ways generate the same volume.

Q: So when to use shell vs. washer?

A: "whichever one is easier"

Washer

$$\pi \int_a^b (\;\;)^2 - (\;\;)^2 \, d?$$

squares could be scary

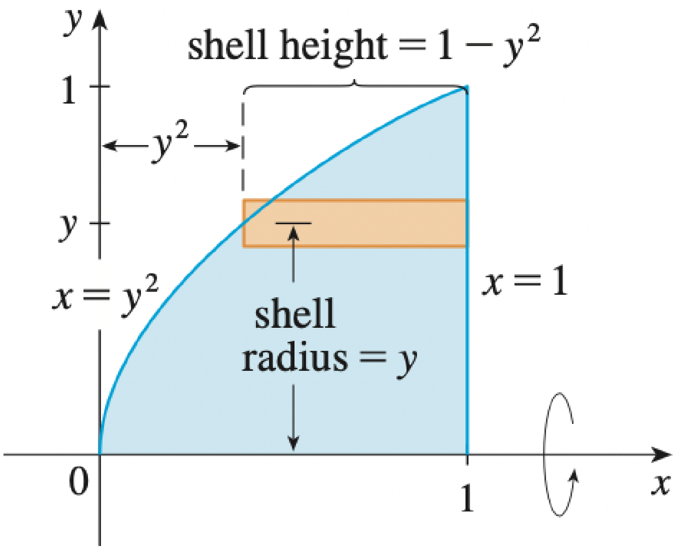
Shell

$$2\pi \int_a^b \underbrace{(\;\;)\cdot(\;\;)}_{} \, d?$$

u sub, maybe IBP who knows

On ME1 I will 100% ask "set up a washer..." AND "setup a shell..."

Example 3: Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

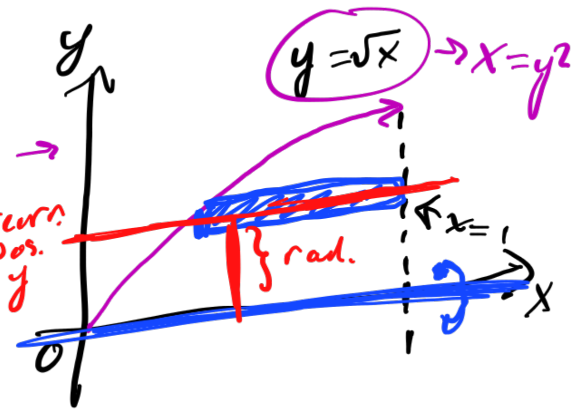


Rotating about x-axis → horiz. line

+ Shell method (parallel slice) → horiz. slice

↳ dy

$$2\pi \int_{\text{start}}^{\text{end}} \text{rad} \cdot \text{height} \, dy$$



height = height in dy = right − left

$= 1 - y^2$

right edge of the area

left edge of the area

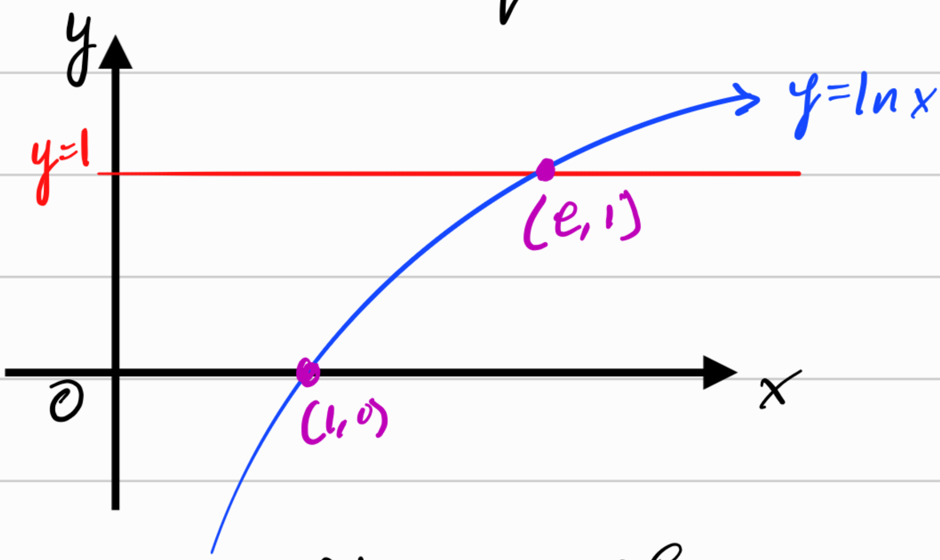
AoR

radius = "current dist to AoR" = top − bot. = $y - 0$

"current pos."

$$2\pi \int_0^1 y(1-y^2)\,dy$$

Consider the region bounded by $y=1$, $y=\ln x$, in the 1st quadrant.

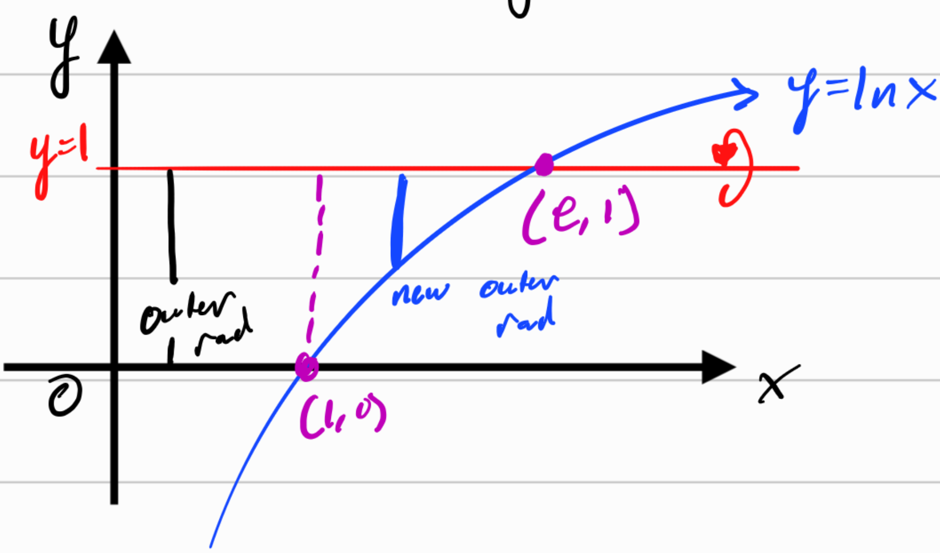


From prev. lecture notes the area of this region is

$$\text{Area via } dx = \int_0^1 dx + \int_1^e 1-\ln x \, dx$$

$$\boxed{\text{Area via } dy = \int_0^1 e^y \, dy}$$

Consider rotating the prev. about $y=1$.



Volume via Washer $\rightarrow \pi \int_{\text{Start}}^{\text{end}} \begin{pmatrix}\text{Outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx$

$$= \pi \int_0^1 \begin{pmatrix}\text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx + \pi \int_1^e \begin{pmatrix}\text{new}\\ \text{outer}\\ \text{rad}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{rad}\end{pmatrix}^2 dx$$

Outer rad on [0, 1] $= 1 - 0$

location AoR

outer edge on [0, 1]

(ii) AoR, inner edge

Inner rad $= 1 - 1 = 0$

(i) b/c disk method

↳ ie. the region has the AoR as an edge

AoR

new outer rad on [1, e] $= 1 - \ln x$

outer edge on [1, e]

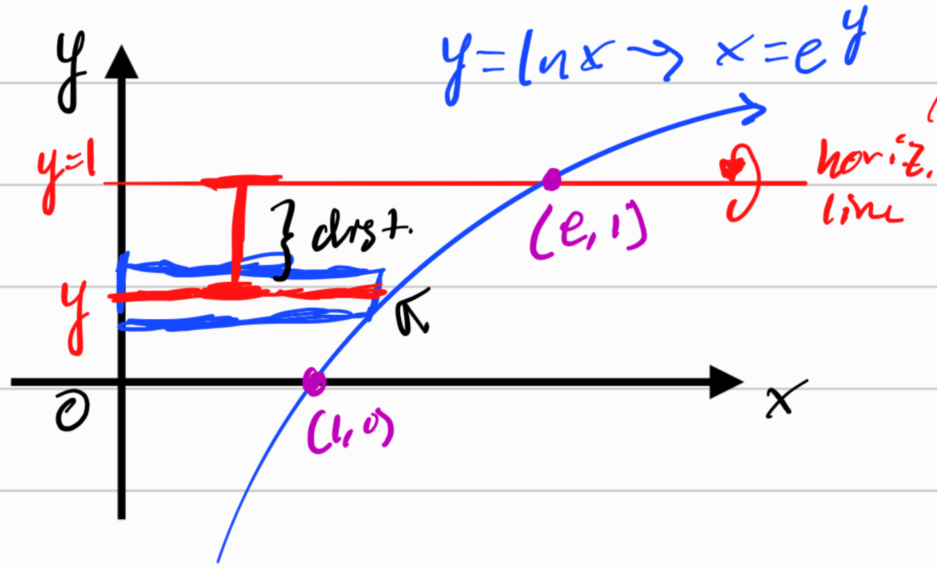
$$\pi \int_0^1 1^2\, dx + \pi \int_1^e (1 - \ln x)^2\, dx$$

$= 1 - 2\ln x + (\ln x)^2$

Via Shell Method → parallel slices

$y = \ln x \rightarrow x = e^y$

↳ dy → need $x = \dots$ eq.



$$2\pi \int_{\text{Start}}^{\text{end}} \text{rad} \cdot \text{height}\, dy$$

$$2\pi \int_0^1 \text{rad} \cdot \text{height}\, dy$$

$$\text{height} = \overbrace{\text{right} - \text{left}}^{dy} = e^y - 0$$

rad = "current dist. to AoR" = top − bot = $1 - y$ ← "current pos."

$\uparrow$ AoR

$$\underbrace{2\pi}\int_0^1 \underbrace{(1-y)}e^y\, dy$$

look area integral $\int_0^1 e^y dy$

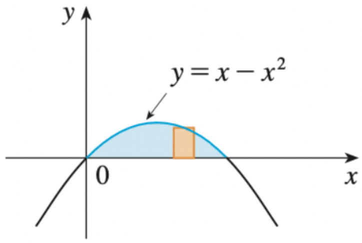
The shell is basically just this integral w/ 2π & the radius term.

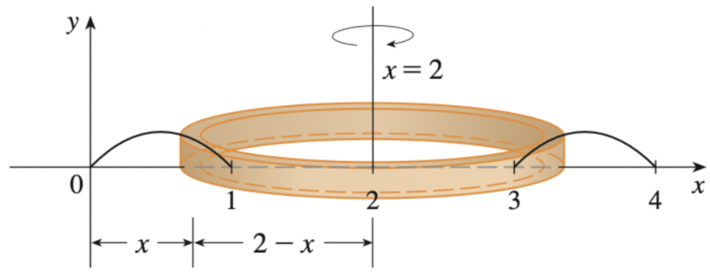
There is not a similarily nice statement for area & the washer method.

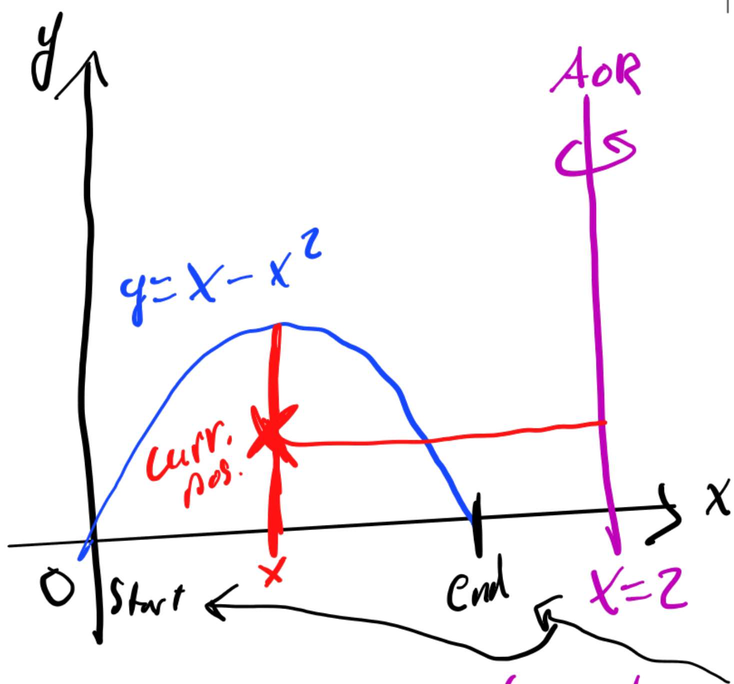
Q: How do approach using 1 method over the other?

A: Look @ the picture & determine of dx or dy is "less of a headache"

Example 4: Set up the volume V of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.







Vert. line + Shell method $\Rightarrow$ parallel to vert. $\Rightarrow dx$

$$2\pi \int_{\text{start}}^{\text{end}} \text{rad} \cdot \text{height} \, dx$$

To find start & end need to find

i.e. where $x - x^2 = 0 \Rightarrow x(1-x) = 0$

@ $\boxed{x = 0 \ \& \ x = 1}$ Start & end

$$\text{height} = \underbrace{\text{top} - \text{bot}}_{dx \text{ integral}} = \underbrace{x - x^2}_{\text{top of region}} - \underbrace{0}_{\text{bot of region}}$$

$$\text{rad} = \underbrace{\text{right} - \text{left}}_{dx \text{ integral}} = \overset{\text{AoR}}{2} - \overset{\text{curr. pos.}}{x} \Rightarrow \boxed{2\pi \int_0^1 (2-x)(x-x^2)\,dx}$$