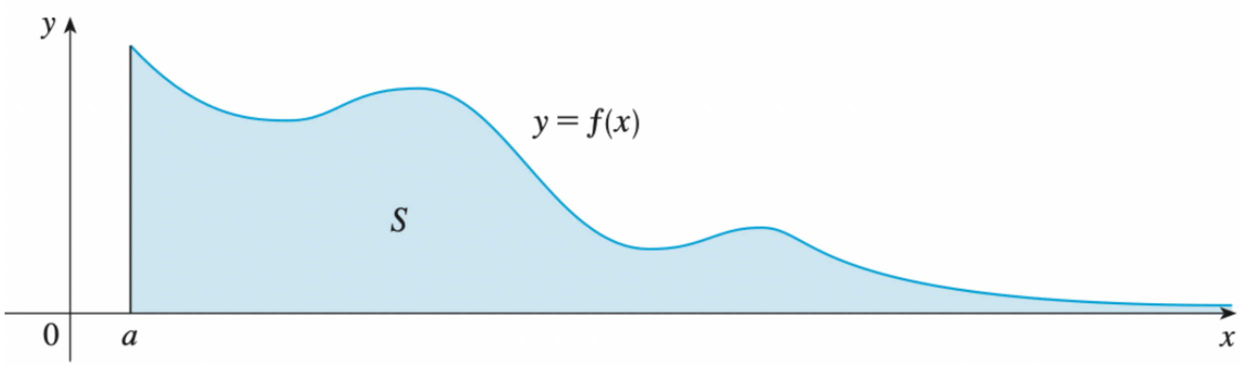
7.8: Improper Integrals

Laplace
Fourier
Trans-forms

Consider a function $f(x)$. Often we talk about the area under the curve $f(x)$ as an integral over some kind of interval $[a, b]$ where a and b are real numbers. But how to we talk about the area under a curve $f(x)$ if our interval goes on forever?

7.8



That is, what happens if we take the interval $[a, b]$ and we send b to ∞? (i.e. $b \to \infty$) This gives us the idea of Improper Integrals. For some reason in the US, we say there are two types of improper integrals, and so, we'll follow the conventions:

In general, when we say "improper integral" we mean that $\int f(x)\, dx$ has something "bad" happening within the domain of integration. These can be thought of coming in two flavours (which can be combined).

Type I Improper Integral: For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the integral $\displaystyle\int_a^t f(x)\, dx$ exists for every number $t \geq a$ then we define

$$\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$$

  provided the **limit** exists.

- If the integral $\displaystyle\int_t^b f(x)\, dx$ exists for every number $t \leq b$ then we define

$$\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx$$

  provided the **limit** exists.

In other words: A **Type I Improper Integral** is an integral where **one or both** bounds is $\pm\infty$.

Some Important teminology:

- If an improper integral gives you a number i.e.

$$\int_a^\infty f(x)\,dx = \# \qquad \text{or} \qquad \int_{-\infty}^b f(x)\,dx = \#$$

  Then we say the Improper Integral is **convergent.**

- If we DON'T get a number (so every other case) we say Improper Integral is **divergent.**

Lastly, if both $\int_{-\infty}^t f(x)\,dx$ and $\int_t^\infty f(x)\,dx$ are convergent then we can write:

$$\boxed{\int_{-\infty}^\infty f(x)\,dx} = \boxed{\int_{-\infty}^t f(x)\,dx} + \boxed{\int_t^\infty f(x)\,dx}$$

Example 1. Determine the convergence of $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$.

① Recognise the "improperness"

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \boxed{\int_1^b \frac{1}{x}\,dx}$$

② Do the indefinite or definite integral

(definite) $\int_1^b \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^b = \ln|b| - \ln|1| = \ln|b|.$

$\ln|1| \to 0$

③ Take the limit

$$\lim_{b\to\infty} \ln|b| = \infty$$

In this case we say $\int_1^\infty \frac{1}{x}\,dx$ is divergent

(often I write div.)

(Indefinite) $\int \frac{1}{x}dx = \ln|x| + C$

FTC $\int_1^b \frac{1}{x}dx = [\ln|x|]_1^b$

$$\int_1^\infty \frac{1}{x^2}dx$$

(i) → (ii)

$$\int_1^b \frac{1}{x^2}dx = -\left[\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}$$

(ii) $\lim\limits_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1 - 0 = 1$

So, b/c $\int_1^\infty \frac{1}{x^2}dx = 1$, the integral is convergent

Example 2. Determine the convergence of $\int_1^\infty \frac{1}{x^p}\,dx$ for all values of p.

Even though there are ∞'ly many p's to choose from only $p=1$ & $p \neq 1$ matter.

For $p=1$ we already saw $\int_1^\infty \frac{1}{x^1}\,dx$ is div.

For $p \neq 1$
$$\int_1^b \frac{1}{x^p}\,dx = \frac{1}{1-p}\left[x^{-p+1}\right]_1^b$$
$$= \frac{1}{1-p}\left[b^{-p+1} - 1\right]$$

$\lim_{b\to\infty} b^{-p+1}$ has two behaviours depending on $-p+1$

$-p+1<0$ (neg #)

$\lim_{b\to\infty} b^{\text{neg}\#} = \lim_{b\to\infty} \frac{1}{b^{\text{pos}\#}} = 0$

$\to 1<p$

$-p+1>0$ (pos #) $\to 1>p$

$\lim_{b\to\infty} b^{(\text{pos}\#)} = \infty$

div.

We can summarise this result as "p-test"

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for $p>1$ and **diverges** for $p \leq 1$

Example 3. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{0} \frac{x}{(x^2+4)^3}\,dx$$

(1) Recognize "improperness"

$$\lim_{a\to-\infty} \underbrace{\int_a^0 \frac{x}{(x^2+4)^3}\,dx}$$

(2) Do definite integral (u-subs)

$u = x^2+4$ @ $x=a \mapsto u = a^2+4$

$du = 2x\,dx$ @ $x=0 \mapsto u = 4$

$$\to \frac{1}{2}\int_{a^2+4}^{4} \frac{du}{u^3}$$

$$= \frac{1}{2}\left[-2u^{-2}\right]_{a^2+4}^{4} = -\left[\frac{1}{4^2} - \frac{1}{(a^2+4)^2}\right]$$

$$= \frac{1}{(a^2+4)^2} - \frac{1}{16}$$

(3) Take a limit:

$$\lim_{a\to-\infty} (\text{prev. step}) = \lim_{a\to-\infty}\left[\frac{1}{(a^2+4)^2} - \frac{1}{16}\right] = 0 - \frac{1}{16}$$

Since this limit exists & is a # the integral is conv. ie. $\int_{-\infty}^{0} \frac{x}{(x^2+4)^3}\,dx = -\frac{1}{16}$ ✓

Example 4. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

For t his kind of improper integral we have to split the integral into 2 parts as such

$$\int_{-\infty}^{\#} \frac{1}{1+x^2}dx \quad \& \quad \int_{\#}^{\infty} \frac{1}{1+x^2}dx$$

Any # works so let's pick a nice one like 0

$$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx = \lim_{b\to 0}\left[\arctan(x)\right]_0^b$$

$$= \lim_{b\to\infty}\left[\arctan(b) - \xcancel{\arctan(0)}^{\,0}\right] = \lim_{b\to 0} \arctan(b)$$

$$= \pi/2$$

$\int_0^{\infty} \frac{1}{1+x^2}dx$ conv. to $\frac{\pi}{2}$

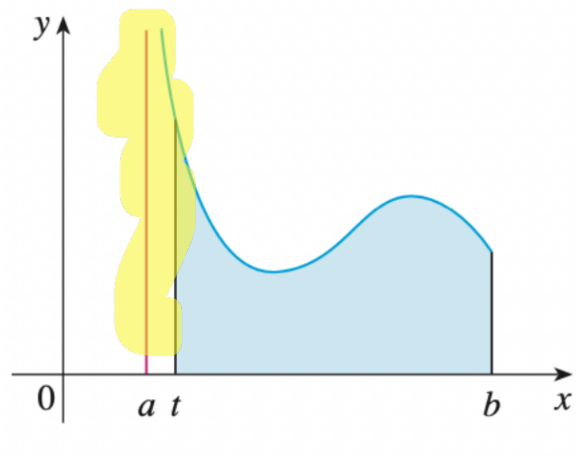
$$\int_{-\infty}^{0} \frac{1}{1+x^2}dx = \lim_{a\to-\infty}\left[\arctan(x)\right]_a^0 = \lim_{a\to-\infty} -\arctan(a) = -\left(-\frac{\pi}{2}\right)$$

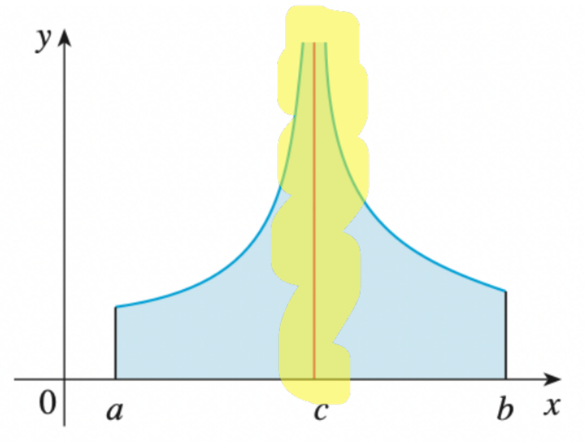
$\Rightarrow$ $\int_{-\infty}^{0} \frac{1}{1+x^2}dx$ conv. to $\frac{\pi}{2}$

So $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

is conv. to π

So now let's look at the area of functions that "blow up:"





Type II Improper Integral: For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the function $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then we define
$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$
provided the **limit** exists.

- If the function $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then we define
$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$$
provided the **limit** exists.

Just like Type I Improper Integrals, we use the words **convergent** and **divergent** if the limits exists. Another important Type II Improper Integral is:

- If the function $f(x)$ has a discontinuity at c where $a < c < b$ and the integrals $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are both convergent, then we define:
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$
provided the **limit** exists.

In other words: A **Type II Improper Integral** is an integral where there is a **discontinuity of the function somewhere** in the domain of integration.

Example 5. Determine if the convergence of $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$. Find its value if it converges.

Since we know about improper integrals we have to ask "Is there an x-value that 'breaks' the function in the integral"

$$\boxed{\frac{1}{\sqrt{x-2}} \text{ "breaks" @ } x=2} \to \text{Improper Integral}$$

(i) $\int_2^5 \frac{1}{\sqrt{x-2}} = \lim_{a\to 2^+} \int_a^5 \frac{1}{\sqrt{x-2}}\,dx$

(ii) $\int_a^5 \frac{1}{\sqrt{x-2}}\,dx = 2\left[\sqrt{x-2}\right]_a^5 = 2\left(\sqrt{3} - \sqrt{a-2}\right)$

(iii) $\lim_{a\to 2^+} 2\left(\sqrt{3} - \sqrt{a-2}\right) = 2\sqrt{3} - 0 = 2\sqrt{3}$

$$\boxed{\int_2^5 \frac{1}{\sqrt{x-2}}\,dx \text{ converges to } 2\sqrt{3}}$$

Example 6 $\int_0^1 \frac{1}{x^2}\,dx$

Note @ $x=0$ $\frac{1}{x^2}$ "breaks"

So $\int_0^1 \frac{1}{x^2}\,dx$ is improper since 0 is in the bounds of integration

(i) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2}\,dx$

(ii) $\int_a^1 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_a^1 = -1 + \frac{1}{a}$

(iii) $\lim_{a \to 0^+} \left(\frac{1}{a} - 1\right) = \infty$

So $\int_0^1 \frac{1}{x^2}\,dx$ diverges

Sometimes, evaluating an integral is hard, but if all we care about is a question about convergence, then we have a handy theorem that takes care of things for us:

The Comparison Theorem:

Suppose that f and g are continuous functions where $f(x) \geq g(x) \geq 0$ for all $x \geq a$ where a is some constant number.

- **IF** $\displaystyle\int_a^\infty f(x)\,dx$ is **convergent**, **then** this implies $\displaystyle\int_a^\infty g(x)\,dx$ is **convergent**.
- **IF** $\displaystyle\int_a^\infty g(x)\,dx$ is **divergent**, **then** this implies $\displaystyle\int_a^\infty f(x)\,dx$ is **divergent**.

This theorem can be thought of as:

- If the **BIGGER thing** converges then so does the smaller one.
- If the **SMALLER thing** diverges then so does the bigger one.

Example 7. Show that $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.