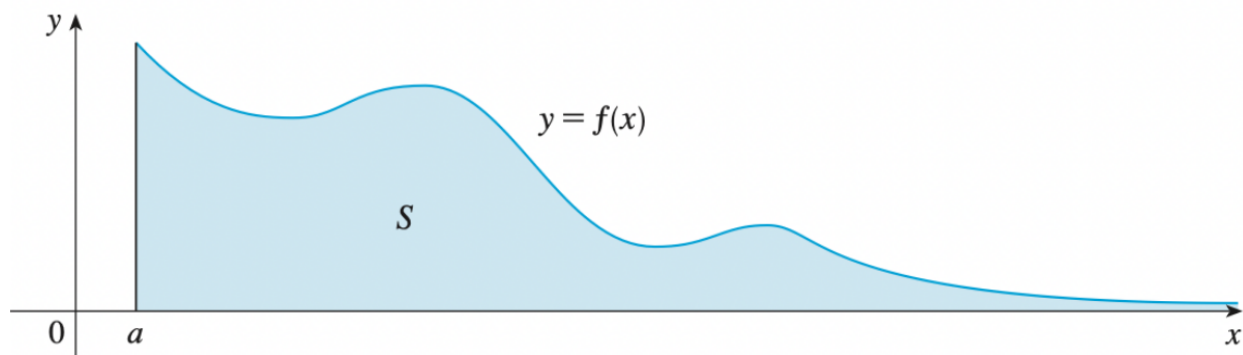# 7.8: Improper Integrals

Consider a function $f(x)$. Often we talk about the area under the curve $f(x)$ as an integral over some kind of interval $[a, b]$ where a and b are real numbers. But how to we talk about the area under a curve $f(x)$ if our interval goes on forever?

That is, what happens if we take the interval $[a, b]$ and we send b to ∞? (i.e. $b \to \infty$) This gives us the idea of Improper Integrals. For some reason in the US, we say there are two types of improper integrals, and so, we'll follow the conventions:

In general, when we say "improper integral" we mean that $\int f(x)\, dx$ has something "bad" happening within the domain of integration. These can be thought of coming in two flavours (which can be combined).

Type I Improper Integral: For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the integral $\displaystyle\int_a^t f(x)\, dx$ exists for every number $t \geq a$ then we define

$$\int_a^\infty f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$

   provided the **limit** exists.

- If the integral $\displaystyle\int_t^b f(x)\, dx$ exists for every number $t \leq b$ then we define

$$\int_{-\infty}^b f(x)\, dx = \lim_{t \to -\infty} \int_t^b f(x)\, dx$$

   provided the **limit** exists.

In other words: A **Type I Improper Integral** is an integral where **one or both** bounds is $\pm\infty$.

Some Important teminology:

- If an improper integral gives you a number i.e.

$$\int_a^\infty f(x)\, dx = \# \qquad \text{or} \qquad \int_{-\infty}^b f(x)\, dx = \#$$

Then we say the Improper Integral is **convergent.**

- If we DON'T get a number (so every other case) we say Improper Integral is **divergent.**

Lastly, if both $\int_{-\infty}^t f(x)\, dx$ and $\int_t^\infty f(x)\, dx$ are convergent then we can write:

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^t f(x)\, dx + \int_t^\infty f(x)\, dx$$

Example 1. Determine the convergence of $\int_1^\infty \frac{1}{x}\, dx$ and $\int_1^\infty \frac{1}{x^2}\, dx$.

Step 1: Recognise the "improperness"

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{b \to \infty} \boxed{\int_1^b \frac{1}{x}\, dx}$$

Step 2: Do the Indefinite or definite Integral

$$\int_1^b \frac{1}{x}\, dx = \left[\ln |x| \right]_1^b = \ln |b| - \ln(1)^{\,0} = \ln |b|$$

Step 3: Take the limit

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx = \lim_{b \to \infty} \ln|b| = \infty$$

(often I write div)

So $\int_1^\infty \frac{1}{x}\, dx$ is divergent

Now for $\int_1^\infty \frac{1}{x^2} dx$

(i)→(ii) $\int_1^b \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^b = 1 - \frac{1}{b}$

(ii) $\lim\limits_{b \to \infty} \left(1 - \frac{1}{b} \right) = 1 < \infty$

So $\int_1^\infty \frac{1}{x^2} dx$ is convergent

(often I write Conv.)

Example 2. Determine the convergence of $\int_1^\infty \frac{1}{x^p}\, dx$ for all values of p.

Note $\int \frac{1}{x^p}\, dx$ has 2 possibilities

If $p = 1$ then $\int \frac{1}{x^p}\, dx = \ln|x| + c$

If $p \neq 1$ then $\int \frac{1}{x^p}\, dx = \int x^{-p}\, dx = \frac{1}{-p+1} x^{-p+1} + c$

$$\int \frac{1}{x^p}\, dx = \frac{1}{1-p} x^{-p+1} + C$$

If $p > 1$ then $0 > -p + 1$

So $\lim_{x \to \infty} x^{-p+1} = \lim_{x \to \infty} x^{(neg\#)}$

$= \lim_{x \to \infty} \frac{1}{x^{(pos\#)}} = 0$

$\rightarrow$ Conv.

If $p < 1$ then $0 < -p + 1$ So $x^{-p+1} = x^{(pos\#)}$

$\lim_{x \to \infty} x^{(pos\#)} = \infty$

$\rightarrow$ dv.

We can summarise this result as

$$\int_1^\infty \frac{1}{x^p}\, dx$$

converges for ___$p > 1$___ and **diverges** for ___$p \leq 1$___

Call this the "p-test"

Example 3. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{0} \frac{x}{(x^2 + 4)^3} \, dx$$

Example 4. Determine if the following integral is convergent or divergent. If it is convergent, find its value

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$$

To do "doubly improper" integrals we have
to split it up into 2 improper integrals.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

$$\int_{-\infty}^{\#} \frac{1}{1+x^2} dx \qquad \int_{\#}^{\infty} \frac{1}{1+x^2} dx$$

But what # to choose?

↳ Any. So pick one convenient to the fxn.

logs ~ 1 & e etc.
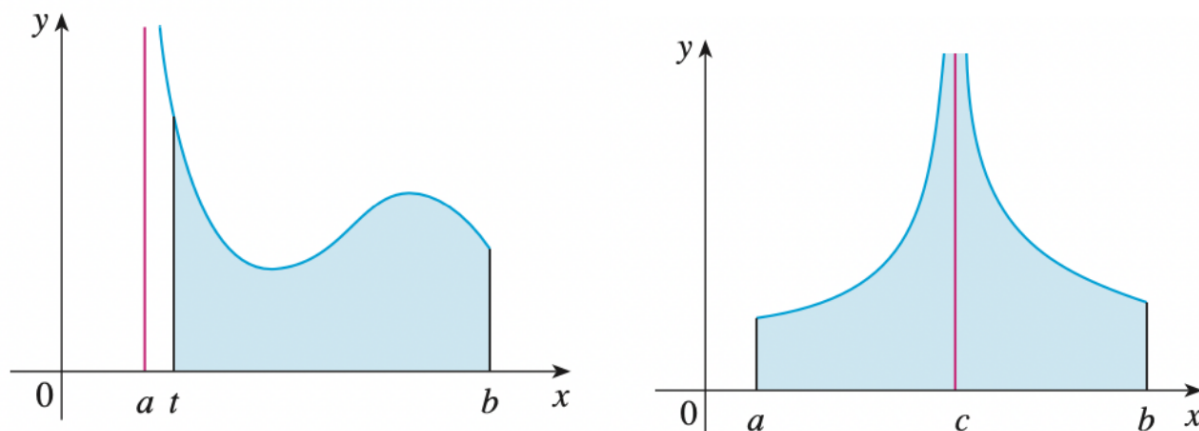
trig ~ $\overline{\pi}/\#$ etc.

all else idk 0 or 1 are cool.

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \to \infty} \left[arctan(x) \right]_0^b$$

$$= \lim_{b \to \infty} \left[arctan(b) - arctan(0) \right]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

So now let's look at the area of functions that "blow up:"

Type II Improper Integral:
For this description, a and b are fixed numbers (i.e. constant) and t is allowed to vary.

- If the function $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then we define

$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

  provided the **limit** exists.

- If the function $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then we define

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

  provided the **limit** exists.

Just like Type I Improper Integrals, we use the words **convergent** and **divergent** if the limits exists. Another important Type II Improper Integral is:

- If the function $f(x)$ has a discontinuity at c where $a < c < b$ and the integrals $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are both convergent, then we define:

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

  provided the **limit** exists.

In other words: A **Type II Improper Integral** is an integral where there is a **discontinuity of the function somewhere** in the domain of integration.

Example 5. Determine if the convergence of $\int_2^5 \boxed{\dfrac{1}{\sqrt{x-2}}} \, dx$ Find its value if it converges.

Since we know about improper integrals we always need to check now to see if the integral has "any problems"

Is there a # where $\dfrac{1}{\sqrt{x-2}} = \dfrac{1}{0}$?

@ $x=2$ which is in our bounds

$\Rightarrow$ This integral is improper (Type II)

①→ⅱ $\displaystyle\int_a^5 \frac{dx}{\sqrt{x-2}} = 2\left[\sqrt{x-2}\right]_a^5 = 2\left(\sqrt{3} - \sqrt{a-2}\right)$

ⅲ $\displaystyle\lim_{a \to 2^+} 2\left(\sqrt{3} - \sqrt{a-2}\right) = 2\sqrt{3}$

Example 6. Determine if the convergence of $\int_0^1 \dfrac{1}{x^2} \, dx$. Find its value if it converges.
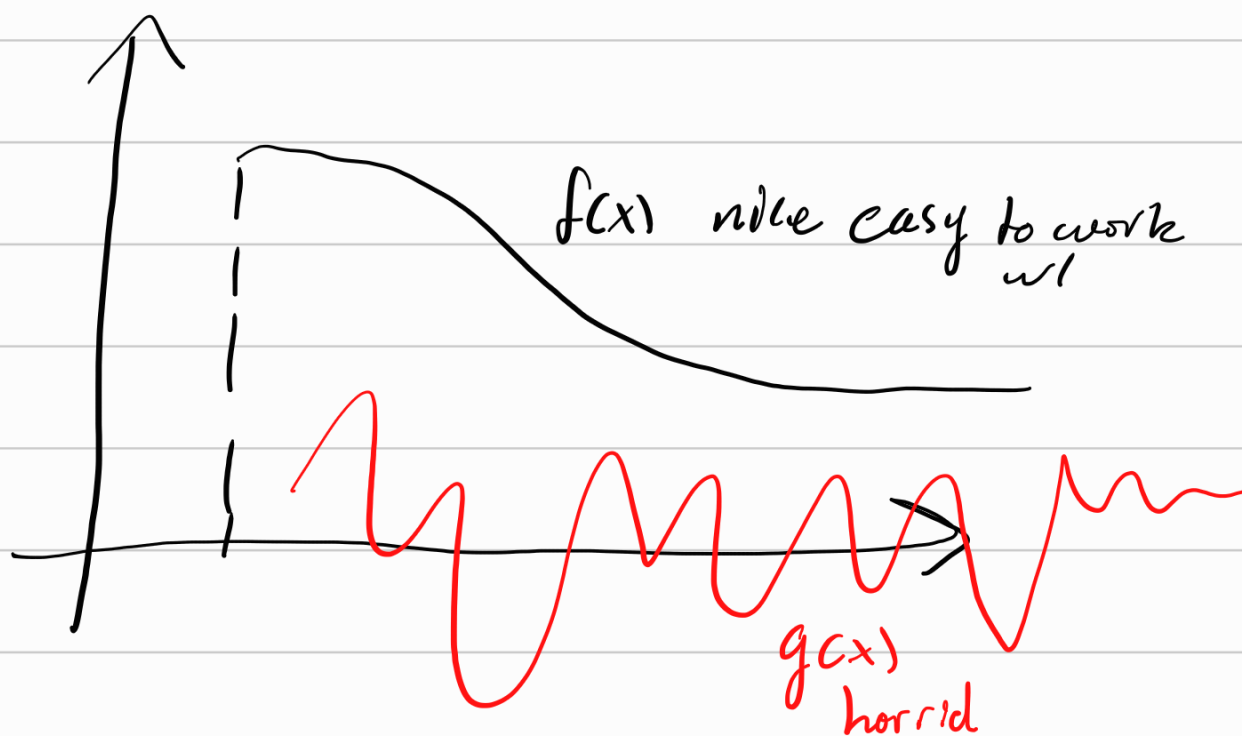
problem @ $x=0$ so improper

$\displaystyle\int_a^1 \frac{1}{x^2} dx = 1 - \frac{1}{a}$

$\displaystyle\lim_{a \to 0^+} \frac{1}{a} - 1 = \infty$ (div)

The next topic is ALL theory, but we care about for making sure our useful tools make sense (such as the Laplace Transform.
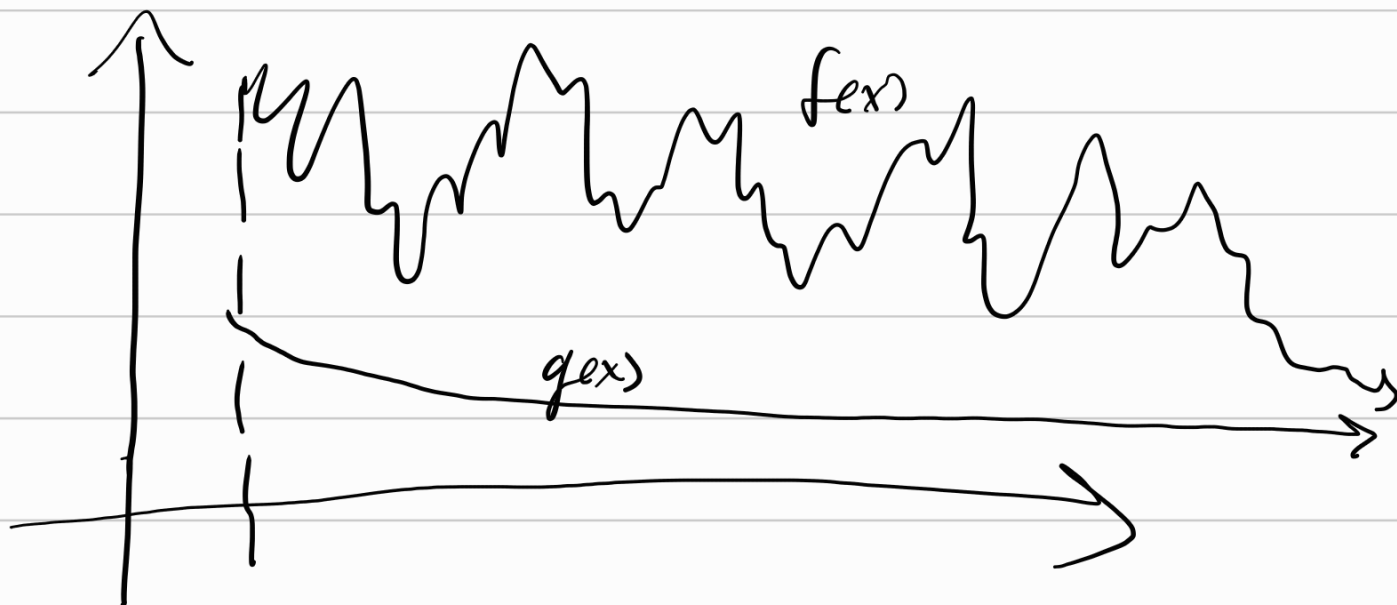
The idea is consider:

$f(x)$ nice easy to work w/

$g(x)$ horrid

If $|g(x)| \leq f(x)$ & $\int_{\#}^{\infty} f \, dx$ conv.

Then so does $\int_{\#}^{\infty} |g(x)| \, dx$

If we instead have

If $\displaystyle\int_1^\infty g(x)\,dx$ dv. then so does $\displaystyle\int_1^\infty f(x)\,dx$

Sometimes, evaluating an integral is hard, but if all we care about is a question about convergence, then we have a handy theorem that takes care of things for us:

The Comparison Theorem:

Suppose that f and g are continuous functions where $f(x) \geq g(x) \geq 0$ for all $x \geq a$ where a is some constant number.

- **IF** $\int_a^\infty f(x)\, dx$ is **convergent**, **then** this implies $\int_a^\infty g(x)\, dx$ is **convergent**.

- **IF** $\int_a^\infty g(x)\, dx$ is **divergent**, **then** this implies $\int_a^\infty f(x)\, dx$ is **divergent**.

This theorem can be thought of as:

- If the **BIGGER thing** **converges** then so does the smaller one.

- If the **SMALLER thing** **diverges** then so does the bigger one.

Example 7. Show that $\int_1^\infty \dfrac{1 + e^{-x}}{x}\, dx$ is divergent.

$$e^{-x} > 0 \qquad \text{for all } x \text{ always}$$

$$\text{So} \quad 1 + e^{-x} > 1 + 0 = 1$$

$$\Rightarrow \quad \frac{1 + e^{-x}}{x} > \frac{1}{x} \qquad \text{for } x > 0$$

$$\text{So} \quad \int_1^\infty \frac{1 + e^{-x}}{x}\, dx \geq \int_1^\infty \frac{1}{x}\, dx$$

div. by comparison w/

div. by p-test