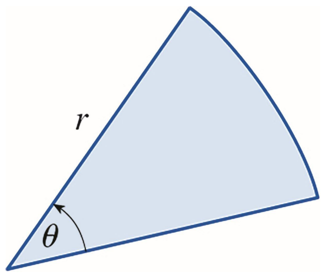
10.4: Calculus in Polar Coordinates

1 Area

The interesting thing about Polar Coordinates, is this is our first example of transforming the coordinate system. The main advantage of doing this is to simplify some integral calculations. In some sense, a transformation of the coordinate system is a generalization u-sub. This idea is further explored in Calc 3. We like polar instead of cartesian when we need to integrate things that are "more circular and less box-y." However, since transforming the coordinate system would mean changing 2 variables, this is more-so applicable to *double integrals* which again is a calc 3 topic as we've only talked about working with a single integral.

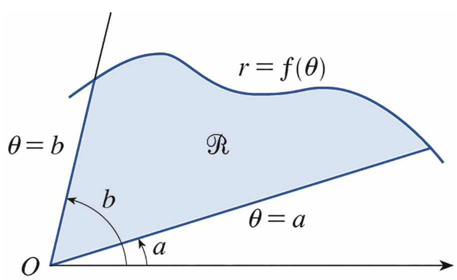
Area in Polar Coordinates:

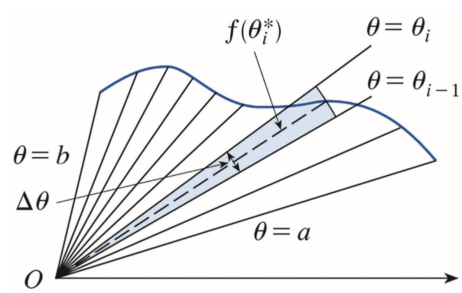
Recall that the area of the section of a circle with radius r swept out by angle θ is:



$$\text{Area of Section of Circle} = \frac{1}{2}r^2\theta$$

So, if we have a polar equation $r = f(\theta)$ and we are interested in integrating f against θ we do so with the same "simple shapes approximating integral" idea we've done in the past.





Using sections of a circle (as that's how polar coordinates tells us to measure our shapes / distances) we get

$$\text{Area between } \theta = a \text{ and } \theta = b \text{ of } f(\theta) \approx \sum_{j=1}^{n} \frac{1}{2}\left[f(\theta_j)\right]^2 \Delta\theta$$

Taking the limit as $n \to \infty$ we get

$$\text{Area between } \theta = a \text{ and } \theta = b \text{ of } f(\theta) = \int_a^b \frac{1}{2}\Big[f(\theta)\Big]^2\, d\theta = \int_a^b \frac{1}{2}r^2\, d\theta$$

Example 1. Find the area between the polar equations $r = 2$ and $0 \leq \theta \leq 2\pi$.

Example 2. Find the area inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$.

2 Arc Length

Arc Length in Polar Coordinates:

Using $x = r\cos\theta$ and $y = r\sin\theta$, if r can be written as a function of θ (i.e. $r = f(\theta)$), then we have a parametric description of (x, y) in terms of θ. By the chain rule we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\Big[r\cos\theta\Big] = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\Big[r\sin\theta\Big] = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Some algebra shows that

$$\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

So, using the arc length equation for parametric equations from 10.2, and the above identity we obtain:

$$\text{Length of Arc of } r = f(\theta) \text{ on } a \leq \theta \leq b = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 1. Find the length of the cardioid $r = 1 + \sin\theta$.

3 Derivatives

Derivatives in Polar Coordinates:

Using $x = r\cos\theta$ and $y = r\sin\theta$, if r can be written as a function of θ (i.e. $r = f(\theta)$), then we have a parametric description of (x, y) in terms of θ. By the chain rule / the arc length block on the previous page we get and the derivative for a parametric system equation:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Hence,

- Horizontal Tangents occur where $\dfrac{dy}{d\theta} = 0$ (if $\dfrac{dx}{d\theta} \neq 0$)
- Vertical Tangents occur where $\dfrac{dx}{d\theta} = 0$ (if $\dfrac{dy}{d\theta} \neq 0$)

Note you find that both $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ are zero at the same time then you have to take a limit to resolve what kind of tangent you have.

At the ORIGIN/POLE (i.e. $r = 0$) then tangents are given by

$$\frac{dy}{dx} = \tan\theta \qquad \text{if } \frac{dr}{d\theta} \neq 0$$

Example 1. Consider the cardioid $1+\sin\theta$. Find the slope of the tangent line when $\theta = \dfrac{\pi}{3}$

Example 2. Consider the cardioid $1 + \sin\theta$. Find the locations of horizontal and vertical tangents.