7.2: Trigonometric Integrals

The goal is to now build techniques to handle integrals specifically involving trig function. This will give us the tools needed to do 7.3 - trig sub. The idea of 7.2 can be boiled down to 2 things:

- Trig-Identities
- u-sub

We either do one of the above things or both for integrals in this section (and whenever you see trig integrals in general).

Trig Identities

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

Double Angle Identities:

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

Half Angle Identities:

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

Product-to-Sum Formulas

$$\sin A \cos B = \frac{1}{2}\Big[\sin(A-B) + \sin(A+B)\Big]$$
$$\sin A \sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$
$$\cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$

Note in this class we won't make use of the product-to-sum formulas much, but for many who are going into engineering these are useful to have in your back pocket.

Example 1. $\int \sin^2 x \cos^3 x \, dx$

Try u-sub:

$u = \cos x$

$du = -\underbrace{\sin x \, dx}$

$\boxed{\text{This won't work}}$

$$\int \underbrace{\sin' x}_{?} \underbrace{\cos^3}_{u^3} x \cdot \underbrace{\sin x \, dx}_{-du}$$

$\boxed{\text{This will work}}$

Try u-sub

$u = \sin x$

$du = \cos x \, dx$

$$\int \sin^2 x \cos^3 x \, dx$$

$$\int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos^2 x}_{1-\sin^2 x = 1-u^2} \cdot \underbrace{\cos x \, dx}_{du}$$

$$\int u^2 (1-u^2) \, du$$

Example 2. $\int \sin^5 x \cos^2 x \, dx = \int \underbrace{\sin^4 x}_{u^4} \cdot \underbrace{\cos^2 x}_{1-\sin^2 x = 1-u^2} \cdot \underbrace{\sin x \, dx}_{-du}$

$u = \cos x$

$du = -\underbrace{\sin x \, dx}$

$$= \int u^4 \cdot (1-u^2) \cdot (-du)$$

$$= -\int u^4 - u^6 \, du = \int u^6 - u^4 \, du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

Example 3. $\int \tan^6 x \sec^4 x \, dx \longrightarrow \int \underbrace{\tan^5 x}_{\text{can't deal w/}} \cdot \underbrace{\sec^3 x}_{u^3} \cdot \underbrace{\sec x \tan x \, dx}_{du}$

Try $u = \sec x$ (won't work)

$du = \sec x \tan x \, dx$

Try $u = \tan x$

$du = \sec^2 x \, dx$

$$\to \int \underbrace{\tan^6 x}_{u^6} \; \underbrace{\sec^2 x}_{\tan^2 x - 1 = u^2 - 1} \cdot \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int u^6 (u^2 - 1) \, du = \int u^8 - u^6 \, du$$

$$= \frac{1}{9} u^9 - \frac{1}{7} u^7 + C$$

$$= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C$$

Example 4. $\int_0^{\pi/2} (1 - \sin x)^2 \, dx$

Example 5. $\int \sin x \sec^5 x \, dx$ $= \int \frac{\sin x}{\cos^5 x} dx$

$u = \cos x$

$du = -\sin x \, dx$

$= -\int \frac{du}{u^5} = -\frac{1}{-4} u^{-4} + C = \frac{1}{4} \frac{1}{u^4} + C$

$= \frac{1}{4} \frac{1}{\cos^4 x} + C = \frac{1}{4} \sec^4 x + C$

Example 6. $\int \sqrt{1 + \cos(2x)} \, dx$ $= \int \sqrt{2\cos^2 x} \, dx = \int \sqrt{2} \cdot \sqrt{\cos^2 x} \, dx$

$= \sqrt{2} \int |\cos x| \, dx$

$\frac{1 + \cos(2x)}{2} = \cos^2(x)$

$1 + \cos(2x) = 2\cos^2 x$

Example 7. $\int \sin(5x) \cos(4x) \, dx$ $=$

$\sin(5x)\cos(4x) = \frac{1}{2}[\sin(5x - 4x) + \sin(5x + 4x)] = \frac{1}{2}[\sin x + \sin 9x]$

$= \frac{1}{2} \int \sin x + \sin 9x \, dx$

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7.2 Trigonometric Integrals

In this section, we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

Example 3: Evaluate $\displaystyle\int \tan^6 x \sec^4 x \, dx$

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7.2 Trigonometric Integrals Continued

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

$$\text{(a)}\quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)}\quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)}\quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 1: Evaluate $\displaystyle\int \sin(5x)\, \cos(4x)\, dx$