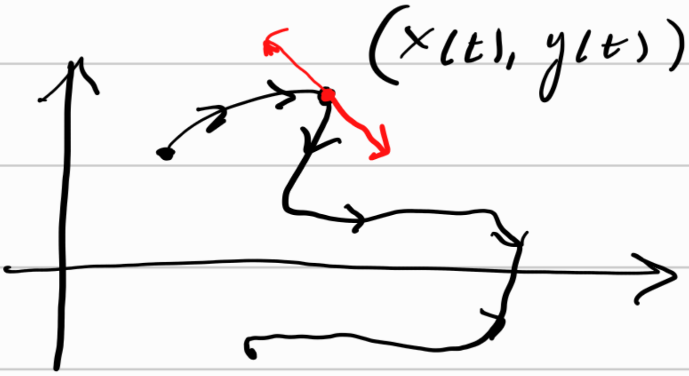
§ 10.2 Calculus w/ Parametric eq.



Consider a curve given parametrically by t. How can we talk about the line tangent to a pt. (x,y)?

We would like some kind of rise-over-run type situation. Let's consider, if we happen to have y as a function of x like we normally do (ie. $y(x)$)

If x is still given parametrically (ie. $x(t)$) Then we have $y(x(t))$ & its derivative with respect to t is

$$\frac{d}{dt}\left[y(x(t))\right] = y'(x(t)) \cdot x'(t)$$ [Newton's notation]

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$ [Leibniz' notation]

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

$\hookrightarrow \frac{d}{dx}[y]$

The (x,y) slope given parametric descriptions for (x,y).

For the 2^{nd} derivative it gets crazier

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{y'(t)}{x'(t)}\right]$$

$$= \frac{x'(t)\cdot y''(t) - y'(t)\cdot x''(t)}{[x'(t)]^2}$$

$\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{x'\cdot y'' - y'\cdot x''}{(x')^3}$$

I don't reccomend committing this to memory

It's better to know $\dfrac{d}{dx}(☺) = \dfrac{\frac{d}{dt}[☺]}{dx/dt}$

where (☺) is whatever symbol is relavent i.e. y, $\frac{dy}{dx}$, etc.

Using this form of the derivative we can answer questions about the tangent line.

- If $y'(t)=0$ & $x'(t)\neq 0$ then we have a horiz. tangent
- If $x'(t)=0$ & $y'(t)\neq 0$ then we have a vert. tangent
- If both $x'(t)$ & $y'(t)$ are zero then we have to do a limit arguement to classify what we have

<u>example</u> The cycloid given parametrically by

$$x(\theta) = r(\theta - \sin\theta)$$
$$y(\theta) = r(1-\cos\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{r(0 + \sin\theta)}{r(1 - \cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

we see if $\sin\theta = 0$ we have POTENTIAL horiz. tangents

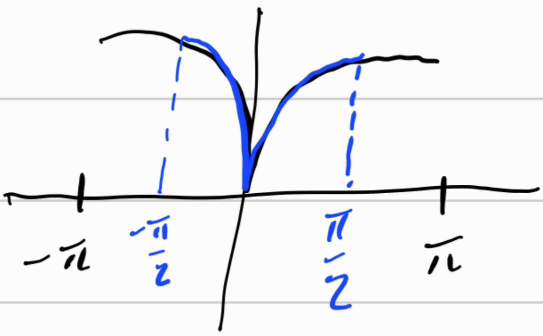
↳ @ $\theta = \pi k$ for integer k (i.e. $k \in \mathbb{Z}$)

If $1-\cos\theta = 0$ we have POTENTIAL vert. tangents

↳ $1 = \cos\theta \Rightarrow$ @ $\theta = 2\pi k$ for integer k

Simplify the problem on the interval $(-\pi/2, \pi/2)$

we have



Note @ $\theta = 0$

$$\text{“} \frac{\sin(0)}{1-\cos(0)} = \frac{0}{0} \text{”}$$

Using quotes b/c "$\frac{0}{0}$" is not a proper mathematical expression

So to classify the tangent @ $\theta = 0$ we look

@

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \text{"} \frac{0}{0} \text{"}$$

quotes b/c these are not a mathematically correct statement

L'Hôpital's

$$= \lim_{\theta \to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = \text{"} \frac{1}{0^+} = +\infty \text{"}$$

Sim. $\displaystyle \lim_{\theta \to 0^-} \frac{\sin\theta}{1-\cos\theta} \overset{LH}{=} \text{"} \frac{1}{0^-} = -\infty \text{"}$

So since $\theta = 0$ gives some kind of "$\frac{1}{0}$" for dy/dx the tangent line is vertical @ $\theta = 0$. //

10.2: Calculus with Parametric Curves

1 Derivatives

Now that we are talking about curves that are described parametrically, that is, $(x(t), y(t))$ is some collection of points in a (x, y)-grid that we can trace out "forwards and backwards in time," it's time to do some calculus with these objects. A lot of it comes as a consequence of the chain rule:

Derivatives of Parametric Curves:

Consider the parametric curve $(x(t), y(t))$. If y is also a differentiable funciton of x (that is, we can write $y(x)$ and take a derivative), then by the chain rule:

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ y'(t) &= y'(x(t))x'(t) \end{aligned}$$

and if $x'(t)$ is nonzero then we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

Geometrically (using calc 3 concepts), the point on the curve $(x(t), y(t))$ a tangential velocity vector given by $(x'(t), y'(t))$. This vector lies in the line tangent to the curve at a given point. i.e. the slope of that vector (a rise over run of sorts) gives the slope of the curve at that point.

Consequently we also see that that if we replace y with $\frac{dy}{dx}$ we get

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{\dfrac{d}{dt}\left(\dfrac{dx}{dy}\right)}{\dfrac{dx}{dt}} \\ &= \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3} \end{aligned}$$

Example 1. Consider the curve give by $x = t^2$ and $y = t^3 - 3t$. We will

(a) Determine where the tangents are vertical and horizontal

(b) Determine where the curve is concave up or down

(c) Show that a parametric curve can have multiple tangent lines at a point (but not for a t value), specifically at $(x, y) = (3, 0)$

(cont. from prev page) $x = t^2$ and $y = t^3 - 3t$

$x'(t) = 2t$

$0 = 2t$

$\Rightarrow t = 0$

potential vert. tangent

$y'(t) = 3t^2 - 3$

$0 = 3t^2 - 3$

$t = \pm 1$

potential horiz. tangent

No overlap means @ $t=0$ we have vert. tangents

@ $t=1$ & $t=-1$ we have horiz. tangents.

For concavity we want $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-3}{2t} = \frac{3}{2}t - \frac{3}{2}\frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dt}\right]}{x'(t)} = \frac{\frac{d}{dt}\left[\frac{3}{2}t - \frac{3}{2}\frac{1}{t}\right]}{x'(t)}$$

$$= \frac{\frac{3}{2} + \frac{3}{2}\frac{1}{t^2}}{2t} = \frac{3}{4t} + \frac{3}{4t^3}$$

$$\frac{d^2y}{dx^2} = \frac{3t^2+3}{4t^3}$$

Set equal to zero for inflection pt.

=> no such pt.

But $t=0$ is when bot $=0$.

setup sign chart

t	-1	0	1
$3t^2+3$	+	+	+
$4t^3$	−	0	+
d^2y/dx^2	−	DNE	+
concavity	C.D.	inflection pt.	C.U

So the curve is C.D. on $(-\infty, 0)$

& C.U. on $(0, \infty)$

2 Area lmao no

Area with parametric curves is an incredibly large and complicated topic. Most if not all introductory textbooks don't cover this material well, and we *certainly* don't have the time to also cover this material well. For now, we will only consider the simplest case and note this cane still produce *weird, but* ***valid*** results.

Area with Parametric Curves (Simplified):

Consider the parametric curve $(x(t), y(t))$ and the integral $\int_a^b y\,dx$ (i.e. **using the idea that an integral collects together the width*height of a bunch of rectangles of width** dx **and height** y), then using the substitution $y = y(t)$ and $x = x(t)$ then

$$\int_a^b y\,dx = \int_\alpha^\beta y(t)x'(t)\,dt \qquad \left(\text{or} \quad \int_\beta^\alpha y(t)x'(t)\,dt\right)$$

where $x(\alpha)$ is either a or b and $x(\beta)$ is the remaining one. Similarly, by a similar idea:

$$\int_c^d x\,dy = \int_\gamma^\delta x(t)y'(t)\,dt \qquad \left(\text{or} \quad \int_\delta^\gamma x(t)y'(t)\,dt\right)$$

But why is this simplified? Well, the textbook only covers this case, but it is possible to have negative areas within this framework and it is still valid. The details are quite messy and not suitable for calc 2.

Example 1. Finding the area under the first arch of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$.

Recall when we built the arc length formula we use triangles

w/ Approx $\approx \sqrt{\Delta x^2 + \Delta y^2}$

If x & y are given parametrically the construction is simplified & thus is good enough for our approx.

$\Rightarrow$ Since x & y depend on t if x or y changes the t is changing ie.

Approx. (parametric) $\approx \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta t^2\left[\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right]}$

$$= \Delta t \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$\Rightarrow$ Arc length for a parametric system over $[\alpha, \beta]$ in t is

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3 Arc Length

Arc Length of Parametric Curves:

Consider the parametric curve $(x(t), y(t))$ and recall that (one version of) arc length is given by the integral $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. If we follow the same substitution idea from before we get

$$\text{Arc Length} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Note that this means that **we don't need to write y in terms of x or vise-versa. This version of Arc Length** (and the version used in calc 3 and beyond) **can be used for any parametric description of a curve**. In fact, we could've derived this version of Arc Length back in section 8.1 if we knew about parametric equations, and then we could have ignored the mean value theorem step.

Writing $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ we get that arc length is by definition $\int ds$ (if you go back in the notes, this ds is present for the 8.1 material).

Define the **arc length function** for a parametrized curve as

$$s(t) := \int_\alpha^t ds = \int_\alpha^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Note this means $s'(t) = \sqrt{x'(t)^2 + y'(t)^2}$

Example 1. Finding the length of the first arch of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1-\cos\theta)$.

$[0, 2\pi]$

$$\boxed{x'(\theta) = r(1-\cos\theta)}$$

$$y'(\theta) = r\sin\theta$$

$$\int_0^{2\pi} \sqrt{[r(1-\cos\theta)]^2 + [r\sin\theta]^2}\,d\theta$$

$$\int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

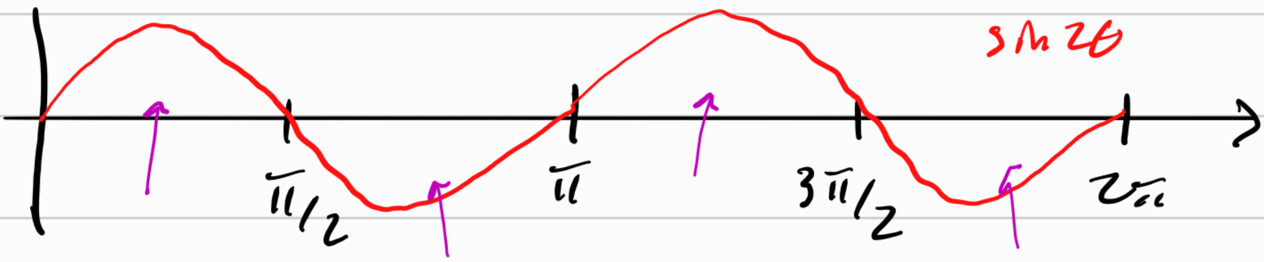
$$= \int_0^{2\pi} r\sqrt{1-2\cos\theta + \underbrace{\cos^2\theta+\sin^2\theta}_{=1}}\, d\theta$$

$$= r\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$

Half angle $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= r\int_0^{2\pi}\sqrt{4\sin^2(2\theta)}\,d\theta$$

$$= 2r\int_0^{2\pi}|\sin 2\theta|\,d\theta$$



$$= 2r\left[\int_0^{\pi/2}\sin 2\theta\,d\theta - \int_{\pi/2}^{\pi}\sin 2\theta\,d\theta + \int_{\pi}^{3\pi/2}\sin 2\theta\,d\theta - \int_{3\pi/2}^{2\pi}\sin 2\theta\,d\theta\right]$$

OR by exploiting symmetry note all regions have the same area so they are all equal i.e.

$$= 2r\cdot 4\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= 8r \cdot \left(-\frac{1}{2}\right)\left[\cos 2\theta\right]_0^{\pi/2}$$

$$= -4r\left[\cos(\pi) - \cos(0)\right]$$

$$= -4r\left[-1 - 1\right] = 8r$$

4 Hard Problems

Problem 1. **(Hard)** Find the length of the curve given by $x = t\sin t$ and $y = t\sin t$ on $0 \leq t \leq \dfrac{\pi}{2}$.