7.4: Integration of Rational Functions and Partial Fractions

The goal of this lesson is to develop another technique of integration to handle integrals of **rational functions**.

Recall that a **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 + a_0$$

i.e. x^2+1 $\quad x \quad$ $x-1$ $\quad 7 \quad$ $4x^{62}+7x^7+3x-2.$

The highest ordered power is called the **degree** of the polynomial. So, in the above examples they are (in order) 2,1,1,0, and 62. A **rational function** $f(x)$ is a function that is a fraction involving two polynomials $P(x)$ and $Q(x)$ on the top and bottom:

$$f(x) = \frac{P(x)}{Q(x)}.$$

Put another way. A rational function is a fraction that has a polynomial on top and a polynomial on the bottom. Some examples of this would be

$$\frac{x^2+1}{x-1} \qquad \frac{x-1}{x^2+1} \qquad \frac{x-1}{x+1} \qquad x$$

We say that a rational function is **improper** if the degree of the bottom is less than or equal to the degree of the top. We say that a rational function is **proper** if the bottom has a bigger degree than the top. In the above examples, only the red expression is proper.

To integrate **improper rational functions** we first need to do polynomial long division:

Example 1. $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

Skip. Even during Spring/Fall needing to do Long division is not tested on

The way we integrate a "proper rational function" is to break it up via PFD & integrate the components

$$\int \frac{dx}{x(x+1)} \xrightarrow{PFD} \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

This is an algebraic technique.

The idea is that when we add fractions:

we start w/ 2 (or more) fractions

$$\frac{1}{x} + \frac{2}{y} = \frac{1}{x}\cdot\frac{y}{y} + \frac{2}{y}\cdot\frac{x}{x}$$

$$= \frac{y+2x}{xy}$$

Then end up w/ 1 fraction

PFD starts $\boxed{\text{here}}$ & goes back to the individual fractions.

To integrate a proper rational function, we need to "break up" the integral is a clever way. This is called **partial fraction decomposition**. The idea comes from wanting to combine fractions with different denominators we have to multiply by what's missing:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\frac{d}{d} + \frac{c}{d}\frac{b}{b} = \frac{ad + cb}{bd}$$

Partial fraction decomposition is about reversing this process. That is, we start with the fraction $\frac{E}{bd}$ and we want to find numbers A and C that let's us decompose it into

$$\frac{E}{bd} = \frac{A}{b} + \frac{C}{d}.$$

Example 2. Decompose the rational function into partial fractions: $\dfrac{x+5}{x^2+x-2}$.

x^2+x-2 should be converted into a product since PFD requires us to identify the factors in the denominator

$$x^2+x-2 = (x+2)(x-1)$$

unknown unknown

$$\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

As you might expect with rational function integration based off of partial fractions, we'll need to integrate $\dfrac{1}{ax+b}$ a lot, so, in short:

$$\int \frac{dx}{ax+b} =$$

$$(x+2)(x-1)\left(\frac{x+5}{(x+2)(x-1)}\right) = \left(\frac{\overset{\text{unknown}}{A}}{x+2} + \frac{\overset{\text{unknown}}{B}}{x-1}\right)(x+2)(x-1)$$

$$x+5 = \frac{A\cancel{(x+2)}(x-1)}{\cancel{(x+2)}} + \frac{B(x+2)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$x+5 = A(x-1) + B(x+2)$$

There are then 3 ways to solve for A & B.

(i) Plug in values for x useful for simple PFD.

(ii) Set up a system of eq. & solve useful for complicated PFD

(iii) Both (i) + (ii) (Fastest for more complicated ones)

Method (i) $x+5 = A(x-1) + B(x+2)$

Plug in x-values

↳ We can use ANY x-value, but <u>sometimes</u> we have "smart choices"

@ $x=1 \Rightarrow 1+5 = A \cdot 0 + B(1+2)$

$$6 = 3B$$

$$2 = B$$

$$@\ x = -2 \Rightarrow -2+5 = A(-2-1) + 2\cancel{(-2+2)}^{\,0}$$

$$3 = -3A$$

$$-1 = A$$

$$\Rightarrow \frac{x+5}{(x+2)(x-1)} = \frac{-1}{x+2} + \frac{2}{x-1}$$

Method (ii) $x+5 = A(x-1) + B(x+2)$

↳ Setup a system of equations

like terms

$$x+5 = Ax - A + Bx + 2B$$

like terms

$\Rightarrow$ The like terms ONLY interact & see each other so

$$x = Ax + Bx \xrightarrow{\text{simplifies}} 1 = A + B$$

$$5 = -A + 2B \qquad\qquad 5 = -A + 2B$$

$$6 = 0 + 3B$$

$$B = 2$$

$$1 = A + 2 \Rightarrow A = -1$$

We can't show (iii) on this example.

Case I: The denominator $f(x) = \frac{P(x)}{Q(x)}$ is a product of distinct linear factors (ie. a bunch of degree 1 polynomials):

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

i.e. $(x+1)(x-2)$ $x(x-2)$ $4x(3x+1)(5x-2)$ *etc.*

The partial fraction decomposition (for $Q(x)$) is:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_n}{a_nx + b_n}$$

Example 2.5. Evaluate $\int \frac{x+5}{x^2+x-2}dx$

by prev. example

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{-1}{x+2} + \frac{2}{x-1}\,dx$$

$$= -\ln|x+2| + \underbrace{2\ln|x-1|} + C$$

$$= -\ln|x+2| + \ln\left[(x-1)^2\right] + C$$

$$= \ln\left|\frac{(x-1)^2}{x+2}\right| + C$$

Case II: The denominator $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor (a_1x+b_1) is repeated r times, that is

$$Q(x) = (a_1x+b_1)^r(a_2x+b_2)\cdots(a_kx+b_k)$$

In this case, there exist constants $A_1, A_2, \ldots A_k$ to be determined such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r} + \frac{A_{r+1}}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Example 3: Consider $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$.

(a) Write out the form of the partial fraction decomposition of the function $\dfrac{4x}{(x-1)^2(x+1)}$.

$$\frac{4x}{\underbrace{(x-1)^2}(x+1)} = \underbrace{\frac{A}{(x-1)} + \frac{B}{(x-1)^2}} + \frac{C}{x+1}$$

A repeated factor has to show up that many times in the PFD.

example $\dfrac{1}{(x-1)^3(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{x+1} + \dfrac{E}{(x+1)^2}$

(b) Evaluate the integral.

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Chap 7 Techniques of Integration

UD ID: ______________________ Name: ______________________

Irreducible quadratics have complex or imaginary roots

7.4 Integration of Rational Functions by Partial Fractions: Continued

Case III: The denominator $Q(x)$ contains irreducible factors, none of which is repeated. If $Q(x)$ has the factor $(ax^2 + bx + c)$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in Case I and II, the expression $P(x)/Q(x)$ will have a partial fraction of the form

eg. x^2+1

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

How to solve:

A given problem $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$.

Step 1: Write out the form of the partial fraction decomposition of the function $\dfrac{P(x)}{Q(x)}$.

Step 2: Evaluate the integral.

Recall:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

(Proof by trig-sub)

Example 1: Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

$$x^3 + 4x = x(x^2+1)$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx + C)x$$

Using method (iii)

$$@\ x = 0 \Rightarrow 0 - 0 + 4 = A(0+4) + (\text{idc}) \cdot 0$$

$$4 = A4$$

$$\Rightarrow 1 = A$$

$$\Rightarrow 2x^2 - x + 4 = x^2 + 4 + (Bx + C)x$$

$$-4 \qquad\qquad -4$$

$$2x^2 - x = x^2 + Bx^2 + Cx$$

The only "x" terms so $C = -1$

$$2 = 1 + B \Rightarrow B = 1$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{1}{x} + \frac{x - 1}{x^2+4}$$

$$\int (\text{that})\,dx = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$$

$$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$\hookrightarrow \ln|x|$ $\quad$ $\hookrightarrow$ u-sub $\quad$ $\hookrightarrow \frac{1}{4}\arctan\left(\frac{x}{4}\right)$

$$= \int \frac{1}{x} - \frac{1}{x^2+4}\,dx + \int \frac{x}{x^2+4}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$$= \ln|x| - \frac{1}{4}\arctan\left(\frac{x}{4}\right) + \frac{1}{2}\int \frac{du}{u}$$

$$= \ln|x| - \frac{1}{4}\arctan\left(\frac{x}{4}\right) + \frac{1}{2}\ln|x^2+4| + C$$

Case IV: The denominator $Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in Case III, we should have partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example 2: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$$\frac{x^3 + x^2 + 1}{x(x-1)\underbrace{(x^2 + x + 1)}(x^2+1)^2}$$

$\hookrightarrow \dfrac{-1 \pm \sqrt{1-4\cdot 1\cdot 1}}{2} \in \mathbb{C}$

irreducible

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}$$

Example 3: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$$\frac{x^4 + x^2 + 1}{(x^2+1)(2x^2+3)^2}$$