7.4: Integration of Rational Functions and Partial Fractions

The goal of this lesson is to develop another technique of integration to handle integrals of **rational functions**.

Recall that a **polynomial** is a function of the form

$\dots + a_1x$

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cancel{a_1} + a_0$$

i.e. $\quad x^2 + 1 \qquad x \qquad x - 1 \qquad 7 \qquad 4x^{62} + 7x^7 + 3x - 2.$

The highest ordered power is called the **degree** of the polynomial. So, in the above examples they are (in order) 2,1,1,0, and 62. A **rational function** $f(x)$ is a function that is a fraction involving two polynomials $P(x)$ and $Q(x)$ on the top and bottom:

$$f(x) = \frac{P(x)}{Q(x)}.$$

Put another way. A rational function is a fraction that has a polynomial on top and a polynomial on the bottom. Some examples of this would be

$$\frac{x^2+1}{x-1} \qquad \boxed{\frac{x-1}{x^2+1}} \qquad \frac{x-1}{x+1} \qquad x$$

We say that a rational function is $\boxed{\textbf{improper}}$ if the degree of the bottom is less than or equal to the degree of the top. We say that a rational function is **proper** if the bottom has a bigger degree than the top. In the above examples, only the red expression is proper.

To integrate **improper rational functions** we first need to do polynomial long division:

Example 1. $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

Skip. Even during Spring/Fall needing to do Long division is not tested on

$$\begin{array}{r} x^2 - x + 2 \\ x+1 \overline{\big) x^3 + x} \\ -(x^3 + x^2) \\ \hline -x^2 + x \\ -(-x^2 - x) \\ \hline 2x \\ -(2x+2) \\ \hline 2 \end{array}$$

$(x+1)\cdot x^2 = x^3 + x^2$

$(x+1)\cdot(-x) = -x^2 - x$

$(x+1)\cdot 2 = 2x + 2$

degree < degree of $x+1$

$$\Rightarrow \frac{x^3+x}{x+1} = x^2 - x + 2 + \underbrace{\frac{2}{x+1}}_{\text{remainder}}$$

$$\int \frac{x^3+x}{x+1}\,dx = \int x^2 - x + 2 + \frac{2}{x+1}\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + 2\ln|x+1| + C$$

To integrate a proper rational function

$\boxed{\text{eg } \frac{x-1}{x(x+1)} \text{ or } \frac{1}{(x^2+1)x}}$

we'll use an algebraic technique called Partial Fraction Decomposition (PFD)

$$\underbrace{\int \frac{1}{x(x+1)}\,dx}_{\text{Hard to integrate}} \xrightarrow{\text{PFD}} \underbrace{\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx}_{\text{nice to Integrate}}$$

$\Rightarrow$

PFD comes from "undoing" the process of adding 2 fractions

If we want these as 1 fraction we do the following

$$\underbrace{\frac{1}{x} + \frac{2}{y}}_{} = \frac{1}{x}\cdot\frac{y}{y} + \frac{2}{y}\cdot\frac{x}{x}$$

$$= \frac{y}{xy} + \frac{2x}{yx} = \underbrace{\frac{y+2x}{xy}}_{}$$

PFD starts here w/ 1 fraction & ends up here w/ multiple fractions

To integrate a proper rational function, we need to "break up" the integral is a clever way. This is called **partial fraction decomposition**. The idea comes from wanting to combine fractions with different denominators we have to multiply by what's missing:

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\frac{d}{d}+\frac{c}{d}\frac{b}{b}=\frac{ad+cb}{bd}$$

Partial fraction decomposition is about reversing this process. That is, we start with the fraction $\frac{E}{bd}$ and we want to find numbers A and C that let's us decompose it into

$$\frac{E}{bd}=\frac{A}{b}+\frac{C}{d}.$$

Example 2. Decompose the rational function into partial fractions: $\dfrac{x+5}{x^2+x-2}$.

To use PFD we need to identify the factors of the denominator. $x^2+x-2=(x+2)(x-1)$

$$\cancel{(x+2)}\cancel{(x-1)}\left[\frac{x+5}{\cancel{(x+2)}\cancel{(x-1)}}\right]=\left[\frac{A}{x+2}+\frac{B}{x-1}\right](x+2)(x-1)$$

As you might expect with rational function integration based off of partial fractions, we'll need to integrate $\dfrac{1}{ax+b}$ a lot, so, in short:

$$\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|+C$$

$$x+5 = \frac{A\cancel{(x+2)}(x-1)}{\cancel{x+2}} + \frac{B(x+2)\cancel{(x-1)}}{\cancel{x-1}}$$

$$x+5 = A(x-1) + B(x+2)$$

I'll 3 ways to solve for A & B.

(i) Plug in x-values

↳ You any x-values

Sometimes there are "nice" choices

(ii) Setup a system of eq. using like terms

(iii) Combo of (i) & (ii)

Method (i) $\quad x+5 = A(x-1) + B(x+2)$

@ $x=1 \Rightarrow \quad 1+5 = A\cdot 0 + B(1+2)$

$$6 = 3B$$

$$2 = B$$

$$x+5 = A(x-1) + 2(x+2)$$

@ $x=-2 \Rightarrow \quad -2+5 = A(-2-1) + 2\cdot 0$

$$3 = -3A$$

$A = -1$

(A info) (B info)

$$\frac{x+5}{(x+2)(x-1)} = \frac{-1}{x+2} + \frac{2}{x-1}$$

Method (ii) $x+5 = A(x-1) + B(x+2)$

↳ Setup a sys. of eq.

like terms

$x+5 = Ax - A + Bx + 2B$

like terms

⇒ like terms can ONLY interact & "see" other like terms
(add/sub)

x-info ⇒ $x = Ax + Bx \overset{\text{simplify}}{\Rightarrow} 1 = A + B$

no x-info ⇒ $5 = -A + 2B$

$$\begin{array}{r} 1 = A + B \\ + \quad 5 = -A + 2B \\ \hline 6 = 0 + 3B \end{array}$$

$2 = B$

$1 = A + 2$

$-1 = A$

Exactly what we got before

Our integration techniques are (suggested order)

(i) "Do it"

(ii) u-Sub

(iii) IBP

(iv) Is the function Rational? $\rightarrow$ Long Div. + $\boxed{\text{PFD.}}$

(v) Trig-sub

Case I: The denominator $f(x) = \frac{P(x)}{Q(x)}$ is a product of distinct linear factors (ie. a bunch of degree 1 polynomials):

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

i.e. $(x + 1)(x - 2)$ $x(x - 2)$ $4x(3x + 1)(5x - 2)$ *etc.*

The partial fraction decomposition (for $Q(x)$) is:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_n}{a_nx + b_n}$$

Example 2.5. Evaluate $\int \frac{x + 5}{x^2 + x - 2} dx$ This is quite difficult, but

$$\int \frac{x+5}{(x+2)(x-1)}\,dx \overset{\text{PFD}}{=} \int \frac{-1}{x+2} + \frac{2}{x-1}\,dx$$

$$= -\ln|x+2| + 2\ln|x-1| + C$$

But if for some reason you want to limit things w/ this you should simplify

$$2\ln|x-1| = \ln[(x-1)^2]$$

$$2\ln|x-1| - \ln|x+2| = \ln[(x-1)^2] - \ln|x+2| = \ln\left|\frac{(x-1)^2}{x+2}\right|$$

As you might expect. $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

Case II: The denominator $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor (a_1x+b_1) is repeated r times, that is

$$Q(x) = (a_1x+b_1)^r(a_2x+b_2)\cdots(a_kx+b_k)$$

In this case, there exist constants $A_1, A_2, ...A_k$ to be determined such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r} + \frac{A_{r+1}}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Example 3: Consider $\int \frac{4x}{(x-1)^2(x+1)}\,dx$.

(a) Write out the form of the partial fraction decomposition of the function $\frac{4x}{(x-1)^2(x+1)}$.

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

eg $$\boxed{\frac{4x}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

@ $x=1$ $\quad 4 = 0 + B(2) + 0$

$\Rightarrow B = 2$

@ $x=-1$ $\quad -4 = 0 + 0 + C(-2)^2$

$$-4 = 4C$$
$$-1 = C$$

$\Rightarrow$

$$4x = A(x-1)(x+1) + 2(x+1) - (x-1)^2$$

@ $x=0$ $\quad 0 = A(-1)(1) + 2 - (-1)^2$

$$= -A + 2 - 1 = -A + 1$$

$$\Rightarrow A = 1$$

$\Rightarrow$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

MATH 242 SPRING 2025
Chap 7 Techniques of Integration

UD ID: ______________________________ Name: ______________________________

7.4 Integration of Rational Functions by Partial Fractions: Continued

Case III: The denominator $Q(x)$ contains irreducible factors, none of which is repeated. If $Q(x)$ has the factor $(ax^2 + bx + c)$, where $\boxed{b^2 - 4ac < 0}$, then in addition to the partial fractions in Case I and II, the expression $P(x)/Q(x)$ will have a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

ax^2+bx+c

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

For example,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

How to solve:

A given problem $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$.

Step 1: Write out the form of the partial fraction decomposition of the function $\dfrac{P(x)}{Q(x)}$.

Step 2: Evaluate the integral.

Recall:

$$\underbrace{\int \frac{dx}{x^2 + a^2}}_{\text{or do this via trig-sub}} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Q: How do we know if a quad. is irreducible?

A: In Quadratic Formula you @ the "Discriminant" $\boxed{b^2-4ac}$ if this is a neg # The quad. is irreducible

The most "common quad" is $x^2 + \#$

$x \cdot (x^2+4)$ irreducible

Example 1: Find $\int \frac{2x^2 - x + 4}{\boxed{x^3 + 4x}}\, dx$.

PFD via Method (iii)

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$x(x^2+4)\left(\frac{2x^2 - x + 4}{x(x^2+4)}\right) = \left(\frac{A}{x} + \frac{Bx+C}{x^2+4}\right) x(x^2+4)$$

$$2x^2 - x + 4 = A(x^2+4) + \frac{(Bx+C)\, x\cancel{(x^2+4)}}{\cancel{(x^2+4)}}$$

Go through all "nice/smart" choices for x

@ $x = 0$

$$\underbrace{0 - 0}_{0} + 4 = A(0+4) + \cancelto{0}{(Bx+C)\cdot 0}$$

$$4 = A\,4$$

$$1 = A$$

$$2x^2 - x + 4 = x^2 + 4 + (Bx + C)x$$

$$-x^2 \quad -4 \quad -x^2 \quad -4$$

$$x^2 - x = Bx^2 + Cx$$

$B = 1$

Using like terms only Bx^2 sees x^2] → $Bx^2 = x^2$

& Cx sees $-x$] → $Cx = -x$

↳ $C = -1$

$$\int \frac{1}{x} + \frac{x-1}{x^2+4}\, dx$$

$$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\, dx$$

↳ ln|-| ↳ u-sub ↳ arctan -

$u = x^2 + 4$

$du = 2x\,dx$

$$= \int \frac{1}{x} - \frac{1}{x^2+4}\, dx + \int \frac{x}{x^2+4}\, dx$$

$$= \ln|x| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\int \frac{du}{u}$$

$$= \ln|x| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|x^2+4| + C$$

Case IV: The denominator $Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in Case III, we should have partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

b^2-4ac

Example 2: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$1-4\cdot1\cdot1=-3<0$

$$\frac{x^3 + x^2 + 1}{x(x-1)\underbrace{(x^2 + x + 1)}_{\text{irreducible}}\boxed{(x^2 + 1)^2}}$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}$$

Example 3: Write out the form of the partial fraction decomposition of the function. Do NOT determine the numerical values of the coefficients.

$$\frac{x^4 + x^2 + 1}{(x^2 + 1)(2x^2 + 3)^2}$$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{2x^2+3} + \frac{Ex+F}{(2x^2+3)^2}$$