Section 6.3: Shell Method

Background: More Than One Way to Slice an Onion

Suppose we want to find the volume of a solid (think: an onion). There is more than one way to "count volume":

- **Disk/Washer Method:** slice the solid into **cross-sections** (slices **perpendicular** to the axis of rotation).
- **Shell Method:** "peel" the solid into **layers** (slices **parallel** to the axis of rotation).

Disk/Washer vs. Shell Method (Key Difference)

Disk/Washer Method	**Shell Method**
Slice **perpendicular** to axis of rotation	Slice **parallel** to axis of rotation
Generate **cross-sections** (disks/washers)	Generate **layers** (cylindrical shells)
Area-based: $A(x)$ then integrate	Surface-area-based: $2\pi(\text{radius})(\text{height})$ then integrate

The Shell Method Idea

We divide the rotated region into **thin layers**. Each layer:

- has a **radius** (distance to the axis of rotation),
- has a **height** (top $-$ bottom or right $-$ left),
- is **parallel** to the axis of rotation.

When one layer is rotated, it forms a **(thin) cylindrical shell**.

Surface Area of a Cylindrical Shell

A cylindrical shell has:

$$\text{circumference} = 2\pi r \qquad \text{and} \qquad \text{height} = h$$

So the (lateral) surface area is:

$$\text{Surface Area} = 2\pi r h$$

Volume of One Rotated Layer

Key Volume Statement

$$\boxed{\text{Volume of thin layer} \approx (\text{Surface Area}) \cdot (\text{Width}) = (2\pi r h)\,(\Delta x \text{ or } \Delta y)}$$

Shell Method Formula (General)

$$\boxed{V = \int_a^b 2\pi\, r(\,\cdot\,)\, h(\,\cdot\,)\, d(\text{slice variable})}$$

- $r(\cdot)$ is the **radius** of the shell = "current" distance to the axis of rotation.
- $h(\cdot)$ is the **height** of the shell:
  - for vertical slices (dx): $h(x) = \text{top} - \text{bottom}$
  - for horizontal slices (dy): $h(y) = \text{right} - \text{left}$

Choosing dx vs. dy for Shells

Rule for Shell Method

Shell method uses slices *parallel* to the axis of rotation.

- Rotate about a **vertical** line ($x = c$ or the y-axis) $\Rightarrow$ use **vertical** slices $\Rightarrow dx$.
- Rotate about a **horizontal** line ($y = c$ or the x-axis) $\Rightarrow$ use **horizontal** slices $\Rightarrow dy$.

Shell Method Setup Checklist

1. Sketch the region and label the **axis of rotation (AoR)**.
2. Decide slice direction using: **slices parallel to AoR**.
3. Write the **radius** = distance from "current position" to the AoR.
4. Write the **height** as:

$$\text{top} - \text{bottom} \quad (\text{for } dx) \qquad \text{or} \qquad \text{right} - \text{left} \quad (\text{for } dy)$$

5. Find bounds (a to b) from intersection points or endpoints.
6. Set up:

$$V = \int_a^b 2\pi\, r(\cdot)\, h(\cdot)\, d(\text{variable})$$

Example 1

Find the volume of the solid obtained by rotating about the y-axis the region bounded by:

$$y = 2x^2 - x^3 \qquad \text{and} \qquad y = 0$$

Example 2

Find the volume of the solid obtained by rotating about the line $x = -1$ the region bounded by

$$y = x \qquad \text{and} \qquad y = x^2.$$

Example 3

Use the shell method to find the volume of the solid obtained by rotating about the x-axis the region under the curve

$$y = \sqrt{x}, \qquad 0 \leq x \leq 1.$$

Example 4

Set up (but do not evaluate) the volume of the solid obtained by rotating about the line $x = 2$ the region bounded by

$$y = x - x^2 \qquad \text{and} \qquad y = 0.$$

Sketch, identify $r(x)$ and $h(x)$, determine bounds, and set up the integral.