



## Efficient implicit scheme with positivity preserving and smoothing properties

Mariyan Milev<sup>a,\*</sup>, Aldo Tagliani<sup>b,1</sup>

<sup>a</sup> Department of Informatics and Statistics, University of Food Technologies, bul. Maritsa 26, 4002 Plovdiv, Bulgaria

<sup>b</sup> Department of Computer and Management Sciences, Trento University, Str. Inama 5, 38 100 Trento, Italy

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### ABSTRACT

Using classical finite difference schemes often generates numerical drawbacks such as spurious oscillations in the solution of the famous Black–Scholes partial differential equation. We analyze the fully implicit scheme, frequently used numerical method in Finance, that in the presence of discontinuous payoff and low volatility arises spurious oscillations. We propose a modification of this scheme so that we guarantee a smooth numerical solution, free of spurious oscillations and satisfies the positivity requirement, as is demanded for the financial solution of the Black–Scholes equation. The method is used, within the strategy suggested by Rannacher, only in few initial time steps in the presence of discontinuous initial conditions. As a consequence, although the method is low order accurate, it returns a spurious oscillations free solution. Next, starting from the smooth initial condition obtained, any other family of arbitrary higher order schemes may be used.

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### 1. Introduction and preliminaries

In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option. The solution of the Black–Scholes partial differential equation determines the option price, respectively according to the used initial conditions, [1].

In the absence of a valuation formula for non-standard options (see [2]), the finite difference approach is a powerful tool for pricing. Usually the choice goes toward numerical methods with high order of accuracy (e.g. Crank–Nicolson scheme), and no attention is paid to how the financial provision of the contract can affect the reliability of the numerical solution; see Duffy in [3], Smith in [4], Tavella and Randall in [5], or Zvan et al. in [6]. Indeed these schemes are applied without considering the well-known problems that can arise in the presence of *discontinuities* in the initial conditions of the Black–Scholes equation which deteriorate the numerical approximation.

In general, finite difference schemes have to satisfy simultaneously several properties. They have to be

1. higher order rates of convergence;
2. unconditionally stable;
3. spurious oscillations free, if non-smooth initial conditions are present.

\* Corresponding author. Tel.: +359 32 603701.

E-mail addresses: [mariyan\\_milev@yahoo.com](mailto:mariyan_milev@yahoo.com) (M. Milev), [aldo.tagliani@unitn.it](mailto:aldo.tagliani@unitn.it) (A. Tagliani).

<sup>1</sup> Tel.: +39 0461 882116; fax: +39 0461 882124.

Then it appears hard task to find methods satisfying simultaneously all requirements 1–3. A strategy was suggested in [7] by proposing a scheme in which Crank–Nicolson timestepping is preceded by a finite number of full implicit steps. The rationale is that high frequency error components will be damped by the fully implicit steps leading to a smooth solution. That strategy is shared also in [8]. The expected rate of convergence remains quadratic since only a finite number of implicit steps are taken. Nevertheless, that strategy is based upon the assumption that the fully implicit scheme is free of spurious oscillations even in the presence of low values of volatility. As we will prove in the next section, whenever the financial parameters of the Black–Scholes model  $\sigma$  and  $r$  satisfy the relationship  $\sigma^2 \ll r$ , the fully implicit scheme can arise spurious oscillations. The latter are consequence of complex eigenvalues of the iteration matrix associated. Spurious oscillations appear mainly in the presence of small time steps, contrary to common sense.

In this paper, we propose a modification of fully implicit scheme which, in the first phase of the strategy suggested in [7], operates during the first few time steps in the presence of a discontinuous initial condition. We prove that, even under the severe condition  $\sigma^2 \ll r$ , the modified implicit scheme

1. provides an accurate spurious oscillations free solution;
2. is positivity-preserving and satisfies the discrete maximum principle;
3. it is time and space consuming, being low order accurate. Nevertheless, it is used only in few time steps.

The properties 1–3 of the modified implicit scheme are not trivial. Indeed, both Crank–Nicolson and fully implicit scheme, under the restrictive condition  $\sigma^2 \ll r$ , provide solutions affected by spurious oscillations regardless of small time step used, as a consequence of complex eigenvalues in the corresponding iteration matrix (with Crank–Nicolson scheme, spurious oscillations arise from real negative eigenvalues close to  $-1$  too). In order to make our analysis concrete, we concentrate our attention on a double barrier knock-out call option with a discrete monitoring clause. Such option has a payoff condition equal to  $\max(S - K, 0)$  but the option expires worthless if before the maturity the asset price has fallen outside the corridor  $[L, U]$  at the prefixed monitoring dates. In the intermediate periods the Black–Scholes equation is applied over the real positive domain. The discontinuity in the initial conditions will be renewed at every monitoring date and often the Crank–Nicolson numerical solution is affected by *spurious oscillations* that do not decay quickly, Fusai et al. in [9]. The oscillations derive from an inaccurate approximation of the very sharp gradient produced by the knock-out clause, generating an error that is damped out very slowly. At each monitoring date the Rannacher strategy is restarted, so that the implicit scheme is applied uniquely in the presence of an initial discontinuous condition. The presented analysis can be easily extended to many other exotic contracts (digital, supershare, binary and truncated payoff options, callable bonds and so on).

In Section 2 we discuss the model for *discrete double barrier knock-out call options* and particularly the main drawbacks, like undesired spurious oscillations, arising from centered difference discretization of the Black–Scholes PDE.

In Section 3 we propose a *modification of the fully implicit scheme* that enables us to solve accurately the examined PDE. An important factor for numerical schemes is the *condition of positivity of the solution* that must be satisfied as a consequence of the financial meaning of the involved PDE.

In Section 4 we explore examples of options that are characterized with discontinuities in the payoff and thus having discontinuous initial conditions in the Black–Scholes equation. Such options are digital and discrete barrier knock-out options that are most frequently used in the literature such as Zvan et al. in [6]. We have pointed out the advantages of the modified implicit scheme. In the conclusion, we give some final remarks for our method.

## 2. Mathematical model. The Black–Scholes PDE

We consider as a model for the movement of the asset price under the risk-neutral measure a standard *geometric Brownian motion* diffusion process with constant coefficients  $r$  and  $\sigma$ :

$$dS/S = rdt + \sigma dW_t. \quad (1)$$

The contract to be priced is a discretely monitored double barrier knock-out call option. If  $t$  is the time to expiry  $T$  of the contract,  $0 \leq t \leq T$ , the price  $V(S, t)$  of the option satisfies the Black–Scholes partial differential equation

$$-\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (2)$$

endowed with initial and boundary conditions:

$$V(S, 0) = \max(S - K, 0) \mathbf{1}_{[L,U]}(S) \quad (3)$$

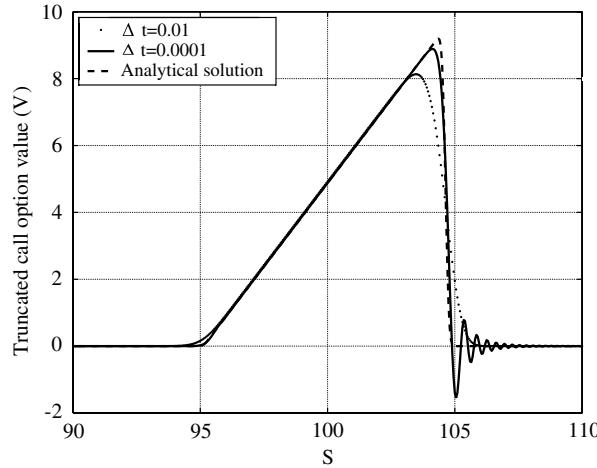
$$V(S, t) \rightarrow 0 \text{ as } S \rightarrow 0 \text{ or } S \rightarrow \infty \quad (4)$$

with updating of the initial condition at the monitoring dates  $t_i$ ,  $i = 1, \dots, F$ :

$$V(S, t_i) = V(S, t_i^-) \mathbf{1}_{[L,U]}(S), \quad 0 = t_0 < t_1 < \dots < t_F = T \quad (5)$$

where  $\mathbf{1}_{[L,U]}(x)$  is the indicator function, i.e.,

$$\mathbf{1}_{[L,U]} = \begin{cases} 1 & \text{if } S \in [L, U] \\ 0 & \text{if } S \notin [L, U]. \end{cases}$$



**Fig. 1.** Truncated call option value just before first monitoring date  $t_1 = T$ . Numerical solutions are obtained using a fully implicit scheme with  $\Delta S = 0.05$ ,  $\Delta t = 0.01$  and  $\Delta t = 0.0001$  respectively. Parameters:  $L = 90$ ,  $K = 100$ ,  $U = 110$ ,  $r = 0.05$ ,  $\sigma = 0.001$ ,  $T = 1$ ,  $S_{\max} = 200$ .

Here  $r = r(t, S)$  and  $\sigma = \sigma(t, S)$  may be considered. The knock-out clause at the monitoring date introduces a discontinuity at the barriers. The presence of undesired spurious oscillations is frequently observed in the Crank–Nicolson solution near the barriers and near the strike, where the Delta  $= \frac{\partial V}{\partial S}$  is discontinuous at  $t = 0$ ; see Milev and Tagliani in [10].

These spikes, which remain well localized, do not reflect instability but rather that the discontinuities that are periodically produced by the barriers at monitoring dates cannot decay fast enough. Unlike the common opinion, the fully implicit solution too is affected by spurious oscillations if the relationship  $\sigma^2 \ll r$  holds, as is illustrated in Fig. 1 for a truncated call option.

In [10] Milev and Tagliani show that mathematically, such spurious oscillations stem from the combined effect of two factors, as

1. positivity of the solution  $V(S, t)$  not preserved;
2. presence of negative or complex eigenvalues in the spectrum of the corresponding matrix originating from the finite difference equation.

Then the fully implicit scheme and a variant of it will be investigated. The latter has the following properties

1. the solution is positive and satisfies conditionally the maximum principle;
2. the spectrum contains only positive eigenvalues;
3. is spurious oscillations free.

### 3. Finite difference approach. Preliminaries.

As usual, in the finite difference approximation the  $S$ -domain is truncated at the value  $S_{\max}$ , sufficiently large such that computed values are not appreciably affected by the upper boundary. The computational domain  $[0, S_{\max}] \times [0, T]$  is discretized by a uniform mesh with steps  $\Delta S$ ,  $\Delta t$ . Therefore we obtain the nodes  $S_j$  and  $t_n$ , where  $(S_j = j\Delta S, t_n = n\Delta t)$ ,  $j = 0, \dots, M$ ,  $n = 0, \dots, X$  so that  $S_{\max} = M\Delta S$ ,  $T = X\Delta t$ ,  $X$  and  $M$  are integer numbers.

1. The choice of a specific numerical scheme is based on its property of convergence. The requirement rests on the Lax equivalence theorem.
2. The parabolic nature of the Black–Scholes equation ensures that being the initial condition  $V(S, 0) = (S - K)^+ 1_{[L,U]}(S)$  square-integrable the solution is smooth in the sense that  $V(\cdot, t) \in C^\infty(\mathbb{R}^+)$ ,  $\forall t \in (t_{i-1}, t_i^-]$ ,  $i = 1, \dots, F$ . Thus rough initial data give rise to smooth solutions in infinitesimal time.

In some cases, as a consequence the solution obeys the maximum principle:

$$\max_{S \in [0, S_{\max}]} |V(S, t_1)| \geq \max_{S \in [0, S_{\max}]} |V(S, t_2)|, \quad t_1 \leq t_2. \quad (6)$$

This inequality means that the maximum value of  $V(S, t)$  should not increase as  $t$  increases. If that condition is violated then the numerical solution may exhibit spurious wiggles near sharp gradients. As a consequence, even though the numerical method converges, it often yields approximate solutions that differ qualitatively from corresponding exact solutions.

#### 3.1. Modification of the fully implicit scheme

The Crank–Nicolson scheme is highly accurate, i.e.  $O(\Delta t^2, \Delta S^2)$ , but whenever the relationship  $\sigma^2 < r$  holds, it requires a prohibitively small time step for large  $M$  to work efficiently, [10,5]. Then in this section we will introduce a less accurate

scheme, i.e.  $O(\Delta t, \Delta S^2)$ , which allows us to choose a more acceptable time step. In the meantime, the latter scheme prevents from undesired spurious oscillations and guarantees a positive solution.

### 3.1.1. The standard fully implicit scheme

First the standard fully implicit scheme is considered, leading to a difference equation (here  $\frac{\partial V}{\partial S}$  is discretized through a centered difference):

$$AV^{n+1} = V^n,$$

where the matrix  $A$  is order  $M$

$$A = \text{tridiag} \left\{ -\frac{\Delta t}{2} \left[ \left( \frac{\sigma S_j}{\Delta S} \right)^2 - r \frac{S_j}{\Delta S} \right]; 1 + \Delta t \left[ \left( \frac{\sigma S_j}{\Delta S} \right)^2 + r \right]; -\frac{\Delta t}{2} \left[ \left( \frac{\sigma S_j}{\Delta S} \right)^2 + r \frac{S_j}{\Delta S} \right] \right\}.$$

The following two distinct cases  $\sigma^2 > r$  and  $\sigma^2 < r$  are considered.

△ Let  $\sigma^2 > r$

- $A$  is a Jacobi matrix, so that theoretical results about its spectrum are available in the literature ([11], p. 24);
- $A$  is a row diagonally dominant  $M$ -matrix, so that  $A^{-1} > 0$  and then  $V^{n+1} > 0$ ;
- $\|A^{-1}\|_\infty \leq \frac{1}{1+r\Delta t}$ , ([11], p. 8);
- $\|V^{n+1}\|_\infty = \|A^{-1}V^n\|_\infty \leq \|A^{-1}\|_\infty \|V^n\|_\infty \leq \frac{1}{1+r\Delta t} \|V^n\|_\infty < \|V^n\|_\infty$  and then maximum principle is satisfied;
- $A$  is similar to a symmetric tridiagonal matrix ([11], p. 24), so that the eigenvalues  $\lambda_i(A)$  are real;
- From Gershgorin theorem  $\lambda_i(A^{-1}) \in \left[ \frac{1}{1+\Delta t[r+2(\sigma M)^2]}, \frac{1}{1+r\Delta t} \right] \subset (0, 1)$  and then spurious oscillations are avoided.

△ Let  $\sigma^2 < r$ .

In the next analysis we follow ([11], pp. 67–69). Here we distinguish further two subcases  $\frac{r}{M} < \sigma^2 < r$  and  $\sigma^2 < \frac{r}{M}$ .

★ Let  $\frac{r}{M} < \sigma^2 < r$ .

The subdiagonal of  $A$  has some positive entries, so that  $A$  is no longer an  $M$ -matrix. By using the Sherman–Morrison formula ([11], p. 15) for rank-one perturbations of a nonsingular matrix, there exists  $A^{-1}$ . The sign pattern of the entries of  $A^{-1}$  is described in ([11], p. 68). As some entries of  $A^{-1}$  are negative, from  $V^{n+1} = A^{-1}V^n$  negative values of  $V^{n+1}$  can be expected and then spurious oscillations. A deeper analysis becomes cumbersome, as the subdiagonal entries have different sign, and deferred to next case.

★ Let  $\sigma^2 < \frac{r}{M}$ .

In this case  $A$  has positive subdiagonal entries. If briefly we denote  $A = \text{tridiag}\{a, c, -b\}$ , with  $a, c, b > 0$  and  $a = [a_1, \dots, a_{M-1}]$ ,  $c = [c_1, \dots, c_M]$  and  $b = [b_1, \dots, b_{M-1}]$  then the leading principal minors  $A_k$  of  $A$  satisfy the relationship ([11], p. 40)

$$A_k = c_k A_{k-1} + a_{k-1} b_{k-1} A_{k-2}, \quad k = 1, \dots, M, \quad A_{-1} = 0, \quad A_0 = 1.$$

Then there exist  $A_k > 0, \forall k$  and  $A^{-1}$ .

The sign pattern of the entries of  $A^{-1}$  is as follows ([11], formula (4.33))

$$A^{-1} = \begin{pmatrix} + & + & + & \cdots & \cdots & \cdots & + \\ - & + & + & & & & \vdots \\ + & - & + & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ (-1)^{M+1} & \dots & \dots & \dots & + & - & + \end{pmatrix}.$$

Now we consider  $V^{n+1} = A^{-1}V^n$  which is oscillating because of the sign pattern of the entries of  $A^{-1}$ . The positivity of  $V^{n+1}$  is not guaranteed, even if  $V^n$  is eventually positive, and spurious oscillations can arise. A deeper insight about time step  $\Delta t$  which avoids spurious oscillations requires to investigate the eigenvalues of  $A$ .

Let  $D = [d_{jj}]$  be a diagonal matrix, with the entries  $d_{11} = 1$  and  $d_{jj} = i^{j-1} \sqrt{\frac{a_1 \cdots a_{j-1}}{(-b_1) \cdots (-b_{j-1})}}, j = 2, \dots, M, i = \sqrt{-1}$ . Similarity transformation with the diagonal matrix  $D$  leads to  $A' = D^{-1}AD$ , with ([11], p. 24)

$$iA' = iD^{-1}AD = \begin{pmatrix} ic_1 & \sqrt{(-b_1 a_1)} & & \\ \sqrt{(-b_1 a_1)} & ic_2 & \sqrt{(-b_2 a_2)} & \\ & \ddots & \ddots & \ddots \\ & & \sqrt{(-b_{M-1} a_{M-1})} & \sqrt{(-b_{M-1} a_{M-1})} \\ & & & ic_M \end{pmatrix}.$$

Then  $iA' = \text{tridiag}\{\sqrt{(-b_j a_j)}, ic_j, \sqrt{(-b_j a_j)}\}$ , is a tridiagonal symmetric matrix with complex entries. Its characteristic polynomial  $p(\lambda) = |\lambda I - iA'|$  has leading principal minors  $p_k(\lambda), k = 1, \dots, M$  satisfying the relationship

$$p_k = (\lambda - ic_k)p_{k-1} - (-b_{k-1} a_{k-1})p_{k-2}, \quad k = 1, \dots, M, \quad p_{-1} = 0, \quad p_0 = 1. \quad (7)$$

The monic complex polynomials  $p_k(\lambda)$  are referred to as ‘orthogonal polynomials on the semicircle’ relative to symmetric weight function (being in (7) the term  $(-b_{k-1} a_{k-1})$  positive) ([12], p. 45). Then the zeros of  $p_k(\lambda), \forall k \geq 1$  are distinct. The latter coincide with the eigenvalues of  $iA'$  and then, up the constant, of  $A$ . Equivalently,  $A$  admits  $M$  distinct complex eigenvalues. To the  $M$  distinct eigenvalues  $\lambda_j(A^{-1})$  are associated  $M$  linearly independent eigenvectors  $v_j$ . Such eigenvectors can be used as a basis for the  $M$  dimensional space of the initial condition  $V^0 = \sum_{j=1}^M w_j v_j$ , where  $w_j$  are proper weights. Then

$$V^{n+1} = (A^{-1})^n V^0 = (A^{-1})^n \sum w_j v_j = \sum w_j (A^{-1})^n v_j = \sum w_j \lambda_j^n v_j. \quad (8)$$

From Gershgorin theorem  $\Re e(\lambda_j(A)) \leq 1 + \Delta t[r(1 + M) + (\sigma M)^2]$  and then the iteration matrix  $A^{-1}$  is such that

$$\Re e(\lambda_j(A^{-1})) \geq \frac{1}{1 + \Delta t[r(1 + M) + (\sigma M)^2]}.$$

Taking not too small  $\Delta t$ , all the eigenvalues  $\lambda_j(A^{-1})$  are such that  $|\lambda_j(A^{-1})| < 1$  and  $|\lambda_j(A^{-1})|$  far from 1. From (8), increasing  $n$ , all  $\lambda_j^n(A^{-1}) \rightarrow 0$  and then spurious oscillations do not occur. Nevertheless, because of low accuracy of fully implicit scheme, the found solution is inaccurate, (see Fig. 1).

On the other hand, with very small  $\Delta t$  all the eigenvalues  $\lambda_j(A^{-1})$  are such that  $|\lambda_j(A^{-1})| \simeq 1$ . From (8), increasing  $n$ , all the terms  $\lambda_j^n(A^{-1})$  are slowly damped and then spurious oscillations occur. We are in the paradoxical case that, with  $\Delta t \rightarrow 0$  in order to guarantee convergence, persistent spurious oscillations are enforced (see Fig. 1). In conclusion, under the assumed condition  $\sigma^2 < \frac{r}{M}$  (as it occurs in certain regions of the grid for stochastic volatility models) the fully implicit scheme is unsuitable for solving (2) with discontinuous payoff in the framework of Rannacher time-stepping. Then a proper variant of it is required and we illustrate such one in the next Section 3.1.2.

Another viable approach is considering a first-order forward finite difference to approximate the convective term  $\frac{\partial V}{\partial S}$ . The corresponding finite difference equation is  $AV^{n+1} = V^n$ , with

$$A = \text{tridiag}\left\{-\frac{\Delta t}{2} \left[\left(\frac{\sigma S_j}{\Delta S}\right)^2\right]; 1 + \Delta t \left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{r S_j}{\Delta S} + r\right]; -\frac{\Delta t}{2} \left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{2r S_j}{\Delta S}\right]\right\}$$

where a  $A$  is row diagonally dominant  $M$ -matrix, similar to a symmetric matrix ([11], p. 24), with real and positive eigenvalues. Then the scheme is positivity-preserving and spurious oscillations free.

Through a standard analysis of consistency, it introduces numerical diffusion  $\frac{1}{2}rS\Delta S \frac{\partial^2 V}{\partial S^2}$ , so that the scheme is the approximation of the following partial differential equation

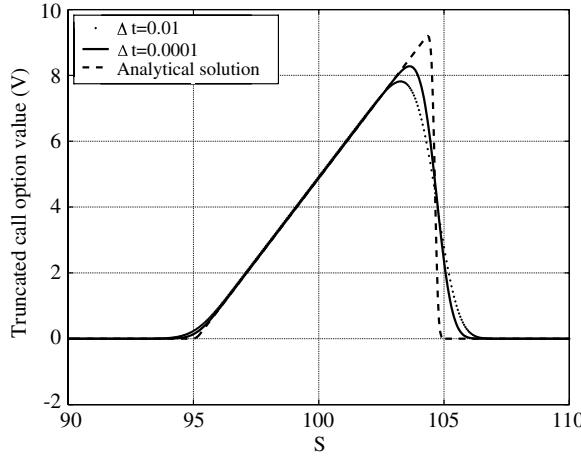
$$-\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \left(rS \frac{\Delta S}{2} + \frac{1}{2}\sigma^2 S^2\right) \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

The numerical diffusion term  $rS \frac{\Delta S}{2} \frac{\partial^2 V}{\partial S^2}$  artificially smears the numerical solution on any realistic spatial grid, particularly when the volatility is small. An accurate solution demands for small  $\Delta S$  step.

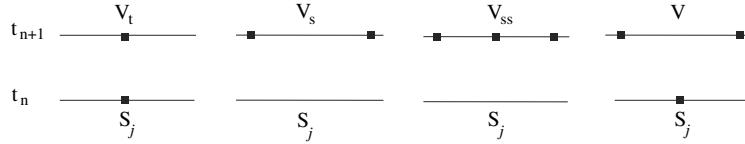
Summarizing we conclude that suitable implicit schemes unaffected by spurious oscillations may be obtained discretizing the convective term  $\frac{\partial V}{\partial S}$  as follows

- $\sigma^2 > r$  centered difference;
- $\sigma^2 < r$  forward difference.

Therefore, it should be desirable a unique scheme positivity preserving and spurious oscillations free regardless the relationship between  $r$  and  $\sigma^2$ .



**Fig. 2.** Truncated call option value just before first monitoring date  $t_1 = T$ . Numerical solutions are obtained using an exponentially fitted scheme, with  $\Delta S = 0.05$ ,  $\Delta t = 0.01$  and  $\Delta t = 0.0001$  respectively. Parameters:  $L = 90$ ,  $K = 100$ ,  $U = 110$ ,  $r = 0.05$ ,  $\sigma = 0.001$ ,  $T = 1$ ,  $S_{\max} = 200$ .



**Fig. 3.** Involved nodes in the fully implicit scheme variant.

**Remark 3.1.** We recall in the literature that usually numerical experiments assume  $\sigma^2 < r$  (avoiding the condition  $\sigma^2 \ll r$ ) without the raising spurious oscillations. As a consequence the implicit scheme is adopted as first low accurate scheme in Rannacher time-stepping in the presence of discontinuous payoff.

**Remark 3.2.** Under the condition  $\sigma^2 \ll r$  more suitable schemes are the exponentially fitted schemes [3]. These schemes are implicit, uniformly convergent and first order accurate, i.e.  $O(\Delta t, \Delta S)$ . Nevertheless, under the restrictive condition  $\sigma^2 \ll r$  they arise numerical diffusion  $\frac{1}{2}rS\Delta S \frac{\partial^2 V}{\partial S^2}$ , (see Fig. 2, where numerical diffusion and slow convergence are evident), requiring severe restrictions on  $\Delta S$  step and losing their peculiarity [13].

### 3.1.2. A modified implicit scheme

The proposed scheme differs from the standard fully implicit one in the discretization of the reaction term  $-rV$  in the Black–Scholes equation (2) by a bivariate approximation. Through a standard procedure we have

$$V(S, t + \Delta t) = a(V_{j+1}^{n+1} + V_{j-1}^{n+1}) + (1 - 2a)V_j^n \quad (9)$$

with discretization error  $O(\Delta S^2, \Delta t)$  and  $a$  arbitrary constant to be determined below. The term  $\frac{\partial V}{\partial S}$  is discretized through a centered difference, as well as  $\frac{\partial^2 V}{\partial S^2}$ .

The stencil of the involved nodes of the scheme is displayed in Fig. 3.

The corresponding finite difference equation is

$$AV^{n+1} = \left[ \frac{1}{\Delta t} - r(1 - 2a) \right] V^n, \quad (10)$$

where

$$A = \text{tridiag} \left\{ ra + \frac{r}{2} \frac{S_j}{\Delta S} - \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2; \frac{1}{\Delta t} + \left( \frac{\sigma S_j}{\Delta S} \right)^2; ra - \frac{r}{2} \frac{S_j}{\Delta S} - \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 \right\}.$$

Parameters  $a$  and  $\Delta t$  are chosen according to the following criteria:

1.  $A$  is an  $M$ -matrix ([11], p. 16). Then we have to put  $ra + \frac{r}{2} \frac{S_j}{\Delta S} - \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 \leq 0$ , from which  $a \leq -\frac{r}{8\sigma^2}$  (as a consequence  $ra - \frac{r}{2} \frac{S_j}{\Delta S} - \frac{1}{2} \left( \frac{\sigma S_j}{\Delta S} \right)^2 \leq 0$  holds too);

2.  $\frac{1}{\Delta t} - r(1 - 2a) > 0$ , from which  $\Delta t < \frac{1}{r(1-2a)}$ .

By taking

$$a = -\frac{r}{8\sigma^2} \quad \text{and} \quad \Delta t < \frac{1}{r\left(1 + \frac{r}{4\sigma^2}\right)} \quad (11)$$

the following properties follow

- $A = [a_{ij}]$  is row diagonally dominant with  $\delta = |a_{ii}| - \sum_{j \neq i} |a_{ij}| = \frac{1}{\Delta t} - (\frac{r}{2\sigma})^2 > 0$ .

- $A$  is similar to a symmetric tridiagonal matrix ([11], p. 24), so that  $\lambda_i(A)$  are real.

Being  $A$  an  $M$ -matrix then

- $A^{-1} > 0$  and from (10) follows  $V^{n+1} > 0$ , i.e. the modified implicit scheme is positivity-preserving;

- $\|A^{-1}\|_\infty \leq \frac{1}{\frac{1}{\Delta t} - (\frac{r}{2\sigma})^2}$ , ([11], p. 8), being  $A$  row diagonally dominant.

- The iteration matrix  $A_{\text{iter}} = \left[ \frac{1}{\Delta t} - r\left(1 + \frac{r}{4\sigma^2}\right) \right] A^{-1}$  has spectral radius  $\rho(A_{\text{iter}})$  satisfying

$$\rho(A_{\text{iter}}) \leq \left\| \left[ \frac{1}{\Delta t} - r\left(1 + \frac{r}{4\sigma^2}\right) \right] A^{-1} \right\|_\infty \leq \frac{\frac{1}{\Delta t} - r - (\frac{r}{2\sigma})^2}{\frac{1}{\Delta t} - (\frac{r}{2\sigma})^2} < 1$$

so that, under (11), the scheme is *conditionally stable*.

- $\|V^{n+1}\|_\infty = \|A_{\text{iter}} V^n\|_\infty \leq \frac{\frac{1}{\Delta t} - r - (\frac{r}{2\sigma})^2}{\frac{1}{\Delta t} - (\frac{r}{2\sigma})^2} \|V^n\|_\infty < \|V^n\|_\infty$ , so that the solution satisfies *conditionally* the maximum principle.

- From Gershgorin theorem we have

$$\lambda_i(A_{\text{iter}}) \in \left[ \frac{\frac{1}{\Delta t} - r - (\frac{r}{2\sigma})^2}{\frac{1}{\Delta t} + (\frac{r}{2\sigma})^2 + 2(\sigma M)^2}, \frac{\frac{1}{\Delta t} - r - (\frac{r}{2\sigma})^2}{\frac{1}{\Delta t} - (\frac{r}{2\sigma})^2} \right] \subset (0, 1).$$

Then spurious oscillations are avoided.

The solution accuracy is defined by analyzing the error component introduced by the bivariate approximation of the reaction term  $-rV$ . The term  $V_j^{n+1}$  in the standard fully implicit scheme is replaced by the expression (9) quoted above. By Taylor expansion about the time level  $(n + 1)\Delta t$  we have

$$a(V_{j-1}^{n+1} + V_{j+1}^{n+1}) + (1 - 2a)V_j^n = V_j^{n+1} + \Delta t(-1 + 2a)\frac{\partial V}{\partial t} + (\Delta S)^2 \frac{1}{8} \left(\frac{r}{\sigma}\right)^2 \frac{\partial^2 V}{\partial S^2}.$$

Then the scheme is *consistent* with Eq. (2), with local truncation error  $O(\Delta t, \Delta S^2)$ . Nevertheless, when  $\sigma^2 \ll r$ , both error terms  $r\Delta t(1 + \frac{r}{4\sigma^2})$  and  $\frac{1}{8}(\Delta S \frac{r}{\sigma})^2$  become influent and the only constraint (11) provides a poor solution. Then an accurate solution requires

$$\Delta t \left(r + \frac{r^2}{4\sigma^2}\right) \ll 1 \quad \text{and} \quad \frac{1}{8} \left(\frac{r}{\sigma}\Delta S\right)^2 \ll 1. \quad (12)$$

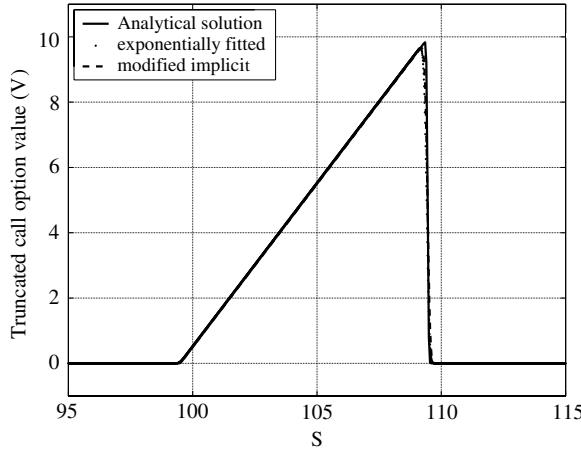
Then, under (12), the proposed scheme guarantees an accurate solution being positivity-preserving and spurious oscillations free. Performances are illustrated in Fig. 4, where a truncated call option is considered. According to Rannacher strategy a small expiry time  $T$  is chosen. Here, the analytical and numerical solutions, obtained by exponentially fitted scheme and modified implicit scheme, are compared. The numerical solutions obtained by the two different methods are practically indistinguishable. Then the modified implicit scheme may be considered as equivalent to implicit exponentially fitted one.

#### 4. Numerical results

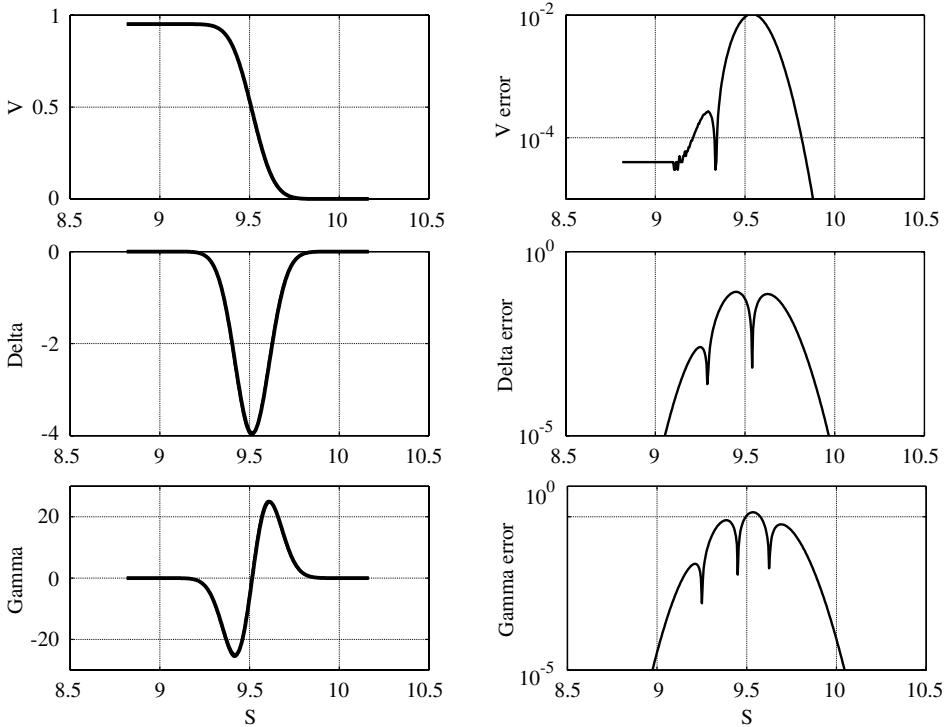
In this section, we have chosen an example of digital options that has exact *smooth solution* of the Black–Scholes equation. Such options have discontinuity in the payoff function at the strike price  $K$  and this produces oscillations in the numerical solution when standard finite difference schemes are applied.

The same drawback is observed when the Greek letters Delta (the first partial derivative of option price with respect to asset price) and Gamma (the second derivative) are computed. They are very important quantities to measure the sensitivity of the option prices for hedging.

**Example 4.1.** Let us price a digital put option that has a discontinuous payoff, specified by  $V(0, S) = A$  for  $S \leq K$ ,  $V(0, S) = 0$  otherwise and boundary conditions  $V(t, 0) = A \exp(-rt)$ ,  $V(t, S_{\max}) = 0$  for large  $S_{\max}$ .



**Fig. 4.** Truncated call option value before first monitoring date. Numerical solutions are obtained using (1) an exponentially fitted scheme, with  $\Delta S = 0.01$ ,  $\Delta t = 10^{-4}$ ; (2) a modified implicit scheme, with  $\Delta S = 0.01$ ,  $\Delta t = 10^{-6}$ . Parameters:  $L = 90$ ,  $K = 100$ ,  $U = 110$ ,  $r = 0.05$ ,  $\sigma = 0.001$ ,  $T = 0.01$ ,  $S_{\max} = 120$ .



**Fig. 5.** Pricing  $V^{\text{ex}}$ ,  $V^{\text{imp}}$  and their difference  $|V^{\text{ex}} - V^{\text{imp}}|$  (left and right upper); Delta $^{\text{ex}}$ , Delta $^{\text{imp}}$  and Delta difference  $|\Delta^{\text{ex}} - \Delta^{\text{imp}}|$  (left and right median); Gamma $^{\text{ex}}$ , Gamma $^{\text{imp}}$  and Gamma difference  $|\Gamma^{\text{ex}} - \Gamma^{\text{imp}}|$  (left and right lower). The approximate solution through the modified implicit scheme is done with  $\Delta S = 0.05$  and  $\Delta t = 0.001$ . The parameters are:  $A = 1$ ,  $K = 10$ ,  $r = 0.05$ ,  $\sigma = 0.01$ ,  $T = 1$ ,  $S_{\max} = 20$ .

The used parameters are  $A = 1$ ,  $K = 10$ ,  $r = 0.05$ ,  $\sigma = 0.01$ ,  $T = 1$ ,  $S_{\max} = 20$ . The analytical solution  $V^{\text{ex}}(T, S) = A \exp(-rT)(1 - N(d_2))$ , with  $N(\cdot)$  standard normal cumulative distribution and  $d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ , and the numerical solution  $V^{\text{imp}}(T, S)$  obtained by modified fully implicit scheme presented in Section 3.1 are compared in Fig. 5.

In the same figure we compare (a) the exact Delta, i.e.  $\Delta^{\text{ex}} = \frac{\partial V^{\text{ex}}}{\partial S} = -\frac{A \exp(-rT) N'(d_2)}{\sigma S \sqrt{T}}$ , and an approximate Delta $^{\text{imp}}$ , obtained from  $V^{\text{imp}}(T, S)$  through a centered difference; (b) the exact Gamma, i.e.  $\Gamma^{\text{ex}} = \frac{\partial^2 V^{\text{ex}}}{\partial S^2} = \frac{A \exp(-rT) d_1 N'(d_2)}{\sigma^2 S^2 T}$ ,  $d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$  and an approximate Gamma $^{\text{imp}}$  obtained from  $V^{\text{imp}}(T, S)$  through a centered difference scheme.

Here  $\sigma^2 \ll r$  is assumed and standard schemes such as fully implicit and Crank–Nicolson ones fail arising spurious oscillations and negative prices.

The exact  $V$  and approximate solution  $V^{\text{imp}}$  are practically indistinguishable. Their difference (and Delta difference) are illustrated in Fig. 5. Observing the error, the discontinuity effect is evident in the neighborhood of  $K$ .

We can note the following advantages of the proposed *modified implicit scheme* in Section 3.1.2:

- This implicit scheme is *positivity-preserving* and *oscillations free*.
- The modification of the implicit scheme works successfully both for  $\sigma^2 < r$  and  $\sigma^2 > r$ , i.e. the accuracy of the scheme is not deteriorated for *low values of the volatility parameter  $\sigma$* .
- The proposed scheme satisfies the *discrete maximum principle*.

## 5. Conclusions

Within the strategy suggested by Rannacher, consisting of a combined use of different finite difference schemes in order to satisfy all the severe requirements of the problem, we have presented an alternative scheme to the classical fully implicit one that does not suffer from spurious oscillations originating from *discontinuous* boundary conditions. This is due to the fact that the scheme has an iteration matrix characterized by real and positive spectrum which allows a fast damping of errors of any order. The proposed *implicit scheme* has lower accuracy, i.e.  $O(\Delta S^2, \Delta t)$ , but it is positivity preserving and satisfies the discrete maximum principle. The scheme operates uniquely during first few time steps, next replaced by a higher order one. The latter starts from a smooth initial condition thanks to the contribution of the first scheme used.

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