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Laplace Transform and finite difference methods for the Black–Scholes equation

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ABSTRACT

In this paper we explore discrete monitored barrier options in the Black–Scholes framework. The discontinuity arising at each monitoring data requires a careful numerical method to avoid spurious oscillations when low volatility is assumed. Here a technique mixing the Laplace Transform and the finite difference method to solve Black–Scholes PDE is considered. Equivalence between the Post–Widder inversion formula joint with finite difference and the standard finite difference technique is proved. The mixed method is positivity-preserving, satisfies the discrete maximum principle according to financial meaning of the involved PDE and converges to the underlying solution. In presence of low volatility, equivalence between methods allows some physical interpretations.

1. Introduction

One of the main concerns about financial options is what the exact values of the options are. In absence of evaluation formula for non-standard options, numerical technique is required. Usually the choice goes toward numerical methods with high order of accuracy (for instance in the finite difference method the Crank–Nicolson scheme) and no attention is paid to the fact how the financial provision of the contract can affect the reliability of the numerical solution. Special options, as discretely monitored barrier options are characterized by discontinuities that are renewed at each monitoring date. In presence of *low volatility* the Black–Scholes PDE becomes *convection dominated*. As a consequence, numerical diffusion or spurious oscillations may arise, so that special numerical techniques have to be employed. A viable route to circumvent discontinuity issues is considering an Integral Transforms method. If we assume that the volatility σ of the underlying asset price S and the risk-free interest rate r of the market depends only on S, i.e., $\sigma = \sigma(S), r = r(S)$, then the Laplace Transform becomes a useful tool. The Black–Scholes equation is solved by the Laplace Transform method for time t discretization. The resulting ordinary differential equation (ODE) is solved by a finite difference scheme and, as a final result, a Laplace Transform of the solution is obtained. Hereinafter we call that method a *'mixed method'*. The crucial issue is the Laplace Transform inversion. A lot of methods are available in literature [1]. They can be roughly classified into two categories: the ones using *complex values* of the Laplace Transform and the ones using only *uniquely real values*. Here, rather than proposing a new method for the Laplace Transform inversion on the real axis, we consider the well-known Post–Widder inversion formula [1, p. 37 and 141–3]. Next we prove the equivalence between *standard finite difference schemes* and the *mixed method* of Laplace Transform with the Post–Widder inversion formula jointly with special finite difference schemes that solve the resulting ODE. We prove that *the mixed method is positivity-preserving, satisfies the discrete maximum principle, is spurious oscillations free, convergent to exact solution and finally provides a physical meaning to Post–Widder inversion formula.*

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In order to make our analysis concrete, we concentrate our attention on a double barrier knock-out call option with a discrete monitoring clause. Such option has a payoff condition equal to $\max(S-K,0)$, where K denotes the strike price, but the option expires worthless if before the maturity T the asset price has fallen outside the corridor $[L,U]$ at the prefixed monitoring dates $0=t_0<t_1<\cdots<t_F=T$. In the intermediate periods the Black Scholes equation is applied over the real positive domain. The discontinuity in the initial conditions will be renewed at every monitoring date and often the Crank–Nicolson numerical solution is affected by spurious oscillations that do not decay quickly [2–4]. The presented analysis can be easily extended to many other exotic contracts (digital, supershare, binary and truncated payoff options, callable bonds and so on).

2. Some background

2.1. The Black–Scholes PDE

Let $V(S,t)$ be the value of an option, where S is the current value of the underlying asset and t is the time to expiry T. The value of the option is related to the current value of the underlying asset via two stochastic parameters, the volatility $\sigma=\sigma(S)$ and the interest rate $r=r(S)$ of the Black–Scholes equation. The price $V(S,t)$ of the option satisfies the Black–Scholes partial differential equation [5]

$$-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{2.1}$$

endowed with initial and boundary conditions:

$$V(S,0)=\max(S-K,0)\,1_{[L,U]}(S) \tag{2.2}$$

$$V(S,t)\to 0 \text{ as } S\to 0 \text{ or } S\to\infty, \tag{2.3}$$

with updating of the initial condition at the monitoring dates $t_i, i=1,\cdots,F$:

$$V(S,t_i)=V(S,t_i^-)1_{[L,U]}(S),\quad 0=t_0<t_1<\cdots<t_F=T, \tag{2.4}$$

where $\mathbf{1}_{[L,U]}(S)$ is the indicator function, i.e.,

$$\mathbf{1}_{[L,U]}=\begin{cases}1 & \text{if}\quad S\in[L,U]\\ 0 & \text{if}\quad S\notin[L,U].\end{cases}$$

Here K represents the strike price. The knock-out clause at the monitoring date introduces a *discontinuity* at the barriers set at $S=L$ and $S=U$ respectively. The presence of undesired spurious oscillations is frequently observed near the barriers and near the strike, if unsuitable finite difference schemes are used. These spikes, which remain well localized, do not reflect instability but rather that the discontinuities that are periodically produced by the barriers at monitoring dates. The spikes cannot decay fast enough.

The parabolic nature of the Black–Scholes equation ensures that, being the initial condition $V(S,0)=(S-K)^+1_{[L,U]}(S)$ square–integrable, the solution is smooth in the sense that $V(\cdot,t)\in C^\infty(\mathbb{R}^+),\forall t\in(t_{i-1},t_i^-],\ i=1,\dots,F$. Thus rough initial data give rise to smooth solutions in infinitesimal time.

The smoothness of $V(S,t), t>0$, allows us to invert the Laplace Transform via Post–Widder inversion formula for calculating $V(S,T)$.

As a consequence of the parabolic nature of Black–Scholes equation, the solution obeys the maximum principle:

$$\max_{S\in[0,+\infty]}|V(S,t_1)|\geqslant\max_{S\in[0,+\infty]}|V(S,t_2)|,\quad t_1\leqslant t_2. \tag{2.5}$$

This inequality means that the maximum value of $V(S,t)$ should not increase as t increases.

2.2. The Laplace Transform

Here some particular definitions used in the sequel are given. If for a function $f:\mathbb{R}^+\to\mathbb{R}^+, f:t\to f(t)$ there exists a $\lambda_0\in\mathbb{R}$, such that the Laplace Transform of the function $f(t)$ defined by

$$F(\lambda)=\mathcal{L}[f](\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt, \tag{2.6}$$

exists for all $\lambda\in\mathbb{R},\ \lambda>\lambda_0$, then $F(\lambda)$ is the Laplace Transform of f.

If $f:\mathbb{R}^+\to\mathbb{R}^+, f:t\to f(t)$ is of exponential order, i.e., for some $\lambda_0\in\mathbb{R}$

$$\sup_{t>0}f(t)e^{-\lambda_0 t}<\infty, \tag{2.7}$$

then the Laplace Transform (2.6) exists for all $\lambda>\lambda_0$ and it is infinitely differentiable with respect to λ for $\lambda>\lambda_0$.

From $f(t) \geqslant 0\ \forall t \in [0, +\infty)$ the k-th derivative

$$F^{(k)}(\lambda) = (-1)^k \int_0^\infty t^k e^{-\lambda t} f(t)dt, \tag{2.8}$$

satisfies

$$(-1)^k F^{(k)}(\lambda) \geqslant 0 \tag{2.9}$$

for $\forall k \geqslant 0$. Then $F(\lambda)$ is a completely monotonic function [6].

2.2.1. The inverse Laplace Transform

For a function $F(\lambda) = \mathcal{L}[f](\lambda)$ in the transformed λ-space, the inverse of the Laplace Transform is denoted by $\mathcal{L}^{-1}[F](t) = f(t)$. Post and Widder [1] presented the original function $f(t)$ as a limit of some sequence, involving $F^{(n)}(\lambda)$ (i.e., the n-th derivative of $F(\lambda)$) on the real axis. This is more convenient for the numerical computation of the inverse Laplace Transform than trying to compute the integral on the complex plane by the Bromwich integral

$$\mathcal{L}^{-1}[F](t) = f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(\lambda)e^{\lambda t}d\lambda, \quad \gamma > \lambda_0.$$

The result is formulated by the following [1, p. 37].

Theorem (Post–Widder inversion formula). If for a continuous function $f(t)$ the integral $F(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ converges for every $\lambda > \lambda_0$ (sufficient for this is the growth condition (2.7)) then

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right) \tag{2.10}$$

for every point $t > 0$ of continuity of $f(t)$.

The advantage of (2.10) lies in the fact that $f(t)$ is expressed in terms of values of $F(\lambda)$ and its derivatives on the real axis. There are mainly two difficulties when this particular approach is used:

1. The need of differentiation of $F(\lambda)$ a large number of times especially when it is a complicated function. However with the general availability of Maple or Mathematica that drawback may be circumvented;
2. The convergence to the limit is very slow, even if the convergence can be speeded up using an appropriate extrapolation technique.

3. The Laplace Transform method for the Black–Scholes equation

We apply the Laplace Transform method to the Black–Scholes equation that depends on one stock asset S. Let us consider the function $V(S,t)$, its Laplace Transform $U(S,\lambda) = \int_0^\infty e^{-\lambda t} V(S,t)dt$ and its k-th derivative $\frac{d^k U(S,\lambda)}{d\lambda^k} =: U^{(k)}(S,\lambda) = (-1)^k \int_0^\infty t^k e^{-\lambda t} V(S,t)dt$. From (2.1), multiplying each term by $t^k e^{-\lambda t}$ and integrating over $[0,\infty)$, after some algebra one has the following ordinary differential equation (ODE)

$$-\frac{1}{2}\sigma^2 S^2 \frac{d^2 U^{(k)}}{dS^2} - rS\frac{dU^{(k)}}{dS} + (r+\lambda)U^{(k)} = \begin{cases} V(S,0), & k = 0 \\ -kU^{(k-1)}, & k = 1,2,\ldots \end{cases} \tag{3.1}$$

The financial model leads to the boundary conditions $U^{(k)}(0,\lambda) = 0$ and $U^{(k)}(+\infty,\lambda) = 0$. Formula (3.1) is a recursive relationship, relating two consecutive derivatives $U^{(k-1)}$ and $U^{(k)}$. Then all higher derivatives $U^{(k)}(S,\lambda)$ are obtained by an iterative procedure involving the ODE (3.1). Numerical solution of (3.1) by a finite difference method requires that the S-domain is truncated at the value S_{max}, which is the position of the so-called far field, sufficiently large that computed values are not appreciably affected by the upper boundary. The computational domain $[0, S_{max}]$ is discretized by uniform mesh with step ΔS. Therefore we obtain the nodes $S_j = j\Delta S, j = 0,\ldots,M$, so that $S_{max} = M\Delta S$, and M is an integer number. The corresponding boundary conditions become $U^{(k)}(0,\lambda) = 0$ and $U^{(k)}(S_{max},\lambda) = 0, \forall k \geqslant 0$. The choice of a proper numerical scheme is suggested by the nature of $U(S,\lambda)$, the Laplace Transform of a positive function $V(S,t)$. From $V(S,t) \geqslant 0$ then $U^{(k)}(S,\lambda)$ is a completely monotonic function [6]. The numerical scheme preserves such a property, for instance, if

- $\frac{d^2 U^{(k)}}{dS^2}$ is discretized by a standard centered difference;
- $\frac{dU^{(k)}}{dS}$ is discretized in a different way, according to the relationship between σ^2 and r. More specifically.
- If $\sigma^2 > r$, a centered difference is used;
- If $\sigma^2 < r$, a forward difference is used.

Indeed (3.1) leads to

$$\begin{cases} A_{PW}U^{(k)} = V(S,0) & k = 0, \\ A_{PW}U^{(k)} = -kU^{(k-1)} & k = 1,2,\ldots, \end{cases} \tag{3.2}$$

with

$$A_{PW} = \text{tridiag}\left\{-\frac{1}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\frac{S_j}{\Delta S}\right]; (r+\lambda) + \left(\frac{\sigma S_j}{\Delta S}\right)^2; -\frac{1}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\frac{S_j}{\Delta S}\right]\right\}, \tag{3.3}$$

if $\sigma^2 > r$ and

$$A_{PW} = \text{tridiag}\left\{-\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2; (r+\lambda) + r\frac{S_j}{\Delta S} + \left(\frac{\sigma S_j}{\Delta S}\right)^2; -\left[\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\frac{S_j}{\Delta S}\right]\right\} \tag{3.4}$$

if $\sigma^2 < r$.

In both cases, from $V(0,t) = V(S_{max},t) = 0,\ \forall t > 0$ we have the boundary conditions $U^{(k)}(0,\lambda) = U^{(k)}(S_{max},\lambda) = 0,\ \forall k \geqslant 0$. Regardless of the relationship between σ^2 and r, the matrix A_{PW} is an M-matrix [7, p. 16] and then $A_{PW}^{-1} \geqslant 0$. From (3.2) the property (2.9) is guaranteed according to the financial meaning of $V(S,t)$, i.e., $V(S,t) \geqslant 0$.

As a final step, if in (3.2) $k = 1,\ldots,N$ and $\lambda = \frac{N}{T}$ are assumed, the approximation $V_N(S_j,t)$ of $V(S_j,t)$ is obtained from the Post–Widder formula (2.10)

$$V_N(S_j,T) = \frac{(-1)^N}{N!}\left(\frac{N}{T}\right)^{N+1} U^{(N)}\left(S_j,\frac{N}{T}\right), \tag{3.5}$$

with $\lim_{N\to\infty} V_N(S_j,t) = V(S_j,t)$, as we prove in the sequel.

In summarizing, (3.5) provides an approximation $V_N(S_j,T)$ of $V(S_j,T)$. Each $U^{(k)}(S_j,\frac{N}{T}), k = 0,\ldots,N$ is numerically obtained by solving the linear system (3.2). The matrix A_{PW} is independent on k, so that the system is solved only one time calculating LU factorization. An explicit form for $V_N(S_j,T)$ is obtained combining (3.2) with (3.5)

$$V_N(S_j,T) = \left(\frac{N}{T}A_{PW}^{-1}\right)^{N+1} V(S_j,0), \tag{3.6}$$

so that $\frac{N}{T}A_{PW}^{-1}$ is the iteration matrix.

4. Equivalence between methods

Here we prove that the approximate solution $V_N(S_j,T)$, obtained by mixed methods, is equivalent to one obtained through a fully implicit finite difference scheme directly from PDE (2.1) in both cases $\sigma^2 > r$ and $\sigma^2 < r$. Discretizing the term $\frac{\partial V}{\partial t}$ by a forward difference and time step Δt one has

$$A_{FD}V^{(n)} = V^{(n-1)}, \quad n = 1,2,\ldots, \tag{4.1}$$

with

$$A_{FD} = \text{tridiag}\left\{-\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\frac{S_j}{\Delta S}\right]; 1 + \Delta t\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\right]; -\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\frac{S_j}{\Delta S}\right]\right\},$$

if $\sigma^2 > r$

$$A_{FD} = \text{tridiag}\left\{-\frac{\Delta t}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2; 1 + \Delta t\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{rS_j}{\Delta S} + r\right]; -\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{2rS_j}{\Delta S}\right]\right\},$$

if $\sigma^2 < r$.

From which after $N+1$ time steps

$$V^{(N+1)}(S_j,T) = \left(A_{FD}^{-1}\right)^{N+1} V(S_j,0). \tag{4.2}$$

Identifying $\frac{1}{\Delta t} = \lambda = \frac{N}{T}$ then $N = T/\Delta t$ and $\frac{1}{\Delta t}A_{FD} = A_{PW}$ hold. As a consequence 3.6 coincides with (4.2), i.e., the approximate solutions, obtained by two distinct methods, coincide. From the two equivalent methods we conclude that

1. the solution $V^{(N+1)}(S_j,T)$ obtained by $N+1$ steps of finite difference scheme with step Δt,
2. the solution $V_N(S_j,T)$ obtained calculating the N-th derivative of Laplace Transform of $V(S,T)$, with $\lambda = 1/\Delta t$ and $N = T/\Delta t$,

are equivalent, i.e. $V_N(S_j,T) = V^{(N+1)}(S_j,T)$.

In standard finite difference method the *positivity-preserving* is realized imposing A_{FD} to be an M-matrix. On the other hand in the mixed method positivity-preserving is obtained by imposing that the Laplace Transform $F(S,\lambda) = U^{(0)}(S,\lambda)$ is a completely monotonic function [6], i.e., $(-1)^k U^{(k)}(S,\lambda) \geqslant 0,\ \forall k \geqslant 0$. Taking into account (3.2), the latter condition may be realized if the matrix A_{PW} is an M-matrix.

We make the following remarks:

1. The equivalence between the mixed method and the standard finite difference one allows us to get the following result. As the fully implicit finite difference scheme is convergent, being consistent and unconditionally stable, the numerical solution $V_N(S_j,T)$ obtained by the mixed method converges to $V(S_j,T)$ too, when $\Delta S \to 0$ (equivalently $M \to \infty$) and $\Delta t \to 0$ (equivalently $N \to \infty$).
2. Since the solutions of (2.1) at intermediate steps $V(S, t < T)$ are not usually of interest, for large T values the Laplace Transform is more convenient than the finite difference method. Indeed, the Laplace Transform method reduces (2.1) to a number of mutually independent boundary value problems (see (3.2), with $k = 0$). The latter may be solved concurrently by applying a distributed algorithm [8] if a particular Laplace Transform inversion formula is used. Along the procedure adopted in [8] the Stehfest method [9] for inversion leads to an approximate solution $V_N(S,T) = \frac{\ln 2}{T}\sum_{j=1}^{N}\omega_j U^{(0)}(S,\lambda_j)$, with $U^{(0)}(S,\lambda_j)$ given in (3.2) and ω_j proper weighting factors. Here $\{\lambda_j\}$ is a finite set of transformation parameters defined by $\lambda_j = j\frac{\ln 2}{T},\ j = 1,\dots,N$. The authors mentioned above state that this approximate inverse Laplace Transform is by no means the most accurate one. Indeed, $V_N(S_j,T)$ is not positivity-preserving and the convergence is not proved. Essentially the method exploits the solution of ODE (3.1) concurrently by means of a distributed algorithm.
3. The condition $\sigma^2 < r$ is severe and as a result some high order finite difference schemes turn out to be unsuitable (e.g., the Crank–Nicolson one), as they suffer from undesired spurious oscillations.

Indeed, the requested condition of M-matrix, in order to be avoided spurious oscillations, leads to a *low order scheme*, as the convection term $\frac{\partial V}{\partial S}$ is discretized through a *forward difference*. Such method introduces *numerical diffusion* when (3.2) is solved. By standard analysis of consistency such a term amounts to $rS\frac{\Delta S}{2}\frac{\partial^2 V}{\partial S^2}$. Then in (3.2) the coefficient $\frac{1}{2}\sigma^2 S^2$ is replaced by $\left(\frac{1}{2}\sigma^2 S^2 + rS\frac{\Delta S}{2}\right)$, so that under the condition $\sigma^2 \ll r$ the numerical diffusion becomes particularly significant. The equivalence between the two methods allows us to interpret the numerical solution $V_N(S_j,T)$ obtained by means of the Post–Widder inversion formula. Under the condition $\sigma^2 < r$ the discrepancy between exact $V(S_j,T)$ and approximate one $V_N(S_j,T)$ is due to the numerical diffusion introduced by the forward difference of the convection term $\frac{\partial V}{\partial S}$. Fig. 1, where two distinct values $\sigma_1 = 0.1$ and $\sigma_2 = 0.001$ are chosen (in the latter case the numerical diffusion becomes particularly significant), confirms the statement in the previous sentence. In Fig. 1, $T = 1$ and $\Delta t = 0.001$ are assumed, so that the numerical solution obtained by finite difference scheme is equivalent to the one obtained by the mixed method with $N = 1000$. Fig. 2, where geometry and parameters coincide with Fig. 1, compares the finite difference solution and the mixed method one for increasing N. According to results in literature, the Post–Widder inversion formula has an *extremely slow convergence rate*. In both Figs. 1 and 2 the spatial domain $[0,\infty)$ has been replaced by $[0, S_{max}]$. The criterion choice of S_{max} will be discussed later. Because of the slow convergence of the sequence $V_N(S_j,T)$ it is natural to seek extrapolation methods to speed up convergence. The latter will be discussed later too.

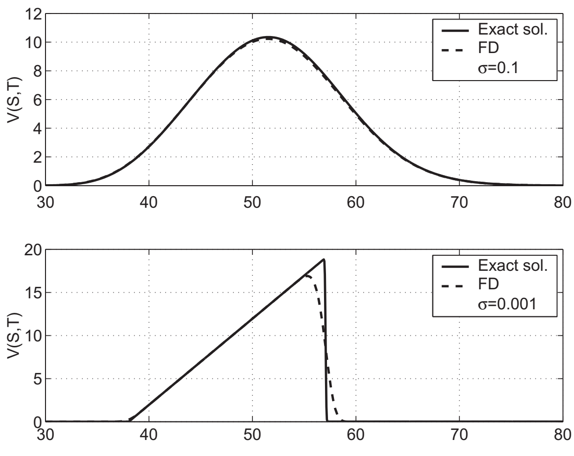

Fig. 1. Truncated call option value just before the first monitoring date $t_1 = T$. Parameters: $L = 0,\ K = 40,\ U = 60,\ r = 0.05,\ \sigma = 0.1$ (the upper graphic) and $\sigma = 0.001$ (the lower graphic), $T = 1$. The numerical solutions are obtained by the fully-implicit scheme (FD) with $\Delta S = 0.2,\ \Delta t = 0.001,\ S_{max} = 160$ (the upper graphic) and $S_{max} = 80$ (the lower graphic), according to (5.2), with $R = 4$ and $R = 2$ respectively.

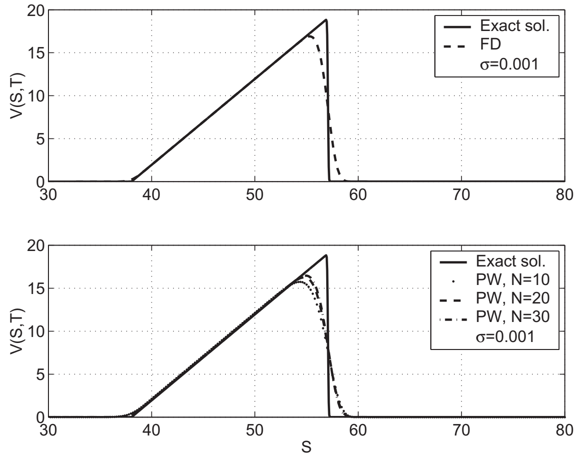

Fig. 2. Truncated call option value just before the first monitoring date $t_1 = T$. Parameters: $L = 0,\ K = 40,\ U = 60,\ r = 0.05,\ \sigma = 0.001,\ T = 1$. The Numerical solutions are obtained by the fully-implicit finite difference scheme (FD), with $\Delta S = 0.2,\ \Delta t = 0.001,\ S_{max} = 80$ (the upper graphic) and mixed method (PW) with $N = 10, 20, 30,\ \Delta S = 0.2,\ S_{max} = 80$ (the lower graphic), according to (5.2), with $R = 2$.

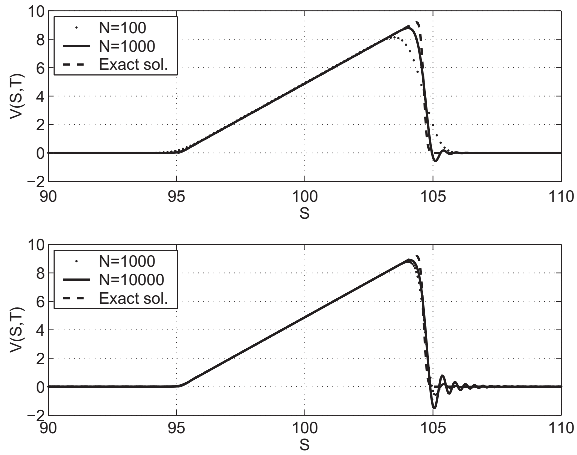

Fig. 3. Truncated call option value just before the first monitoring date $t_1 = T$. The numerical solutions are obtained using the mixed method with $\Delta S = 0.05,\ N = 100,\ N = 1000$ (the upper graphic) and $N = 1000,\ N = 10000$ (the lower graphic), respectively. Parameters: $L = 90,\ K = 100,\ U = 110,\ r = 0.05,\ \sigma = 0.001,\ T = 1,\ S_{max} = 200$, according to (5.2), with $R = 2$.

4. In literature spurious oscillations, arising from discontinuity in initial/boundary conditions are usually associated with finite difference or finite element methods. It is surprising how the mixed method, which combines the Laplace Transform on the real axis with a particular inversion technique and finite difference method (where the term $\frac{dU^{(k)}}{dS}$ is discretized by a centered difference), exhibits an analogous phenomenon (see Fig. 3). A posteriori we can say that arising spurious oscillations is a consequence of the equivalence of the adopted methods. Now we illustrate and prove this statement. In other terms, we ask what happens if an unsuitable finite difference scheme is adopted. More specifically, under the condition $\sigma^2 < r$, what solution does the mixed method, with centered difference for discretizing the convection term $\frac{\partial V}{\partial S}$, arise? For reasons that will be evident in the sequel, the condition $\sigma^2 < \frac{r}{M}$ is assumed. Through mixed method, if the convection term $\frac{\partial V}{\partial S}$ is discretized by a centered difference, we have the matrix A_{PW} given by (3.3).

In the next analysis we will follow [7, p. 67–69]. The subdiagonal of A_{PW} has positive entries as a consequence of the relationship $\sigma^2 < \frac{r}{M}$, so that A_{PW} is no longer an M-matrix. If we denote $A_{PW} = \text{tridiag}\{a, c, -b\}$, with $a, c, b > 0$ and $a = [a_1, \ldots, a_{M-1}],\ c = [c_1, \ldots, c_M]\ b = [b_1, \ldots, b_{M-1}]$ then the leading principal minors $(A_{PW})_k$ of A_{PW} satisfy the relationship [7, p. 40]

$$(A_{PW})_k = c_k (A_{PW})_{k-1} + a_{k-1} b_{k-1} (A_{PW})_{k-2},$$
$$k = 1, \ldots, M, \quad (A_{PW})_{-1} = 0, \quad (A_{PW})_0 = 1.$$

Then $(A_{PW})_k > 0,\ \forall k$ and A_{PW}^{-1} there exists.

The sign pattern of the entries of A_{PW}^{-1} is as follows [7, formula (4.33)]

$$A_{PW}^{-1} = \begin{pmatrix} + & + & + & \cdots & \cdots & \cdots & + \\ - & + & + & & & & \vdots \\ + & - & + & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ (-1)^{M+1} & \cdots & \cdots & \cdots & + & - & + \end{pmatrix}$$

Next we solve (3.2), until an assigned N, obtaining (3.5) and (3.6). Since $A_{PW}^{-1} \geqslant 0$ is not guaranteed, $V_N(S_j, T)$ could assume negative values, which are nothing but spurious oscillations. The latter arises for any value of N, nevertheless they become visible when N assumes high values (see Fig. 3).

As $a(-b) < 0$, the eigenvalues $\mu_j(A_{PW})$ satisfy the relationship $\min(c) \leqslant Re\left(\mu_j(A_{PW})\right) \leqslant \max(c)$ (see [10, Th. 1]) from which

$$\frac{\frac{N}{T}}{r + \frac{N}{T} + (\sigma M)^2} \leqslant Re\left(\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)\right) \leqslant \frac{\frac{N}{T}}{r + \frac{N}{T} + \sigma^2}. \tag{4.3}$$

When $N \to \infty, Re\left(\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)\right) \to 1$. Besides this, A_{PW} is diagonally dominant, so that $\|\frac{N}{T} A_{PW}^{-1}\|_\infty \leqslant \frac{\frac{N}{T}}{r+\frac{N}{T}}$, (see [7, p. 8]). From $\rho\left(\frac{N}{T} A_{PW}^{-1}\right) \leqslant \|\frac{N}{T} A_{PW}^{-1}\|_\infty \leqslant \frac{\frac{N}{T}}{r+\frac{N}{T}} \simeq 1$, where $\rho(\cdot)$ denotes the spectral radius, then all the eigenvalues of $\frac{N}{T} A_{PW}^{-1}$ satisfy $|\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)| \simeq 1$, from which $Im\left(\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)\right) \simeq 0$. As $N \to \infty$ all eigenvalues $\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)$ cluster in the complex plane around the point $(1,0)$. Then in (3.6) the eigenmodes with the largest eigenvalue *'excited'* by discontinuity, are not dampened with the consequent arising of spurious oscillations.

Taking N small integer values, all the eigenvalues $\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)$ are far from 1 and then spurious oscillations do not occur. Nevertheless, the found solution is inaccurate because of the low value N. With large N all the eigenvalues $\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)$ satisfy $|\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right)| \simeq 1$. From (4.5), increasing N, all the terms $(\mu_j\left(\frac{N}{T} A_{PW}^{-1}\right))^{N+1}$ are slowly dampened and then spurious oscillations occur. We are in the paradoxical case that, with $N \to \infty$ in order to guarantee convergence, persistent spurious oscillations are enforced. Fig. 3 illustrates this statement with different values of N. Here the spurious oscillations, arising close to discontinuity, increase as N increases. Nevertheless, disregarding the oscillations issue, as expected, far from barriers the convergence to exact solution is observed as N increases.

In conclusion, under the assumed condition $\sigma^2 < \frac{r}{M}$ (as it occurs in certain regions of the grid for stochastic volatility models) the mixed method, with a convection term discretized by a centered difference, is unsuitable for solving problem (2.1) with a *discontinuous payoff*.

Through an heuristic reasoning we prove that A_{PW} is diagonalizable, showing that A_{PW} admits M distinct eigenvalues. More specifically we prove that the eigenvalues $\mu_j(A_{PW})$ vary continuously with r. The hypothesis that A_{PW} is diagonalizable is basic to invoke some theorems about the perturbation of eigenvalues.

Let us write $A_{PW} = A_1 + A_2$, with

$$A_1 = \frac{\sigma^2}{2} \text{tridiag}\left\{-j^2; \frac{2}{\sigma^2}\frac{N}{T} + 2j^2; -j^2\right\},$$

$$A_2 = \frac{r}{2} \text{tridiag}\{j; 2; -j\}, \text{ with } \|A_2\|_1 = \|A_2\|_\infty = \frac{r}{2}(2M-1).$$

The matrix A_1 is a symmetric Jacobi matrix with M real and distinct eigenvalues $\mu_j(A_1)$, so that A_1 is a symmetric diagonalizable matrix with similarity transformation matrix P. From the Schur decomposition theorem it follows that the similarity transformation matrix P is unitary, so that $\text{Cond}_2(P) = 1$ [11, p. 56–57] (same results if norms $\|\cdot\|_1$ or $\|\cdot\|_\infty$ are used). Then, from Bauer–Fike theorem, the eigenvalues $\mu_j(A_{PW}) = \mu_j(A_1 + A_2)$ lie in the union of the disks

$$\{\mu \in C : |\mu - \mu_j(A_1)| \leqslant \text{Cond}_2(P)\|A_2\|_2 \leqslant 1 \cdot \sqrt{\|A_2\|_1\|A_2\|_\infty} = \|A_2\|_1 = \|A_2\|_\infty = \frac{r}{2}(2M-1)\}, \tag{4.4}$$

being A_2 a perturbation of $A_{PW} = A_1 + A_2$.

As A_1 is symmetric with distinct eigenvalues, for sufficiently small r (4.4) may be improved as follows. The eigenvalues of A_{PW} and A_1 may be ordered so that

$$|\mu_j(A_{PW}) - \mu_j(A_1)| \leqslant \|A_2\|_2 \leqslant \frac{r}{2}(2M-1), \quad j = 1, \ldots, M.$$

As $N \to \infty$, multiple real/complex eigenvalues of A_{PW} can not be allowed and being the maximum principle satisfied then $V_N(S_j, t) \leqslant V(S_j, 0)$ holds. Indeed, a multiple complex eigenvalue, say $\mu_1(A_{PW})$ (and then its conjugate $\overline{\mu_1}(A_{PW})$) introduces

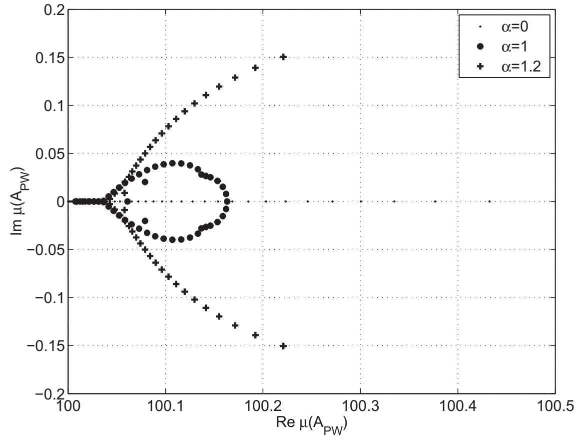

Fig. 4. Real/complex eigenvalues $\mu_j(A_{PW})$ for different values of $r = \alpha M\sigma^2$, with $\alpha = 0, 1, 1.2$, and $\sigma = 0.01,\ M = 50,\ \frac{N}{T} = 100$ fixed.

terms like $N\mu_1^N, N^2\mu_1^N, \ldots$, which amount to $N\rho^N\cos(N\theta), N^2\rho^N\cos(N\theta), \ldots, N\rho^N\sin(N\theta), N^2\rho^N\sin(N\theta), \ldots$ As $N \to \infty$, from (4.3) it follows $\rho \to 1$, from which spurious oscillations increasing with N. Similar analysis if $\mu_1(A_{PW})$ is multiple and real. Eigenvalues pattern of A_{PW} for different r values is illustrated in Fig. 4.

Summarizing, as $N \to \infty, A_{PW}$ admits M distinct real/complex eigenvalues $\mu_j(A_{PW})$ and corresponding M eigenvectors v_j. The latter allows us to write the initial condition as $V(S_j, 0) = \sum_{j=1}^{M} w_j v_j$, where w_j are proper weights. Then from (3.6)

$$V_N(S_j, T) = \left(\frac{N}{T}A_{PW}^{-1}\right)^{N+1} V(S_j, 0) = \left(\frac{N}{T}A_{PW}^{-1}\right)^{N+1}\sum_{j=1}^{M} w_j v_j = \sum_{j=1}^{M} w_j\left(\frac{N}{T}A_{PW}^{-1}\right)^{N+1} v_j = \sum_{j=1}^{M} w_j\left(\frac{N}{T}\mu_j(A_{PW}^{-1})\right)^{N+1} v_j. \tag{4.5}$$

Formula (4.5) allows us the above conclusions for spurious oscillations.

5. Numerical aspects

1. As we have mentioned before, the Post–Widder inversion formula involves high derivatives $U^{(k)}(S_j, \lambda)$ which require the need to differentiate $U^{(0)}(S_j, \lambda)$ a large number of times. Besides, it is well known that high order derivatives are sensible to round-off errors, causing thereby instability. In (3.2) the instabilities for calculating $U^{(k)}(S_j, \lambda)$ may arise from the ill-conditioning of matrix A_{PW}. Now we estimate its conditioning number $\text{Cond}_\infty(A_{PW}) = \|A_{PW}\|_\infty \|A_{PW}^{-1}\|_\infty$. In both cases $\sigma^2 > r$ and $\sigma^2 < r$ one has $\|A_{PW}\|_\infty \leqslant 2(\sigma M)^2 + 2rM + r + \frac{N}{T}$. A_{PW} is diagonally dominant, so that $\|A_{PW}^{-1}\|_\infty \leqslant \frac{1}{r+\lambda} = \frac{1}{r+\frac{N}{T}}$ holds [7, p. 8]. Then $\text{Cond}_\infty(A_{PW}) \leqslant 1 + \frac{2(\sigma M)^2 + 2rM}{r+\frac{N}{T}}$, with $\lim_{N\to\infty}\text{Cond}_\infty(A_{PW}) = 1$ and then, for large value of N A_{PW} is well conditioned, guaranteeing an accurate value $U^{(N)}(S_j, \lambda)$.

   Besides, the approximate solution $V_N(S_j, t)$ satisfies the discrete maximum principle, i.e., if $t_1 < t_2 < T$ are arbitrary values of t and then $t_1 = N_1\Delta t < t_2 = N_2\Delta t$, it holds

$$\begin{aligned} \|V_{N_2}(S, t_2)\|_\infty &= \left\|\left(\frac{N}{T}A_{PW}^{-1}\right)^{N_2+1} V(S, 0)\right\|_\infty = \left\|\left(\frac{N}{T}A_{PW}^{-1}\right)^{N_2-N_1}\left(\frac{N}{T}A_{PW}^{-1}\right)^{N_1+1} V(S, 0)\right\|_\infty \\ &\leqslant \left\|\left(\frac{N}{T}A_{PW}^{-1}\right)^{N_2-N_1}\right\|_\infty \|V_{N_1}(S, t_1)\|_\infty \leqslant \left(\frac{\frac{N}{T}}{r+\frac{N}{T}}\right)^{N_2-N_1} \|V_{N_1}(S, t_1)\|_\infty \leqslant \|V_{N_1}(S, t_1)\|_\infty. \end{aligned} \tag{5.1}$$

In both cases $\sigma^2 > r$ and $\sigma^2 < r$ the matrix A_{PW} has subdiagonal and superdiagonal strictly negative entries, so that A_{PW} is an M-matrix similar to a symmetric tridiagonal matrix [7, p. 24]. Then A_{PW} has real and positive eigenvalues $\mu_j(A_{PW})$. From the Gerschgorin theorem we obtain that the iteration matrix $\frac{N}{T}A_{PW}^{-1}$ has eigenvalues $\mu\left(\frac{N}{T}A_{PW}^{-1}\right) \in \left[\frac{\frac{N}{T}}{r+\frac{N}{T}+2rM+2(\sigma M)^2}; \frac{\frac{N}{T}}{r+\frac{N}{T}}\right] \subset (0, 1)$. Then the approximate solution $V_N(S_j, t)$ is positivity-preserving and not affected by spurious oscillations close to barriers.

2. S_{max} is a sufficiently large positive constant so that the boundary conditions at the end of the infinite interval can be applied with sufficiently accuracy. Usually, the boundary condition on the artificial boundary $S = S_{max}$ is imposed by extending a given payoff. In our case, as truncated call options are mainly considered, it will be assumed $V(S_{max}, t) = 0,\ \forall t \geqslant 0$. From which, in the Laplace transformed domain it corresponds $U^{(k)}(S_{max}, \lambda) = 0,\ \forall k \geqslant 0$. In [12] the errors caused by Dirichlet boundary conditions on the artificial boundary are estimated and thus one can determine a suitable asset price for the artificial boundary to meet a given tolerance errors. In [12, p. 1367] for an european call option with constant coefficient, the proper size of the domain is proposed after a careful analysis. Under the restrictive

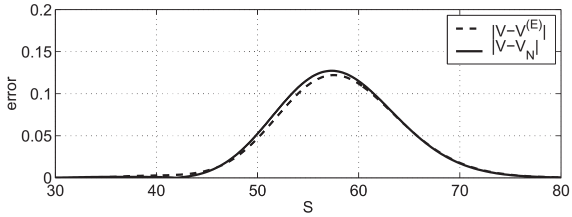

Fig. 5. Truncated call option. Geometry and parameters as upper graphic of Fig. 1. Numerical solution $V_N(S,T)$ and $V_{2N}(S,T)$ from Post–Widder formula, with $N = 1000$. Errors $|V(S,T) - V_N(S,T)|$ and $|V(S,T) - V^{(E)}(S,T)|$.

condition $\sigma^2 \geqslant 2r$, the far field boundary condition reads $S_{max} = K\exp(\sqrt{2T\sigma^2 \ln 100})$, involving strike price K, maturity T and volatility σ. Numerical experiments in literature show that the estimates at least incorporate the usual rule of thumb (e.g., "three or four times the strike price"). So that in the numerical experiments above, S_{max} is assumed according to the widely used criterion.

$$S_{max} = \max\{RK, K\exp(\sqrt{2T\sigma^2 \ln 100})\}, \tag{5.2}$$

with $R \geqslant 2$ and closely correlated to σ. (5.2) has very general character. As $\sigma \ll 1$, it holds $S_{max} = \max\{RK, K\exp(\sqrt{2T\sigma^2 \ln 100})\} = RK$, where R has to be chosen with the usual rule of thumb. In the numerical examples above is evident that R in (5.2) depends on N (equivalently Δt) too significantly. Indeed, as $\sigma^2 \ll r$, numerical diffusion arises, so that the term $\frac{1}{2}\sigma^2 S^2$ is replaced by $\left(\frac{1}{2}\sigma^2 S^2 + rS\frac{\Delta S}{2}\right)$ (see Fig. 2). If N assumes low values (equivalently Δt large values), the entire solution is smeared, so that in (5.2) large R values have to be chosen. On the contrary, as $\sigma^2 \ll r$ and large values of N are chosen, spurious persistent oscillations arise close to the barrier. As a consequence, (5.2) requires large R values (see Fig. 3). In conclusion, (5.2) has general character and an accurate value of S_{max} has to include both the geometry conditions of the problem (the barriers) and the values of the parameter N (equivalently Δt).

As a second alternative for accurate truncation of the solution domain we add the following remark. Instead of such boundary condition (5.2), a transparent boundary condition is introduced in [13] with which one can evaluate the solution in the truncated domain without any truncation error. However, the transparent boundary condition [13] is an integro–differential one, which needs some suitable numerical schemes to approximate it that will produce other possibly significant errors. For more details see [13] where the boundary condition is implemented in the Laplace transformed setting instead of the usual space–time setting.

3. Post–Widder method has very slow convergence rate. On the other hand it retains essential structural characteristics of the original solution, as positivity and discrete maximum principle. The construction of a sequence that is more rapidly convergent than the approximation sequence (3.5) is given in [14]. Once T and N are fixed, (3.2) is solved two times
(a) taking $\lambda = \frac{N}{T}$, one solves N times (3.2) from which one has $V_N(S_j,T)$ according to (3.5);
(b) taking $\lambda = \frac{2N}{T}$, one solves $2N$ times (3.2) from which one has $V_{2N}(S_j,T)$ according to (3.5).

An extrapolate solution $V^{(E)}(S_j,T)$ is obtained as [14, Eq. (11)]

$$V^{(E)}(S_j,T) = (2+\frac{1}{N})V_{2N}(S_j,T) - (1+\frac{1}{N})V_N(S_j,T), \tag{5.3}$$

having error $o\left(\frac{1}{N^2}\right)$ [14, Eq. (13)]. (5.3) improves the convergence rate but usually does not preserve positivity. As N assumes high values, (5.3) is replaced by

$$V^{(E)}(S_j,T) \simeq 2V_{2N}(S_j,T) - V_N(S_j,T). \tag{5.4}$$

(5.4) is reminiscent with an extrapolation method for the implicit finite difference scheme [15, p. 127]. Indeed, such an analogy comes from equivalence between mixed method and finite difference scheme above proved. If we call $A_{FD} = I - \Delta tA$, with evident meaning of the symbols, (4.1) may be solved as follows. If $t < T$ is an arbitrary instant and $V(S_j,t)$ the numerical solution

(a) $V^{(1)}(S_j,t+2\Delta t)$ is calculated from $V(S_j,t)$ by solving the linear system $(I - 2\Delta tA)V^{(1)}(S_j,t+2\Delta t) = V(S_j,t)$;
(b) $V^{(2)}(S_j,t+2\Delta t)$ is calculated from $V(S_j,t)$ by solving the linear system $(I - \Delta tA)^2 V^{(2)}(S_j,t+2\Delta t) = V(S_j,t)$.

Naturally, the extrapolate solution values are used as starting values for the extrapolation procedure over the next two time-levels. From the above procedure (a) and (b), the extrapolate solution $V_{FD}^{(E)}(S_j,T)$ is obtained. The latter is unlike from (5.3); here the extrapolation happens uniquely one time at $t = T$. At the contrary, in finite difference method the extrapolation happens at every time $t + 2\Delta t$. Using the geometry and parameters of Fig. 1(top), for a fixed N, $V_N(S_j,T)$ and $V_{2N}(S_j,T)$ are calculated, from which the extrapolate solution $V^{(E)}(S_j,T)$, according to (5.4). In Fig. 5 errors $|V(S,T) - V_N(S,T)|$ and $|V(S,T) - V^{(E)}(S,T)|$ are reported. The improvement coming from extrapolation is modest.

4. From above analysis it appears evident that the mixed method here illustrated runs well whether the condition $\sigma^2 > r$ is satisfied. It is evident too that the mixed method cannot be extended to solve all the special geometries, parameters and so on, arising from Black–Scholes equation. For instance, issues related to nonlinear volatility, and considered in [8], will be disregarded here. If we require that special properties of the solution are preserved, we admit that mixed method has to be replaced with other more appropriate methods, chosen among the ones that in natural way take into account nonlinear volatility.

6. Conclusions

Options having discontinuity in the initial/boundary conditions and low volatility have been considered. Laplace Transform with the Post–Widder inversion formula jointly with the finite difference method has been proved to be equivalent to standard fully-implicit finite difference scheme. The mixed adopted method, in order to save some physical properties of the solution as positivity and maximum principle, has low order of accuracy and is convergent. Unlike analogous mixed methods proposed in literature, the investigated method causes a set of ordinary differential equations that cannot be solved concurrently in a distributed environment. On the other hand, the main goal of this paper is to prove the equivalence between two different methods of solution. From the latter equivalence the N-th approximation of the solution obtained by the Post–Widder inversion formula received a physical meaning. That approximation is nothing but the one obtained through a finite difference scheme with time step Δt related to N by $\Delta t = \frac{T}{N}$.

It is well known that Laplace Transform technique is competitive compared to grid methods whenever maturity T assumes high values. Nevertheless, the explored Laplace technique is not a universal tool as the presence of barriers jointly with low volatility causes numerical drawbacks such as spurious oscillations and numerical diffusion. The latter are significant especially whenever $\sigma^2 \ll r$, that is a condition scarcely considered in literature. Laplace Transform technique requires some care, because special geometry settings (e.g., barriers) and involved parameters (mainly concerning σ) may destroy the solution accuracy. Then we dont propose a new method to solve the Black–Scholes equation. Instead, we explore the Laplace Transform in a concrete case in order to prove and point out the fact that, in analogy with grid methods (i.e., finite difference schemes and etc.), a mechanical use of Laplace Transform may lead to misleading results.

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