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MAT 243: Discrete Math Structures Project #2 Waves

While this class is mostly concerned with the interaction of discrete mathematical structures, the methods are also useful when working with more traditional, real-valued functions. By adding and subtracting simple waveforms of different integer frequencies, it is possible to get very complex patterns of constructive and destructive interference. With a sufficient, though possibly infinite, number of frequencies, it is possible to exactly match any function, from polynomials and exponentials, to the complex orbits of the stars, and even "orbits" that draw pictures of Homer Simpson. Solutions involving multiple waveforms are often the solution to Differential Equations, found using Fourier Transforms.

In this Project you will use Mathematical Induction to prove an Identity involving SIN and COS functions. You will want to review Trigonometric Identities involving sums, differences and products of SIN and COS. (You may also find it helpful to look at Lagrange's Trigonometric Identity.) If you do any external research be sure to properly cite your sources. If you work with another student(s), please include their name(s) in the write-up as well. If you use any electronic resources to help you visualize the waveforms or perform calculations, name them in the report. [Note: There may be other ways to prove the inequality, but you must use mathematical induction for this Project.]

Prove:

$$\sum_{j=1}^{n} cos(jx) = \frac{cos(\frac{(n+1)x}{2})sin(\frac{nx}{2})}{sin(\frac{x}{2})}$$

$$\forall n \in \mathbb{Z}^{+}, \forall x(sin(\frac{x}{2}) \neq 0)$$

Let P(n) be the proposition that $\sum_{j=1}^{n} cos(jx) = \frac{cos(\frac{(n+1)x}{2})sin(\frac{nx}{2})}{sin(\frac{x}{2})}$ $, \forall n \in \mathbb{Z}^{+}, \forall x(sin(\frac{x}{2}) \neq 0)$

is correct.

We know that $\sum_{j=1}^{n} cos(jx) = cos(x) + cos(2x) + cos(3x) + \ldots + cos(nx)$.

To complete the basis step, we find P(1).

$$P(1) = \frac{cos(\frac{(1+1)x}{2})sin(\frac{1x}{2})}{sin(\frac{x}{2})} = cos(\frac{2x}{2}) = cos(x)$$

Using the proposed formula, P(1) is consistent with $\sum_{j=1}^{n} cos(jx)$ when n = 1.

Inductive Step: The inductive hypothesis is that P(k) is the statement

$$\mathrm{P(k)} = \sum_{j=1}^{k} cos(jx) = \frac{cos(\frac{(k+1)x}{2})sin(\frac{kx}{2})}{sin(\frac{x}{2})}$$

If P(k) is true, then P(k+1) is also true.

P(k+1) would be the statement

$$\mathrm{P(k+1)} = \sum_{j=1}^{k+1} cos(jx) = \frac{cos(\frac{((k+1)+1)x}{2})sin(\frac{(k+1)x}{2})}{sin(\frac{x}{2})}$$

To determine whether P(k) implies P(k+1), we begin by simplifying P(k+1).

$$P(k+1) = \frac{cos(\frac{(kx+2x)}{2})sin(\frac{(kx+x)}{2})}{sin(\frac{x}{2})}$$

$$P(k+1) = \frac{cos(\frac{kx}{2}+\frac{2x}{2})sin(\frac{kx}{2}+\frac{x}{2})}{sin(\frac{x}{2})}$$

Using the product identity for Sine and Cosine, we can rewrite P(k+1) in the following way:

$$P(k+1) = \frac{sin(\frac{kx}{2}+\frac{x}{2}-\frac{kx}{2}-\frac{2x}{2}) + sin(\frac{kx}{2}+\frac{x}{2}+\frac{kx}{2}+\frac{2x}{2})}{2sin(\frac{x}{2})}$$

We can simplify further:

$$P(k+1) = \frac{sin(-\frac{x}{2}) + sin(\frac{2kx}{2} + \frac{3x}{2})}{2sin(\frac{x}{2})}$$

Using the identity for negative angles, we can rewrite P(k+1) as

$$P(k+1) = \frac{-sin(\frac{x}{2}) + sin(\frac{2kx}{2} + \frac{3x}{2})}{2sin(\frac{x}{2})}$$

We can simplify further:

$$P(k+1) = -\frac{1}{2} + \frac{sin(\frac{2kx}{2} + \frac{3x}{2})}{2sin(\frac{x}{2})}$$

$$P(k+1) = -\frac{1}{2} + \frac{sin(kx + \frac{3x}{2})}{2sin(\frac{x}{2})}$$

Now that P(k+1) has been rewritten and simplified, let's consider Lagrange's Trigonometric Identity.

Lagrange's Trigonometric Identity states that

$$\sum_{n=1}^{N} cos(n\theta) = -\frac{1}{2} + \frac{sin((N + \frac{1}{2})\theta)}{2sin(\frac{\theta}{2})}$$

Using our terms, we can let L(k) represent this identity as

$$\mathrm{L(k)} = \sum_{j=1}^{k} cos(jx) = -\frac{1}{2} + \frac{sin((k + \frac{1}{2})x)}{2sin(\frac{x}{2})}$$

Given Lagrange's Trigonometric Identity, we can perform induction here as well to help show that

L(k) implies L(k+1).

Knowing that

$$\sum_{j=1}^{k} cos(jx) = \cos(\mathrm{x}) + \cos(2\mathrm{x}) + \ldots + \cos(\mathrm{kx})$$

it follows that

$$\sum_{j=1}^{k+1} cos(jx) = \cos(\mathrm{x}) + \ldots + \cos(\mathrm{kx}) + \cos((\mathrm{k}+1)\mathrm{x})$$

Since L(k+1) would appear as the following

$$\mathrm{L}(\mathrm{k}+1) = \sum_{j=1}^{k+1} cos(jx) = -\frac{1}{2} + \frac{sin((k+1+\frac{1}{2})x)}{2sin(\frac{x}{2})}$$

we can show this to be true by adding cos((k+1)x) to L(x), which is representing Lagrange's Identity, in the following way:

$$\sum_{j=1}^{k+1} cos(jx) = \cos(\mathrm{x}) + \ldots + \cos((\mathrm{k}+1)\mathrm{x}) = -\frac{1}{2} + \frac{sin((k+\frac{1}{2})x)}{2sin(\frac{x}{2})} + \cos((\mathrm{k}+1)\mathrm{x})$$

We can rewrite the far right side of the equation like so:

$$-\frac{1}{2} + \frac{sin(k\mathrm{x}+\frac{\mathrm{x}}{2}) + 2\sin(\frac{\mathrm{x}}{2})\cos(\mathrm{kx}+\mathrm{x})}{2sin(\frac{x}{2})}$$

We can use the trigonometric product identities to rewrite the expression like so (some simplification is also occurring):

$$-\frac{1}{2} + \frac{sin(k\mathrm{x}+\frac{\mathrm{x}}{2}) + \sin(\frac{\mathrm{x}}{2}+\mathrm{kx}+\mathrm{x}) + \sin(\frac{\mathrm{x}}{2} - \mathrm{kx} - \mathrm{x})}{2sin(\frac{x}{2})}$$

$$-\frac{1}{2} + \frac{\sin(\frac{\mathrm{x}}{2}+\mathrm{kx}+\mathrm{x}) + \sin(\mathrm{kx}+\frac{\mathrm{x}}{2}) + \sin(-\mathrm{kx} - \frac{\mathrm{x}}{2})}{2sin(\frac{x}{2})}$$

Using the identities for negative angles, we can rewrite the expression like so:

$$-\frac{1}{2}+\frac{\sin(\frac{x}{2}+kx+x)+\sin(kx+\frac{x}{2}) - \sin(kx+\frac{x}{2})}{2sin(\frac{x}{2})}$$

$$-\frac{1}{2}+\frac{\sin(\frac{x}{2}+kx+x)}{2sin(\frac{x}{2})}$$

$$-\frac{1}{2}+\frac{\sin((k+1+\frac{1}{2})x)}{2sin(\frac{x}{2})}$$

This expression matches the hypothetical form of L(k+1). To finish this induction, we find L(1).

$$-\frac{1}{2}+\frac{sin((1+\frac{1}{2})x)}{2sin(\frac{x}{2})}$$

$$-\frac{1}{2}+\frac{sin(x+\frac{x}{2})}{2sin(\frac{x}{2})}$$

By the sum identity for Sine, we can rewrite this as

$$-\frac{1}{2}+\frac{sin(x)\cos(\frac{x}{2})+\cos(x)\sin(\frac{x}{2})}{2sin(\frac{x}{2})}$$

$$-\frac{1}{2}+\frac{sin(x)\cos(\frac{x}{2})}{2sin(\frac{x}{2})}+\frac{1}{2}\cos(x)$$

$$-\frac{1}{2}+\frac{sin(x)}{2}(\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})})+\frac{1}{2}\cos(x)$$

$$-\frac{1}{2}+\frac{sin(x)}{2}(\cot(\frac{x}{2}))+\frac{1}{2}\cos(x)$$

By the half angle identity for Tangent, which is the reciprocal of Cotangent, we can rewrite this as

$$-\frac{1}{2}+\frac{sin(x)}{2}(\frac{1+\cos(x)}{\sin(x)})+\frac{1}{2}\cos(x)$$

$$-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cos(x)+\frac{1}{2}\cos(x)$$

$$\cos(x)$$

Thus, L(1) is consistent with $\sum_{j=1}^{n} cos(\mathrm{k}x)$ when n = 1, proving that L(k+1) is true.

Knowing that L(k+1) is true and knowing that L(k) is equal to the summation that we are trying to prove is also equal to P(k), we can try to rewrite P(k+1) to match the form of L(k+1) to show that P(k+1) = L(k+1).

$$P(k+1) = -\frac{1}{2} + \frac{sin(kx + \frac{3x}{2})}{2sin(\frac{x}{2})}$$

$$P(k+1) = -\frac{1}{2} + \frac{sin((k + \frac{3}{2})\mathrm{x})}{2sin(\frac{x}{2})}$$

$$P(k+1) = -\frac{1}{2} + \frac{sin((k + 1 + \frac{1}{2})\mathrm{x})}{2sin(\frac{x}{2})} = \mathrm{L(k+1)}$$

Because we have shown that P(k+1) = L(k+1) and because L(k+1) is a true statement that follows from L(k), which is a representation of Lagrange's Trigonometric Identity, we have demonstrated that if P(k) is true, then P(k+1) must also be true. Additionally, because P(1) is consistent with $\sum_{j=1}^{n} cos(jx)$ when n = 1 since P(1) = cos(x), we have shown by mathematical induction that P(n) is true, proving that

$$\sum_{j=1}^{n} cos(jx) = \frac{cos(\frac{(n+1)x}{2})sin(\frac{nx}{2})}{sin(\frac{x}{2})}$$

$$\forall n \in \mathbb{Z}^{+}, \forall x(sin(\frac{x}{2}) \neq 0)$$

Sources

List of Trigonometric Identities:

https://www2.clarku.edu/faculty/djoyce/trig/identities.html

Desmos Graphing Calculator:

https://www.desmos.com/calculator

Lagrange’s Trigonometric Identity:

https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Lagrange's_trigonometric_identities

https://aapt.scitation.org/doi/10.1119/1.1933371