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MAT 243 Project 5

For this project, we were tasked with validating the ISBN-13 of the class textbook using the given congruence and demonstrate various qualities of an ISBN-13. We have to demonstrate both that the check digit of an ISBN-13 can always detect a single digit error and that there exist transposition errors of two digits that are not detected by the ISBN-13.

To begin, we recall the congruence involving the 13 digits of an ISBN-13, namely, a_1, a_2, …, a_{13}, where a_{13} is the check digit. The given congruence is shown in Figure 1.

$$(a_1 + a_3 + \ldots + a_{13}) + 3(a_2 + a_4 + \ldots + a_{12}) \equiv 0 \pmod{10}$$

Figure 1.

As Figure 1 demonstrates, each ISBN-13 digit with an odd index (a_1, a_3, a_5, etc.) is grouped with the other odd-indexed digits while each ISBN-13 digit with an even index (a_2, a_4, a_6, etc.) is grouped with the other even-indexed digits. The sum of the even-indexed digits is then multiplied by 3. The resulting product is then added to the sum of the odd-indexed digits. The final result should then be congruent to 0 (mod 10).

As defined in Section 4.1 page 240 of the class textbook, “If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides a - b”. It is clear that, because each digit of an ISBN-13 is a non-negative integer and because we are simply adding the integers together and multiplying the sum of the even-indexed integers by the integer 3, the final result will be an integer. This final integer result can be substituted for a. Additionally, 0 is also an integer, so 0 may be substituted for b and 10 may be substituted for k. Therefore, if 10 divides the difference between the final result and 0, that is, if 10 divides the final result, then the proposition that the final result is congruent to 0 (mod 10) would be valid.

The textbook’s ISBN-13 is provided in Figure 2, which also arranges its digits as shown in the given congruence. The red numbers represent the odd-indexed digits from a_1 to a_{13} while the green numbers represent the even-indexed digits from a_2 to a_{12}.

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$$(9 + 8 + 0 + 3 + 8 + 0 + 5) + 3(7 + 0 + 7 + 3 + 3 + 9) \equiv 0 \pmod{10}$$

Figure 2

We can simplify the congruence to help determine if the expression in Figure 2 is valid. Figure 3 demonstrates this simplification.

$$33 + 3(29) \equiv 0 \pmod{10}$$

$$33 + 87 \equiv 0 \pmod{10}$$

$$120 \equiv 0 \pmod{10}$$

Figure 3

We now have the claim that 120 is congruent to 0 (mod 10). As explained earlier, we can determine the validity of this statement if we can show that 10 divides (120 - 0), that is, (120 - 0) divided by 10 has a remainder of 0. Figure 4 demonstrates that, because 120 divided by 10 is 12 with a remainder of 0, 10 divides 120, which means that 10 would also divide (120 - 0).

$$(120 - 0)/10$$

$$120/10$$

12 Remainder 0.

Figure 4

Therefore, we have confirmed that the class textbook's ISBN-13 is valid since the sum of the sum of the odd-indexed digits and three times the sum of the even-indexed digits is congruent to 0 (mod 10).

We must now show that the check digit of an ISBN-13 can always detect a single digit error. We have found that the check digit of the original ISBN-13 number is a value of 5 since a_{13}, the last element in the ISBN-13, is 5 for the class textbook. In order to show that the check digit of an ISBN-13 can always detect a single digit error we must first change one digit from the original ISBN-13 and check if the result of the operations performed on the digits, including the check digit 5, remains congruent to 0 (mod 10). Figure 5 shows the original ISBN-13 with the tenth digit changed from a 3 to a 2 and the process of checking the number's congruence.

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$$(9 + 8 + 0 + 3 + 8 + 0) + 3(7 + 0 + 7 + 3 + 2 + 9) \equiv 0 \pmod{10}$$

$$112 \equiv 0 \pmod{10}$$

$$(112 - 0)/10$$

$$112/10$$

11 Remainder 2.

The result 112 is not congruent to 0 (mod 10)

Figure 5

We can see that when we try to check if the modified ISBN-13 is congruent, the remainder isn't 0 like before. This indicates that the 10th digit of the modified ISBN-13 is invalid. To help demonstrate why the modification of the 10th digit results in an invalid congruence, Figure 6 demonstrates all the possible variations of the check digit for the ISBN-13 of the textbook. It will also show that only one check digit produces a congruence. Since the check digit is an odd-indexed digit, Figure 6 will also show the possible results for even-indexed digit where the check digit remains 5.

When check digit is 0, result is 115. Not congruent with 0 (mod 10).
When check digit is 1, result is 116. Not congruent with 0 (mod 10).
When check digit is 2, result is 117. Not congruent with 0 (mod 10).
When check digit is 3, result is 118. Not congruent with 0 (mod 10).
When check digit is 4, result is 119. Not congruent with 0 (mod 10).
When check digit is 5, result is 120. Congruent with 0 (mod 10).
When check digit is 6, result is 121. Not congruent with 0 (mod 10).
When check digit is 7, result is 122. Not congruent with 0 (mod 10).
When check digit is 8, result is 123. Not congruent with 0 (mod 10).
When check digit is 9, result is 124. Not congruent with 0 (mod 10).

When digit a_2 is 0, result is 99. Not congruent with 0 (mod 10).
When digit a_2 is 1, result is 102. Not congruent with 0 (mod 10).
When digit a_2 is 2, result is 105. Not congruent with 0 (mod 10).
When digit a_2 is 3, result is 108. Not congruent with 0 (mod 10).
When digit a_2 is 4, result is 111. Not congruent with 0 (mod 10).
When digit a_2 is 5, result is 114. Not congruent with 0 (mod 10).
When digit a_2 is 6, result is 117. Not congruent with 0 (mod 10).
When digit a_2 is 7, result is 120. Congruent with 0 (mod 10).
When digit a_2 is 8, result is 123. Not congruent with 0 (mod 10).
When digit a_2 is 9, result is 126. Not congruent with 0 (mod 10).

Figure 6

Figure 6 demonstrates that, whether the index of the digit be even or odd, there is only one correct digit that, when added with the other correct digits in the ISBN-13, produces a result that is congruent with 0 (mod 10). This shows that the check digit successfully identifies single digit errors for the textbook ISBN-13. To express the error checking in another way, we can consider the result of the operations on the digits of the ISBN-13 as x + 3y, where x is the sum of the odd-indexed digits and y is the sum of the even-indexed digits. This sum of even-indexed digits is then multiplied by 3.

We can let x and y represent values that, when the operations on them are finished, produce a value that is congruent with 0 (mod 10). We can then modify x by increasing x by 1 ten times. This causes the final result to be incongruent with 0 (mod 10) until it is increased by 1 the tenth time. This is because by the tenth time, you have essentially added 10 to the result, which will remain congruent with 0 (mod 10). This would be the demonstration that, if an integer value is divisible by 10, only adding a multiple of 10 to that value will produce a new value also divisible by 10.

Similarly, if you alter y by adding 1 to it ten times, only by the tenth addition will you have a y value that, when multiplied by 3 and increased by x, produces a new value that is divisible by 10. This is because an addition of 10 to y results in 3(y + 10), which is 3y + 30. Practically speaking, multiples of 30 are added to the entire result when multiples of 10 are added to y. Since multiples of 30 are divisible by 10, if the rest of the value is divisible by 10, then the sum of the two will also be divisible by 10.

The example using x + 3y is all to say that, when considering a digit that can contain values from 0 to 9, only one of those values for a specific digit, when the other operations are performed on the correct digits, produce a result that is divisible by 10. Because of this, an ISBN-13 can always detect single digit errors.

Finally, we must show that there are two-digit transposition errors that are not detected by an ISBN-13. We can do so by considering two separate groups, one of odd-indexed digits, and the other of even-indexed digits. By commutativity, we know that a + b = b + a, where a and b can stand in for integers. Therefore, when adding a group of numbers, given that they are all of even indices or of odd indices, transposition errors will not be detected by an ISBN-13. Figure 7 demonstrates grouping odd and even indexed digits separately and using the commutative property to show transposition between two values of odd indices and even indices. In other words, when two commonly-indexed digits are swapped, the transposition error will remain undetected due to the commutative property. If two odd-indexed digits are swapped, no error is detected. If two even-indexed digits are swapped, no error is detected.

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$$(9 + 8 + 0 + 3 + 8 + 0 + 5) + 3(7 + 0 + 7 + 3 + 3 + 9) \equiv 0 \pmod{10}$$
$$120 \equiv 0 \pmod{10}$$

Odd-indexed digit swap: 8790073383095

$$(8 + 9 + 0 + 3 + 8 + 0 + 5) + 3(7 + 0 + 7 + 3 + 3 + 9) \equiv 0 \pmod{10}$$
$$120 \equiv 0 \pmod{10}$$

Even-indexed digit swap: 9980073383075

$$(9 + 8 + 0 + 3 + 8 + 0 + 5) + 3(9 + 0 + 7 + 3 + 3 + 7) \equiv 0 \pmod{10}$$
$$120 \equiv 0 \pmod{10}$$

Figure 7

One can see that both groupings of even-indexed digits and odd-indexed digits are commutative. This allows digits within these groupings to be swapped, which will not change the result due to this commutativity. Each result will be congruent with 0 (mod 10), therefore demonstrating that the transposition of two commonly-indexed digits will not be detected by the ISBN-13.

Sources:

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http://www.hahnlibrary.net/libraries/isbncalc.html

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https://www.luther.edu/bergerr/assets/Math_260_Check_digits.pdf

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