Jacob Hreshchyshyn
Rithvik Arun

MAT 243 Project 4

Mortgages and Monthly Payments

In this project our goal was to examine the financial mortgage and come up with a recurrence relation for the monthly payment of the mortgage. Using this recurrence relation we had to solve the relation for the value as a function of the current month as well as solve for M, the monthly payment. With that we also had to find how the monthly payment changed as the Principal increased, as the rate increased, as the number of months in the loan increased and as the number of months approaches infinity.

We were provided with a model that gave us a simple explanation of what a mortgage is. In order to create a recurrence relation we had to understand what the model meant. As we were examining the model, we noticed that the principal is the value that we start with. In the case of the recurrence relation, the principal would be our first term or P_0. This meant that in order to get the principal for the next month, we have to know the principal for the previous month. Therefore there must be a P_0 because we need somewhere to start, and P_{n-1}, because we need to know what the principal currently is before the monthly payment is made (See Figure 1).

$$P_n = P_{n-1}$$

Figure 1: Our recurrence relation so far. The reason that we have n-1 is because we must know the principal for the previous month in order to get the principal for the current month.

Our next step was to look at the interest rate. In the model it said the interest should be a percentage of the principal and should be added back to the current value. In our case the current value is P_{n-1}. Thus, a percentage of the current value can be represented as rP_{n-1}, where r is the percentage, which is being multiplied by P_{n-1}. Our recurrence relation now resembles Figure 2.

$$P_n = P_{n-1} + rP_{n-1}$$

Figure 2: Our recurrence relation so far. We now add the current value with a percentage of the current value.

Finally, our model instructs us to subtract a fixed amount, the Monthly Payment. We can represent the Monthly Payment as M. Additionally, we must remember the initial condition of this recurrence relation, which we stated earlier can be represented as P_0. Thus, our completed recurrence relation is shown in Figure 3.

$$P_n = P_{n-1} + rP_{n-1} - M;\ P_0$$

Figure 3: Our completed recurrence relation. It is worth noting that the initial condition, P_0, would normally have a value associated with it. However, because we are describing a general recurrence relation, we are not given a specific value for the principal. We thus leave P_0 as is to indicate that there is an initial condition, the principal.

Having found our recurrence relation in terms of P, r, and M, we now seek to solve the recurrence relation for the value as a function of the current month.

This means that we are searching for a closed form expression based on n, the number of months, which will produce the remaining mortgage payment.

To help us find this closed form expression, we can try to identify patterns by determining sequentially what the next remaining payment is, starting with the first payment on the Principal. Since P_0 is our Principal, the next payment would be represented by P_1. Figure 4 demonstrates how we can represent P_1.

$$P_1 = P_0 + rP_0 - M$$

or

$$P_1 = P_0(1 + r) - M$$

Figure 4: P_1, the next remaining mortgage value after a monthly payment is made on the Principal, P_0, is demonstrated.

Similarly, the remaining mortgage value after a monthly payment is made on P_1 can be represented as shown in Figure 5.

$$P_2 = P_1(1 + r) - M$$

Figure 5: P_2 would be represented by making a monthly payment on the mortgage amount that results from a monthly payment on the Principal.

One can see that the expression needs a preceding value to get the next value just like our recurrence relation. However, because it may be unclear what that preceding value is, we can also represent this expression in terms of the starting value, which is our Principal, P_0. Figure 6 demonstrates how one can represent P_2 using P_0 instead of P_1. Since $P_1 = P_0(1 + r) - M$, then:

$$P_2 = (P_0(1 + r) - M)(1 + r) - M$$

$$P_2 = P_0 + P_0r + P_0r + P_0r^2 - M - Mr - M$$

$$P_2 = P_0(1 + 2r + r^2) - M (1 +r) - M$$

$$P_2 = P_0(1 + r)^2 - M (1 +r) - M$$

Figure 6: A representation of P_2 in terms of the Principal.

We can then find P_3 in terms of the Principal. First, P_3 would follow P_2. Thus, P_2 can be represented as shown in Figure 7.

$$P_3 = P_2(1 + r) - M$$

Figure 7: A representation of P_3 in terms of P_2.

We can then substitute what we found in Figure 6 for P_2 and rewrite the equation. Figure 8 represents P_3 in terms of the Principal.

$$P_3 = (P_0(1 + r)^2 - M (1 +r) - M)(1 + r) - M$$

After distributing (1 + r)

$$P_3 = P_0(1 + r)^3 - M(1 + r)^2 - M(1 + r) - M$$

Figure 8: A representation of P_3 in terms of the Principal.

Comparing P_1, P_2 and P_3, we begin to notice a pattern. The resulting mortgage amount after the first month can be calculated by multiplying the Principal with (1 + r), then subtracting by M. The resulting mortgage amount after the second month can be calculated by multiplying the Principal with $(1 + r)^2$, subtracting by M(1 + r), and subtracting again by M. Finally, the resulting mortgage amount after the third month can be calculated by multiplying the Principal with $(1 + r)^3$, subtracting by $M(1 + r)^2$, subtracting again by M(1 + r), and subtracting again by M.

Thus, we can see a relation between the number of months after the first payment and the degree of the polynomial, when treating (1 + r) as an individual term. After 1 month, the remaining amount can be represented by a first degree polynomial. After 2 months, the remaining amount can be represented by a second degree polynomial. After 3 months, the remaining amount can be represented by a third degree polynomial. Additionally, we see that P_0 is multiplied with the highest degree term while -M is multiplied by the remaining degrees down to the 0th degree, where $-M(1 + r)^0 = -M$.

Given this pattern for 1, 2, and 3 months, we can intuit a pattern for n months. Figure 9 demonstrates what P_n would be after n months.

$$P_n = P_0(1 + r)^n - M(1 + r)^{n-1} . . . - M(1 + r) - M$$

If we factor out -M

$$P_n = P_0(1 + r)^n - M((1 + r)^{n-1} . . . + (1 + r) + 1)$$

Figure 9: A representation of P_n in terms of the Principal. n represents the number of months.

We can now see after factoring out -M that there exists a sequence of partial sums. The sequence of partial sums looks like $(1 + r)^{n-1} + (1 + r)^{n-2} + (1 + r)^{n-3} + \ldots + 1$. We can represent this as Sn, as shown in Figure 10.

$$Sn = (1 + r)^{n-1} + (1 + r)^{n-2} + \ldots + 1$$

Figure 10: Sn, which represents the sequence of partial sums starting from $(1 + r)^{n-1}$ to 1.

To represent Sn without an infinite number of terms, we begin by multiplying Sn with (1 + r). The result is demonstrated in Figure 11.

$$Sn(1 + r) = (1 + r)^{n} + (1 + r)^{n-1} + \ldots + (1 + r)$$

Figure 11: Every term within Sn multiplied by (1 + r).

Next, to remove the infinite number of like terms, we can subtract Sn(1 + r) by Sn. Figure 12 demonstrates the result.

$$Sn(1 + r) = (1 + r)^{n} + (1 + r)^{n-1} + \ldots + (1 + r)$$
$$-\ Sn = -\ (1 + r)^{n-1} - (1 + r)^{n-2} - \ldots - 1$$
$$Sn(1 + r) - Sn = (1 + r)^{n} - 1$$

Figure 12: Sn(1 + r) - Sn

Next, we rewrite the equation to find Sn by itself. Figure 13 demonstrates this process, picking up where Figure 12 left off.

$$Sn((1 + r) - 1) = (1 + r)^{n} - 1$$
$$Sn(r) = (1 + r)^{n} - 1$$
$$Sn = ((1 + r)^{n} - 1)/r$$

Figure 13: We have successfully defined Sn without using an infinite number of terms.

Since we have successfully defined Sn without using an infinite number of terms, we can now define Pn without using an infinite number of terms. Using the equation for Pn from Figure 9, Figure 14 demonstrates our closed form expression, which represents the mortgage value as a function of the current month.

$$P_n = P_0(1 + r)^n - M(Sn)$$

$$Pn = P_0(1 + r)^n - M(((1 + r)^n - 1)/r)$$

Figure 14: The complete closed form expression that represents the value of the mortgage for the current month.

Next, we are asked to solve for M, the Monthly Payment. To do so, we remind ourselves that the last value of the mortgage must be zero, since we repeatedly make monthly payments on the current value "until the value reaches zero." n, while acting as a variable that allows one to represent the current month by substituting a value for n, also generally represents the last month in which a payment on the mortgage is paid. After a payment is made on the last month, the mortgage is theoretically paid off. Thus, we can solve for M after setting Pn equal to zero, as Figure 15 demonstrates.

$$0 = P_0(1 + r)^n - M(((1 + r)^n - 1)/r)$$

$$M(((1 + r)^n - 1)/r) = P_0(1 + r)^n$$

$$M = P_0(1 + r)^n/(((1 + r)^n - 1)/r)$$

$$M = P_0r(1 + r)^n/((1 + r)^n - 1)$$

Figure 15: Representation of the Monthly Payment in terms of the Principal, the current month, and the rate.

Now that we figured out the equation for the monthly payment we can find how the monthly payment is affected when the principal is increased. In order to find this we must know the principal, interest rate, and the amount of time we have to pay the mortgage. We can use our closed form expression to express this with a graph. Figure 16 represents a graph that shows how the monthly payment increases when the principal increases.

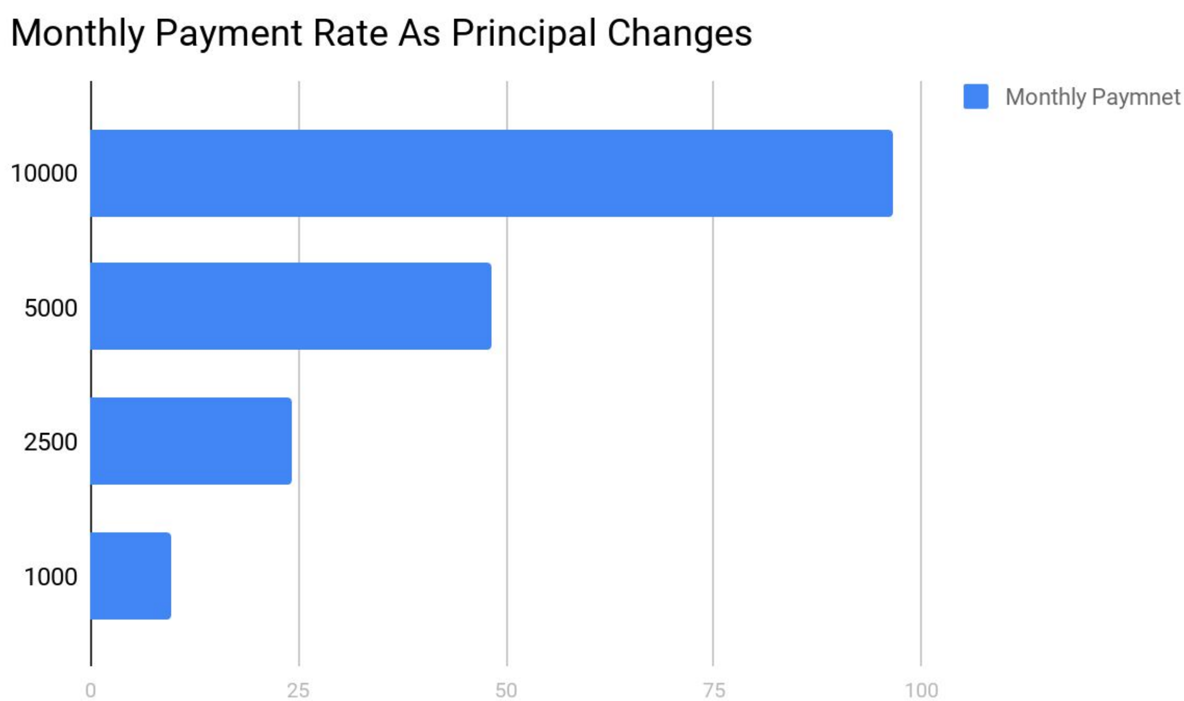

Figure 16: This graph shows how the monthly payment is affected when the principal is changed. We used 3% as our yearly rate and 120 months to pay off the mortgage. Note that the rate is yearly so we have to divide by 12 to get the monthly rate. We plugged these values in to our monthly payment expression and showed it as a bar graph. The chosen Principal values are shown on the left side of the graph. As one can see, as the principal increases the monthly payment increases.

Next, we can take a look at how the monthly payment is affected when the rate is increased. Like before, we will use our monthly payment expression to see how it changes. Figure 17 represents a graph that shows how the monthly payment increases when the rate increases.

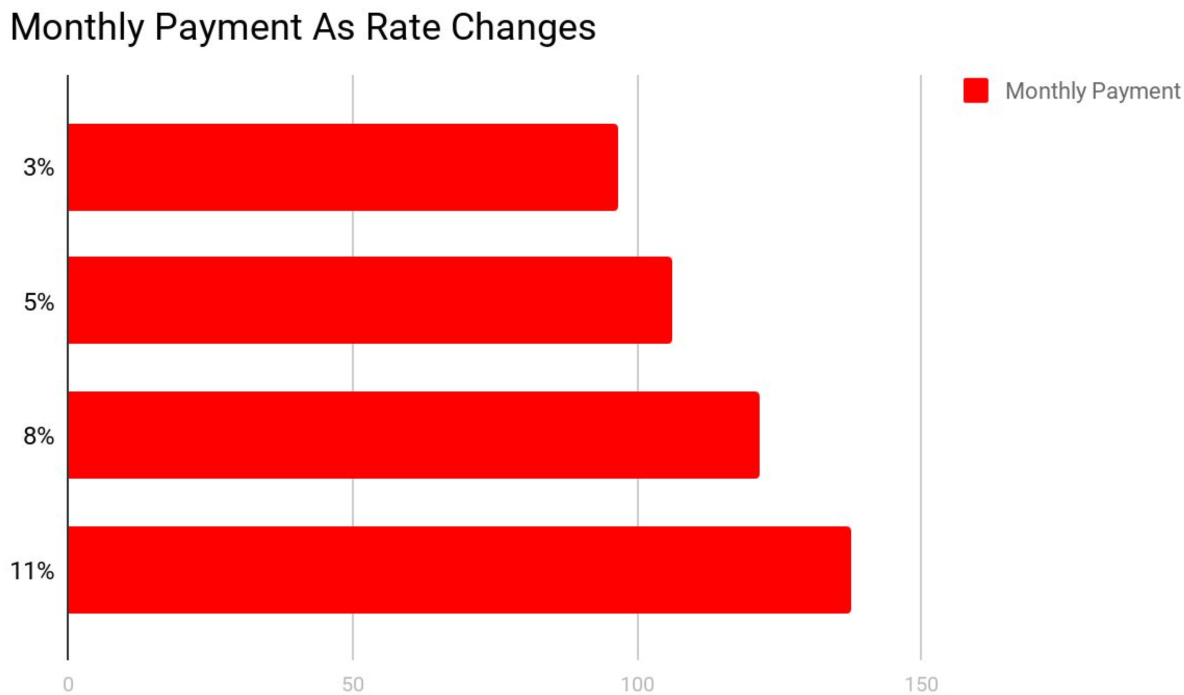

Figure 17: This graph shows how the monthly payment is affected when the rate is changed. The rate is an annual rate, which means that we have to divide it by 12 to get the value for one month. We used 10,000 as our principal and 120 months to pay off the mortgage. We plugged these values into our monthly payment expression and showed it as a bar graph. As one can see, as the rate increases, the monthly payment increases.

Next, we can take a look at how the monthly payment is affected when the number of months for the loan is increased. Like before, we will use our monthly payment expression to see how it changes. Figure 18 represents a graph that shows how the monthly payment decreases when the number of months on the mortgage increases.

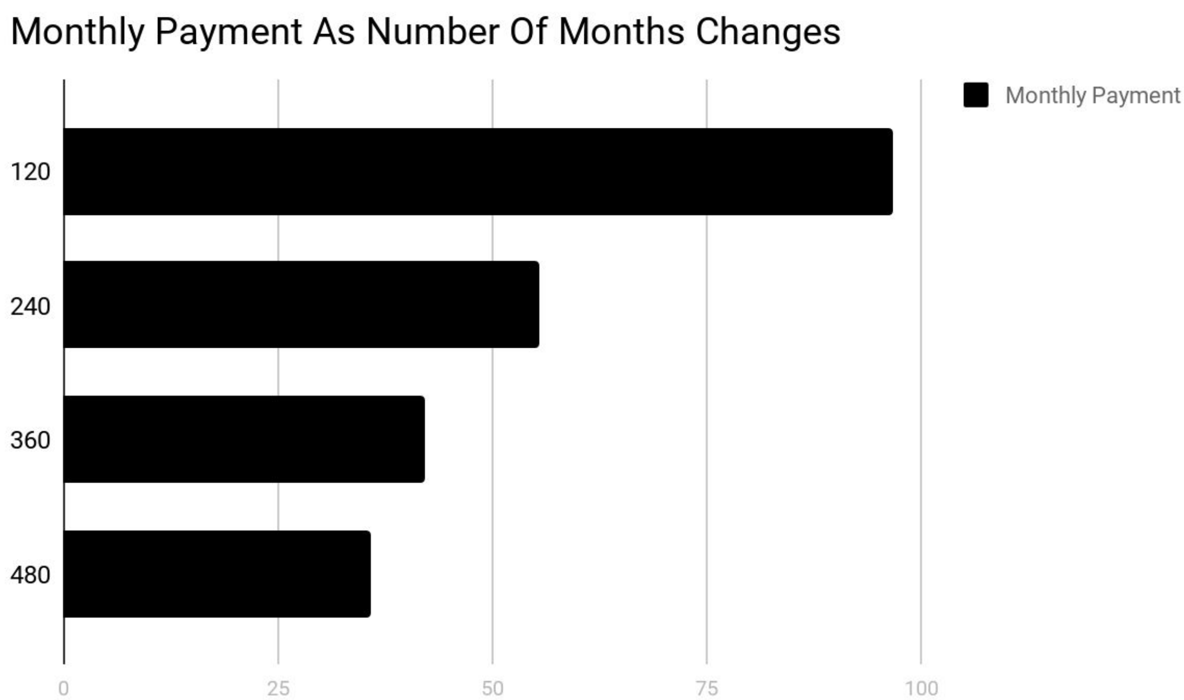

Figure 18: This graph shows the how the monthly payment is affected when the number of months on the mortgage is changed. We used 10,000 as our principal and 3% yearly as the rate. Note that the rate is yearly so we have to divide by 12 to get the monthly rate. We plugged these values in to our monthly payment expression and showed it as a bar graph. The left side of the graph shows the different values for the number of months. As one can see, as the number of months on the mortgage increases the monthly payment decreases.

Next, we can show how the monthly payment is affected when the number of months approaches infinity. As we have shown in the graph before, as the number of months get larger, the monthly payment decreases. So we know that the monthly payment will get smaller as the number of months gets bigger. As the number of months approaches infinity we also know that it will never reach zero, since the number of months will never be defined. In order to show this we take the limit of our monthly payment expression and graph it. Figure 19 represents a graph that shows the limit of our closed form expression.

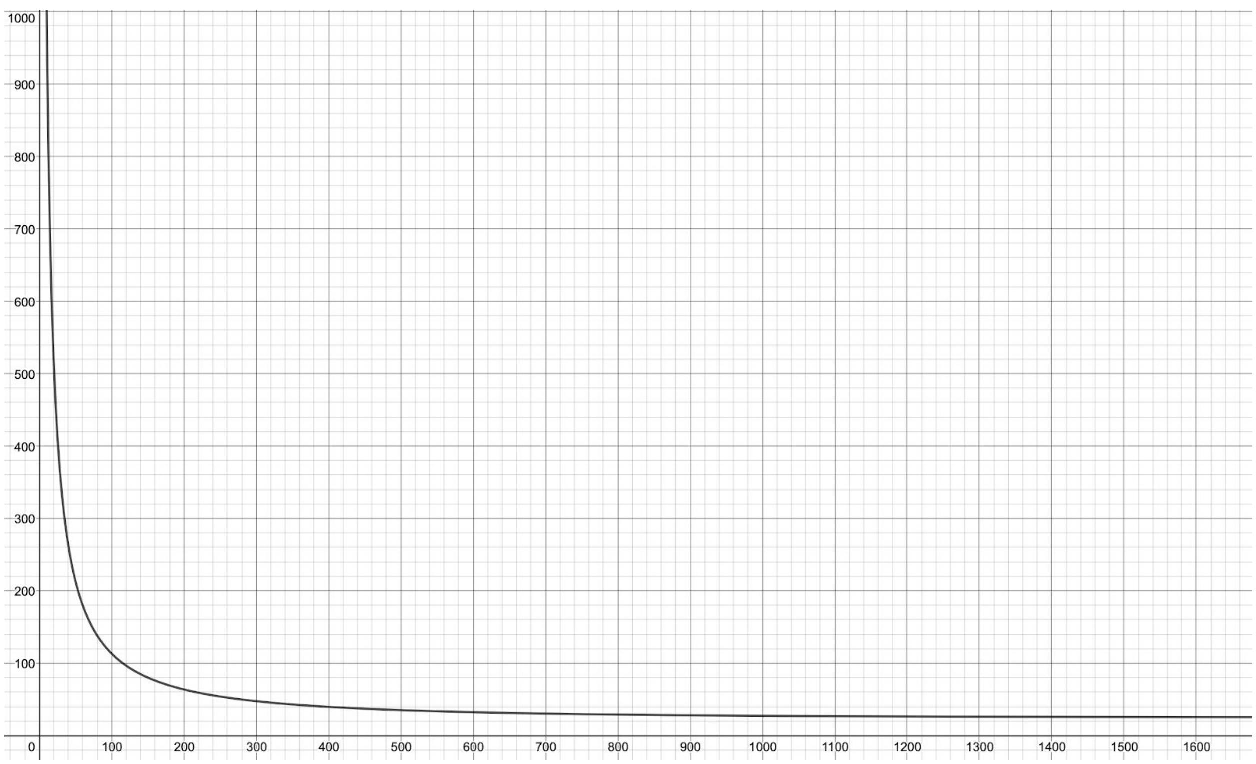

Figure 19: Graph that shows our closed form expression and the number of months approaching infinity. As one can see, the graph will never touch zero as it approaches infinity. The values used for the Principal and yearly rate are 10000 and 3%, respectively. The graph indicates that, as n approaches infinity, the monthly payment approaches 25.

To prove that the monthly payment approaches 25 in this case, we can take the limit as n approaches infinity of our closed form expression where P_0 is 10000 and r is (3/100)(1/12), which is done to convert the yearly rate into a monthly rate. Figure 20 illustrates the process of finding the limit.

$$lim\ n \to \infty\ (10000(3/100)(1/12)(1 + (3/100)(1/12))^n)/((1 + (3/100)(1/12))^n - 1)$$
$$25lim\ n \to \infty\ (1 + (1/400))^n)/((1 + (1/400))^n - 1)$$
$$25lim\ n \to \infty\ (1 + (1/400))^n)/((1 + (1/400))^n(1 - (1/(1 + (1/400))^n)))$$
$$25lim\ n \to \infty\ 1/(1 - (1/(1 + (1/400))^n))$$
$$25lim\ n \to \infty\ 1/(1 - (1/(401/400)^n))$$
$$25lim\ n \to \infty\ 1/(1 - (400/401)^n)$$
$$25(1/1)$$
$$25$$

Figure 20: Demonstration that the limit as n approaches infinity given that P_0 is 10000 and the yearly rate is 3% is equal to 25.

Interestingly, if one were to divide the Principal by the limit of the monthly payment as n approaches infinity, then you would find the reciprocal of the rate. For example the Principal 10000 divided by the limit 25 results in 400, whose reciprocal is 1/400 or (3/100)(1/12), which is the monthly rate. This holds true for different Principals and rates.