Classical Hydrostatic Equilibrium Equation Derivation

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Hydrostatic equilibrium in stars deals with a balance of the inward force of gravity with the outward push of pressure. Hydrostatic equilibrium is the reason stars do not implode or explode at random. Classically hydrostatic equilibrium is described as

$$\frac{dP}{dr} = -\frac{\epsilon(r)M(r)}{r^2} \ , \tag{1}$$

where P is the pressure, ϵ is the energy-density, M is the mass, and r is the radius and where the natural mathematical units of $G = c = 1$. The derivation of Eq. (1) is a great way to understand some of the basic fundamentals of stellar structure.

Lets begin by considering a volume element with area dA, length dr, a distance r fromthe center of a start shown in the figure below.

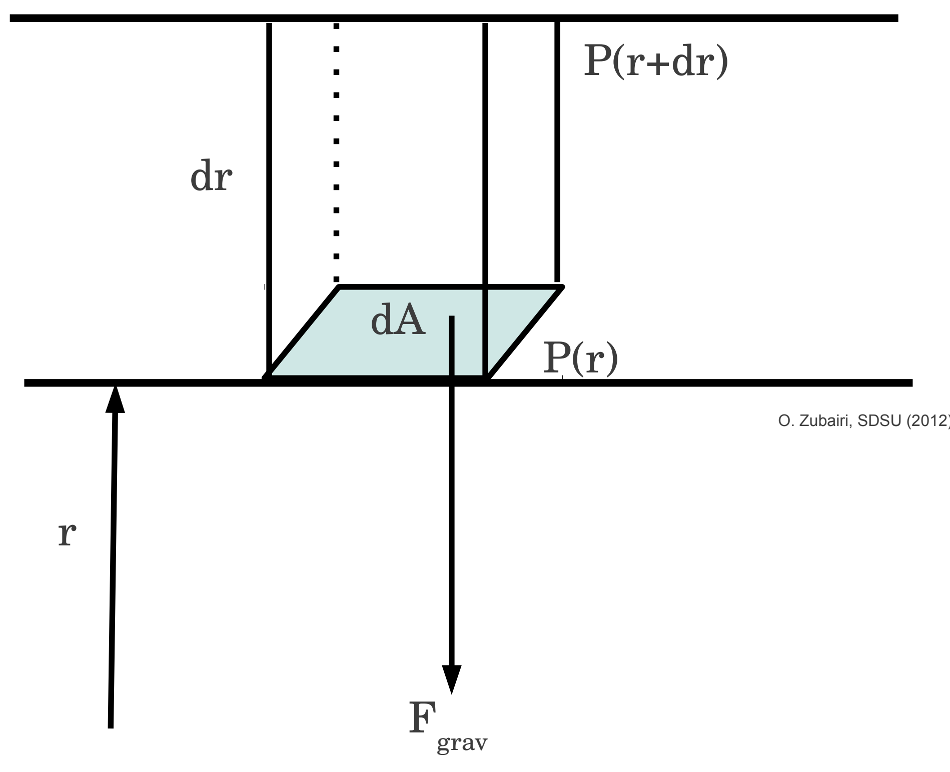

Figure 1: Volume Element

The properties (i.e. Density, Volume, and Mass) of this volume element are as follows:

- Density: $\rho(r)$

- Volume: $dAdr$
- Mass: $dm = \rho(r)dAdr$ (remember that $\rho = \frac{m}{V}$)

Next, we need to consider what is $\vec{F}_{grav}$ (force of gravity) acting on the volume element... Its is simply

$$F_{grav} = -\frac{GM(r)dm}{r^2} \ . \tag{2}$$

Substitute dm into Eq. (2) to obtain

$$F_{grav} = -\frac{GM(r)\rho(r)dAdr}{r^2} \ . \tag{3}$$