Homework Policies:
When showing a theoretical statement or computation by hand is part of the exercise, the clarity and completeness of your arguments/steps will be as important as their correctness.

Your homework assignments can be handwritten or typeset, but all graphs should be included as images from the graphing tool that you use.

The submission of homework that requires numerical work on a computer should include the following: printout of the code used to solve the problems, its inputs, and outputs. The code should be written clearly and should be commented in such a way that at least the inputs and outputs of the code are clear. The specific outputs requested by the exercise should be discussed in your write-up as needed in order to answer the questions in the problems.

The filenames for the code should be indicative of the exercise. For example, if it is a Jupyter notebook for homework 1 exercise 2, then it could be named `hw1Ex2.ipynb`.

Figures: Each required plot should have the following features:

- Clearly label the x- and y-axes. All the labels should have fontsize 18.
- Add a title with fontsize 20.
- Add legends to clearly label the plots. The fontsize for the labels should be 18.
- All the ticks in the figures should have fontsize 16.
- The markersize should be 10 pts, and the line-widths should be 4 pts.

Submission:

- Please submit your solutions in a *PDF file*, together with *a .zip file containing all the code needed to reproduce your results*. Mention the students with whom you discussed the homework.
- For the computer problems, include the printout of the code, inputs, outputs, required plots, and discussions needed to answer the questions (when appropriate).

Before working on the exercises:

Study and practice Python. You may want to use the resources provided on the course Moodle page. The best way to learn would be by looking at existing code and writing a few programs yourself. If you are not sure how to write Python code properly or have never programmed before, please come to my office hours and discuss.

Exercises:

1. (*40 pts*) In this problem you will determine a cubic spline approximation that fits exactly through the five data points below:

$$P_1(0,2), \quad P_2(0.5,0), \quad P_3(1,1), \quad P_4(1.5,2), \quad P_5(2,0).$$

The spline will be built using two cubic functions:

$$y_1(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3,$$

$$y_2(x) = a_5 + a_6 x + a_7 x^2 + a_8 x^3.$$

(a) Formulate three linear algebraic equations ensuring that $y_1(x)$ goes exactly through the three points P_1, P_2, and P_3.

(b) Formulate three linear algebraic equations ensuring that $y_2(x)$ goes exactly through the three points P_3, P_4, and P_5.

(c) Formulate two linear algebraic equations ensuring that the slope and curvature (measured via the second derivative) are both continuous at the point P_3.

(d) By assembling the equations found in parts (a) – (c), determine the matrix $\mathbf{A}$ and vector $\mathbf{b}$ in the linear system

$$\mathbf{A}\mathbf{a} = \mathbf{b}, \text{ where } \mathbf{a} = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]^T.$$

(e) Solve the linear system in part (d) to determine the two cubic functions $y_1(x)$ and $y_2(x)$, respectively. You may solve the linear system using Python packages.

(f) On the same graph, plot y_1 on the interval $[0,1]$, y_2 on the interval $[1,2]$, and the five data points $P_1, \ldots, P_5$.

2. (*30 pts*) Consider the linear regression model

$$\hat{f}(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2, \qquad \text{where } \mathbf{x} = [x_1, x_2],$$

for the given data

$$\mathbf{x}_1 = [1,1], y_1 = 0.1;\ \mathbf{x}_2 = [-1,-1], y_2 = 5.95;\ \mathbf{x}_3 = [0,1], y_3 = 0.8;$$

$$\mathbf{x}_4 = [1,0], y_4 = 2.1;\ \mathbf{x}_5 = [1,2], y_5 = -1.8;\ \mathbf{x}_6 = [2,1], y_6 = -1.05.$$

(a) Derive the normal equation for this least squares linear regression problem.

(b) Set up the normal equation in Python and use it to determine the regression model.

(c) On the same graph, plot the approximation $\hat{f}$ via a surface plot (using `plot_surface` function) and plot the six data points.

(d) What is the residual sum of squares of this model?

3. (*30 pts*) Let $\mathbf{X} \in \mathbb{R}^{n\times(m+1)}$ and $\mathbf{y} \in \mathbb{R}^n$. Consider the least squares problem with Tikhonov regularization for $\lambda > 0$,

$$\min_{\beta\in\mathbb{R}^{m+1}} \|\mathbf{X}\beta - \mathbf{y}\|_2^2 + \lambda^2\|\beta\|_2^2.$$

   (a) Derive the normal equations for this least squares problem.
   Hint: Define $J(\beta) = \|\mathbf{X}\beta - \mathbf{y}\|_2^2 + \lambda^2\|\beta\|_2^2.$, and consider its gradient and Hessian matrices.

   (b) Consider the linear regression model

$$\hat{f}(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2, \qquad \text{where } \mathbf{x} = [x_1, x_2],$$

   for the given data

$$\mathbf{x}_1 = [1,1], y_1 = 0.1;\ \mathbf{x}_2 = [-1,-1], y_2 = 5.95;\ \mathbf{x}_3 = [0,1], y_3 = 0.8;$$

$$\mathbf{x}_4 = [1,0], y_4 = 2.1;\ \mathbf{x}_5 = [1,2], y_5 = -1.8;\ \mathbf{x}_6 = [2,1], y_6 = -1.05.$$

   Find the parameters β_0, β_1, and β_2 such that the following regularized squared error loss function with $\lambda = 1$ is minimized:

$$\min_{\beta\in\mathbb{R}^3} \sum_{i=1}^{6} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})^2 + \lambda^2(\beta_0^2 + \beta_1^2 + \beta_2^2).$$

   (c) Repeat the process in part (b) with $\lambda = 2$.

   (d) What is the residual sum of squares when $\lambda = 1$ and when $\lambda = 2$ from parts (b) – (c)?