FORMALISING $\det(e^A) = e^{\mathrm{tr}(A)}$ IN LEAN

1. The non-formal proof

We list some facts that the proof follows from. The matrix A is assumed to have entries in $\mathbb{C}$ and P is an invertible matrix with entries in $\mathbb{C}$.

(1) For all A, P, we have $\det(e^{PAP^{-1}}) = \det(e^A)$.
(2) For every matrix A, there is an invertible P such that $PAP^{-1} = U$, with U upper-triangular.
(3) $\det(U) = \prod(U)_{(i,i)}$.
(4) If U is upper-triangular, then $(e^U)_{(i,i)} = e^{(U)_{(i,i)}}$.
(5) For complex numbers a and b, we have $e^{a+b} = e^a e^b$.
(6) For all A, P, we have $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$.

The proof now follows in the following argument:

Proof.

$$
\begin{aligned}
\det(e^A) &= \det(e^{PAP^{-1}}) && \text{by 1} \\
&= \det(e^U) && \text{by 2} \\
&= \prod e^{(U)_{(i,i)}} && \text{by 3 and 4} \\
&= e^{\sum U_{(i,i)}} && \text{by 5} \\
&= e^{\mathrm{tr}(U)} && \text{by definition of trace} \\
&= e^{\mathrm{tr}(PAP^{-1})} && \text{by 2 again} \\
&= e^{\mathrm{tr}(A)} && \text{by 6}
\end{aligned}
$$

□

We list the difficulties of the various parts.

(1) This already exists in Lean, in `Matrix.exp_conj`
(2) This will need proving (and might be hard)
(3) This should follow from `Matrix.det_succ_column`
(4) This will need proving
(5) This exists somewhere (to do: find it!)
(6) This should follow from `Matrix.trace_mul_cycle`