FORMALISING $\det(e^A) = e^{\mathrm{tr}(A)}$ IN LEAN

1. The non-formal proof

We list some facts that the proof follows from. The matrix A is assumed to have entries in $\mathbb{C}$ and P is an invertible matrix with entries in $\mathbb{C}$.

(1) For all M, P, we have $\det(e^M) = \det(Pe^M P^{-1})$.
(2) For every matrix A, there is an invertible P such that $A = PUP^{-1}$, with U upper-triangular.
(3) $\det(U) = \prod(U)_{(i,i)}$.
(4) If U is upper-triangular, then $(e^U)_{(i,i)} = e^{(U)_{(i,i)}}$.
(5) For complex numbers a and b, we have $e^{a+b} = e^a e^b$.
(6) For all M, P, we have $\mathrm{tr}(PMP^{-1}) = \mathrm{tr}(M)$.
(7) For all M, P, we have $\det(PMP^{-1}) = \det(M)$.

The proof now follows in the following argument:

Proof.

$$\begin{aligned}
\det(e^A) &= \det(Pe^U P^{-1}) && \text{by 1 and 2} \\
&= \det(e^U) && \text{by 7} \\
&= \prod e^{(U)_{(i,i)}} && \text{by 3 and 4} \\
&= e^{\sum U_{(i,i)}} && \text{by 5} \\
&= e^{\mathrm{tr}(U)} && \text{by definition of trace} \\
&= e^{\mathrm{tr}(PUP^{-1})} && \text{by 6} \\
&= e^{\mathrm{tr}(A)} && \text{by 2 again}
\end{aligned}$$

□

We list the difficulties of the various parts.

(1) This already exists in Lean, in `Matrix.exp_conj`.
(2) This will need proving. This can be done by showing that no minimal counterexample exists, using the fact that every matrix over with entries in $\mathbb{C}$ has an eigenvector.
(3) This can be proven inductively using `Matrix.det_succ_column`.
(4) This will need proving.
(5) This should follow from `Matrix.trace_mul_cycle`, since applying one cyclic permutation allows us to cancel P with its inverse.
(6) This should be in Lean (what is it called?).