5 MONTH INTERIM REPORT

JAKE SAUNDERS

I am investigating whether the Euler genus can be realised as a map of highly structured ring spectra.

1. Euler homology and spectra

Let $\chi(M)$ denote the Euler characteristic of a manifold. Unoriented bordism is a (co)homology theory represented by a spectrum denoted MO. My current aim is to find a spectrum EO and a map $e : MO \to EO$ of highly-structured ring spectra which descends to the (graded) Euler characteristic map when taking homotopy groups: $\pi_*(e) : \pi_*(MO) \to \pi_*(EO)$. That is, the map

$$[M] \mapsto \chi(M) \cdot t^{\dim M/2}.$$

A ring homomorphism with domain a bordism ring is known as a *genus*, hence the name Euler genus. A map of homology theories realising this was constructed in [11].

The unoriented bordism ring is well-known to be the graded polynomial ring over $\mathbb{F}_2$ with a generator in each degree n such that n is not one less than a power of 2, i.e.

$$\pi_*(MO) = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots].$$

I showed that the kernel of this Euler characteristic map is the ideal generated by elements

- x_{2k+1} for $2k + 1 \neq 2^i - 1$,
- $[\mathbb{RP}^{2k}] - [\mathbb{RP}^2]^k$ for $k > 1$,

and furthermore that this is generated by a regular sequence.

If one has a commutative S-algebra such as MO or MU (the representing spectrum of complex cobordism), results of [3] allow us to define quotients by ideals generated by regular sequences, and in some cases give us highly structued ring spectra. The highly-structued results, however, require that the S-algebra is *even*, that is, its homotopy groups of odd degree vanish. MU, however, has homotopy groups $\mathbb{Z}[a_2, a_4, a_6, \dots]$ (one polynomial generator in each positive even degree), and since complex manifolds are orientable and by [4], orientable manifolds of dimension $4k+2$ have vanishing mod 2 Euler characteristic, we can define a spectrum EU which admits an associative, commutative MU-algebra structure, and such that the natural map $MU \to EU$ descends to the mod-2 Euler characteristic. It appears that different results are required for MO.

2. Stratifold homology and ad theories

In between the first and second semesters, I re-read [11] and became interested in the construction of the Euler homology theory, which uses the theory of stratifolds introduced by Kreck [5]. I read [6] which gives a summary of similar such theories. Another paper [9] uses stratifolds to realise the signature of a manifold as a homology theory, which is another bordism invariant. This led me to read about Quinn spectra induced by *bordism-type theories* introduced in [10] and refined to the notion of an *ad theory* in [7, 8]. After discussing this with my supervisor, we established that this may be a route to solving the problem, albeit quite an overpowered and un-enlightening one, and so will focus on using different methods.

3. Obstruction theory for E_∞ structures

In the last few weeks I have been reading about obstruction theories for highly structured ring spectra [1], called Hochschild homology, gamma homology, and topological Andre-Quillen homology. In particular, the case I am working on is similar to the previously solved case of complex K-theory, and so I am studying the paper [2], where the gamma homology of KU is shown to vanish, and therefore KU has a unique E_∞ structure. I expect to be able to apply similar ideas to EO, and show that it also has a unique E_∞ structure.

4. Misc

In the past 5 months, I have given three talks:

- Sheaf Seminar: Matrix Algebras, Matrix Varieties and the Utility of Weyr Matrices (talk on my masters project).
- Galois Cohomology Reading Group: Groups with cohomological dimension bounded above by 1, and dualizing modules.
- Infinity Categories Reading Group: Defining Infinity Categories.

I am an assistant demonstrator for MAS005. I am also an assistant demonstrator for the AMSP Y12 Enrichment Sessions.

References

[1] A. Baker and B. Richter, eds., *Structured ring spectra*, vol. 315 of Lond. Math. Soc. Lect. Note Ser., Cambridge: Cambridge University Press, 2004.

[2] ———, *On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory*, Comment. Math. Helv., 80 (2005), pp. 691–723.

[3] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, and M. Cole, *Rings, Modules, and Algebras in Stable Homotopy Theory*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, Apr. 2007.

[4] R. S. Hoekzema, *Manifolds with odd euler characteristic and higher orientability*, International Mathematics Research Notices, 2020 (2018), pp. 4496–4511.

[5] M. Kreck, *Differential algebraic topology. From stratifolds to exotic spheres*, vol. 110 of Grad. Stud. Math., Providence, RI: American Mathematical Society (AMS), 2010.

[6] ———, *A zoo of geometric homology theories*, 2018.

[7] G. Laures and J. E. McClure, *Multiplicative properties of quinn spectra*, 2009.

[8] ———, *Commutativity properties of quinn spectra*, 2013.

[9] A. Minatta, *Signature homology*, J. Reine Angew. Math., 592 (2006), pp. 79–122.

[10] F. Quinn, *Assembly maps in bordism-type theories*, in Novikov conjectures, index theorems and rigidity. Vol. 1. Based on a conference of the Mathematisches Forschungsinstitut Oberwolfach held in September 1993, Cambridge: Cambridge University Press, 1995, pp. 201–271.

[11] J. Weber, *Euler homology*, 2006.