Linear Algebr

MAT244 Slides

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

Exercise 2

Let

$$\begin{aligned} &\text{(Birth Rate)} && K = 1.1 \text{ children per starfish per year} \\ &\text{(Initial Pop.)} && P_0 = 10 \text{ star fish} \end{aligned}$$

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

Exercise 3

Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

Exercise 4

In the model $\mathbf{M}_1$, we assumed the starfish had K children at one point during the year.

4.1 Create a model $\mathbf{M}_n$ where the starfish are assumed to have K/n children n times per year (at regular intervals).

4.2 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.3 What happens to $\mathbf{M}_n$ as $n \to \infty$?

Exercise 5

Exploring $\mathbf{M}_n$

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t+1/n)$ there will be (on average) K/n new children per starfish.

5.1 Write an expression for $P_n(t+1/n)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP, the change in population from time t to $t+\Delta t$.

5.3 Write an expression for $\frac{\Delta P}{\Delta t}$.

5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_n(t)$.

Exercise 6

Define the model $\mathbf{M}_\infty$ by

- $P(0) = 10$
- $P'(t) = kP(t)$

and recall the model $\mathbf{M}_1$ defined by

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

6.2 Suppose that $\mathbf{M}_1$ accurately predicts the population. Can you find a value of k so that $\mathbf{M}_\infty$ accurately predicts the population?

6.3 What are some advantages and disadvantages of the models $\mathbf{M}_1$ and $\mathbf{M}_\infty$?

Exercise 7

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

7.1 Produce a graph of population vs. time for $k = 1.1$.

7.2 How do **S** and $\mathbf{M}_\infty$ compare (with the same k)?

7.3 Can you come up with a discrete model for **S** with a time-step of one year? Why or why not?