

Linear Algebra

MAT244 Slides



A diagram illustrating vector projections. A magenta line represents a subspace. A yellow vector \vec{u} is shown originating from the bottom left. Three white arrows show the orthogonal projections of \vec{u} onto the magenta line at different points along the line.

\vec{u}

Exercise 1

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\text{\#new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

Exercise 2

Let

(Birth Rate) $K = 1.1$ children per starfish per year

(Initial Pop.) $P_0 = 10$ star fish

and define the model \mathbf{M}_1 to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

Exercise 3

Recall the model \mathbf{M}_1 (from the previous question).

Define the model \mathbf{M}_1^* to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are \mathbf{M}_1 and \mathbf{M}_1^* different models or the same?

3.2 Which of \mathbf{M}_1 or \mathbf{M}_1^* is better?

3.3 List an advantage and a disadvantage for each of \mathbf{M}_1 and \mathbf{M}_1^* .

Exercise 4

In the model \mathbf{M}_1 , we assumed the starfish had K children at one point during the year.

- 4.1 Create a model \mathbf{M}_n where the starfish are assumed to have K/n children n times per year (at regular intervals).
- 4.2 Simulate the models \mathbf{M}_1 , \mathbf{M}_2 , \mathbf{M}_3 in Excel. Which grows fastest?
- 4.3 What happens to \mathbf{M}_n as $n \rightarrow \infty$?

Exercise 5

Exploring \mathbf{M}_n

We can rewrite the assumptions of \mathbf{M}_n as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t + 1/n)$ there will be (on average) K/n new children per starfish.

5.1 Write an expression for $P_n(t + 1/n)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP , the change in population from time t to $t + \Delta t$.

5.3 Write an expression for $\frac{\Delta P}{\Delta t}$.

5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n \rightarrow \infty} P_n(t)$.

Exercise 6

Define the model \mathbf{M}_∞ by

- $P(0) = 10$
- $P'(t) = kP(t)$

and recall the model \mathbf{M}_1 defined by

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

6.1 If $k = K = 1.1$, does the model \mathbf{M}_∞ produce the same population estimates as \mathbf{M}_1 ?

6.2 Suppose that \mathbf{M}_1 accurately predicts the population. Can you find a value of k so that \mathbf{M}_∞ accurately predicts the population?

6.3 What are some advantages and disadvantages of the models \mathbf{M}_1 and \mathbf{M}_∞ ?

Exercise 7

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

7.1 What can you tell about the population (without trying to compute it)?

7.2 Assuming $k = 1.1$, estimate the population after 10 years.

7.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

Consider the following argument:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(1) + P'(1) \cdot 1 = P(1) = 10.$$

And so on.

So, the population of starfish remains constant.

8.1 Do you believe this argument? Can it be improved?

Exercise 9

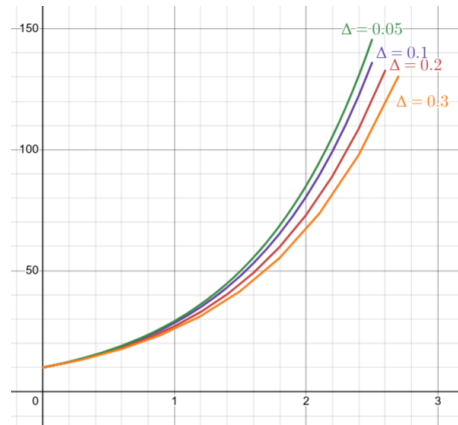
(Simulating M_∞ with different Δ s)

Time	Pop. ($\Delta = 0.1$)	Time	Pop. ($\Delta = 0.2$)
0.0	10	0.0	10
0.1	11.1	0.2	12.2
0.2	12.321	0.4	14.884
0.3	13.67631	0.6	18.15848
0.4	15.1807041	0.8	22.1533456

- 9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?
- 9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?
- 9.3 What Δ s give the largest estimate for the population at time t ?
- 9.4 Is there a limit as $\Delta \rightarrow 0$?

Exercise 9

(Simulating M_∞ with different Δ s)



- 9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?
- 9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?
- 9.3 What Δ s give the largest estimate for the population at time t ?
- 9.4 Is there a limit as $\Delta \rightarrow 0$?

Exercise 10

Consider the following models for starfish growth

M # new children per year \sim current population

N # new children per year \sim resources available per individual

O # new children per year \sim current population times the fraction of total resources remaining

10.1 Guess what the population vs. time curves look like for each model.

10.2 Create a differential equation for each model.

10.3 Simulate population vs. time curves for each model (but pick a common initial population).

Exercise 11

Recall the models

M # new children per year \sim current population

N # new children per year \sim resources available per individual

O # new children per year \sim current population times the fraction of total resources remaining

- 11.1 Determine which population grows fastest in the short term and which grows fastest in the long term.
- 11.2 Are some models more sensitive to your choice of Δ when simulating?
- 11.3 Are your simulations for each model consistently underestimates? Overestimates?
- 11.4 Compare your simulated results with your guesses from question 10.1. What did you guess correctly? Where were you off the mark?

Exercise 12

A simple model for population growth has the form

$$P'(t) = kP(t)$$

where k is the *birth rate*.

12.1 Create a better model for population that includes both births and deaths.

Exercise 13

The *Lotka-Volterra Predator-Prey* models two populations, F (foxes) and R (rabbits), simultaneously. It takes the form

$$F'(t) = (B_F - D_F) \cdot F(t)$$

$$R'(t) = (B_R - D_R) \cdot R(t)$$

where B_i stands for births and D_i stands for deaths.

- 13.1 Come up with appropriate formulas for B_F , B_R , D_F , and D_R , given your knowledge about how foxes and rabbits might interact.
- 13.2 When would B_F and D_F be at their max/min?
- 13.3 When would B_R and D_R be at their max/min?

Exercise 14

Suppose the population of F (foxes) and R (rabbits) evolves over time following the rule

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

- 14.1 Simulate the population of foxes and rabbits with a spreadsheet.
- 14.2 Do the populations continue to grow/shrink forever? Are they cyclic?
- 14.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

Exercise 15

Open the spreadsheet

<https://uoft.me/foxes-and-rabbits>

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

15.1 Is the max population of the rabbits being over/under estimated? Sometimes over, sometimes under?

15.2 What about the foxes?

15.3 What about the min populations?

Exercise 16

Open the spreadsheet

<https://uoft.me/foxes-and-rabbits>

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Component Graph & Phase Plane. For a differential equation involving the functions F_1, F_2, \dots, F_n , and the variable t , the **component graphs** are the n graphs of $(t, F_1), (t, F_2), \dots$.

The **phase plane** or **phase space** associated with the differential equation is the n -dimensional space with axes corresponding to the values of F_1, F_2, \dots, F_n .

- 16.1 Plot the Fox vs. Rabbit population in the *phase plane*.
- 16.2 Should your plot show a closed curve or a spiral?
- 16.3 What “direction” do points move along the curve as time increases? Justify by referring to the model.
- 16.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

Exercise 17

Open the spreadsheet

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which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Equilibrium Solution. An *equilibrium solution* to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

- 17.1 By changing initial conditions, what is the “smallest” curve you can get in the phase plane? What happens at those initial conditions?
- 17.2 What should F' and R' be if F and R are *equilibrium solutions*?
- 17.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.
- 17.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

Recall the logistic model for starfish growth:

- # new children per year \sim current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time t
- k is a constant of proportionality

- R is the total number of resources
- R_i is the resources that one starfish consumes

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

- 18.1 What are the equilibrium solutions for model **O**?
- 18.2 What does a “phase plane” for model **O** look like? What do graphs of equilibrium solutions look like?
- 18.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

Classification of Equilibria. An equilibrium solution f is called

- **attracting** if solutions locally converge to f
- **repelling** if solutions locally diverge from f
- **stable** if solutions do not locally diverge from f
- **unstable** if solutions do not locally converge to f
- **semi-stable** if solutions locally converge to f from one side and locally diverge from f on another.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

- 19.1 Draw an example of what solutions might look like if f is *attracting*.
- 19.2 Draw an example of what solutions might look like if f is *repelling*.
- 19.3 Draw an example of what solutions might look like if f is *stable*.
- 19.4 Could f be stable but *not* attracting?

Classification of Equilibria. An equilibrium solution f is called

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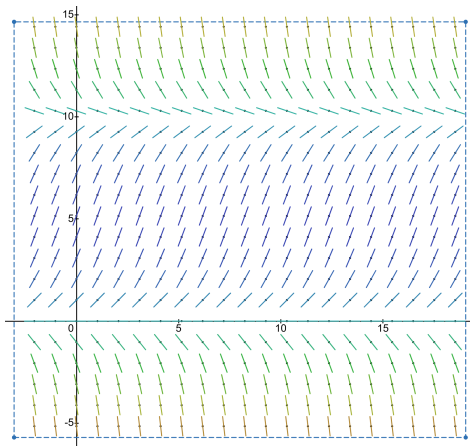
Recall the starfish population model **O** given by

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

- 20.1 Classify the equilibrium solutions for model **O** as attracting/repelling/stable/unstable/semi-stable.
- 20.2 Does changing k change the nature of the equilibrium solutions? How can you tell?

Exercise 21



A *slope field* is a plot of small segments of would-be tangent lines to solutions of a differential equation at different initial conditions.

On the left is a slope field for model **O**, available at

<https://www.desmos.com/calculator/ghavqzqqjn>

- 21.1 If you were sketching the slope field for model **O** by hand, what line would you sketch (a segment of) at $(5, 3)$? Write an equation for the line.
- 21.2 How can you recognize equilibrium solutions in a slope field?
- 21.3 Describe different solutions to the *differential equation* using words. Do all of those solutions make sense in terms of *model O*?