# Linear Algebr

MAT244 Slides

## Exercise 1

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

## Exercise 2

Let

$$\begin{aligned} &\text{(Birth Rate)} && K = 1.1 \text{ children per starfish per year} \\ &\text{(Initial Pop.)} && P_0 = 10 \text{ star fish} \end{aligned}$$

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

# Exercise 3

Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

# Exercise 4

In the model $\mathbf{M}_1$, we assumed the starfish had $K$ children at one point during the year.

4.1 Create a model $\mathbf{M}_n$ where the starfish are assumed to have $K/n$ children $n$ times per year (at regular intervals).

4.2 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.3 What happens to $\mathbf{M}_n$ as $n \to \infty$?

Exploring $\mathbf{M}_n$

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time $t$ there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t + 1/n)$ there will be (on average) $K/n$ new children per starfish.

5.1 Write an expression for $P_n(t + 1/n)$ in terms of $P_n(t)$.

5.2 Write an expression for $\Delta P$, the change in population from time $t$ to $t + \Delta t$.

5.3 Write an expression for $\frac{\Delta P}{\Delta t}$.

5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_n(t)$.

# Exercise 6

Define the model $\mathbf{M}_\infty$ by

- $P(0) = 10$
- $P'(t) = kP(t)$

and recall the model $\mathbf{M}_1$ defined by

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

6.2 Suppose that $\mathbf{M}_1$ accurately predicts the population. Can you find a value of $k$ so that $\mathbf{M}_\infty$ accurately predicts the population?

6.3 What are some advantages and disadvantages of the models $\mathbf{M}_1$ and $\mathbf{M}_\infty$?

# Exercise 7

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

7.1 What can you tell about the population (without trying to compute it)?

7.2 Assuming $k = 1.1$, estimate the population after 10 years.

7.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

## Exercise 8

Consider the following argument:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(2) + P'(2) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

8.1 Do you believe this argument? Can it be improved?

## Exercise 9

(Simulating $\mathbf{M}_\infty$ with different $\Delta$s)

| Time | Pop. ($\Delta = 0.1$) | Time | Pop. ($\Delta = 0.2$) |
|---|---|---|---|
| 0.0 | 10 | 0.0 | 10 |
| 0.1 | 11.1 | 0.2 | 12.2 |
| 0.2 | 12.321 | 0.4 | 14.884 |
| 0.3 | 13.67631 | 0.6 | 18.15848 |
| 0.4 | 15.1807041 | 0.8 | 22.1533456 |

9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

9.3 What $\Delta$s give the largest estimate for the population at time $t$?

9.4 Is there a limit as $\Delta \to 0$?

# Exercise 9

(Simulating $\mathbf{M}_\infty$ with different $\Delta$s)


9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

9.3 What $\Delta$s give the largest estimate for the population at time $t$?

9.4 Is there a limit as $\Delta \to 0$?

## Exercise 10

Consider the following models for starfish growth

**M** # new children per year $\sim$ current population

**N** # new children per year $\sim$ resources available per individual

**O** # new children per year $\sim$ current population times the fraction of total resources remaining

10.1 Guess what the population vs. time curves look like for each model.

10.2 Create a differential equation for each model.

10.3 Simulate population vs. time curves for each model (but pick a common initial population).

# Exercise 11

Recall the models

**M** # new children per year $\sim$ current population

**N** # new children per year $\sim$ resources available per individual

**O** # new children per year $\sim$ current population times the fraction of total resources remaining

11.1 Determine which population grows fastest in the short term and which grows fastest in the long term.

11.2 Are some models more sensitive to your choice of $\Delta$ when simulating?

11.3 Are your simulations for each model consistently underestimates? Overestimates?

11.4 Compare your simulated results with your guesses from question 10.1. What did you guess correctly? Where were you off the mark?

## Exercise 12

A simple model for population growth has the form

$$P'(t) = kP(t)$$

where $k$ is the *birth rate*.

12.1 Create a better model for population that includes both births and deaths.

# Exercise 13

The *Lotka-Volterra Predator-Prey* models two populations, $F$ (foxes) and $R$ (rabbits), simultaneously. It takes the form

$$F'(t) = (B_F - D_F) \cdot F(t)$$
$$R'(t) = (B_R - D_R) \cdot R(t)$$

where $B_?$ stands for births and $D_?$ stands for deaths.

13.1 Come up with appropriate formulas for $B_F$, $B_R$, $D_F$, and $D_R$, given your knowledge about how foxes and rabbits might interact.

13.2 When would $B_F$ and $D_F$ be at their max/min?

13.3 When would $B_R$ and $D_R$ be at their max/min?

## Exercise 14

Suppose the population of $F$ (foxes) and $R$ (rabbits) evolves over time following the rule

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

14.1 Simulate the population of foxes and rabbits with a spreadsheet.

14.2 Do the populations continue to grow/shrink forever? Are they cyclic?

14.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

# Exercise 15

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

15.1 Is the max population of the rabbits being over/under estimated? Sometimes over, sometimes under?

15.2 What about the foxes?

15.3 What about the min populations?

# Exercise 16

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

**Component Graph & Phase Plane.** For a differential equation involving the functions $F_1, F_2, \ldots, F_n$, and the variable $t$, the ***component graphs*** are the $n$ graphs of $(t, F_1), (t, F_n), \ldots$.
The ***phase plane*** or ***phase space*** associated with the differential equation is the $n$-dimensional space with axes corresponding to the values of $F_1, F_2, \ldots, F_n$.

16.1 Plot the Fox vs. Rabbit population in the *phase plane*.

16.2 Should your plot show a closed curve or a spiral?

16.3 What "direction" do points move along the curve as time increases? Justify by referring to the model.

16.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

## Exercise 17

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

**Equilibrium Solution.** An *equilibrium solution* to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

17.1 By changing initial conditions, what is the "smallest" curve you can get in the phase plane? What happens at those initial conditions?

17.2 What should $F'$ and $R'$ be if $F$ and $R$ are *equilibrium solutions*?

17.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.

17.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

## Exercise 18

Recall the logistic model for starfish growth:

**O** # new children per year ~ current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time $t$
- $k$ is a constant of proportionality
- $R$ is the total number of resources
- $R_i$ is the resources that one starfish consumes

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

18.1 What are the equilibrium solutions for model **O**?

18.2 What does a "phase plane" for model **O** look like? What do graphs of equilibrium solutions look like?

18.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

# Exercise 19

**Classification of Equilibria.** An equilibrium solution $f$ is called

- *attracting* if solutions locally converge to $f$
- *repelling* if solutions locally diverge from $f$
- *stable* if solutions do not locally diverge from $f$
- *unstable* if solutions do not locally converge to $f$
- *semi-stable* if solutions locally converge to $f$ from one side and locally diverge from $f$ on another.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

19.1 Draw an example of what solutions might look like if $f$ is *attracting*.

19.2 Draw an example of what solutions might look like if $f$ is *repelling*.

19.3 Draw an example of what solutions might look like if $f$ is *stable*.

19.4 Could $f$ be stable but *not* attracting?

# Exercise 20

**Classification of Equilibria.** An equilibrium solution $f$ is called

- *attracting* if solutions locally converge to $f$
- *repelling* if solutions locally diverge from $f$
- *stable* if solutions do not locally diverge from $f$
- *unstable* if solutions do not locally converge to $f$
- *semi-stable* if solutions locally converge to $f$ from one side and locally diverge from $f$ on another.

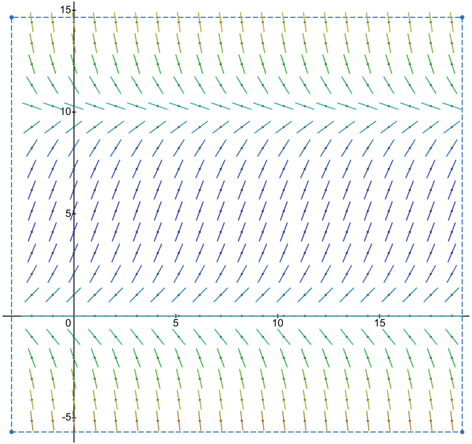
Recall the starfish population model **O** given by

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

20.1 Classify the equilibrium solutions for model **O** as attracting/repelling/stable/unstable/semi-stable.

20.2 Does changing $k$ change the nature of the equilibrium solutions? How can you tell?




A *slope field* is a plot of small segments of would-be tangent lines to solutions of a differential equation at different initial conditions.

On the left is a slope field for model **O**, available at

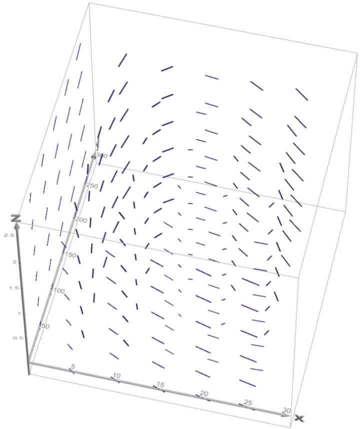
`https://www.desmos.com/calculator/ghavqzqqjn`

21.1 If you were sketching the slope field for model **O** by hand, what line would you sketch (a segment of) at $(5,3)$? Write an equation for the line.

21.2 How can you recognize equilibrium solutions in a slope field?

21.3 Describe different solutions to the *differential equation* using words. Do all of those solutions make sense in terms of *model **O***?

# Exercise 22




3d slope fields are possible, but hard to interpret.

On the left is a slope field for the Foxes–Rabbits model.

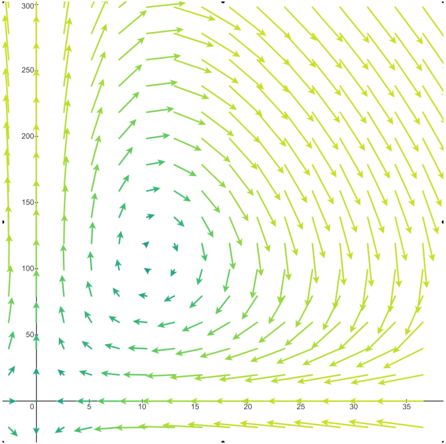
`https://www.desmos.com/3d/fsfbhvy2h9`

22.1 What are the three dimensions in the plot?

22.2 What should the graph of an equilibrium solution look like?

22.3 What should the graph of a typical solution look like?

22.4 What are ways to simplify the picture so we can more easily analyze solutions?




> **Phase Portrait.** A *phase portrait* or *phase diagram* is the plot of a vector field in phase space where each vector rooted at $(x, y)$ is tangent to a solution curve passing through $(x, y)$ and its length is given by the speed of a solution when passing through $(x, y)$.

On the left is a phase portrait for the Foxes–Rabbits model.

`https://www.desmos.com/calculator/vrk0q4espx`

23.1 What do the $x$ and $y$ axes correspond to?

23.2 Identify the equilibria in the phase portrait. What are the lengths of the vectors at those points?

23.3 Classify each equilibrium as stable/unstable.

23.4 Why is the vectors at $(30, 300)$ longer than the vector at $(5, 50)$? Justify numerically.

## Exercise 24

Use the phase portrait plotter

`https://www.desmos.com/calculator/vrk0q4espx`

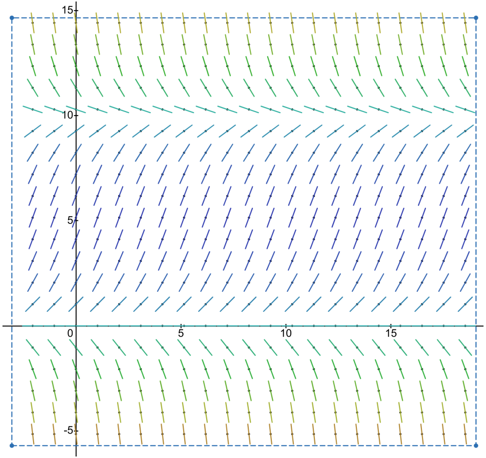
to create a system of differential equations where:

24.1 There is an attracting equilibrium solution.

24.2 There is a repelling equilibrium solution.

24.3 There are no equilibrium solutions.

# Exercise 25




Recall the slope field for model **O**.

25.1 What would a phase portrait for model **O** look like? Draw it.

25.2 Where are the arrows the longest? Shortest?

25.3 How could you tell from a 1d phase portrait whether an equilibrium solution is attracting/repelling/etc.?

## Exercise 26

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time $t$
- $A(t)$ = combined surface area (in square meters) of all leaves at time $t$

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 < b < 2$

26.1 Modify

`https://www.desmos.com/calculator/vrk0q4espx`

to make a phase portrait for the tree model.

26.2 What do equilibrium solutions mean in terms of tree growth?

26.3 For $b = 1$ what are the equilibrium solution(s)?

## Exercise 27

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time $t$
- $A(t)$ = combined surface area (in square meters) of all leaves at time $t$

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

27.1 Fix a value of $b$ and use a spreadsheet to simulate some solutions with different initial conditions. Plot the results on your phase portrait from 26.1.

27.2 What will happen to a tree with $(H(0), A(0)) = (20, 10)$? Does this depend on $b$?

27.3 What will happen to a tree with $(H(0), A(0)) = (10, 10)$? Does this depend on $b$?

# Exercise 28

The tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

was based on the premises

$P_{\text{height 1}}$ $CO_2$ is absorbed by the leaves and turned directly into trunk height.

$P_{\text{height 2}}$ The tree is in a swamp and constantly sinks at a speed proportional to its height.

$P_{\text{leaves 1}}$ Leaves grow proportionality to the energy available.

$P_{\text{energy 1}}$ The tree absorbs energy from the sun proportionality to the leaf area.

$P_{\text{energy 2}}$ It costs energy proportional to the square of the height for the tree to maintain its current size.

28.1 How are the premises expressed in the differential equations?

28.2 What does the parameter $b$ represent?

28.3 Simulations show solutions with negative values for $H$ and $A$. What actually happens to the tree when this occurs?

# Exercise 29

Recall the tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

29.1 Find all equilibrium solutions for $0 \leq b \leq 2$.

29.2 For which $b$ does a tree have the possibility of living forever? If the wind occasionally blew off a few random leaves, would that change your answer?