Linear Algebr

MAT244 Slides

Exercise 1

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

Exercise 2

Let

$$\begin{aligned} &\text{(Birth Rate)} && K = 1.1 \text{ children per starfish per year} \\ &\text{(Initial Pop.)} && P_0 = 10 \text{ star fish} \end{aligned}$$

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

Exercise 3

Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

Exercise 4

In the model $\mathbf{M}_1$, we assumed the starfish had K children at one point during the year.

4.1 Create a model $\mathbf{M}_n$ where the starfish are assumed to have K/n children n times per year (at regular intervals).

4.2 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.3 What happens to $\mathbf{M}_n$ as $n \to \infty$?

Exercise 5

Exploring $\mathbf{M}_n$

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t+1/n)$ there will be (on average) K/n new children per starfish.

5.1 Write an expression for $P_n(t+1/n)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP, the change in population from time t to $t+\Delta t$.

5.3 Write an expression for $\frac{\Delta P}{\Delta t}$.

5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_n(t)$.

Exercise 6

Define the model $\mathbf{M}_\infty$ by

- $P(0) = 10$
- $P'(t) = kP(t)$

and recall the model $\mathbf{M}_1$ defined by

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

6.2 Suppose that $\mathbf{M}_1$ accurately predicts the population. Can you find a value of k so that $\mathbf{M}_\infty$ accurately predicts the population?

6.3 What are some advantages and disadvantages of the models $\mathbf{M}_1$ and $\mathbf{M}_\infty$?

Exercise 7

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

7.1 What can you tell about the population (without trying to compute it)?

7.2 Assuming $k = 1.1$, estimate the population after 10 years.

7.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

Exercise 8

Consider the following argument:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(2) + P'(2) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

8.1 Do you believe this argument? Can it be improved?

Exercise 9

(Simulating $\mathbf{M}_\infty$ with different Δs)

Time	Pop. ($\Delta = 0.1$)	Time	Pop. ($\Delta = 0.2$)
0.0	10	0.0	10
0.1	11.1	0.2	12.2
0.2	12.321	0.4	14.884
0.3	13.67631	0.6	18.15848
0.4	15.1807041	0.8	22.1533456

9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

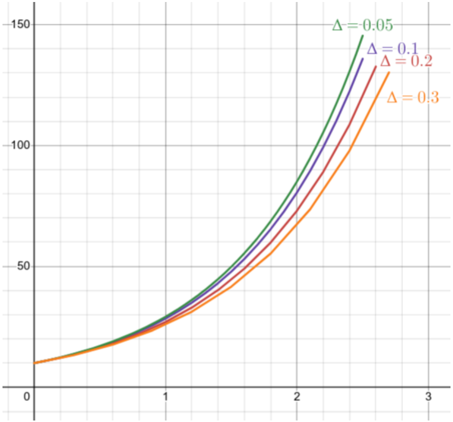
9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

9.3 What Δs give the largest estimate for the population at time t?

9.4 Is there a limit as $\Delta \to 0$?

Exercise 9

(Simulating $\mathbf{M}_\infty$ with different Δs)



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Exercise 10

Consider the following models for starfish growth

M # new children per year $\sim$ current population

N # new children per year $\sim$ resources available per individual

O # new children per year $\sim$ current population times the fraction of total resources remaining

10.1 Guess what the population vs. time curves look like for each model.

10.2 Create a differential equation for each model.

10.3 Simulate population vs. time curves for each model (but pick a common initial population).

Exercise 11

Recall the models

M # new children per year $\sim$ current population

N # new children per year $\sim$ resources available per individual

O # new children per year $\sim$ current population times the fraction of total resources remaining

11.1 Determine which population grows fastest in the short term and which grows fastest in the long term.

11.2 Are some models more sensitive to your choice of Δ when simulating?

11.3 Are your simulations for each model consistently underestimates? Overestimates?

11.4 Compare your simulated results with your guesses from question 10.1. What did you guess correctly? Where were you off the mark?