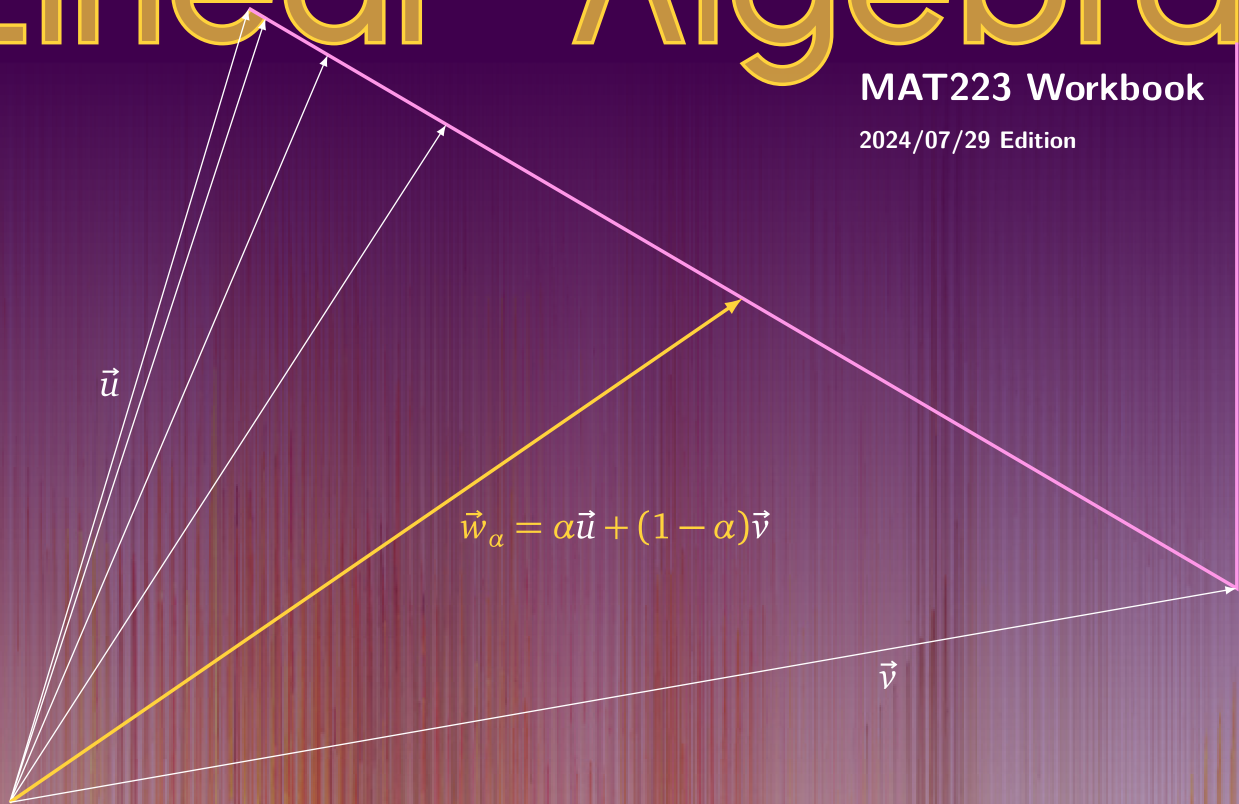
Linear Algebra

MAT223 Workbook

2024/07/29 Edition



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Linear Algebra

About this Book

For the student

This book is your introductory guide to linear algebra. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of linear algebra. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas and linear algebra terminology.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of linear algebra by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle with you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

So many definitions. A big part of linear algebra is learning precise and technical language.[1] There are many terms and definitions you need to learn, and by far the best way to successfully learn these terms is to understand where they come from, why they're needed, and practice using them. That is, don't try to memorize definitions word for word. Instead memorize the idea and *reconstruct* the definition; go through the core exercises and identify which definitions appear where; and explain linear algebra to others using these technical terms.

Contributing to the book. Did you find an error? Do you have a better way to explain a linear algebra concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/siefkenj/IBLLinearAlgebra`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory linear algebra course course with a focus on geometry (MAT223 at the University of Toronto). It has not been designed for an "intro to proofs"-style course, but could be adapted for one.

Unlike a traditional textbook that is grouped into chapters and sections by subject, this book is grouped into modules. Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related

[1] Beyond three dimensions, things get very confusing very quickly. Having precise definitions allows us to make arguments that rely on logic instead of intuition; and logic works in all dimensions.

concepts, but the modules have been designed to facilitate learning linear algebra rather than to serve as a reference. For example, information about change-of-basis is spread across several non-consecutive modules; each time change-of-basis is readdressed, more detail is added.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.

2. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.

5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a geometric lean. Vectors are initially introduced with familiar coordinate notation, but eventually, coordinates are understood to be *representations* of vectors rather than "true" geometric vectors, and objects like the determinant are defined via oriented volumes rather than formulas involving matrix entries.

Specifically lacking are exercises focusing on the mechanical skills of row reduction and computing matrix inverses. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.

2. **Work through the core exercises.**

3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.

4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 2–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

License

Included in this text are tasks created by the Inquiry-Oriented Linear Algebra (IOLA) project. Details about these tasks can be found on their website `http://iola.math.vt.edu/`. Also included are some practice problems from Beezer's *A First Course in Linear Algebra* (marked with the symbol B next to the problem), and from Hefferon's *Linear Algebra* (marked with the symbol H next to the problem).

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/siefkenj/IBLLinearAlgebra/`

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

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Dedication

This book is dedicated to Dr. Bob Burton—friend and mentor.

"Sometimes you have to walk the mystical path with practical feet."

Contents

Modeling

In this module you will learn

- ??

Modeling

Suppose you are observing some *green* ants walking on the sidewalk. In the first minute you record 10 ants. In the second minute you record 20. In the third minute, you record 40 ants. This continues until there are too many ants for you to count.

Minute	#Green Ants
1	10
2	20
3	40
4	80
⋮	⋮

Since you lost count of the ants, you decide to use mathematics to try and figure out how many ants walked by on minutes 5, 6, ... You notice the pattern that

$$\text{Green ants per minute } n = 2^{n-1} \cdot 10.$$

Stupendous! Mathematics now predicts there were 160 ants during minute 5. But something else catches your eye. Across the sidewalk are *brown* ants. You count these ants every minute.

Minute	#Brown Ants
1	3
2	6
3	12
4	24
⋮	⋮

The pattern is slightly different. This time,

$$\text{Green ants per minute } n = 2^{n-1} \cdot 3.$$

Your friend, who was watching you the whole time, looks confused. "Why come up with two complicated equations when you can describe both types of ant at once?" they declare.

$$\begin{aligned} \#\text{Ants at minute } n &= 2 \cdot (\#\text{Ants at minute } n-1) \\ \#\text{Green ants at minute } 1 &= 10 \\ \#\text{Brown ants at minute } 1 &= 3 \end{aligned}$$

Your friend has a point. Their model is elegant, but *your* model can predict how many ants pass by at minute 3.222! Though, your friend would probably complain that 46.654 is not a number of ants....

You and your friend have just come up with two different *mathematical models* for the number of ants that walk across the sidewalk. They happen to make similar predictions for each minute and each have their strengths and weakenesses. In this course, we will be focused on a particular type of mathematical model—one that uses *differential equations* at its core.

Types of Models

Mathematical Model. A ***mathematical model*** is a description of the world

1. created in the service of answering a question, and

2. where the complexity of the world has been abstracted away to numbers, quantities, and their relationships.[a]

[a]Other mathematical objects are also allowed.

In the previous situation, the *question* you were trying to answer was "how many ants are there at a given minute?". We sidestepped difficult issues like, "Is an ant that is missing three legs still an ant?" by using the common-sense convention that "the number of ants is a whole number and one colored blob that moves under its own power corresponds to one ant"; thus, we could use single numbers to represent our quantity of interest (the ants).

You and your friend already came up with two types of models.

- An **analytic** model based on known functions.
- A **recursive** model where subsequent terms are based on previous terms and initial conditions.

If we define $A(n)$ to be the number of ants corssing the sidewalk at minute n, the *analytic* model presented for green ants is

$$A(n) = 2^{n-1} \cdot 10$$

and the *recursive* model presented is

$$\begin{aligned} A(1) &= 10 \\ A(n) &= 2 \cdot A(n-1). \end{aligned}$$

Each type of model has pros and cons. For example, the analytic model allows you to calculate the number of ants at any minute with few button presses on a calculator, whereas the recursive model is more difficult to calculate but makes it clear that the number of ants is doubling every minute.

Often times recursive models are easier to write down than analytic models,[2] but they maybe harder to analyze. A third type of model has similarities to both analytic and recursive models, and brings the power of calculus to modeling.

- A **differential-equations** model is a model based on a relationship between a function's derivative(s), its values, and initial conditions.

Differential-equations models are useful because derivatives correspond to rates of change—and things in the world are always changing. Let's try to come up with a differential equations model for the ants.

We'd like an equation relating $A(n)$, the number of ants at minute n, to $A'(n)$, the *instantaneous rate of change* of the number of ants at minute n. Making a table, we see

Minute	#Brown Ants	Change (from prev. minute)
1	3	?
2	6	3
3	12	6
4	24	12

or

Minute	#Brown Ants	Change (from next minute)
1	3	3
2	6	6
3	12	12
4	24	?

depending on whether we record the change from the previous minute or up to the subsequent minute. Neither table gives the *instantaneous* rate of change, but in both tables, the change is proportional to the number of ants. So, we can set up a model

$$A'(n) = kA(n)$$

where k is a constant of proportionality that we will try to determine later. We've just written down a *differential equation* with an undetermined parameter, k.

[2]In fact, in many real-world situations, an analytic model doesn't exist

Differential Equation. A ***differential equation*** is an equation relating a function to one or more of its derivatives.

We'd like to figure out what k is. One way to do so is to solve the differential equation and find the values of k so that our model correctly predicts the data. This is called *fitting* the model to data.

Fitting a Model. Given a model M with parameters $k_1, k_2, \ldots$ and data D, ***fitting the model M to the data D*** is the process of finding values for the parameters $k_1, k_2, \ldots$ so that M most accurately predicts the data D.

Note that, in general, fitting a model to data doesn't necessarily produce *unique* values for the unknown parameters, and a fitted model (especially when the data comes from real-world observations) usually doesn't reproduce the data exactly. However, in the case of these ants, we just might get lucky.

Solving Differential Equations

In general, *there is no algorithm for solving differential equations*. Fortunately, it is easy to check whether any particular function is a solution to a differential equation, since there *is* an algorithm to differentiate functions.[3] Because of this, *guess and check* will be our primary method for solving differential equations.

Example. Use educated guessing to solve $A'(n) = kA(n)$.

Since $A' \approx A$, we might start with a function that is equal to its own derivative. There is a famous one: e^n. Testing, we see

$$\frac{\mathrm{d}}{\mathrm{d}n} e^n = e^n = ke^n$$

if $k = 1$, but it doesn't work for other k's. Trying e^{kn} instead yields

$$\frac{\mathrm{d}}{\mathrm{d}n} e^{kn} = ke^{kn}$$

which holds for all k. Thus $A(n) = e^{kn}$ is *a* solution to $A'(n) = kA(n)$. However, there are other solutions, because

$$\frac{\mathrm{d}}{\mathrm{d}n} Ce^{kn} = C\left(ke^{kn}\right) = k\left(Ce^{kn}\right),$$

and so for every C, the function $A(n) = Ce^{kn}$ is a solution to $A'(n) = kA(n)$.

By guessing-and-checking, we have found an infinite number of solutions to $A'(n) = kA(n)$. It's now time to fit our model to the data.

Example. Find values of C and k so that $A(n) = Ce^{kn}$ best models brown ants.

Taking two rows from our brown ants table, we see

$$A(1) = Ce^{k} = 3$$
$$A(2) = Ce^{2k} = 6.$$

Since e^k can never be zero, from the first equation we get $C = 3/e^k$. Combining with the second equation we find

$$Ce^{2k} = \frac{3}{e^k} e^{2k} = 3e^k = 6$$

and so $e^k = 2$. In other words $k = \ln 2$. Plugging this back in, we find $C = 3/2$. Thus our fitted model is

$$A(n) = \frac{3}{2} e^{n \ln 2}.$$

Upon inspection, we can see that $\frac{3}{2} e^{n \ln 2} = 3 \cdot 2^{n-1}$, which is the analytic model that was first guessed for brown ants.

[3] More specifically, there is an algorithm to differentiate the *elementary* functions, those functions formed by compositions, sums, products, and quotients of polynomials, trig, exponentials, and logs.

The Magic Carpet Ride

1 You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 km East and 64 km North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

The Magic Carpet Ride, Hide and Seek

2 You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

3

3.1 Which of the following statements are true?

(a) $3 \in \{1, 2, 3\}$.

(b) $1.5 \in \{1, 2, 3\}$.

(c) $4 \in \{1, 2, 3\}$.

(d) "b"$\in \{x : x \text{ is an English letter}\}$.

(e) "ò"$\in \{x : x \text{ is an English letter}\}$.

(f) $\{1, 2\} \subseteq \{1, 2, 3\}$.

(g) For some $a \in \{1, 2, 3\}$, $a \geq 3$.

(h) For any $a \in \{1, 2, 3\}$, $a \geq 3$.

(i) $1 \subseteq \{1, 2, 3\}$.

(j) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

(k) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

4 Write the following in set-builder notation

4.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

4.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

DEFINITION

Unions & Intersections

Let X and Y be sets. The ***union*** of X and Y and the ***intersection*** of X and Y are defined as follows.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

5 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

5.1 $X \cup Y$

5.2 $X \cap Y$

5.3 $X \cup Y \cup Z$

5.4 $X \cap Y \cap Z$

6 Draw the following subsets of $\mathbb{R}^2$.

6.1 $V = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

6.2 $H = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

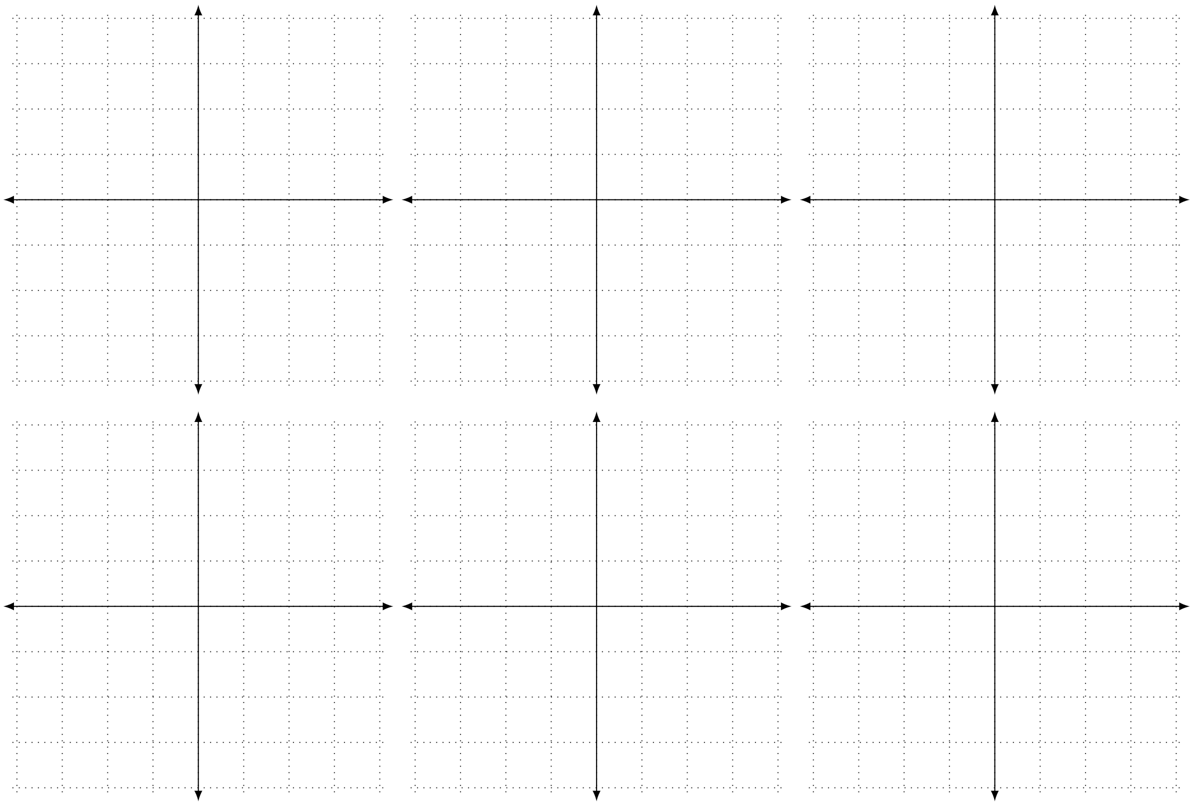
6.3 $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

6.4 $N = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}.$

6.5 $V \cup H$.

6.6 $V \cap H$.

6.7 Does $V \cup H = \mathbb{R}^2$?



Vector Combinations

Linear Combination

DEFINITION

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the ***coefficients*** of the linear combination.

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7 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

7.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

7.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

8 fd

Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

8.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

8.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

8.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

Systems of Linear Equations I

In this appendix you will learn

- What a system of linear equations is.
- What the solution set to a system of equations is, and what it means for a system of equations to be consistent or inconsistent.
- How augmented matrices can be used to solve systems of linear equations.
- How to apply row reduction to find a unique solution to a system of linear equations and to determine if a system of linear equations is consistent or inconsistent.

An *equation* encodes a relationship between quantities. For example, writing

$$\underbrace{\text{Slices of cake}}_{C}=\underbrace{\text{Slices you ate}}_{M}+\underbrace{\text{Slices your brother ate}}_{B}$$

specifies a precise relationship between the quantities C, M, and B. Without more information, C, M, and B could be almost anything. As such, we call C, M, and B *variables* or *unknowns*. However, the relationship *between* them is precisely defined.

Additional relationships give rise to additional equations, which we express concisely as a *system of equations*, that is, a list of equations. For example, suppose you know the cake had six pieces and your brother ate twice as many pieces as you. We might now write the system

$$\begin{aligned} C&=M+B\\ B&=2M\\ C&=6 \end{aligned}$$

which should be interpreted as: "the relationship $C=M+B$ holds *and* the relationship $B=2M$ holds *and* the relationship $C=6$ holds." All this information, taken together, is enough to deduce the unknowns: $M=2$, $B=4$, and $C=6$.

Systems of equations naturally appear in linear algebra through vector equations. Suppose $\vec u=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec v=\begin{bmatrix}2\\3\end{bmatrix}$, and $\vec w=\begin{bmatrix}1\\1\end{bmatrix}$. You might wonder if $\vec w$ was a linear combination of $\vec u$ and $\vec v$. The answer is *yes* if and only if the vector equation

$$\vec w=a\vec u+b\vec v$$

has a solution for some a and b. Written in coordinates, this equation is equivalent to

$$\begin{bmatrix}1\\1\end{bmatrix}=a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}a+2b\\2a+3b\end{bmatrix}.$$

Equating coordinates, a system of equations appears:

$$\begin{cases} a+2b=1\\ 2a+3b=1 \end{cases}$$

Every vector equation, by way of coordinates, corresponds to a system of equations. And, fortunately for us, there is an *algorithm* to find all solutions to these systems.[4]

Systems of Linear Equations

There's no guarantee that a general equation, like $x^4-e^x+7=0$, has a solution, and it might be impossible to decide if an arbitrary equation has a solution, let alone what the solutions are![5] However, for *linear* equations and systems of linear equations we can *always* tell whether there is a solution and what the solution(s) are.

[4] Saying there is an *algorithm* for "X" means that there is a specific set of (non-random) rules that *always* accomplishes "X". As a consequence, doing "X" never requires special insight. For example, there *is* an algorithm for multiplying numbers, but there *is not* an algorithm for factoring polynomials of degree 5 or greater.

[5] Fermat's Last Theorem famously claimed that $a^n+b^n=c^n$ has no positive integer solutions for $n\geq 3$. However, it took 350 years of human ingenuity before anyone was able rigorously back up the claim.

Linear Equation. A ***linear equation*** in the variables $x_1, \ldots, x_n$ is one that can be expressed as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

for constants $a_1, \ldots, a_n$ and c. A ***system of linear equations*** is a system of equations consisting of one or more linear equations.

Every vector equation corresponds to an *equivalent* system of linear equations and vice versa, where equivalent means "expresses the same relationships between variables".

Example. Write down the vector equation corresponding to the system of linear equations $\begin{cases} 2x+3y+z=2 \\ \qquad\quad y-z=-1 \end{cases}$ and the system of linear equations corresponding to the vector equation $t\vec{w}+\vec{u}=r\vec{v}$ where $\vec{w}=\begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$, and $\vec{v}=\begin{bmatrix}4\\4\end{bmatrix}$.

The system $\begin{cases} 2x+3y+z=2 \\ 0x+\ \ y-z=-1 \end{cases}$ corresponds to the vector equation

$$x\begin{bmatrix}2\\0\end{bmatrix}+y\begin{bmatrix}3\\1\end{bmatrix}+z\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}.$$

As for the vector equation $t\vec{w}+\vec{u}=r\vec{v}$, rewriting each vector in coordinates gives us a corresponding system of linear equations:

$$t\vec{w}+\vec{u}=r\vec{v} \to t\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=r\begin{bmatrix}4\\4\end{bmatrix}$$
$$\begin{bmatrix}t+2\\-t+3\end{bmatrix}=\begin{bmatrix}4r\\4r\end{bmatrix} \to \begin{cases} -4r+t=-2 \\ -4r-t=-3 \end{cases}.$$

Takeaway. Every vector equation corresponds to a system of linear equations and every system of linear equations corresponds to a vector equation.

Solution Sets

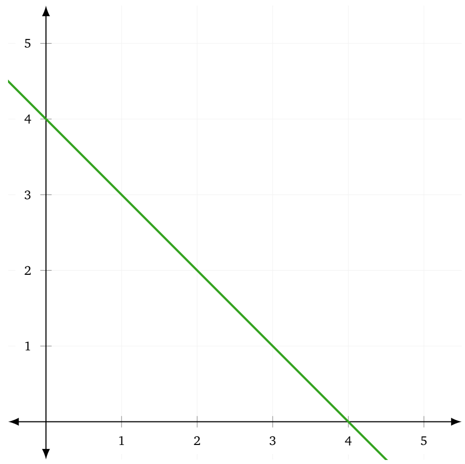
Before looking at how to solve systems of linear equations, let's agree on some terminology.

A *solution* to an equation is a particular choice of values for the variables that satisfy (i.e. make true) the equation. For example

$$x+y=4 \tag{1}$$

has a solution $x=y=2$. However, $x=y=2$ is just one of *many* possible solutions; we also have $x=4$ and $y=0$ or $x=-2$ and $y=6$. The *solution set*, also called the *complete solution*, to an equation (or system of equations) is the set of all possible solutions. For example, the solution set to Equation (1) is $S=\{(x,y) : y=4-x\}$. In this case, S contains infinitely many solutions, including $(x,y)=(2,2)$, the first solution we found.

Solution sets look a lot like sets of vectors: the set $S=\{(x,y) : y=4-x\}$ could be thought of as a subset of $\mathbb{R}^2$ where we identify a solution $x=a$ and $y=b$ with the column vector $\begin{bmatrix}a\\b\end{bmatrix}$. Drawing S as a subset of $\mathbb{R}^2$, we see a familiar picture.



It's the graph of the line given in $y = mx + b$ form by $y = -x + 4$. In other words, *via solution sets, equations and systems of equations can represent geometric objects.*

Consistent & Inconsistent Systems

Consider the following equations (as separate equations, not as a system):

$$x^2 = 0 \qquad \text{with solution set} \qquad S_x \subseteq \mathbb{R},$$

$$y^2 = 4 \qquad \text{with solution set} \qquad S_y \subseteq \mathbb{R},$$

and

$$z^2 = -1 \qquad \text{with solution set} \qquad S_z \subseteq \mathbb{R}.$$

$S_x = \{0\}$ consists of a single number. $S_y = \{2, -2\}$ consists of two numbers, and $S_z = \{\}$ consists of no numbers.[6] In this case, we would call the first two equations *consistent* and the third equation *inconsistent*.

> **Consistent & Inconsistent.** An equation or system of equations is called ***consistent*** if it has at least one solution. That is, its solution set is non-empty. Otherwise, an equation or system of equations is called ***inconsistent***.

Why the word *consistent*? This comes from the term *logically consistent* which means "able to be true". An equation is an assertion that the left hand side equals the right hand side. If that can never happen, the assertion is not logically consistent.

This terminology becomes more clear with systems. Consider the system

$$\begin{cases} x - y = 0 \\ x - y = 1 \end{cases}.$$

From the first equation, we deduce $y = x$. From the second equation, we deduce $x = 1 + y$. Since $x = x$, we know that $y = x = 1 + y$ and therefore $y = 1 + y$. However, this is never true! There is a logical inconsistency between the two equations. In isolation they're fine, but taken together they're not.

Equivalent Systems

Two systems of equations are logically equivalent if they express the same relationships between their variables. For example, the equations $x = 2y$ and $\frac{1}{2}x = y$ express the exact same relationship between the variables x and y. This can be formalized using solution sets.

> **Equivalent Systems.** Two equations or systems of equations are called ***equivalent*** if they have the same solution sets.

Again, $x = 2y$ and $\frac{1}{2}x = y$ both have the same solution set (a line through the origin of slope $\frac{1}{2}$), and so they are equivalent.

Philosophical note: the process of "doing algebra" can be viewed as the process of *manipulating equations/systems into easier to understand equivalent equations/systems*. When you're asked to algebraically solve $x^2 - 4 = 0$.

[6] We're not allowing complex numbers at the moment.

You might first factor to get the equivalent equation $(x-2)(x+2)=0$. Then, since non-zero numbers cannot multiply to give zero, we know $x-2=0$ or $x+2=0$, which in turn is equivalent to $x=\pm 2$. It's always been about equivalent systems![7]

Row Reduction

Consider the vector equation

$$t\vec{u}+s\vec{v}+r\vec{w}=\vec{p} \qquad \text{where} \qquad \vec{u}=\begin{bmatrix}1\\2\\1\end{bmatrix},\ \vec{v}=\begin{bmatrix}2\\1\\-4\end{bmatrix},\ \vec{w}=\begin{bmatrix}-2\\-5\\1\end{bmatrix},\ \vec{p}=\begin{bmatrix}-15\\-21\\18\end{bmatrix}.$$

By expanding in terms of coordinates, we get an equivalent system of linear equations.

$$\begin{cases} t+2s-2r=-15 & \text{row}_1\\ 2t+s-5r=-21 & \text{row}_2\\ t-4s+r=18 & \text{row}_3\end{cases} \tag{2}$$

The most general way to solve any system is by *substitution*. For System (2), we could solve the first equation for t, substitute the result in the remaining equations, solve the next equation for s, etc.. However, because System (2) is a *linear* system, there's an alternate method: *row reduction*.[8]

Study the following manipulations of System (2) and convince yourself that each operation produces a system equivalent to the one it came from.

$$\begin{cases} t+2s-2r=-15\\ 2t+s-5r=-21\\ t-4s+r=18\end{cases} \xrightarrow{\text{row}_3\mapsto\text{row}_3-\text{row}_1} \begin{cases} t+2s-2r=-15\\ 2t+s-5r=-21\\ -6s+3r=33\end{cases}$$

$$\xrightarrow{\text{row}_2\mapsto\text{row}_2-2\text{row}_1} \begin{cases} t+2s-2r=-15\\ -3s-r=9\\ -6s+3r=33\end{cases}$$

$$\xrightarrow{\text{row}_3\mapsto\text{row}_3-2\text{row}_2} \begin{cases} t+2s-2r=-15\\ -3s-r=9\\ 5r=15\end{cases} \tag{3}$$

From the final system, System (3), it's easy to see that $r=3$. From there, we can substitute $r=3$ into the second row of System (3) to find $s=-4$ and we can substitute both r and s into the first row of System (3) to find $t=-1$.

By adding and subtracting rows, we "reduced" the number of variables from some equations until they were easy to solve. As an added benefit, every system along the way to System (3) was nicely organized and formatted. In fact, the systems are so well organized that we can save time by not writing the variables and keeping track of the numbers in an *augmented matrix*.[9] That is, instead of writing

$$\begin{cases} t+2s-2r=-15\\ 2t+s-5r=-21\\ t-4s+r=18\end{cases}$$

we will write

$$\left[\begin{array}{rrr|r} 1 & 2 & -2 & -15\\ 2 & 1 & -5 & -21\\ 1 & -4 & 1 & 18\end{array}\right].$$

We call the matrix an *augmented matrix* to stress that it contains two types of information: the *coefficients* of the variables t, s, and r and the *constants* on the right hand side of the equations. An (optional) vertical line separates the two types of numbers.

[7] Technically, up to this point we've been talking about *conjunctive* systems, which means that a solution must hold for all equations of a system. The system $x=\pm 2$ is a *disjunctive* system, which means a solution only needs to hold for *one* of the equations ($x=2$ or $x=-2$), but the idea is the same.

[8] Row reduction is sometimes referred to as *Gaussian elimination*, *Gauss-Jordan elimination*, or just *elimination*; though there are subtle differences between Gaussian and Gauss-Jordan elimination, they aren't important, and we'll refer to all similar methods as *row reduction*.

[9] A *matrix* is a box of numbers. An *augmented matrix* is a matrix with extra information associated with it.

Now, the process of row reduction looks like this:

$$\begin{cases} t+2s-2r=-15 \\ 2t+\ \ s-5r=-21 \\ t-4s+\ \ r=18 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 2 & 1 & -5 & -21 \\ 1 & -4 & 1 & 18 \end{array}\right] \xrightarrow{\text{row}_3\mapsto\text{row}_3-\text{row}_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 2 & 1 & -5 & -21 \\ 0 & -6 & 3 & 33 \end{array}\right]$$

$$\xrightarrow{\text{row}_2\mapsto\text{row}_2-2\text{row}_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & -6 & 3 & 33 \end{array}\right]$$

$$\xrightarrow{\text{row}_3\mapsto\text{row}_3-2\text{row}_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & 0 & 5 & 15 \end{array}\right] \to \begin{cases} t+2s-2r=-15 \\ \quad -3s-\ \ r=9 \\ \qquad\quad 5r=15 \end{cases}$$

The operations are identical, but we write augmented matrices instead of equations.

Takeaway. Augmented matrices are a notational tool that makes the process of doing row reduction more efficient.

The Rules of Row Reduction

So far, we've been able to row reduce systems by adding a multiple of one row to another,[10] but to fully solve *any* system, we need additional operations.[11]

Elementary Row Operations. The three ***elementary row operations***, which can be performed on a matrix or system of equations, are

- swapping two rows (written $\text{row}_i \leftrightarrow \text{row}_j$),
- multiplying a row by a non-zero scalar (written $\text{row}_i \mapsto k\,\text{row}_i$), and
- adding a multiple of one row to another (written $\text{row}_i \mapsto \text{row}_i + k\,\text{row}_j$).

Notice that each elementary row operation can be undone. For example, if you perform $\text{row}_i \mapsto k\,\text{row}_i$, you can undo it with $\text{row}_i \mapsto \frac{1}{k}\text{row}_i$. Therefore, applying an elementary row operation to a system is guaranteed to produce an equivalent system.

The strategy for solving a system is now summarized as:

1. Rewrite the system as an augmented matrix.
2. Use elementary row operations to zero-out the lower triangle of the augmented matrix.
3. Convert the matrix back to a system of equations.
4. Read off the solution (substituting when necessary).

Example. Find a solution to the following system:

$$\begin{cases} a+3b+2c=1 \\ 2a+7b+5c=2 \\ -a-4b\qquad\ =11 \end{cases}.$$

To do so, we rewrite the system as an augmented matrix then row reduce.

$$\begin{cases} a+3b+2c=1 \\ 2a+7b+5c=2 \\ -a-4b\qquad\ =11 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 7 & 5 & 2 \\ -1 & -4 & 0 & 11 \end{array}\right] \xrightarrow{\text{row}_2\mapsto\text{row}_2-2\text{row}_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -4 & 0 & 11 \end{array}\right]$$

$$\xrightarrow{\text{row}_3\mapsto\text{row}_3+\text{row}_1+\text{row}_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 12 \end{array}\right] \to \begin{cases} a+3b+2c=1 \\ \quad b+\ \ c=0 \\ \qquad\ 3c=12 \end{cases}$$

[10] Technically, we subtracted, but that's just adding a negative.

[11] If you're clever, you can think up alternatives to the elementary row operations that work just as well, but there's good reason to favor the three elementary row operations. We'll see them when discussing matrix decompositions.

The third row reveals that $c = 4$; substituting into the second row, we find $b = -4$. Now we can substitute $b = -4$ and $c = 4$ into the first row and we obtain $a = 5$.

Thus, the solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 4 \end{bmatrix}$. Since this is the only solution to the system, the solution set is $\left\{ \begin{bmatrix} 5 \\ -4 \\ 4 \end{bmatrix} \right\}$.

Example. Solve the system

$$\begin{cases} 3t + s + 13r = -2 \\ t \quad + 5r = 1 \\ -t + s - 6r = -8 \\ t + s + 4r = -6 \end{cases}.$$

Again, we row reduce the corresponding augmented matrix to find an equivalent system from which we can more easily compute the solution.

$$\begin{cases} 3t + s + 13r = -2 \\ t \quad + 5r = 1 \\ -t + s - 6r = -8 \\ t + s + 4r = -6 \end{cases} \to \left[\begin{array}{ccc|c} 3 & 1 & 13 & -2 \\ 1 & 0 & 5 & 1 \\ -1 & 1 & -6 & -8 \\ 1 & 1 & 4 & -6 \end{array}\right] \xrightarrow{\text{row}_1 \leftrightarrow \text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 3 & 1 & 13 & -2 \\ -1 & 1 & -6 & -8 \\ 1 & 1 & 4 & -6 \end{array}\right]$$

$$\xrightarrow{\text{row}_2 \mapsto \text{row}_2 - 3\text{row}_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -2 & -5 \\ -1 & 1 & -6 & -8 \\ 1 & 1 & 4 & -6 \end{array}\right]$$

$$\xrightarrow{\text{row}_3 \mapsto \text{row}_3 + \text{row}_1 - \text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 4 & -6 \end{array}\right]$$

$$\xrightarrow{\text{row}_4 \mapsto \text{row}_4 - \text{row}_1 - \text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{\text{row}_4 \mapsto \text{row}_4 - \text{row}_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{cases} t \quad + 5r = 1 \\ s - 2r = -5 \\ r = -2 \\ 0 = 0 \end{cases}$$

Our equivalent system reveals $r = -2$, which we can substitute back into the first and second rows to find that $t = 11$ and $s = -9$.

As a vector, the solution is $\begin{bmatrix} t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \\ -2 \end{bmatrix}$ and so the solution set is $\left\{ \begin{bmatrix} 11 \\ -9 \\ -2 \end{bmatrix} \right\}$.

In the examples so far, we've stopped row reducing when the equations became simple enough to solve by inspection. However, we could continue row reducing until the system is as simple as possible.

Example. Solve the system

$$\begin{cases} a + 3b + 2c = 1 \\ 2a + 7b + 5c = 2 \\ -a - 4b \quad = 11 \end{cases}$$

Notice that we solved this system using a combination of row reduction and substitution in a previous example. This time, let us use only row reduction.
The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 7 & 5 & 2 \\ -1 & -4 & 0 & 11 \end{array}\right].$$

Based on the work from the previous example, we know it can be reduced to

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 12 \end{array}\right].$$

Now let us continue row reducing.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 12 \end{array}\right] \xrightarrow{\text{row}_3 \mapsto \frac{1}{3}\text{row}_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{\text{row}_2 \mapsto \text{row}_2 - \text{row}_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{\text{row}_1 \mapsto \text{row}_1 - 3\text{row}_2 - 2\text{row}_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{array}\right] \to \begin{cases} a & = 5 \\ \quad b & = -4 \\ \qquad c & = 4 \end{cases}$$

The solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 4 \end{bmatrix}$ and the solution set is $\left\{ \begin{bmatrix} 5 \\ -4 \\ 4 \end{bmatrix} \right\}$, which is the same as we got before.

What happens when you apply row reduction to an inconsistent system? Let's see. Consider the system

$$\begin{cases} x + y = 1 \\ 4x + 4y = 7 \end{cases}. \tag{4}$$

Before continuing, convince yourself that this system is inconsistent. The augmented matrix for System (4) is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 4 & 7 \end{array}\right].$$

We apply the row operation $\text{row}_2 \mapsto \text{row}_2 - 4\text{row}_1$ to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right],$$

which corresponds to the system

$$\begin{cases} x + y = 1 \\ 0x + 0y = 3 \end{cases}.$$

But, the last equation says $0x + 0y = 3$, which is not true for any choice of x and y! Thus, we see applying row reduction to an inconsistent system reveals its inconsistency.

Example. Find a solution to the following system:

$$\begin{cases} x + z = 4 \\ x + y + 2z = -8 \\ x + 3y + 4z = -18 \end{cases}.$$

As usual we rewrite the system as an augmented matrix and then row reduce.

$$\begin{cases} x \phantom{{}+3y} + z = 4 \\ x + y + 2z = -8 \\ x + 3y + 4z = -18 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & -8 \\ 1 & 3 & 4 & -18 \end{array}\right] \xrightarrow{\text{row}_3 \mapsto \text{row}_3 - \text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & -8 \\ 0 & 2 & 2 & -10 \end{array}\right]$$

$$\xrightarrow{\text{row}_2 \mapsto \text{row}_2 - \text{row}_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 2 & 2 & -10 \end{array}\right]$$

$$\xrightarrow{\text{row}_3 \mapsto \text{row}_3 - 2\text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 0 & 0 & 14 \end{array}\right] \to \begin{cases} x \phantom{{}+0y} + z = 4 \\ \phantom{0x+{}} y + z = -12 \\ 0x + 0y + 0z = 14 \end{cases}$$

The equation $0x + 0y + 0z = 14$ is never true and so the system is inconsistent. Since there are no values of x, y, and z that satisfy the system, the solution set is $\{\}$, the empty set.

Practice Problems

1 For each equation given below, determine if it is a linear equation. If not, explain what makes it nonlinear.

(a) $\cos(4)x_1 + \mathrm{e}y_2 + \pi z_3 = \mathrm{e}^{\pi}$

(b) $4x_1 + 2x_2 + 5x_4 = 4x_2 + 4x_5 + 5$

(c) $5x + 2y + 8z = \cos(y)$

(d) $12x + 3xy + 5z = 2$

(e) $\cos(4)x + \sin(4)y = \tan(4)x$

(f) $\frac{x}{y} = 1$

2 Convert each vector equation given below to a system of linear equations.

(a) $x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix}$

(b) $x\begin{bmatrix} 7 \\ 16 \end{bmatrix} + y\begin{bmatrix} 8 \\ 13 \end{bmatrix} = \begin{bmatrix} 11 \\ 30 \end{bmatrix}$

(c) $\vec u + t\vec u - s(\vec v + \vec w) = \vec 0$ where $\vec u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\vec w = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

3 Convert each system of linear equations given below to a vector equation.

(a) $\begin{cases} \phantom{x_1 +{}} 4x_2 + 2x_3 = 0 \\ x_1 \phantom{{}+4x_2} + 2x_3 = 0 \\ \phantom{x_1 +{}} 9x_2 + 2x_3 = 1 \end{cases}$

(b) $\begin{cases} 0x + 0y + 0z = 0 \\ x + y + z = 3 \end{cases}$

4 Consider the vector equation $x\begin{bmatrix} 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} 8 \\ 16 \end{bmatrix} = \vec b$ where $\vec b$ is unknown.

(a) Show that if $\vec b = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$, the system is consistent.

(b) Are there other vectors $\vec b$ that make the system consistent? If so, how many? Justify your answer.

(c) Show that if $\vec b = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, the system is inconsistent.

(d) Are there other vectors $\vec b$ that make the system inconsistent? If so, how many? Justify your answer.

5 On Kokoro's farm, there is a cage with 35 animals, some of which are chickens and some of which are rabbits. Kokoro counted the total number of legs in the cage and found that there were 94 legs in all (notably, each chicken has exactly two legs and each rabbit has four legs). Kokoro decides to use this information to figure out how many chickens and how many rabbits there are.[12]

(a) Set up a system of linear equations that you could solve to answer Kokoro's question.

(b) Is the system consistent? If so, answer Kokoro's question.

(c) Kokoro wants to set up three other cages. For each described cage below, explain using complete English sentences, whether such a configuration is possible. Justify your answers using linear algebra.

i. Kokoro wants to set up a cage with *cats* and *dogs* (notably, each cat has exactly four legs and each dog has four legs) so that there are 35 animals in total, and the total number of legs is 94.

ii. Kokoro wants to set up a cage with *cats* and *dogs* so that there are 35 animals in total, and the total number of legs is 140.

iii. Kokoro wants to set up a cage with *chickens* and *rabbits* so that there are 42 animals in total, and the total number of legs is 77.

6 For each statement below, determine whether it is true or false. Justify your answer.

(a) A system of linear equations of 4 variables with 3 equations is always consistent.

(b) Any system of linear equation with $0x_1 + 0x_2 + \cdots + 0x_n = 0$ being one of the equations must be consistent.

(c) There are $m, c \in \mathbb{R}$ so that the y-axis is the solution set to the equation $y = mx + c$.

(d) There are $m, c \in \mathbb{R}$ so that the x-axis is the solution set to the equation $y = mx + c$.

[12] This problem based on a classical Chinese problem from the ancient Chinese treatise *Mathematical Classic of Master Sun* (or *Sunzi Suanjing*) written during 3rd to 5th centuries A.D.

(e) There are $m_1, m_2, c \in \mathbb{R}$ so that the x-axis (in $\mathbb{R}^3$) is the solution set to the equation $z = m_1 x + m_2 y + c$.

(f) A system of exactly one equation can have an empty solution set.

Practice Problem Solutions

Solutions for Appendix 1

1 (a) Linear equation.

(b) Linear equation.

(c) Not a linear equation because of the $\cos(y)$ term.

(d) Not a linear equation because of the $3xy$ term.

(e) Linear equation.

(f) Not a linear equation because of the $\frac{x}{y}$ term. Note that it is *almost* equivalent to the equation $x=y$, but they are not equivalent because $x=0, y=0$ is a solution to the latter equation but not the former.

2 (a) $\begin{cases} x \quad\quad + 4z = 2 \\ -x + y + 6z = -5 \\ \quad\quad\quad z = 2 \end{cases}$

(b) $\begin{cases} 7x + \;\; 8y = 11 \\ 16x + 13y = 30 \end{cases}$

(c) $\begin{cases} -5s + t = -1 \\ -3s + t = -1 \end{cases}$

3 (a) $x_1\begin{bmatrix}0\\1\\0\end{bmatrix} + x_2\begin{bmatrix}4\\0\\9\end{bmatrix} + x_3\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

(b) $x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + z\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$

4 (a) If $\vec{b} = \begin{bmatrix}7\\14\end{bmatrix}$, then the vector equation becomes

$$x\begin{bmatrix}2\\4\end{bmatrix} + y\begin{bmatrix}8\\16\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix}.$$

Converting it to a system of linear equations and row reducing we get

$$\begin{cases} 2x + \;\; 8y = 7 \\ 4x + 16y = 14 \end{cases} \to \begin{cases} x + 4y = 3.5 \\ 0x + 0y = 0 \end{cases}.$$

The solution to this system is then

$$\begin{cases} x = 3.5 - 4t \\ y = t \end{cases} \quad (t \in \mathbb{R}).$$

This system is consistent.

(b) There are vectors $\vec{b}$ that makes the system consistent. For instance, any vector $\vec{b} = \vec{b} = \begin{bmatrix}t\\2t\end{bmatrix}$ where $t \in \mathbb{R}$ makes the system consistent. Since there are infinitely many real numbers, we conclude that there are infinitely many vectors $\vec{b}$ that makes the system consistent.

(c) If $\vec{b} = \begin{bmatrix}5\\14\end{bmatrix}$, then the vector equation becomes

$$x\begin{bmatrix}2\\4\end{bmatrix} + y\begin{bmatrix}8\\16\end{bmatrix} = \begin{bmatrix}5\\12\end{bmatrix}.$$

Converting it to a system of linear equations and row reducing we get

$$\begin{cases} 2x + \;\; 8y = 5 \\ 4x + 16y = 12 \end{cases} \to \begin{cases} x + 4y = 2.5 \\ 0x + 0y = 2 \end{cases}.$$

This system is inconsistent.

(d) There are vectors $\vec{b}$ that makes the system inconsistent. For instance, $\begin{bmatrix}10\\24\end{bmatrix}$ is is such a vector. In general, any vector $\vec{b}$ with $\vec{b} = \begin{bmatrix}5t\\12t\end{bmatrix}$ where $t \in \mathbb{R}$ ($t \neq 0$) makes the system inconsistent. Since there are infinitely many real numbers, we conclude that there are infinitely many vectors $\vec{b}$ that makes the system inconsistent.

5 (a) Let x be the number of chickens, and let y be the number of rabbits. Using the information given in the problem, we have

$$\begin{cases} x + \;\; y = 35 \\ 2x + 4y = 94 \end{cases}.$$

(b) Row reducing

$$\begin{cases} x + \;\; y = 35 \\ 2x + 4y = 94 \end{cases},$$

we get

$$\begin{cases} x + y = 35 \\ \quad\;\; y = 12 \end{cases}.$$

This shows that the system is consistent. The solution to this system is $x = 23, y = 12$. Thus, there are 23 chickens and 12 rabbits in the farm.

(c) Before discussing each configuration, we point out that a configuration is possible if there exists a natural number solution to the system of linear equations associated with the configuration.

i. For the first configuration, let x be the number of cats, and let y be the number of dogs. Using the information given in the problem, we have

$$\begin{cases} x + \;\; y = 35 \\ 4x + 4y = 94 \end{cases}.$$

Row reducing this system, we get

$$\begin{cases} x + \;\; y = 35 \\ 0x + 0y = -46 \end{cases}.$$

This system is inconsistent, which means there's no solution to this system. Therefore, Kokoro's first configuration is not possible.

ii. For the second configuration, let x be the number of cats, and let y be the number of dogs. Using the information given in the problem, we have

$$\begin{cases} x + \;\; y = 35 \\ 4x + 4y = 140 \end{cases}.$$

Row reducing this system, we get

$$\begin{cases} x + \;\; y = 35 \\ 0x + 0y = 0 \end{cases}.$$

Appendix 2

This system is consistent, and the complete solution is given by

$$\begin{cases} x \quad\; = 35 - t \\ \quad\; y = t \end{cases} \quad (t \in \mathbb{R}).$$

Take $t = 1$, and we get a natural number solution $x = 34, y = 1$. (In fact, there is more than one natural number solution.) Therefore, Kokoro's second configuration is possible.

iii. For the third configuration, let x be the number of chickens, and let y be the number of rabbits. Using the information given in the problem, we have

$$\begin{cases} x + y = 42 \\ 2x + 4y = 77 \end{cases}.$$

Row reducing this system, we get

$$\begin{cases} x + y = 42 \\ \quad\; y = -\frac{7}{2} \end{cases}.$$

This system is consistent and the unique solution is $x = \frac{91}{2}, y = -\frac{7}{2}$ However, there cannot be $91/2$ of a chicken, so Kokoro's third configuration is not possible.

6 (a) False. A counterexample is given by

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + x_2 + x_3 + x_4 = 3 \end{cases}.$$

(b) False. A counterexample is given by

$$\begin{cases} 0x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 1 \end{cases}.$$

(c) False. Assume the y-axis can be represented as the complete solution to $y = mx + c$ for some m, c. Since $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$ are both on the y axis, we know $0 = 0m + c$ and $1 = 0m + c$. This gives $0 = 1$, which is false. Therefore, there's no $m, c \in \mathbb{R}$ so that the y-axis is the solution set to the equation $y = mx + c$.

(d) True. Take $m = 0$, $c = 0$. The equation then becomes $y = 0$. A complete solution to this equation is given by $\begin{bmatrix} t \\ 0 \end{bmatrix}$ $(t \in \mathbb{R})$, which is exactly the x-axis.

(e) False. The x-axis in $\mathbb{R}^3$ can be described as $\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3 : x \in \mathbb{R} \right\}$. Assume the x-axis can be represented as the complete solution to $z = m_1x + m_2y + c$ for some m_1, m_2, c. Since $(x, y, z) = (0, 0, 0)$ is on the x axis, we know $c = 0$. Since $(x, y, z) = (1, 0, 0)$ is on the x axis, we know that $m_1 = 0$. The equation then becomes $z = m_2y$. However, for each choice of m_2, $x = 0, y = 1, z = m_2$ is a solution to the system which does not lie in the x-axis. Therefore, there's no there's no $m_1, m_2, c \in \mathbb{R}$ so that the x-axis is the solution set to the equation $z = m_1x + m_2y + c$.

(f) True. An example is given by

$$\{0x + 0y = 1.$$

Indices

Index of Symbols

Index of Terms

Index of Definitions