Differential Equations

MAT244 Slides

2025/09/02 Edition

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Welcome to MAT244 *LEC0101*

Ordinary Differential Equations

Fall 2025

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Exercise 0

- Who took 135-136 vs 137?
- Who took 223&224 vs only 223?
- Who thought about math over the Summer?

Exercise 0

What is the goal of a university education?

Exercise 0

How does someone learn something new?

Exercise 0

What is the value of making mistakes in the learning process?

Exercise 1

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish. You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

Exercise 2

Let

$$\begin{aligned} \text{(Birth Rate)} \quad & K = 1.1 \text{ children per starfish per year} \\ \text{(Initial Pop.)} \quad & P_0 = 10 \text{ star fish} \end{aligned}$$

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

MS Excel Orientation (*appendix 1*)

- Drag to continue a pattern (with numbers)
- Drag to continue formulas
- Copy-paste changes the formulas
- $ sign in formulas
- Formulas re-compute automatically
- Graphing:
  - Use ctrl+click or command+click to select the two columns
  - Select Insert > Chart and pick “Scatter with lines”

- Add labels to cells.

Exercise 3

Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

Exercise 4

In the model $\mathbf{M}_1$, we assumed the starfish had K children at one point during the year.

We want to create a model $\mathbf{M}_n$ where the starfish are assumed to have K/n children n times per year (at regular intervals).

4.1 Let t_0, t_1, t_2, …, be the times that children are born in model $\mathbf{M}_n$. Find expressions for t_0, t_1, t_2, ….

4.2 Find a (recursive) formula that gives the population for model $\mathbf{M}_n$.

4.3 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.4 What happens to $\mathbf{M}_n$ as $n \to \infty$?

Exploring $\mathbf{M}_n$

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $\left(t, t+\frac{1}{n}\right)$ there will be (on average) $\frac{K}{n}$ new children per starfish.

5.1 Write an expression for $P_n\left(t+\frac{1}{n}\right)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP_n, the change in population from time t to $t+\Delta t$.

5.3 Write an expression for $\frac{\Delta P_n}{\Delta t}$.

5.4 Write down a differential equation relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_n(t)$.

Exercise 6

Recall the model $\mathbf{M}_1$ defined by:

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model $\mathbf{M}_\infty$ by:

- $P(0) = 10$
- $P'(t) = kP(t)$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

Exercise 7

Suppose that the estimates produced by $\mathbf{M}_1$ agree with the actual (measured) population of starfish.

Fill out the table with ✓ or ✕ indicating which models have which properties.

Model	Accuracy	Explanatory	(your favourite property)
$\mathbf{M}_1$			
$\mathbf{M}_1^*$			
$\mathbf{M}_\infty$			

Exercise 8

Recall the model $\mathbf{M}_1$ defined by:

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model $\mathbf{M}_\infty$ by:

- $P(0) = 10$
- $P'(t) = kP(t)$.

For this question, we will assume that that $\mathbf{M}_1$ accurately predicts the population.

8.1 If $k = K = 1.1$, does $\mathbf{M}_\infty$ underestimate or overestimate the population?

8.2 If $k = 0.5$, does $\mathbf{M}_\infty$ underestimate or overestimate the population?

8.3 Can you find a value of k so that $\mathbf{M}_\infty$ accurately predicts the population?

Exercise 9

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

9.1 What can you tell about the population (without trying to compute it)?

9.2 Assuming $k = 1.1$, estimate the population after 10 years.

9.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

Exercise 10

Consider the following argument for the population model **S** where $P'(t) = P(t) \cdot |\sin(2\pi t)|$ with $P(0) = 10$:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(1) + P'(1) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

10.1 Do you believe this argument? Can it be improved?

10.2 Simulate an improved version using a spreadsheet.

Exercise 11

(Simulating M_∞ from Core Exercise 6 with different Δs)

Time	Pop. ($\Delta = 0.1$)
0.0	10
0.1	11.1
0.2	12.321
0.3	13.67631
0.4	15.1807041

Time	Pop. ($\Delta = 0.2$)
0.0	10
0.2	12.2
0.4	14.884
0.6	18.15848
0.8	22.1533456

11.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

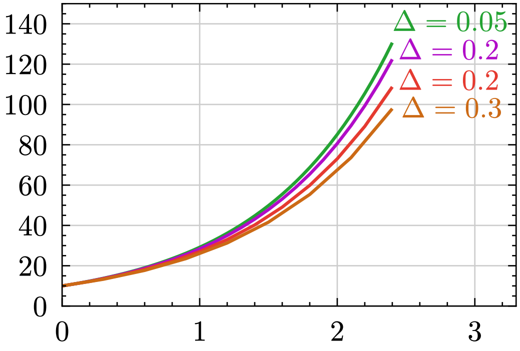
11.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

11.3 What Δs give the largest estimate for the population at time t?

11.4 Is there a limit as $\Delta \to 0$?

Exercise 11

(Simulating M_∞ with different Δs)



11.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

11.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

11.3 What Δs give the largest estimate for the population at time t?

11.4 Is there a limit as $\Delta \to 0$?

Exercise 12

Consider the following models for starfish growth:

M # new children per year $\sim$ current population.

N # new children per year $\sim$ current population times resources available per individual.

O # new children per year $\sim$ current population times the fraction of total resources remaining.

12.1 Model **N** introduces the concept of “resources available per individual”.

(a) Come up with a definition/notation/assumptions for this concept.

(b) Create a differential equation for model **N**.

12.2 Repeat the modelling process for model **O**.

12.3 Simulate population vs. time curves for each model.

Exercise 13

Recall the models

M # new children per year $\sim$ current population.

N # new children per year $\sim$ current population times resources available per individual.

O # new children per year $\sim$ current population times the fraction of total resources remaining.

13.1 Determine which population grows fastest in the short term and which grows fastest in the long term.

13.2 Are some models more sensitive to your choice of Δ when simulating?

13.3 Are your simulations for each model consistently underestimates? Overestimates? Do any results surprise you?

Exercise 14

A simple model for population growth has the form

$$P'(t) = b \cdot P(t)$$

where b is the birth rate.

14.1 Create a better model for population that includes both births and deaths.

Exercise 15

Lotka-Volterra Predator-Prey models predict two populations, F (foxes) and R (rabbits), simultaneously. They take the form

$$F'(t) = (B_F - D_F) \cdot F(t)$$
$$R'(t) = (B_R - D_R) \cdot R(t)$$

where $B_{\{?\}}$ stands for births and $D_{\{?\}}$ stands for deaths.

We will assume:

($\mathrm{P}_{\text{foxes 1}}$) Foxes die at a constant rate.
($\mathrm{P}_{\text{foxes 2}}$) Foxes mate when food is plentiful.
($\mathrm{P}_{\text{rabbits}}$) Rabbits mate at a constant rate.
($\mathrm{P}_{\text{predation}}$) Foxes eat rabbits.

15.1 Speculate on when B_F, D_F, B_R, and D_R would be at their maximum(s)/minimum(s), given our assumptions.

15.2 Come up with appropriate formulas for B_F, B_R, D_F, and D_R.

Exercise 16

Suppose the population of F (foxes) and R (rabbits) evolves over time following the rule

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

16.1 Simulate the population of foxes and rabbits with a spreadsheet.

16.2 Do the populations continue to grow/shrink forever? Are they cyclic?

16.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

Exercise 17

Open and make a copy of the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

17.1 Simulate the rabbit population using different step sizes Δ.

(a) Does the choice of Δ affect the qualitative "shape" of the population curve?

(b) Does it affect the height of the peaks and valleys?

(c) Does it affect the *time* when the peaks and valleys occur?

17.2 We want to know about the peaks and valleys of the *exact* population curve for rabbits.

Do your simulations consistently under or over estimate the population of rabbits?

17.3 Let p_1 and p_2 be the first and second local maxima for the (exact) rabbit population. Is p_1 bigger, smaller, or equal to p_2? Justify with numerical evidence.

Exercise 18

Open and make a copy of the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Component Graph & Phase Plane. For a differential equation involving the functions F_1, F_2, …, F_n, and the variable t, the ***component graphs*** are the n graphs of $(t, F_1(t))$, $(t, F_2(t))$, …, $(t, F_n(t))$.

The ***phase plane*** or ***phase space*** associated with the differential equation is the n-dimensional space with axes corresponding to the values of F_1, F_2, …, F_n.

18.1 Plot the Fox vs. Rabbit population in the phase plane.

18.2 Should your plot show a closed curve or a spiral?

18.3 What "direction" do points move along the curve as time increases? Justify by referring to the model.

18.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

Exercise 19

Open and make a copy of the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Equilibrium Solution. An ***equilibrium solution*** to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

19.1 By changing initial conditions, what is the "smallest" curve you can get in the phase plane? What happens at those initial conditions?

19.2 What should F' and R' be if F and R are *equilibrium solutions*?

19.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.

19.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

Exercise 20

Recall the logistic model for starfish growth (introduced in Core Exercise 12):

O # new children per year $\sim$ current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time t
- k is a constant of proportionality
- R is the total number of resources
- R_i is the resources that one starfish wants to consume

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

20.1 What are the equilibrium solutions for model **O**?

20.2 What does a "phase plane" for model **O** look like? What do graphs of equilibrium solutions look like?

20.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

Exercise 21

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if locally, solutions converge to f;
- *repelling* if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- *stable* if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- *unstable* if f is not stable.

Classification of Equilibria (Formal). An equilibrium solution f is called

- ***attracting at time*** t_0 if there exists $\varepsilon > 0$ such that for all solutions g satisfying $|g(t_0) - f(t_0)| < \varepsilon$, we have $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t)$.
- ***repelling at time*** t_0 if there exists $\varepsilon > 0$ and $\delta > 0$ such that for all solutions g that satisfy $0 < |g(t_0) - f(t_0)| < \varepsilon$ there exists $T \in \mathbb{R}$ so that for all $t > T$ we have $|g(t) - f(t)| > \delta$.
- ***stable at time*** t_0 if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all solutions g satisfying $|g(t_0) - f(t_0)| < \delta$ we have $|g(t) - f(t)| < \varepsilon$ for all $t > t_0$.
- ***unstable at time*** t_0 if f is not stable at time t_0.

f is called attracting/repelling/stable/unstable if it has the corresponding property for all t.

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if locally, solutions converge to f;
- *repelling* if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- *stable* if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- *unstable* if f is not stable.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

21.1 Draw an example of what solutions might look like if f is *attracting*.

21.2 Draw an example of what solutions might look like if f is *repelling*.

21.3 Draw an example of what solutions might look like if f is *stable*.

21.4 Could f be stable but *not* attracting?

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if locally, solutions converge to f;
- *repelling* if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- *stable* if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- *unstable* if f is not stable.

Recall the starfish population model **O** given by

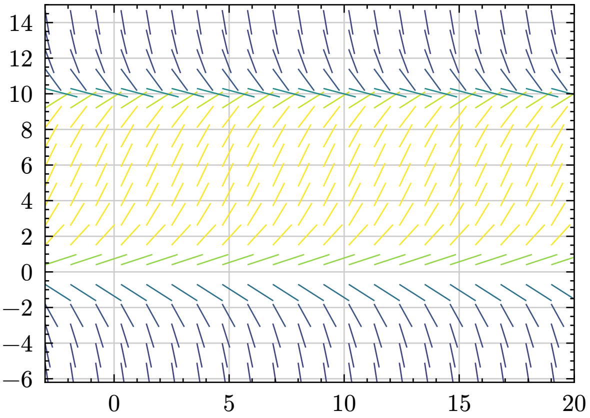
$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

22.1 Classify the equilibrium solutions for model **O** as attracting, repelling, stable, or unstable.

22.2 Does changing k change the nature of the equilibrium solutions? How can you tell?

Exercise 23



A *slope field* is a plot of small segments of tangent lines to solutions of a differential equation at different initial conditions.

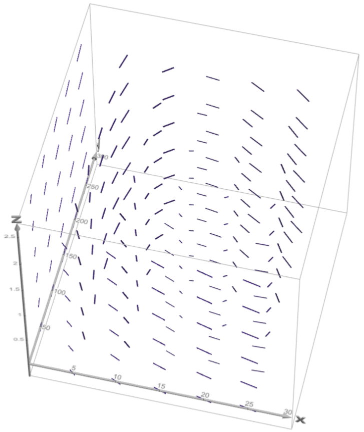
On the left is a slope field for model **O**, available at

`https://www.desmos.com/calculator/ghavqzqqjn`

23.1 If you were sketching the slope field for model **O** by hand, what (straight) line would you sketch a segment of at $(5, 3)$? Write an equation for that line.

23.2 How can you recognize equilibrium solutions in a slope field?

23.3 Give qualitative descriptions of different solutions to the *differential equation* used in model **O** (i.e., use words to describe them). Do all of those solutions make sense in terms of *model* **O**?



https://www.desmos.com/3d/kvyztvmp0g

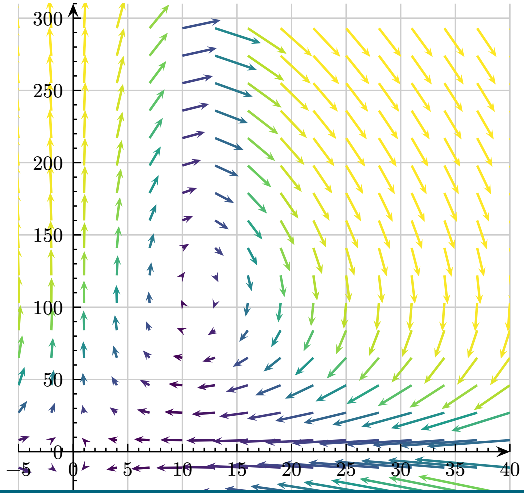
Three dimensional slope fields are possible, but hard to interpret. This is a slope field for the Foxes–Rabbits model.

24.1 What are the three dimensions in the plot?

24.2 What should the graph of an equilibrium solution look like?

24.3 What should the graph of a typical solution look like?

24.4 What are ways to simplify the picture so we can more easily analyze solutions?



Phase Portrait. A ***phase portrait*** or ***phase diagram*** is the plot of a vector field in phase space where each vector rooted at (x, y) is tangent to a solution curve passing through (x, y) and its length is given by the speed of a solution passing through (x, y).

This is a phase portrait for the Foxes–Rabbits model (introduced in Core Exercise 15).

`https://www.desmos.com/calculator/vrk0q4espx`

25.1 What do the x and y axes correspond to?

25.2 Identify the equilibria in the phase portrait. What are the lengths of the vectors at those points?

25.3 Classify each equilibrium as stable/unstable.

25.4 Copy and paste data from your simulation spreadsheet into the Desmos plot. Does the resulting curve fit with the picture?

Exercise 26

The unknown (continuous) system of differential equations $x' = \ldots$, $y' = \ldots$ has an *attracting* equilibrium solution

$$x_{\text{eq}}(t) = 2$$
$$y_{\text{eq}}(t) = 4$$

26.1 (a) Sketch component graphs for the equilibrium solution.

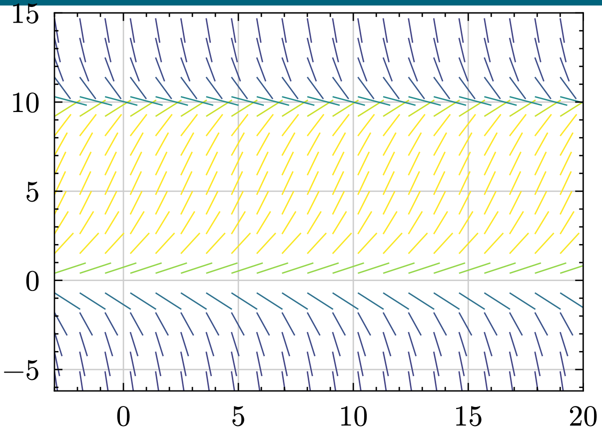
(b) Sketch the equilibrium in *phase space*.

26.2 Suppose $(x(t), y(t))$ is a solution that satisfies $(x(0), y(0)) = (3, 3)$. Sketch a possible graph for this solution. Make sketches in both component and phase spaces.

26.3 Draw a possible phase portrait for this system that agrees with your answer to the previous parts.

26.4 Sketch a phase portrait for a *new* system of differential equations that has a repelling equilibrium solution.

26.5 Sketch a phase portrait for a *new* system of differential equations that has no equilibrium solutions.



Recall the slope field for model **O**.

27.1 What would a phase portrait for model **O** look like? Draw it.

27.2 Where are the arrows the longest? Shortest?

27.3 How could you tell from a 1d phase portrait whether an equilibrium solution is attracting/repelling/etc.?

Exercise 28

The following differential equation models the life cycle of a tree. In the model

- $H(t) =$ height (in meters) of tree trunk at time t
- $A(t) =$ surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$.

28.1 Modify

`https://www.desmos.com/calculator/vrk0q4espx`

to make a phase portrait for the tree model.

28.2 What do equilibrium solutions mean in terms of tree growth?

28.3 For $b = 1$ what are the equilibrium solution(s)?

Exercise 29

The following differential equation models the life cycle of a tree. In the model

- $H(t) =$ height (in meters) of tree trunk at time t
- $A(t) =$ surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \le b \le 2$.

29.1 Fix a value of b and use a spreadsheet to simulate some solutions with different initial conditions. Plot the results on your phase portrait from 28.1.

29.2 What will happen to a tree with $(H(0), A(0)) = (20, 10)$? Does this depend on b?

29.3 What will happen to a tree with $(H(0), A(0)) = (10, 10)$? Does this depend on b?

Exercise 30

The tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

was based on the premises

($P_{\text{height 1}}$) CO_2 is absorbed by the leaves and turned directly into trunk height.

($P_{\text{height 2}}$) The tree is in a swamp and constantly sinks at a speed proportional to its height.

(P_{leaves}) Leaves grow proportionality to the energy available.

($P_{\text{energy 1}}$) The tree gains energy from the sun proportionally to the leaf area.

($P_{\text{energy 2}}$) The tree loses energy proportionally to the square of its height.

30.1 How are the premises expressed in the differential equations?

30.2 What does the parameter b represent (in the real world)?

30.3 Applying Euler's method to this system shows solutions that pass from the 1st to 4th quadrants of the phase plane. Is this realistic? Describe the life cycle of such a tree?

Exercise 31

Recall the tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$

$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

31.1 Find all equilibrium solutions for $0 \leq b \leq 2$.

31.2 For which b does a tree have the possibility of living forever? If the wind occasionally blew off a few random leaves, would that change your answer?

31.3 Find a value b_5 of b so that there is an equilibrium with $H = 5$.

Find a value b_{12} of b so that there is an equilibrium with $H = 12$.

31.4 Predict what happens to a tree near equilibrium (but not at equilibrium) when $b = b_5$. What about when $b = b_{12}$.

Exercise 32

Consider the system of differential equations

$$x'(t) = x(t)$$
$$y'(t) = 2y(t)$$

32.1 Make a phase portrait for the system.

`https://www.desmos.com/calculator/h3wtwjghv0`

32.2 What are the equilibrium solution(s) of the system?

32.3 Find a formula for $x(t)$ and $y(t)$ that satisfy the initial conditions $(x(0), y(0)) = (x_0, y_0)$.

32.4 Let $\vec{r}(t) = (x(t), y(t))$. Find a matrix A so that the differential equation can be equivalently expressed as

$$\vec{r}'(t) = A\vec{r}(t).$$

32.5 Write a solution to $\vec{r}' = A\vec{r}$ (where A is the matrix you came up with).

Hint: you already did most of the work!

Exercise 33

The *superposition principle* states that solutions to the matrix equation $\vec{r}' = A\vec{r}$ form a subspace.

33.1 Justify that if $\vec{p}$ and $\vec{q}$ are solutions to $\vec{r}' = A\vec{r}$, then so is $\vec{s}(t) = \vec{p}(t) + \vec{q}(t)$. Does this show that solutions to $\vec{r}' = A\vec{r}$ form a subspace? What is left to show?

33.2 Recall the differential equation $\vec{r}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\vec{r}$ from Core Exercise 32. Express the solutions you found as a span.

33.3 Let $\mathcal{S}$ be the set of all solutions to $\vec{r}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\vec{r}$ and consider the following theorem:

Theorem (Solution Space Dimension)

Let M be an $n \times n$ matrix and let $\mathcal{S}$ be the set of all solutions to $\vec{r}'(t) = M\vec{r}(t)$. Then $\dim(\mathcal{S}) = n$.

Use this theorem to justify that your span from 33.2 is equal to $\mathcal{S}$.

33.4 Let $\mathcal{K} = \operatorname{span}\left\{ \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}, \begin{bmatrix} 7e^t \\ 7e^{2t} \end{bmatrix} \right\}$. Is $\mathcal{K} = \mathcal{S}$?

Let $\mathcal{J} = \operatorname{span}\left\{ \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}, \begin{bmatrix} e^t \\ 7e^{2t} \end{bmatrix} \right\}$. Is $\mathcal{J} = \mathcal{S}$?

Justify your answers.

Exercise 34

Consider the system

$$x'(t) = 2x(t)$$
$$y'(t) = 3y(t)$$

34.1 Rewrite the system in matrix form.

34.2 Classify the following as solutions or non-solutions to the system.

$$\vec{r}_1(t) = e^{2t} \qquad \vec{r}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

$$\vec{r}_3(t) = \begin{bmatrix} e^{2t} \\ 4e^{3t} \end{bmatrix} \qquad \vec{r}_4(t) = \begin{bmatrix} e^{3t} \\ e^{2t} \end{bmatrix}$$

$$\vec{r}_5(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

34.1 State the definition of an eigenvector for the matrix M.

34.2 What should the definition of an *eigen solution* be for this system?

34.3 Which functions from 34.2 are eigen solutions?

34.4 Find an eigen solution $\vec{r}_6$ that is linearly independent from $\vec{r}_2$.

34.5 Let $S = \text{span}\{\vec{r}_2, \vec{r}_6\}$. Does S contain *all* solutions to the system? Justify your answer.

Exercise 35

Recall the system

$$x'(t) = 2x(t)$$
$$y'(t) = 3y(t)$$

has eigen solutions $\vec{r}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{r}_6(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$

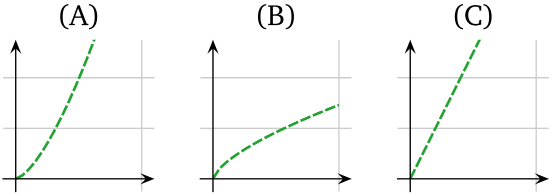
35.1 Sketch $\vec{r}_2$ and $\vec{r}_6$ in the phase plane.

35.2 Use

`https://www.desmos.com/calculator/h3wtwjghv0`

to make a phase portrait for the system.

35.3



In which phase plane above is the dashed (green) curve the graph of a solution to the system? Explain.

Exercise 36

Suppose A is a 2×2 matrix and $\vec{s}_1$ and $\vec{s}_2$ are eigen solutions to $\vec{r}' = A\vec{r}$ with eigenvalues 1 and -1, respectively.

36.1 Write possible formulas for $\vec{s}_1(t)$ and $\vec{s}_2(t)$.

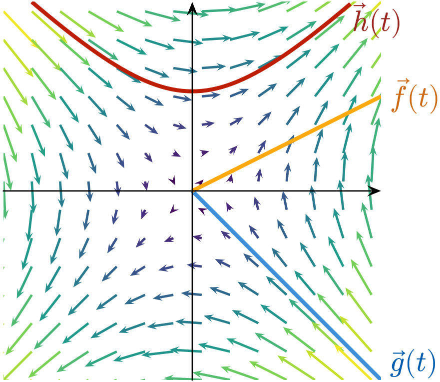
36.2 Sketch a phase plane with graphs of $\vec{s}_1$ and $\vec{s}_2$ on it.

36.3 Add a non-eigen solution to your sketch.

36.4 Sketch a possible phase portrait for $\vec{r}' = A\vec{r}$. Can you extend your phase portrait to all quadrants?

Exercise 37

Consider the following phase portrait for a system of the form $\vec{r}' = A\vec{r}$ for an unknown matrix A.



37.1 Identify which of $\vec{f}$, $\vec{g}$, and $\vec{h}$ are *solutions* to the differential equation. Which are *eigen solutions*?

37.2 Graph an additional eigen solution.

37.3 What can you say about the eigenvalues of A? What are their signs?

Exercise 38

Consider the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

38.1 Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors for M. What are the corresponding eigenvalues?

38.2 (a) Is $\vec{r}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

(b) Is $\vec{r}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

(c) Is $\vec{r}_3(t) = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

38.3 Find an eigen solution for the system corresponding to the eigenvalue -2. Write your answer in vector form.

38.4 Let $\vec{v}$ be an eigenvector for M with eigenvalue λ. Explain how to write down an eigen solution to $\vec{r}'(t) = M\vec{r}(t)$ with eigenvalue λ.

38.5 Let $\vec{v} \neq \vec{0}$ be a non-eigenvector for M. Could $\vec{r}(t) = e^{\lambda t}\vec{v}$ be a solution to $\vec{r}'(t) = M\vec{r}(t)$ for some λ? Explain.

Exercise 39

Recall the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

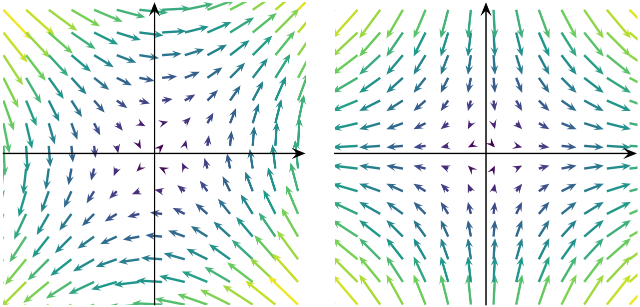
39.1 Write down a general solution to the differential equation.

39.2 Write down a solution to the initial value problem $\vec{r}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

39.3 Are your answers to the first two parts the same? Do they contain the same information?

Exercise 40

The phase portrait for a differential equation arising from the matrix $\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}$ (left) and $\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$ (right) are shown.



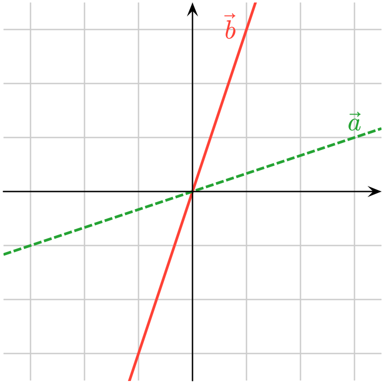
Both have eigenvalues ± 2, but they have different eigenvectors.

40.1 How are the phase portraits related to each other?

40.2 Suppose P is a 2×2 matrix with eigenvalues ± 2. In what ways could the phase portrait for $\vec{r}'(t) = P\vec{r}(t)$ look *different* from the above portraits? In what way(s) must it look the same?

Exercise 41

The lines with directions $\vec{a}$ (dashed green) and $\vec{b}$ (red) are graphs of eigen solutions to a differential equation.



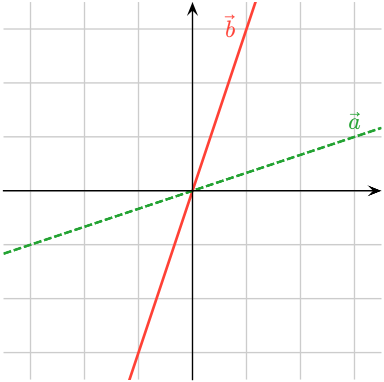
41.1 Suppose the eigenvalue for $\vec{a}$ is -1 and the eigenvalue for $\vec{b}$ is 1.

(a) Sketch five possible solutions to the differential equation and mark where each solution curve is traced out fastest vs. slowest.

(b) Sketch a phase portrait for the differential equation. Pay close attention to when the arrows are long vs. short.

Exercise 41

The lines with directions $\vec{a}$ (dashed green) and $\vec{b}$ (red) are graphs of eigen solutions to a differential equation.



41.2 Sketch a phase portrait where the eigenvalues associated with $\vec{a}$ and $\vec{b}$ are:

	sign for $\vec{a}$	sign for $\vec{b}$
1	neg	pos
2	pos	neg
3	pos	pos
4	neg	neg
5	pos	zero

41.3 Classify the solution at the origin for situations (1)–(5) as stable or unstable.

41.4 Would any of your classifications in the previous part change if the directions of $\vec{a}$ and $\vec{b}$ changed?

Exercise 42

You are examining a differential equation $\vec{r}'(t) = M\vec{r}(t)$ for an unknown 2×2 matrix M.

You would like to determine whether $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is stable, unstable, attracting, or repelling.

42.1 Come up with a rule to determine the nature of the equilibrium solution $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ based on the eigenvalues of M (provided there exist two linearly independent eigen solutions).

42.2 Consider the system of differential equations

$$x'(t) = x(t) + 2 \cdot y(t)$$
$$y'(t) = 3 \cdot x(t) - 4 \cdot y(t)$$

(a) Classify the stability of the equilibrium solution $(x(t), y(t)) = (0, 0)$ using any method you want.

(b) Justify your answer analytically using eigenvalues.

Exercise 43

Consider the following model of Social Media Usage where

$$P(t) = \text{millions of social media posts at year } \ t$$
$$U(t) = \text{millions of social media users at year } \ t$$

- ($P1_P$) Ignoring all else, each year posts decay proportionally to the current number of posts with proportionality constant 1.
- ($P2_P$) Ignoring all else (independent of decay), posts grow by a constant amount of 2 million posts every year.
- ($P1_U$) Ignoring all else, social media users increase/decrease in proportion to the number of posts.
- ($P2_U$) Ignoring all else, social media users increase/decrease in proportion to the number of users.
- ($P3_U$) Ignoring all else, 1 million people stop using the platform every year.

A school intervention is described by the parameter $a \in \left[-\frac{1}{2}, 1\right]$:

- After the intervention, the proportionality constant for ($P1_U$) is $1 - a$.
- After the intervention, the proportionality constant for ($P2_U$) is a.

43.1 Model this situation using a system of differential equations. Explain which parts of your model correspond to which premise(s).

Exercise 44

The **SM** model of Social Media Usage is

$$P' = -P + 2$$
$$U' = (1-a)P + aU - 1$$

where

$$P(t) = \text{millions of social media posts at year } t$$
$$U(t) = \text{millions of social media users at year } t$$
$$a \in \left[-\tfrac{1}{2}, 1\right]$$

44.1 What are the equilibrium solution(s)?

44.2 Make a phase portrait for the system.

https://www.desmos.com/calculator/h3wtwjghv0

44.3 Use phase portraits to conjecture: what do you think happens to the equilibrium solution(s) as a transitions from negative to positive? Justify with a computation.

Exercise 45

The **SM** model of Social Media Usage is

$$\begin{aligned} P' &= -P + 2 \\ U' &= (1-a)P + aU - 1 \end{aligned}$$

where

$$\begin{aligned} P(t) &= \text{millions of social media posts at year } \ t \\ U(t) &= \text{millions of social media users at year } \ t \\ a &\in \left[-\tfrac{1}{2}, 1\right] \end{aligned}$$

45.1 Can you rewrite the system in matrix form? I.e., in the form $\vec{r}\,'(t) = M\vec{r}(t)$ for some matrix M where $\vec{r}(t) = \begin{bmatrix} P(t) \\ U(t) \end{bmatrix}$.

45.2 Define $\vec{s}(t) = \begin{bmatrix} S_{P(t)} \\ S_{U(t)} \end{bmatrix}$ to be the displacement from equilibrium in the **SM** model at time t (provided an equilibrium exists).

(a) Write $\vec{s}$ in terms of P and U.

(b) Find $\vec{s}\,'$ in terms of P and U.

(c) Find $\vec{s}\,'$ in terms of S_P and S_U.

(d) Can one of your differential equations for $\vec{s}$ be written in matrix form? Which one?

(e) Analytically classify the equilibrium solution for your differential equation for $\vec{s}$ when $a = -\frac{1}{2}$, $a = \frac{1}{2}$, and $a = 1$. (You may use a calculator for computing eigenvectors/values.)

Exercise 46

The **SM** model of Social Media Usage is

$$\begin{aligned} P' &= -P + 2 \\ U' &= (1-a)P + aU - 1 \end{aligned}$$

where

$$\begin{aligned} P(t) &= \text{millions of social media posts at year } t \\ U(t) &= \text{millions of social media users at year } t \\ a &\in \left[-\tfrac{1}{2}, 1\right] \end{aligned}$$

Some politicians have been looking at the model. They made the following posts on social media:

1. *The model shows the number of posts will always be increasing. SAD!*
2. *I see the number of social media users always increases. That's not what we want!*
3. *It looks like social media is just a fad. Although users initially increase, they eventually settle down.*
4. *I have a dream! That one day there will be social media posts, but eventually there will be no social media users!*

46.1 For each social media post, make an educated guess about what initial conditions and what value(s) of a the politician was considering.

46.2 The school board wants to limit the number of social media users to fewer than 10 million. Make a recommendation about what value of a they should target.

Exercise 47

Consider the following **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$

- ($P1_F$) Ignoring all else, the number of parasites decays in proportion to its population (with constant 1).
- ($P2_F$) Ignoring all else, parasite numbers grow in proportion to the number of hosts (with constant 1 $\frac{\text{mega flea}}{\text{kilo dog}}$).
- ($P1_D$) Ignoring all else, hosts numbers grow in proportion to their current number (with constant 1).
- ($P2_D$) Ignoring all else, host numbers decrease in proportion to the number of parasites (with constant 2 $\frac{\text{kilo dog}}{\text{mega flea}}$).
- ($P1_c$) Anti-flea collars remove 2 million fleas per year.
- ($P2_c$) Constant dog breeding adds 1 thousand dogs per year.

47.1 Write a system of differential equations for the **FD** model.

47.2 Can you rewrite the system in matrix form $\vec{r}' = M\vec{r}$? What about in *affine* form $\vec{r}' = M\vec{r} + \vec{b}$?

47.3 Make a phase portrait for your model.

47.4 What should solutions to the system look like in the phase plane? What are the equilibrium solution(s)?

Exercise 48

Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix}$$

and

$$\vec{r}'(t) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \vec{r}(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Define $\vec{s}(t)$ to be the displacement of $\vec{r}(t)$ from equilibrium at time t.

48.1 Find a formula for $\vec{s}$ in terms of $\vec{r}$.

48.2 Can you find a matrix M so that $\vec{s}'(t) = M\vec{s}(t)$?

48.3 What are the eigenvalues of M?

48.4 Find an eigenvector for each eigenvalue of M.

48.5 What are the eigen solutions for $\vec{s}' = M\vec{s}$?

Exercise 49

Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}\,'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

This equation has eigen solutions

$$\vec{s}_1(t) = e^{it}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$
$$\vec{s}_2(t) = e^{-it}\begin{bmatrix} 1+i \\ 2 \end{bmatrix}.$$

49.1 Recall Euler's formula $e^{it} = \cos(t) + i\sin(t)$.

(a) Use Euler's formula to expand $\vec{s}_1 + \vec{s}_2$. Are there any imaginary numbers remaining?

(b) Use Euler's formula to expand $i(\vec{s}_1 - \vec{s}_2)$. Are there any imaginary numbers remaining?

49.2 Verify that your formulas for $\vec{s}_1 + \vec{s}_2$ and $i(\vec{s}_1 - \vec{s}_2)$ are solutions to $\vec{s}\,'(t) = M\vec{s}(t)$.

49.3 Can you give a third *real* solution to $\vec{s}\,'(t) = M\vec{s}(t)$?

Exercise 50

Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}\,'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

50.1 What is the dimension of the space of solutions to $\vec{s}\,'(t) = M\vec{s}(t)$?

50.2 Give a basis for all solutions to $\vec{s}\,'(t) = M\vec{s}(t)$.

50.3 Find a solution satisfying $\vec{s}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

50.4 Using what you know, find a general formula for $\vec{r}(t)$.

50.5 Find a formula for $\vec{r}(t)$ satisfying $\vec{r}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

Exercise 51

Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$

$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$

$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

Some research is being done on a shampoo for the dogs. It affects flea and dog reproduction:

- ($\mathrm{P}S_F$) Ignoring all else, the number of parasites decays in proportion to its population with constant $1 + a$.
- ($\mathrm{P}S_D$) Ignoring all else, hosts numbers grow in proportion to their current number with constant $1 - a$.
- $-1 \leq a \leq 1$.

These premises replace ($P1_F$) and ($P1_D$).

51.1 Modify the previous **FD** model to incorporate the effects of the shampoo.

51.2 Make a phase portrait for the **FD Shampoo** model.

51.3 Find the equilibrium solutions for the **FD Shampoo** model.

51.4 For each equilibrium solution determine its stability/instability/etc.

51.5 Analytically justify your conclusions about stability/instability/etc.

Exercise 52

Consider the differential equation

$$\vec{s}'(t) = M\vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$$

52.1 Make a phase portrait. Based on your phase portrait, classify the equilibrium solution.

`https://www.desmos.com/calculator/h3wtwjghv0`

52.1 Find eigen solutions for this differential equation (you may use a calculator/computer to assist).

52.2 Find a general *real* solution.

52.3 Analytically classify the equilibrium solution.

Exercise 53

Recall the tree model from Core Exercise 28:

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$

$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

A phase portrait for this model is available at `https://www.desmos.com/calculator/tvjag852ja`

53.1 Visually classify the stability of each equilibrium solution as attracting/repelling/etc. Does the stability depend on b? Are you confident in your visual assessment?

53.2 Can you rewrite the system in matrix/affine form? Why or why not?

Exercise 54

A simple logistic model $\mathbf{L}$ for a population is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(t) \cdot \left(1 - \frac{P(t)}{2}\right)$$

where $P(t)$ represents the population at time t.

We will focus on finding a simpler version of model $\mathbf{L}$ that works when $P \approx \frac{1}{2}$.

54.1 Define $f(P) = P \cdot \left(1 - \frac{P}{2}\right)$ and let $A_{\frac{1}{2}}(P)$ denote the affine approximation[1] to f centered at $P = \frac{1}{2}$. Find $A_{\frac{1}{2}}(P)$.

54.2 Notice that $\frac{\mathrm{d}P}{\mathrm{d}t} = f(P(t))$. Use this observation to create a "simple" model $\mathbf{L}_{\frac{1}{2}}$ that approximates $\mathbf{L}$ when $P \approx \frac{1}{2}$.

54.3 Model $\mathbf{L}_{\frac{1}{2}}$ is called an *affine approximation of model* $\mathbf{L}$ *centered at* $P = \frac{1}{2}$.

Find additional affine approximations to model $\mathbf{L}$ centered at each equilibrium solution.

[1]In calculus, this is called a "linear approximation".

Exercise 55

Based on our calculations from Core Exercise 54, we have several different affine approximations.

$$
\begin{aligned}
&(\text{Original } \mathbf{L}) && P' = P\left(1 - \tfrac{P}{2}\right) && (\texttt{https://www.desmos.com/calculator/v1coz4shtw})\\
&(\mathbf{L}_{\frac{1}{2}}) && P' \approx \tfrac{3}{8} + \tfrac{1}{2}\left(P - \tfrac{1}{2}\right) && (\texttt{https://www.desmos.com/calculator/zsb2apxhqs})\\
&(\mathbf{L}_0) && P' \approx P && (\texttt{https://www.desmos.com/calculator/vw48bvqgrc})\\
&(\mathbf{L}_2) && P' \approx -(P-2) && (\texttt{https://www.desmos.com/calculator/i2utk6vnqh})
\end{aligned}
$$

55.1 What are the similarities/differences in the Desmos plots of solutions to the original equation vs. the other equations?

55.2 Does the nature of the equilibrium solutions change when using an affine approximation?

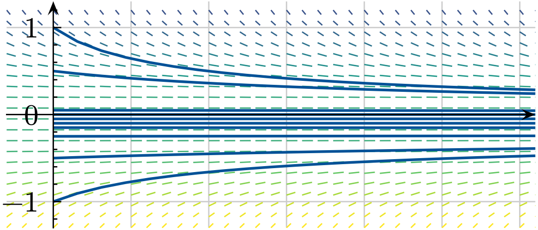
55.3 Classify each equilibrium solution of the original equation by using affine approximations.

Exercise 56

Consider the differential equation whose slope field is sketched below.

$$P'(t) = -P(t)\cdot(0.1+P(t))\cdot(0.2+P(t))$$
$$= -(P(t))^3 - 0.3\cdot(P(t))^2 - 0.02\cdot P(t)$$

`https://www.desmos.com/calculator/ikp9rgo0kv`



56.1 Find all equilibrium solutions.

56.2 Use affine approximations to classify the equilibrium solutions as stable/unstable/etc.

Exercise 57

To make a 1d affine approximation of a function f at the point E we have the formula

$$f(x) \qquad \approx \qquad f(E) + f'(E)(x - E).$$

To make a 2d approximation of a function $\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$ at the point $\vec{E}$, we have a similar formula

$$\vec{F}(x, y) \qquad \approx \qquad \vec{F}\left(\vec{E}\right) + D_{\vec{F}}\left(\vec{E}\right)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \vec{E}\right)$$

where $D_{\vec{F}}\left(\vec{E}\right)$ is the *total derivative* of $\vec{F}$ at $\vec{E}$, which can be expressed as the matrix

$$D_{\vec{F}}\left(\vec{E}\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

evaluated at $\vec{E}$.

Exercise 57

Recall our model from Exercise Core Exercise 28 for the life cycle of a tree where $H(t)$ was height, $A(t)$ was the leaves' surface area, and t was time:

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

with $0 \leq b \leq 2$

We know the following:

- The equations cannot be written in matrix form.
- The equilibrium points are $(0,0)$ and $\left(\frac{100}{9}b, \frac{1000}{27}b^2\right)$.

We want to find an affine approximation to the system.

Define $\vec{F}(H,A) = (H', A')$

57.1 Find the matrix for $D_{\vec{F}}$, the total derivative of $\vec{F}$.

57.2 Create an affine approximation to $\vec{F}$ around $\vec{e} = (0,0)$ and use this to write an approximation to the original system.

57.3 In the original system, the equilibrium $(0,0)$ is unstable and not repelling. Justify this using your affine approximation.

57.4 Create an affine approximation to $\vec{F}$ around $\vec{e} = \left(\frac{100}{9}b, \frac{1000}{27}b^2\right)$ and use this to write an approximation to the original system.

57.5 Make a phase portrait for the original system and your approximation from part 57.4. How do they compare?

57.6 Analyze the nature of the equilibrium solution in part 57.4 using eigen techniques. Relate your analysis to the original system.

Exercise 58

Define $\vec{F}(x,y) = \begin{bmatrix} y \\ -xy+x^2-x-y \end{bmatrix}$ and consider the differential equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \vec{F}(x,y).$$

58.1 Make a phase portrait for this differential equation. Based on your phase portrait, can you determine the nature of the equilibrium at $(0,0)$?

`https://www.desmos.com/calculator/peby3xd7jj`

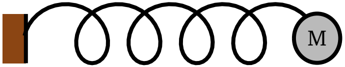
58.2 Find an affine approximation to $\vec{F}$ centered at $(0,0)$.

58.3 Write down a differential equation that approximates the original equation near $(0,0)$.

58.4 Analyze the nature of the equilibrium solution $\vec{r}(t) = (0,0)$ using eigen techniques. (You may use a computer to assist in eigen computations.) Relate your analysis to the original system.

Exercise 59

Consider a spring with a mass attached to the end.



Let $x(t) =$ displacement to the right of the spring from equilibrium at time t.

Recall from Physics the following laws:

- (HL) Hooke's Law: For an elastic spring, force is proportional to negative the displacement from equilibrium.
- (NL) Newton's Second Law: Force is proportional to acceleration (the proportionality constant is called mass).
- (ML) Laws of Motion: Velocity is the time derivative of displacement and acceleration is the time derivative of velocity.

59.1 Model $x(t)$ with a differential equation.

For the remaining parts, assume the elasticity of the spring is $k = 1$ and the mass is 1.

59.2 Suppose the spring is stretched 0.5m from equilibrium and then let go (at time $t = 0$).

(a) At $t = 0$, what are x, x', and x''?

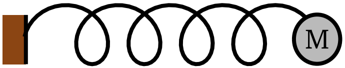
(b) Modify Euler's method to approximate a solution to the initial value problem.

59.3 Introduce the auxiliary equation $y = x'$. Can the second-order spring equation be rewritten as a first-order system involving x' and y'? If so, do it.

59.4 Simulate the *system* you found in the previous part using Euler's method.

Exercise 60

Recall a spring with a mass attached to the end.



$x(t)$ = displacement to the right of the spring from equilibrium at time t

We have two competing models

$$x'' = -kx \tag{A}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{B}$$

where $y = x'$

60.1 Make a phase portrait for system (B). What are the axes on the phase portrait? What do you expect general solutions to look like?

60.2 Use eigenvalues/eigenvectors to find a general solution to (B). (You may use a computer to compute eigenvalues/vectors.)

60.3 Use your solution to (B) to find a general solution to (A).

Exercise 61

Consider the second-order differential equation

$$x'' = -(1+x)\cdot x' + x^2 - x$$

61.1 Rewrite the second-order differential equation as a system of first-order differential equations. (Hint: you may need to introduce an auxiliary equation.)

61.2 The following Desmos link plots a phase portrait and draws an Euler approximation on the phase portrait:

`https://www.desmos.com/calculator/fvqxqp6eds`

Use the link to make a phase portrait for your system and answer the following questions:

(a) Are there initial conditions with $x(0) < 0$ so that a solution $x(t)$ is always increasing?

(b) Are there initial conditions with $x(0) < 0$ so that a solution $x(t)$ first decreases and then increases?

61.3 Show that $x(t) = 0$ is an equilibrium solution for this equation.

61.4 Use linearization and eigenvalues to classify the equilibrium $(x, x') = (0, 0)$ in phase space.

61.5 Let $x(t)$ be a solution to the original equation and suppose $x(0) = \delta_1 \approx 0$.

(a) If $x'(0) = \delta_2 \approx 0$, speculate on the long term behaviour of $x(t)$.

(b) If we put no conditions on $x'(0)$ will your answer be the same? Explain.

Boundary Value Problems

Recall the spring-mass system modeled by

$$x'' = -x$$

We would like to use the spring-mass system to ring a bell at regular intervals, so we put a hammer at the end of the spring. Whenever the displacement is maximal, the hammer strikes a bell producing a ring.

62.1 Convert the spring-mass system into a system of differential equations. Make a phase portrait for the system using the following Desmos link:

`https://www.desmos.com/calculator/fvqxqp6eds`

62.2 In the *Options Euler* on Desmos, adjust Δ and the number of steps so that simulated solutions are only shown for $t \in [0, 1]$.

Use simulations to answer the remaining questions.

62.3 You start by displacing the hammer by 1m and letting go. Is it possible that the bell rings every 1 second?

62.4 You start by displacing the hammer by 1m and giving the hammer a push. Is it possible that the bell rings every 1 second?

62.5 What is the smallest amount of time between consecutive rings (given a positive displacement)?

Boundary Value Problems

Recall the spring-mass system modeled by

$$x'' = -x$$

We would like to use the spring-mass system to ring a bell at regular intervals, so we put a hammer at the end of the spring. Whenever the displacement is maximal, the hammer strikes a bell producing a ring.

The general solution to the spring-mass system can also be written as

$$x(t) = A\cos(t+d)$$

where $A, d \in \mathbb{R}$ are parameters.

Analytically answer the remaining questions.

63.1 You start by displacing the hammer by 1m and letting go. Is it possible that the bell rings every 1 second?

63.2 You start by displacing the hammer by 1m and giving the hammer a push. Is it possible that the bell rings every 1 second?

63.3 What is the smallest amount of time between consecutive rings (given a positive displacement)?

Boundary Value Problems

A boundary value problem is a differential equation paired with two conditions at different values of t.

Consider the following boundary value problems:

(i)	(ii)	(iii)
$x'' = -x$	$x'' = -x$	$x'' = -x$
$x(0) = 1$	$x(0) = 1$	$x(0) = 1$
$x(\pi) = 1$	$x(\pi) = -1$	$x(\frac{\pi}{2}) = 0$

64.1 Using phase portraits and simulations, determine how many solutions each boundary value problem has.

64.2 Can you find analytic arguments to justify your conclusions?

Existence and Uniqueness

Whether a solution to a differential equation exists or is unique is a *hard* question with many partial answers.

Theorem (Existence and Uniqueness II)

Let $F(t,x,x')=0$ with $x(t_0)=x_0$ describe an initial value problem.

- IF $F(t,x,x')=x'(t)+p(t)x(t)+g(t)$ for some functions p and g
- AND p and g are continuous on an open interval I containing t_0
- THEN the initial value problem has a unique solution on I.

65.1 The theorem expresses differential equations in the form $F(t,x,x',x'',\ldots)=0$ (i.e. as a level set of some function F).

Rewrite the following differential equations in the form $F(t,x,x',x'',\ldots)=0$:

(a) $x''=-kx$

(b) $x''=-x\cdot x'+x^2$

(c) $x'''=(x')^2-\cos x$

65.2 Which of the following does the theorem say *must* have a unique solution on an interval containing 0?

(a) $y'=\frac{3}{2}y^{\frac{1}{3}}$ with $y(0)=0$

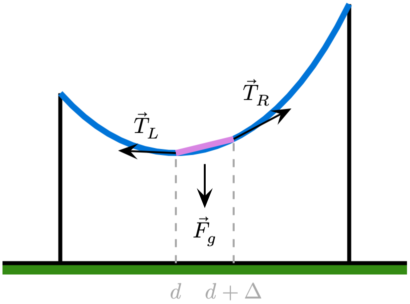
(b) $x'(t)=\lfloor t\rfloor x(t)$ with $x(0)=0$

(c) $x'(t)=\left\lfloor t-\frac{1}{2}\right\rfloor x(t)+t^2$ with $x(0)=0$

Note: $\lfloor x\rfloor$ is the *floor* of x, i.e., the largest integer less than or equal to x.

Exercise 66

Consider a rope hanging from two poles.



$H(d) =$ height of the rope above ground at position d.

We will consider the following premises and physics laws:

- (P_D) The linear density of the rope is constant: ρ kg/m
- (P_G) Gravity pulls downwards in proportion to mass (the proportionality constant is called g)
- (P_T) Tension pulls tangentially to the rope
- (P_{NL}) Newton's First Law: a body at rest will remain at rest unless it is acted upon by a force

To model the rope, imagine it is made of **small rigid rods**. We will focus on one such rod, S, (drawn in the figure) from d to $d+\Delta$.

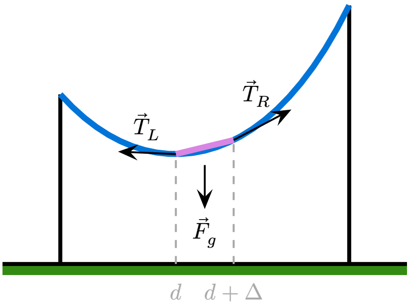
66.1 Given (P_{NL}), find a relation between the force vectors $\vec{T}_L$, $\vec{T}_R$, $\vec{F}_g$.

66.2 Approximate the length of the segment **S** and its mass. Approximate the vector $\vec{F}_g$.

66.3 Find a vector $\vec{V}_L$ in the direction of $\vec{T}_L$ (the magnitude doesn't matter at this point).

Exercise 67

Consider a rope hanging from two poles.



The only forces acting on the rope are gravity and tension.

Similarly to the previous exercise, we can find a vector $\vec{V}_R = \begin{bmatrix} 1 & H'(d+\Delta) \end{bmatrix}$ in the direction of $\vec{T}_R$, but with possibly different magnitude.

So far we have:

- $\vec{T}_L = \alpha \vec{V}_L$ for some $\alpha > 0$, and
- $\vec{T}_R = \beta \vec{V}_R$ for some $\beta > 0$.

67.1 Can you find approximations of the vectors $\vec{F}_g$, $\vec{T}_L$, $\vec{T}_R$ that only use $H(d)$, $H'(d)$, and $H''(d)$?

Hint:

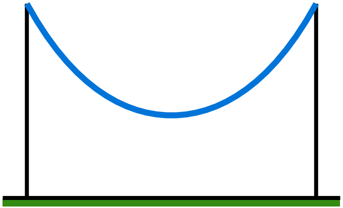
- $H(d+\Delta) \approx H(d) + \Delta \cdot H'(d)$,
- $H'(d+\Delta) \approx H'(d) + \Delta \cdot H''(d)$.

67.2 Put everything together to find a (second order) differential equation for H.

67.3 Do α or β depend on d? Explain.

Exercise 68

Recall a rope hanging from two poles.



$H(d) =$ the height of the rope at position d.

We have the following model for it:

$$H''(d) = k\sqrt{1 + (H'(d))^2}$$

Toronto Hydro is stringing some wire. The posts are 20m apart and at a height of 10m. At the lowest point, the wire is 5m above the ground.

68.1 Set up a boundary value problem that can be used to find the total length of the wire.

68.2 Using the following Desmos link, can you find a solution to the boundary value problem?

`https://www.desmos.com/calculator/l4fair6454`

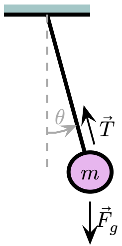
68.3 It happens that $k = \frac{\rho g}{T}$ where T is the minimum tension in the rope.

Suppose Toronto Hydro hung the wires so that they were at minimum 9m above the ground. Would the tension be higher or lower? By how much?

68.4 Should the difference between maximum and minimum tension be higher or lower for low-hanging wires? What does your intuition say? What does the phase portrait say?

Exercise 69

Consider a pendulum as in the figure below.



$\theta(t)$ = the angle the pendulum makes with the vertical axis (positive in the counterclockwise direction and negative in the clockwise direction).

Assume the pendulum is composed of a weightless rigid rod of length 1m and a mass of 1kg at its end. In addition assume:

- (P_G) Gravity pulls downwards in proportion to mass (the proportionality constant is called g).
- (P_T) Tension pulls the mass in the direction of the rod.
- (P_{NL}) Newton's Second Law: Force is proportional to acceleration (the proportionality constant is called mass).
- (P_{ML}) Laws of Motion: Velocity is the time derivative of displacement and acceleration is the time derivative of velocity.

69.1 Let $\theta(t)$ be the angle at time t and let $\vec{r}(t)$ be the mass's position at time t.

Find $\vec{r}(t)$ and $\vec{r}''(t)$ in terms of $\theta(t)$.

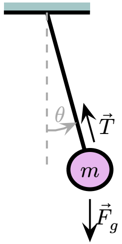
69.2 Find the vector $\vec{F}_g$.

69.3 Find a vector $\vec{T}_d$ so that $\vec{T} = \alpha\vec{T}_d$ for some $\alpha > 0$.

69.4 Find a second-order differential equation for the pendulum. *Hint: (P_{NL}) gives you an equation for each coordinate. Solve one for $(\theta')^2$ and substitute it into the other equation.*

Exercise 70

Consider a pendulum as in the figure below.



$\theta(t)$ = the angle the pendulum makes with the vertical axis (positive in the counterclockwise direction and negative in the clockwise direction).

If we had preserved length and mass in our derivation, we would have the following model:

$$\theta''(t) = -\left(\frac{g}{L}\right)\sin(\theta(t))$$

Let (P) be the corresponding system of first-order differential equations. The following Desmos link is already set up with (P).

`https://www.desmos.com/calculator/acmiingcqf`

70.1 If $L = 3\text{m}$, and you set the pendulum in motion at $\theta = 0$ by giving it a **small** push, what does the motion look like?

70.2 If $L = 3\text{m}$, and you set the pendulum in motion at $\theta = 0$ by giving it a **big** push, what does the motion look like?

70.3 Why are there infinitely many equilibrium solutions? Based on your physical intuition, which equilibria are stable and which are unstable?

70.4 Find an affine approximation to (P) around $(\theta, \theta') = (0, 0)$.

70.5 Physicists often claim that $\theta(t)$ oscillates like a sine wave with period $2\pi\sqrt{\frac{L}{g}}$. Under what conditions are the (mostly) correct?

