

Differential Equations

MAT244 Notes

Jason Siefken
Bernardo Galvão-Sousa



1 You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\text{\#new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

2 Let

(Birth Rate) $K = 1.1$ children per starfish per year

(Initial Pop.) $P_0 = 10$ star fish

and define the model \mathbf{M}_1 to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

3 Recall the model \mathbf{M}_1 (from the previous question).

Define the model \mathbf{M}_1^* to be

$$P(t) = P_0 e^{0.742t}$$

- 3.1 Are \mathbf{M}_1 and \mathbf{M}_1^* different models or the same?
- 3.2 Which of \mathbf{M}_1 or \mathbf{M}_1^* is better?
- 3.3 List an advantage and a disadvantage for each of \mathbf{M}_1 and \mathbf{M}_1^* .

4 In the model \mathbf{M}_1 , we assumed the starfish had K children at one point during the year.

- 4.1 Create a model \mathbf{M}_n where the starfish are assumed to have K/n children n times per year (at regular intervals).
- 4.2 Simulate the models \mathbf{M}_1 , \mathbf{M}_2 , \mathbf{M}_3 in Excel. Which grows fastest?
- 4.3 What happens to \mathbf{M}_n as $n \rightarrow \infty$?

5 Exploring \mathbf{M}_n

We can rewrite the assumptions of \mathbf{M}_n as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t + 1/n)$ there will be (on average) K/n new children per starfish.

- 5.1 Write an expression for $P_n(t + 1/n)$ in terms of $P_n(t)$.
- 5.2 Write an expression for ΔP_n , the change in population from time t to $t + \Delta t$.
- 5.3 Write an expression for $\frac{\Delta P_n}{\Delta t}$.
- 5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n \rightarrow \infty} P_n(t)$.

6 Recall the model \mathbf{M}_1 defined by

- $P_1(0) = 10$
- $P_1(t + 1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model \mathbf{M}_∞ by

- $P(0) = 10$
- $P'(t) = kP(t)$.

6.1 If $k = K = 1.1$, does the model \mathbf{M}_∞ produce the same population estimates as \mathbf{M}_1 ?

7 Suppose that the estimates produced by \mathbf{M}_1 agree with the actual (measured) population of starfish.

Fill out the table indicating which models have which properties.

Model	Accuracy	Explanatory	(your favourite property)
\mathbf{M}_1			
\mathbf{M}_1^*			
\mathbf{M}_∞			

8 Recall the model \mathbf{M}_1 defined by

- $P_1(0) = 10$
- $P_1(t + 1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model \mathbf{M}_∞ by

- $P(0) = 10$
- $P'(t) = kP(t)$.

8.1 Suppose that \mathbf{M}_1 accurately predicts the population. Can you find a value of k so that \mathbf{M}_∞ accurately predicts the population?

9 After more observations, scientists notice a seasonal effect on starfish. They propose a new model called \mathbf{S} :

- $P(0) = 10$

- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

9.1 What can you tell about the population (without trying to compute it)?

9.2 Assuming $k = 1.1$, estimate the population after 10 years.

9.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

10

Consider the following argument for the population model **S** where $P'(t) = P(t) \cdot |\sin(2\pi t)|$ with $P(0) = 10$:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(1) + P'(1) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

10.1 Do you believe this argument? Can it be improved?

10.2 Simulate an improved version using a spreadsheet.

11

(Simulating M_∞ with different Δ s)

Time	Pop. ($\Delta = 0.1$)	Time	Pop. ($\Delta = 0.2$)
0.0	10	0.0	10
0.1	11.1	0.2	12.2
0.2	12.321	0.4	14.884
0.3	13.67631	0.6	18.15848
0.4	15.1807041	0.8	22.1533456

11.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

11.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

11.3 What Δ s give the largest estimate for the population at time t ?

11.4 Is there a limit as $\Delta \rightarrow 0$?

(Simulating M_∞ with different Δ s)



- 11.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?
- 11.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?
- 11.3 What Δ s give the largest estimate for the population at time t ?
- 11.4 Is there a limit as $\Delta \rightarrow 0$?

12 Consider the following models for starfish growth

M # new children per year \sim current population

N # new children per year \sim current population times resources available per individual

O # new children per year \sim current population times the fraction of total resources remaining

- 12.1 Guess what the population vs. time curves look like for each model.
- 12.2 Create a differential equation for each model.
- 12.3 Simulate population vs. time curves for each model (but pick a common initial population).

13 Recall the models

M # new children per year \sim current population

N # new children per year \sim current population times resources available per individual

O # new children per year \sim current population times the fraction of total resources remaining

- 13.1 Determine which population grows fastest in the short term and which grows fastest in the long term.
- 13.2 Are some models more sensitive to your choice of Δ when simulating?
- 13.3 Are your simulations for each model consistently underestimates? Overestimates?
- 13.4 Compare your simulated results with your guesses from question 12.1. What did you guess correctly? Where were you off the mark?

14 A simple model for population growth has the form

$$P'(t) = bP(t)$$

where b is the *birth rate*.

- 14.1 Create a better model for population that includes both births and deaths.

The *Lotka-Volterra Predator-Prey* models two populations, F (foxes) and R (rabbits), simultaneously. It takes the form

$$\begin{aligned}F'(t) &= (B_F - D_F) \cdot F(t) \\R'(t) &= (B_R - D_R) \cdot R(t)\end{aligned}$$

where B_γ stands for births and D_γ stands for deaths.

We will assume:

- Foxes die at a constant rate.
- Foxes mate when food is plentiful.
- Rabbits mate at a constant rate.
- Foxes eat rabbits.

15.1 Speculate on when B_F , D_F , B_R , and D_R would be at their maximum(s)/minimum(s), given our assumptions.

15.2 Come up with appropriate formulas for B_F , B_R , D_F , and D_R .

Suppose the population of F (foxes) and R (rabbits) evolves over time following the rule

$$\begin{aligned}F'(t) &= (0.01 \cdot R(t) - 1.1) \cdot F(t) \\R'(t) &= (1.1 - 0.1 \cdot F(t)) \cdot R(t)\end{aligned}$$

16.1 Simulate the population of foxes and rabbits with a spreadsheet.

16.2 Do the populations continue to grow/shrink forever? Are they cyclic?

16.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

Open the spreadsheet

<https://uoft.me/foxes-and-rabbits>

which contains an Euler approximation for the Foxes and Rabbits population.

$$\begin{aligned}F'(t) &= (0.01 \cdot R(t) - 1.1) \cdot F(t) \\R'(t) &= (1.1 - 0.1 \cdot F(t)) \cdot R(t)\end{aligned}$$

17.1 Is the max population of the rabbits over/under estimated? Sometimes over, sometimes under?

17.2 What about the foxes?

17.3 What about the min populations?

18

Open the spreadsheet

<https://uoft.me/foxes-and-rabbits>

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Component Graph & Phase Plane

DEFINITION

For a differential equation involving the functions F_1, F_2, \dots, F_n , and the variable t , the **component graphs** are the n graphs of $(t, F_1(t)), (t, F_2(t)), \dots$

The **phase plane** or **phase space** associated with the differential equation is the n -dimensional space with axes corresponding to the values of F_1, F_2, \dots, F_n .

- 18.1 Plot the Fox vs. Rabbit population in the *phase plane*.
- 18.2 Should your plot show a closed curve or a spiral?
- 18.3 What “direction” do points move along the curve as time increases? Justify by referring to the model.
- 18.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

19

Open the spreadsheet

<https://uoft.me/foxes-and-rabbits>

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$

$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Equilibrium Solution

DEF

An **equilibrium solution** to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

- 19.1 By changing initial conditions, what is the “smallest” curve you can get in the phase plane? What happens at those initial conditions?
- 19.2 What should F' and R' be if F and R are *equilibrium solutions*?
- 19.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.
- 19.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

20

Recall the logistic model for starfish growth:

- # new children per year \sim current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time t
- k is a constant of proportionality
- R is the total number of resources

- R_i is the resources that one starfish wants to consume

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

- 20.1 What are the equilibrium solutions for model **O**?
- 20.2 What does a “phase plane” for model **O** look like? What do graphs of equilibrium solutions look like?
- 20.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

21

Classification of Equilibria

An equilibrium solution f is called

DEFINITION

- **attracting** if solutions locally converge to f
- **repelling** if solutions locally diverge from f
- **stable** if solutions do not locally diverge from f
- **unstable** if solutions do not locally converge to f
- **semi-stable** if solutions locally converge to f from one side and locally diverge from f on another.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

- 21.1 Draw an example of what solutions might look like if f is *attracting*.
- 21.2 Draw an example of what solutions might look like if f is *repelling*.
- 21.3 Draw an example of what solutions might look like if f is *stable*.
- 21.4 Could f be stable but *not* attracting?

22

Classification of Equilibria

An equilibrium solution f is called

DEFINITION

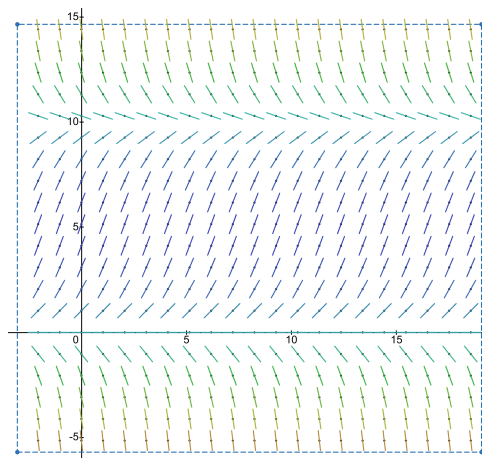
- **attracting** if solutions locally converge to f
- **repelling** if solutions locally diverge from f
- **stable** if solutions do not locally diverge from f
- **unstable** if solutions do not locally converge to f
- **semi-stable** if solutions locally converge to f from one side and locally diverge from f on another.

Recall the starfish population model **O** given by

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

- 22.1 Classify the equilibrium solutions for model **O** as attracting/repelling/stable/unstable/semi-stable.
- 22.2 Does changing k change the nature of the equilibrium solutions? How can you tell?

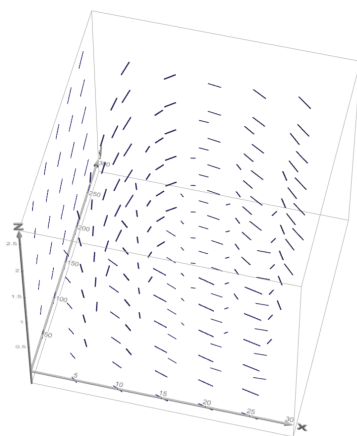


A *slope field* is a plot of small segments of tangent lines to solutions of a differential equation at different initial conditions.

On the left is a slope field for model **O**, available at

<https://www.desmos.com/calculator/ghavqzqqjn>

- 23.1 If you were sketching the slope field for model **O** by hand, what line would you sketch (a segment of) at $(5, 3)$? Write an equation for that line.
- 23.2 How can you recognize equilibrium solutions in a slope field?
- 23.3 Describe different solutions to the *differential equation* using words. Do all of those solutions make sense in terms of *model O*?

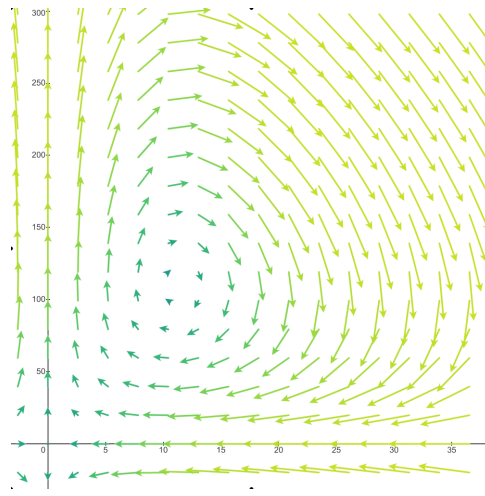


3d slope fields are possible, but hard to interpret.

On the left is a slope field for the Foxes–Rabbits model.

<https://www.desmos.com/3d/fsfbhvy2h9>

- 24.1 What are the three dimensions in the plot?
- 24.2 What should the graph of an equilibrium solution look like?
- 24.3 What should the graph of a typical solution look like?
- 24.4 What are ways to simplify the picture so we can more easily analyze solutions?



Phase Portrait

DEF

A **phase portrait** or **phase diagram** is the plot of a vector field in phase space where each vector rooted at (x, y) is tangent to a solution curve passing through (x, y) and its length is given by the speed of a solution passing through (x, y) .

On the left is a phase portrait for the Foxes–Rabbits model.

<https://www.desmos.com/calculator/vrk0q4espx>

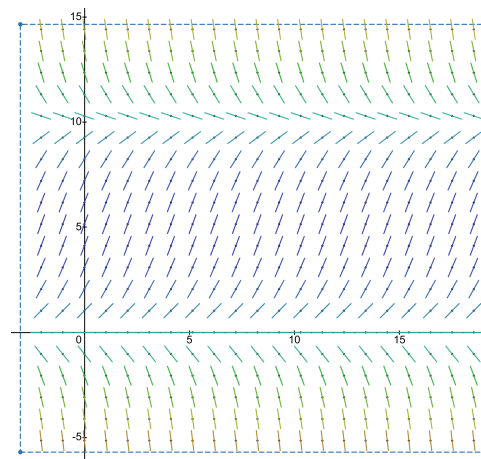
- 25.1 What do the x and y axes correspond to?
- 25.2 Identify the equilibria in the phase portrait. What are the lengths of the vectors at those points?
- 25.3 Classify each equilibrium as stable/unstable.
- 25.4 Why is the vector at $(5, 100)$ longer than the vector at $(10, 100)$? Justify numerically.

26

Sketch your own vector field where the corresponding system of differential equations:

- 26.1 Has an attracting equilibrium solution.
- 26.2 Has a repelling equilibrium solution.
- 26.3 Has no equilibrium solutions.

27



Recall the slope field for model **O**.

- 27.1 What would a phase portrait for model **O** look like? Draw it.
- 27.2 Where are the arrows the longest? Shortest?
- 27.3 How could you tell from a 1d phase portrait whether an equilibrium solution is attracting/repelling/etc.?

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$\begin{aligned}H'(t) &= 0.3 \cdot A(t) - b \cdot H(t) \\A'(t) &= -0.3 \cdot (H(t))^2 + A(t)\end{aligned}$$

and $0 \leq b \leq 2$

28.1 Modify

<https://www.desmos.com/calculator/vrk0q4espx>

to make a phase portrait for the tree model.

28.2 What do equilibrium solutions mean in terms of tree growth?

28.3 For $b = 1$ what are the equilibrium solution(s)?

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$\begin{aligned}H'(t) &= 0.3 \cdot A(t) - b \cdot H(t) \\A'(t) &= -0.3 \cdot (H(t))^2 + A(t)\end{aligned}$$

and $0 \leq b \leq 2$

29.1 Fix a value of b and use a spreadsheet to simulate some solutions with different initial conditions. Plot the results on your phase portrait from 28.1.

29.2 What will happen to a tree with $(H(0), A(0)) = (20, 10)$? Does this depend on b ?

29.3 What will happen to a tree with $(H(0), A(0)) = (10, 10)$? Does this depend on b ?

The tree model

$$\begin{aligned}H'(t) &= 0.3 \cdot A(t) - b \cdot H(t) \\A'(t) &= -0.3 \cdot (H(t))^2 + A(t)\end{aligned}$$

was based on the premises

$P_{\text{height } 1}$ CO_2 is absorbed by the leaves and turned directly into trunk height.

$P_{\text{height } 2}$ The tree is in a swamp and constantly sinks at a speed proportional to its height.

$P_{\text{leaves } 1}$ Leaves grow proportionality to the energy available.

$P_{\text{energy } 1}$ The tree absorbs energy from the sun proportionality to the leaf area.

P_{energy} 2 It costs energy proportional to the square of the height for the tree to maintain its current size.

- 30.1 How are the premises expressed in the differential equations?
- 30.2 What does the parameter b represent?
- 30.3 Applying Euler's method to this system shows solutions that pass from the 1st to 4th quadrants of the phase plane. Is this realistic? Describe the life cycle of such a tree?

31

Recall the tree model

$$\begin{aligned}H'(t) &= 0.3 \cdot A(t) - b \cdot H(t) \\A'(t) &= -0.3 \cdot (H(t))^2 + A(t)\end{aligned}$$

- 31.1 Find all equilibrium solutions for $0 \leq b \leq 2$.
- 31.2 For which b does a tree have the possibility of living forever? If the wind occasionally blew off a few random leaves, would that change your answer?
- 31.3 Find a value b_5 of b so that there is an equilibrium with $H = 5$.
Find a value b_{12} of b so that there is an equilibrium with $H = 12$.
- 31.4 Predict what happens to a tree near equilibrium in condition b_5 and a tree near equilibrium in condition b_{12} .

32

Consider the system of differential equations

$$\begin{aligned}x'(t) &= x(t) \\y'(t) &= 2y(t)\end{aligned}$$

- 32.1 Make a phase portrait for the system.
- 32.2 What are the equilibrium solution(s) of the system?
- 32.3 Find a formula for $x(t)$ and $y(t)$ that satisfy the initial conditions $(x(0), y(0)) = (x_0, y_0)$.
- 32.4 Let $\vec{r}(t) = (x(t), y(t))$. Find a matrix A so that the differential equation can be equivalently expressed as

$$\vec{r}'(t) = A\vec{r}(t).$$

- 32.5 Write a solution to $\vec{r}' = A\vec{r}$ (where A is the matrix you came up with).

33

Let A be an unknown matrix and suppose \vec{p} and \vec{q} are solutions to $\vec{r}' = A\vec{r}$.

- 33.1 Is $\vec{s}(t) = \vec{p}(t) + \vec{q}(t)$ a solution to $\vec{r}' = A\vec{r}$? Justify your answer.
- 33.2 Can you construct other solutions from \vec{p} and \vec{q} ? If yes, how so?

34

Recall from MAT223:

Linearly Dependent & Independent (Algebraic)

DEF

The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are **linearly dependent** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

Define

$$\vec{p}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} \quad \vec{q}(t) = \begin{bmatrix} 4e^t \\ 0 \end{bmatrix} \quad \vec{h}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \quad \vec{z}(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

- 34.1 Are \vec{p} and \vec{q} linearly independent or linearly dependent? Justify with the definition.
- 34.2 Are \vec{p} and \vec{h} linearly independent or linearly dependent? Justify with the definition.
- 34.3 Are \vec{h} and \vec{z} linearly independent or linearly dependent? Justify with the definition.
- 34.4 Is the set of three functions $\{\vec{p}, \vec{h}, \vec{z}\}$ linearly independent or linearly dependent? Justify with the definition.

$$\vec{p}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} \quad \vec{q}(t) = \begin{bmatrix} 4e^t \\ 0 \end{bmatrix} \quad \vec{h}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \quad \vec{z}(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

- 35.1 Intuitively, describe $\text{span}\{\vec{p}, \vec{h}\}$. What is its dimension? What is a basis for it?
- 35.2 Let S be the set of all solutions to $\vec{r}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{r}(t)$. w (You've seen this equation before.)
Intuitively, is S a subspace? If so, what is its dimension?
- 35.3 Provided S is a subspace, give a basis for S .

Consider the differential equation

$$y'(t) = 2 \cdot y(t).$$

- 36.1 Write a solution whose graph passes through the point $(t, y) = (0, 3)$.
- 36.2 Write a solution whose graph passes through the point $(t, y) = (0, y_0)$.
- 36.3 Write a solution whose graph passes through the point $(t, y) = (t_0, y_0)$.
- 36.4 Consider the following argument:

For every point (t_0, y_0) , there is a corresponding solution to $y'(t) = 2 \cdot y(t)$.

Since $\{(t_0, y_0) : t_0, y_0 \in \mathbb{R}\}$ is two dimensional, this means the set of solutions to $y'(t) = 2 \cdot y(t)$ is two dimensional.

Do you agree? Explain.

THM

For an **autonomous** ordinary differential equation (whose solutions are defined on all of \mathbb{R}), a solution that passes through (t_0, y_0) also passes through $(0, y_0^*)$ for some y_0^* .

(Uniqueness 1)

THM

The differential equation $y'(t) = a \cdot y(t) + b$ has a unique solution passing through every point.

- 37.1 Explain why the *autonomous* condition is important for the first theorem.
- 37.2 Suppose that f and g are solutions to $y' = a \cdot y + b$. If the graph of f passes through $(0, 1)$ and the graph of g passes through $(1, 0)$, does the second theorem (Uniqueness 1) say that $f \neq g$? Explain.
- 37.3 Consider the following argument:

For every point (t_0, y_0) , there is a corresponding solution to $y'(t) = 2 \cdot y(t)$.

Since $\{(t_0, y_0) : t_0, y_0 \in \mathbb{R}\}$ is two dimensional, this means the set of solutions to $y'(t) = 2 \cdot y(t)$ is two dimensional.

Apply the above theorems to decide if the argument is true or false.

THM

For an **autonomous** ordinary differential equation (whose solutions are defined on all of \mathbb{R}), a solution that passes through (t_0, y_0) also passes through $(0, y_0^*)$ for some y_0^* .

(Uniqueness 1)

THM

The differential equation $y'(t) = a \cdot y(t) + b$ has a unique solution passing through every point.

Let S be the set of all solutions to $\vec{r}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{r}(t)$.

- 38.1 What is the dimension of S ? Justify your answer.

Consider the system

$$\begin{aligned}x'(t) &= 2x(t) \\ y'(t) &= 3y(t)\end{aligned}$$

39.1 Rewrite the system in matrix form.

39.2 Classify the following as solutions or non-solutions to the system.

$$\begin{aligned}\vec{r}_1(t) &= e^{2t} & \vec{r}_2(t) &= \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \\ \vec{r}_3(t) &= \begin{bmatrix} e^{2t} \\ 4e^{3t} \end{bmatrix} & \vec{r}_4(t) &= \begin{bmatrix} 4e^{3t} \\ e^{2t} \end{bmatrix} \\ \vec{r}_5(t) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

39.3 State the definition of an eigenvector for the matrix M .

39.4 What should the definition of an *eigen solution* be for this system?

39.5 Which functions from 39.2 are eigen solutions?

39.6 Find an eigen solution \vec{r}_6 that is linearly independent from \vec{r}_2 .

39.7 Let $S = \text{span } \vec{r}_2, \vec{r}_6$. Does S contain *all* solutions to the system? Justify your answer.

Recall the system

$$\begin{aligned}x'(t) &= 2x(t) \\ y'(t) &= 3y(t)\end{aligned}$$

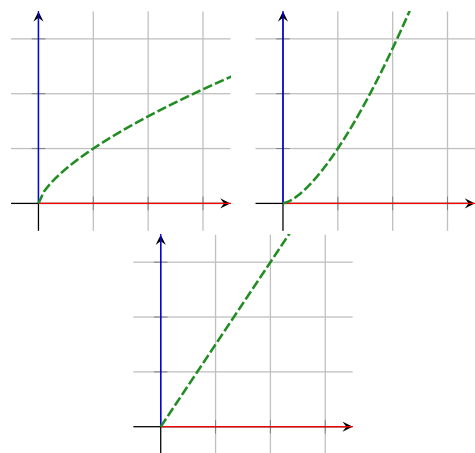
has eigen solutions $\vec{r}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{r}_6(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$.

40.1 Sketch \vec{r}_2 and \vec{r}_6 in the phase plane.

40.2 Use

<https://www.desmos.com/calculator/h3wtwjghv0>
to make a phase portrait for the system.

40.3



In which phase plane above is the dashed (green)

curve the graph of a solution to the system? Explain.

Suppose \vec{s}_1 and \vec{s}_2 are eigen solutions to $\vec{r}' = A\vec{r}$ with eigenvalues 1 and -1 , respectively.

41.1 Write possible formulas for $\vec{s}_1(t)$ and $\vec{s}_2(t)$.

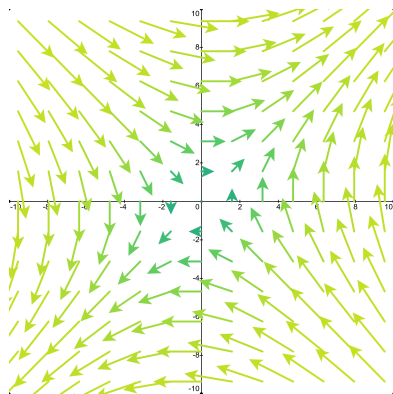
41.2 Sketch a phase plane with graphs of \vec{s}_1 and \vec{s}_2 on it.

41.3 Add a non-eigen solution to your sketch.

41.4 Sketch a possible phase portrait for $\vec{r}' = A\vec{r}$. Can you extend your phase portrait to all quadrants?

42

Consider the following phase portrait for a system of the form $\vec{r}' = A\vec{r}$ for an unknown matrix A .



42.1 Can you identify any eigen solutions?

42.2 What are the eigenvalues of A ? What are their sign(s)?

43

Consider the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

43.1 Find the eigenvectors and eigenvalues for M .

43.2 Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors for M . What are the corresponding eigenvalues?

43.3 (a) Is $\vec{r}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ a solution to the differential equation?

(b) Is $\vec{r}_2(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ a solution to the differential equation?

(c) Is $\vec{r}_3(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a solution to the differential equation?

43.4 Find an eigen solution for the system corresponding to the eigenvalue -1 . Write your answer in vector form.

44

Recall the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

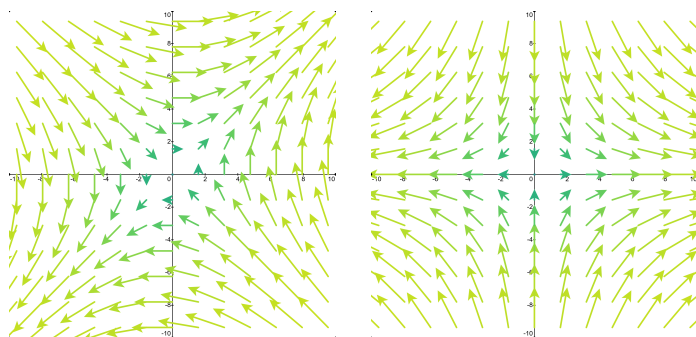
44.1 Write down a general solution to the differential equation.

44.2 Write down a solution to the initial value problem $\vec{r}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

44.3 Are your answers to the first two parts the same? Do they contain the same information?

45

The phase portrait for a differential equation arising from the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (left) and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (right) are shown.



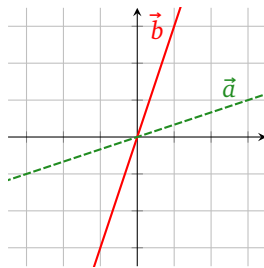
Both have eigenvalues ± 1 , but they have different eigenvectors.

45.1 How are the phase portraits related to each other?

45.2 Suppose P is a 2×2 matrix with eigenvalues ± 1 . In what ways could the phase portrait for $\vec{r}'(t) = P \vec{r}(t)$ look *different* from the above portraits? In what way(s) must it look the same?

46

Consider the following phase plane with lines in the direction of \vec{a} (red) and \vec{b} (dashed green).



46.1 Sketch a phase portrait where the directions \vec{a} and \vec{b} correspond to eigen solutions with eigenvalues that are

	sign for \vec{a}	sign for \vec{b}
(1)	pos	pos
(2)	neg	neg
(3)	neg	pos
(4)	pos	neg
(5)	pos	zero

46.2 Classify the solution at the origin for situations (1)-(5) as stable or unstable.

46.3 Would any of your classifications in 46.2 change if the directions of \vec{a} and \vec{b} changed?

47

You are examining a differential equation $\vec{r}'(t) = M \vec{r}(t)$ for an unknown matrix M .

You would like to determine whether $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is stable/unstable/etc.

47.1 Come up with a rule to determine the nature of the equilibrium solution $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ based on the eigenvalues of M .

47.2 Consider the system of differential equations

$$\begin{aligned} x'(t) &= x(t) + 2y(t) \\ y'(t) &= 3x(t) - 4y(t) \end{aligned}$$

(a) Classify the stability of the equilibrium solution $(x(t), y(t)) = (0, 0)$ using any method you want.

(b) Justify your answer analytically using eigenvalues.

48

Consider the following model of Social Media Usage where

$x(t)$ = number of social media posts at year t

$y(t)$ = number of social media users at year t

(P1_x) Ignoring all else, each year posts decay proportionally to the current number of posts with proportionality constant 1.

(P2_x) Ignoring all else, social media users increase/decrease in proportion to the number of posts.

(P1_y) Ignoring all else (independent of decay), posts grow by a constant amount of 2 million posts every year.

(P2_y) Ignoring all else, social media users increase/decrease in proportion to the number of users.

(P3_y) Ignoring all else, 1 million people stop using the platform every year.

A school intervention is described by the parameter $a \in [-1/2, 1]$:

- After the intervention, the proportionality constant for (P1_y) is $1 - a$.
- After the intervention, the proportionality constant for (P2_y) is a .

48.1 Model this situation using a system of differential equations. Explain which parts of your model correspond to which premise(s).

49

The **SM** model of Social Media Usage is

$$\begin{aligned}x' &= -x + 2 \\ y' &= (1 - a)x + ay - 1\end{aligned}$$

where

$$\begin{aligned}x(t) &= \text{number of social media posts at year } t \\ y(t) &= \text{number of social media users at year } t \\ a &\in [-1/2, 1]\end{aligned}$$

49.1 What are the equilibrium solution(s)?

49.2 Make a phase portrait for the system.

49.3 Use phase portraits to conjecture: what do you think happens to the equilibrium solution(s) as a transitions from negative to positive? Justify with a computation.

50

The **SM** model of Social Media Usage is

$$\begin{aligned}x' &= -x + 2 \\ y' &= (1 - a)x + ay - 1\end{aligned}$$

where

$$\begin{aligned}x(t) &= \text{number of social media posts at year } t \\ y(t) &= \text{number of social media users at year } t \\ a &\in [-1/2, 1]\end{aligned}$$

50.1 Can you rewrite the system in matrix form? (I.e., in the form $\vec{r}'(t) = M \vec{r}(t)$ for some matrix M .)

50.2 Define $\vec{s}(t)$ to be the displacement from equilibrium in the **SM** model at time t .

- Write \vec{s} in terms of x and y .
- Write a differential equation governing \vec{s} .
- Can your differential equation governing \vec{s} be written in matrix form?
- Analytically classify the equilibrium solution for your differential equation for \vec{s} when $a = -1/2, 1/2$, and 1 . (You may use a calculator for computing eigenvectors/values.)

The **SM** model of Social Media Usage is

$$\begin{aligned}x' &= -x + 2 \\ y' &= (1-a)x + ay - 1\end{aligned}$$

where

$$\begin{aligned}x(t) &= \text{number of social media posts at year } t \\ y(t) &= \text{number of social media users at year } t \\ a &\in [-1/2, 1]\end{aligned}$$

Some politicians have been looking at the model. They made the following posts on social media:

1. *The model shows the number of posts will always be increasing. SAD!*
2. *I see the number of social media users always increases. That's not what we want!*
3. *It looks like social media is just a fad. Although users initially increase, they eventually settle down.*
4. *I have a dream! That one day there will be social media posts, but eventually there will be no social media users!*

51.1 For each social media post, make an educated guess about what initial conditions and what value(s) of a the politician was considering.

51.2 The school board wants to limit the number of social media users to fewer than 10 million. Make a recommendation about what value of a they should target.

Consider the following **DF** model of Dogs and Fleas where

$$\begin{aligned}x(t) &= \text{number of parasites (fleas) at year } t \text{ (in millions)} \\ y(t) &= \text{number of hosts (dogs) at year } t \text{ (in thousands)}\end{aligned}$$

(P1_x) Ignoring all else, the number of parasites decays in proportion to its population (with constant 1).

(P2_x) Ignoring all else, parasite numbers grow in proportion to the number of hosts (with constant 1).

(P1_y) Ignoring all else, hosts numbers grow in proportion to their current number (with constant 1).

(P2_y) Ignoring all else, host numbers decrease in proportion to the number of parasites (with constant 2).

(P1_c) Anti-flea collars remove 2 million fleas per year.

(P1_c) Constant dog breeding adds 1 thousand dogs per year.

52.1 Write a system of differential equations for the **DF** model.

52.2 Can you rewrite the system in matrix form $\vec{r}' = M \vec{r}$? What about in *affine* form $\vec{r}' = M \vec{r} + \vec{b}$?

52.3 Make a phase portrait for your mode.

52.4 What should solutions to the system look like in the phase plane? What are the equilibrium solutions?

53 Recall the **DF** model of Dogs and Fleas where

$x(t)$ = number of parasites (fleas) at year t (in millions)

$y(t)$ = number of hosts (dogs) at year t (in thousands)

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

and

$$\vec{r}'(t) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \vec{r}(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Define $\vec{s}(t)$ to be the displacement of $\vec{r}(t)$ from equilibrium at time t .

53.1 Find a formula for \vec{s} in terms of \vec{r} .

53.2 Can you find a matrix M so that $\vec{s}'(t) = M \vec{s}(t)$?

53.3 What are the eigen solutions for $\vec{s}' = M \vec{s}$?

54 Recall the **DF** model of Dogs and Fleas where

$x(t)$ = number of parasites (fleas) at year t (in millions)

$y(t)$ = number of hosts (dogs) at year t (in thousands)

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}'(t) = M \vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

This equation has eigen solutions

$$\begin{aligned} \vec{s}_1(t) &= \begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{it} \\ \vec{s}_2(t) &= \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{-it} \end{aligned}$$

54.1 Recall Euler's formula $e^{it} = \cos(t) + i \sin(t)$.

(a) Use Euler's formula to expand $\vec{s}_1 + \vec{s}_2$. Are there any imaginary numbers remaining?

(b) Use Euler's formula to expand $\vec{s}_1 - \vec{s}_2$. Are there any imaginary numbers remaining?

54.2 Verify that your formulas for $\vec{s}_1 + \vec{s}_2$ and $\vec{s}_1 - \vec{s}_2$ are solutions to $\vec{s}'(t) = M \vec{s}(t)$.

54.3 Can you give a third *real* solution to $\vec{s}'(t) = M \vec{s}(t)$?

55 Recall the **DF** model of Dogs and Fleas where

$x(t)$ = number of parasites (fleas) at year t (in millions)

$y(t)$ = number of hosts (dogs) at year t (in thousands)

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}'(t) = M \vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

55.1 What is the dimension of the space of solutions to $\vec{s}'(t) = M \vec{s}(t)$?

55.2 Give a basis for all solutions to $\vec{s}'(t) = M \vec{s}(t)$.

55.3 Find a solution satisfying $\vec{s}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

55.4 Using what you know, find a general formula for $\vec{r}(t)$.

55.5 Find a formula for $\vec{r}(t)$ satisfying $\vec{r}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

56

Consider the differential equation

$$\vec{s}'(t) = M\vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$$

56.1 Find eigen solutions for this differential equation (you may use a calculator/computer to assist).

56.2 Find a general *real* solution.

56.3 Make a phase portrait. What do sketches of your solutions look like in phase space?

57

Recall the **DF** model of Dogs and Fleas where

$x(t)$ = number of parasites (fleas) at year t (in millions)

$y(t)$ = number of hosts (dogs) at year t (in thousands)

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}'(t) = M\vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

Some research is being done on a shampoo for the dogs. It affects flea and dog reproduction:

(PS_x) Ignoring all else, the number of parasites decays in proportion to its population with constant $1 + a$.

(PS_y) Ignoring all else, hosts numbers grow in proportion to their current number with constant $1 - a$.

57.1 Modify the previous **DF** model to incorporate the effects of the shampoo.

57.2 Make a phase portrait for the **DF Shampoo** model.

57.3 Find the equilibrium solutions for the **DF Shampoo** model.

57.4 For each equilibrium solution determine its stability/instability/etc..

57.5 Analytically justify your conclusions about stability/instability/etc..

58

Recall the tree model from Question 28:

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$

$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

A phase portrait for this model is available at

<https://www.desmos.com/calculator/tvjag852ja>

58.1 Find explicit formulas for equilibrium solutions of the tree model.

58.2 Visually classify the nature of each equilibrium solution as attracting/repelling/etc..

58.3 Can you rewrite the system in matrix/affine form? Why or why not?

A simple logistic model for a population is

$$\frac{dP}{dt} = P(t) \cdot \left(1 - \frac{P(t)}{2}\right)$$

where $P(t)$ represents the population at time t .

We'd like to approximate dP/dt when $P \approx 1/2$.

- 59.1 What is the value of dP/dt when $P = 1/2$?
- 59.2 What is the approximate value of dP/dt when $P = 1/2 + \Delta$ when Δ is small?
- 59.3 Write down a linear approximation $S(\Delta)$ that approximates dP/dt when P is Δ away from $1/2$.
- 59.4 Let $A_{1/2}(t)$ be an *affine* approximation to dP/dt that is a good approximation when $P \approx 1/2$. Find a formula for $A_{1/2}(t)$ expressed in terms of $P(t)$.
- 59.5 Find additional affine approximations to dP/dt centered at each equilibrium solution.

Based on our calculations from last time, we have several different equations.

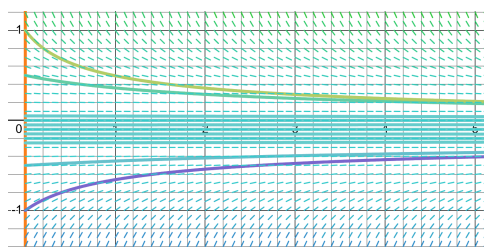
(Original)	$P' = P(1 - P/2)$	(https://www.desmos.com/calculator/v1coz4shtw)
$(A_{1/2})$	$P' \approx \frac{3}{8} + \frac{1}{2}(P - \frac{1}{2})$	(https://www.desmos.com/calculator/zsb2apxhqs)
(A_0)	$P' \approx P$	(https://www.desmos.com/calculator/vw48bvqgrc)
(A_2)	$P' \approx -(P - 2)$	(https://www.desmos.com/calculator/i2utk6vnqh)

- 60.1 What do you notice about the solutions sketched on the different slope fields (in Desmos)?
- 60.2 Does the nature of equilibrium solutions change when using an affine approximation?

Consider the differential equation whose slope field is sketched below.

$$P'(t) = -P(t) \cdot (0.1 + P(t)) \cdot (0.2 + P(t)).$$

<https://www.desmos.com/calculator/ikp9rgo0kv>



- 61.1 Find all equilibrium solutions.
- 61.2 Use linear approximations to classify the equilibrium solutions as stable/unstable/etc..

To make a 1d affine approximation of a function f at the point E we have the formula

$$f(x) \approx f(E) + f'(E)(x - E).$$

To make a 2d approximation of a function $\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$ at the point \vec{E} , we have a similar formula

$$\vec{F}(x, y) \approx \vec{F}(\vec{E}) + D_{\vec{F}}(\vec{E})((x, y) - \vec{E})$$

where $D_{\vec{F}}(\vec{E})$ is the *total derivative* of \vec{F} at \vec{E} , which can be expressed as the matrix

$$D_{\vec{F}}(\vec{E}) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

evaluated at \vec{E} .

Recall our model from Question 28 for the life cycle of a tree where $H(t)$ was height, $A(t)$ was the leaves' surface area, and t was time:

$$\begin{aligned} H'(t) &= 0.3 \cdot A(t) - b \cdot H(t) \\ A'(t) &= -0.3 \cdot (H(t))^2 + A(t) \end{aligned}$$

with $0 \leq b \leq 2$

We know the following:

- The equations cannot be written in matrix form.
- The equilibrium points are $(0, 0)$ and $(\frac{100}{9}b, \frac{1000}{27}b^2)$.

We want to find an affine approximation to the system.

Define $\vec{F}(H, A) = (H', A')$

- 62.1 Find the matrix for $D_{\vec{F}}$, the total derivative of \vec{F} .
- 62.2 Create an affine approximation to \vec{F} around $(0, 0)$ and use this to write an approximation to the original system.
- 62.3 Create an affine approximation to \vec{F} around $(\frac{100}{9}b, \frac{1000}{27}b^2)$ and use this to write an approximation to the original system.
- 62.4 Make a phase portrait for the original system and your approximation from part 3. How do they compare?
- 62.5 Analyze the nature of the equilibrium solution in part 3 using eigen techniques. Relate your analysis to the original system.