Linear Algebr

MAT244 Slides

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Exercise 1

You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish.

You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

Exercise 2

Let

$$\begin{aligned} &\text{(Birth Rate)} && K = 1.1 \text{ children per starfish per year} \\ &\text{(Initial Pop.)} && P_0 = 10 \text{ star fish} \end{aligned}$$

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

Exercise 3

Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

Exercise 4

In the model $\mathbf{M}_1$, we assumed the starfish had K children at one point during the year.

4.1 Create a model $\mathbf{M}_n$ where the starfish are assumed to have K/n children n times per year (at regular intervals).

4.2 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.3 What happens to $\mathbf{M}_n$ as $n \to \infty$?

Exercise 5

Exploring $\mathbf{M}_n$

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $(t, t+1/n)$ there will be (on average) K/n new children per starfish.

5.1 Write an expression for $P_n(t+1/n)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP, the change in population from time t to $t+\Delta t$.

5.3 Write an expression for $\frac{\Delta P}{\Delta t}$.

5.4 Write down a *differential equation* relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_n(t)$.

Exercise 6

Define the model $\mathbf{M}_\infty$ by

- $P(0) = 10$
- $P'(t) = kP(t)$

and recall the model $\mathbf{M}_1$ defined by

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

6.2 Suppose that $\mathbf{M}_1$ accurately predicts the population. Can you find a value of k so that $\mathbf{M}_\infty$ accurately predicts the population?

6.3 What are some advantages and disadvantages of the models $\mathbf{M}_1$ and $\mathbf{M}_\infty$?

Exercise 7

After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

7.1 What can you tell about the population (without trying to compute it)?

7.2 Assuming $k = 1.1$, estimate the population after 10 years.

7.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

Exercise 8

Consider the following argument:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(2) + P'(2) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

8.1 Do you believe this argument? Can it be improved?

Exercise 9

(Simulating $\mathbf{M}_\infty$ with different Δs)

Time	Pop. ($\Delta = 0.1$)	Time	Pop. ($\Delta = 0.2$)
0.0	10	0.0	10
0.1	11.1	0.2	12.2
0.2	12.321	0.4	14.884
0.3	13.67631	0.6	18.15848
0.4	15.1807041	0.8	22.1533456

9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

9.3 What Δs give the largest estimate for the population at time t?

9.4 Is there a limit as $\Delta \to 0$?

Exercise 9

(Simulating $\mathbf{M}_\infty$ with different Δs)

9.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

9.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

9.3 What Δs give the largest estimate for the population at time t?

9.4 Is there a limit as $\Delta \to 0$?

Exercise 10

Consider the following models for starfish growth

M # new children per year $\sim$ current population

N # new children per year $\sim$ resources available per individual

O # new children per year $\sim$ current population times the fraction of total resources remaining

10.1 Guess what the population vs. time curves look like for each model.

10.2 Create a differential equation for each model.

10.3 Simulate population vs. time curves for each model (but pick a common initial population).

Exercise 11

Recall the models

M # new children per year $\sim$ current population

N # new children per year $\sim$ resources available per individual

O # new children per year $\sim$ current population times the fraction of total resources remaining

11.1 Determine which population grows fastest in the short term and which grows fastest in the long term.

11.2 Are some models more sensitive to your choice of Δ when simulating?

11.3 Are your simulations for each model consistently underestimates? Overestimates?

11.4 Compare your simulated results with your guesses from question 10.1. What did you guess correctly? Where were you off the mark?

Exercise 12

A simple model for population growth has the form

$$P'(t) = kP(t)$$

where k is the *birth rate*.

12.1 Create a better model for population that includes both births and deaths.

Exercise 13

The *Lotka-Volterra Predator-Prey* models two populations, F (foxes) and R (rabbits), simultaneously. It takes the form

$$F'(t) = (B_F - D_F) \cdot F(t)$$
$$R'(t) = (B_R - D_R) \cdot R(t)$$

where $B_?$ stands for births and $D_?$ stands for deaths.

13.1 Come up with appropriate formulas for B_F, B_R, D_F, and D_R, given your knowledge about how foxes and rabbits might interact.

13.2 When would B_F and D_F be at their max/min?

13.3 When would B_R and D_R be at their max/min?

Exercise 14

Suppose the population of F (foxes) and R (rabbits) evolves over time following the rule

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

14.1 Simulate the population of foxes and rabbits with a spreadsheet.

14.2 Do the populations continue to grow/shrink forever? Are they cyclic?

14.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

Exercise 15

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

15.1 Is the max population of the rabbits being over/under estimated? Sometimes over, sometimes under?

15.2 What about the foxes?

15.3 What about the min populations?

Exercise 16

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Component Graph & Phase Plane. For a differential equation involving the functions $F_1, F_2, \ldots, F_n$, and the variable t, the ***component graphs*** are the n graphs of $(t, F_1(t))$, $(t, F_2(t))$, $\ldots$.
The ***phase plane*** or ***phase space*** associated with the differential equation is the n-dimensional space with axes corresponding to the values of $F_1, F_2, \ldots, F_n$.

16.1 Plot the Fox vs. Rabbit population in the *phase plane*.

16.2 Should your plot show a closed curve or a spiral?

16.3 What "direction" do points move along the curve as time increases? Justify by referring to the model.

16.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

Exercise 17

Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Equilibrium Solution. An ***equilibrium solution*** to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

17.1 By changing initial conditions, what is the "smallest" curve you can get in the phase plane? What happens at those initial conditions?

17.2 What should F' and R' be if F and R are *equilibrium solutions*?

17.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.

17.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

Exercise 18

Recall the logistic model for starfish growth:

O # new children per year $\sim$ current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time t
- k is a constant of proportionality
- R is the total number of resources
- R_i is the resources that one starfish wants to consume

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

18.1 What are the equilibrium solutions for model **O**?

18.2 What does a "phase plane" for model **O** look like? What do graphs of equilibrium solutions look like?

18.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if solutions locally converge to f
- *repelling* if solutions locally diverge from f
- *stable* if solutions do not locally diverge from f
- *unstable* if solutions do not locally converge to f
- *semi-stable* if solutions locally converge to f from one side and locally diverge from f on another.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

19.1 Draw an example of what solutions might look like if f is *attracting*.

19.2 Draw an example of what solutions might look like if f is *repelling*.

19.3 Draw an example of what solutions might look like if f is *stable*.

19.4 Could f be stable but *not* attracting?

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if solutions locally converge to f
- *repelling* if solutions locally diverge from f
- *stable* if solutions do not locally diverge from f
- *unstable* if solutions do not locally converge to f
- *semi-stable* if solutions locally converge to f from one side and locally diverge from f on another.

Recall the starfish population model **O** given by

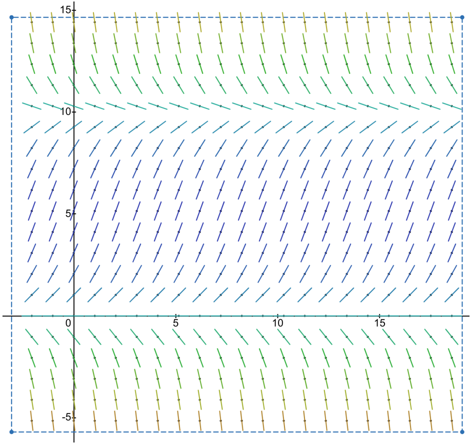
$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

20.1 Classify the equilibrium solutions for model **O** as attracting/repelling/stable/unstable/semi-stable.

20.2 Does changing k change the nature of the equilibrium solutions? How can you tell?

Exercise 21



A *slope field* is a plot of small segments of tangent lines to solutions of a differential equation at different initial conditions.

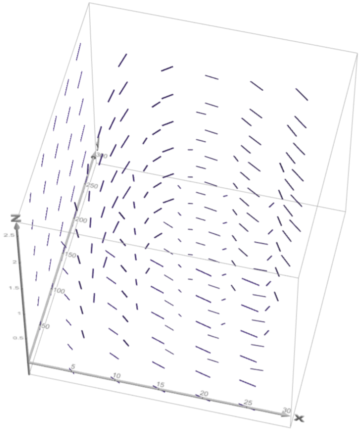
On the left is a slope field for model **O**, available at

`https://www.desmos.com/calculator/ghavqzqqjn`

21.1 If you were sketching the slope field for model **O** by hand, what line would you sketch (a segment of) at $(5,3)$? Write an equation for that line.

21.2 How can you recognize equilibrium solutions in a slope field?

21.3 Describe different solutions to the *differential equation* using words. Do all of those solutions make sense in terms of *model* ***O***?



3d slope fields are possible, but hard to interpret.

On the left is a slope field for the Foxes–Rabbits model.

`https://www.desmos.com/3d/fsfbhvy2h9`

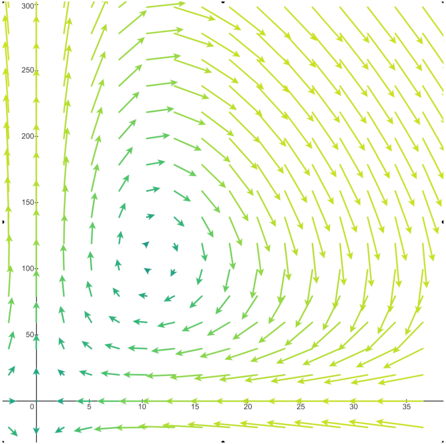
22.1 What are the three dimensions in the plot?

22.2 What should the graph of an equilibrium solution look like?

22.3 What should the graph of a typical solution look like?

22.4 What are ways to simplify the picture so we can more easily analyze solutions?

Exercise 23



Phase Portrait. A ***phase portrait*** or ***phase diagram*** is the plot of a vector field in phase space where each vector rooted at (x, y) is tangent to a solution curve passing through (x, y) and its length is given by the speed of a solution passing through (x, y).

On the left is a phase portrait for the Foxes–Rabbits model.

`https://www.desmos.com/calculator/vrk0q4espx`

23.1 What do the x and y axes correspond to?

23.2 Identify the equilibria in the phase portrait. What are the lengths of the vectors at those points?

23.3 Classify each equilibrium as stable/unstable.

23.4 Why is the vector at $(5, 100)$ longer than the vector at $(10, 100)$? Justify numerically.

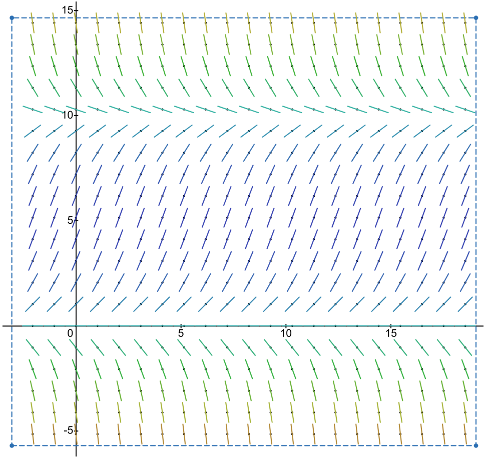
Exercise 24

Sketch your own vector field where the corresponding system of differential equations:

24.1 Has an attracting equilibrium solution.

24.2 Has a repelling equilibrium solution.

24.3 Has no equilibrium solutions.



Recall the slope field for model **O**.

25.1 What would a phase portrait for model **O** look like? Draw it.

25.2 Where are the arrows the longest? Shortest?

25.3 How could you tell from a 1d phase portrait whether an equilibrium solution is attracting/repelling/etc.?

Exercise 26

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

26.1 Modify

`https://www.desmos.com/calculator/vrk0q4espx`

to make a phase portrait for the tree model.

26.2 What do equilibrium solutions mean in terms of tree growth?

26.3 For $b = 1$ what are the equilibrium solution(s)?

Exercise 27

The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

27.1 Fix a value of b and use a spreadsheet to simulate some solutions with different initial conditions. Plot the results on your phase portrait from 26.1.

27.2 What will happen to a tree with $(H(0), A(0)) = (20, 10)$? Does this depend on b?

27.3 What will happen to a tree with $(H(0), A(0)) = (10, 10)$? Does this depend on b?

Exercise 28

The tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

was based on the premises

$P_{\text{height 1}}$ CO_2 is absorbed by the leaves and turned directly into trunk height.

$P_{\text{height 2}}$ The tree is in a swamp and constantly sinks at a speed proportional to its height.

$P_{\text{leaves 1}}$ Leaves grow proportionality to the energy available.

$P_{\text{energy 1}}$ The tree absorbs energy from the sun proportionality to the leaf area.

$P_{\text{energy 2}}$ It costs energy proportional to the square of the height for the tree to maintain its current size.

28.1 How are the premises expressed in the differential equations?

28.2 What does the parameter b represent?

28.3 Applying Euler's method to this system shows solutions that pass from the 1st to 4th quadrants of the phase plane. Is this realistic? Describe the life cycle of such a tree?

Exercise 29

Recall the tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

29.1 Find all equilibrium solutions for $0 \le b \le 2$.

29.2 For which b does a tree have the possibility of living forever? If the wind occasionally blew off a few random leaves, would that change your answer?

29.3 Find a value b_5 of b so that there is an equilibrium with $H = 5$.

Find a value b_{12} of b so that there is an equilibrium with $H = 12$.

29.4 Predict what happens to a tree near equilibrium in condition b_5 and a tree near equilibrium in condition b_{12}.

Exercise 30

Consider the system of differential equations

$$x'(t) = x(t)$$
$$y'(t) = 2y(t)$$

30.1 Make a phase portrait for the system.

30.2 What are the equilibrium solution(s) of the system?

30.3 Find a formula for $x(t)$ and $y(t)$ that satisfy the initial conditions $(x(0), y(0)) = (x_0, y_0)$.

30.4 Let $\vec{r}(t) = (x(t), y(t))$. Find a matrix A so that the differential equation can be equivalently expressed as

$$\vec{r}'(t) = A\vec{r}(t).$$

30.5 Write a solution to $\vec{r}' = A\vec{r}$ (where A is the matrix you came up with).

Exercise 31

Let A be an unknown matrix and suppose $\vec{p}$ and $\vec{q}$ are solutions to $\vec{r}' = A\vec{r}$.

31.1 Is $\vec{s}(t) = \vec{p}(t) + \vec{q}(t)$ a solution to $\vec{r}' = A\vec{r}$? Justify your answer.

31.2 Can you construct other solutions from $\vec{p}$ and $\vec{q}$? If yes, how so?

Exercise 32

Recall from MAT223:

Linearly Dependent & Independent (Algebraic).
The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

Define

$$\vec{p}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} \qquad \vec{q}(t) = \begin{bmatrix} 4e^t \\ 0 \end{bmatrix} \qquad \vec{h}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \qquad \vec{z}(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

32.1 Are $\vec{p}$ and $\vec{q}$ linearly independent or linearly dependent? Justify with the definition.

32.2 Are $\vec{p}$ and $\vec{h}$ linearly independent or linearly dependent? Justify with the definition.

32.3 Are $\vec{h}$ and $\vec{z}$ linearly independent or linearly dependent? Justify with the definition.

32.4 Is the set of three functions $\{\vec{p}, \vec{h}, \vec{z}\}$ linearly independent or linearly dependent? Justify with the definition.

Exercise 33

Recall

$$\vec{p}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} \qquad \vec{q}(t) = \begin{bmatrix} 4e^t \\ 0 \end{bmatrix} \qquad \vec{h}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \qquad \vec{z}(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

33.1 Intuitively, describe $\operatorname{span}\{\vec{p}, \vec{h}\}$. What is its dimension? What is a basis for it?

33.2 Let S be the set of all solutions to $\vec{r}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{r}(t)$. w (You've seen this equation before.) Intuitively, is S a subspace? If so, what is its dimension?

33.3 Provided S is a subspace, give a basis for S.

Exercise 34

Consider the differential equation

$$y'(t) = 2 \cdot y(t).$$

34.1 Write a solution whose graph passes through the point $(t, y) = (0, 3)$.

34.2 Write a solution whose graph passes through the point $(t, y) = (0, y_0)$.

34.3 Write a solution whose graph passes through the point $(t, y) = (t_0, y_0)$.

34.4 Consider the following argument:

> For every point (t_0, y_0), there is a corresponding solution to $y'(t) = 2 \cdot y(t)$. Since $\{(t_0, y_0) : t_0, y_0 \in \mathbb{R}\}$ is two dimensional, this means the set of solutions to $y'(t) = 2 \cdot y(t)$ is two dimensional.

Do you agree? Explain.

> **Theorem.** For an ***autonomous*** ordinary differential equation (whose solutions are defined on all of $\mathbb{R}$), a solution that passes through (t_0, y_0) also passes through $(0, y_0^*)$ for some y_0^*.

> **Theorem (Uniqueness 1).** The differential equation $y'(t) = a \cdot y(t) + b$ has a unique solution passing through every point.

35.1 Explain why the *autonomous* condition is important for the first theorem.

35.2 Suppose that f and g are solutions to $y' = a \cdot y + b$. If the graph of f passes through $(0, 1)$ and the graph of g passes through $(1, 0)$, does the second theorem (Uniqueness 1) say that $f \neq g$? Explain.

35.3 Consider the following argument:

> For every point (t_0, y_0), there is a corresponding solution to $y'(t) = 2 \cdot y(t)$.
>
> Since $\{(t_0, y_0) : t_0, y_0 \in \mathbb{R}\}$ is two dimensional, this means the set of solutions to $y'(t) = 2 \cdot y(t)$ is two dimensional.

Apply the above theorems to decide if the argument is true or false.

Exercise 36

Theorem. For an ***autonomous*** ordinary differential equation (whose solutions are defined on all of $\mathbb{R}$), a solution that passes through (t_0, y_0) also passes through $(0, y_0^*)$ for some y_0^*.

Theorem (Uniqueness 1). The differential equation $y'(t) = a \cdot y(t) + b$ has a unique solution passing through every point.

Let S be the set of all solutions to $\vec{r}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{r}(t)$.

36.1 What is the dimension of S? Justify your answer.