mat240 tut 1-21

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1 Introduction

Let V be a vector space. If $u, v, w \in V$ are such that $u + v = w + v$, then $u = w$.

1. $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. $\begin{pmatrix} 2 & -2 & -3 & 0 \\ 3 & -3 & -2 & 5 \\ 1 & -1 & -2 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & -2 & -1 \\ 2 & -2 & -3 & 0 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. $\begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & -3 & -5 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & -3 & -5\\ 0 & 1 & -3 & 0\\ 1 & 1 & 5 & 5\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0 & 5\\ 0 & 1 & -3 & 0\\ 1 & 1 & 5 & 5\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 8 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

4. $\begin{pmatrix}1 & 1 & -1 & 1\\ 1 & 2 & 0 & 1\\ 0 & 1 & 1 & -1\\ 2 & 4 & 0 & 1\end{pmatrix}$

$$\begin{pmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

1. $\begin{pmatrix}1 & 0 & 2 & -3 & | & 4\\ 0 & 1 & 2 & 1 & | & -1\\ 0 & 0 & 0 & 0 & | & 0\end{pmatrix}$

$$x_1 + 2x_3 - 3x_4 = 4 \implies x_1 = -2x_3 + 3x_4 - 4$$

$$x_2 + 2x_3 + x_4 = -1 \implies x_2 = -2x_3 - x_4 + 1$$

$$x_1 = -2s + 3t - 4$$

$$x_2 = -2s - t + 1$$

$$x_3 = s$$

$$x_4 = t$$

$$(-2s + 3t - 4, -2s - t + 1, s, t)$$

$$x = s\begin{pmatrix}-2\\ -2\\ 1\\ 0\end{pmatrix} + t\begin{pmatrix}3\\ -1\\ 0\\ 1\end{pmatrix} + \begin{pmatrix}-4\\ 1\\ 0\\ 0\end{pmatrix}$$

2. $\begin{pmatrix}0 & 1 & 0 & 2 & -3 & | & 4\\ 0 & 0 & 1 & 2 & 1 & | & -1\\ 0 & 0 & 0 & 0 & 0 & | & 0\end{pmatrix}$

$$x_2 + 2x_4 - 3x_5 = 4 \implies x_2 = -2x_4 + 3x_5 - 4$$

$$x_3 + 2x_4 + x_5 = -1 \implies x_3 = -2x_4 - x_5 + 1$$

- $x_1 = a$

- $x_2 = -2b + 3c - 4$
- $x_3 = -2b - c + 1$
- $x_4 = b$
- $x_5 = c$

3. $\begin{pmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$

No solution due to last row.

4. $\begin{pmatrix} 1 & 0 & 1 & -4 & 2 & | & -1 & \\ 0 & 1 & -2 & -4 & 0 & | & -1 & \\ 0 & 0 & 0 & 0 & & 0 & | & 0 \end{pmatrix}$

$$x_1 + x_3 - 4x_4 + 2x_5 = -1 \implies x_1 = -x_3 + 4x_4 - 2x_5 - 1$$

$$x_2 - 2x_3 - 4x_4 = -1 \implies x_2 = 2x_3 + 4x_4 - 1$$

2 Writing Activity

Theorem: Let $\mathbf{F}$ be a field, and V a vector space over $\mathbf{F}$. A subset $U \subseteq V$ is a subspace if and only if

1. $0 \in U$ and
2. for all $x, y \in U$, we have $x + y \in U$ for all $a \in \mathbf{F}$

2.0.1 Only if

Given that U is a subspace of $\mathbf{F}$, we know that U is non-empty. Take an arbitrary element $w \in U$ and multiply it by 0.

$$0w = 0$$

Since U is closed under scaling we know that the product, which is the zero vector, must be in U.

$$0 \in U$$

Next, take an arbitrary $x \in U$ and $a \in \mathbf{F}$, then $ax \in U$ because U is closed under scaling.

$$\forall x \in U, \forall a \in \mathbf{F}, ax \in U$$

Furthermore, taking another arbitrary vector $y \in U$, we know that $ax + y \in U$ because U is closed under vector addition.

$$\forall y \in U, \forall x \in U, \forall a \in \mathbf{F}, ax + y \in U$$

2.0.2 if

Given that the zero vector is in U, U is non-empty.

$$0 \in U \implies U \neq \emptyset$$

If we set a to 1, we find that for any two vectors $x, y \in U$, we know that $x + y = U$. We know $1 \in \mathbf{F}$ because $\mathbf{F}$ is a field.

$$\forall x, y \in U, \forall a \in \mathbf{F}, ax + y \in U \wedge (1 \in \mathbf{F})$$

$$\implies \forall x, y \in U, x + y \in U$$

If we set y to 0, then for any $a \in \mathbf{F}$, we know that $ax \in U$.

$$\forall x, y \in U, \forall a \in \mathbf{F}, ax + y \in U \wedge (0 \in U)$$

$$\implies \forall x \in U, \forall a \in \mathbf{F}\, ax \in U$$

Thus, we have satisfied the conditions to say that U is a subspace of V.