# mat159 1-23

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## 1 Week 3 Review:

- Indefinite Integral $\int f(x)dx$
  - Definition
  - Technique
  - Rationalization $R(x,y), P(x,y) = 0$

### 1.1 Definite Integral

Def: Partition
Let $[a,b]$ be a closed interval.
Denote by

$$T : a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

a partition of $[a,b]$

$$\bar{T}\bar{T} := \max_{a \leq i \leq n-1} x_{i-1} - x_i$$

We use

$$\Omega[a,b] \text{ to denote all partitions on } [a,b]$$

Def: Marked Partition

$$\Omega^*[a,b]$$

### 1.2 Riemann Sum

Let $f : [a,b] \to \mathbb{R}$
The Riemann Sum

$$\sigma(f,\Gamma,\eta) = \sum_{i=0}^{n-1} f(\eta_i)\Delta x_i,\ \Delta x_i = x_{i+1} - x_i$$

Def: Riemann Integral
If $\int \in \mathbb{R}$ is the limit of the Riemann Sum, in the sense.

$$\lim_{\bar{T} \to 0} \sigma(f, \Gamma, \eta) = S$$

Then we call $\int$ the Riemann integral of f on $[a, b]$ denoted by $\int_a^b f(x)dx$. This is just a real number.

$$\to \forall \epsilon > 0, \exists \delta > 0, \forall (\Gamma, \eta) \in \Omega^*[a, b], |\Gamma| < \delta \implies |\sigma(f, \Gamma, \eta) - S| < \epsilon$$

### 1.2.1 Remark

- We don't care about how we make the partition
- We don't care about the marked point.

### 1.2.2 Example: (Affine Function)

$$f : [a, b] \to \mathbb{R}, \; f(x) = kx + l$$

For any given partition, we pick some special marked point.

$$\eta_i = \frac{x_i + x_{i+1}}{2}$$

$$\sigma(f, \Gamma, \eta) = \sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2})\Delta x_i$$

$$= \sum_{i=0}^{n-1} (k(\frac{x_i + x_{i+1}}{2}) + l)(x_{i+1} - x_i)$$

$$= \sum_{i=0}^{n-1} k\frac{x_{i+1}^2 - x_i^2}{2} + l(x_{i+1} - x_i)$$

$$= \frac{k(b^2 - a^2)}{2} + l(b - a)$$

Now let $(\tilde{\Gamma}, \tilde{\eta})$ be an arbitrary marked partition.

$$\sigma(f, \tilde{\Gamma}, \tilde{\eta}) = \sum_{i=0}^{m-1} f(\tilde{\eta})\Delta x_i$$

$$|\sigma(f, \tilde{\Gamma}, \tilde{\eta}) - (\frac{k}{2}(b^2 - a^2) + l(b - a)|$$

$$|\sum_{i=0}^{m-1} (f(\tilde{\Gamma}_i) - f(\Gamma_i))\Delta x_i|$$

$$\leq \sum_{i=0}^{m-1} |k\Delta x_i| * \Delta x_i \leq (\sum_{i=0}^{m-1} \Delta x_i)|k||\Gamma| = (b - a)|k||\Gamma|$$

### 1.2.3 Remark

What if we had Cauchy Criterion, but for Riemann integral?

$$\forall \epsilon > 0, \exists \delta > 0, st\forall (\Gamma_1, \eta_1), |\Gamma_2, \eta_2| \in \Omega^*[a, b]$$

$$|\Gamma, 1|, |\Gamma_2| < \delta \implies |\sigma(f, \Gamma_1, \eta_1) - \sigma(f, \Gamma_2, \eta_2)| < \epsilon$$

## 1.3 Existence of Riemann Integral

Necessary Condition

$$f \in \mathbb{R}[a, b] \implies \exists M \geq 0,\ st|f| \leq M \text{ on } [a, b]$$

Proof:
If f is unbounded on [a,b], we can divide this into partitions, say $\eta_1, \eta_2, \eta_3, \ldots, \eta_6$. At least one of these partitions must be unbounded as well, maybe multiple. We then pick the unbounded partition, say $\eta_3$. Let $M$ be the sum of all other partitions which are bounded.

$$|\Delta x_3 f(\eta_3) + M| > 1$$

Now, we can continue to add more partitions to shrink them more and get a larger sum each time.

$$|\Delta x_n f(\eta_n) + M| \to \infty$$

However, this is the definition of Riemann integral, thus our Riemann integral is also unbounded. and does not exist.

### 1.3.1 Remark

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$.

$$V_f(D) = \sup_{x,y \in D} |f(x) - f(y)|$$

Given a partition $\Gamma$, the total oscillation of f on $\Gamma$ is defined as.

$$\sum_{i=0}^{n-1}$$

### 1.3.2 Proposition

Let f be the bounded function of $[a, b]$. Then $f \in \mathbb{R}[a, b]$ if

$$\forall \epsilon > 0, \exists \delta > 0 \forall \Gamma \in \Omega[a, b], |\Gamma| < S \implies |\sum_{i=0}^{n-1} V_f([x_i, x_{i+1}])\Delta x_i| < \epsilon$$

### 1.3.3 Proof

Now, let $\epsilon > 0$, pick $\delta$, st $\forall(\Gamma) \in \Omega^*[a,b], \sum_{i=0}^{n-1} V_f([x_i, x_{i+1}])\Delta x_i < \frac{\epsilon}{2}$.

$$\forall(\Gamma_1, \eta_1), (\Gamma_2, \eta_2) \in \Omega^*[a,b]$$

$$|\sigma(f, \Gamma_1, \eta_1) - \sigma(f, \Gamma_2, \eta_2)|$$

Now, take the union of $\Gamma_1$ and $\Gamma_2$, this is a refinement on both of them.

$$\leq |\sigma(f, \Gamma_1 \cup \Gamma_2, \eta_1) - \sigma(f, \Gamma_2, \eta_2)| + |\sigma(f, \Gamma_1, \eta_1) - \sigma(f, \Gamma_1 \cup \Gamma_2, \eta_1)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

### 1.3.4 Corollary 1

$$f \in C[a,b] \implies f \in \mathbb{R}[a,b]$$

### 1.3.5 Corollary 2

f is bounded and f has finitely many discontinuities, then $f \in \mathbb{R}[a,b]$ Proof: Assume there are k many discontinuities, numbered $y_1, y_2, \ldots y_k$. Choose a small interval around the discontinuities.
Now, take $[a,b]/\bigcup_{i=1}^{k} I_{\delta_i}(y+i)$. This is an open set which we can take a partition on.
Either a partition does not overlap with any discontinuity or it does. The parts which do not overlap have a max of $(b-a)\epsilon$.
A part which does overlap might lose $V_f([a,b])(2\delta_1 + 2\delta_2)k$.
Now, as we reduce $\delta_1, \delta_2$, $(b-a)\epsilon$ does not so we have a value.

### 1.3.6 Corollary 3

If a function is monotonically increasing on $[a,b] \implies f \in \mathbb{R}[a,b]$.
Variation is controlled.

## 1.4 Refinement

Let $\Gamma$ be a partition. A refinement of $\Gamma$ is obtained by adding new points in $\Gamma$.

### 1.4.1 Remark

A typical idea is "refinement makes things more stable".
For example, refinement decreases the total oscillation on a partition.
$\Gamma$ : Partition
$\tilde{\Gamma}$ : Refinement of $\Gamma$
For example, we take

$$x_{ij}, i = 1, 2, 3 \ldots$$

which are new points in $[x_i, x_{i+1}]$.

$$\begin{aligned}
&\sigma(f,\Gamma,\eta) - \sigma(f,\tilde{\Gamma},\tilde{\eta}) \\
&= \sum_{i=0}^{n-1} f(\eta_i)\Delta x_i - \sum_{i=0}^{n-1}\sum_j f(\eta_i j)\Delta x_{ij}
\end{aligned}$$

We can take the sum of the small intervals within each interval.

$$= \sum_{i=0}^{n-1}\sum_j f(\eta_i)\Delta x_i - \sum_{i=0}^{n-1}\sum_j f(\eta_i j)\Delta x_{ij}$$

$$= \sum_{i=0}^{n-1}\sum_j |f(\eta_i) - f(\eta_i j)|\Delta x_{ij}$$

$$\leq \sum_{i=0}^{n-1}\sum_j |V_f([x_i, x_{i+1}])|\Delta x_{ij}$$

$$= \sum_{i=0}^{n+1} V_f([x_i, x_{i+1}])\Delta x_i$$