Solutions to $a^x = a^y$ where x and y are unique natural numbers

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Introduction

We want to find all real values of a such that $a^x = a^y$ where x and y are unique natural numbers.

Theorem 1

Let a and b be real numbers and let n be a natural number, then:

$$a^n - b^n$$

$$= a^n + a^{n-1}b - a^{n-1}b + a^{n-2}b^2 - a^{n-2}b^2 + \cdots + ab^{n-1} - ab^{n-1} - b^n \qquad (1)$$

$$= a(a^{n-1} + a^{n-2}b \cdots + ab^{n-2} + b^{n-1}) - b(a^{n-1} + a^{n-2}b \cdots + ab^{n-2} + b^{n-1}) \qquad (2)$$

$$= (a-b)(a^{n-1} + a^{n-2}b \cdots + ab^{n-2} + b^{n-1}) \qquad (3)$$

In the first line we add and subtract terms of the form $a^x b^y$ such that $x + y = n$, this makes no change to the value of the expression because each new term is eliminated by its corresponding term. In the second line we factor an a from each positive term and a $-b$ from each negative term. Then in the final line the entire expression is factored.

Theorem 2

let a be a real number and n be a positive odd number number, then:

$$a^n + a^{n-1} + a^{n-2} + \cdots + a + 1$$

$$= a^{n-1}(a+1) + a^{n-3}(a+1) + \cdots + a^2(a+1) + (a+1) \qquad (1)$$

$$= (a+1)(a^{n-1} + a^{n-3} + \cdots + a^2 + 1) \qquad (2)$$

In line 1 we factor each pair of terms, since consecutive terms differ by a factor of a we can take out the entirety of the second term in the pair leaving an a for the first term and this is done for the whole polynomial. since n is odd and there are n terms followed by a $+1$ there are an even number of terms which is why we are able to pair them up. Then in line 2 we factor the $a+1$ from each term.

Main proof

let a be a real number and x and y be natural numbers with $x > y$, then if:

$$a^x = a^y$$

$$\frac{a^x}{a^y} = 1 \qquad (a^y \neq 0)$$

$$a^{x-y} = 1$$

$$a^{x-y} - 1 = 0$$

$$a^{x-y} - 1^{x-y} = 0$$

$$(a-1)(a^{x-y-1} + a^{x-y-2} + \cdots + a + 1) = 0 \qquad \text{let } x - y - 1 = n$$

$$(a-1)(a^n + a^{n-1} + \cdots + a + 1) = 0$$

From here we will investigate two cases, in the first case x and y are of the same parity so $x - y$ will be even hence n is odd, in the second case x and y will be of different parity so $x - y$ is odd and n is even.

Case 1:

$$(a-1)(a^n + a^{n-1} + \cdots + a + 1) = 0$$

$$(a-1)(a+1)(a^{n-1} + a^{n-3} + \cdots + a^2 + 1) = 0$$

Since n is odd in this case we can see that in the third bracket each algebraic term is being raised to an even power and it is therefore greater than or equal to zero, since we have $+1$ too that means the whole polynomial will be greater than 0 for all a so will have no root. This means our solutions come from $a-1 = 0$ and $a+1 = 0$ which give $a = 1$ and $a = -1$ respectively.

Case 2:

$$(a-1)(a^n + a^{n-1} + \cdots + a + 1) = 0$$

In this case it will be shown that $a^n + a^{n-1} + \cdots + a + 1 = 0$ has no real roots. This is clearly true for $a > 0$ since each term would be positive hence their sum would be too, so any possible roots would need to be negative. Now consider the sum like this, $(a^n + a^{n-1}) + (a^{n-2} + a^{n-3}) + \cdots + (a^2 + a) + 1 = 0$, for $a < -1$, $|a^n| > |a^{n-1}| > |a^{n-2}| > \cdots > |a^2| > |a|$. This means that for each set of brackets in the sum the first term has a greater magnitude than the second but since n is even the first term in each bracket will be positive (since its power is a multiple of two) and the second term will be odd, these facts imply that the total value of each bracket is greater than zero so the whole sum is too. Next consider the sum written like this $a^n + (a^{n-1} + a^{n-2}) + \cdots + (a+1)$, for $-1 < a < 0$, $|a^n| < |a^{n-1}| < |a^{n-2}| < \cdots < |a^2| < |a|$. This means that for each set of brackets in the sum the first term has a lesser magnitude than the second but since n is even the first term will be negative and the second will be positive. This implies that each bracket is positive and since a^n is also positive, the whole sum is positive too. It is clear that at 0 and -1 the sum evaluates to 1, if we put all this together we see that the sum is greater than zero for all real numbers and hence has no real roots as we intended to show. This implies that the only solution will be given by $a - 1 = 0$ which gives us $a = 1$

A note on zero:

It is important to note that early in the proof we divided by a^y which means we lost any solution in which this takes on the value of zero, clearly the only such case is $a = 0$. By considering zero separately we can see that, since $0^y = 0$ for all y, zero will be a solution to the equation and will work regardless of the parity of x and y.

Conclusion

In conclusion we have found that zero and one will satisfy the equation with the given conditions and that negative one will also satisfy the equation given that x and y are of the same parity, furthermore, we have shown that no other real numbers satisfy the equation. For further investigation, it may be interesting to try to expand this result to negative, rational and perhaps real values of x and y, in addition to this I conjecture that an evaluation of the same equation but considering complex solutions will yield new solutions and possibly even an infinite number of them.