Area of regular polygons

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1 Introduction

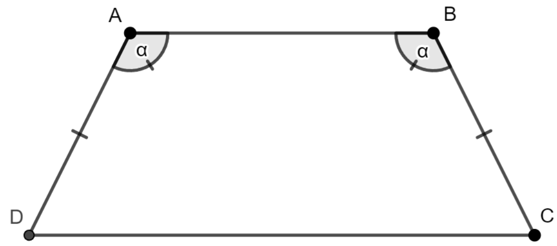
In this paper we will derive a formula to calculate the area of a regular n-gon (a polygon with n sides), This will be done by dividing the n-gon into a series of trapeziums, and for odd values of n a triangle too, then we will use recurrence relations between the trapezium to generate three functions to give some of their properties. These function will be used to build the formulae for the areas of polygons (A much easier proof is possible by cutting the polygon into congruent isosceles triangle but I think the method presented here is more fun). In the paper the following things will be used without proof:

- $\sin(n\pi) = 0 \quad n \in \mathbb{N}$
- $\cos(n\pi) = (-1)^n \quad n \in \mathbb{N}$
- the t-formulae

2 Geometry

Trapezium lemma

We will begin by showing that if a quadrilateral $ABCD$ has $AD = BC$ and $\angle DAB = \angle CBA$ then it must be the case that AB and CD are parallel.

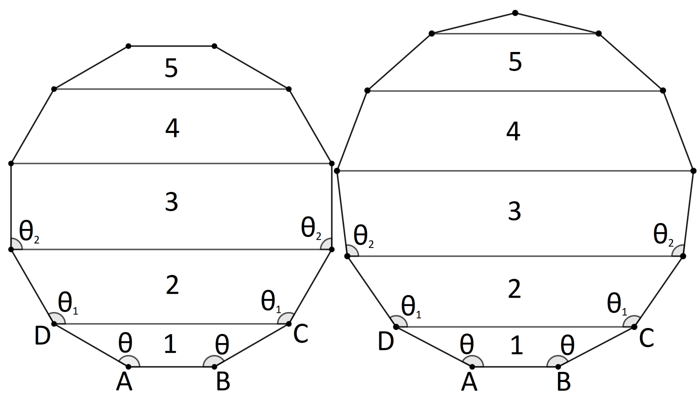


First, we assume the angle α is not right, then let AD and BC be extended until they cross, which they must since they make co-interior angles that sum to 2α which isn't equal to 180 since $\alpha \neq 90$. Call the point at which they intersect E, since $\angle DAB = \angle CBA$ it must be that $\angle BAE = \angle ABE$ which means ΔAEB is isosceles. This means that $AE = BE$ and since $AD = BC$ it is true that $DE = CE$, but this means ΔDCE is also isosceles. From this it follows that $\angle DCE = \angle CDE$, let this angle be β, and since the sum of internal angles of $ABCD$ is 360 we have that $2\alpha + 2\beta = 360$. This implies that $\alpha + \beta = 180$ but $\alpha + \beta$ is a sum of co-interior angles, and since it is equal to 180 it must be the case that AB and CD are parallel.

Should $\alpha = 90$ then it is clear that AD and BC are parallel since the co-interior angles sum to 180. Construct the line segment BD and consider ΔDAB and ΔDCB, BD is common, it is given that $AD = BC$, and $\angle ADB = \angle CBD$ (alternate), thus we conclude that ΔDAB and ΔDCB are congruent. We then have $\angle DCB = \alpha = 90$ which means that $\angle ABC + \angle BCD = 180$ therefore AB and CD are parallel. Hence we have shown that if a quadrilateral $ABCD$ has $AD = BC$ and $\angle DAB = \angle CBA$ then it must be the case that AB and CD are parallel.

Application to polygons

We can use the trapezium lemma to show that any regular polygon can be split into trapeziums and, in the case of polygons with an odd number of sides, a triangle.

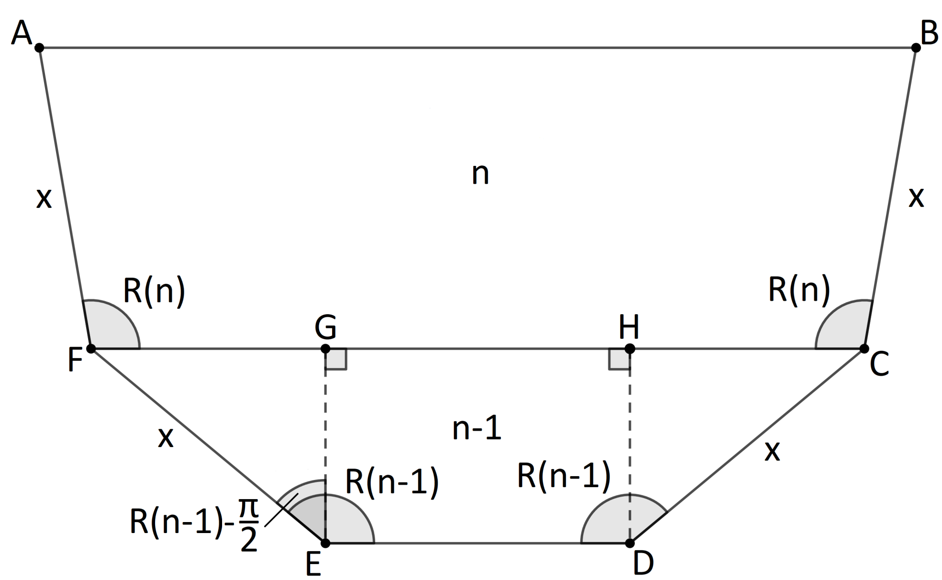


We know that for all regular polygons every side will be equal in length and every angle will be equal in size. In the diagrams above a 12-gon (left) and a 13-gon(right) are shown split into smaller shapes, it is clear that in both of them the first region (labeled 1) has three sides and two angles in common with the polygon. By considering the trapezium lemma we can conclude that AB is parallel to DC hence region 1 is a trapezium, another important thing to note as a consequence of this lemma is that $\angle ADC = \angle DCB$. Since the internal angles of the polygon are equal and the small angles in region 1 are equal it must be the case that the two angles marked θ_1 must be equal (this is of course only true for each diagram separately and the angles marked the same in different diagrams will not be equal). If we now turn our attention to region 2 we see that, by the same lemma, it must too be a trapezium and that yet again the angles marked θ_2 must be equal. This reasoning cascades up the shape and we get that the whole thing, regardless of how many sides it has, can be divided in this way nearly to the top. What happens when you get to the last few region depends on the parity of the number of sides the polygon has, to understand why this is we will consider the 12-gon. When the fist region is created by drawing DC, three original sides of the shape are lost and one new side is gained, this means that the remaining shape has nine sides of equal length, an odd number, and one side of a new length. Each subsequent region that is cut off takes two sides of the original polygon, so the number of sides of equal length remaining will stay odd until it reaches three which becomes the final trapezium. In the case of odd sided polygons, like the 13-gon, it will reach a point where it has two sides of equal length remaining so instead of the final region being a trapezium it is a triangle.

3 Useful functions

Definitions

The trapezium that makes up region 1 is nice because it has two angles and three sides in common with the polygon, unfortunately the rest of them only share two sides with the polygon. However, since each trapezium is connected to the next we can use the properties of the first to determine the properties of the rest. We will define three functions, $R(n)$ which will give the value of the two angles of the nth trapezium that are on the side shared with the previous region, $T(n)$ which will give the length of the side of the nth trapezium that is not shared with the previous region (you can think of this as the top of the trapezium), and $A(n)$ which gives the area of the nth region. If the first two definitions are confusing then refer back to the previous diagram, for the 12-gon (left) $R(1) = \theta$, $R(2) = \theta_1$, $R(3) = \theta_2$ and so on, and $T(1) = |DC|$ and $T(2)$ would be equal to the length of the line segment separating regions 2 and 3.



The diagram shows an example of two regions taken from a polygon with side length x and internal angle θ so we know that $\angle EFA = \angle DCB = \theta$. since ED and FC are parallel $\angle CFE = \pi - R(n-1)$ which implies $R(n) = \theta - (\pi - R(n-1))$ and we know $R(1) = \theta$ for all polygons since the first region shares these angles with the polygon.

$$\begin{aligned} R(n) &= \theta - (\pi - R(n-1)) \\ &= \theta - \pi + R(n-1) \\ &= 2(\theta - \pi) + R(n-2) \\ &= 3(\theta - \pi) + R(n-3) \\ &\vdots \\ &= (n-1)(\theta - \pi) + R(1) \\ &= (n-1)(\theta - \pi) + \theta \\ &= (n-1)\theta - (n-1)\pi + \theta \\ &= n\theta - (n-1)\pi \end{aligned}$$

Since the bottom of region $n-1$ would have been the top of the previous region we know that ED must be $T(n-2)$, we can also use this reasoning when considering the first region to conclude that $T(0) = x$. The two dashed perpendicular lines in the diagram make it clear that $T(n-1)$ we be the sum of $T(n-2)$ and two other lengths that are clearly equal, namely FG and HC. By considering the right triangle EFG it can be determined that $|FG| = x \sin\left(R(n-1) - \frac{\pi}{2}\right)$, from this we get

$$T(n-1) = T(n-2) + 2x\sin\left(R(n-1) - \frac{\pi}{2}\right)$$

$$\begin{aligned} T(n) &= T(n-1) + 2x\sin\left(R(n) - \frac{\pi}{2}\right) \\ &= T(n-1) - 2x\cos(R(n)) \\ &= T(n-2) - 2x[\cos(R(n)) + \cos(R(n-1))] \\ &= T(n-3) - 2x[\cos(R(n)) + \cos(R(n-1)) + \cos(R(n-2))] \\ &\vdots \\ &= T(0) - 2x\sum_{i=1}^{n}\cos(R(i)) \end{aligned}$$

$$= x - 2x \sum_{i=1}^{n} \cos(R(i))$$

To find $A(n-1)$ we would need to sum the areas of the two triangles EFG and DCH and the rectangle $EDHG$, by using simple trigonometry on the right triangle EFG we find that it has a height of $x\cos\left(R(n-1) - \frac{\pi}{2}\right)$ and a base of $x\sin\left(R(n-1) - \frac{\pi}{2}\right)$. Multiplying these give the combined area of the triangles since they are congruent and by multiplying the height by $T(n-2)$ we get the area of the rectangle, summing these gives

$$A(n-1) = \left[x\cos\left(R(n-1) - \frac{\pi}{2}\right)\right]\left[x\sin\left(R(n-1) - \frac{\pi}{2}\right)\right] + T(n-2)\left[x\cos\left(R(n-1) - \frac{\pi}{2}\right)\right]$$

$$A(n) = [x\sin(R(n))][-x\cos(R(n))] + T(n-1)[x\sin(R(n))]$$

$$= x\sin(R(n))[T(n-1) - x\cos(R(n))]$$

In conclusion we have

$$R(n) = n\theta - (n-1)\pi$$

$$T(n) = x - 2x \sum_{i=1}^{n} \cos(R(i))$$

$$A(n) = x\sin(R(n))[T(n-1) - x\cos(R(n))]$$

Closed form for $T(n)$

We have that $T(n) = x - 2x \sum_{i=1}^{n} \cos(R(i))$, to find a closed form we will consider the sum in isolation. The first step is to sub in $R(i)$ and simplify the sum to get a series of cosines.

$$\text{let } C = \sum_{i=1}^{n} \cos(R(i))$$

$$= \sum_{i=1}^{n} \cos(i\theta - (i-1)\pi)$$

$$= \sum_{i=1}^{n} \cos(i\theta)\cos((i-1)\pi) + \sin(i\theta)\sin((i-1)\pi)$$

$$= \sum_{i=1}^{n} \cos(i\theta)\cos((i-1)\pi)$$

$$= \sum_{i=1}^{n} (-1)^{i-1}\cos(i\theta)$$

$$= \cos(\theta) - \cos(2\theta) + \cos(3\theta) - \cdots + (-1)^{n-1}\cos(n\theta)$$

To find the value of this series we will define a new series, S, similar in form to C but using sine instead of cosine. We do this with the intent of finding $C + iS$ and using Euler's relation to write the series in terms of $e^{i\theta}$ then dealing with it as a geometric series and finally splitting it into its real and imaginary parts because we know that the real part is C, which we are looking for.

$$C = \cos(\theta) - \cos(2\theta) + \cos(3\theta) - \cdots + (-1)^{n-1}\cos(n\theta)$$

$$S := \sin(\theta) - \sin(2\theta) + \sin(3\theta) - \cdots + (-1)^{n-1}\sin(n\theta)$$

$$C+iS = [\cos(\theta)+i\sin(\theta)]-[\cos(2\theta)+i\sin(2\theta)]+\cdots+(-1)^{n-1}[n\cos(\theta)+i\sin(n\theta)]$$

$$-C-iS = -[\cos(\theta)+i\sin(\theta)]+[\cos(2\theta)+i\sin(2\theta)]-\cdots+(-1)^{n}[n\cos(\theta)+i\sin(n\theta)]$$

$$= -e^{i\theta} + \left(-e^{i\theta}\right)^2 + \left(-e^{i\theta}\right)^3 + \cdots + \left(-e^{i\theta}\right)^n$$

$$= -e^{i\theta}\left(1 + \left(-e^{i\theta}\right) + \left(-e^{i\theta}\right)^2 + \cdots + \left(-e^{i\theta}\right)^{n-1}\right)$$

$$= -e^{i\theta}\left(1 + [-C - iS] - \left(-e^{i\theta}\right)^n\right)$$

$$= -e^{i\theta} - e^{i\theta}[-C - iS] - \left(-e^{i\theta}\right)^{n+1}$$

$$[-C - iS] + e^{i\theta}[-C - iS] = -e^{i\theta} - \left(-e^{i\theta}\right)^{n+1}$$

$$C + iS = \frac{e^{i\theta} + \left(-e^{i\theta}\right)^{n+1}}{1 + e^{i\theta}}$$

$$= \frac{e^{i\theta} + (-1)^{n+1}e^{i(n+1)\theta}}{e^{i\frac{1}{2}\theta}\left(e^{-i\frac{1}{2}\theta} + e^{i\frac{1}{2}\theta}\right)}$$

$$= \frac{e^{i\frac{1}{2}\theta} + (-1)^{n+1}e^{i\left(n+\frac{1}{2}\right)\theta}}{e^{-i\frac{1}{2}\theta} + e^{i\frac{1}{2}\theta}}$$

$$= \frac{\cos\left(\frac{1}{2}\theta\right) + i\sin\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right) + i(-1)^{n+1}\sin\left(n\theta + \frac{1}{2}\theta\right)}{\cos\left(-\frac{1}{2}\theta\right) + i\sin\left(-\frac{1}{2}\theta\right) + \cos\left(\frac{1}{2}\theta\right) + i\sin\left(\frac{1}{2}\theta\right)}$$

$$= \frac{\left[\cos\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right)\right] + i\left[\sin\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\sin\left(n\theta + \frac{1}{2}\theta\right)\right]}{2\cos\left(\frac{1}{2}\theta\right)}$$

$$= \frac{\cos\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right)}{2\cos\left(\frac{1}{2}\theta\right)} + i\frac{\sin\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\sin\left(n\theta + \frac{1}{2}\theta\right)}{2\cos\left(\frac{1}{2}\theta\right)}$$

$$\implies C = \frac{\cos\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right)}{2\cos\left(\frac{1}{2}\theta\right)}$$

Having found C we now have

$$\begin{aligned} T(n) &= x - 2x\left(\frac{\cos\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right)}{2\cos\left(\frac{1}{2}\theta\right)}\right) \\ &= x\left(1 - \frac{\cos\left(\frac{1}{2}\theta\right) + (-1)^{n+1}\cos\left(n\theta + \frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}\right) \\ &= x\left(\frac{(-1)^n\cos\left(\theta\left(n + \frac{1}{2}\right)\right)}{\cos\left(\frac{1}{2}\theta\right)}\right) \end{aligned}$$

Closed form for $A(n)$

We have that $A(n) = x\sin(R(n))[T(n-1) - x\cos(R(n))]$, now that we have a more workable form for $T(n)$ we can substitute it into this formula and simplify.

$$\begin{aligned} A(n) &= x\sin(R(n))[T(n-1) - x\cos(R(n))] \\ &= x\sin(R(n))[T(n-1) - x\cos(n\theta - (n-1)\pi)] \\ &= x\sin(R(n))[T(n-1) - x[\cos(n\theta)\cos((n-1)\pi) + \sin(n\theta)\sin((n-1)\pi)]] \\ &= x\sin(R(n))[T(n-1) - x\cos(n\theta)(-1)^{n-1}] \\ &= x\sin(n\theta - (n-1)\pi)[T(n-1) - x\cos(n\theta)(-1)^{n-1}] \\ &= x[\sin(n\theta)\cos((n-1)\pi) - \sin((n-1)\pi)cos(n\theta)][T(n-1) - x\cos(n\theta)(-1)^{n-1}] \\ &= x\sin(n\theta)(-1)^{n-1}[T(n-1) - x\cos(n\theta)(-1)^{n-1}] \end{aligned}$$

$$
\begin{aligned}
&= x\sin(n\theta)(-1)^{n-1}\left[x\frac{(-1)^{n-1}\cos\left(\theta\left(n-\frac{1}{2}\right)\right)}{\cos\left(\frac{1}{2}\theta\right)} - x\cos(n\theta)(-1)^{n-1}\right] \\
&= \left((-1)^{n-1}\right)^2 x^2\sin(n\theta)\left[\frac{\cos\left(n\theta-\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)} - \cos(n\theta)\right] \\
&= x^2\sin(n\theta)\left[\frac{\cos(n\theta)\cos\left(\frac{1}{2}\theta\right)+\sin(n\theta)\sin\left(\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)} - \cos(n\theta)\right] \\
&= x^2\sin(n\theta)\left[\frac{\sin(n\theta)\sin\left(\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}\right] \\
&= x^2\sin^2(n\theta)\tan\left(\frac{1}{2}\theta\right)
\end{aligned}
$$

This is one formula for $A(n)$ but by it can be rewritten using the t-formulae.

$$
\begin{aligned}
\text{let }\ \tan\left(\frac{1}{2}\theta\right) &= t \\
&= \frac{2t}{2} \\
&= \frac{\frac{2t}{1+t^2}}{\frac{2}{1+t^2}} \\
&= \frac{\frac{2t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}} \\
&= \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \\
&= \frac{\sin(\theta)}{1+\cos(\theta)} \\
\therefore A(n) &= x^2\frac{\sin(\theta)\sin^2(n\theta)}{1+\cos(\theta)}
\end{aligned}
$$

4 Finding areas

Closed form for $\sum_{i=1}^{n} A(i)$

By considering $\sum_{i=1}^{n} A(i)$ we can sum the areas of the trapeziums to get the area of the polygon.

$$\sum_{i=1}^{n} A(i) = \sum_{i=1}^{n} x^2 \frac{\sin(\theta)\sin^2(n\theta)}{1+\cos(\theta)}$$

$$= x^2 \frac{\sin(\theta)}{1+\cos(\theta)} \sum_{i=1}^{n} \sin^2(n\theta)$$

$$\text{let } C = \sum_{i=1}^{n} \sin^2(n\theta)$$

$$C = \sin^2(\theta) + sin^2(2\theta) + \cdots + sin^2(n\theta)$$

$$-2C = -2\sin^2(\theta) - 2sin^2(2\theta) - \cdots - 2sin^2(n\theta)$$

$$n - 2C = [1 - 2\sin^2(\theta)] + [1 - 2sin^2(2\theta)] + \cdots + [1 + 2sin^2(n\theta)]$$

$$= \cos(2\theta) + \cos(4\theta) + \cdots + \cos(2n\theta)$$

$$S := \sin(2\theta) + \sin(4\theta) + \cdots + \sin(2n\theta)$$

$$(n-2C)+iS = [\cos(2\theta)+i\sin(2\theta)]+[\cos(4\theta)+i\sin(4\theta)]+\cdots+[\cos(2n\theta)+i\sin(2n\theta)]$$

$$= e^{i2\theta} + e^{i4\theta} + \cdots + e^{i2n\theta}$$

$$= e^{i2\theta}\left(1 + e^{i2\theta} + e^{i4\theta} + \cdots + e^{i2(n-1)\theta}\right)$$

$$= e^{i2\theta}\left(1 + [(n-2C)+iS] - e^{i2n\theta}\right)$$

$$= e^{i2\theta} + e^{i2\theta}[(n-2C)+iS] - e^{i2(n+1)\theta}$$

$$[(n-2C)+iS] - [e^{i2\theta}[(n-2C)+iS]] = e^{i2\theta} - e^{i2(n+1)\theta}$$

$$(n-2C)+iS = \frac{e^{i2\theta} - e^{i2(n+1)\theta}}{1 - e^{i2\theta}}$$

$$= \frac{e^{i\theta}\left[e^{i\theta} - e^{i(2n+1)\theta}\right]}{e^{i\theta}\left[e^{-i\theta} - e^{i\theta}\right]}$$

$$= \frac{\cos(\theta) + i\sin(\theta) - \cos(2n\theta+\theta) - i\sin(2n\theta+\theta)}{\cos(-\theta) + i\sin(-\theta) - \cos(\theta) - i\sin(\theta)}$$

$$
\begin{aligned}
&= \frac{\cos(\theta) - \cos(2n\theta + \theta)}{-2i\sin(\theta)} + \frac{\sin(\theta) - \sin(2n\theta + \theta)}{-2\sin(\theta)} \\
&\implies n - 2C = \frac{\sin(\theta) - \sin(2n\theta + \theta)}{-2\sin(\theta)} \\
&C = \frac{\sin(\theta) - \sin(2n\theta + \theta)}{4\sin(\theta)} + \frac{n}{2} \\
&\therefore \sum_{i=1}^{n} A(i) = x^2 \frac{\sin(\theta)}{1 + \cos(\theta)} \left[\frac{\sin(\theta) - \sin(2n\theta + \theta)}{4\sin(\theta)} + \frac{n}{2} \right] \\
&= x^2 \frac{\sin(\theta) - \sin(2n\theta + \theta) + 2n\sin(\theta)}{4(1 + \cos(\theta))} \\
&= x^2 \frac{(2n+1)\sin(\theta) - \sin((2n+1)\theta)}{4(1 + \cos(\theta))}
\end{aligned}
$$

Even sided

Our formula for $\sum_{i=1}^{n} A(i)$ will give the sum of each section of a polygon, for even-sided polygons this is enough to find their area immediately without adding anything else. For a polygon with s sides, where s is even, the upper limit of the sum will be $\frac{s-2}{2}$, this is because each section of the polygon will have two sides from the original shape except for the ones at either end which have three. If we subtract 2 from the number of sides then we are left with two for each of the regions, dividing by two will then give us the number of regions and hence the upper limit of our sum. It is also true that the internal angle of a polygon with s sides is $\pi - \frac{2\pi}{s}$. By substituting and simplifying we get

$$
\begin{aligned}
\sum_{i=1}^{\frac{s-2}{2}} A(i) &= x^2 \frac{(s-1)\sin\left(\pi - \frac{2\pi}{s}\right) - \sin\left((s-1)\left(\pi - \frac{2\pi}{s}\right)\right)}{4\left(1 + \cos\left(\pi - \frac{2\pi}{s}\right)\right)} \\
&= x^2 \frac{(s-1)\sin\left(\frac{2\pi}{s}\right) - \sin\left((s-1)\frac{2\pi}{s}\right)}{4\left(1 - \cos\left(\frac{2\pi}{s}\right)\right)} \\
&= x^2 \frac{(s-1)\sin\left(\frac{2\pi}{s}\right) + \sin\left(\frac{2\pi}{s}\right)}{4\left(1 - \cos\left(\frac{2\pi}{s}\right)\right)} \\
&= x^2 \frac{s}{4} \left(\frac{\sin\left(\frac{2\pi}{s}\right)}{1 - \cos\left(\frac{2\pi}{s}\right)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= x^2 \frac{s}{4}\left(\frac{2\sin\left(\frac{\pi}{s}\right)\cos\left(\frac{\pi}{s}\right)}{2\sin^2\left(\frac{\pi}{s}\right)}\right)\\
&= x^2 \frac{s}{4}\left(\frac{\cos\left(\frac{\pi}{s}\right)}{\sin\left(\frac{\pi}{s}\right)}\right)\\
&= x^2 \frac{s}{4}\cot\left(\frac{\pi}{s}\right)
\end{aligned}
$$

Hence this is the formula for the area of a polygon with s sides and side length x, where s is even.

Odd sided

For a polygon with s sides, where s is odd, the upper bound of the sum would be $\frac{s-3}{2}$, this is justified through similar reasoning to that used to find the corresponding limit for even values of s. By considering the sum of the regions and the triangle we get that the area is given by

$$
\begin{aligned}
&x^2\frac{(s-2)\sin\left(\pi-\frac{2\pi}{s}\right)-\sin\left((s-2)\left(\pi-\frac{2\pi}{s}\right)\right)}{4\left(1+\cos\left(\pi-\frac{2\pi}{s}\right)\right)}+\frac{1}{2}x^2\sin\left(\pi-\frac{2\pi}{s}\right)\\
&= x^2\frac{(s-2)\sin\left(\frac{2\pi}{s}\right)-\sin\left((s-2)\frac{2\pi}{s}\right)}{4\left(1-\cos\left(\frac{2\pi}{s}\right)\right)}+\frac{1}{2}x^2\sin\left(\frac{2\pi}{s}\right)\\
&= x^2\frac{(s-2)\sin\left(\frac{2\pi}{s}\right)+\sin\left(\frac{4\pi}{s}\right)}{4\left(1-\cos\left(\frac{2\pi}{s}\right)\right)}+\frac{1}{2}x^2\sin\left(\frac{2\pi}{s}\right)\\
&= x^2\left[\frac{(s-2)\sin\left(\frac{2\pi}{s}\right)+\sin\left(\frac{4\pi}{s}\right)}{4\left(1-\cos\left(\frac{2\pi}{s}\right)\right)}+\frac{2\sin\left(\frac{2\pi}{s}\right)\left[1-\cos\left(\frac{2\pi}{s}\right)\right]}{4\left(1-\cos\left(\frac{2\pi}{s}\right)\right)}\right]\\
&= x^2\left[\frac{(s-2)\sin\left(\frac{2\pi}{s}\right)+2\sin\left(\frac{2\pi}{s}\right)}{4\left(1-\cos\left(\frac{2\pi}{s}\right)\right)}\right]\\
&= x^2\frac{s}{4}\left[\frac{\sin\left(\frac{2\pi}{s}\right)}{1-\cos\left(\frac{2\pi}{s}\right)}\right]\\
&= x^2\frac{s}{4}\cot\left(\frac{\pi}{s}\right)
\end{aligned}
$$

This is the same formula as before therefore we know it works for all polygons regardless of the number of sides.

5 Infinity

The following section uses maths that I am not confident in and may be wrong, I intend on revisiting it at some point.

It would be interesting to consider what would happen if we took the limit of the formula as s approaches infinity. As the number of sides a polygon has increases it becomes a better and better approximation of a circle and as such we would expect the formula to become an approximation of the area of a circle πr^2 and to eventually equal it at infinity. One problem we run into is that we have an x^2 term in the formula so we would have to set a side length but of course this doesn't make sense for a circle. If x is a constant then it's clear our limit would diverge since as s increases the polygon would get bigger and bigger, from this we conclude that as s tends to infinity x must tend to zero. I don't know how to use two limits and I don't think it would help anyway, instead I think a better idea is to describe x as a function of s such that the formula as a whole converges but then the trouble is choosing a function that isn't just arbitrary. We also have the problem that we have no idea what the diameter or radius of the polygon is so how can we compare it to the formula for the area of a circle. To solve both of these problems we can make x a function of s such that the diameter of the shape is constant. However this skims over the fact that polygons don't have constant diameters (that is when defining a diameter as a straight line through the shape form one point on the perimeter to another, passing through the center). I don't think this matters though because as s gets larger the range of possible lengths of diameters gets smaller, that is to say that all the diameters will converge to be the same anyway so along a we take a valid diameter in the first place it will be fine. We will start by considering only the polygons with s sides such that $s \equiv 2 \mod 4$, since this means s will be even these polygons will have $\frac{s-2}{2}$ regions. $T(\frac{s-2}{4})$ will be the length of a line passing through the center of the polygon. By setting this equal to a constant we can find the necessary function of s for x. Let r be a constant

$$T\left(\frac{s-2}{4}\right) = 2r$$

$$x\left(\frac{(-1)^{\frac{s-2}{4}}\cos\left(\theta\left(\left(\frac{s-2}{4}\right)+\frac{1}{2}\right)\right)}{\cos\left(\frac{1}{2}\theta\right)}\right) = 2r$$

$$x\left(\frac{(-1)^{\frac{s-2}{4}}\cos\left(\left(\pi-\frac{2\pi}{s}\right)\left(\left(\frac{s-2}{4}\right)+\frac{1}{2}\right)\right)}{\cos\left(\frac{\pi}{2}-\frac{\pi}{s}\right)}\right)=2r$$

$$x\left(\frac{(-1)^{\frac{s-2}{4}}\cos\left(\pi\left(\frac{s-2}{4}\right)\right)}{\sin\left(\frac{\pi}{s}\right)}\right)=2r$$

$$x\left(\frac{1}{\sin\left(\frac{\pi}{s}\right)}\right)=2r$$

$$x=2r\sin\left(\frac{\pi}{s}\right)$$

Now that we have this we can take the limit

$$\lim_{s\to\infty}\left(2r\sin\left(\frac{\pi}{s}\right)\right)^2\frac{s}{4}\cot\left(\frac{\pi}{s}\right)=\pi r^2$$

$$\lim_{s\to\infty}4r^2\sin^2\left(\frac{\pi}{s}\right)\frac{s}{4}\cot\left(\frac{\pi}{s}\right)=\pi r^2$$

$$\lim_{s\to\infty}\sin\left(\frac{\pi}{s}\right)s=\pi$$

$$\lim_{s\to\infty}\frac{s}{\pi}\sin\left(\frac{\pi}{s}\right)=1$$

By now making the substitution $\frac{\pi}{s}=x$ (this x is not the side length it's just a random variable) and making sure to adjust the limit accordingly, we get

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

6 Conclusion

Closing remarks

The first thing worth mentioning is that this is certainly not the most direct way of finding the formula for the area of regular polygons. By connecting the center of a polygon to each vertex it will be cut into congruent triangle, the area of these is not hard to find and using this to find a formula for the area is pretty straight forward, and this is just the most direct method I found, there my be better ways of doing it. The point of this paper was not really to find the formula but rather to be a practice in problem solving and proof construction which is true for all my work. I am satisfied with the majority of this work and see no need to revisit it with one exception, the section on infinity doesn't feel fully explored and may not be very rigorous.

I believe the result that I got in this section is right because I think I have seen it before but It feels like an anticlimactic point to stop. In the future, if I find there is more to be done then I will write a follow up to correct any mistakes and fully explore the idea.

List of results

- The area of a polygon with s sides, each of length x is given by $x^2 \frac{s}{4} \cot\left(\frac{\pi}{s}\right)$
- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$