

Machine Learning Course - CS-433

K-Means Clustering

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Clustering

Clusters are groups of points whose inter-point distances are small compared to the distances outside the cluster.

The goal is to find “prototype” points $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K$ and cluster assignments $z_n \in \{1, 2, \dots, K\}$ for all $n = 1, 2, \dots, N$ data vectors $\mathbf{x}_n \in \mathbb{R}^D$.

K-means clustering

Assume K is known.

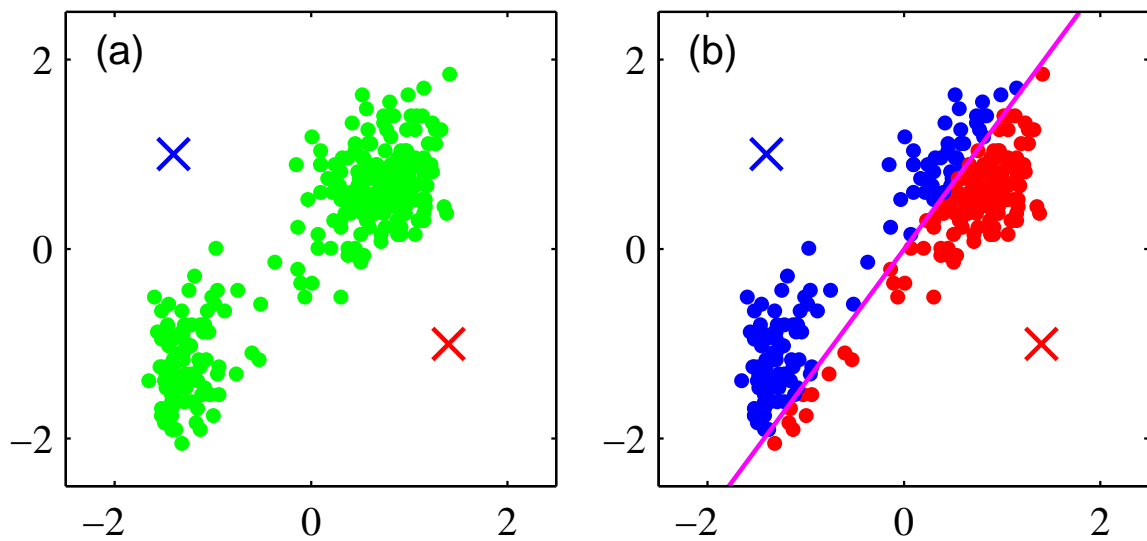
$$\begin{aligned} \min_{\mathbf{z}, \boldsymbol{\mu}} \mathcal{L}(\mathbf{z}, \boldsymbol{\mu}) &= \sum_{n=1}^N \sum_{k=1}^K z_{nk} \|\mathbf{x}_n - \boldsymbol{\mu}_k\|_2^2 \\ \text{s.t. } \boldsymbol{\mu}_k &\in \mathbb{R}^D, z_{nk} \in \{0, 1\}, \sum_{k=1}^K z_{nk} = 1, \\ \text{where } \mathbf{z}_n &= [z_{n1}, z_{n2}, \dots, z_{nK}]^\top \\ \mathbf{z} &= [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N]^\top \\ \boldsymbol{\mu} &= [\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K]^\top \end{aligned}$$

Is this optimization problem easy?

Algorithm: Initialize $\boldsymbol{\mu}_k \forall k$,
then iterate:

1. For all n , compute \mathbf{z}_n given $\boldsymbol{\mu}$.
2. For all k , compute $\boldsymbol{\mu}_k$ given \mathbf{z} .

Step 1: For all n , compute \mathbf{z}_n given $\boldsymbol{\mu}$.

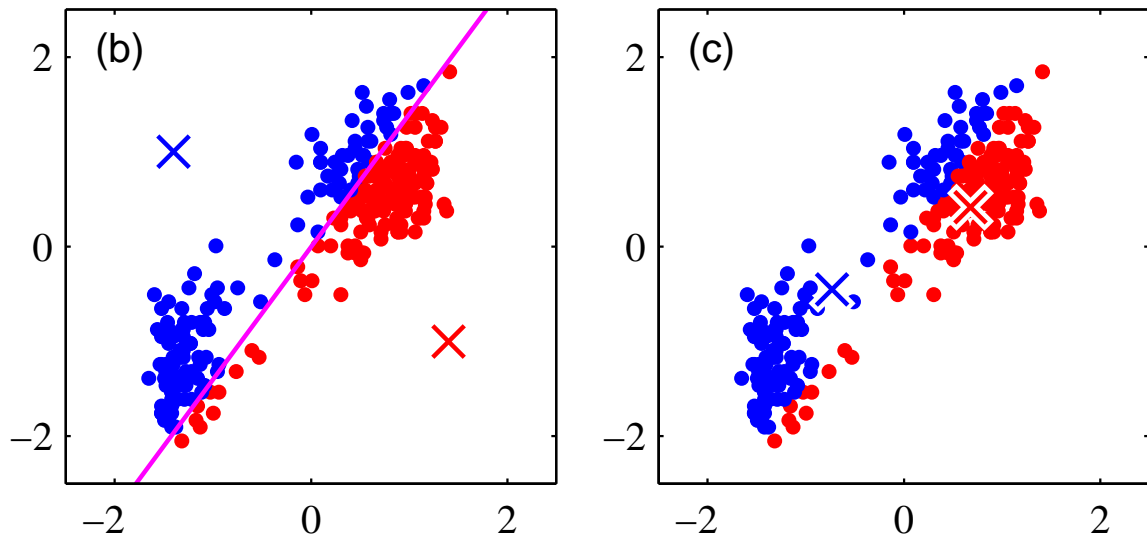


$$z_{nk} = \begin{cases} 1 & \text{if } k = \arg \min_{j=1,2,\dots,K} \|\mathbf{x}_n - \boldsymbol{\mu}_j\|_2^2 \\ 0 & \text{otherwise} \end{cases}$$

Step 2: For all k , compute $\boldsymbol{\mu}_k$ given \mathbf{z} .
Take derivative w.r.t. $\boldsymbol{\mu}_k$ to get:

$$\boldsymbol{\mu}_k = \frac{\sum_{n=1}^N z_{nk} \mathbf{x}_n}{\sum_{n=1}^N z_{nk}}$$

Hence, the name '**K-means**'.



Summary of K-means

Initialize $\mu_k \forall k$, then iterate:

1. For all n , compute z_n given μ .

$$z_{nk} = \begin{cases} 1 & \text{if } k = \arg \min_j \|\mathbf{x}_n - \mu_j\|_2^2 \\ 0 & \text{otherwise} \end{cases}$$

2. For all k , compute μ_k given \mathbf{z} .

$$\mu_k = \frac{\sum_{n=1}^N z_{nk} \mathbf{x}_n}{\sum_{n=1}^N z_{nk}}$$

Convergence to a local optimum is assured since each step decreases the cost (see Bishop, Exercise 9.1).

Coordinate descent

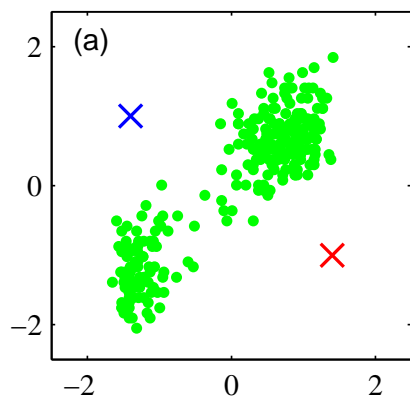
K-means is a coordinate descent algorithm, where, to find $\min_{\mathbf{z}, \boldsymbol{\mu}} \mathcal{L}(\mathbf{z}, \boldsymbol{\mu})$, we start with some $\boldsymbol{\mu}^{(0)}$ and repeat the following:

$$\mathbf{z}^{(t+1)} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{z}, \boldsymbol{\mu}^{(t)})$$

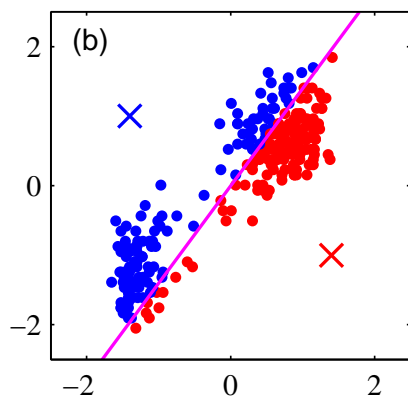
$$\boldsymbol{\mu}^{(t+1)} := \arg \min_{\boldsymbol{\mu}} \mathcal{L}(\mathbf{z}^{(t+1)}, \boldsymbol{\mu})$$

Examples

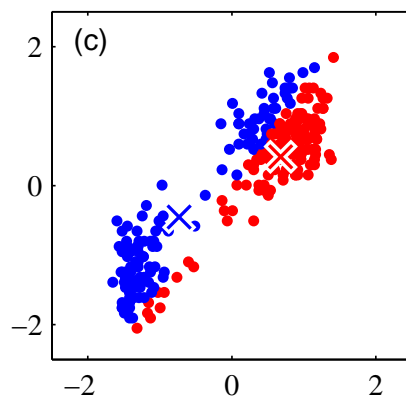
K-means for the “old-faithful” dataset (Bishop’s Figure 9.1)



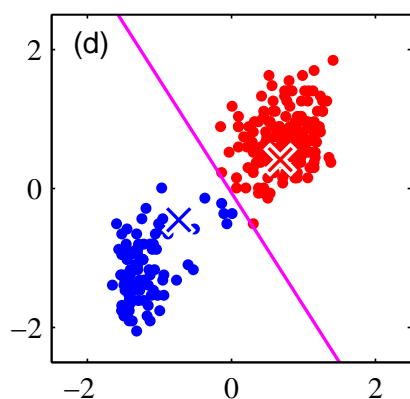
(e) Iteration 0



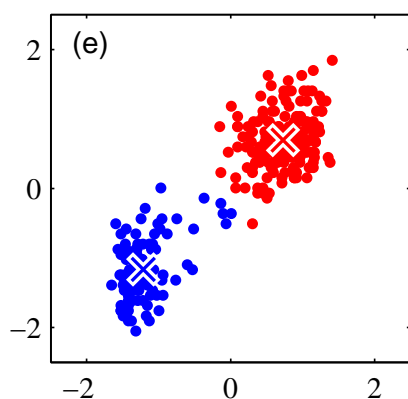
(f) Iteration 1



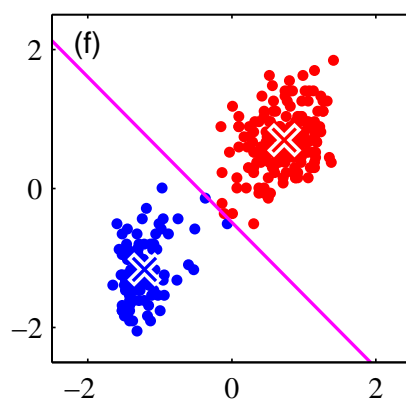
(g) Iteration 1



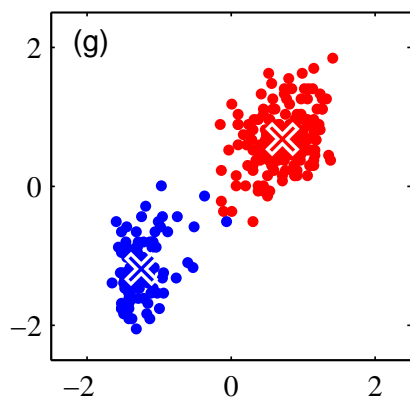
(h) Iteration 2



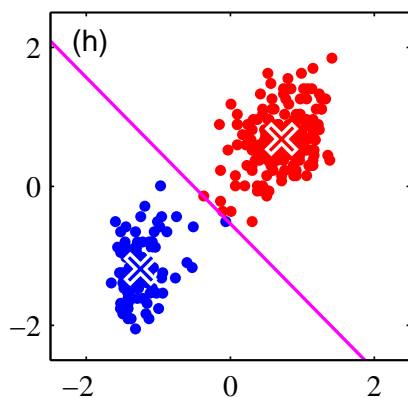
(i) Iteration 2



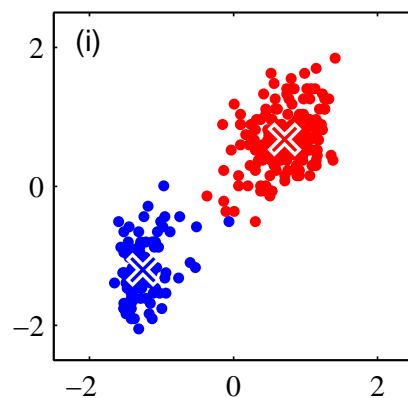
(j) Iteration 3



(k) Iteration 3

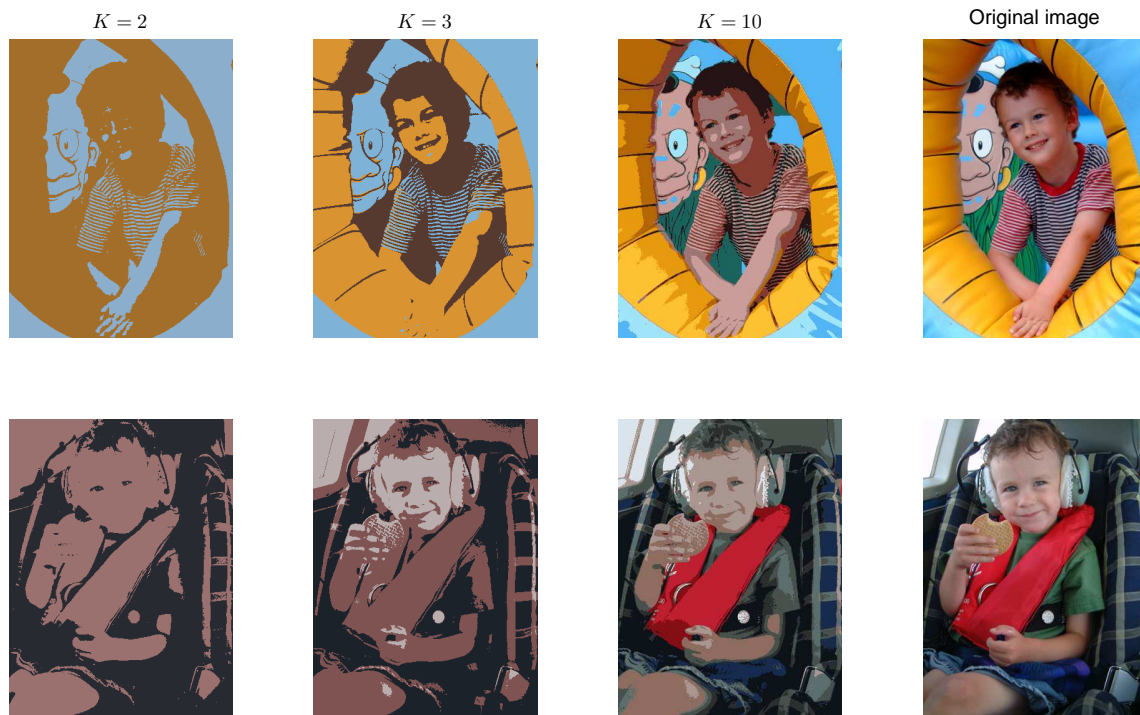


(l) Iteration 4



(m) Iteration 4

Data compression for images (this is also known as vector quantization).



Probabilistic model for K-means

K-means as a Matrix Factorization

Recall the objective

$$\begin{aligned}\min_{\mathbf{z}, \boldsymbol{\mu}} \mathcal{L}(\mathbf{z}, \boldsymbol{\mu}) &= \sum_{n=1}^N \sum_{k=1}^K z_{nk} \|\mathbf{x}_n - \boldsymbol{\mu}_k\|_2^2 \\ &= \|\mathbf{X}^\top - \mathbf{M}\mathbf{Z}^\top\|_{\text{Frob}}^2\end{aligned}$$

$$\text{s.t. } \boldsymbol{\mu}_k \in \mathbb{R}^D,$$

$$z_{nk} \in \{0, 1\}, \sum_{k=1}^K z_{nk} = 1.$$

Issues with K-means

1. Computation can be heavy for large N , D and K .
2. Clusters are forced to be spherical (e.g. cannot be elliptical).
3. Each example can belong to only one cluster (“hard” cluster assignments).