DIFFERENTIAL EQUATIONS, EULER'S METHOD AND STABILITY

Week 7

Dr. Luca Manneschi

OBJECTIVES for this second part

UNDERSTAND AND SIMULATE EBMs

Euler's Method

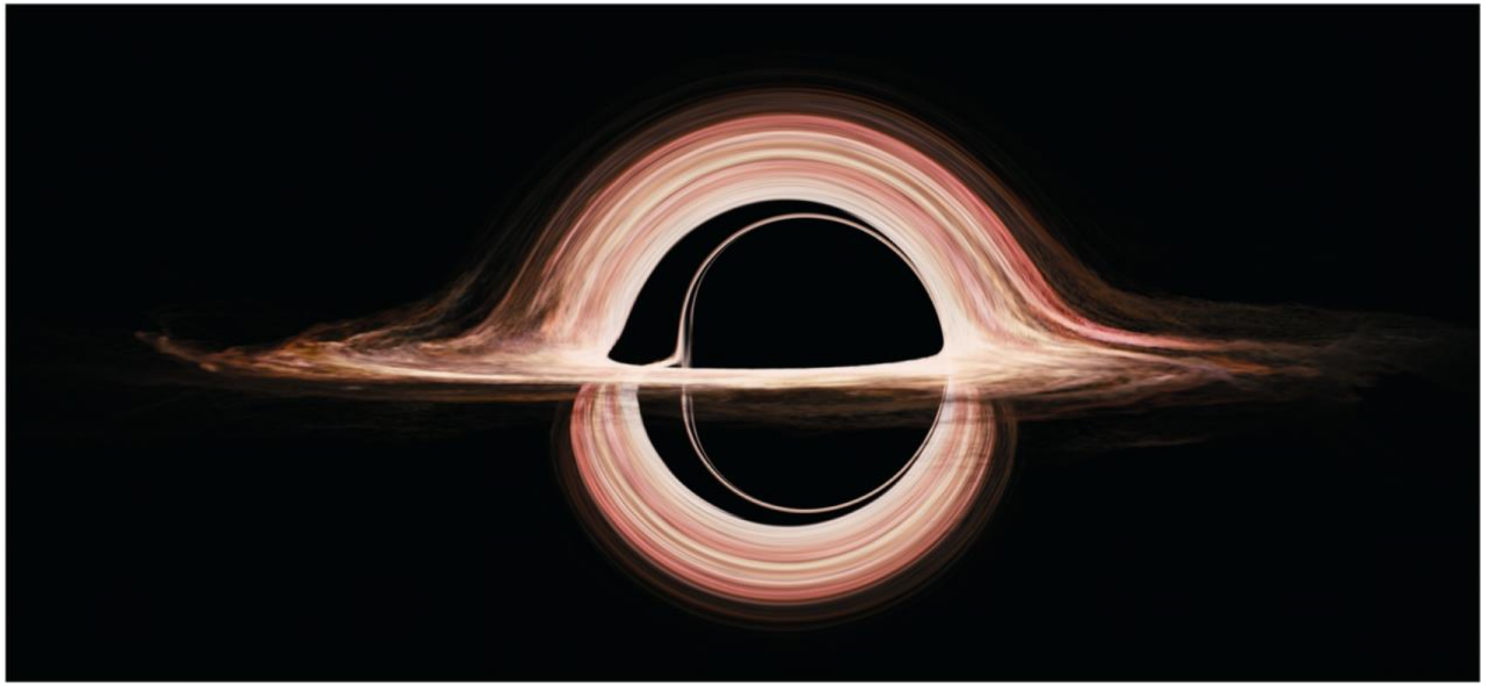
Equilibrium points

Runge-Kutta methods

Higher-order systems

Examples in
Machine Learning/Neuroscience

Why EBMs? An example



James, Oliver, et al. "Gravitational lensing by spinning black holes in astrophysics, and in the movie Interstellar." *Classical and Quantum Gravity* 32.6 (2015): 065001.

Differential Equations: An ignorant introduction

In an equation that is a relation between functions, the solution corresponds to an ensemble of values

$$x^2 - 5x + 6 = 0$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(t) - t^2 = 0$$

In differential equations, derivatives appear as terms of the equations
The solution is the family of functions that satisfies the relation considered

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(1)}(t)$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(2)}(t)$$

Differential Equations: An ignorant introduction

In an equation that is a relation between functions, the solution corresponds to an ensemble of values

$$x^2 - 5x + 6 = 0$$

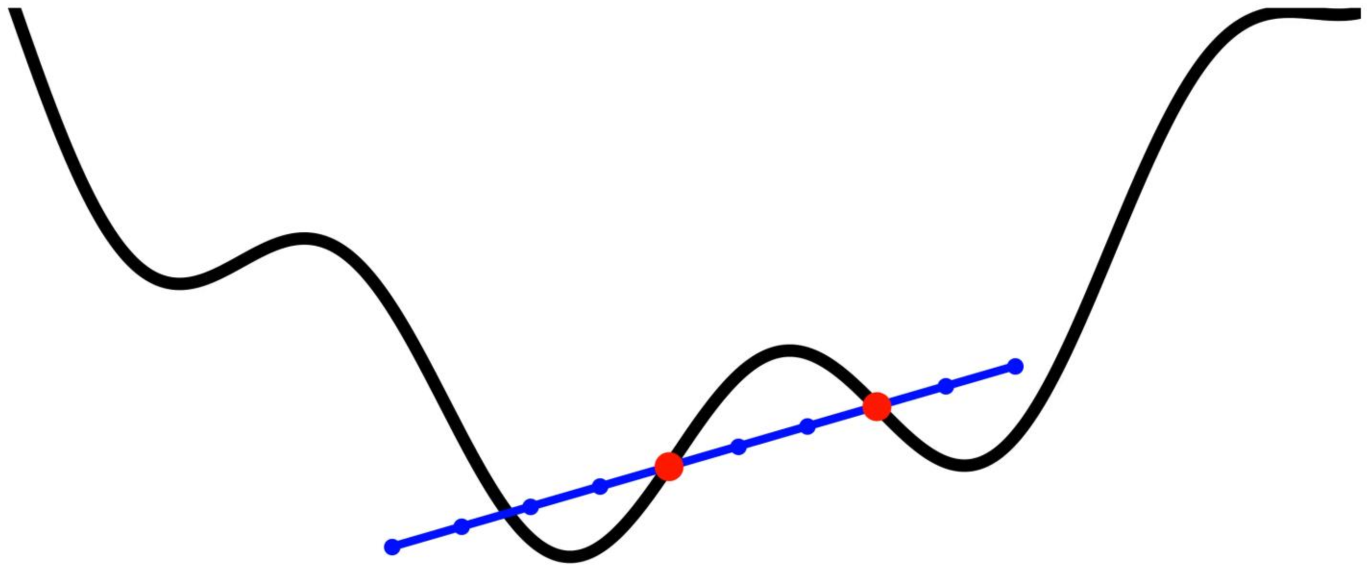
$$x^{(2)}(t) + 5x^{(1)}(t) - x(t) - t^2 = 0$$

In differential equations, derivatives appear as terms of the equations
The solution is the family of functions that satisfies the relation considered

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(1)}(t) \qquad \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(2)}(t)$$

Derivative: Recap

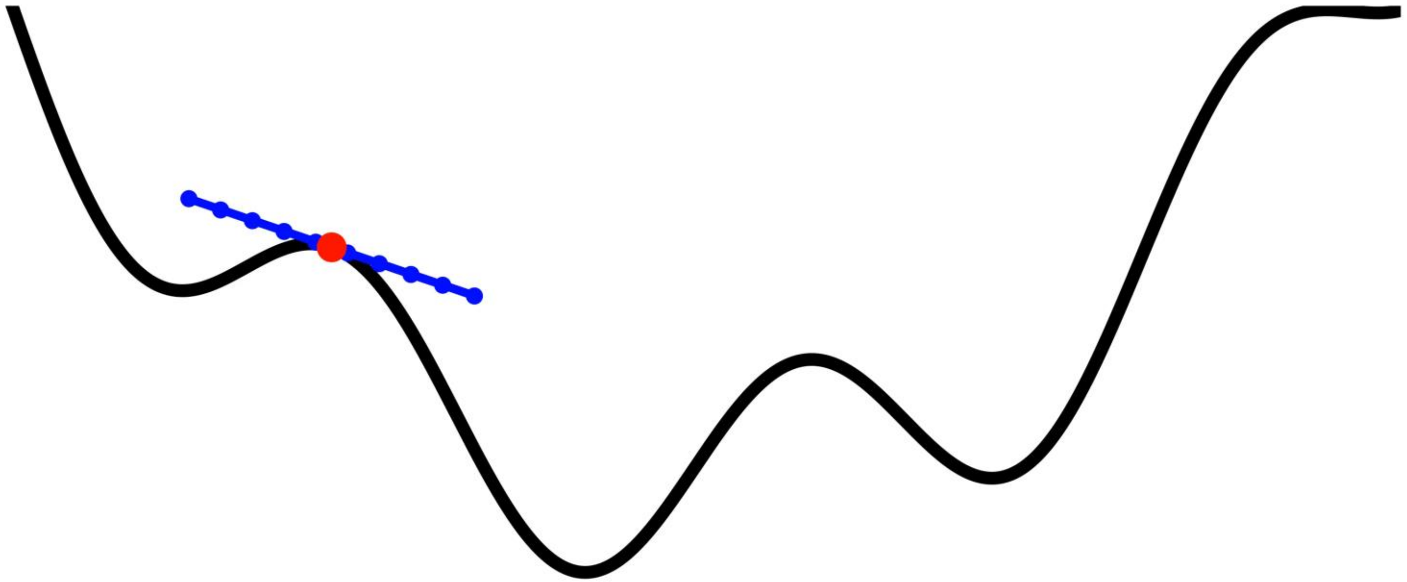
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t}$$



The derivative is the slope of the blue line

Derivative: Recap

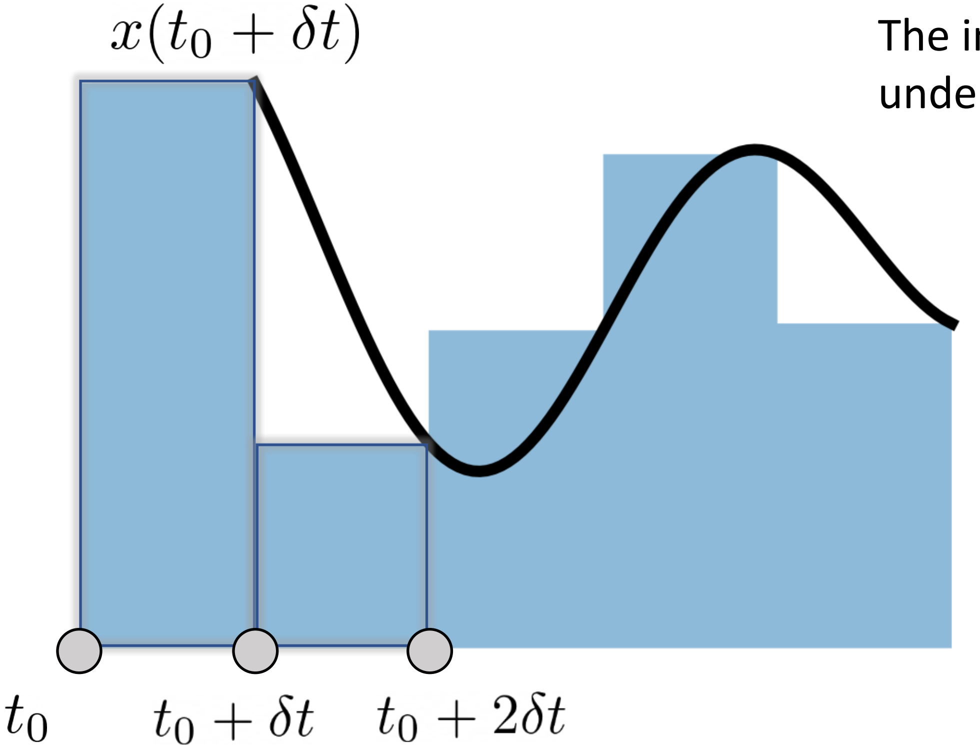
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t}$$



The derivative is the slope of the blue line

Integral: Recap

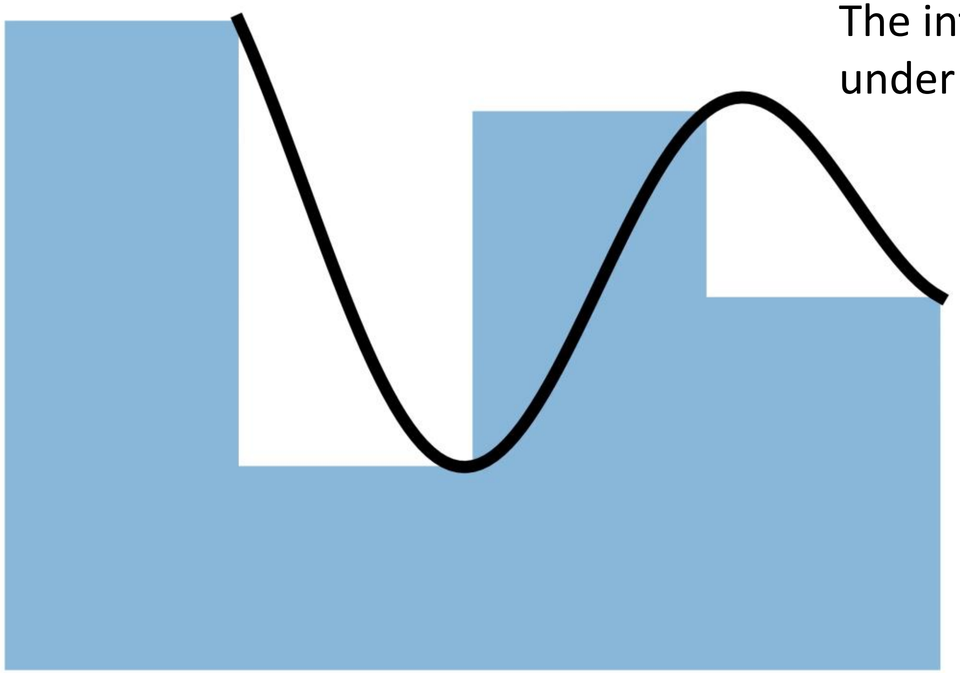
$$\int \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t = x(t) \qquad \int_{t_0}^{t} x(t)\mathrm{d}t = \lim_{\delta t \to 0} \sum_{n=1}^{\frac{t-t_0}{\delta t}} x(t + n\delta t)\delta t$$



The integral can be thought of as the area under the function considered (black line)

Integral: Recap

$$\int \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t = x(t) \qquad \int_{t_0}^{t} x(t)\mathrm{d}t = \lim_{\delta t \to 0} \sum_{n=1}^{\frac{t-t_0}{\delta t}} x(t + n\delta t)\delta t$$



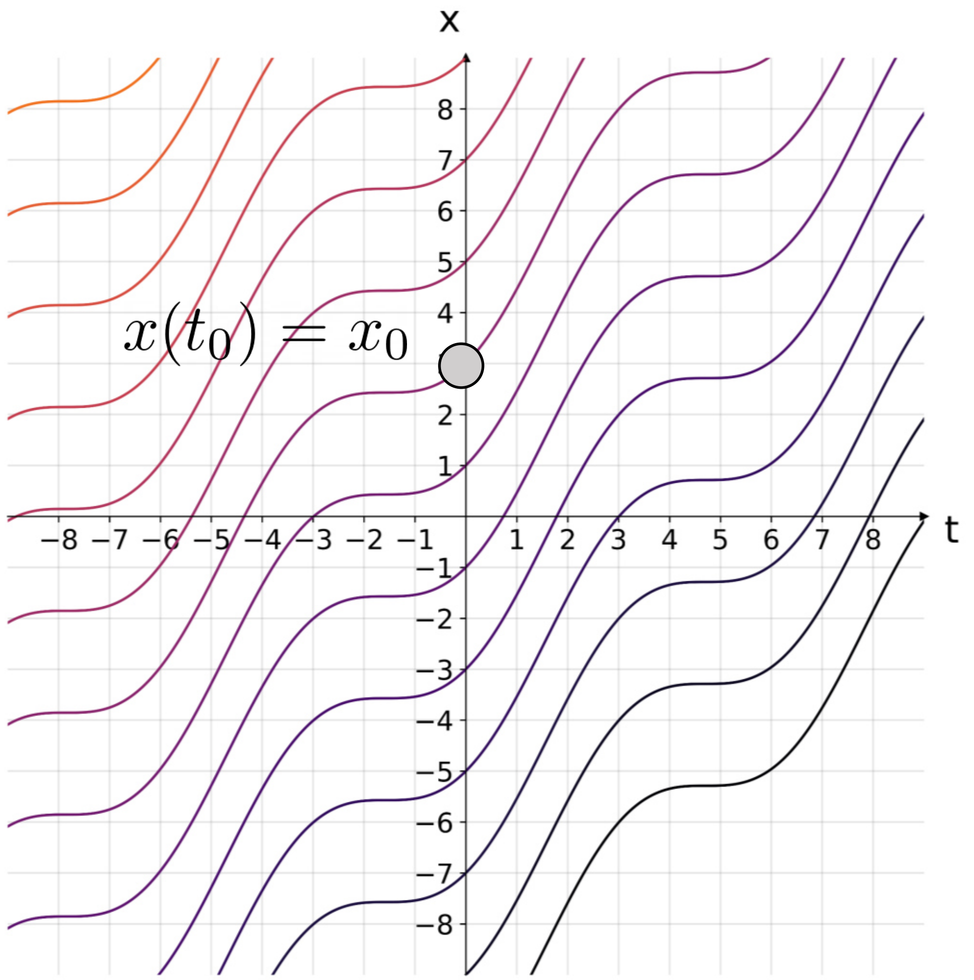
The integral can be thought of as the area under the function considered (black line)

Some simple examples

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t)$$

Can be integrated directly

$$\int \mathrm{d}x = \int f(t)\mathrm{d}t$$



$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin(t) + 1$$

$$x(t) = \int \big(\sin(t) + 1\big)\mathrm{d}t = -\cos(t) + t + k$$

How do we know which one? Initial condition

For this first-order differential equation, we can specify simply $x(t_0) = x_0$ and find *k*

Some simple examples

For instance, given $x(t) = -\cos(t) + t + k$,

for $x(0) = 1$ we find $1 = -1 + k \qquad k = 2$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = f(t)$$

Can also be integrated directly, we need to integrate twice

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = \int f(t)\mathrm{d}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int f(t)\mathrm{d}t$$

Some simple examples

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = t$$

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = \int t\mathrm{d}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^2}{2} + k_1$$

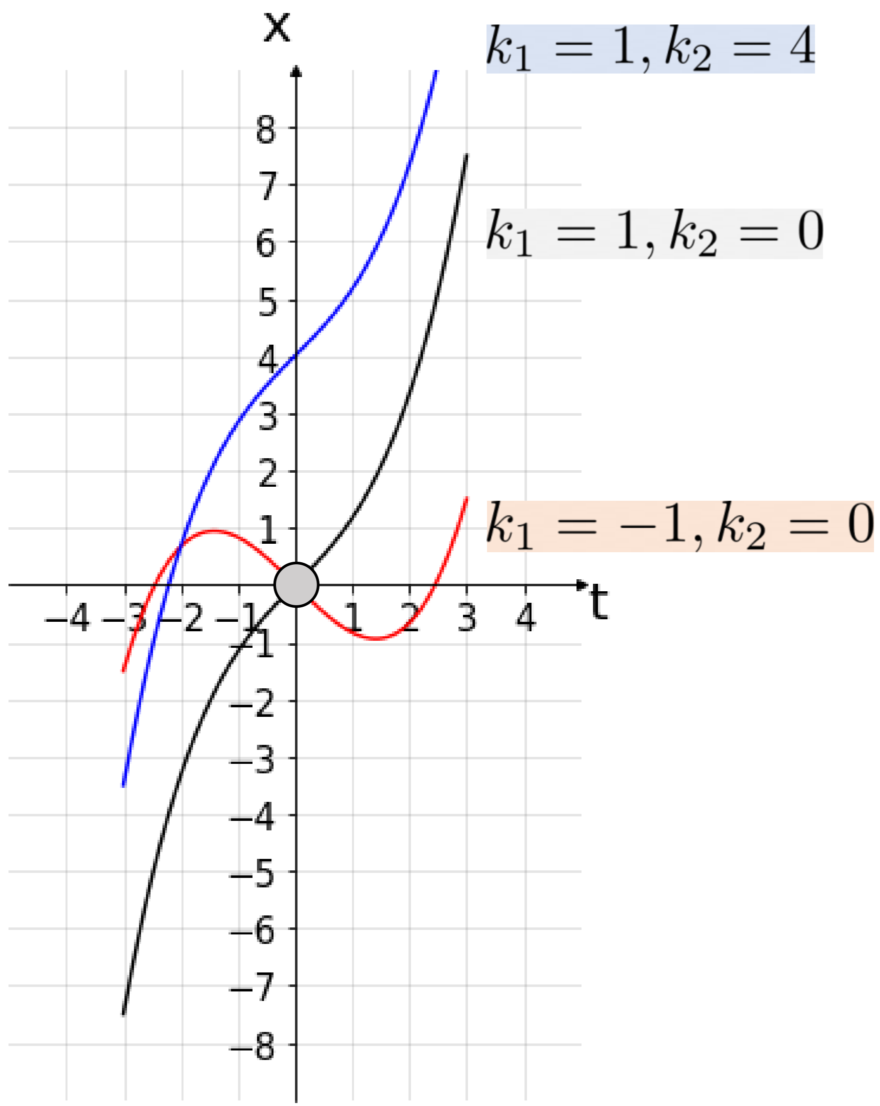
$$\int \mathrm{d}x = \int \frac{t^2}{2} + k_1 \mathrm{d}t$$

$$x(t) = \frac{t^3}{6} + k_1 t + k_2$$

The solutions intersect,
we need to specify

$$x(t_0) = x_0$$

$$x^{(1)}(t_1) = x_1$$



Cauchy problem

A differential equation is of order n if it contains the n-th derivative at maximum

For a differential equation of order n, the task of finding the function that satisfies the equation with the initial conditions

$$x(t_0) = x_0$$

$$x^{(1)}(t_1) = x_1$$

$$\vdots$$

$$x^{(n)}(t_n) = x_n$$

Is called a Cauchy problem

Differential Equations

When solving analytically, we will mostly consider the case

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

$$\frac{\mathrm{d}x(t)}{f\big(x(t)\big)} = g(t)\mathrm{d}t$$

$$\int \frac{\mathrm{d}x}{f(x)} = \int g(t)\mathrm{d}t$$

Examples

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

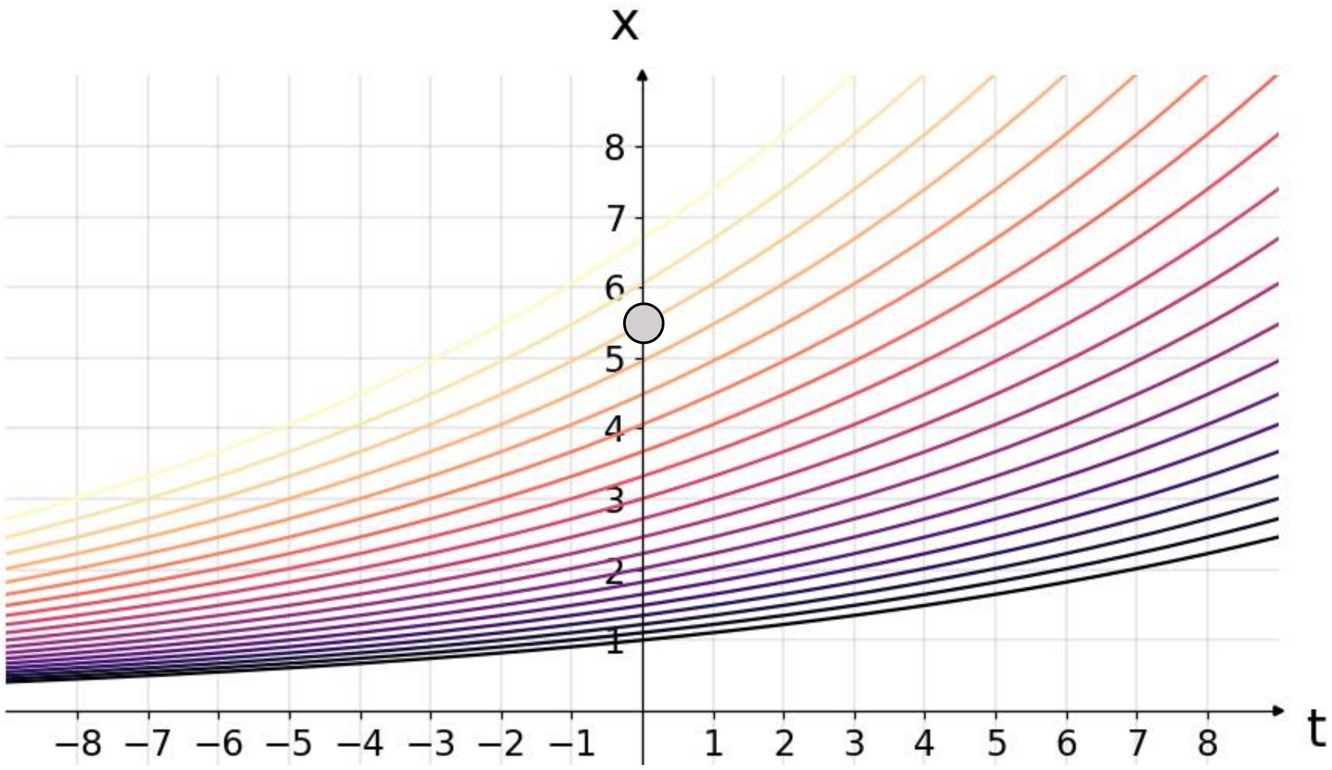
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$$f\big(x(t)\big) = x(t),\;\; g(t) = \alpha$$

$$\int \frac{\mathrm{d}x}{x} = \int \alpha \mathrm{d}t$$

$$\log(x) = \alpha t + k$$

$$x(t) = e^{\alpha t + k}$$



Initial condition

$$x(t_0) = x_0$$

Examples

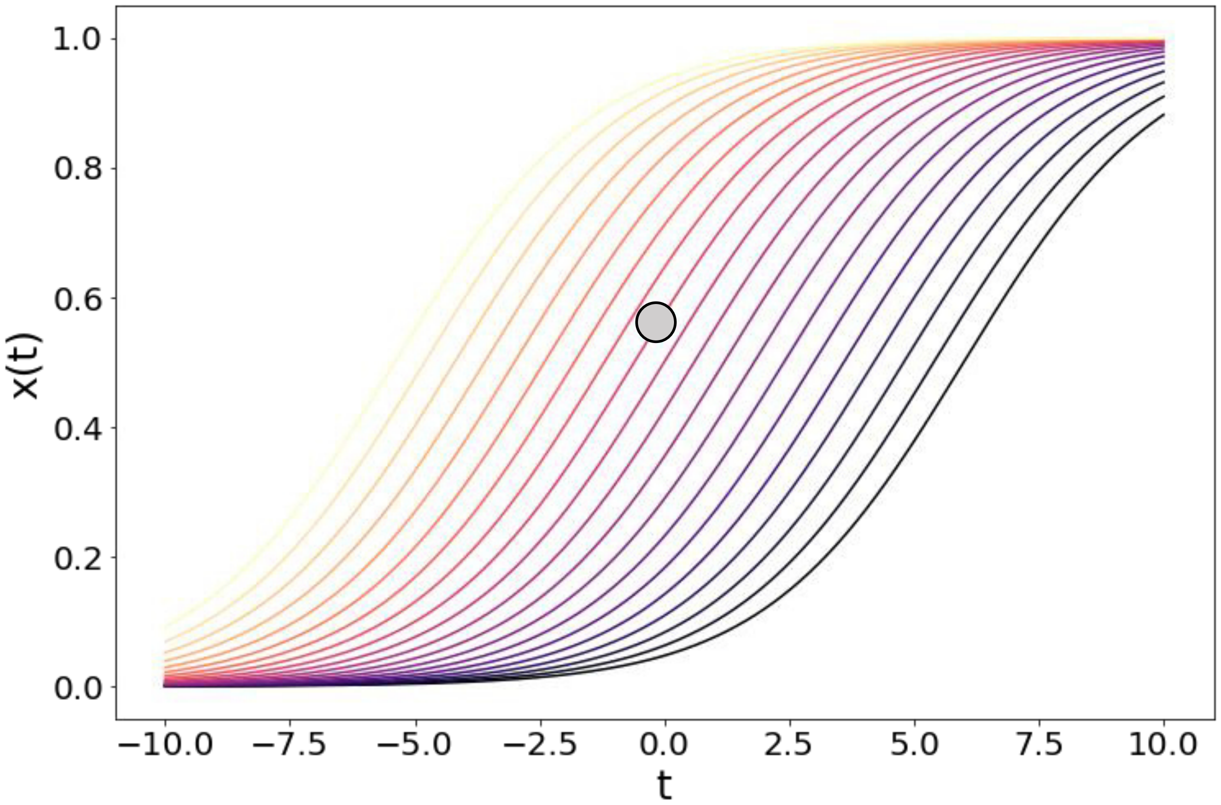
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

$$\int \frac{\mathrm{d}x}{x(1-x)} = \int \alpha \mathrm{d}t$$

$$\log\left(\frac{x}{1-x}\right) = \alpha t + k$$

$$x(t) = \frac{e^{\alpha t + k}}{1 + e^{\alpha t + k}}$$



Initial condition

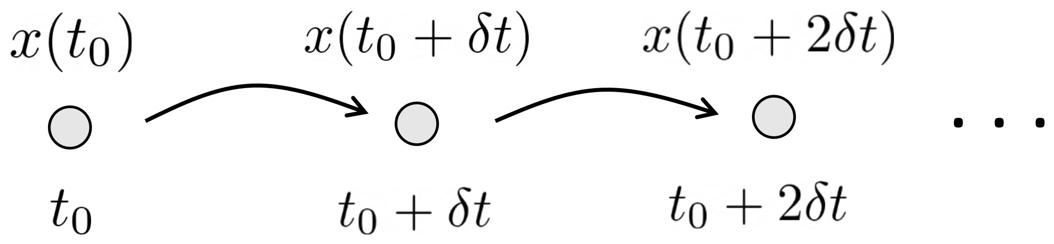
$$x(t_0) = x_0$$

Euler's method: recap

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\frac{x(t+\delta t) - x(t)}{\delta t} = f\big(x(t), t\big)$$

$$x(t+\delta t) = x(t) + \delta t f\big(x(t), t\big)$$



The approach is an approximation…why?

We are not actually taking a limit

Population growth

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \alpha N(t) \qquad N(0) = N_0$$

Analytical

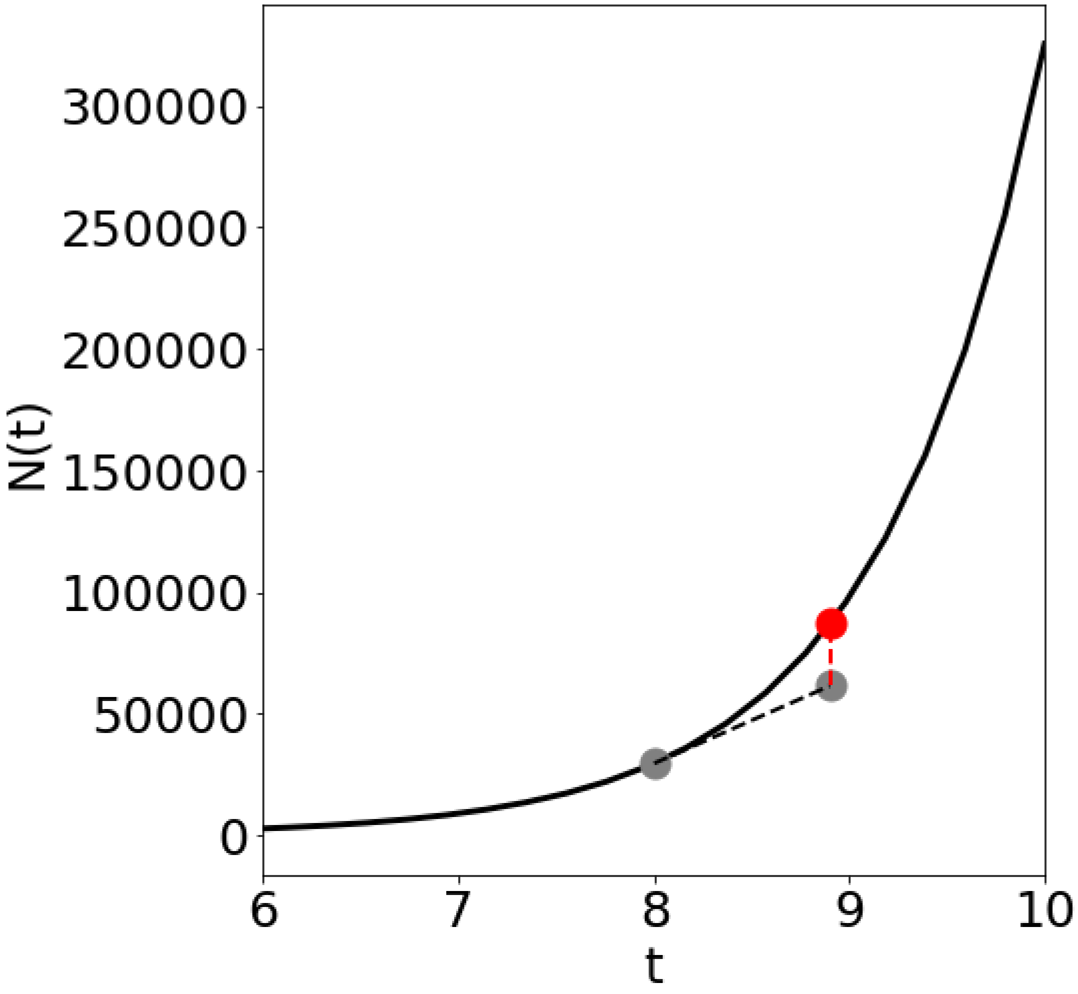
$$N(t) = N_0 e^{\alpha t}$$

Euler's method

$$N(t + \delta t) = N(t)(1 + \alpha \delta t)$$

What is the error introduced? Let's measure…

Error Analysis



Let's consider $N(8) \approx 3 \times 10^4$ as initial condition

$$N(8+\delta t) = N(8)e^{\alpha \delta t}$$

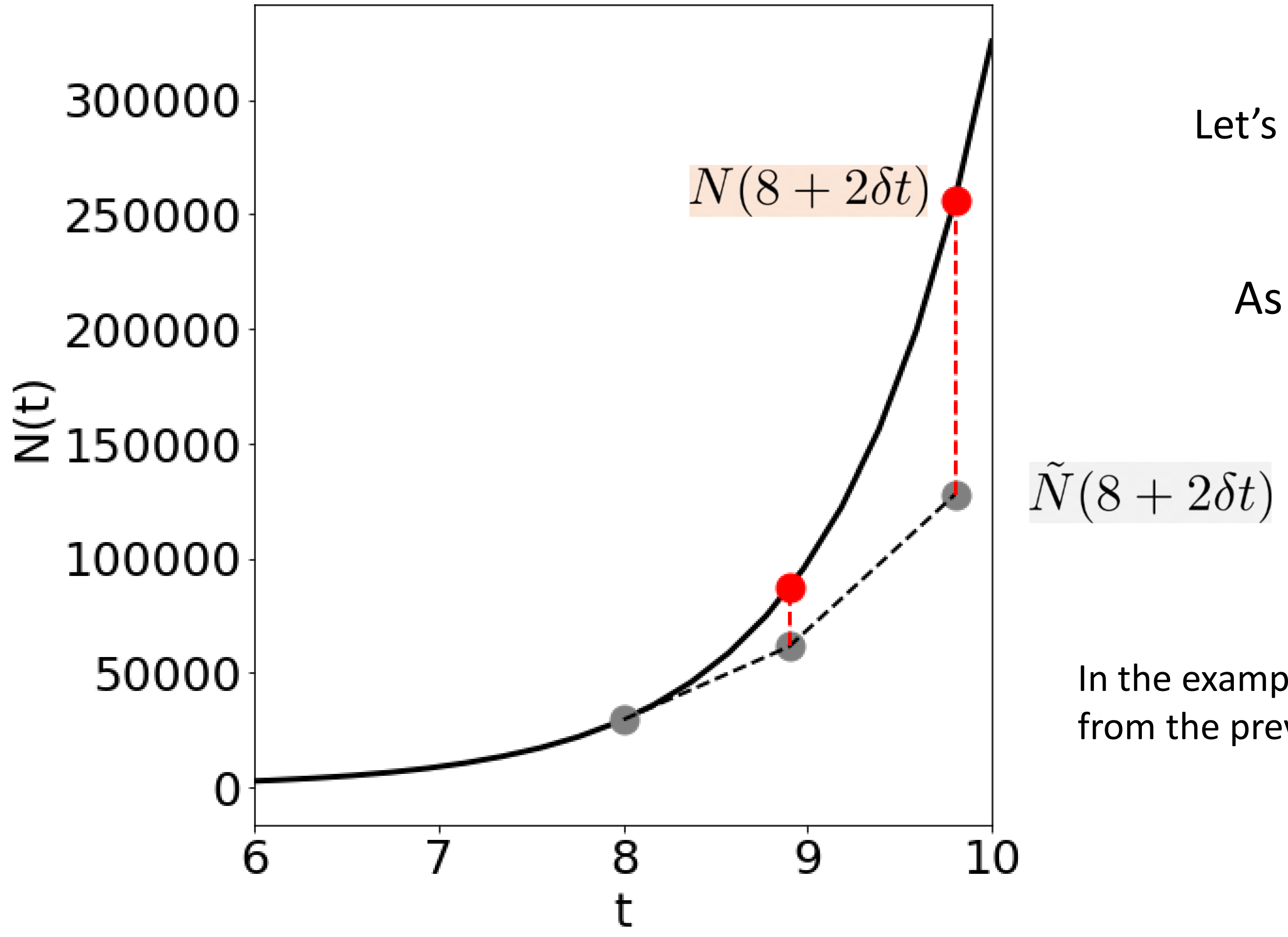
$$\tilde{N}(8+\delta t) = N(8)(1+\alpha \delta t)$$

$$\text{error} = \left| \frac{True - Estimate}{True} \right|$$

$$\text{error} = \left| \frac{N(8+\delta t) - \tilde{N}(8+\delta t)}{N(8+\delta t)} \right| \approx 0.3$$

Local error of the order $\mathcal{O}(\delta t^2)$

Error Analysis



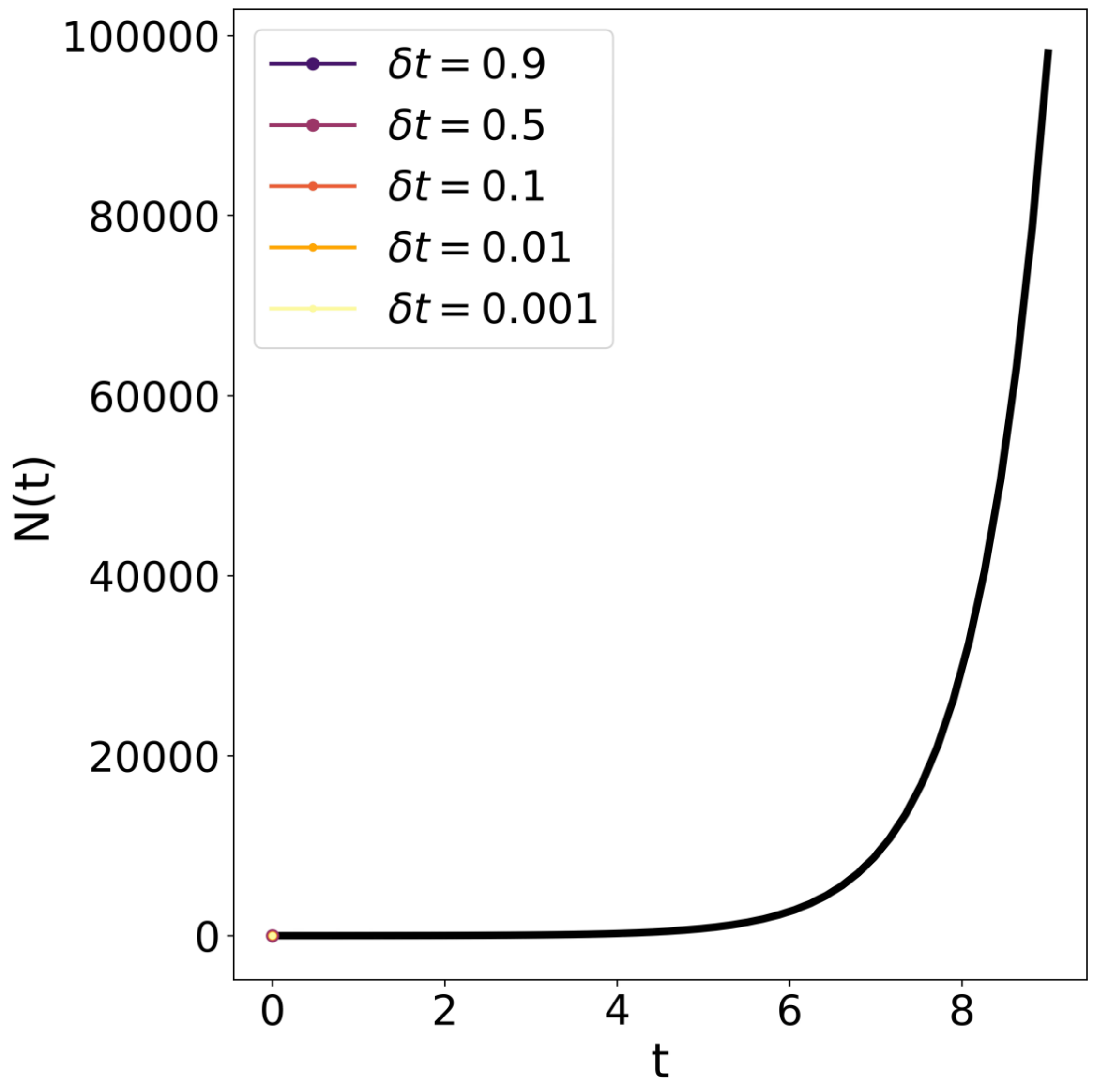
Let’s consider $N(8) \approx 3 \times 10^4$ as initial condition

As we evolve the system across time, previous errors get propagated

In the example on the left, the new update contains the error from the previous time step

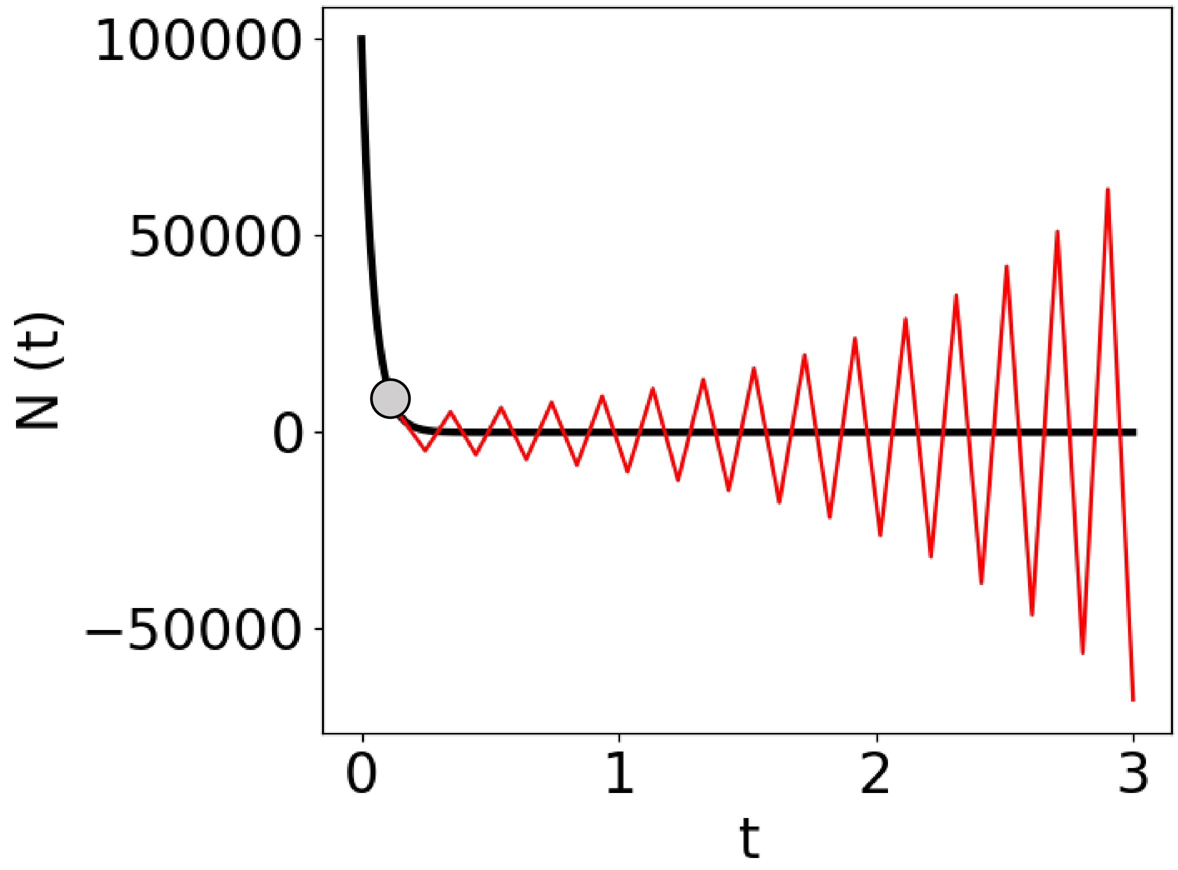
Global error of the order $\mathcal{O}(\delta t)$

Error Analysis



Euler, unstable behavior

$$\frac{\mathrm{d}N(t)}{dt} = -\alpha N(t) \qquad \alpha > 0$$



Analytical

$$N(t) = N_0 e^{-\alpha t}$$

Euler's method

$$N(t + \delta t) = N(t)(1 - \alpha \delta t)$$

If $|1 - \alpha \delta t| > 1$

Euler is unstable

Equilibria

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

The set of points x_{eq} for which

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x_{eq}} = 0 \qquad\qquad f(x_{eq}) = 0$$

The system is not changing over time at those points

They can be stable or unstable

Equilibria

Let's consider an equilibrium point x_{eq} $\quad f(x_{eq}) = 0$

We perturb the equilibrium by applying a small variation

$$\tilde{x} = x_{eq} + \delta x$$

Does the system go back to the equilibrium or does it run away?

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$$

Has the opposite sign of δx . The system goes back. Stable

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$$

Has the same sign of δx . The system goes away. Unstable

Equilibria, example

$$\frac{\mathrm{d}N(t)}{dt} = \alpha N(t) \qquad \alpha > 0$$

$$N_{eq} = 0$$

$$\tilde{N} = N_{eq} + \delta N = \delta N$$

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right|_{\tilde{N}} = \alpha \delta N$$

The same sign as the perturbation, unstable

$$\frac{\mathrm{d}N(t)}{dt} = -\alpha N(t) \qquad \alpha > 0$$

Equilibrium point stable or unstable?

Stability, derivatives

Asking if $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$ has the same/opposite sign as the perturbation δx

Is equivalent to see if the following is positive or negative

$$\frac{f(\tilde{x})}{\delta x} = \frac{f(\tilde{x}) - f(x_{eq})}{\delta x} =$$

$$= \frac{f(x_{eq} + \delta x) - f(x_{eq})}{\delta x}$$

Taking the limit…

$$\lim_{\delta x \to 0} \frac{f(x_{eq} + \delta x) - f(x_{eq})}{\delta x} = f^{(1)}(x_{eq})$$

> 0 Unstable

< 0 Stable

$= 0$ Look at $f^{(2)}(x_{eq})$

Stability, Example

$$\frac{\mathrm{d}N(t)}{dt} = f\big(N(t)\big) = \alpha N(t)\big(1 - N(t)\big) \qquad \alpha > 0$$

$$N_{eq} = 0, 1$$

$$\frac{\mathrm{d}}{\mathrm{d}N}\Big\{\alpha N(1 - N)\Big\} = \alpha(1 - N) - \alpha N$$

$f^{(1)}(0) = \alpha$ Unstable

$f^{(1)}(1) = -\alpha$ Stable

Thank you

Python, jupyter notebook installed for the Lab

Supplementary Material

1. Recap of derivative definition and examples (slide 30).
2. Table of derivatives (slide 31).
3. Table of Integrals (slide 32).
4. Solution of the equation in slide 16 (slide 33).
5. Understanding the error of Euler's method (slide 34-36). For the brave students…

Supplementary Material: Derivative definition and examples

We won't ask for this slide in the exam but understanding it is highly recommended.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t}$$

$$x(t) = k$$

$$\lim_{\delta t \to 0} \frac{k - k}{\delta t} = 0$$

$$x(t) = t$$

$$\lim_{\delta t \to 0} \frac{t + \delta t - t}{\delta t} = 1$$

$$x(t) = t^2$$

$$\lim_{\delta t \to 0} \frac{(t+\delta t)^2 - t^2}{\delta t} =$$

$$\lim_{\delta t \to 0} \frac{2t\delta t + \delta t^2}{\delta t} =$$

$$\lim_{\delta t \to 0} \frac{\delta t}{\delta t} 2t + \delta t = 2t$$

Supplementary Material: Table of derivatives

Function	Derivative	Example
k	0	$\frac{\mathrm{d}}{\mathrm{d}x}1 = 0$
x^k	kx^{k-1}	$\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$
e^{kx}	ke^{kx}	$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} = 2e^{2x}$
$\sin(kx)$	$k\cos(kx)$	$\frac{\mathrm{d}}{\mathrm{d}x}\sin(4x) = 4\cos(4x)$
$\cos(kx)$	$-k\sin(kx)$	$\frac{\mathrm{d}}{\mathrm{d}x}\cos(5x) = -5\sin(5x)$

Important rules	Example
$\frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x) = f^{(1)}(x)g(x) + f(x)g^{(1)}(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(2x)(2x^2+3x)\right] = -2\sin(2x)(2x^2+3x) + \cos(2x)(4x+3)$
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f^{(1)}(x)g(x) - g^{(1)}(x)f(x)}{g^2(x)}$	$\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^{2x}}{x^2} = \frac{2e^{2x}x^2 - 2xe^{2x}}{x^4}$
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = f^{(1)}\big(g(x)\big)g^{(1)}(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}e^{3x^2+2x} = e^{3x^2+2x}(6x+2)$

Supplementary Material: Table of integrals

Function	Integral	Example
k	$kx + k_1$	$\int 1\mathrm{d}x = x + k_1$
x^k	$\frac{x^{k+1}}{k+1} + k_1$	$\int x^3\mathrm{d}x = \frac{x^4}{4} + k_1$
e^{kx}	$\frac{e^{kx}}{k} + k_1$	$\int e^{2x}\mathrm{d}x = \frac{e^{2x}}{2} + k_1$
		$\int e^{2x}\mathrm{d}x = \frac{e^{2x}}{2} + k_1$
$\sin(kx)$	$\frac{-\cos(kx)}{k} + k_1$	$\int \sin(kx)\mathrm{d}x = \frac{-\cos(kx)}{k} + k_1$
$\cos(kx)$	$\frac{\sin(kx)}{k} + k_1$	$\int \cos(kx)\mathrm{d}x = \frac{\sin(kx)}{k} + k_1$
$\frac{1}{x}$	$\log(\lvert x\rvert) + k_1$	$\int \frac{1}{1-x}\mathrm{d}x = -\log(\lvert 1-x\rvert) + k_1$

The constant k_1 appears because the integral is indefinite. Indeed, if you derive the Integral you should obtain the starting Function. The derivative of a constant is zero.

Supplementary Material: Derivation of the integral at slide 16

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

$$\int \frac{\mathrm{d}x}{x\big(1-x\big)} = \int \alpha \mathrm{d}t$$

$$\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{1-x} =$$

$$= \frac{a(1-x) + bx}{x(1-x)} = \frac{a + x(b-a)}{x(1-x)}$$

$$a = 1,\ b = a = 1$$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{(1-x)}$$

We compare this numerator with the starting one (=1). Thus, there must be no term in x, which means b=a. Then, a=1.

$$\int \frac{1}{x} + \frac{1}{1-x} \mathrm{d}x = \int \alpha \mathrm{d}t$$

$$\log(x) - \log(1-x) = \alpha t + k$$

$$\log\left(\frac{x}{1-x}\right) = \alpha t + k$$

$$\frac{x}{1-x} = e^{\alpha t + k}$$

$$x = e^{\alpha t + k} - x e^{\alpha t + k}$$

$$x\left(1 + e^{\alpha t + k}\right) = e^{\alpha t + k}$$

$$x(t) = \frac{e^{\alpha t + k}}{1 + e^{\alpha t + k}}$$

Supplementary Material: Understanding the error in Euler's update

For the brave students...

Taylor expansion around t_0 :

$$x(t) = x(t_0) + x^{(1)}(t_0)(t-t_0) + \frac{1}{2}x^{(2)}(t_0)(t-t_0)^2 + \ \dots \ + \frac{1}{n!}x^{(n)}(t_0)(t-t_0)^n + \dots =$$

$$= x(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!}x^{(n)}(t_0)(t-t_0)^n$$

Let's expand the following exponential function around zero

$$x(t) = e^{\alpha t}$$

$x^{(1)}(t) = \alpha e^{\alpha t}$ First derivative

$x^{(2)}(t) = \alpha \frac{d}{dt}\left(e^{\alpha t}\right) = \alpha^2 e^{\alpha t}$ Second derivative

Supplementary Material: Understanding the error in Euler's update

For the brave students...

Taylor expansion around t_0 :

$$x(t) = x(t_0) + x^{(1)}(t_0)(t - t_0) + \frac{1}{2}x^{(2)}(t_0)(t - t_0)^2 + \ \dots \ + \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n + \dots =$$

$$= x(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n$$

Let's expand the following exponential function around zero

$$x(t) = e^{\alpha t} = e^0 + \alpha e^0(t - 0) + \frac{\alpha^2}{2}e^0(t - 0)^2 + \frac{\alpha^3}{6}e^0(t - 0)^3 + \dots =$$

We have used the derivatives from the previous slide

$$= 1 + \alpha t + \frac{\alpha^2}{2}t^2 + + \frac{\alpha^3}{6}t^3 + \dots$$

Supplementary Material: Understanding the error in Euler's update

For the brave students...

We consider now the population growth model below. We will use the analytical solution and Taylor expansion and compare the result with Euler's

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \alpha N(t) \qquad N(0) = N_0$$

Analytical

$$N(t) = N_0 e^{\alpha t}$$

Euler's method

$$N(t + \delta t) = N(t)(1 + \alpha \delta t)$$

$$N(t + \delta t) = N_0 e^{\alpha(t+\delta t)} =$$

$$= N_0 e^{\alpha t} e^{\alpha \delta t} = N(t) e^{\alpha \delta t} =$$

$$= N(t)\left[\underbrace{1 + \alpha \delta t}_{\text{Euler's terms}} + \underbrace{\frac{\alpha^2}{2}(\delta t)^2 + \frac{\alpha^3}{6}(\delta t)^3 + \ldots}_{\text{Terms neglected } \mathcal{O}(\delta t^2)}\right]$$

We expanded

$$e^{\alpha \delta t}$$

around zero

IMPROVED NUMERICAL METHODS, RUNGE-KUTTA

Week 8

Dr. Luca Manneschi

So far…

Differential Equations and initial condition. How to find a unique solution?

Solutions to

Please, learn how to solve these

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$\alpha > 0$ Exponential growth

$\alpha < 0$ Exponential decay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

Saturating

Euler's method and its error $\mathcal{O}(\delta t^2)$ $\mathcal{O}(\delta t)$

Definition of equilibrium points in one dimension

Autonomous and not

Autonomous

If it does not contain t explicitly

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

Non-Autonomous

If it does contain t explicitly

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\alpha x(t)}{t}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + \cos(t)$$

Non-Autonomous

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big) + e^{-x(t)}$$

Autonomous

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + I(t)$$

$I(t) = k$ Autonomous

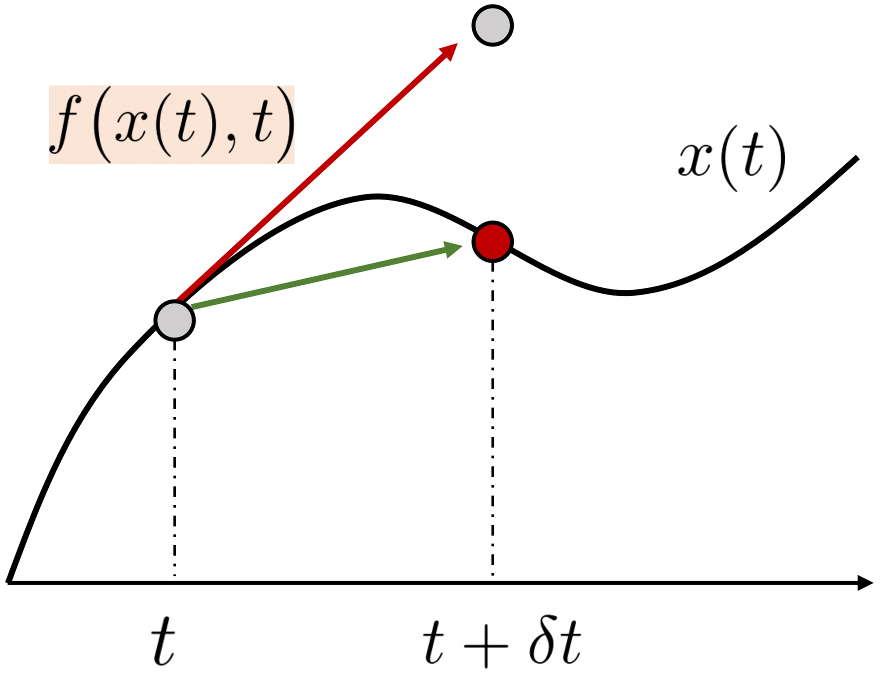
Otherwise Non-Autonomous

Euler's Method: Geometrical Interpretation

Let's consider the differential equation from the point of view of $x(t)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\left(x(t), t\right)$$

Our task is to find the solution $x(t)$



Euler's method

$$x(t + \delta t) = x(t) + \delta t f\left(x(t), t\right)$$

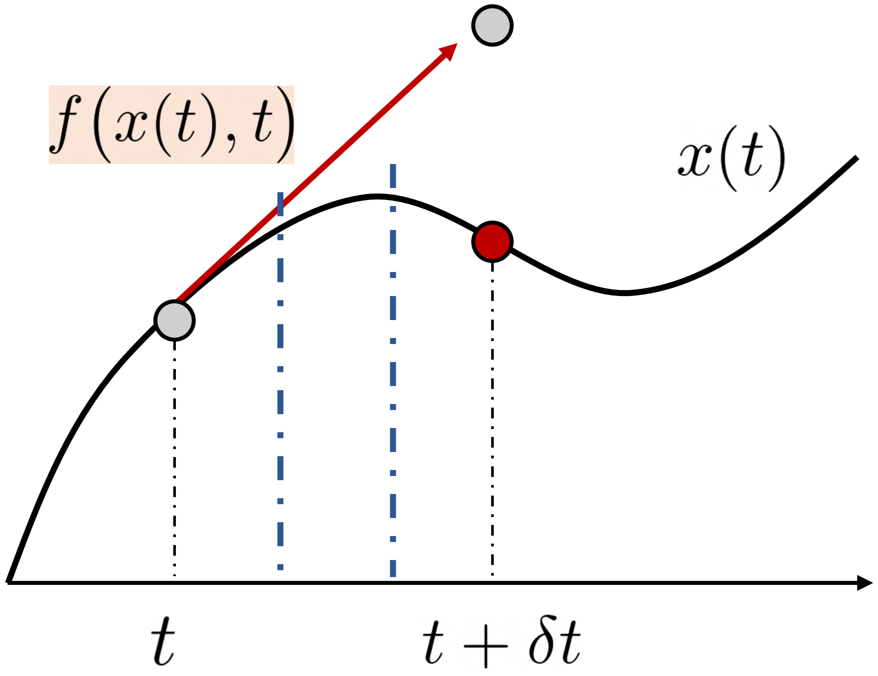
If the function is non-linear between t and t+delta t, Euler's method is not accurate

Euler's Method: Geometrical Interpretation

Let's consider the differential equation from the point of view of $x(t)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\left(x(t), t\right)$$

Our task is to find the solution $x(t)$



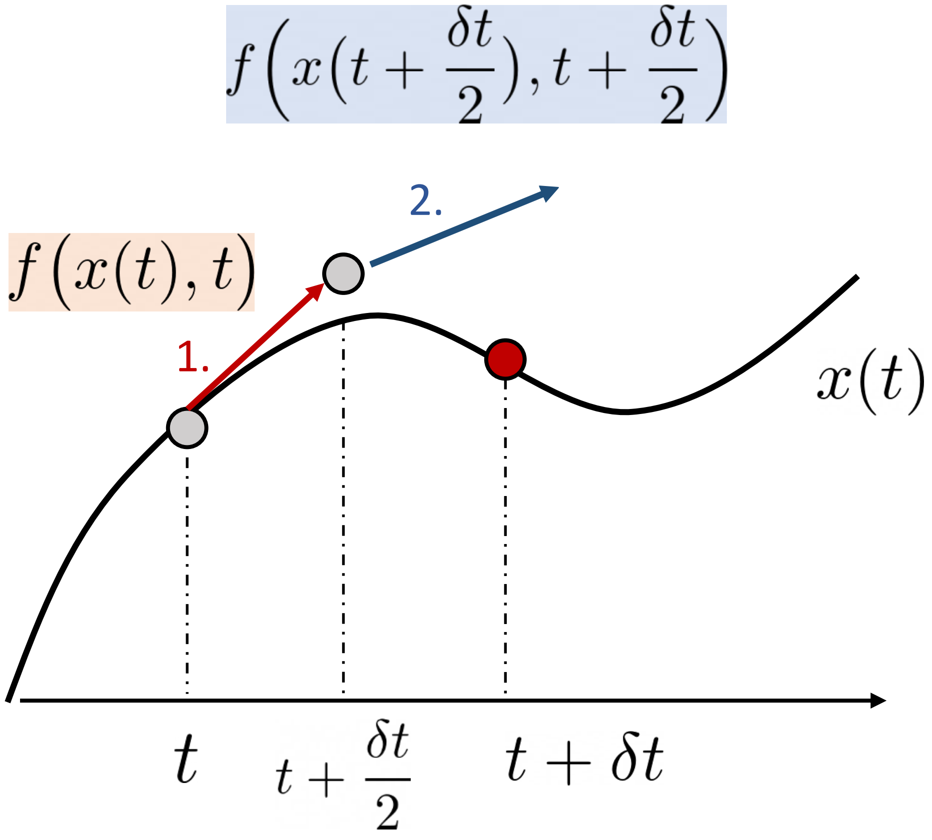
Can we exploit estimates of the derivatives computed at different points in the interval to find a better approximation?

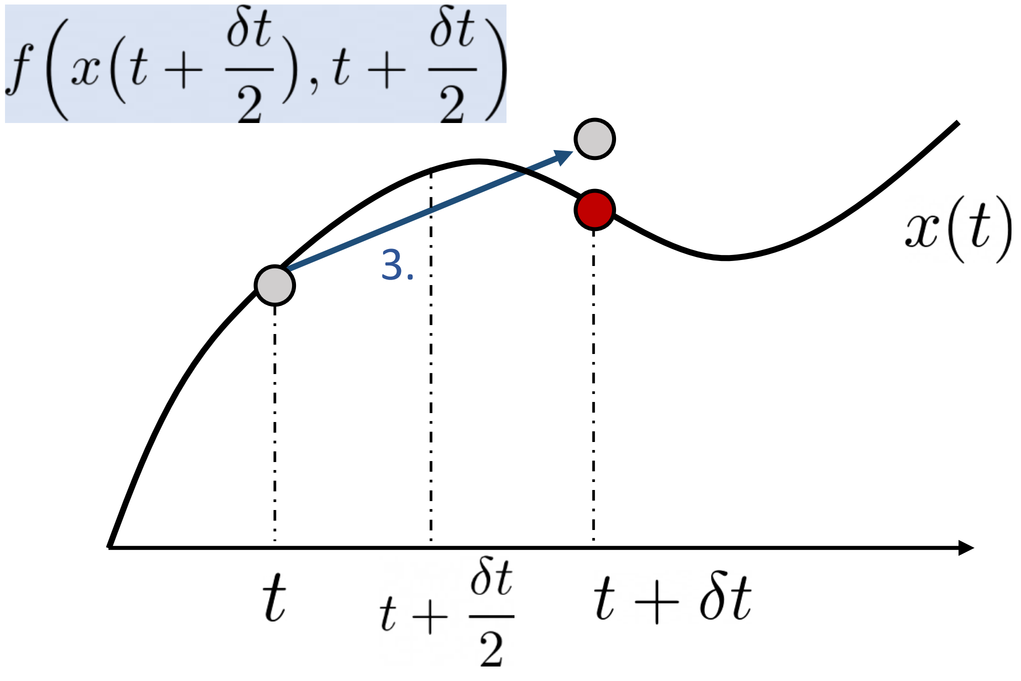
Why estimates? We do not know $x(t)$

Midpoint Method

What if we use the slope at the middle point $t + \frac{\delta t}{2}$?

1. We first use the slope at t to evolve the system for $\frac{\delta t}{2}$
2. We compute the slope at $t + \frac{\delta t}{2}$
3. We apply the slope to perform the update from time t



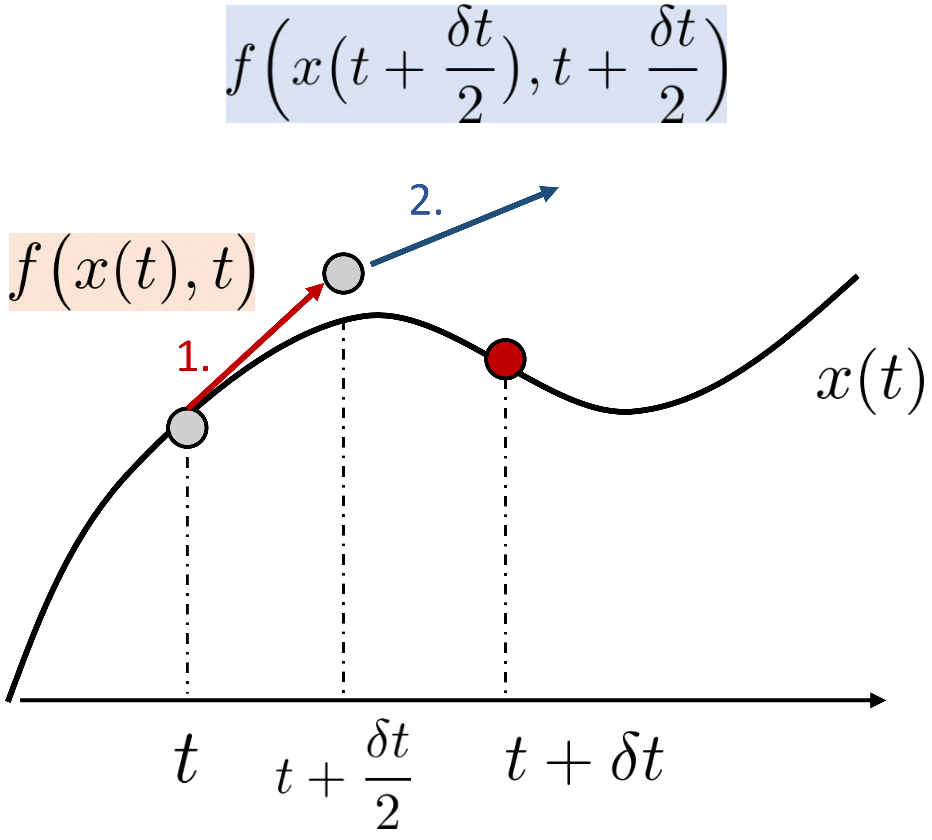


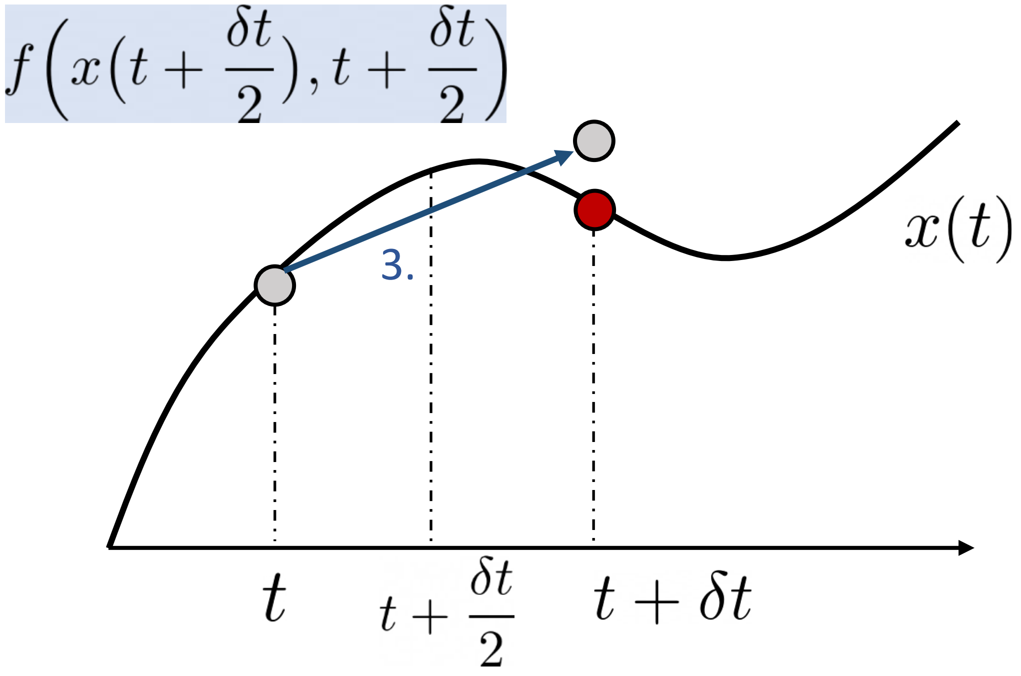
Midpoint Method

1. $x\left(t+\frac{\delta t}{2}\right) = x(t) + f\left(x(t),t\right)\frac{\delta t}{2}$

2. $f_M = f\left(x(t+\frac{\delta t}{2}), t+\frac{\delta t}{2}\right)$

3. $x(t+\delta t) = x(t) + \delta t f_M$





Midpoint Method: How to do it?

Define the function $f(x,t)$ and choose δt

Start from initial condition $x_{\mathrm{p}}, t_{\mathrm{p}}$

```
## YOUR VALUES

dt=...
x_p=...
t_p=...
```

```
def f(x,t):

    ## Y = YOUR FUNCTION

    return Y
```

REPEAT

Use the first slope $f(x_p,t_p)$ to compute middle-point

$$x_{MP} = x_p + f(x_p,t_p)\frac{\delta t}{2}$$

```
x_mp=x_p+f(x_p,t_p)*dt/2
```

Use the slope at the middle-point to perform the update

$$x_{new} = x_p + f\left(x_{MP}, t_p + \frac{\delta t}{2}\right)\delta t$$

```
x_new=x_p+f(x_mp,t_p+dt/2)*dt
```

$$x_p \leftarrow x_{new}, \quad t_p \leftarrow t_p + \delta t$$

Of course, everything can be embedded in a nice class …

Midpoint Method: Explicit analytical form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

$$x\left(t + \frac{\delta t}{2}\right) = x(t) + \alpha x(t)\frac{\delta t}{2}$$

$$f_M = \alpha\left(x(t) + \alpha x(t)\frac{\delta t}{2}\right) = \alpha x(t) + \alpha^2 x(t)\frac{\delta t}{2}$$

$$x(t + \delta t) = x(t) + f_M \delta t$$

$$x(t + \delta t) = x(t)\left[1 + \alpha\delta t + \alpha^2\frac{\delta t^2}{2}\right]$$

Midpoint Method, comparison with Euler

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

Midpoint

$$x(t+\delta t) = x(t)\left[1 + \alpha\delta t + \alpha^2 \frac{\delta t^2}{2}\right]$$

Euler

$$x(t+\delta t) = x(t)\left[1 + \alpha\delta t\right]$$

Look at $\delta t, \delta t^2, \delta t^3, \ldots$

$$x(t) = \underbrace{x(t_0) + x^{(1)}(t_0)\delta t}_{\text{Euler's term}} + \underbrace{\frac{1}{2}x^{(2)}(t_0)\delta t^2 + \frac{1}{6}x^{(3)}(t_0)\delta t^3 + \ldots}_{\text{Terms neglected by Euler}}$$

Local error of the order $\mathcal{O}(\delta t^2)$

Global error of the order $\mathcal{O}(\delta t)$

Midpoint Method, comparison with Euler

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

Midpoint

$$x(t+\delta t) = x(t)\left[1 + \alpha\delta t + \alpha^2 \frac{\delta t^2}{2}\right]$$

Euler

$$x(t+\delta t) = x(t)\big[1 + \alpha\delta t\big]$$

Look at $\delta t, \delta t^2, \delta t^3, \ldots$

$$x(t) = \underbrace{x(t_0) + x^{(1)}(t_0)\delta t + \frac{1}{2}x^{(2)}(t_0)\delta t^2}_{\text{Midpoint terms}} + \underbrace{\frac{1}{6}x^{(3)}(t_0)\delta t^3 + \ldots}_{\text{Terms neglected by Midpoint}}$$

Local error of the order $\mathcal{O}(\delta t^3)$

Global error of the order $\mathcal{O}(\delta t^2)$

RUNGE-KUTTA Methods

$$x(t+\delta t) = x(t) + \phi\delta t$$

$$\phi = a_1 k_1 + a_2 k_2 + ... + a_n k_n$$

ϕ Is an effective slope

Can be written as a linear combination of k's, which are slopes evaluated at different points

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\big(x(t) + k_1 q_{11}\delta t, t + p_1\delta t\big)$$

$$k_3 = f\big(x(t) + k_2 q_{22}\delta t + k_1 q_{21}\delta t, t + p_2\delta t\big)$$

$$\cdots$$

RUNGE-KUTTA Methods

$$x(t+\delta t) = x(t) + \phi\delta t$$

ϕ Is an effective slope

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

Can be written as a linear combination of k's, which are slopes evaluated at different points

$$k_1 = f\left(x(t), t\right)$$

$$k_2 = f\left(x(t) + k_1 q_{11}\delta t, t + p_1\delta t\right)$$

$$k_3 = f\left(x(t) + k_2 q_{22}\delta t + k_1 q_{21}\delta t, t + p_2\delta t\right)$$

$$\cdots$$

The k's are in a recursive form; k2 is function of k1, k3 of k2 and so on

The other constants p's q's and a's can be found by comparing these equations with the Taylor expansion (we will not cover this)

RUNGE-KUTTA Methods

$$x(t+\delta t) = x(t) + \phi \delta t$$

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\big(x(t) + k_1 q_{11} \delta t, t + p_1 \delta t\big)$$

$$k_3 = f\big(x(t) + k_2 q_{22} \delta t + k_1 q_{21} \delta t, t + p_2 \delta t\big)$$

$$\cdots$$

Runge-Kutta n is defined by maintaining the first n k's. For instance, in Runge-Kutta 2 we use only the terms

RUNGE-KUTTA 1

$$x(t+\delta t) = x(t) + \phi\delta t$$

$$\phi = k_1 \qquad k_1 = f\big(x(t), t\big)$$

Does it look familiar?

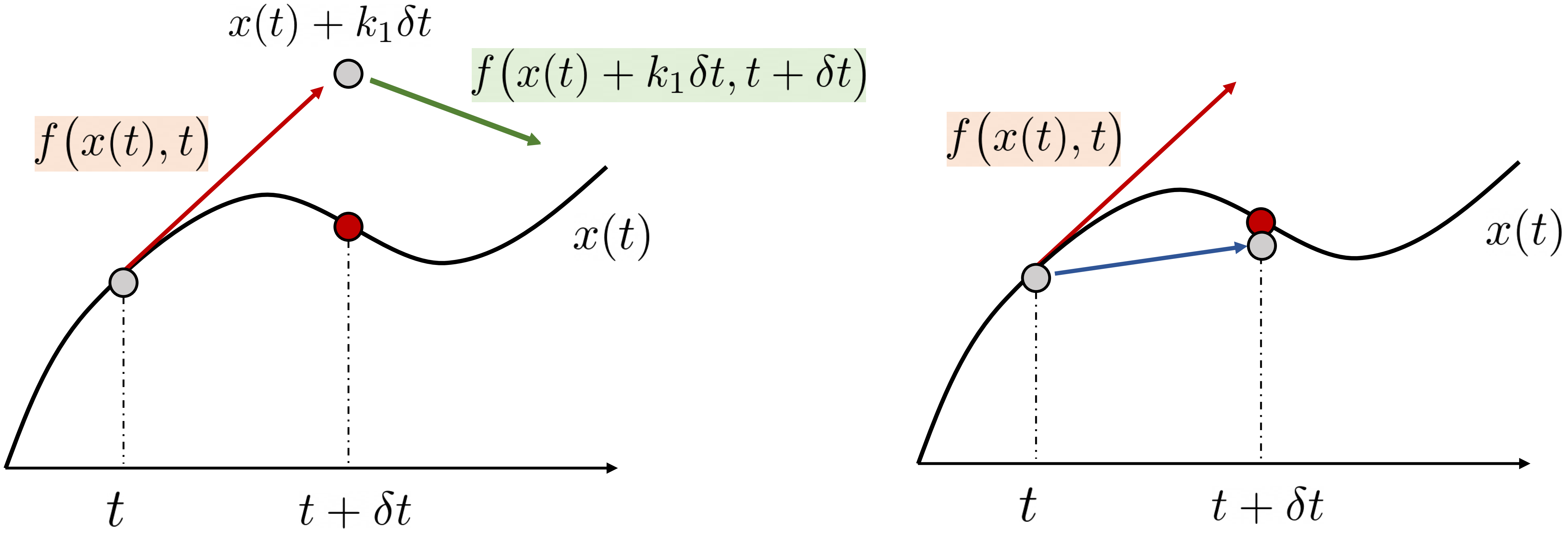
$$x(t+\delta t) = x(t) + \delta t f\big(x(t), t\big)$$

It is Euler!

RUNGE-KUTTA 2

$$x(t+\delta t) = x(t) + \phi \delta t$$

$$\phi = \frac{1}{2}k_1 + \frac{1}{2}k_2 \qquad k_1 = f\big(x(t), t\big) \qquad k_2 = f\big(x(t) + k_1\delta t, t + \delta t\big)$$



RUNGE-KUTTA 2: In Practice

Define the function $f(x,t)$ and choose δt

Start from initial condition $x_{\mathrm{p}}, t_{\mathrm{p}}$

```
## YOUR VALUES

dt=...
x_p=...
t_p=...
```

```
def f(x,t):

    ## Y = YOUR FUNCTION

    return Y
```

REPEAT

Use the first slope $k_1 = f(x_p, t_p)$ to compute k_2

$$k_2 = f\left(x_p + k_1 \delta t, t_p + \delta t\right)$$

Perform the update

$$x_{new} = x_p + \frac{1}{2}\left(k_1 + k_2\right)\delta t$$

$$x_p \leftarrow x_{new}, \quad t_p \leftarrow t_p + \delta t$$

Of course, everything can be embedded in a nice class …

RUNGE-KUTTA 2: Analytically

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

$$k_1 = \alpha x(t)$$

$$\begin{aligned} k_2 = f\big(x(t) + k_1 \delta t, t + \delta t\big) &= \\ f\big(x(t) + \alpha x(t)\delta t, t + \delta t\big) &= \\ \alpha\big[x(t) + \alpha x(t)\delta t\big] = \alpha x(t) + \alpha^2 x(t)\delta t \end{aligned}$$

$$\begin{aligned} x(t+\delta t) = x(t) + \frac{1}{2}\big[k_1 + k_2\big]\delta t &= \\ = x(t) + \frac{1}{2}\big[2\alpha x(t) + \alpha^2 x(t)\delta t\big]\delta t &= \\ = x(t)\Big[1 + \alpha\delta t + \alpha^2\frac{\delta t^2}{2}\Big] \end{aligned}$$

Local error of the order $\mathcal{O}(\delta t^3)$ Global error of the order $\mathcal{O}(\delta t^2)$

RUNGE-KUTTA 4: The 'golden' standard

$$x(t+\delta t) = x(t) + \phi \delta t$$

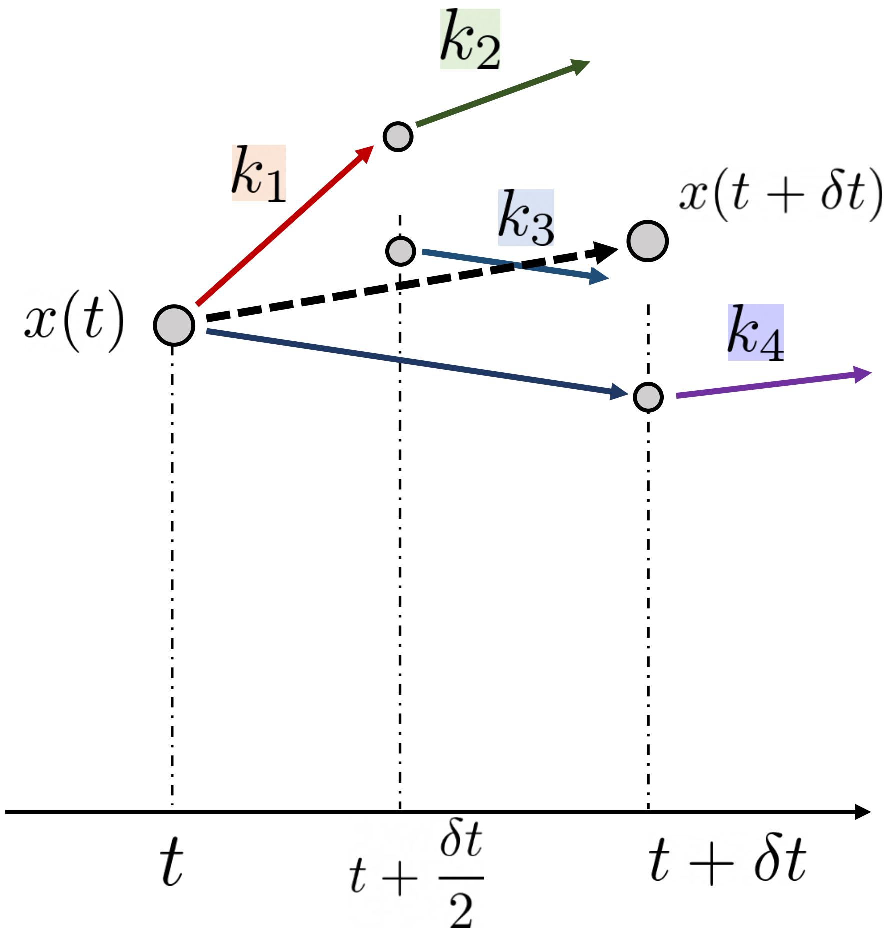
$$\phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\left(x(t) + k_1 \frac{\delta t}{2}, t + \frac{\delta t}{2}\right)$$

$$k_3 = f\left(x(t) + k_2 \frac{\delta t}{2}, t + \frac{\delta t}{2}\right)$$

$$k_4 = f\Big(x(t) + k_3 \delta t, t + \delta t\Big)$$



Thank you

HIGHER ORDER SYSTEMS

Week 9

Dr. Luca Manneschi

So far...

Differential Equations and initial condition. How to find a unique solution?

Solutions to

Look at the exercises

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$\alpha > 0$ Exponential growth

$\alpha < 0$ Exponential decay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

Saturating

Euler's method and its error $\mathcal{O}(\delta t^2)$ $\mathcal{O}(\delta t)$

Definition of equilibrium points in one dimension

Autonomous and non-autonomous: Definition

MidPoint Method, Runge-Kutta 2 and 4

Today…

Before we had a 1-dimensional system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t) \end{cases}$$

In vectorial form

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{F}\big(\mathbf{z}(t), t\big)$$

$$\mathbf{F} = \begin{pmatrix} f\big(\mathbf{z}(t), t\big) \\ g\big(\mathbf{z}(t), t\big) \end{pmatrix}$$

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Euler's method

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t) \\[2em] \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t) \end{cases}$$

$$\begin{cases} x(t + \delta t) = x(t) + \delta t f(x, y, t) \\[2em] y(t + \delta t) = y(t) + \delta t g(x, y, t) \end{cases}$$

Or in vectorial form

$$\mathbf{z}(t + \delta t) = \mathbf{z}(t) + \mathbf{F}\big(\mathbf{z}(t), t\big)\delta t$$

It is analogous to Euler's method in one dimension, but we have vectors

Improved numerical methods in higher dimensions

We will not cover this specifically, you can use premade tools available. However, if you use the vectorial form, the methods (Runke-Kutta and others) appear analogous to the 1 dimensional case…

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{F}\big(\mathbf{z}(t), t\big)$$

In the lab, you will see an example of this

Phase-Plot

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y) \end{cases}$$

We plot the solutions y as a function of x

For the direction? We plot arrows as x and y vary $\mathbf{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$

Example: Lotka-Volterra equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

X: Prey

Y: Predator

$\alpha > 0,$ Reproduction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

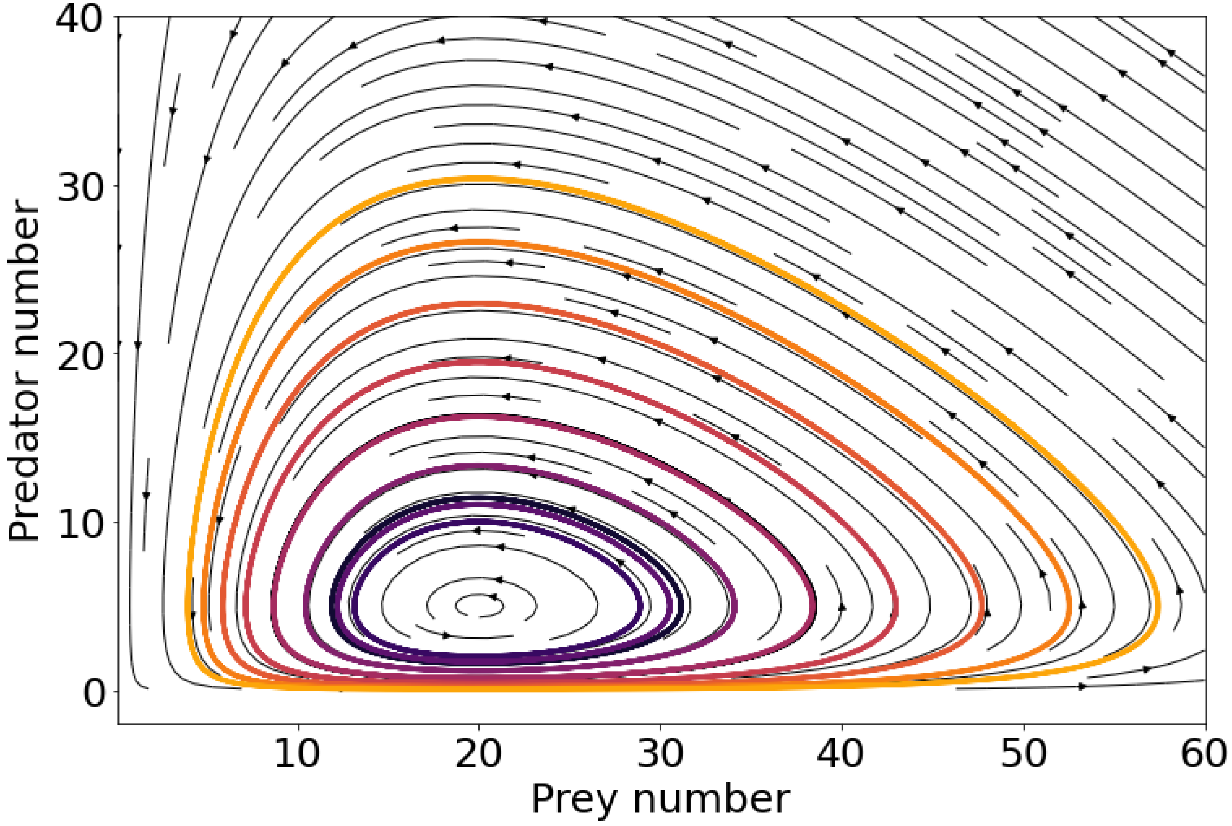
$\delta > 0,$ Reproduction predator

Phase-Plot

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

We plot the solutions y as a function of x

For the direction? We plot arrows as x and y vary $\mathbf{F}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$



Equilibrium Points

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y) \end{cases}$$

$$(x_{eq}, y_{eq})$$

$$f(x_{eq}, y_{eq}) = 0, \text{ and } \; g(x_{eq}, y_{eq}) = 0$$

What are the equilibrium points for Lotka-Volterra?

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$$(0, 0) \qquad \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$$

Stability

Remember: in 1-dimension

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

$$x_{eq},\ f(x_{eq}) = 0$$

We compute the derivative

$$\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x}$$

Evaluate it at the equilibrium and study the sign

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases}$$

$$(x_{eq}, y_{eq})$$

$$f(x_{eq}, y_{eq}) = 0, \text{ and } \ g(x_{eq}, y_{eq}) = 0$$

We compute the Jacobian

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}$$

Evaluate it at the equilibrium and study the signs of the eigenvalues

Stability

Given a matrix $\mathcal{J}$, the eigenvalues and eigenvectors satisfy the relation $\mathcal{J}\mathbf{v} = \lambda\mathbf{v}$

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}$$

$$\lambda_1, \lambda_2, \;\; \mathbf{v}_1, \mathbf{v}_2$$

$\lambda_1 < 0, \;\; \lambda_2 < 0$	Eigenvalues with opposite signs	$\lambda_1 > 0, \;\; \lambda_2 > 0$
Stable	Saddle (Unstable)	Unstable

Stability, Lotka-Volterra

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha > 0,$ Reproduction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

$\delta > 0,$ Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha, \ -\gamma$

Saddle

This is the reason why it is hard for the system to
reach the extinction of the species

Stability, Lotka-Volterra variation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha < 0,$ Extinction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

$\delta > 0,$ Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\Big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha, \ -\gamma$ Stable

Stability, Lotka-Volterra variation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha > 0,$ Reproduction prey

$\beta > 0,$ Predation

$\gamma < 0,$ Reproduction predator

$\delta > 0,$ Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\Big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha,\ -\gamma$ Unstable

Can we find the phase-plot? Analytically?

We consider the linear case

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_{11}x + a_{12}y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = a_{21}x + a_{22}y \end{cases}$$

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\lambda_1, \lambda_2, \ \mathbf{v}_1, \mathbf{v}_2$$

The solution has a form like

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v_2} e^{\lambda_2 t}$$

Exponential functions…Recall that the solution to $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$ is $\propto e^{\alpha t}$

Unstable Node

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x - y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases}$$

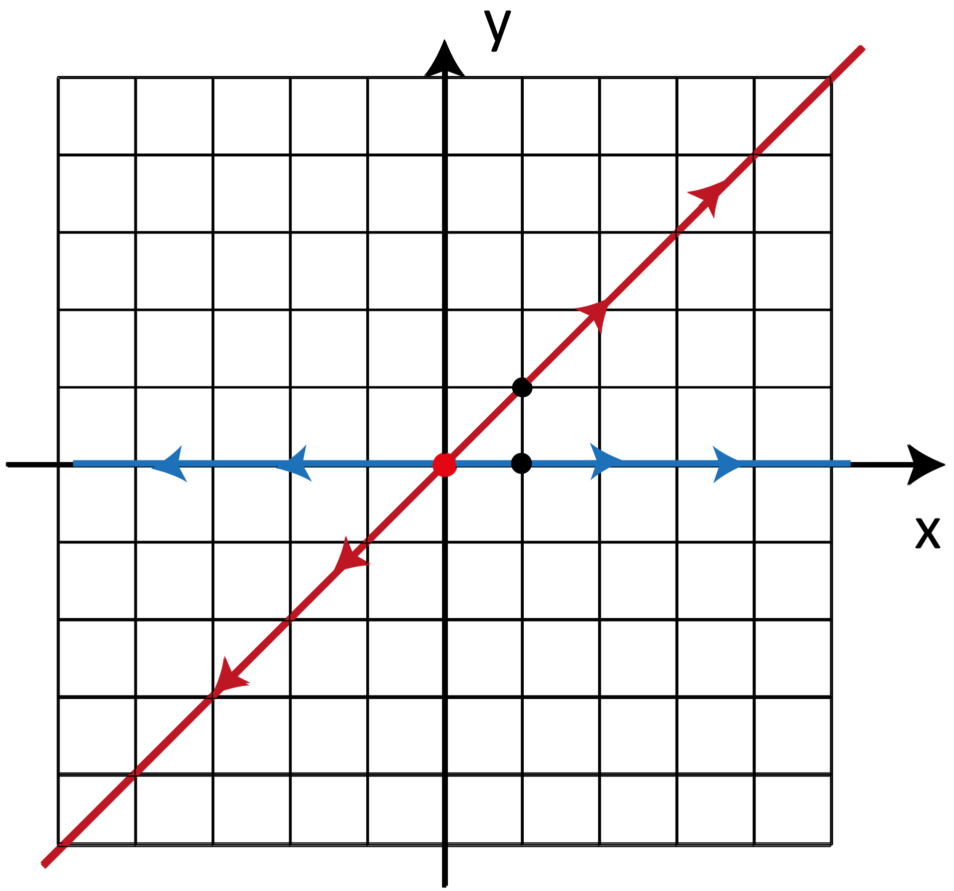
$$\lambda_1 = 2,\ \ \lambda_2 = 1,\ \ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v_2} e^{\lambda_2 t} \qquad \mathbf{z}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Let’s first draw two simple solutions,

$c_1 = 0,\ \ c_2 \neq 0$

$c_1 \neq 0,\ \ c_2 = 0$



Unstable Node

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x - y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases}$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{z}(t) = c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{t}$$

To draw the others, let us consider

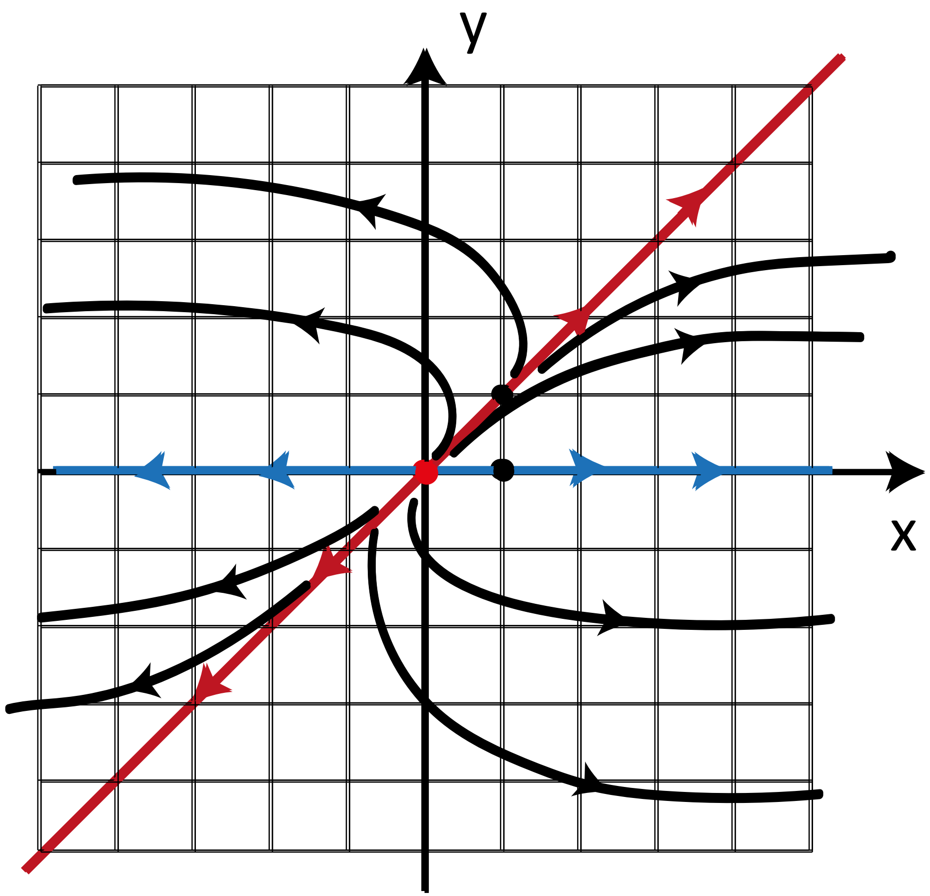
$$t = -\mathrm{M} \qquad \mathbf{z}(t) = \cancel{c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{-2\mathrm{M}}} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{-\mathrm{M}}$$

We are close to the origin and parallel to $\mathbf{v}_2$

$$t = \mathrm{M} \qquad \mathbf{z}(t) = c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{2\mathrm{M}} + \cancel{c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{\mathrm{M}}}$$

We are far from the origin and parallel to $\mathbf{v}_1$

$\mathrm{M} > 0$ and arbitrary big



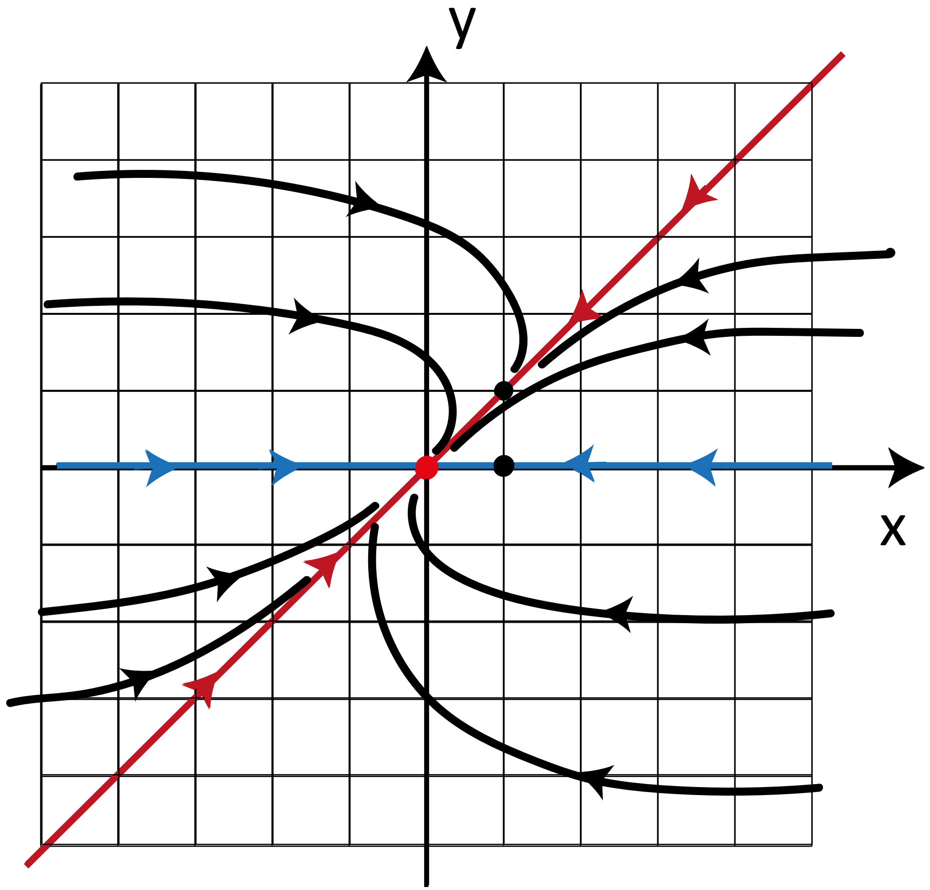
Stable Node

$$\lambda_1 = -2,\ \lambda_2 = -1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{z}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$t = -\mathrm{M}$ We are far from the origin and parallel to $\mathbf{v}_1$

$t = \mathrm{M}$ We are close to the origin and parallel to $\mathbf{v}_2$



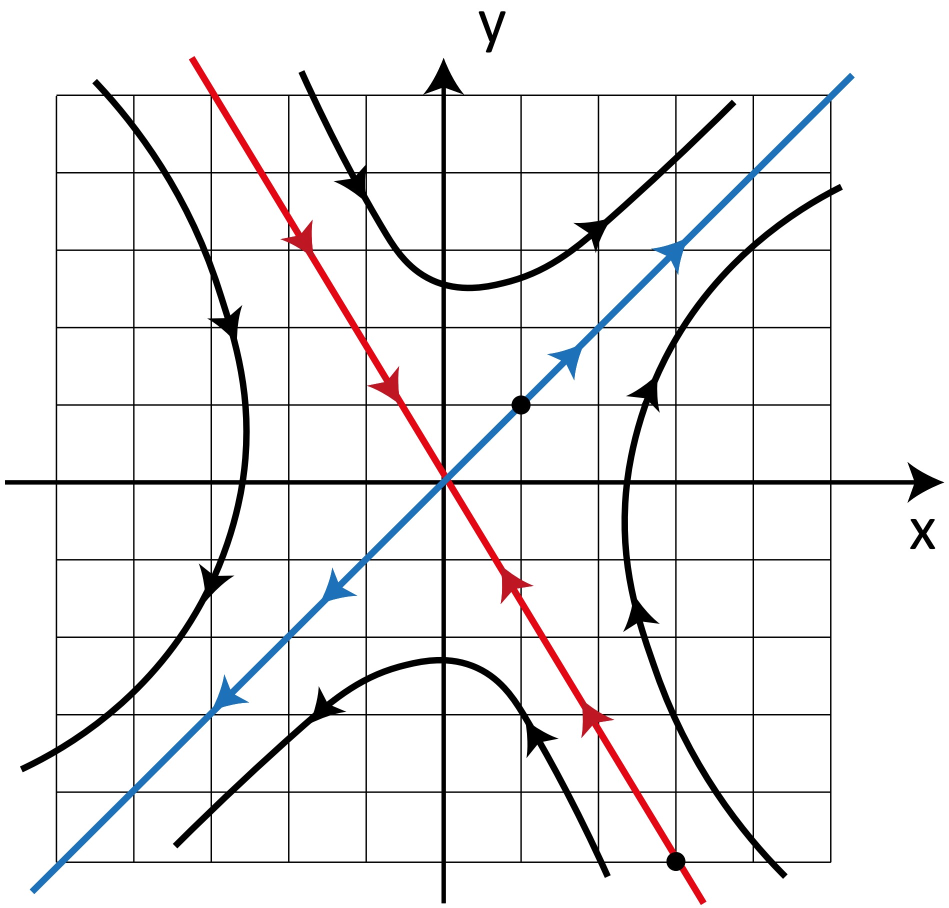
Saddle

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x + 3y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 5x - 3y \end{cases}$$

$$\lambda_1 = -6,\ \lambda_2 = 2,\ \mathbf{v}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is simpler to draw, just follow the directions

If the system starts exactly on the eigenvector with a negative eigenvalue (red on the plot), it goes to zero. If the system is slightly away from that eigenvector, it goes to infinity.



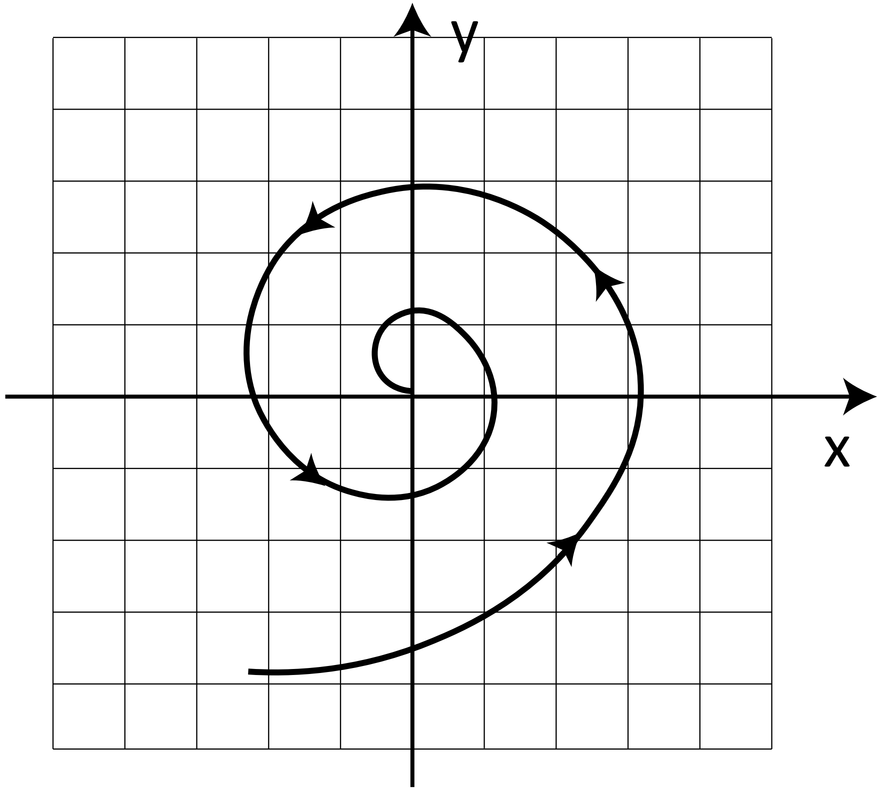
Focus/Spiral, stable

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - 2y \end{cases}$$

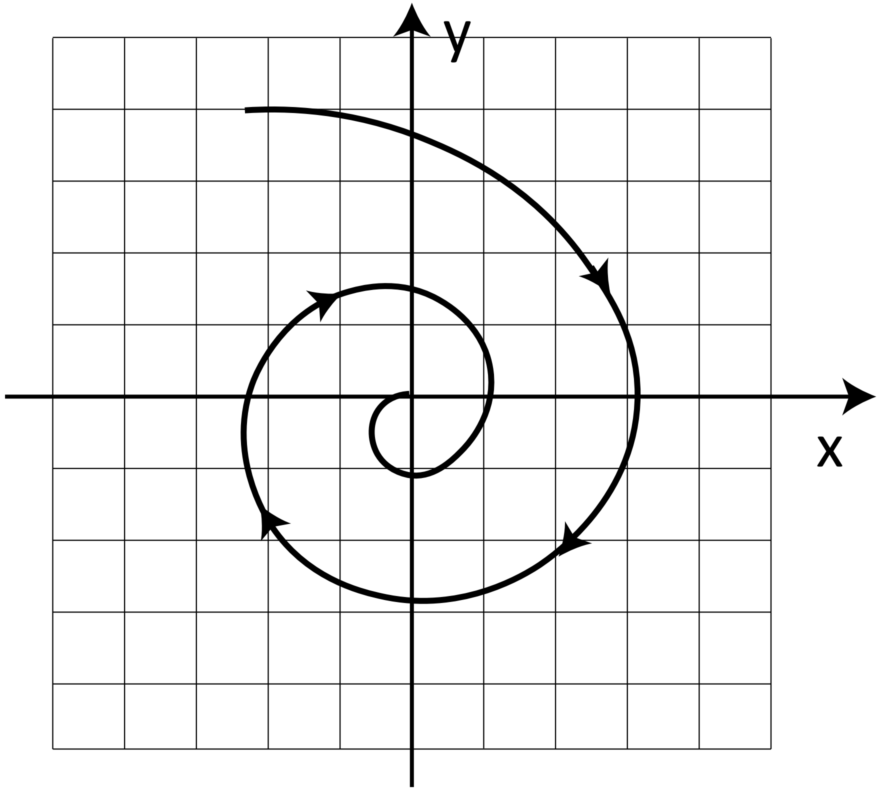
$$\lambda_1 = -1 + \sqrt{2}i, \ \lambda_2 = -1 - \sqrt{2}i$$

Imaginary numbers. Oscillations. Indeed, the solution has terms with sinusoidal functions. You can draw this by noticing the presence of Imaginary values and studying the sign of the real part.

The real part is negative, the system spirals through the origin.



Clockwise or counterclockwise?

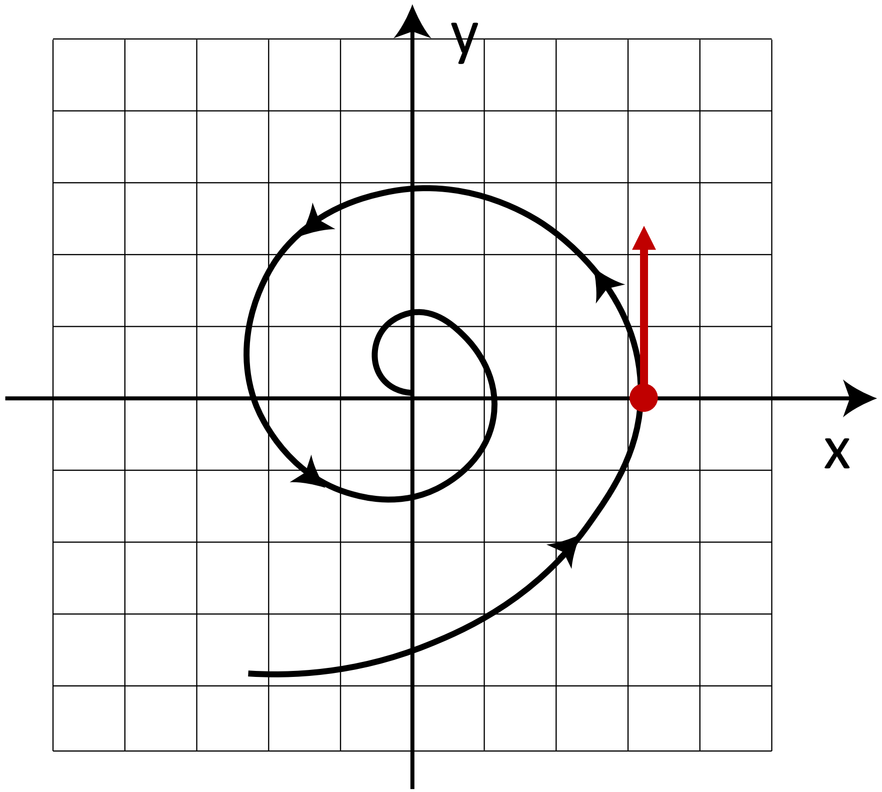


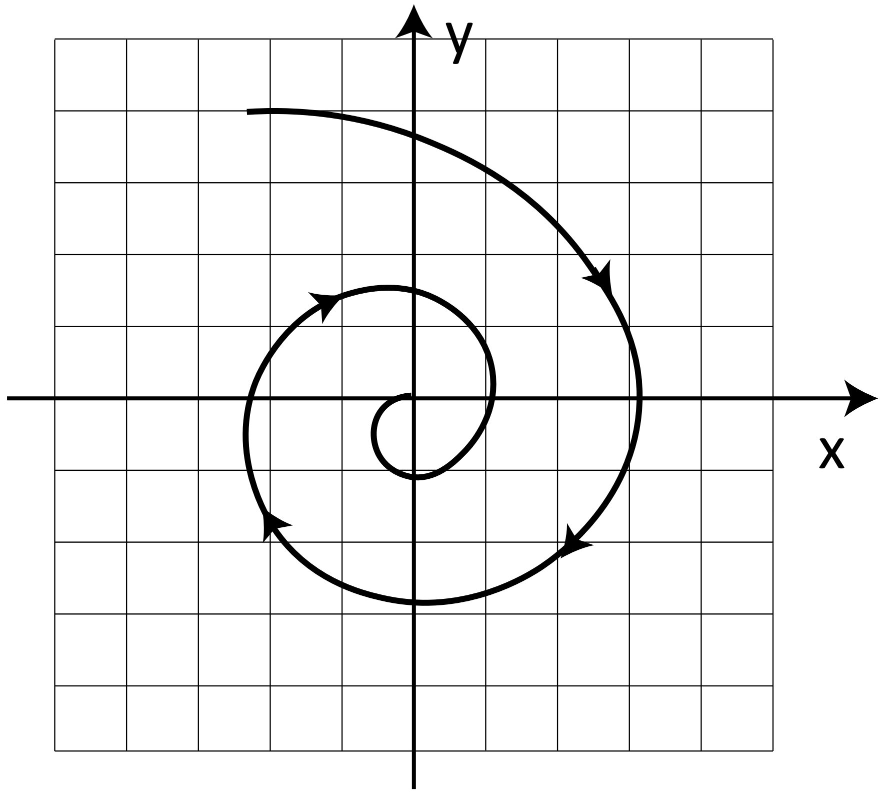
Focus/Spiral, stable

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - 2y \end{cases}$$

$$\lambda_1 = -1 + \sqrt{2}i, \ \lambda_2 = -1 - \sqrt{2}i$$

Clockwise or counterclockwise? Look at the direction at one point.
In this case, in (1,0) the direction is (0,1). Thus, the system goes counterclockwise in the considered example and the left graph is the correct one.





Practical Recipe

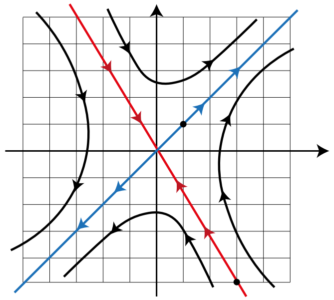
Real eigenvalues

Draw the eigenvectors with the corresponding directions

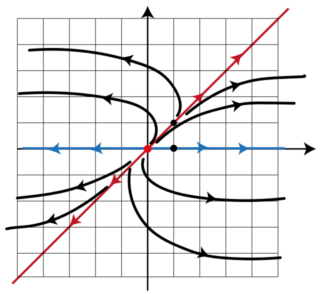
If the two eigenvalues have opposite signs, follow the directions of the eigenvectors

If the two eigenvalues have the same signs, consider the limits

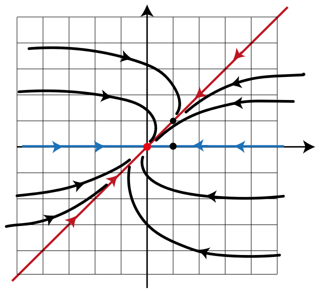
Saddle



Unstable Node



Stable Node



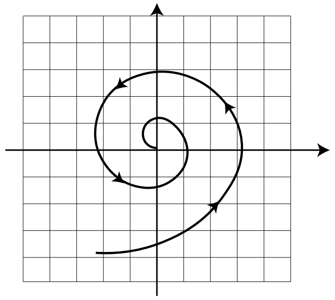
Imaginary eigenvalues

Find the directions at one or more points to understand if the system spirals clockwise or counterclockwise

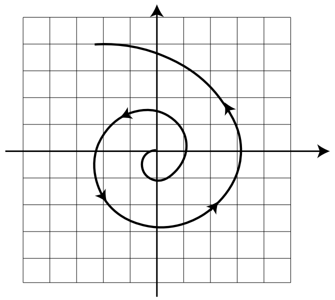
If the real part is negative, the system goes to the origin

If the real part is positive, the system goes away from the origin

Stable Spiral



Unstable Spiral



Notice: these figures are examples and other systems will not have exactly these trends

Thank you

SPIKING NEURONS

Week 10

Dr. Luca Manneschi

So far…

Differential Equations and initial condition. How to find a unique solution?

Solutions to

Look at the exercises

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$\alpha > 0$ Exponential growth

$\alpha < 0$ Exponential decay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

Saturating

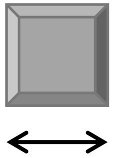
Euler's method and its error $\mathcal{O}(\delta t^2)$ $\mathcal{O}(\delta t)$

Definition of equilibrium points in one and two dimensions. Phase portraits…

Autonomous and non-autonomous: Definition

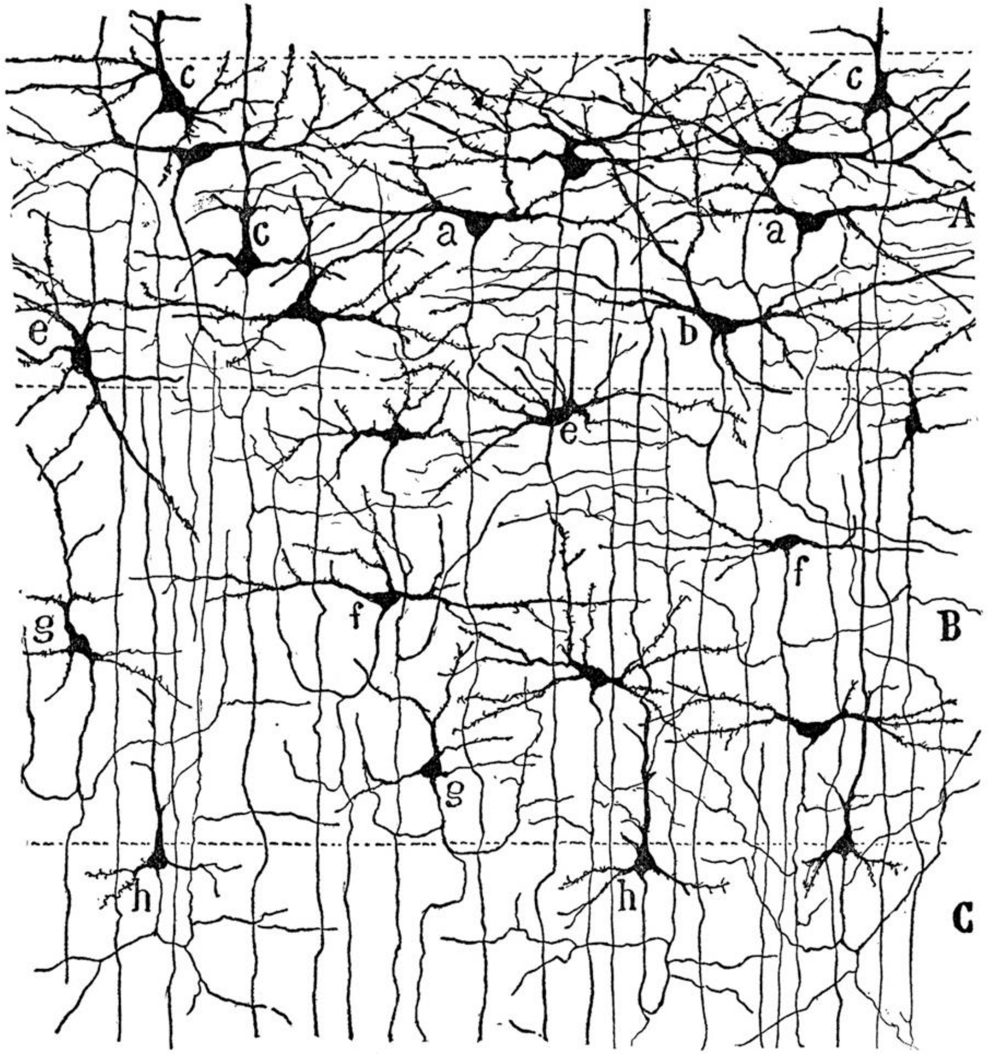
MidPoint Method, Runge-Kutta 2 and 4

Today…

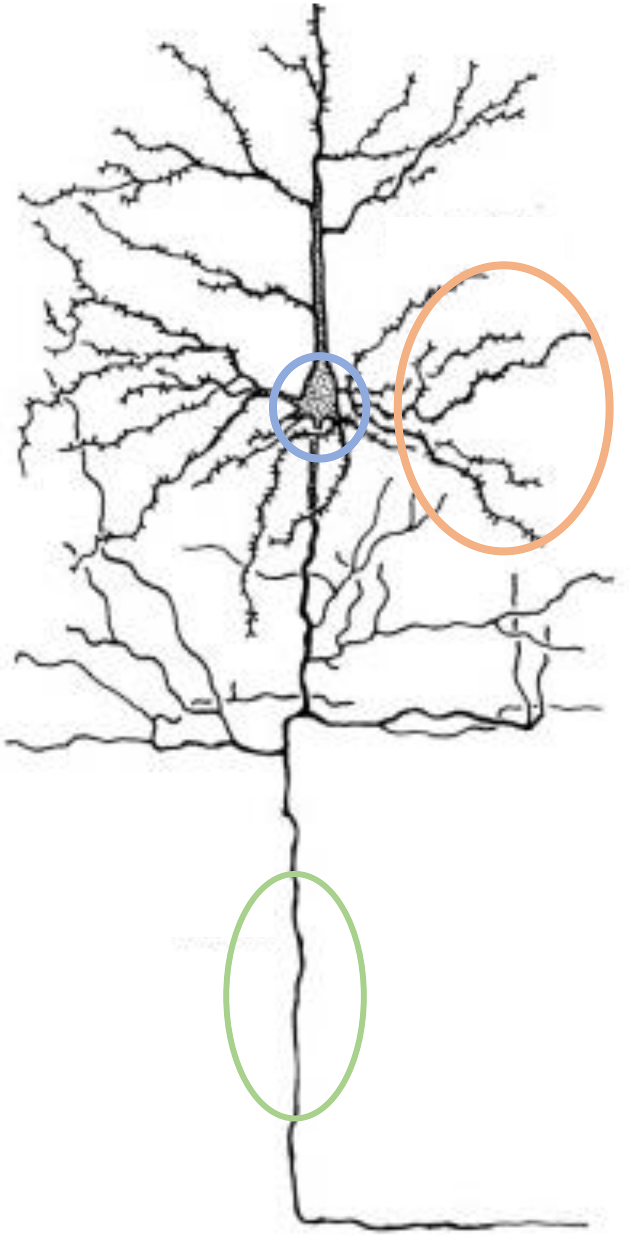


1 mm

`10 000 Neurons
3 Km of wires



Neuron: the main parts

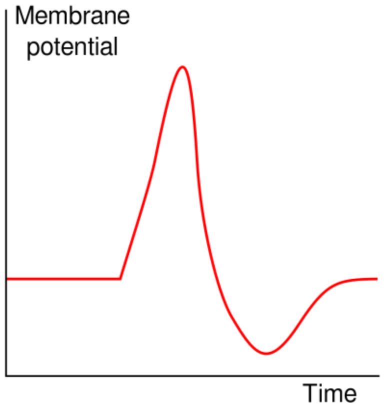


Dendrites: Input device

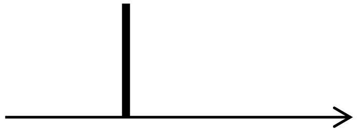
Soma: Central processing unit

Axon: Output

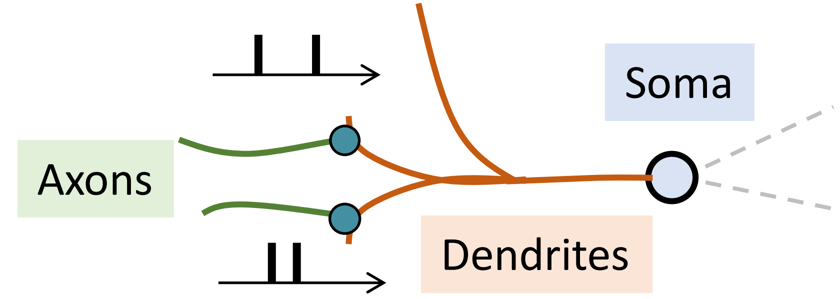
Communication? In a ‘stereotype’ manner…
Action potentials



Often we will treat this
event as instantaneous

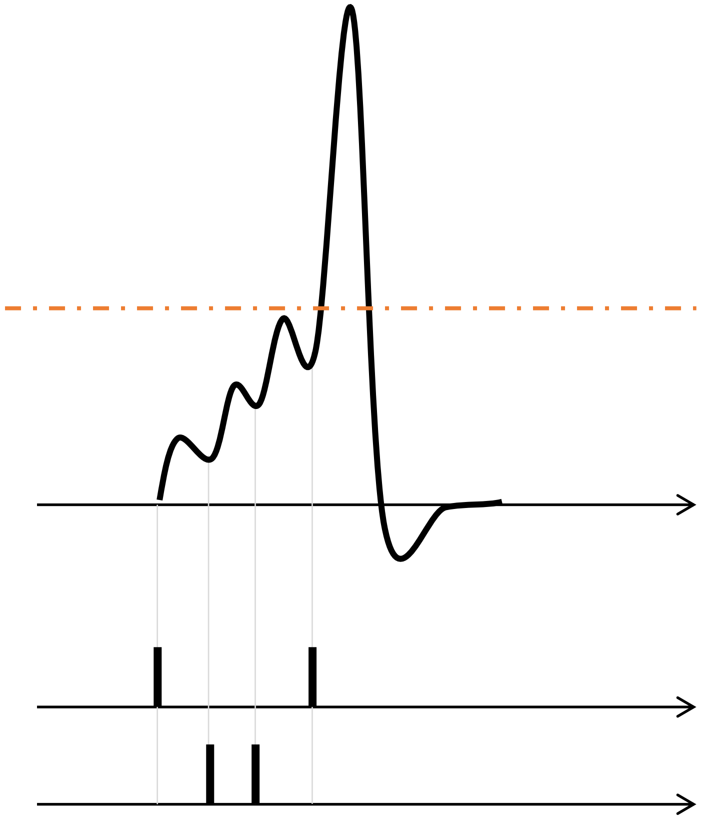


Neuron: the main parts

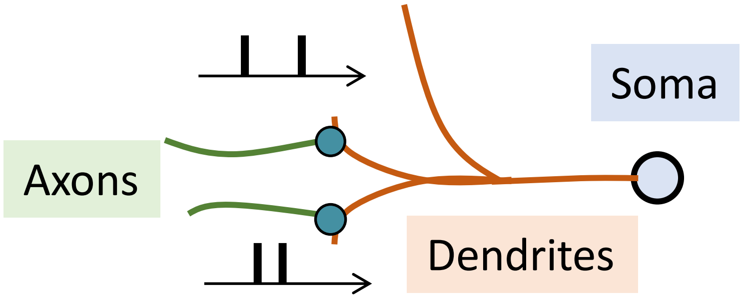


We can measure the potential difference between the inner part and outer part of the soma

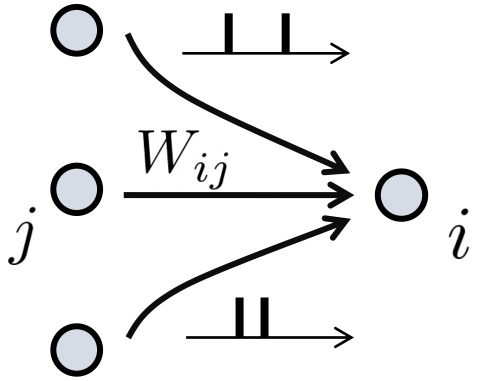
The junction between neurons is called the synapse

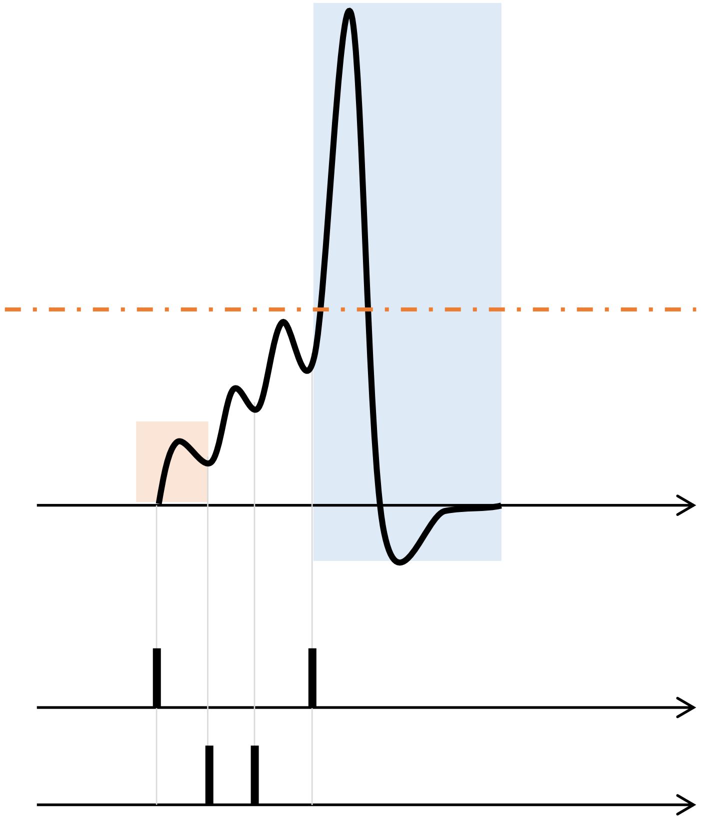


Response neuron model



We simplify the left diagram considering that there are presynaptic neurons and postsynaptic neurons connected (right diagram)





$\epsilon(t - t_j^f)$ Postsynaptic potential: the effect that an input spike has on the potential

$\eta(t - t_i^l)$ Spike of the neuron i considered

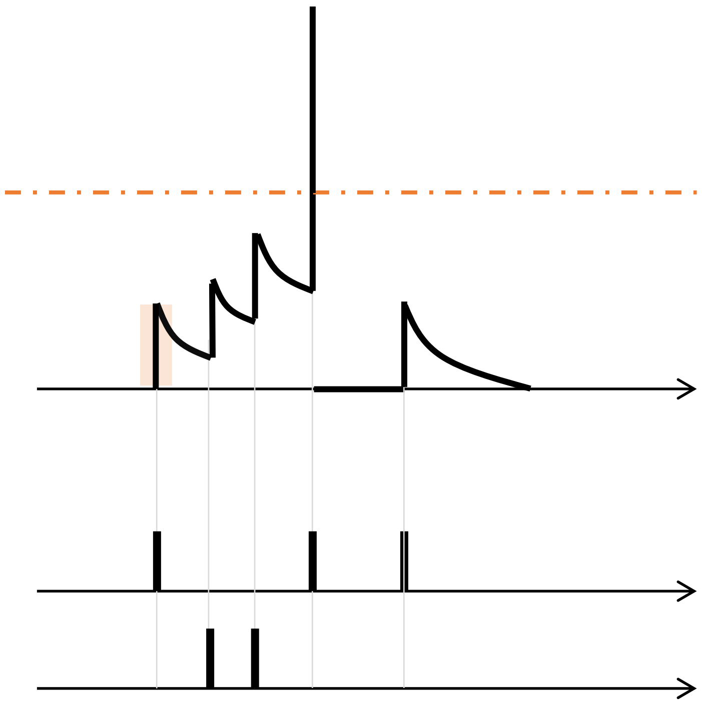
Sum across all presynaptic neurons (j) and all spikes (f)

$$u_i(t) = \eta(t - t_i^l) + \sum_j \sum_f W_{ij}\epsilon(t - t_j^f)$$

$$u_i(t) = \theta, \ \rightarrow t_i^l = t$$

Leaky integrate and fire

If we consider that the postsynaptic potential is linear we can define the leaky-integrate and fire model



Input

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + V(t)$$

$$u_i(t) = \theta \rightarrow$$ Fire + Reset

Does it sound familiar?

Autonomous or not?

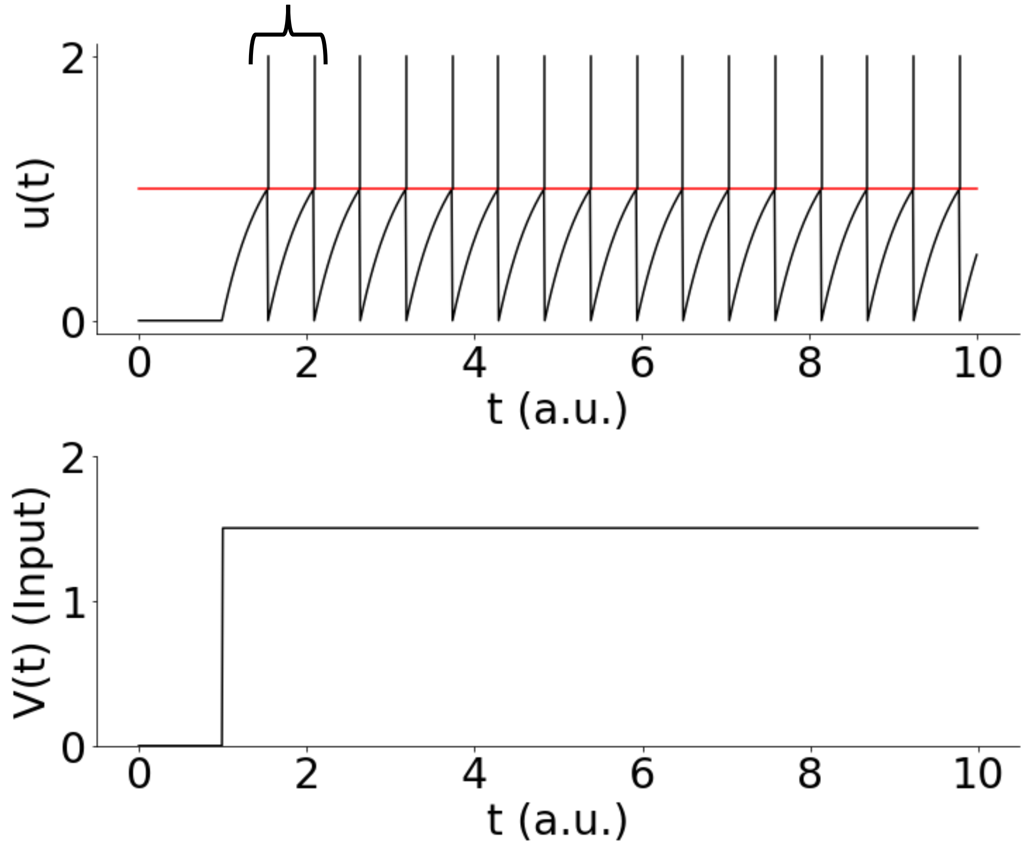
Response Examples

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + V(t)$$

$u_i(t) = \theta \rightarrow$ Fire + Reset

'Constant' Input

Interval between successive spikes

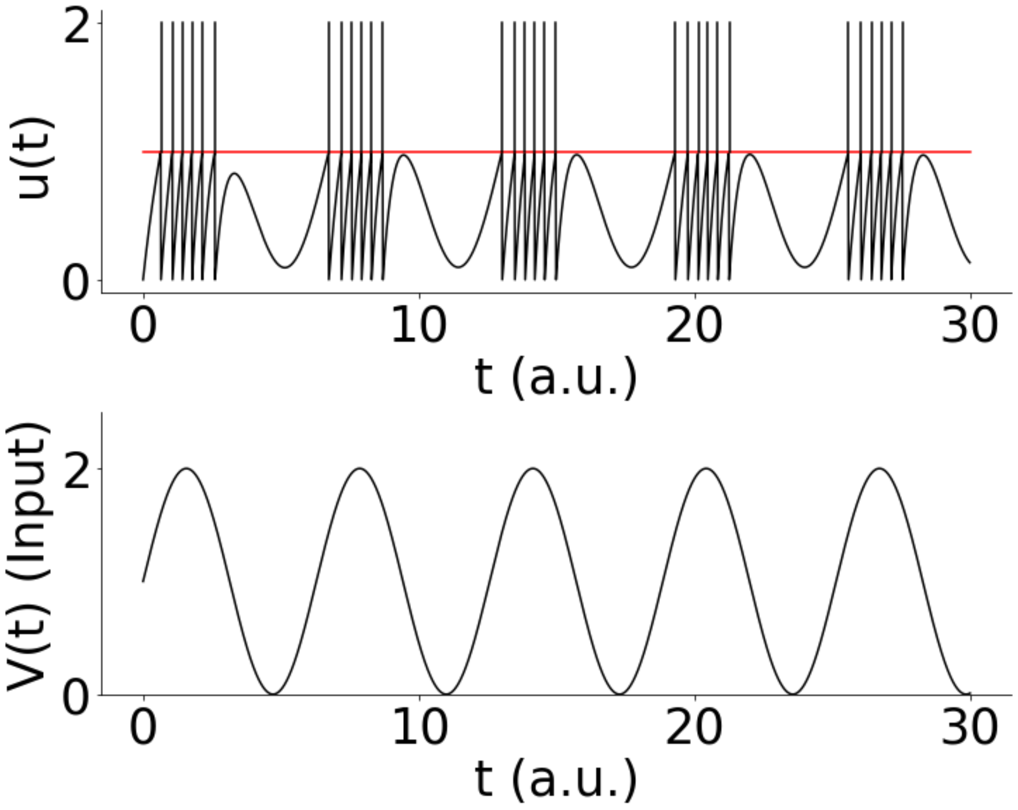


Response Examples

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + V(t)$$

$u_i(t) = \theta \rightarrow$ Fire + Reset

Sinusoidal Input



Response Examples

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + V(t)$$

$u_i(t) = \theta \rightarrow$ Fire + Reset

Poisson spike model

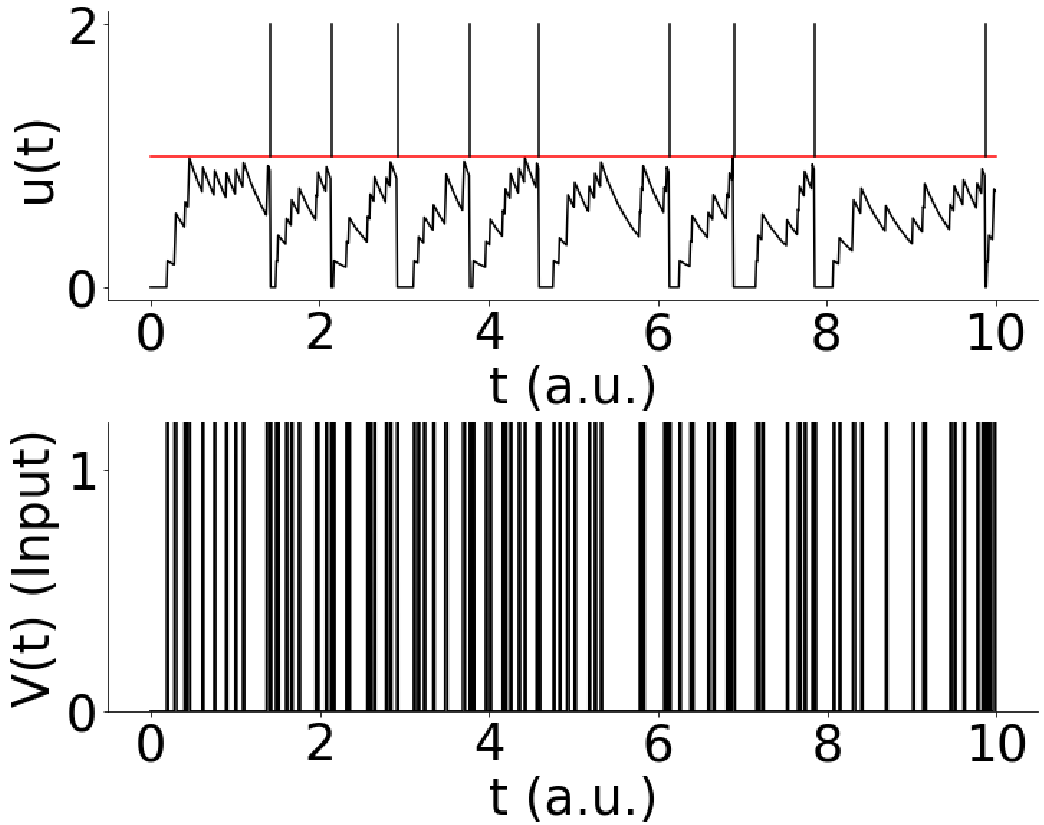
Rate

$$p_i(t) \approx r_i(t)\delta t$$

Probability of firing

Practically, you compute the probability from the rate. Then, generate a random number between [0, 1]. If the random number is less than the probability the neuron fires, otherwise it does not.

Poisson spike model as Input



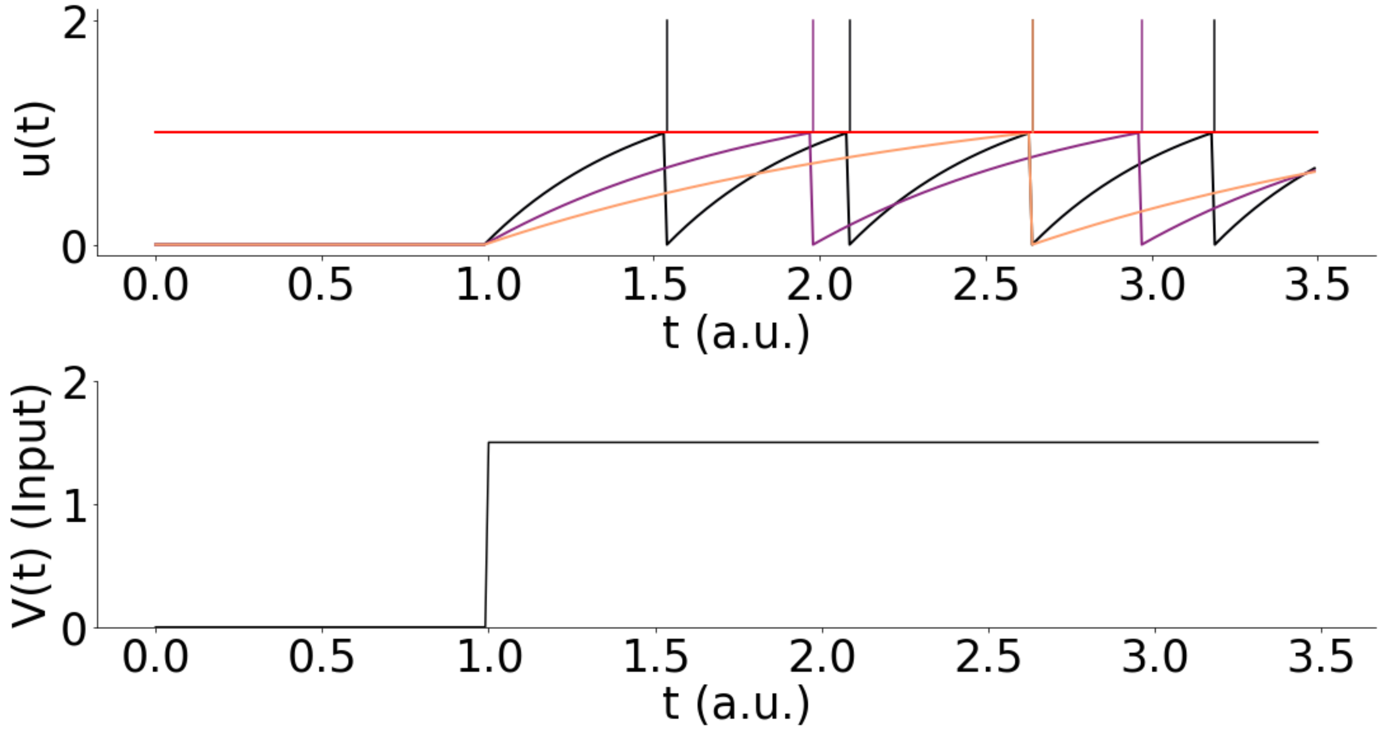
The meaning of tau

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + V(t)$$

It describes the rate at which information is integrated

$u_i(t) = \theta \rightarrow$ Fire + Reset

It increases from darker to brighter colours



Constant Input, Solution

$$\tau \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + C$$

$$\int \frac{\mathrm{d}u_i(t)}{-u_i(t) + C} = \int \frac{\mathrm{d}t}{\tau}$$

$$-\log(-u_i(t) + C) = \frac{t}{\tau} + k$$

$$-u_i(t) + C = e^{-t/\tau + k}$$

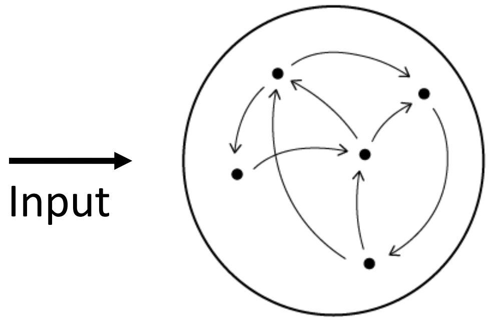
$$-u_i(t) = e^{-t/\tau} e^k - C$$

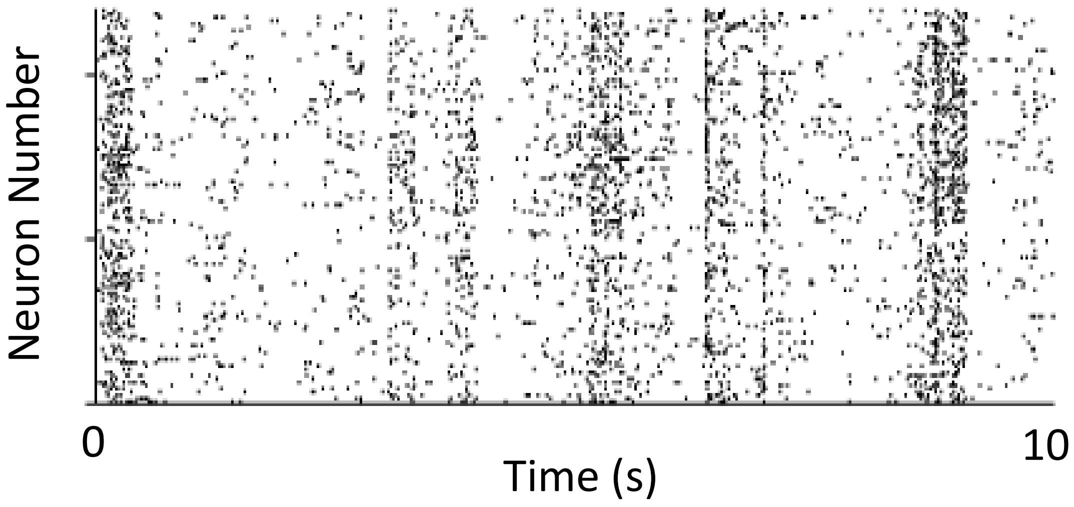
$$u_i(t) = C - e^{-t/\tau} k_1$$

Can you compute the time at which it reaches the threshold?

The problem of neuronal coding

We consider a population of neurons





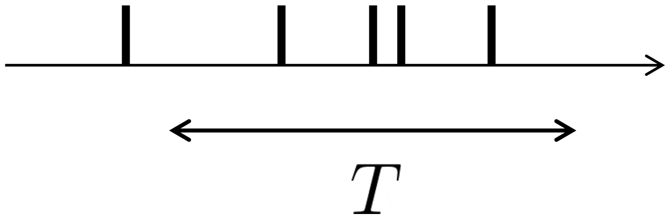
Where is the information?

The problem of neuronal coding

Rate coding. The information is in the rate at which neurons spike

How can we find the rate? We have spikes (binary information), we need to average…

For a single neuron



Temporal average

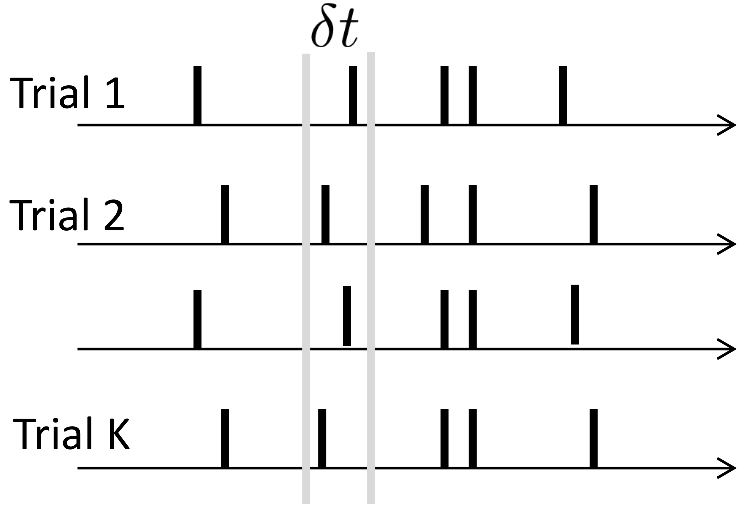
$$\nu = \frac{n(t, t+T)}{T}$$

The problem of neuronal coding

Rate coding. The information is in the rate at which neurons spike

How can we find the rate? We have spikes (binary information), we need to average…

For a single neuron



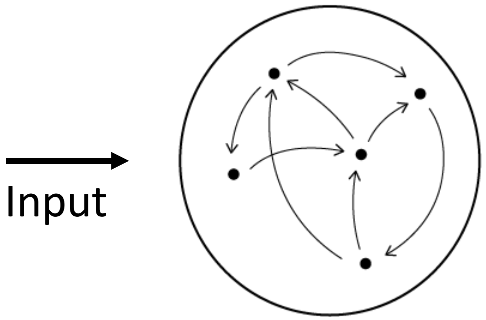
Average across repetitions

$$PSTH(t) = \frac{n(t, t+\delta t)}{K\delta t}$$

K Number of repetitions

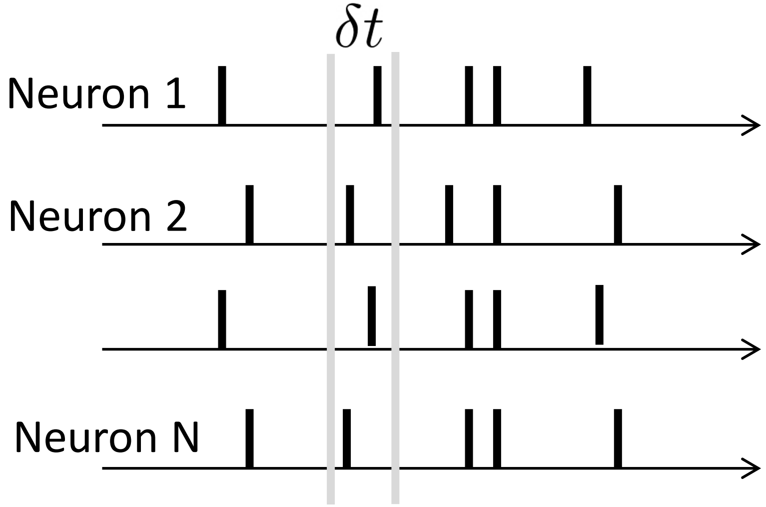
The problem of neuronal coding

Rate coding. The information is in the rate at which neurons spike



How can we find the rate? We have spikes (binary information), we need to average…

For a population



Average across neurons

$$A(t) = \frac{n(t, t + \delta t)}{N \delta t}$$

Number of neurons

The problem of neuronal coding

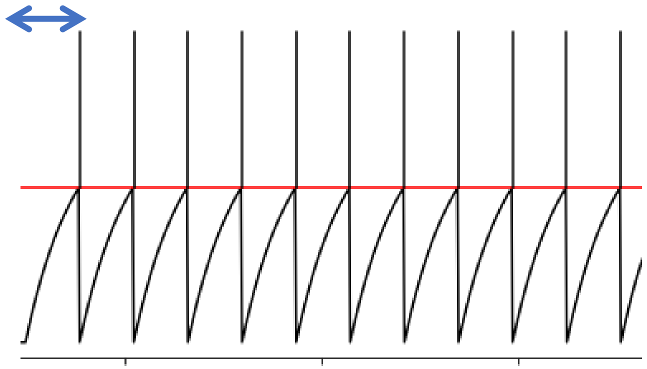
The problem with rate coding on a single neuronal level.

Imagine a frog that catches a fly. The frog reacts in approximately 70 ms. In rate coding we need to average, and averaging takes time. In this activity, there is no time for the considered neuron of the frog to encode the rate across time.

The problem of neuronal coding

Alternative to rate coding: Temporal coding. Examples:

Time-to-first-spike after input

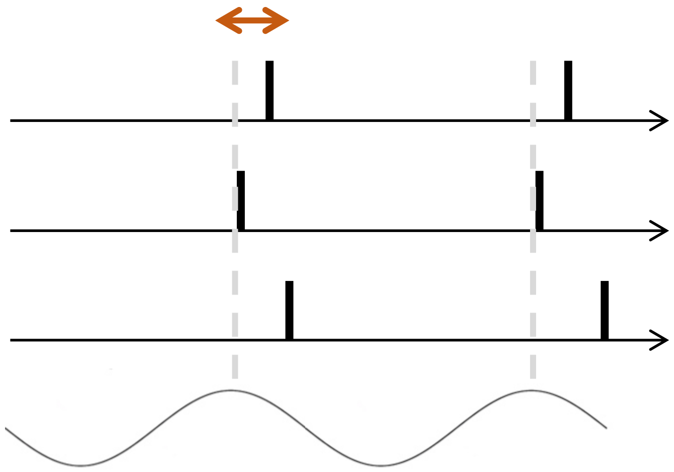


It would need only a single spike to encode this information

The problem of neuronal coding

Alternative to rate coding: Temporal coding. Examples:

Phase with respect to oscillation



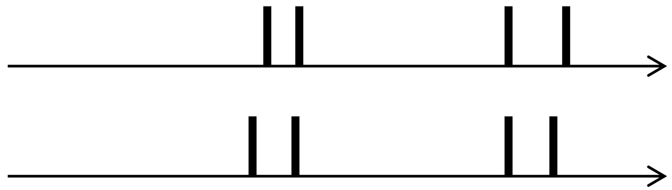
We know that parts of the brain as the hippocampus are characterized by oscillations. We can then measure the 'first spike' relative to a background oscillation

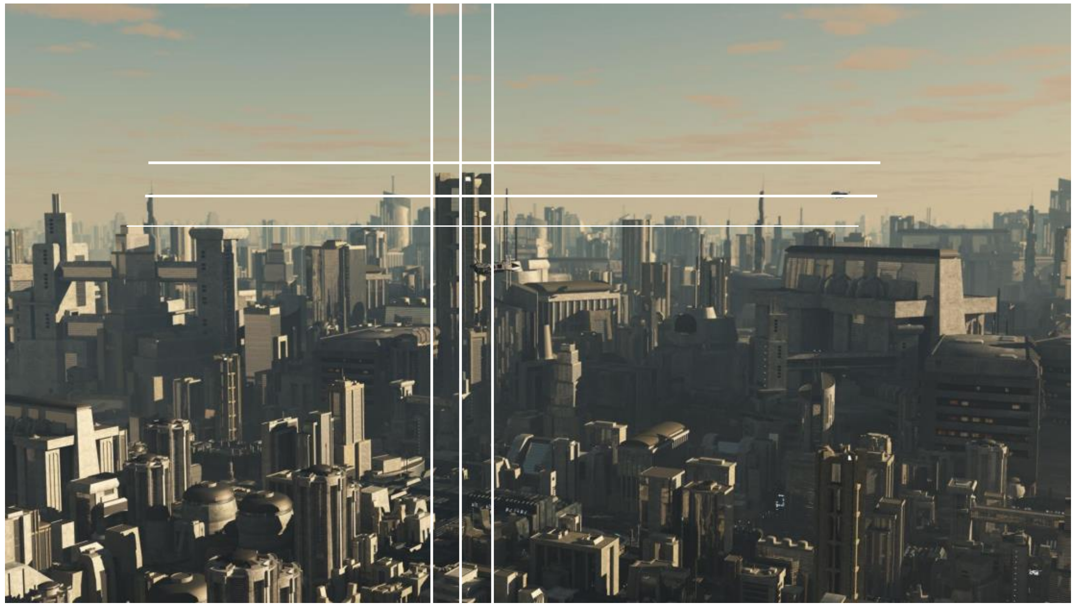
The problem of neuronal coding

Alternative to rate coding: Temporal coding. Examples:

Correlations and synchrony

The fact that a pair or an ensemble of neurons have synchronous activity could mean something. For instance, it could mean that they ‘belong together’…Imagine that neurons encode different locations in an image. Neurons that have synchronous activity could mean that they are encoding the same object





Thank you