DIFFERENTIAL EQUATIONS, EULER'S METHOD AND STABILITY

Week 7

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OBJECTIVES for this second part

UNDERSTAND AND SIMULATE EBMs

Euler's Method

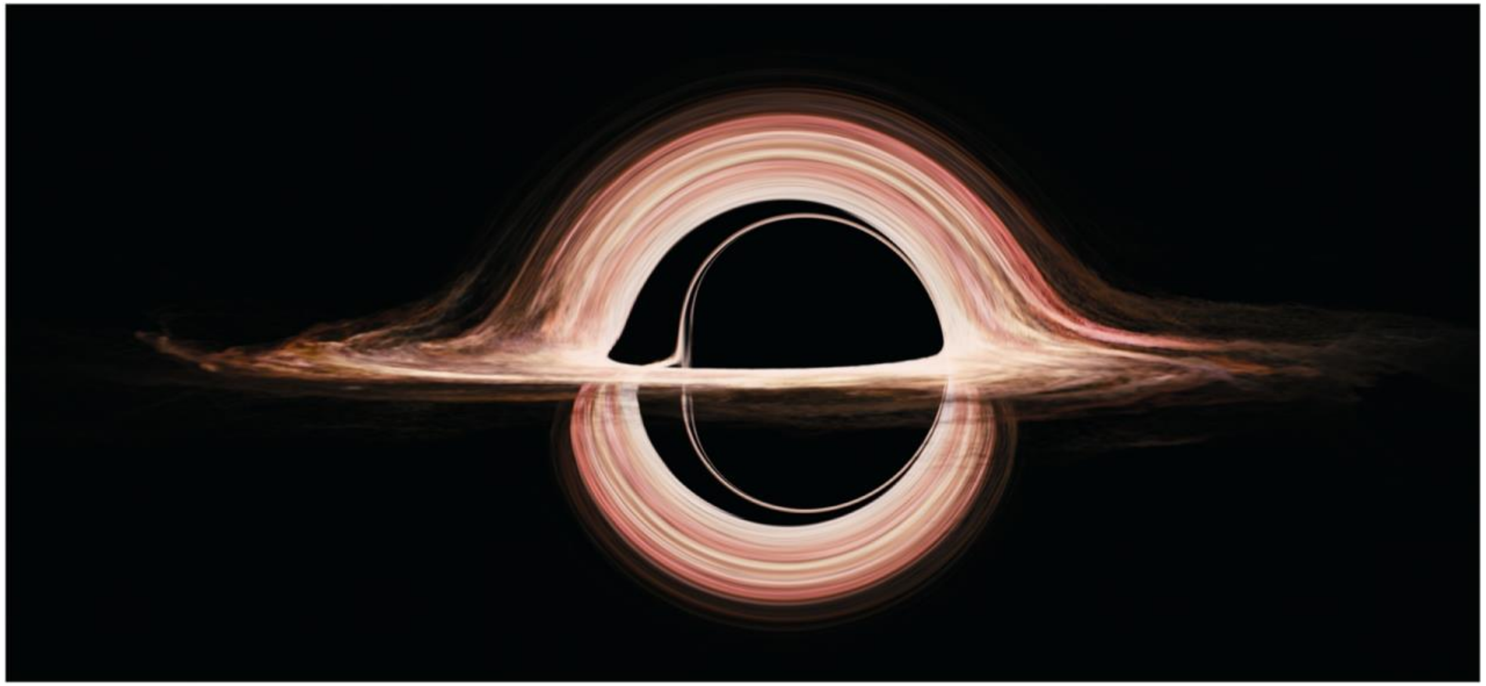
Equilibrium points

Runge-Kutta methods

Higher-order systems

Examples in
Machine Learning/Neuroscience

Why EBMs? An example



James, Oliver, et al. "Gravitational lensing by spinning black holes in astrophysics, and in the movie Interstellar." *Classical and Quantum Gravity* 32.6 (2015): 065001.

Differential Equations: An ignorant introduction

In an equation that is a relation between functions, the solution corresponds to an ensemble of values

$$x^2 - 5x + 6 = 0$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(t) - t^2 = 0$$

In differential equations, derivatives appear as terms of the equations
The solution is the family of functions that satisfies the relation considered

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(1)}(t) \qquad \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(2)}(t)$$

Differential Equations: An ignorant introduction

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$$x^2 - 5x + 6 = 0$$

$$x^{(2)}(t) + 5x^{(1)}(t) - x(t) - t^2 = 0$$

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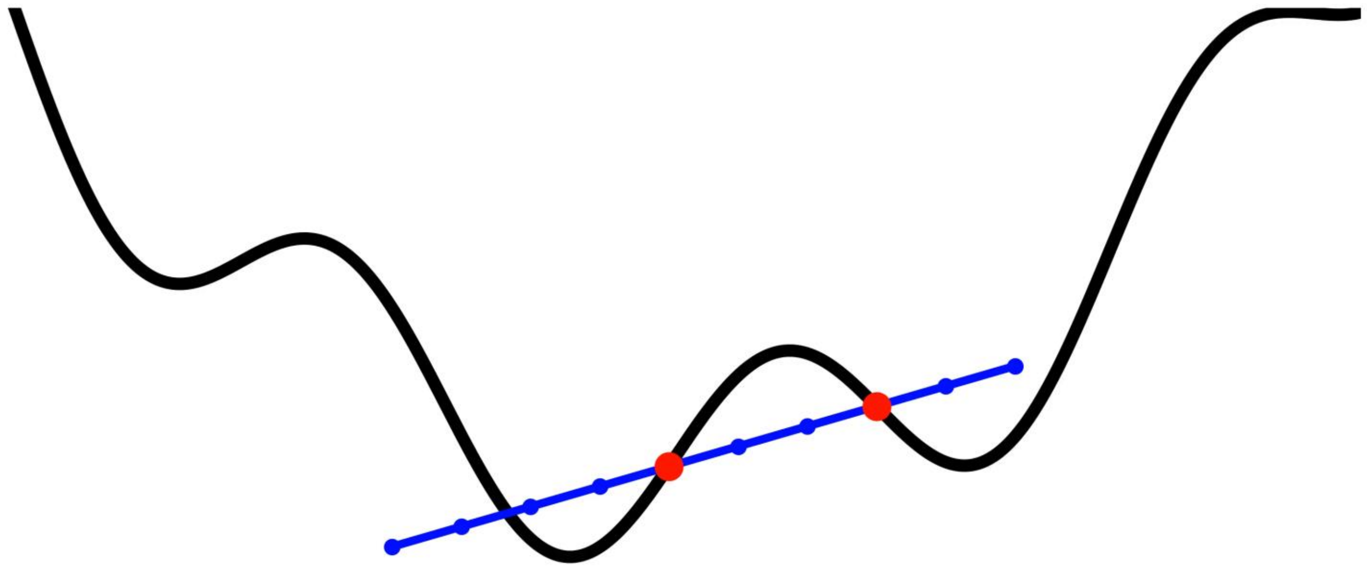
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(1)}(t) \qquad \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x^{(2)}(t)$$

Derivative: Recap

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t}$$



The derivative is the slope of the blue line

Derivative: Recap

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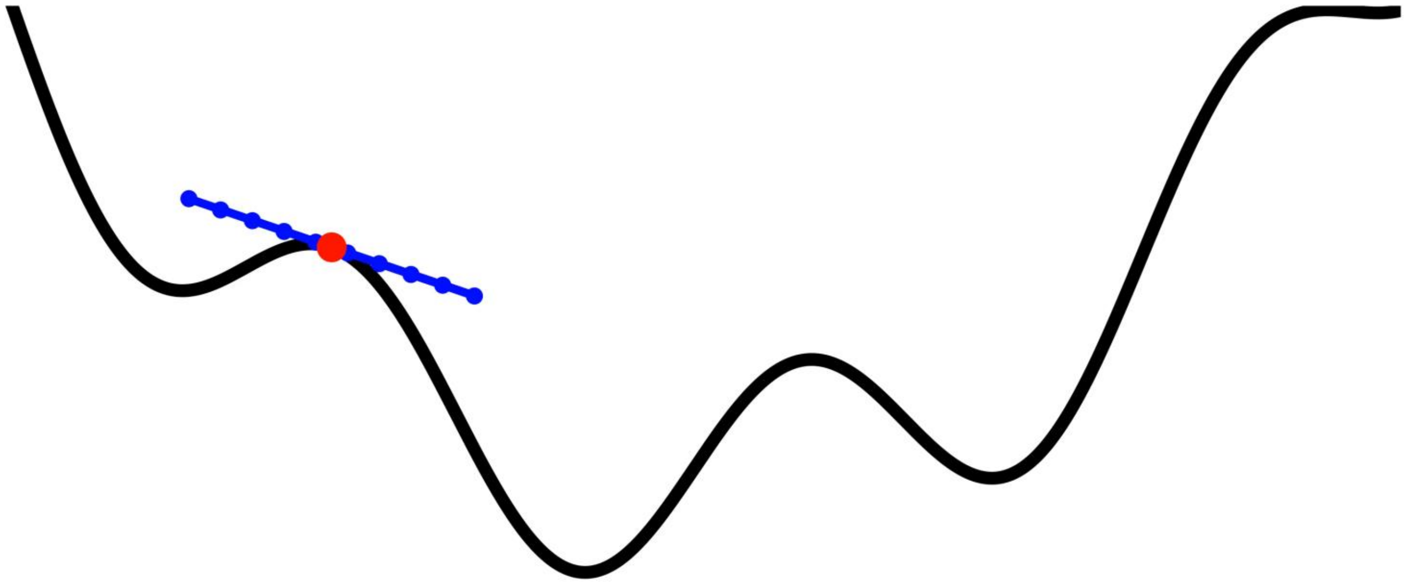
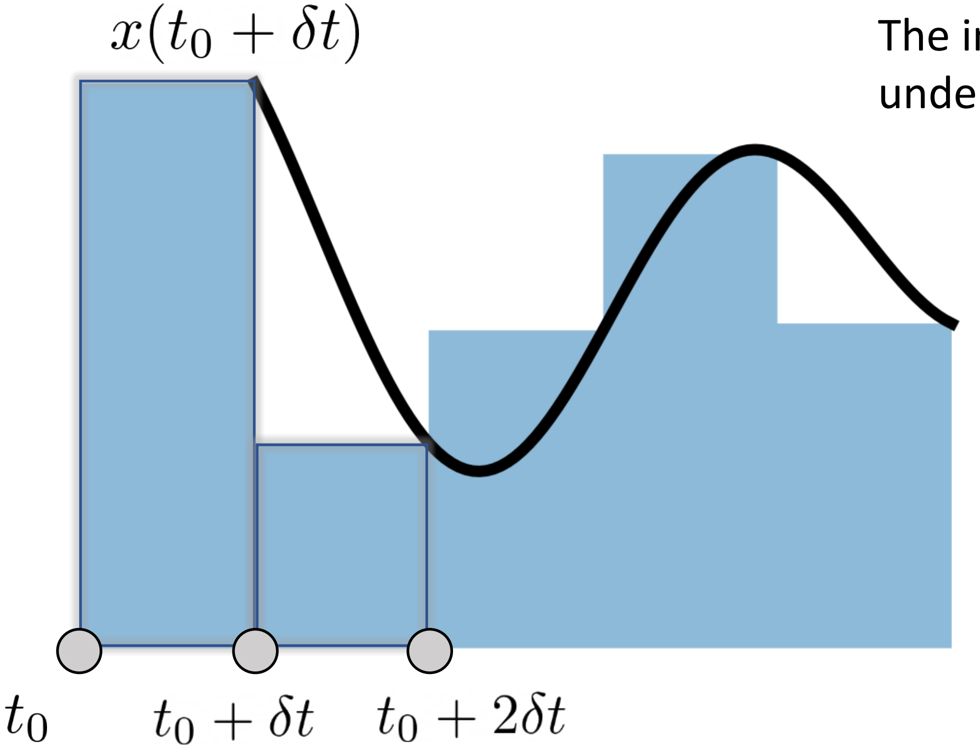


The derivative is the slope of the blue line

Integral: Recap

$$\int \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t = x(t) \qquad \int_{t_0}^{t} x(t)\mathrm{d}t = \lim_{\delta t \to 0} \sum_{n=1}^{\frac{t-t_0}{\delta t}} x(t + n\delta t)\delta t$$



The integral can be thought of as the area under the function considered (black line)

Integral: Recap

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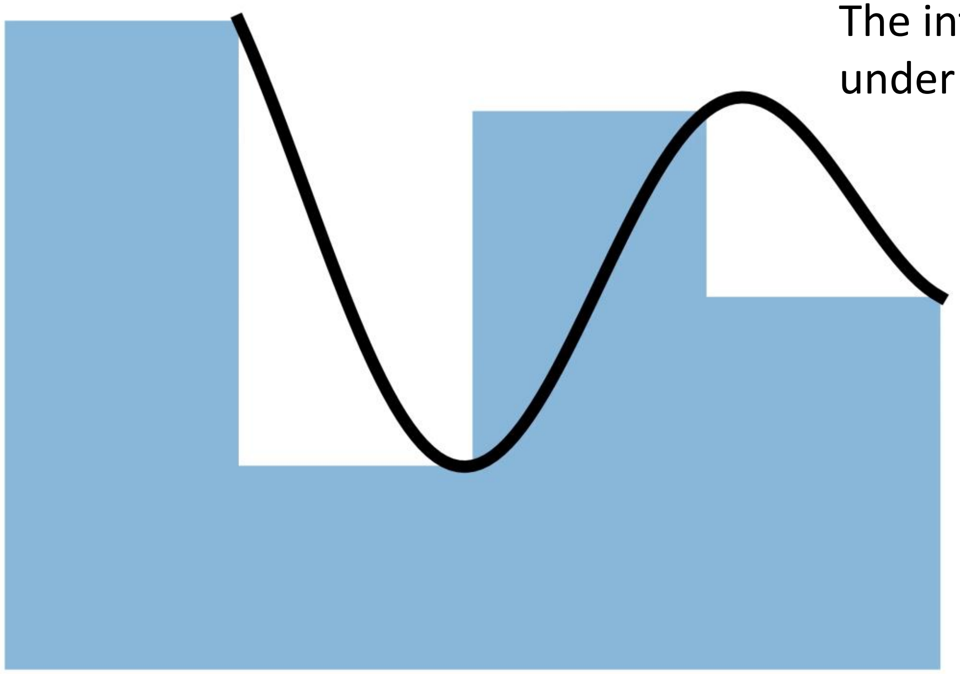


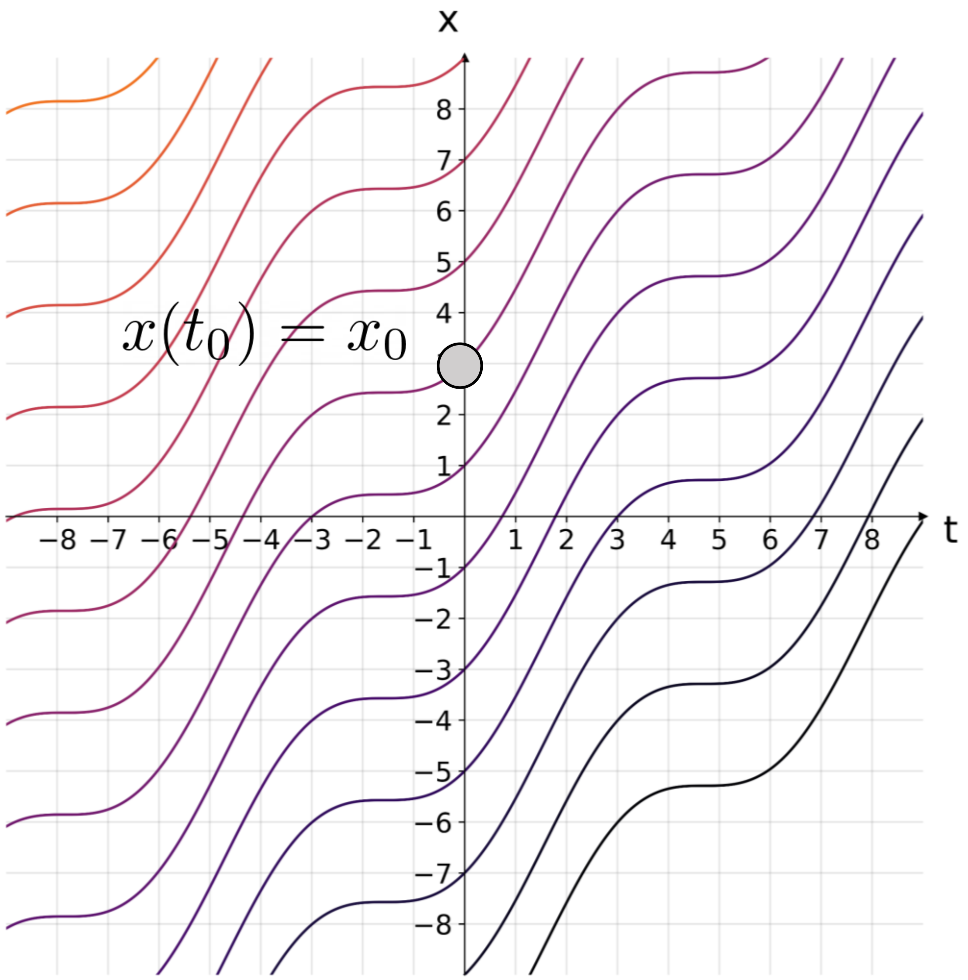
The integral can be thought of as the area under the function considered (black line)

Some simple examples

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t)$$

Can be integrated directly

$$\int \mathrm{d}x = \int f(t)\mathrm{d}t$$



$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin(t) + 1$$

$$x(t) = \int \big(\sin(t) + 1\big)\mathrm{d}t = -\cos(t) + t + k$$

How do we know which one? Initial condition

For this first-order differential equation, we can specify simply $x(t_0) = x_0$ and find *k*

Some simple examples

For instance, given $x(t) = -\cos(t) + t + k$,

for $x(0) = 1$ we find $1 = -1 + k \qquad k = 2$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = f(t)$$

Can also be integrated directly, we need to integrate twice

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = \int f(t)\mathrm{d}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int f(t)\mathrm{d}t$$

Some simple examples

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = t$$

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = \int t\mathrm{d}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^2}{2} + k_1$$

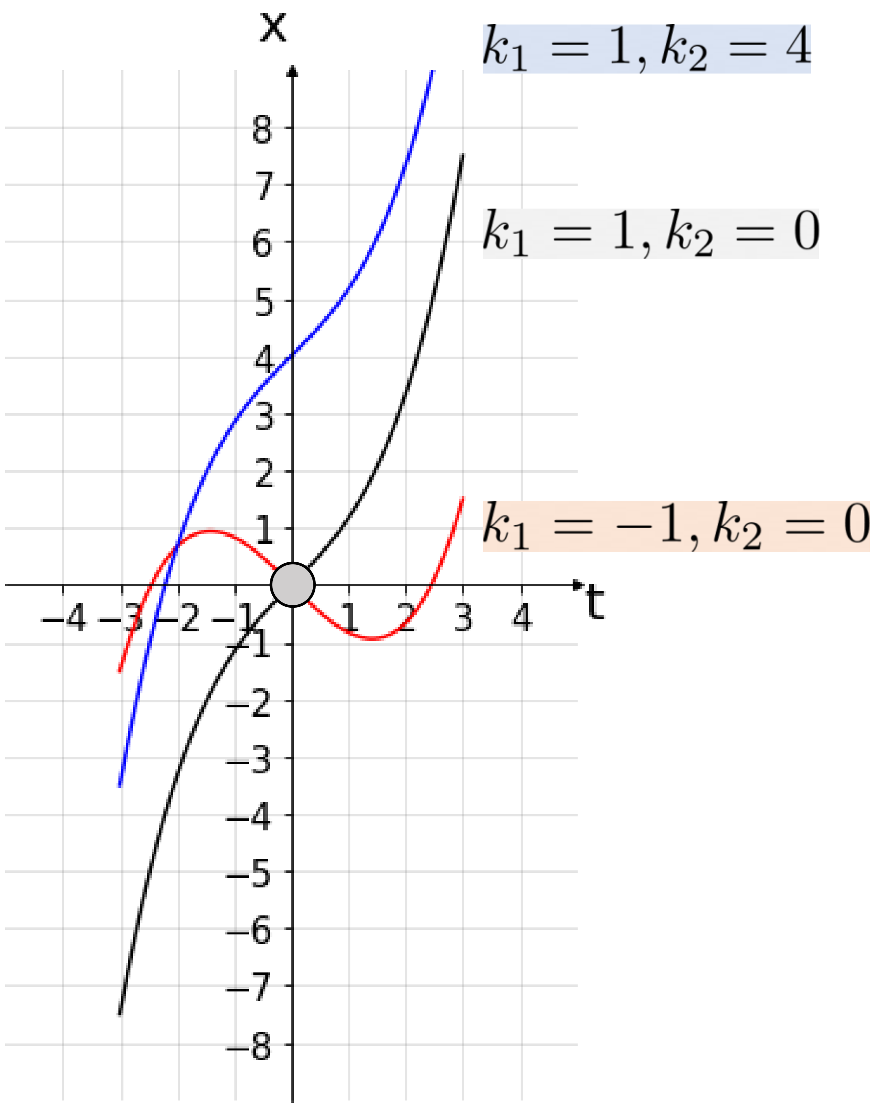
$$\int \mathrm{d}x = \int \frac{t^2}{2} + k_1 \mathrm{d}t$$

$$x(t) = \frac{t^3}{6} + k_1 t + k_2$$

The solutions intersect,
we need to specify

$$x(t_0) = x_0$$

$$x^{(1)}(t_1) = x_1$$



Cauchy problem

A differential equation is of order n if it contains the n-th derivative at maximum

For a differential equation of order n, the task of finding the function that satisfies the equation with the initial conditions

$$x(t_0) = x_0$$

$$x^{(1)}(t_1) = x_1$$

$$\vdots$$

$$x^{(n)}(t_n) = x_n$$

Is called a Cauchy problem

Differential Equations

When solving analytically, we will mostly consider the case

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

$$\frac{\mathrm{d}x(t)}{f\big(x(t)\big)} = g(t)\mathrm{d}t$$

$$\int \frac{\mathrm{d}x}{f(x)} = \int g(t)\mathrm{d}t$$

Examples

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$$f\big(x(t)\big) = x(t), \;\; g(t) = \alpha$$

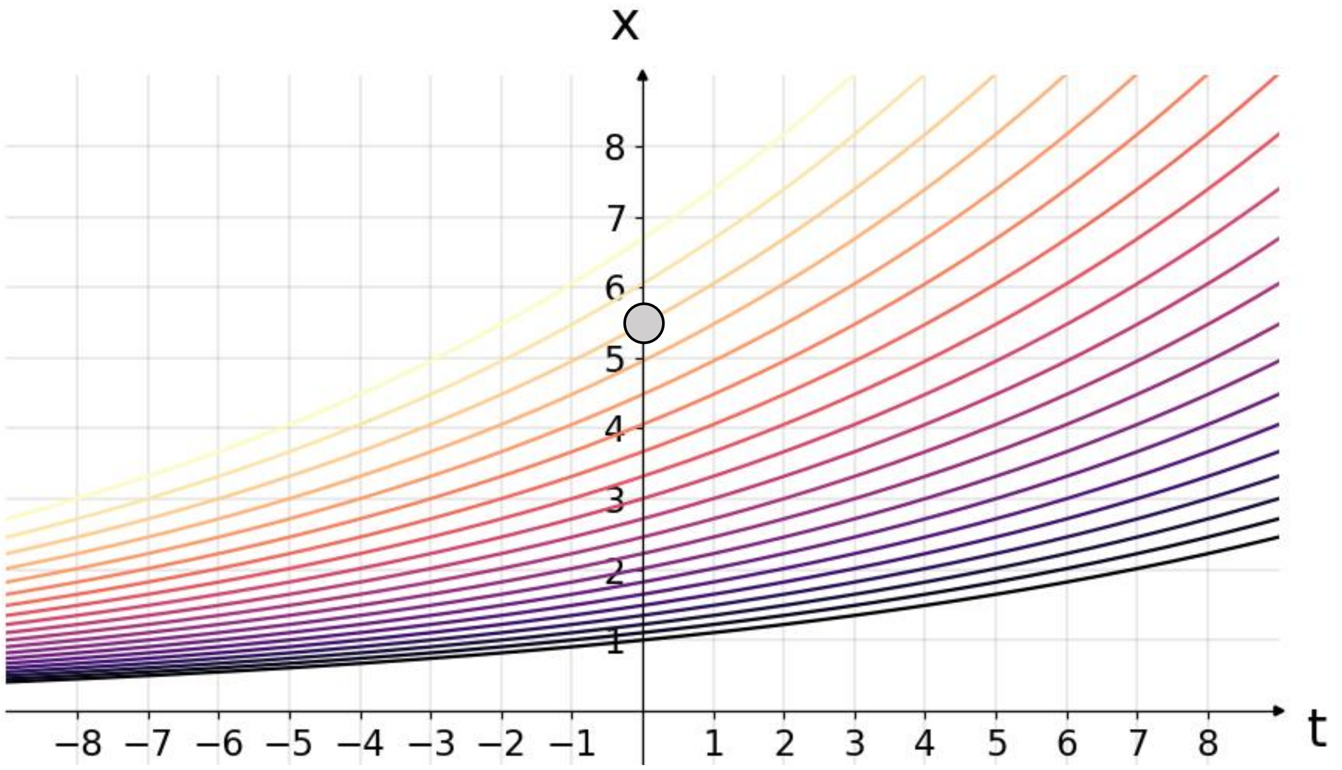
$$\int \frac{\mathrm{d}x}{x} = \int \alpha \mathrm{d}t$$

$$\log(x) = \alpha t + k$$

$$x(t) = e^{\alpha t + k}$$

Initial condition

$$x(t_0) = x_0$$



Examples

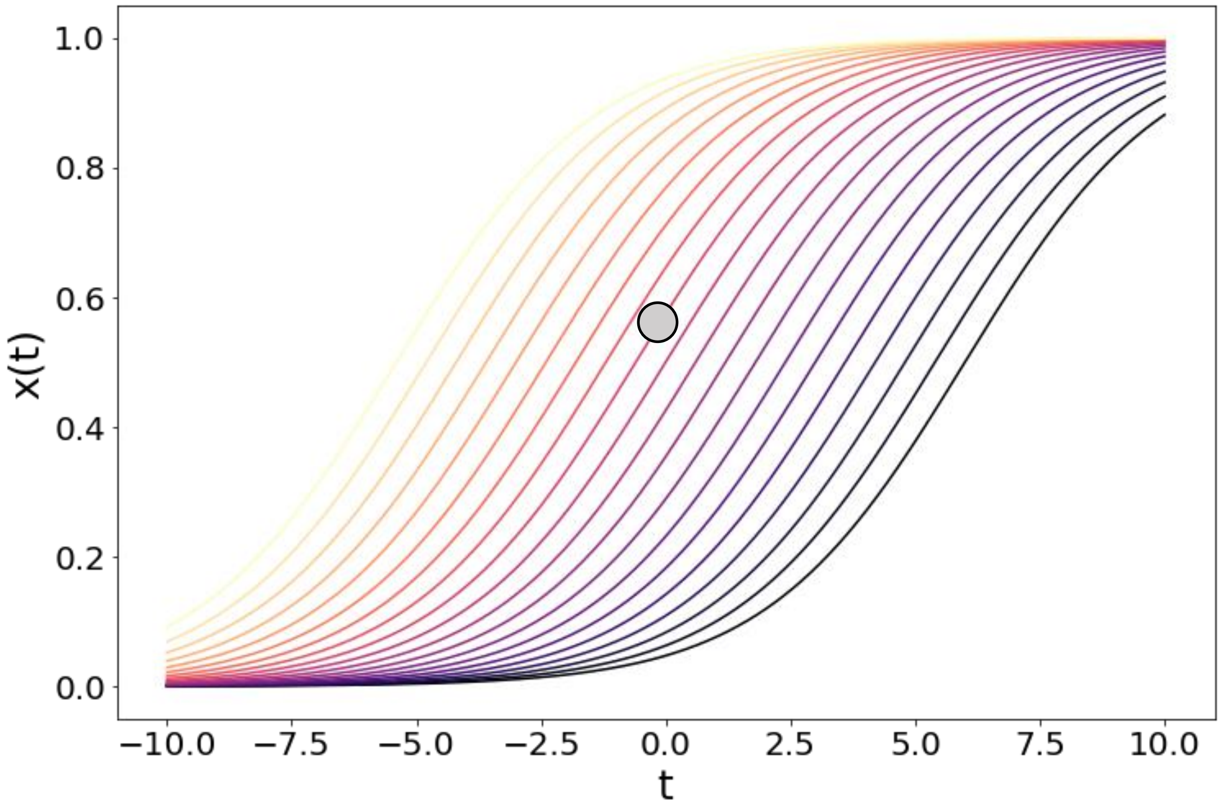
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)g(t)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

$$\int \frac{\mathrm{d}x}{x(1-x)} = \int \alpha \mathrm{d}t$$

$$\log(\frac{x}{1-x}) = \alpha t + k$$

$$x(t) = \frac{e^{\alpha t + k}}{1 + e^{\alpha t + k}}$$



Initial condition

$$x(t_0) = x_0$$

Euler's method: recap

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\frac{x(t+\delta t) - x(t)}{\delta t} = f\big(x(t), t\big)$$

$$x(t+\delta t) = x(t) + \delta t f\big(x(t), t\big)$$

$$\begin{array}{cccccc} x(t_0) & & x(t_0+\delta t) & & x(t_0+2\delta t) & \\ \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc & \cdots \\ t_0 & & t_0+\delta t & & t_0+2\delta t & \end{array}$$

The approach is an approximation…why?

We are not actually taking a limit

Population growth

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \alpha N(t) \qquad N(0) = N_0$$

Analytical

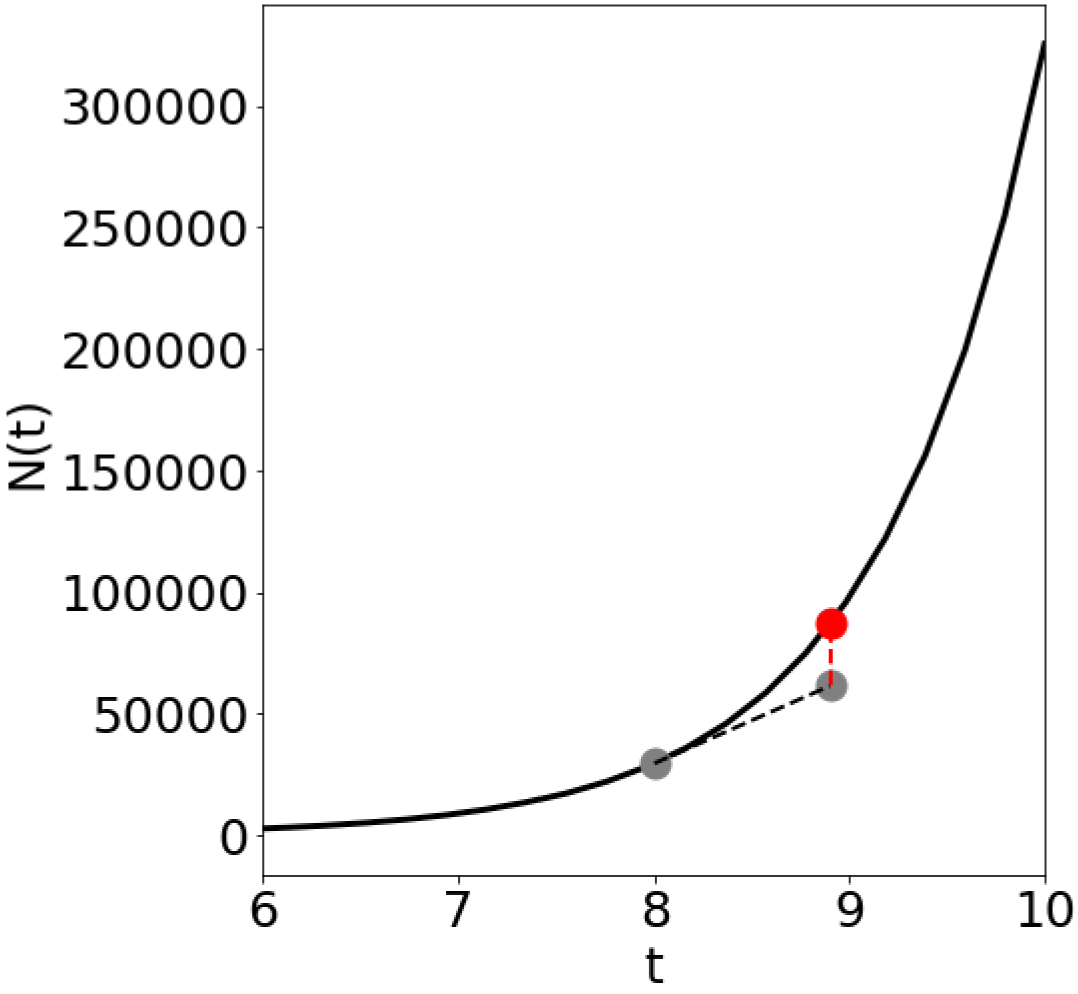
$$N(t) = N_0 e^{\alpha t}$$

Euler's method

$$N(t + \delta t) = N(t)(1 + \alpha \delta t)$$

What is the error introduced? Let's measure…

Error Analysis



Let's consider $N(8) \approx 3 \times 10^4$ as initial condition

$$N(8+\delta t) = N(8)e^{\alpha \delta t}$$

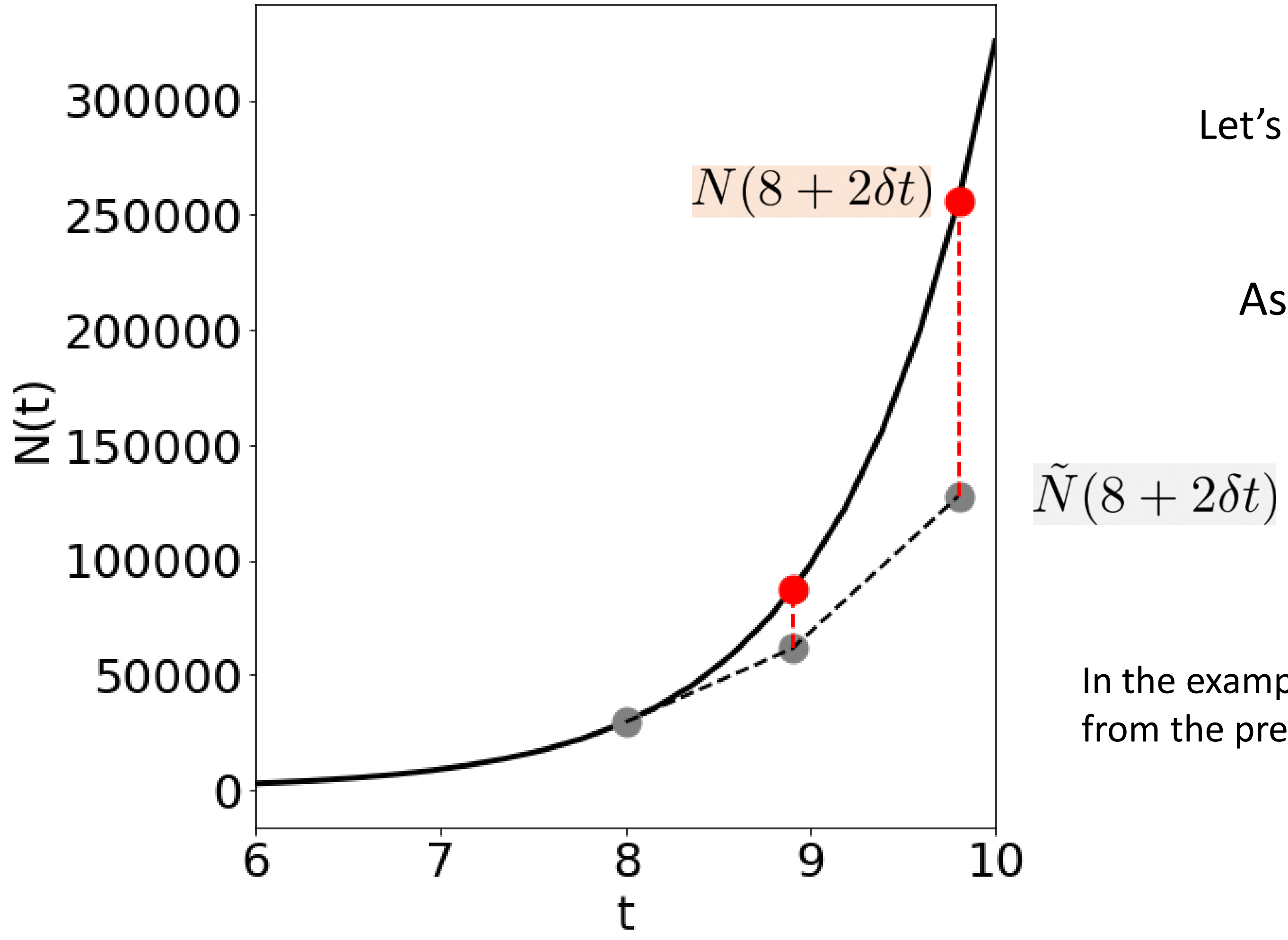
$$\tilde{N}(8+\delta t) = N(8)(1+\alpha \delta t)$$

$$\mathrm{error} = \left| \frac{True - Estimate}{True} \right|$$

$$\mathrm{error} = \left| \frac{N(8+\delta t) - \tilde{N}(8+\delta t)}{N(8+\delta t)} \right| \approx 0.3$$

Local error of the order $\mathcal{O}(\delta t^2)$

Error Analysis



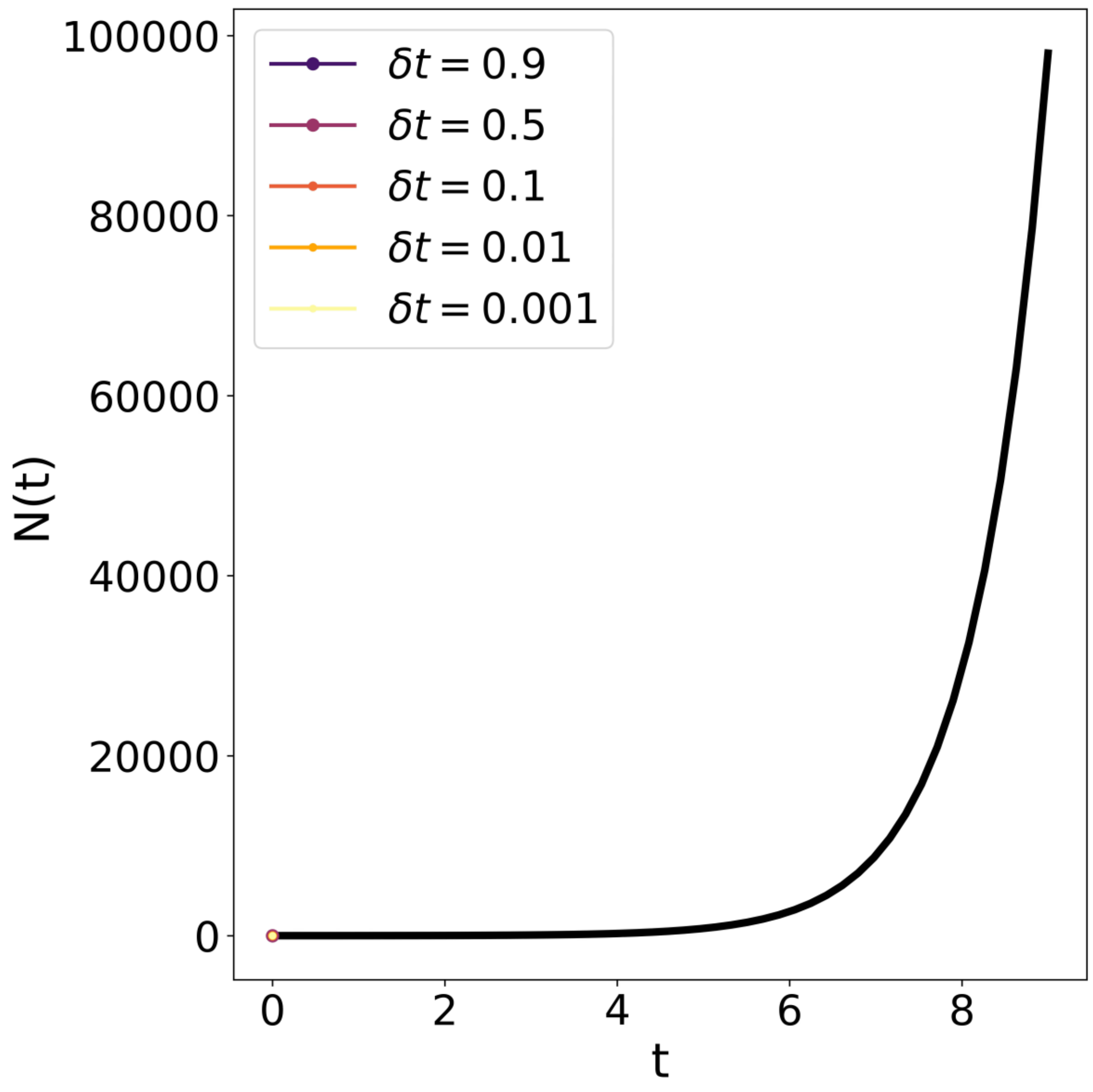
Let's consider $N(8) \approx 3 \times 10^4$ as initial condition

As we evolve the system across time, previous errors get propagated

In the example on the left, the new update contains the error from the previous time step

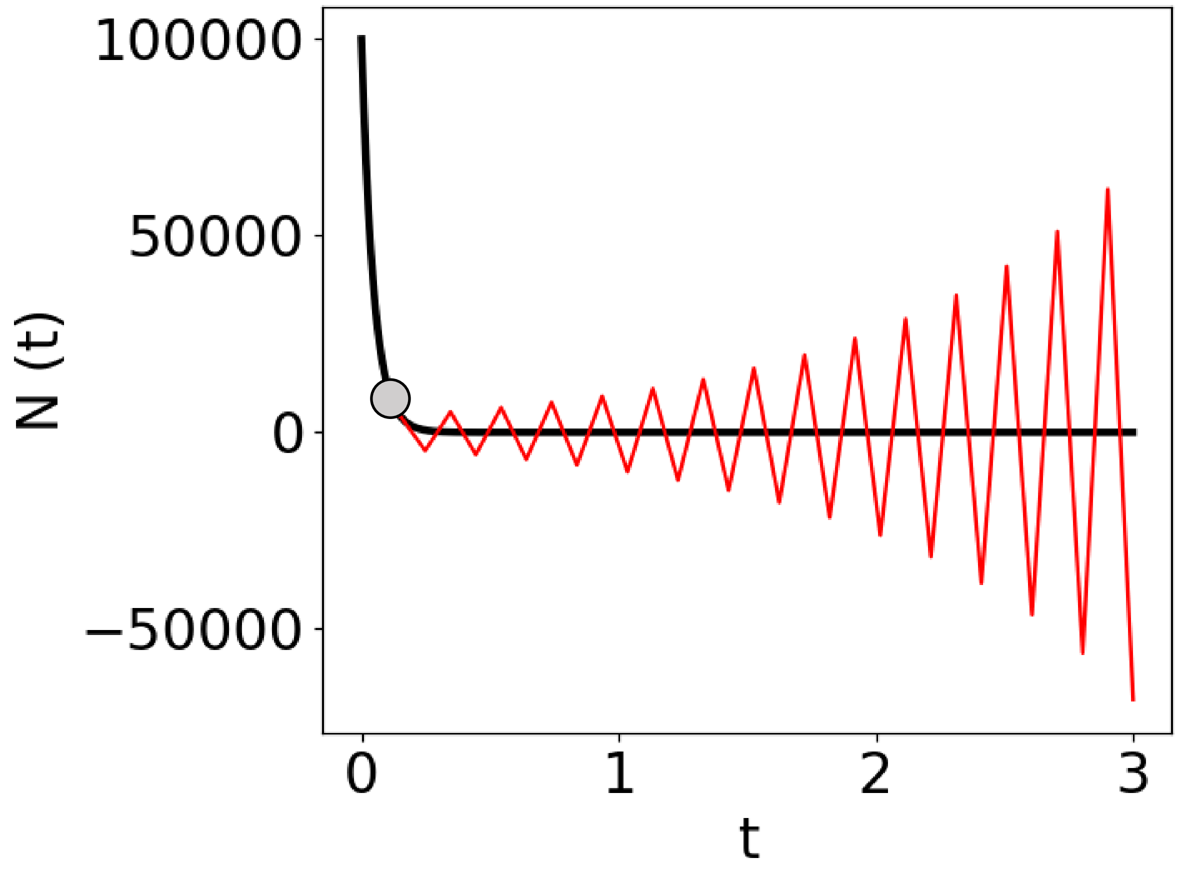
Global error of the order $\mathcal{O}(\delta t)$

Error Analysis



Euler, unstable behavior

$$\frac{\mathrm{d}N(t)}{dt} = -\alpha N(t) \qquad \alpha > 0$$



Analytical

$$N(t) = N_0 e^{-\alpha t}$$

Euler's method

$$N(t+\delta t) = N(t)(1-\alpha\delta t)$$

If $|1-\alpha\delta t| > 1$

Euler is unstable

Equilibria

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

The set of points x_{eq} for which

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x_{eq}} = 0 \qquad\qquad f(x_{eq}) = 0$$

The system is not changing over time at those points

They can be stable or unstable

Equilibria

Let's consider an equilibrium point x_{eq} $f(x_{eq}) = 0$

We perturb the equilibrium by applying a small variation

$$\tilde{x} = x_{eq} + \delta x$$

Does the system go back to the equilibrium or does it run away?

$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$ Has the opposite sign of δx. The system goes back. Stable

$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$ Has the same sign of δx. The system goes away. Unstable

Equilibria, example

$$\frac{\mathrm{d}N(t)}{dt} = \alpha N(t) \qquad \alpha > 0$$

$$N_{eq} = 0$$

$$\tilde{N} = N_{eq} + \delta N = \delta N$$

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right|_{\tilde{N}} = \alpha \delta N$$

The same sign as the perturbation, unstable

$$\frac{\mathrm{d}N(t)}{dt} = -\alpha N(t) \qquad \alpha > 0$$

Equilibrium point stable or unstable?

Stability, derivatives

Asking if $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\tilde{x}} = f(\tilde{x})$ has the same/opposite sign as the perturbation δx

Is equivalent to see if the following is positive or negative

$$\frac{f(\tilde{x})}{\delta x} = \frac{f(\tilde{x}) - f(x_{eq})}{\delta x} =$$
$$= \frac{f(x_{eq} + \delta x) - f(x_{eq})}{\delta x}$$

Taking the limit…

$$\lim_{\delta x \to 0} \frac{f(x_{eq} + \delta x) - f(x_{eq})}{\delta x} = f^{(1)}(x_{eq})$$

> 0 Unstable

< 0 Stable

$= 0$ Look at $f^{(2)}(x_{eq})$

Stability, Example

$$\frac{\mathrm{d}N(t)}{dt} = f\big(N(t)\big) = \alpha N(t)\big(1 - N(t)\big) \qquad \alpha > 0$$

$$N_{eq} = 0, 1$$

$$\frac{\mathrm{d}}{\mathrm{d}N}\Big\{\alpha N(1 - N)\Big\} = \alpha(1 - N) - \alpha N$$

$f^{(1)}(0) = \alpha$ Unstable

$f^{(1)}(1) = -\alpha$ Stable

Thank you

Python, jupyter notebook installed for the Lab

Supplementary Material

1. Recap of derivative definition and examples (slide 30).

2. Table of derivatives (slide 31).

3. Table of Integrals (slide 32).

4. Solution of the equation in slide 16 (slide 33).

5. Understanding the error of Euler's method (slide 34-36). For the brave students…

Supplementary Material: Derivative definition and examples

We won't ask for this slide in the exam but understanding it is highly recommended.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t}$$

$$x(t) = k$$

$$\lim_{\delta t \to 0} \frac{k-k}{\delta t} = 0$$

$$x(t) = t$$

$$\lim_{\delta t \to 0} \frac{t + \delta t - t}{\delta t} = 1$$

$$x(t) = t^2$$

$$\lim_{\delta t \to 0} \frac{(t+\delta t)^2 - t^2}{\delta t} =$$

$$\lim_{\delta t \to 0} \frac{2t\delta t + \delta t^2}{\delta t} =$$

$$\lim_{\delta t \to 0} \frac{\delta t}{\delta t} 2t + \delta t = 2t$$

Supplementary Material: Table of derivatives

Function	Derivative	Example
k	0	$\frac{\mathrm{d}}{\mathrm{d}x}1 = 0$
x^k	kx^{k-1}	$\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$
e^{kx}	ke^{kx}	$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} = 2e^{2x}$
$\sin(kx)$	$k\cos(kx)$	$\frac{\mathrm{d}}{\mathrm{d}x}\sin(4x) = 4\cos(4x)$
$\cos(kx)$	$-k\sin(kx)$	$\frac{\mathrm{d}}{\mathrm{d}x}\cos(5x) = -5\sin(5x)$

Important rules	Example
$\frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x) = f^{(1)}(x)g(x) + f(x)g^{(1)}(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(2x)(2x^2+3x)\right] = -2\sin(2x)(2x^2+3x) + \cos(2x)(4x+3)$
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f^{(1)}(x)g(x) - g^{(1)}(x)f(x)}{g^2(x)}$	$\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^{2x}}{x^2} = \frac{2e^{2x}x^2 - 2xe^{2x}}{x^4}$
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = f^{(1)}\big(g(x)\big)g^{(1)}(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}e^{3x^2+2x} = e^{3x^2+2x}(6x+2)$

Supplementary Material: Table of integrals

Function	Integral	Example
k	$kx + k_1$	$\int 1\mathrm{d}x = x + k_1$
x^k	$\frac{x^{k+1}}{k+1} + k_1$	$\int x^3\mathrm{d}x = \frac{x^4}{4} + k_1$
e^{kx}	$\frac{e^{kx}}{k} + k_1$	$\int e^{2x}\mathrm{d}x = \frac{e^{2x}}{2} + k_1$
		$\int e^{2x}\mathrm{d}x = \frac{e^{2x}}{2} + k_1$
$\sin(kx)$	$\frac{-\cos(kx)}{k} + k_1$	$\int \sin(kx)\mathrm{d}x = \frac{-\cos(kx)}{k} + k_1$
$\cos(kx)$	$\frac{\sin(kx)}{k} + k_1$	$\int \cos(kx)\mathrm{d}x = \frac{\sin(kx)}{k} + k_1$
$\frac{1}{x}$	$\log(\lvert x\rvert) + k_1$	$\int \frac{1}{1-x}\mathrm{d}x = -\log(\lvert 1-x\rvert) + k_1$

The constant k_1 appears because the integral is indefinite. Indeed, if you derive the Integral you should obtain the starting Function. The derivative of a constant is zero.

Supplementary Material: Derivation of the integral at slide 16

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

$$\int \frac{\mathrm{d}x}{x(1-x)} = \int \alpha \mathrm{d}t$$

$$\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{1-x} =$$

$$= \frac{a(1-x)+bx}{x(1-x)} = \frac{a + x(b-a)}{x(1-x)}$$

$$a = 1,\ b = a = 1$$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{(1-x)}$$

We compare this numerator with the starting one (=1). Thus, there must be no term in x, which means b=a. Then, a=1.

$$\int \frac{1}{x} + \frac{1}{1-x}\mathrm{d}x = \int \alpha \mathrm{d}t$$

$$\log(x) - \log(1-x) = \alpha t + k$$

$$\log\left(\frac{x}{1-x}\right) = \alpha t + k$$

$$\frac{x}{1-x} = e^{\alpha t + k}$$

$$x = e^{\alpha t + k} - x e^{\alpha t + k}$$

$$x\left(1 + e^{\alpha t + k}\right) = e^{\alpha t + k}$$

$$x(t) = \frac{e^{\alpha t + k}}{1 + e^{\alpha t + k}}$$

Supplementary Material: Understanding the error in Euler's update

For the brave students...

Taylor expansion around t_0 :

$$x(t) = x(t_0) + x^{(1)}(t_0)(t - t_0) + \frac{1}{2}x^{(2)}(t_0)(t - t_0)^2 + \ \dots \ + \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n + \dots =$$

$$= x(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n$$

Let's expand the following exponential function around zero

$$x(t) = e^{\alpha t}$$

$x^{(1)}(t) = \alpha e^{\alpha t}$ First derivative

$x^{(2)}(t) = \alpha \frac{d}{dt}\left(e^{\alpha t}\right) = \alpha^2 e^{\alpha t}$ Second derivative

Supplementary Material: Understanding the error in Euler's update

For the brave students...

Taylor expansion around t_0 :

$$x(t) = x(t_0) + x^{(1)}(t_0)(t - t_0) + \frac{1}{2}x^{(2)}(t_0)(t - t_0)^2 + \ \dots \ + \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n + \dots =$$

$$= x(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!}x^{(n)}(t_0)(t - t_0)^n$$

Let's expand the following exponential function around zero

$$x(t) = e^{\alpha t} = e^0 + \alpha e^0(t - 0) + \frac{\alpha^2}{2}e^0(t - 0)^2 + \frac{\alpha^3}{6}e^0(t - 0)^3 + \dots =$$

$$= 1 + \alpha t + \frac{\alpha^2}{2}t^2 + + \frac{\alpha^3}{6}t^3 + \dots$$

We have used the derivatives from the previous slide

Supplementary Material: Understanding the error in Euler's update

For the brave students...

We consider now the population growth model below. We will use the analytical solution and Taylor expansion and compare the result with Euler's

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \alpha N(t) \qquad N(0) = N_0$$

Analytical

$$N(t) = N_0 e^{\alpha t}$$

Euler's method

$$N(t+\delta t) = N(t)(1+\alpha\delta t)$$

$$N(t+\delta t) = N_0 e^{\alpha(t+\delta t)} =$$

$$= N_0 e^{\alpha t} e^{\alpha \delta t} = N(t) e^{\alpha\delta t} =$$

$$= N(t)\Big[\underbrace{1+\alpha\delta t}_{\text{Euler's terms}} + \underbrace{\frac{\alpha^2}{2}(\delta t)^2 + \frac{\alpha^3}{6}(\delta t)^3 + \ldots}_{\text{Terms neglected } \mathcal{O}(\delta t^2)}\Big]$$

We expanded $e^{\alpha\delta t}$ around zero