IMPROVED NUMERICAL METHODS, RUNGE-KUTTA

Week 8

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So far…

Differential Equations and initial condition. How to find a unique solution?

Solutions to

Please, learn how to solve these

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$\alpha > 0$ Exponential growth

$\alpha < 0$ Exponential decay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

Saturating

Euler's method and its error $\mathcal{O}(\delta t^2)$ $\mathcal{O}(\delta t)$

Definition of equilibrium points in one dimension

Autonomous and not

Autonomous

If it does not contain t explicitly

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

Non-Autonomous

If it does contain t explicitly

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\alpha x(t)}{t}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + \cos(t)$$

Non-Autonomous

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big) + e^{-x(t)}$$

Autonomous

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\alpha x(t) + I(t)$$

$I(t) = k$ Autonomous

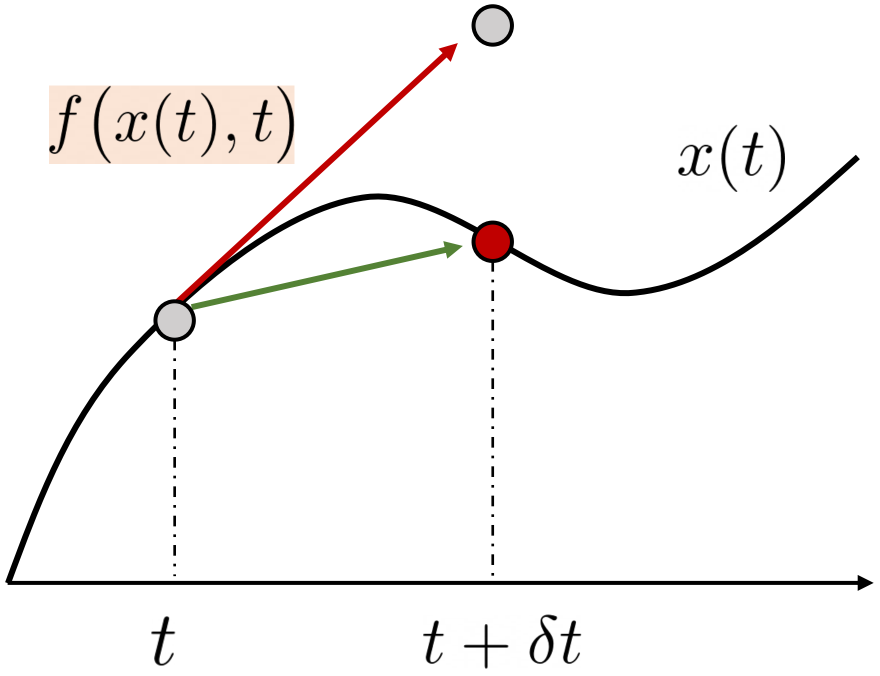
Otherwise Non-Autonomous

Euler's Method: Geometrical Interpretation

Let's consider the differential equation from the point of view of $x(t)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\left(x(t), t\right)$$

Our task is to find the solution $x(t)$



Euler's method

$$x(t + \delta t) = x(t) + \delta t f\left(x(t), t\right)$$

If the function is non-linear between t and t+delta t, Euler's method is not accurate

Euler’s Method: Geometrical Interpretation

Let’s consider the differential equation from the point of view of $x(t)$

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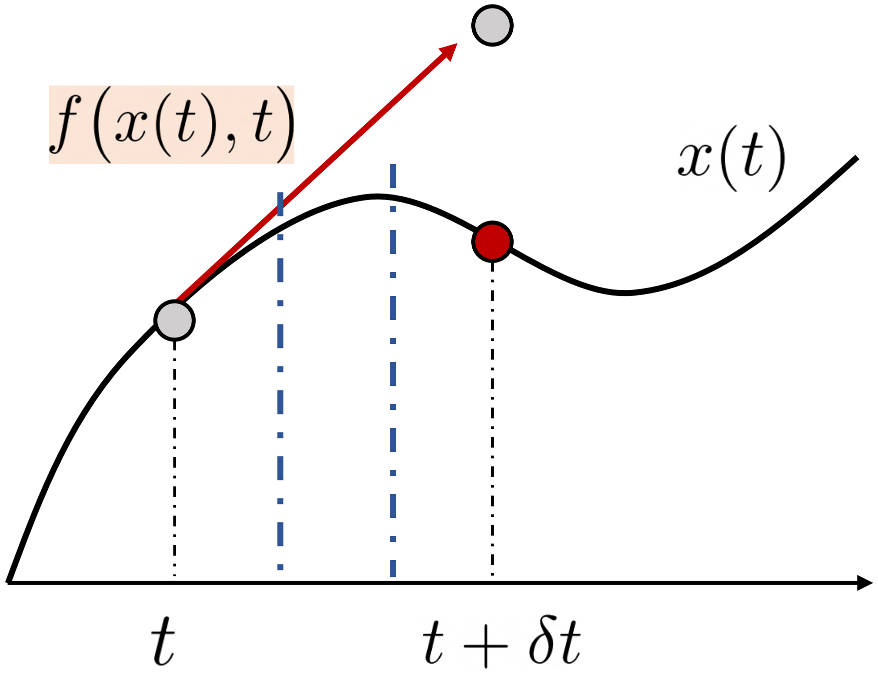


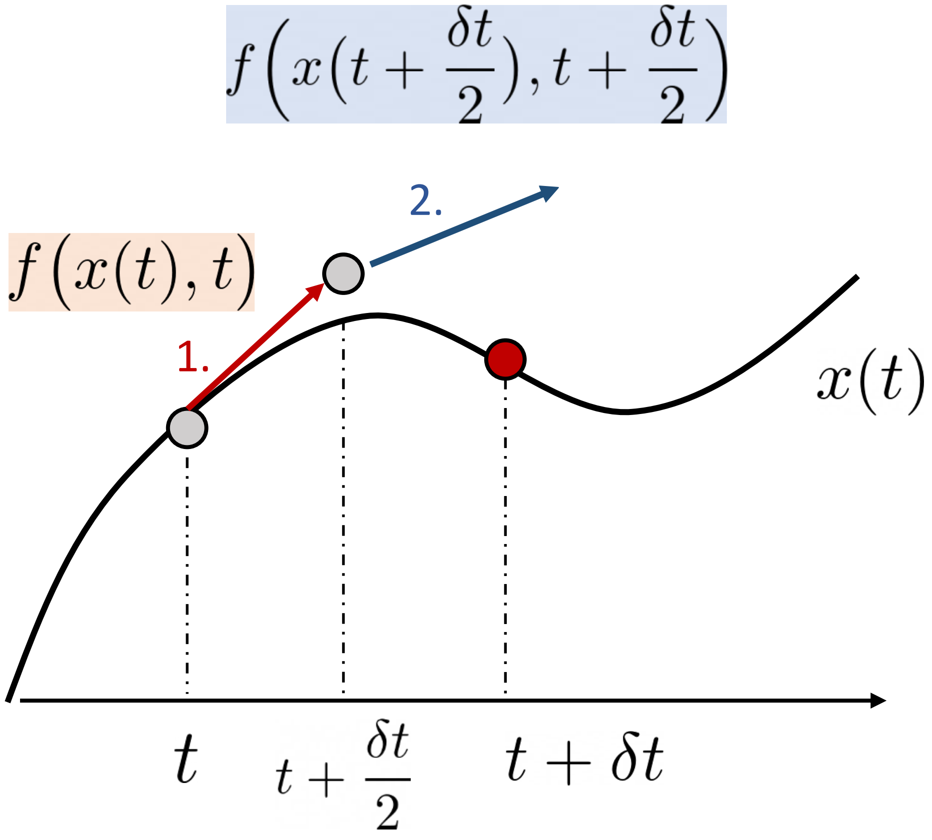
Can we exploit estimates of the derivatives computed at different points in the interval to find a better approximation?

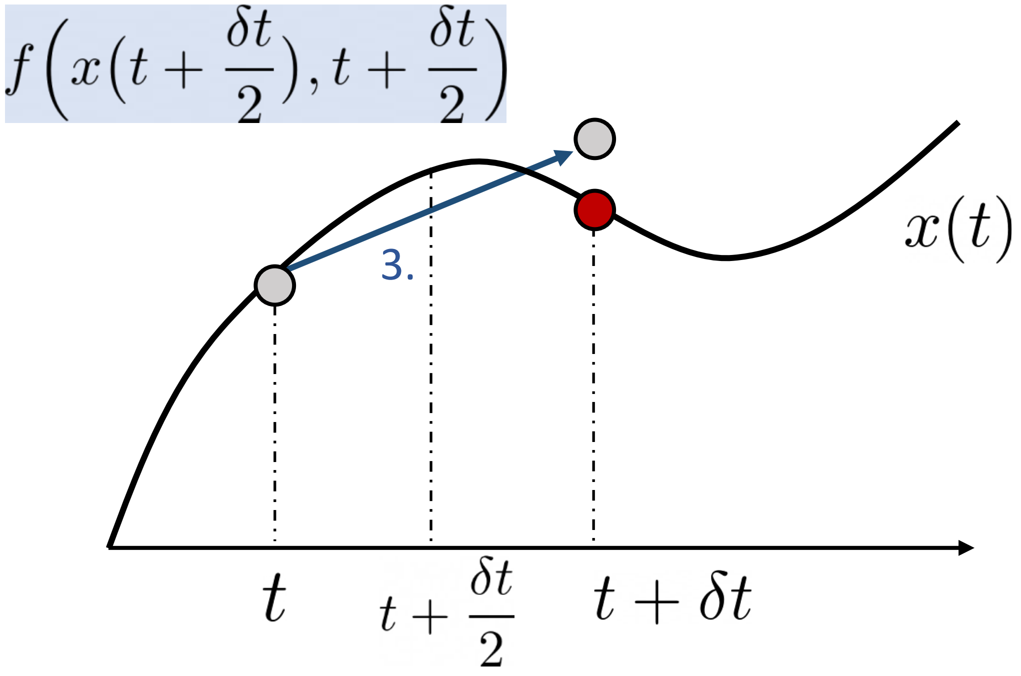
Why estimates? We do not know $x(t)$

Midpoint Method

What if we use the slope at the middle point $t + \frac{\delta t}{2}$?

1. We first use the slope at t to evolve the system for $\frac{\delta t}{2}$
2. We compute the slope at $t + \frac{\delta t}{2}$
3. We apply the slope to perform the update from time t



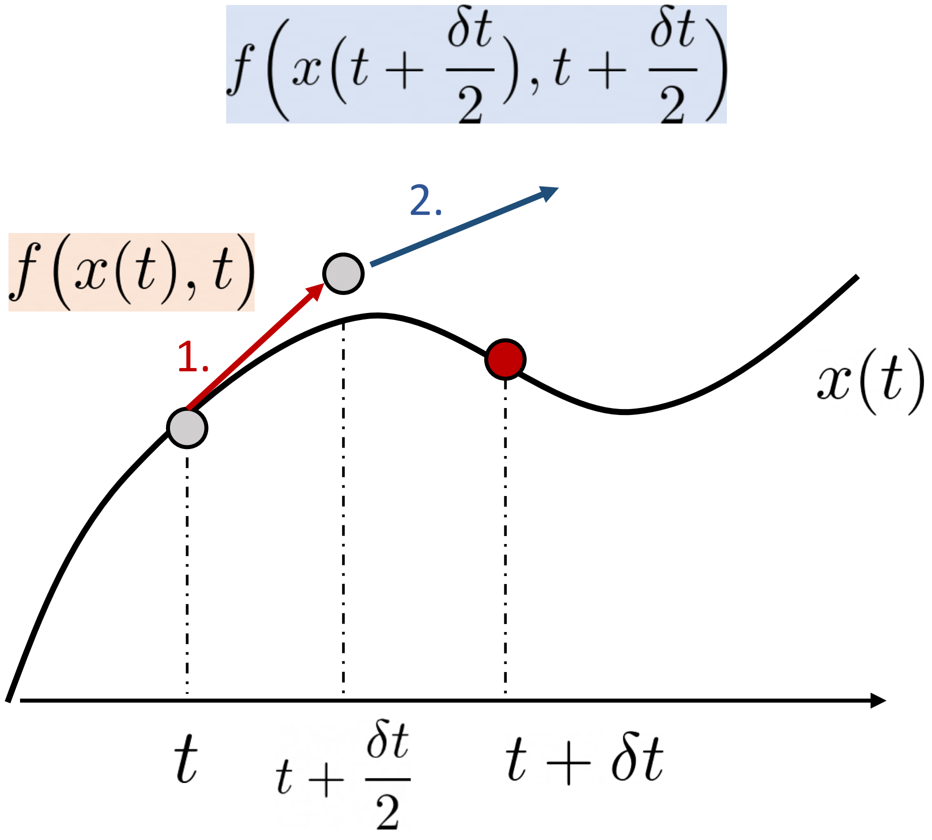


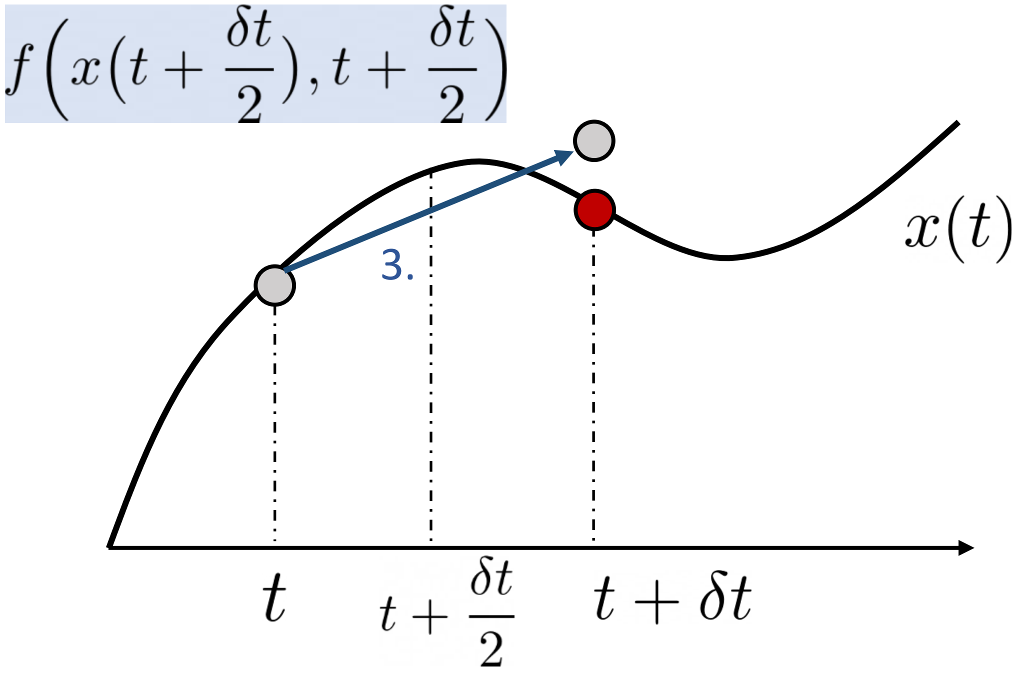
Midpoint Method

1. $x\left(t+\frac{\delta t}{2}\right)=x(t)+f\left(x(t),t\right)\frac{\delta t}{2}$

2. $f_M=f\left(x(t+\frac{\delta t}{2}),t+\frac{\delta t}{2}\right)$

3. $x(t+\delta t)=x(t)+\delta t f_M$





Midpoint Method: How to do it?

Define the function $f(x,t)$ and choose δt

Start from initial condition $x_{\mathrm{p}}, t_{\mathrm{p}}$

```
## YOUR VALUES

dt=...
x_p=...
t_p=...
```

```
def f(x,t):

    ## Y = YOUR FUNCTION

    return Y
```

REPEAT

Use the first slope $f(x_p, t_p)$ to compute middle-point

$$x_{MP} = x_p + f(x_p, t_p)\frac{\delta t}{2}$$

```
x_mp=x_p+f(x_p,t_p)*dt/2
```

Use the slope at the middle-point to perform the update

$$x_{new} = x_p + f\left(x_{MP}, t_p + \frac{\delta t}{2}\right)\delta t$$

```
x_new=x_p+f(x_mp,t_p+dt/2)*dt
```

$$x_p \leftarrow x_{new}, \quad t_p \leftarrow t_p + \delta t$$

Of course, everything can be embedded in a nice class …

Midpoint Method: Explicit analytical form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

$$x\left(t + \frac{\delta t}{2}\right) = x(t) + \alpha x(t)\frac{\delta t}{2}$$

$$f_M = \alpha\left(x(t) + \alpha x(t)\frac{\delta t}{2}\right) = \alpha x(t) + \alpha^2 x(t)\frac{\delta t}{2}$$

$$x(t + \delta t) = x(t) + f_M \delta t$$

$$x(t + \delta t) = x(t)\left[1 + \alpha\delta t + \alpha^2\frac{\delta t^2}{2}\right]$$

Midpoint Method, comparison with Euler

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t),t\big) = \alpha x(t)$$

Midpoint

$$x(t+\delta t) = x(t)\left[1 + \alpha\delta t + \alpha^2\frac{\delta t^2}{2}\right]$$

Euler

$$x(t+\delta t) = x(t)\big[1 + \alpha\delta t\big]$$

Look at $\delta t, \delta t^2, \delta t^3, \ldots$

$$x(t) = x(t_0) + x^{(1)}(t_0)\delta t + \frac{1}{2}x^{(2)}(t_0)\delta t^2 + \frac{1}{6}x^{(3)}(t_0)\delta t^3 + \ldots$$

Euler's term

Terms neglected by Euler

Local error of the order $\mathcal{O}(\delta t^2)$

Global error of the order $\mathcal{O}(\delta t)$

Midpoint Method, comparison with Euler

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

Midpoint

$$x(t+\delta t) = x(t)\Big[1 + \alpha\delta t + \alpha^2\frac{\delta t^2}{2}\Big]$$

Euler

$$x(t+\delta t) = x(t)\big[1 + \alpha\delta t\big]$$

Look at $\delta t, \delta t^2, \delta t^3, \ldots$

$$x(t) = \underbrace{x(t_0) + x^{(1)}(t_0)\delta t + \frac{1}{2}x^{(2)}(t_0)\delta t^2}_{\text{Midpoint terms}} + \underbrace{\frac{1}{6}x^{(3)}(t_0)\delta t^3 + \ldots}_{\text{Terms neglected by Midpoint}}$$

Local error of the order $\mathcal{O}(\delta t^3)$

Global error of the order $\mathcal{O}(\delta t^2)$

RUNGE-KUTTA Methods

$$x(t+\delta t) = x(t) + \phi\delta t$$

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

ϕ Is an effective slope

Can be written as a linear combination of k's, which are slopes evaluated at different points

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\big(x(t) + k_1 q_{11}\delta t, t + p_1\delta t\big)$$

$$k_3 = f\big(x(t) + k_2 q_{22}\delta t + k_1 q_{21}\delta t, t + p_2\delta t\big)$$

$$\cdots$$

RUNGE-KUTTA Methods

$$x(t + \delta t) = x(t) + \phi \delta t$$

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$$k_3 = f\left(x(t) + k_2 q_{22} \delta t + k_1 q_{21} \delta t, t + p_2 \delta t\right)$$

$$\dots$$

The k's are in a recursive form; k2 is function of k1, k3 of k2 and so on

The other constants p's q's and a's can be found by comparing these equations with the Taylor expansion (we will not cover this)

RUNGE-KUTTA Methods

$$x(t+\delta t) = x(t) + \phi\delta t$$

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\big(x(t) + k_1 q_{11}\delta t, t + p_1 \delta t\big)$$

$$k_3 = f\big(x(t) + k_2 q_{22}\delta t + k_1 q_{21}\delta t, t + p_2 \delta t\big)$$

$$\cdots$$

Runge-Kutta n is defined by maintaining the first n k's. For instance, in Runge-Kutta 2 we use only the terms

RUNGE-KUTTA 1

$$x(t+\delta t) = x(t) + \phi\delta t$$

$$\phi = k_1 \qquad k_1 = f\big(x(t), t\big)$$

Does it look familiar?

$$x(t+\delta t) = x(t) + \delta t f\big(x(t), t\big)$$

It is Euler!

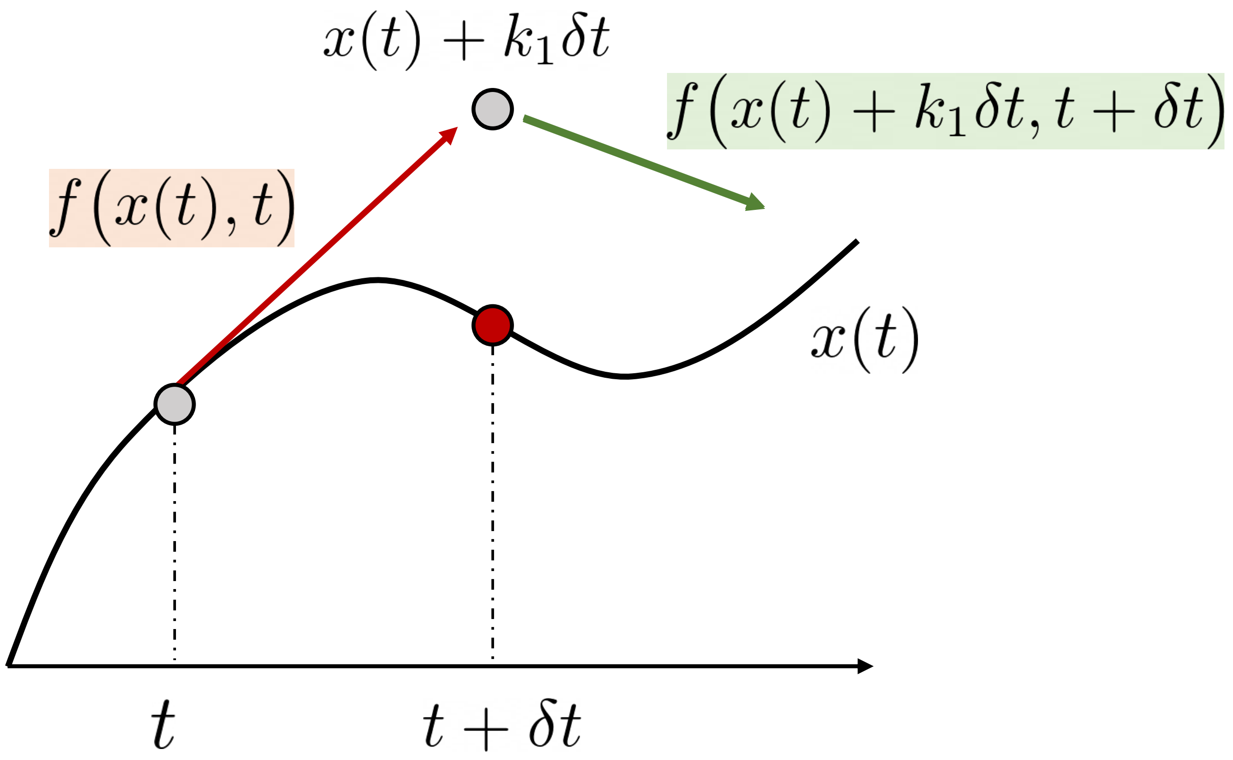
RUNGE-KUTTA 2

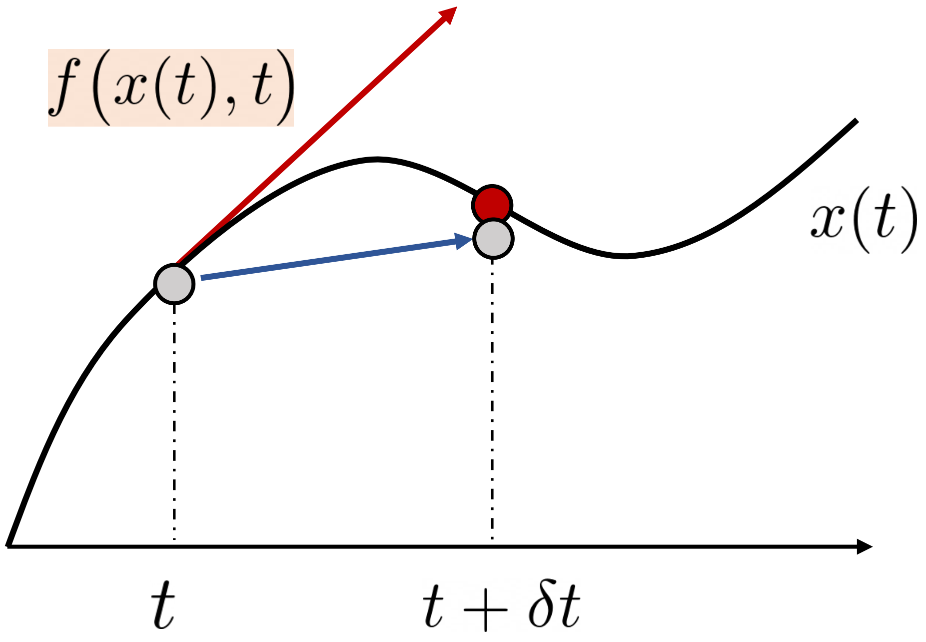
$$x(t+\delta t) = x(t) + \phi \delta t$$

$$\phi = \frac{1}{2}k_1 + \frac{1}{2}k_2$$

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\big(x(t) + k_1 \delta t, t + \delta t\big)$$





RUNGE-KUTTA 2: In Practice

Define the function $f(x,t)$ and choose δt

Start from initial condition $x_\mathrm{p}, t_\mathrm{p}$

```
## YOUR VALUES

dt=...
x_p=...
t_p=...
```

```
def f(x,t):

    ## Y = YOUR FUNCTION

    return Y
```

REPEAT

Use the first slope $k_1 = f(x_p, t_p)$ to compute k_2

$$k_2 = f\left(x_p + k_1\delta t, t_p + \delta t\right)$$

Perform the update

$$x_{new} = x_p + \frac{1}{2}(k_1 + k_2)\delta t$$

$$x_p \leftarrow x_{new}, \quad t_p \leftarrow t_p + \delta t$$

Of course, everything can be embedded in a nice class …

RUNGE-KUTTA 2: Analytically

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t) \qquad f\big(x(t), t\big) = \alpha x(t)$$

$$k_1 = \alpha x(t)$$

$$\begin{aligned} k_2 = f\big(x(t) + k_1 \delta t, t + \delta t\big) = \\ f\big(x(t) + \alpha x(t) \delta t, t + \delta t\big) = \\ \alpha\big[x(t) + \alpha x(t)\delta t\big] = \alpha x(t) + \alpha^2 x(t) \delta t \end{aligned}$$

$$\begin{aligned} x(t + \delta t) = x(t) + \frac{1}{2}\big[k_1 + k_2\big]\delta t = \\ = x(t) + \frac{1}{2}\big[2\alpha x(t) + \alpha^2 x(t) \delta t\big]\delta t = \\ = x(t)\big[1 + \alpha \delta t + \alpha^2 \frac{\delta t^2}{2}\big] \end{aligned}$$

Local error of the order $\mathcal{O}(\delta t^3)$ Global error of the order $\mathcal{O}(\delta t^2)$

RUNGE-KUTTA 4: The 'golden' standard

$$x(t+\delta t) = x(t) + \phi \delta t$$

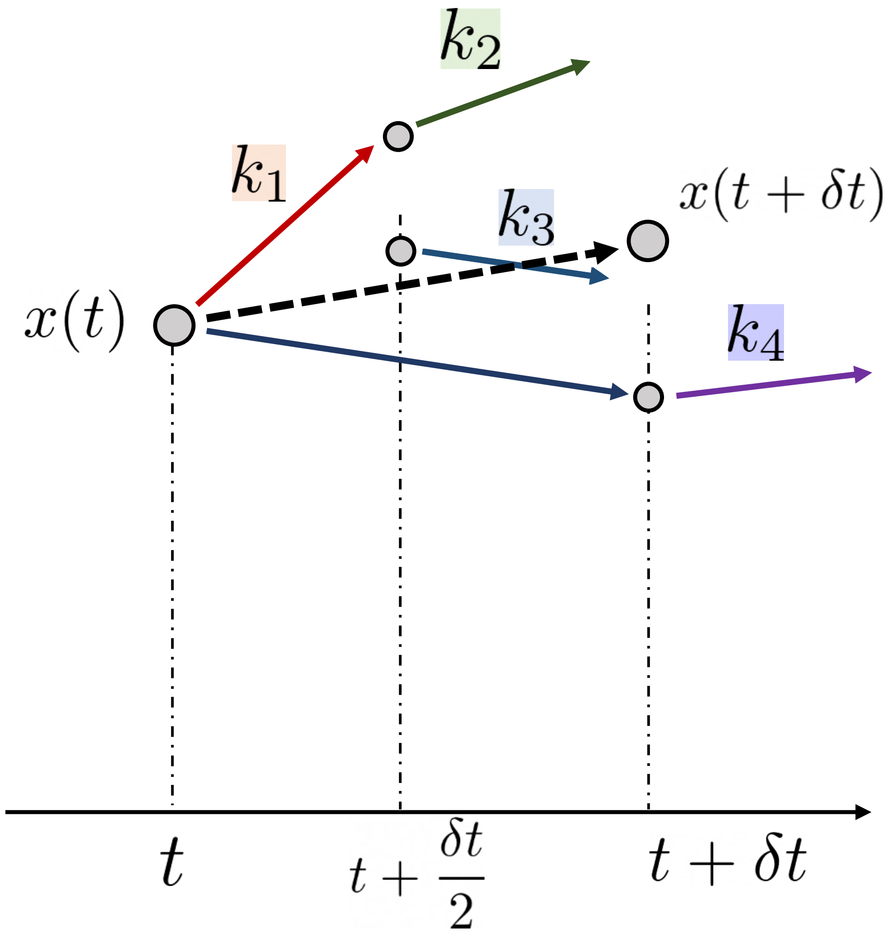
$$\phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f\big(x(t), t\big)$$

$$k_2 = f\left(x(t) + k_1\frac{\delta t}{2}, t + \frac{\delta t}{2}\right)$$

$$k_3 = f\left(x(t) + k_2\frac{\delta t}{2}, t + \frac{\delta t}{2}\right)$$

$$k_4 = f\Big(x(t) + k_3\delta t, t + \delta t\Big)$$



Thank you