HIGHER ORDER SYSTEMS

Week 9

Dr. Luca Manneschi

So far…

Differential Equations and initial condition. How to find a unique solution?

Solutions to

Look at the exercises

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$$

$\alpha > 0$ Exponential growth

$\alpha < 0$ Exponential decay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)\big(1 - x(t)\big)$$

Saturating

Euler's method and its error $\mathcal{O}(\delta t^2)$ $\mathcal{O}(\delta t)$

Definition of equilibrium points in one dimension

Autonomous and non-autonomous: Definition

MidPoint Method, Runge-Kutta 2 and 4

Today…

Before we had a 1-dimensional system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t), t\big)$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t) \end{cases}$$

In vectorial form

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{F}\big(\mathbf{z}(t), t\big)$$

$$\mathbf{F} = \begin{pmatrix} f\big(\mathbf{z}(t), t\big) \\ g\big(\mathbf{z}(t), t\big) \end{pmatrix}$$

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Euler’s method

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t) \end{cases}$$

$$\begin{cases} x(t + \delta t) = x(t) + \delta t f(x, y, t) \\ \\ y(t + \delta t) = y(t) + \delta t g(x, y, t) \end{cases}$$

Or in vectorial form

$$\mathbf{z}(t + \delta t) = \mathbf{z}(t) + \mathbf{F}\big(\mathbf{z}(t), t\big)\delta t$$

It is analogous to Euler’s method in one dimension, but we have vectors

Improved numerical methods in higher dimensions

We will not cover this specifically, you can use premade tools available. However, if you use the vectorial form, the methods (Runke-Kutta and others) appear analogous to the 1 dimensional case…

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{F}\big(\mathbf{z}(t), t\big)$$

In the lab, you will see an example of this

Phase-Plot

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y) \end{cases}$$

We plot the solutions y as a function of x

For the direction? We plot arrows as x and y vary $\mathbf{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$

Example: Lotka-Volterra equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

X: Prey

Y: Predator

$\alpha > 0,$ Reproduction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

$\delta > 0,$ Reproduction predator

Phase-Plot

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

We plot the solutions y as a function of x

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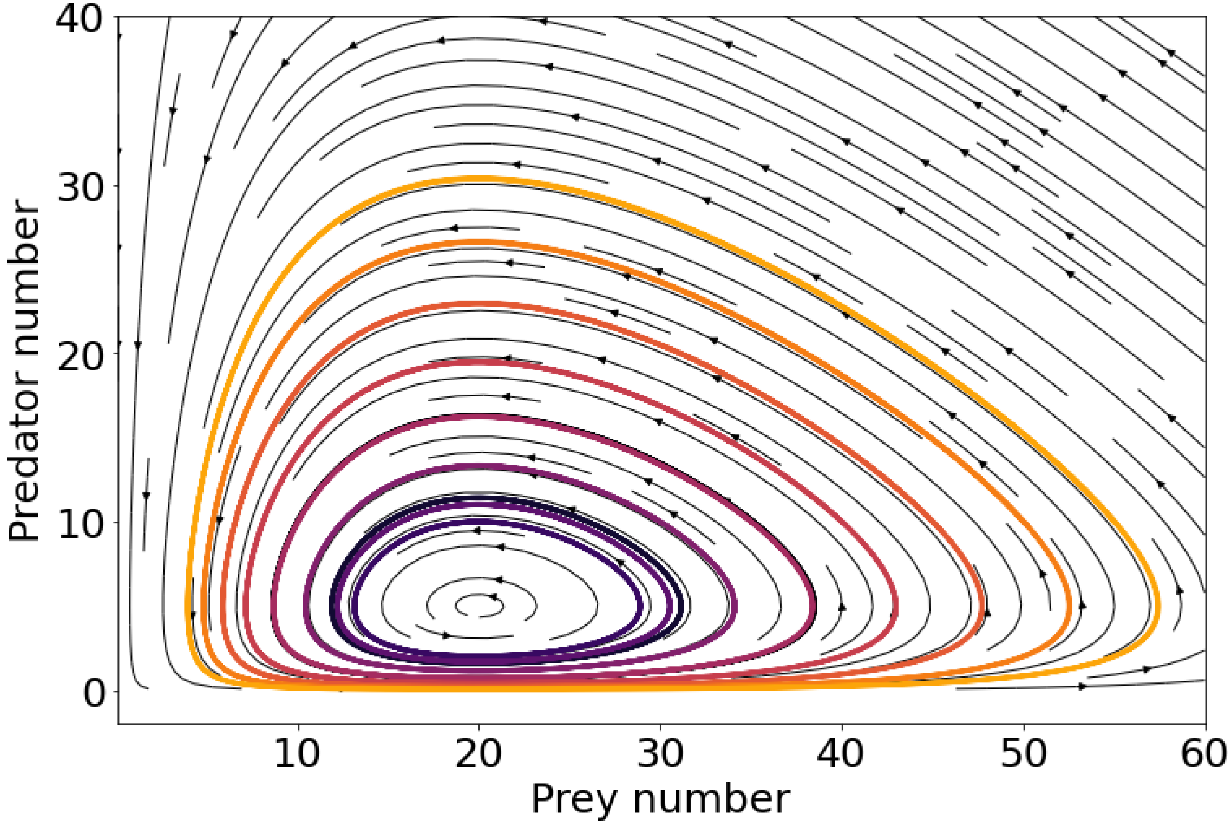


Equilibrium Points

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases}$$

$$(x_{eq}, y_{eq})$$

$$f(x_{eq}, y_{eq}) = 0, \text{ and } \; g(x_{eq}, y_{eq}) = 0$$

What are the equilibrium points for Lotka-Volterra?

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$$(0,0) \qquad \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$$

Stability

Remember: in 1-dimension

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big)$$

$$x_{eq},\ f(x_{eq}) = 0$$

We compute the derivative

$$\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x}$$

Evaluate it at the equilibrium and study the sign

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases}$$

$$(x_{eq}, y_{eq})$$

$$f(x_{eq}, y_{eq}) = 0, \text{ and } \ g(x_{eq}, y_{eq}) = 0$$

We compute the Jacobian

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}$$

Evaluate it at the equilibrium and study the signs of the eigenvalues

Stability

Given a matrix $\mathcal{J}$, the eigenvalues and eigenvectors satisfy the relation $\mathcal{J}\mathbf{v} = \lambda\mathbf{v}$

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

$$\lambda_1, \lambda_2, \ \mathbf{v}_1, \mathbf{v}_2$$

$\lambda_1 < 0, \ \lambda_2 < 0$

Stable

Eigenvalues with opposite signs

Saddle (Unstable)

$\lambda_1 > 0, \ \lambda_2 > 0$

Unstable

Stability, Lotka-Volterra

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha > 0,$ Reproduction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

$\delta > 0,$ Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\Big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha, \ -\gamma$

Saddle

This is the reason why it is hard for the system to
reach the extinction of the species

Stability, Lotka-Volterra variation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha < 0,$ Extinction prey

$\beta > 0,$ Predation

$\gamma > 0,$ Extinction predator

$\delta > 0,$ Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\Big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha, \ -\gamma$ Stable

Stability, Lotka-Volterra variation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y) \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(\delta x - \gamma) \end{cases}$$

$\alpha > 0,$	Reproduction prey
$\beta > 0,$	Predation
$\gamma < 0,$	Reproduction predator
$\delta > 0,$	Reproduction predator

$$\mathcal{J} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \qquad \mathcal{J}\Big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues are $\alpha, \ -\gamma$ Unstable

Can we find the phase-plot? Analytically?

We consider the linear case

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_{11}x + a_{12}y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = a_{21}x + a_{22}y \end{cases}$$

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\lambda_1, \lambda_2, \ \mathbf{v}_1, \mathbf{v}_2$$

The solution has a form like

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v_2} e^{\lambda_2 t}$$

Exponential functions…Recall that the solution to $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \alpha x(t)$ is $\propto e^{\alpha t}$

Unstable Node

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x - y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases}$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

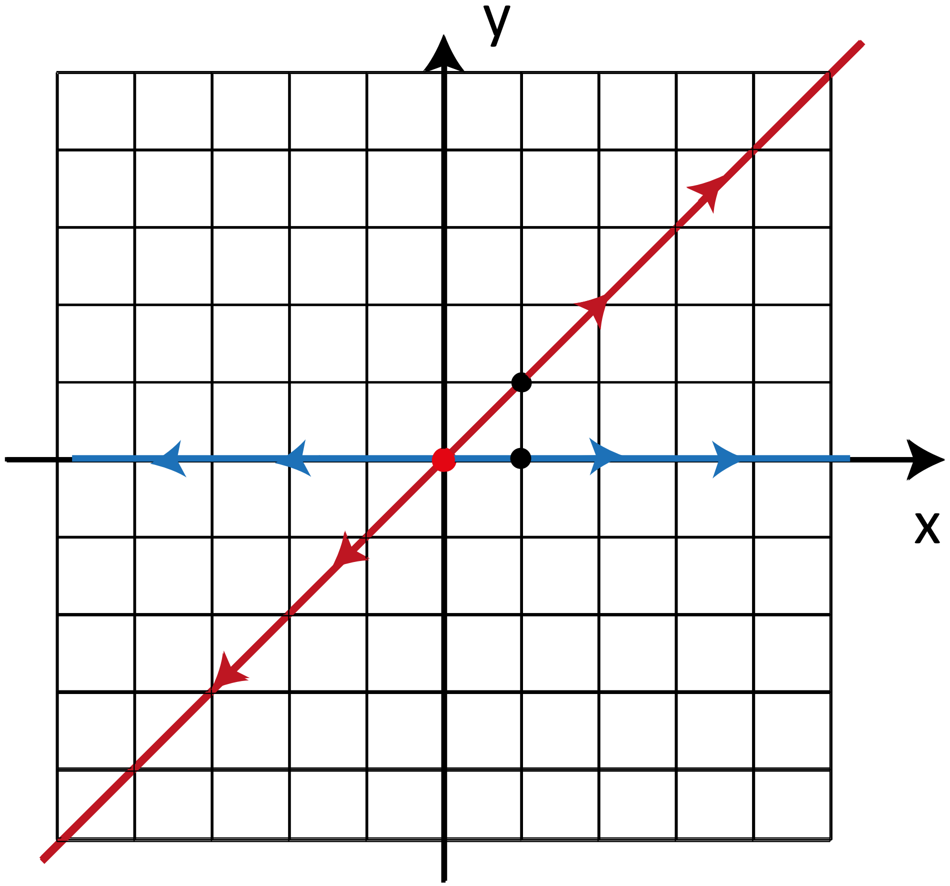
$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v_2} e^{\lambda_2 t}$$

$$\mathbf{z}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Let's first draw two simple solutions,

$c_1 = 0,\ c_2 \neq 0$

$c_1 \neq 0,\ c_2 = 0$



Unstable Node

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x - y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases}$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{z}(t) = c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{t}$$

To draw the others, let us consider

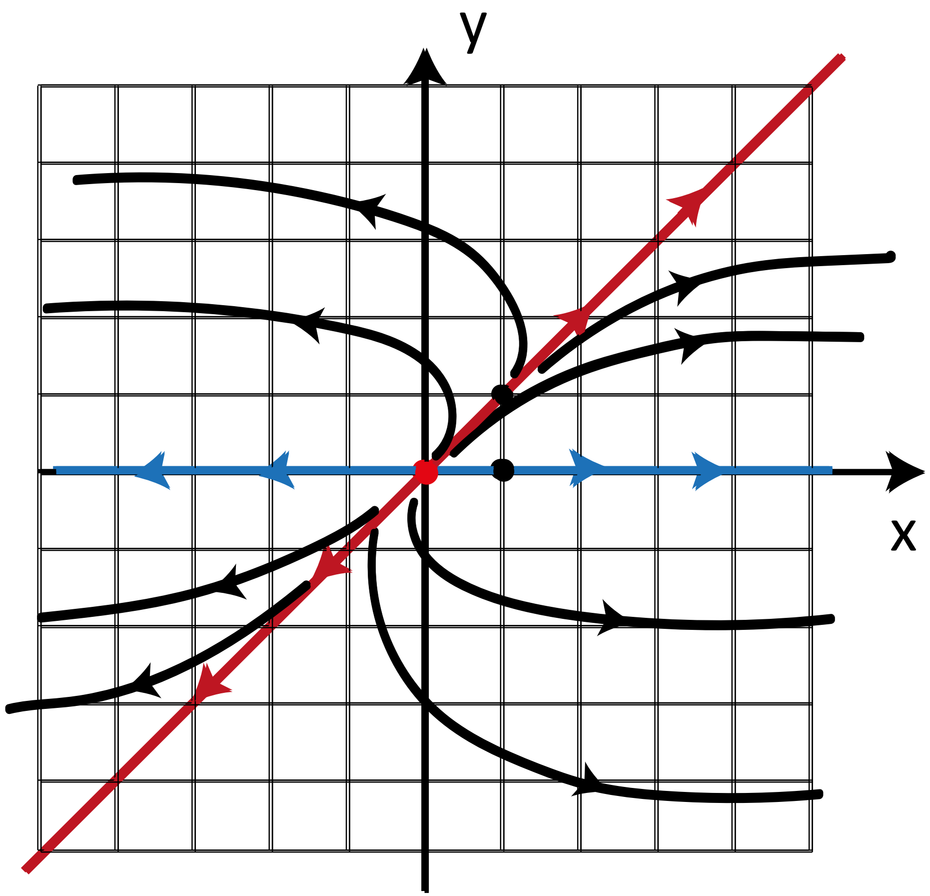
$$t = -\mathrm{M} \qquad \mathbf{z}(t) = \cancel{c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{-2\mathrm{M}}} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{-\mathrm{M}}$$

We are close to the origin and parallel to $\mathbf{v}_2$

$$t = \mathrm{M} \qquad \mathbf{z}(t) = c_1 \begin{pmatrix}1\\0\end{pmatrix} e^{2\mathrm{M}} + \cancel{c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{\mathrm{M}}}$$

We are far from the origin and parallel to $\mathbf{v}_1$

$\mathrm{M} > 0$ and arbitrary big



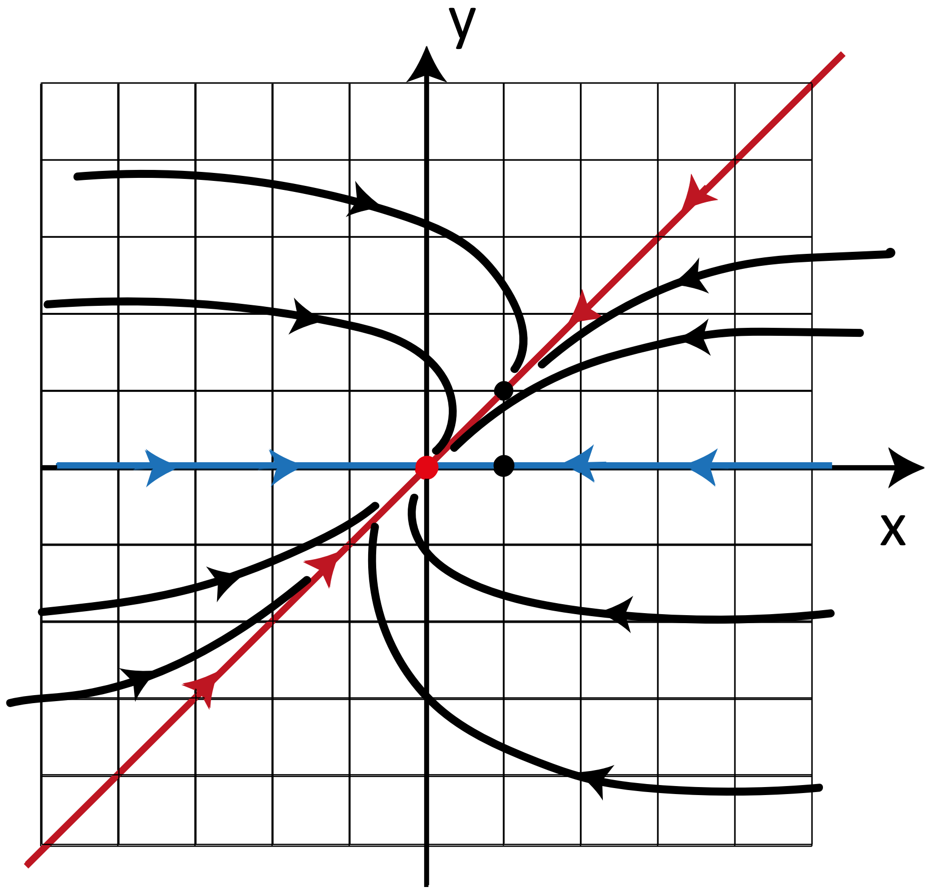
Stable Node

$$\lambda_1 = -2,\ \lambda_2 = -1,\ \mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathbf{z}(t) = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t}$$

$t = -\mathrm{M}$ We are far from the origin and parallel to $\mathbf{v}_1$

$t = \mathrm{M}$ We are close to the origin and parallel to $\mathbf{v}_2$



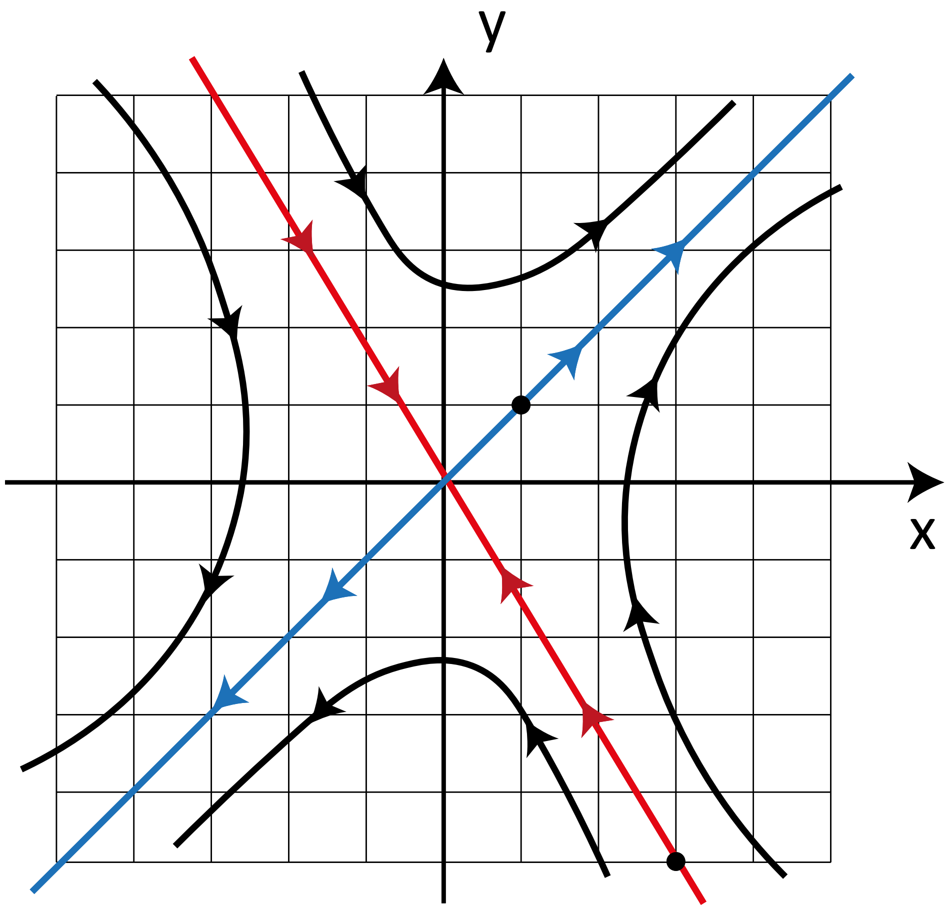
Saddle

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x + 3y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 5x - 3y \end{cases}$$

$$\lambda_1 = -6,\ \lambda_2 = 2,\ \mathbf{v}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is simpler to draw, just follow the directions

If the system starts exactly on the eigenvector with a negative eigenvalue (red on the plot), it goes to zero. If the system is slightly away from that eigenvector, it goes to infinity.



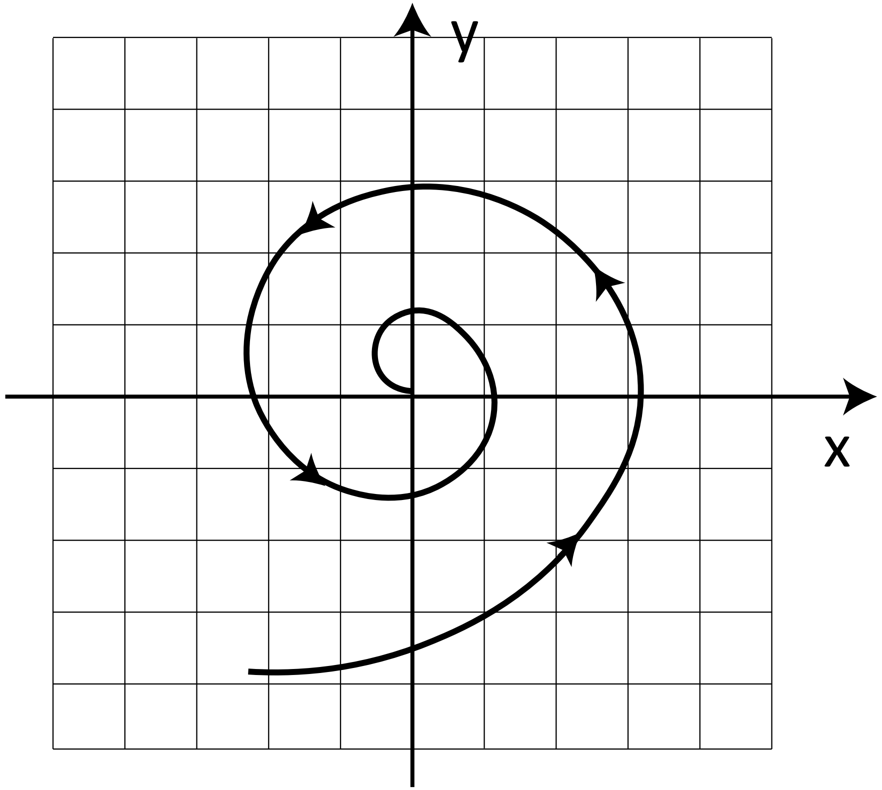
Focus/Spiral, stable

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - 2y \end{cases}$$

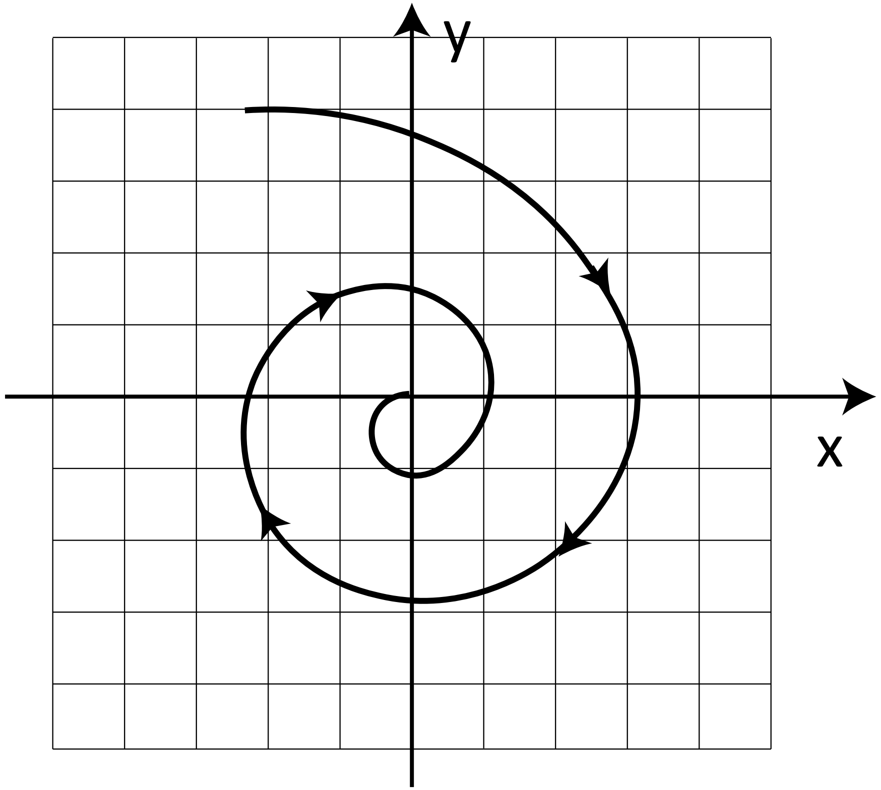
$$\lambda_1 = -1 + \sqrt{2}i,\ \lambda_2 = -1 - \sqrt{2}i$$

Imaginary numbers. Oscillations. Indeed, the solution has terms with sinusoidal functions. You can draw this by noticing the presence of Imaginary values and studying the sign of the real part.

The real part is negative, the system spirals through the origin.



Clockwise or counterclockwise?

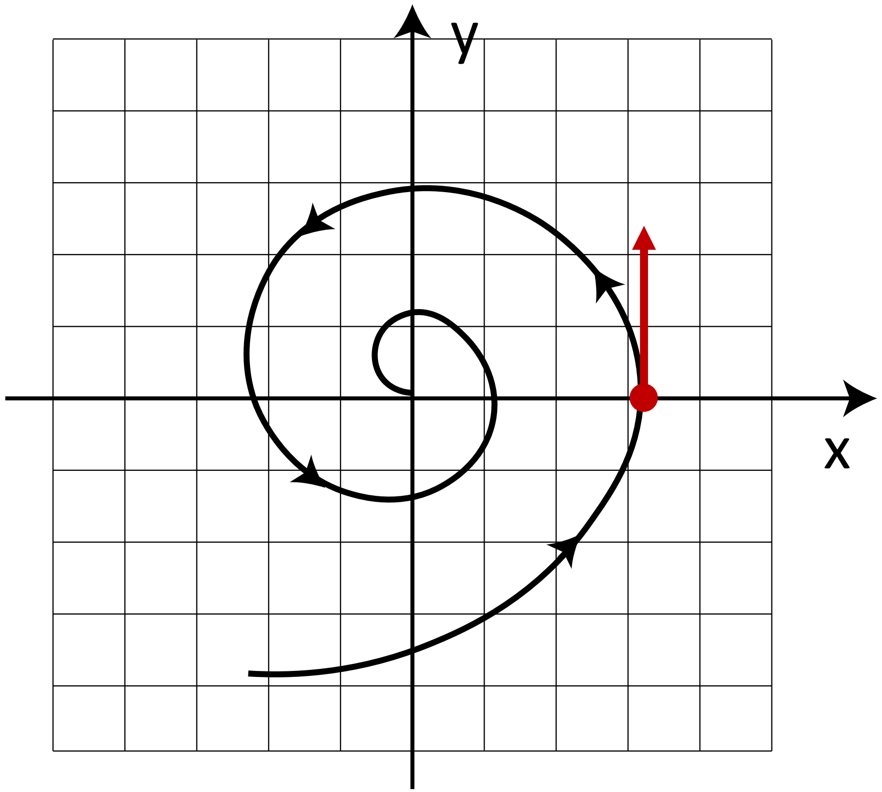


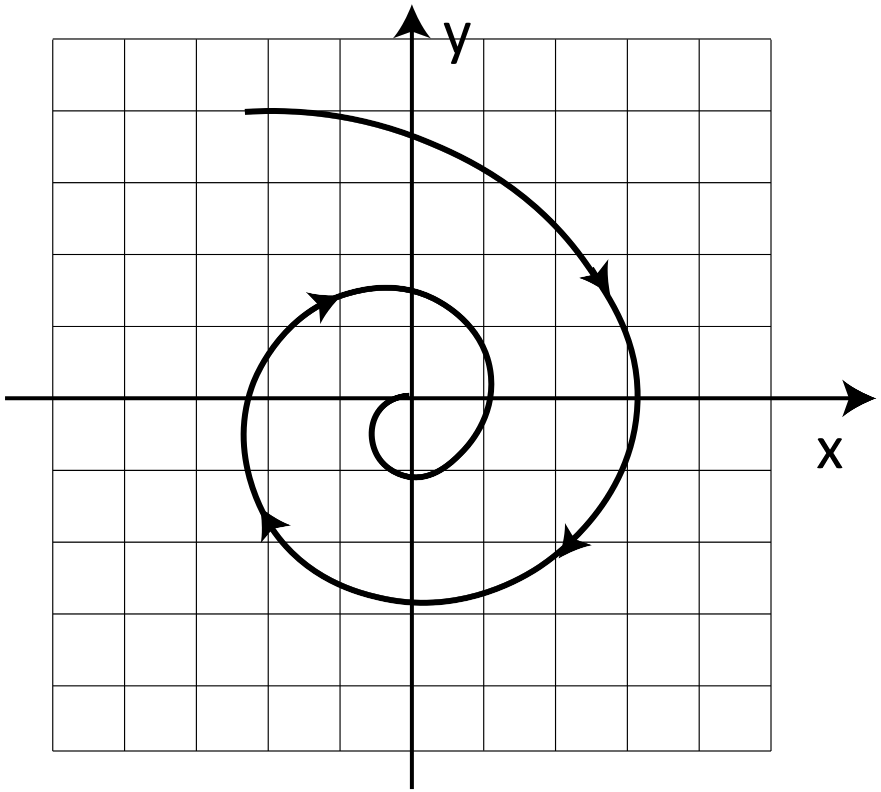
Focus/Spiral, stable

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - 2y \end{cases}$$

$$\lambda_1 = -1 + \sqrt{2}i, \;\; \lambda_2 = -1 - \sqrt{2}i$$

Clockwise or counterclockwise? Look at the direction at one point.
In this case, in (1,0) the direction is (0,1). Thus, the system goes counterclockwise in the considered example and the left graph is the correct one.





Practical Recipe

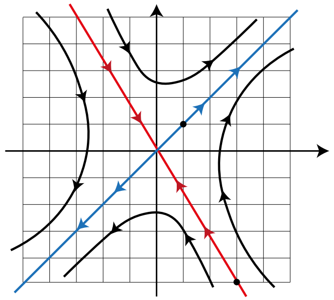
Real eigenvalues

Draw the eigenvectors with the corresponding directions

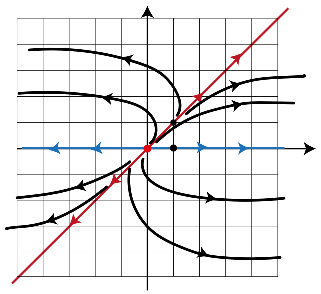
If the two eigenvalues have opposite signs, follow the directions of the eigenvectors

If the two eigenvalues have the same signs, consider the limits

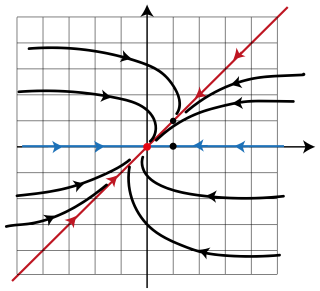
Saddle



Unstable Node



Stable Node



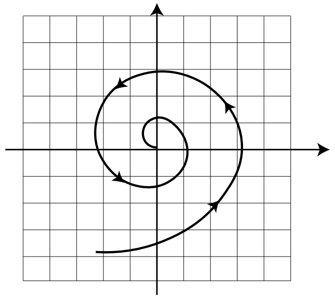
Imaginary eigenvalues

Find the directions at one or more points to understand if the system spirals clockwise or counterclockwise

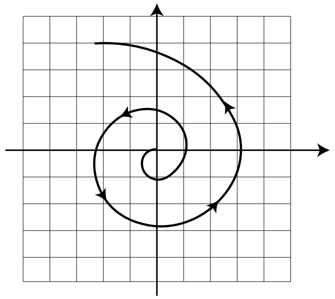
If the real part is negative, the system goes to the origin

If the real part is positive, the system goes away from the origin

Stable Spiral



Unstable Spiral



Notice: these figures are examples and other systems will not have exactly these trends

Thank you