GRP_11: Mastermind

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Course Modelling Project

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Abstract

The goal of the project is to assess whether or not a game of mastermind can have a guaranteed win. The scope only includes the final guess of the game, the first 3 guesses have already been made and are used to determine the goal state. This is based on a game consisting of 4 pegs combinations for each guess with 6 possible colour options per peg (A total of 6^4 possible combinations). The feedback pegs (black and white) correspond to each of the guesses (top left is the first, top right is the second, bottom left is the third, and bottom right is the fourth peg). If a peg is black, then that specific peg is correct. If a feedback peg is white, then the peg is part of the win state but is in the wrong position. If there is no feedback peg then the colour and position of the peg is wrong. Combining the provided information from the previous guesses, we will use a SAT solver to determine whether the correct guess can be determined. It must result that there is only one possible solution and it must match the predetermined solution.

Propositions

- $P_{i,j} \Rightarrow$ A purple peg is in row i, column j
  - This rule follows for all the different colours
  - T= teal, Y = yellow, O = orange, G = green, P = purple, and R = red
- $W_{i,k} \Rightarrow$ One correct colour in row i (a white peg)
- $B_{i,k} \Rightarrow$ One correct colour and position in row i (a black peg)
- $E_{i,k} \Rightarrow$ One incorrect colour (an empty peg)

Constraints

- $1 \leq i \leq 3$; this represents the rows/guess position of both guess pegs and feedback pegs.
- $1 \leq j \leq 4$; this represents the column position of each guess peg.
- $1 \leq k \leq 4$; this represents the column position of feedback pegs.
- $P_{i,j} \longrightarrow (\neg T_{i,j} \wedge \neg Y_{i,j} \wedge \neg O_{i,j} \wedge \neg G_{i,j} \wedge \neg R_{i,j})$; If a peg exists in the position i,j then no other colour peg can exist in that location.
- $W_{i,k} \longrightarrow (\neg B_{i,k} \wedge \neg E_{i,k})$; If the feedback peg for a position is white (correct colour but wrong location) then there cannot be a black or empty peg in that position on the board (which also correlates to a specific location).
- $B_{i,k} \longrightarrow (\neg W_{i,k} \wedge \neg E_{i,k})$; If the feedback peg for a position is black (correct colour and location) then there cannot be a white or empty peg in that position on the board.

- $E_{i,k} \longrightarrow (\neg W_{i,k} \wedge \neg W_{i,k})$; If the feedback peg for a position is empty (wrong position and colour) then there cannot be a black or white feedback peg in that same position.

- Guesses and solutions will not contain any duplicate colours. Example case using P in position i,j as an example but holds for all colours. $P_{i,j} \longrightarrow (\neg P_{i,j+1} \wedge \neg P_{i,j+2} \wedge \neg P_{i,j+3})$

- $G_{i,j} \wedge W_{i,k} \longrightarrow (\neg G_{i_1,j} \wedge \neg G_{i_2,j} \wedge \neg G_{i_3,j})$; if a guess has a white feedback peg, then that same colour should never be guessed again within the same column (j).

Model Exploration

We spent a lot of time trying to determine how to build the board without making it too complex. The main aspect of that was deciding whether to include the feedback pegs with the guess rows or as a separate object. After getting feedback from others and discussing it together, we determined that directly relating the feedback pegs to specific positions in the guess would ensure that the project did not become too difficult.

This also changed our overall goal of the problem. Before we were considering whether to determine whether a potential win could be determined but not guaranteed (multiple possible combinations in the end) or be able to narrow the solution down to a specific combination and if any others existed then the results would be false.

We also quickly realized that we would need to introduce a new proposition that handled no feedback peg (a null value). As we were generating our jape proofs, it was noticed that there was a flaw in our logic since the implication that a black feedback peg does not exist does not imply that a white one does and vice versa. Once the empty peg was introduced, our logic and proofs were logically sound.

At one point, we tried using a new Guess propositional class in order to implement the constraint of having no duplicate colours within a guess. We found all possible valid guesses and tried to initialize new Guess propositions by taking exactly one guess from that collection of valid guesses. What we found was that this approach would lead to added complexity, since we were basically creating the guesses already through our individual colour propositions. In the end, we figured out how to ensure that no two slots of a guess can have the same colour by implementing constraints for each index, for each colour, for each feedback type, not just taking them from an array, as that would still not have helped the model solve for solutions.

During the implementation of our feedback constraints, we realized that we had been forgetting to implement the bijection of feedback, colour, and answer propositions. Not only should colour+feedback propositions imply particular answer propositions, but colour+answer propositions should also imply feedback propositions, and feedback+answer propositions should imply particular colour propositions. Upon doing this we noticed a decrease in our of solutions, bringing our model closer to the real simulation.

Jape Theorems

1. When the color $P_{i,j}$ (purple) is guessed and the feedback peg gives an empty peg at $E_{i,j}$ then we can conclude that P should not be guessed in the answer.
   P represents the purple guess at i,j
   E represents an empty feedback peg at i,j
   A is the color purple guessed at answer j

$$(\neg P \vee E), (P \wedge (E \longrightarrow \neg A)) \vdash \neg A$$

1:	(¬P∨E), (P∧(E→¬A))	premises
2:	E→¬A	∧ elim 1.2
3:	P	∧ elim 1.2
4:	¬P	assumption
5:	⊥	¬ elim 3,4
6:	¬A	contra (constructive) 5
7:	E	assumption
8:	¬A	→ elim 2,7
9:	¬A	∨ elim 1.1,4-6,7-8

2. When the color $G_{i_1,j}$ (green) is guessed and the feedback peg is white at (i_1, j_1) then we know that one of the pegs in that row must be green. So either $G_{i_2,j}$, $G_{i_3,j}$, or $G_{i_4,j}$ must be true.
   G represents the first Green guess at i,j
   x represents a white feedback peg for position i,k
   A,B,C,D represent the 4 possible positions in each row

$$(G \wedge x), (x \longrightarrow A \vee B \vee C \vee D), (G \longrightarrow \neg A) \vdash \neg A \wedge (B \vee C \vee D)$$

1:	G∧x, (x→A∨B∨C∨D), (G→¬A)	premises
2:	x	∧ elim 1.1
3:	A∨B∨C∨D	→ elim 1.2,2
4:	G	∧ elim 1.1
5:	¬A	→ elim 1.3,4
6:	A∨B∨C	assumption
7:	A∨B	assumption
8:	A	assumption
9:	⊥	¬ elim 8,5
10:	B∨C∨D	contra (constructive) 9
11:	B	assumption
12:	B∨C	∨ intro 11
13:	B∨C∨D	∨ intro 12
14:	B∨C∨D	∨ elim 7,8-10,11-13
15:	C	assumption
16:	B∨C	∨ intro 15
17:	B∨C∨D	∨ intro 16
18:	B∨C∨D	∨ elim 6,7-14,15-17
19:	D	assumption
20:	B∨C∨D	∨ intro 19
21:	B∨C∨D	∨ elim 3,6-18,19-20
22:	¬A∧(B∨C∨D)	∧ intro 5,21

3. When a colour (in this case $P_{i,j}$) has a corresponding black feedback peg ($B_{i,k}$) then we know that given the constraint that colours cannot repeat in the answer, that P cannot exist in any of the other locations.
   P represents a purple peg in position i,j
   B represents a black feedback peg in position i,k
   A,B,C,D represent the 4 possible positions per row

$$(P \land B), ((P \land B) \longrightarrow C), ((A \land \neg C \land \neg D \land \neg F) \lor$$
$$(\neg A \land C \land \neg D \land \neg F) \lor (\neg A \land \neg C \land D \land \neg F) \lor$$
$$(\neg A \land \neg C \land \neg D \land F)) \vdash (\neg A \land C \land \neg D \land \neg F)$$

1:	(P∧B), ((P∧B)→C), ((A∧¬C∧¬D∧¬F)∨(¬A∧C∧¬D∧¬F)∨(¬A∧¬C∧D∧¬F)∨(¬A∧¬C∧¬D∧F))	premises
2:	C	→ elim 1.2,1.1
3:	(A∧¬C∧¬D∧¬F)∨(¬A∧C∧¬D∧¬F)∨(¬A∧¬C∧D∧¬F)	assumption
4:	(A∧¬C∧¬D∧¬F)∨(¬A∧C∧¬D∧¬F)	assumption
5:	A∧¬C∧¬D∧¬F	assumption
6:	A∧¬C∧¬D	∧ elim 5
7:	A∧¬C	∧ elim 6
8:	¬C	∧ elim 7
9:	⊥	¬ elim 2,8
10:	¬A∧C∧¬D∧¬F	contra (constructive) 9
11:	¬A∧C∧¬D∧¬F	assumption
12:	¬A∧C∧¬D∧¬F	∨ elim 4,5-10,11-11
13:	¬A∧¬C∧D∧¬F	assumption
14:	¬A∧¬C∧D	∧ elim 13
15:	¬A∧¬C	∧ elim 14
16:	¬C	∧ elim 15
17:	⊥	¬ elim 2,16
18:	¬A∧C∧¬D∧¬F	contra (constructive) 17
19:	¬A∧C∧¬D∧¬F	∨ elim 3,4-12,13-18
20:	¬A∧¬C∧¬D∧F	assumption
21:	¬A∧¬C∧¬D	∧ elim 20
22:	¬A∧¬C	∧ elim 21
23:	¬C	∧ elim 22
24:	⊥	¬ elim 2,23
25:	¬A∧C∧¬D∧¬F	contra (constructive) 24
26:	(¬A∧C∧¬D∧¬F)	∨ elim 1.3,3-19,20-25

First-Order Extension

The implementation of predicate logic into our project would allow us to utilize existential and universal quantifiers. There are a few places where these can come in handy. For example, for all colour peg guesses, there exists one corresponding feedback value. In predicate logic, this would look like: $\forall C_{x,y}(B(C_{x,y}) \vee W(C_{x,y}) \vee E(C_{x,y}))$. B($C_{x,y}$) is true when the guess peg in positions x,y is the correct colour. $W(C_{x,y})$ is true when the guess peg exists in the answer but is in the wrong position. $E(C_{x,y})$) is true when the guess peg is the wrong colour and position. One other place we could have used predicate logic would be to describe the correct sequence of colour pegs. If X represents a position (the 4 possible peg locations), and P(X), T(X), Y(X), G(X), O(X), R(X), represent whether that colour is correct at X position, $\forall X.((P(X) \wedge \neg T(X) \wedge \neg Y(X) \wedge \neg G(X) \wedge \neg O(X) \wedge \neg R(X)) \vee (T(X) \wedge \neg P(X) \wedge \neg Y(X) \wedge \neg G(X) \wedge \neg O(X) \wedge \neg R(X)) \vee (Y(X) \wedge \neg P(X) \wedge \neg T(X) \wedge \neg G(X) \wedge \neg O(X) \wedge \neg R(X)) \vee (G(X) \wedge \neg P(X) \wedge \neg T(X) \wedge \neg Y(X) \wedge \neg O(X) \wedge \neg R(X)) \vee (O(X) \wedge \neg P(X) \wedge \neg T(X) \wedge \neg Y(X) \wedge \neg G(X) \wedge \neg R(X)) \vee (R(X) \wedge \neg P(X) \wedge \neg T(X) \wedge \neg Y(X) \wedge \neg G(X) \wedge \neg O(X)))$. This says that for each position, ONLY one of the colours are correct.