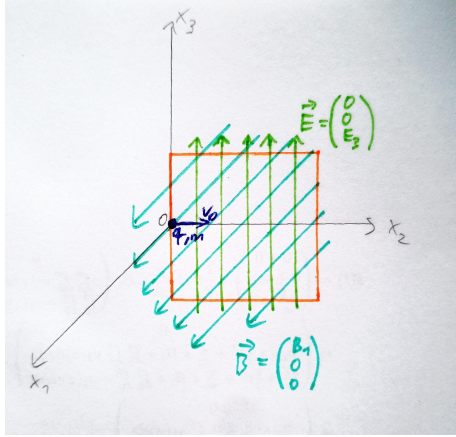


# Trajectory of a particle through a Wien filter

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What path does a particle take through a Wien filter if its initial velocity is not the filter's pass-through velocity?



$$\text{given: } q; m; \vec{E} = \begin{pmatrix} 0 \\ 0 \\ E_3 \end{pmatrix}; \vec{B} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}; \vec{v}(0) = \begin{pmatrix} 0 \\ v_{2;0} \\ 0 \end{pmatrix}; \vec{s}(0) = \vec{0}$$

$$\vec{F}_{el} = q\vec{E} \quad (1)$$

$$\vec{F}_L = q\vec{v}(t) \times \vec{B} \quad (2)$$

$$\begin{aligned} \vec{F}(t) &= \vec{F}_{el} + \vec{F}_L(t) \\ &= q\vec{E} + q\vec{v}(t) \times \vec{B} \\ &= q(\vec{E} + \vec{v}(t) \times \vec{B}) \end{aligned} \quad (3)$$

$$\vec{a}(t) = \frac{\vec{F}(t)}{m} = \frac{q}{m}(\vec{E} + \vec{v}(t) \times \vec{B}) \quad (4)$$

$$\begin{aligned} \vec{v}(t) &= \int_0^t \vec{a}(x) dx + \vec{v}_0 \\ &= \int_0^t \frac{q}{m}(\vec{E} + \vec{v}(t) \times \vec{B}) dx + \vec{v}_0 \end{aligned} \quad (5)$$

$$\begin{aligned}
\vec{s}(t) &= \int_0^t \vec{v}(x) dx \\
&= \int_0^t \left( \int_0^x \frac{q}{m} (\vec{E} + \vec{v}(z) \times \vec{B}) dz + \vec{v}_0 \right) dx \\
&= \frac{q}{m} \int_0^t \int_0^x (\vec{E} + \vec{v}(z) \times \vec{B}) dz dx + \int_0^t \vec{v}_0 dx \\
&= \frac{q}{m} \int_0^t \int_0^x \vec{E} dz dx + \frac{q}{m} * \int_0^t \int_0^x \vec{v}(z) \times \vec{B} dz dx + \vec{v}_0 * t \\
&= \frac{q}{m} * \vec{E} * \frac{1}{2} t^2 + \frac{q}{m} * \int_0^t \int_0^x \vec{v}(z) \times \vec{B} dz dx + \vec{v}_0 t \\
&= \frac{q\vec{E}}{2m} t^2 + \vec{v}_0 t + \frac{q}{m} \int_0^t \int_0^x \vec{v}(z) \times \vec{B} dz dx
\end{aligned} \tag{6}$$

$$\begin{aligned}
\int_0^t \int_0^x \vec{v}(z) \times \vec{B} dz dx &= \int_0^t \int_0^x \begin{pmatrix} v_2(z)0 - v_3(z)0 \\ v_3(z)B_1 - v_1(z)0 \\ v_1(z)0 - v_2(z)B_1 \end{pmatrix} dz dx \\
&= \int_0^t \int_0^x \begin{pmatrix} 0 \\ v_3(z)B_1 \\ -v_2(z)B_1 \end{pmatrix} dz dx \\
&= B_1 \int_0^t \int_0^x \begin{pmatrix} 0 \\ v_3(z) \\ -v_2(z) \end{pmatrix} dz dx
\end{aligned} \tag{7}$$

$$\begin{aligned}
\vec{s}(t) &= \begin{pmatrix} 0 \\ 0 \\ \frac{qE_3}{2m} \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ v_{2;0} \\ 0 \end{pmatrix} t + \frac{q}{m} * B_1 \begin{pmatrix} 0 \\ \int_0^t \int_0^x v_3(z) dz dx \\ \int_0^t \int_0^x -v_2(z) dz dx \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ v_{2;0}t + \frac{q}{m} * B_1 * \int_0^t \int_0^x v_3(z) dz dx \\ \frac{qE_3}{2m} t^2 + \frac{q}{m} * B_1 * \int_0^t \int_0^x -v_2(z) dz dx \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ v_{2;0}t + \frac{qB_1}{m} \int_0^t s_3(x) dx \\ \frac{qE_3}{2m} t^2 + \frac{qB_1}{m} \int_0^t -s_2(x) dx \end{pmatrix}
\end{aligned} \tag{8}$$

$s_2$  in  $s_3$ :

$$\begin{aligned}
s_3(t) &= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m} \int_0^t - \left( v_{2;0}x + \frac{qB_1}{m} \int_0^x s_3(z)dz \right) dx \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m} \int_0^t -v_{2;0}x - \frac{qB_1}{m} \int_0^x s_3(z)dz dx \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m} \left( - \int_0^t v_{2;0}x dx - \int_0^t \frac{qB_1}{m} \int_0^x s_3(z)dz dx \right) \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m} \left( -v_{2;0} \int_0^t x dx - \frac{qB_1}{m} \int_0^t \int_0^x s_3(z)dz dx \right) \quad (9) \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1 v_{2;0}}{m} \int_0^t x dx - \frac{q^2 B_1^2}{m^2} \int_0^t \int_0^x s_3(z)dz dx \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1 v_{2;0}}{m} * \frac{1}{2}t^2 - \frac{q^2 B_1^2}{m^2} \int_0^t \int_0^x s_3(z)dz dx \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1 v_{2;0}}{2m}t^2 - \frac{q^2 B_1^2}{m^2} \int_0^t \int_0^x s_3(z)dz dx
\end{aligned}$$

$$\text{subst.: } g(t) = \int_0^t \int_0^x s_3(z)dz dx \Rightarrow \ddot{g}(t) = s_3(t)$$

$$\begin{aligned}
\ddot{g}(t) &= \frac{qE_3}{2m}t^2 - \frac{qB_1 v_{2;0}}{2m}t^2 - \frac{q^2 B_1^2}{m^2}g(t) \\
&= \frac{qE_3 - qB_1 v_{2;0}}{2m}t^2 - \frac{q^2 B_1^2}{m^2}g(t) \quad (10)
\end{aligned}$$

This differential equation of the form

$$\ddot{f}(x) = a * x^2 - b * f(t) \quad (11)$$

where  $a = \frac{qE_3 - qB_1 v_{2;0}}{2m}$  and  $b = \frac{q^2 B_1^2}{m^2}$   
has the solution

$$\begin{aligned}
f(t) &= -\frac{2a}{b^2} + \frac{at^2}{b} + c_2 \sin(\sqrt{bt}) + c_1 \cos(\sqrt{bt}) \\
\dot{f}(t) &= \frac{2at}{b} - \sqrt{b}c_1 \sin(\sqrt{bt}) + \sqrt{b}c_2 \cos(\sqrt{bt}) \\
\ddot{f}(t) &= \frac{2a}{b} - bc_2 \sin(\sqrt{bt}) - bc_1 \cos(\sqrt{bt}) \quad (12) \\
\ddot{f}(t) &= b^{\frac{3}{2}} \left( c_1 \sin(\sqrt{bt}) - c_2 \cos(\sqrt{bt}) \right)
\end{aligned}$$

resubst.:

$$\begin{aligned}
s_3(t) = \ddot{g}(t) &= \frac{(qE_3 - qB_1 v_{2;0})m}{q^2 B_1^2} - \frac{q^2 B_1^2}{m^2}c_2 \sin\left(\frac{qB_1}{m}t\right) - \frac{q^2 B_1^2}{m^2}c_1 \cos\left(\frac{qB_1}{m}t\right) \\
&= \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} - \frac{q^2 B_1^2}{m^2} \left( c_2 \sin\left(\frac{qB_1}{m}t\right) + c_1 \cos\left(\frac{qB_1}{m}t\right) \right) \quad (13)
\end{aligned}$$

$$\begin{aligned}
v_3(t) &= \left( \frac{q^2 B_1^2}{m^2} \right)^{\frac{3}{2}} \left( c_1 \sin \left( \frac{q B_1}{m} t \right) - c_2 \cos \left( \frac{q B_1}{m} t \right) \right) \\
&= \frac{q^3 B_1^3}{m^3} \left( c_1 \sin \left( \frac{q B_1}{m} t \right) - c_2 \cos \left( \frac{q B_1}{m} t \right) \right)
\end{aligned} \tag{14}$$

We know that  $v_3(0) = 0$ :

$$\begin{aligned}
v_3(0) &= \underbrace{\frac{q^3 B_1^3}{m^3}}_{\neq 0} (c_1 \sin(0) - c_2 \cos(0)) \stackrel{!}{=} 0 \\
&\Rightarrow c_1 \sin(0) - c_2 \cos(0) = 0 \\
&\Leftrightarrow c_1 * 0 - c_2 * 1 = 0 \\
&\Rightarrow c_2 = 0
\end{aligned} \tag{15}$$

This simplifies  $s_3(t)$  and  $v_3(t)$ :

$$s_3(t) = \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} - \frac{q^2 B_1^2}{m^2} c_1 \cos \left( \frac{q B_1}{m} t \right) \tag{16}$$

$$v_3(t) = \frac{q^3 B_1^3}{m^3} c_1 \sin \left( \frac{q B_1}{m} t \right) \tag{17}$$

We know that  $s_3(0) = 0$ :

$$\begin{aligned}
s_3(t) &= \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} - \frac{q^2 B_1^2}{m^2} c_1 \cos(0) \stackrel{!}{=} 0 \\
&\Leftrightarrow \frac{q^2 B_1^2}{m^2} c_1 = \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} \\
&\Leftrightarrow c_1 = \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} * \frac{m^2}{q^2 B_1^2}
\end{aligned} \tag{18}$$

This simplifies  $s_3(t)$  and  $v_3(t)$ :

$$\begin{aligned}
s_3(t) &= \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} - \frac{E_3 m - B_1 v_{2;0} m}{q B_1^2} \cos \left( \frac{q B_1}{m} t \right) \\
&= m \frac{E_3 - B_1 v_{2;0}}{q B_1^2} \left( 1 - \cos \left( \frac{q B_1}{m} t \right) \right) \\
&= \frac{m}{q B_1} \left( \frac{E_3}{B_1} - v_{2;0} \right) \left( 1 - \cos \left( \frac{q B_1}{m} t \right) \right)
\end{aligned} \tag{19}$$

$$\begin{aligned}
s_3(t) &= -\frac{m}{q B_1} \left( v_{2;0} - \frac{E_3}{B_1} \right) \left( 1 - \cos \left( \frac{q B_1}{m} t \right) \right) \\
v_3(t) &= -\left( v_{2;0} - \frac{E_3}{B_1} \right) \sin \left( \frac{q B_1}{m} t \right)
\end{aligned} \tag{20}$$

$s_3$  in  $s_2$ :

$$\begin{aligned}
s_2(t) &= v_{2;0}t + \frac{qB_1}{m} \int_0^t s_3(x) dx \\
&= v_{2;0}t + \frac{qB_1}{m} \int_0^t m \frac{E_3 - B_1 v_{2;0}}{qB_1^2} \left( 1 - \cos \left( \frac{qB_1}{m} x \right) \right) dx \\
&= v_{2;0}t + \frac{E_3 - B_1 v_{2;0}}{B_1} \int_0^t \left( 1 - \cos \left( \frac{qB_1}{m} x \right) \right) dx \\
&= v_{2;0}t + \frac{E_3 - B_1 v_{2;0}}{B_1} \left( t - \frac{\sin \left( \frac{qB_1}{m} t \right) m}{qB_1} \right) \\
&= v_{2;0}t - \left( v_{2;0} - \frac{E_3}{B_1} \right) \left( t - \frac{m}{qB_1} \sin \left( \frac{qB_1}{m} t \right) \right) \\
s_2(t) &= \frac{E_3}{B_1} t + \frac{m}{qB_1} \left( v_{2;0} - \frac{E_3}{B_1} \right) \sin \left( \frac{qB_1}{m} t \right)
\end{aligned} \tag{21}$$

$$v_2(t) = \frac{E_3}{B_1} + \left( v_{2;0} - \frac{E_3}{B_1} \right) \cos \left( \frac{qB_1}{m} t \right) \tag{22}$$

For  $E = 0$ :

$$\begin{aligned}
s_2(t) &= v_{2;0}t - v_{2;0} \left( t - \frac{\sin \left( \frac{qB_1}{m} t \right) m}{qB_1} \right) \\
&= \frac{mv_{2;0}}{qB_1} \sin \left( \frac{qB_1}{m} t \right)
\end{aligned} \tag{23}$$

$$\begin{aligned}
s_3(t) &= -m \frac{B_1 v_{2;0}}{qB_1^2} \left( 1 - \cos \left( \frac{qB_1}{m} t \right) \right) \\
&= -\frac{mv_{2;0}}{qB_1} + \frac{mv_{2;0}}{qB_1} \cos \left( \frac{qB_1}{m} t \right)
\end{aligned} \tag{24}$$

As expected, this is the parametric form of a circle with  $r = \frac{mv_{2;0}}{qB_1}$ .

Figure 1: Example with  $v_0 = \frac{E}{B}$

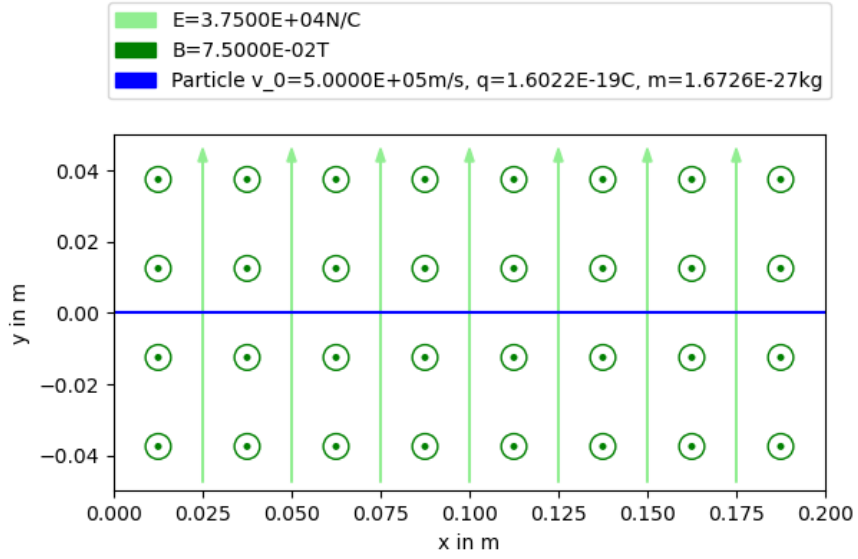


Figure 2: Example with  $v_0 > \frac{E}{B}$

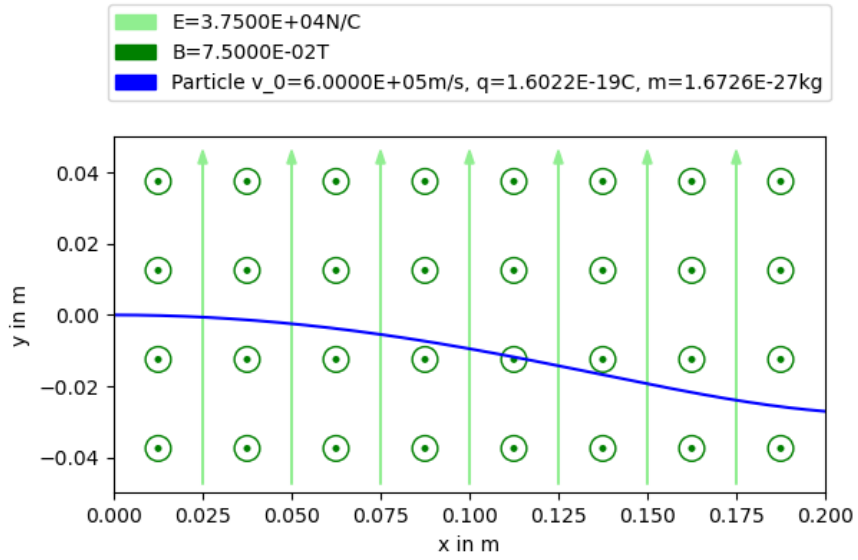


Figure 3: Example with  $v_0 < \frac{E}{B}$

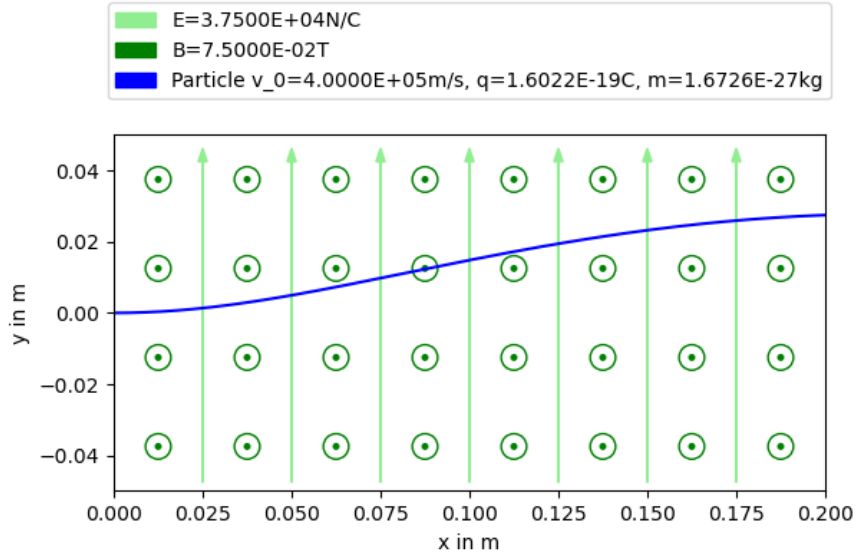


Figure 4: Example with  $v_0 \gg \frac{E}{B}$

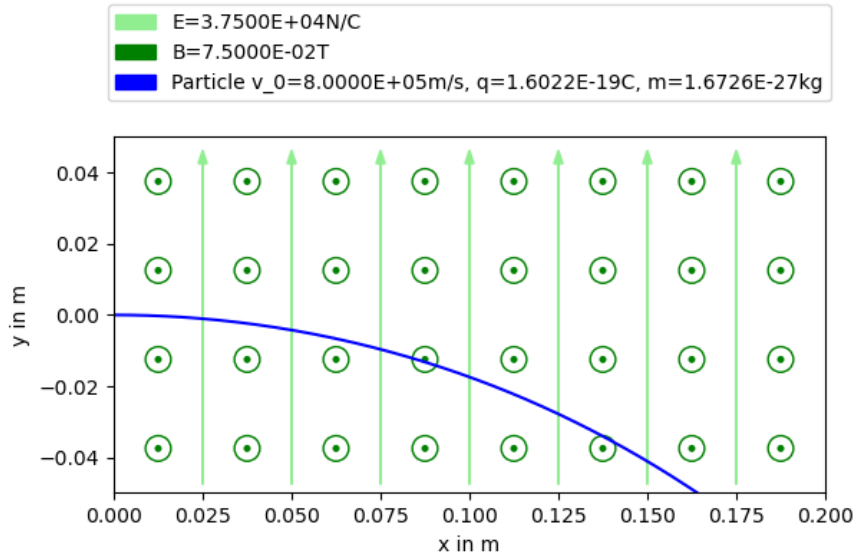


Figure 5: Example with  $v_0 \ll \frac{E}{B}$

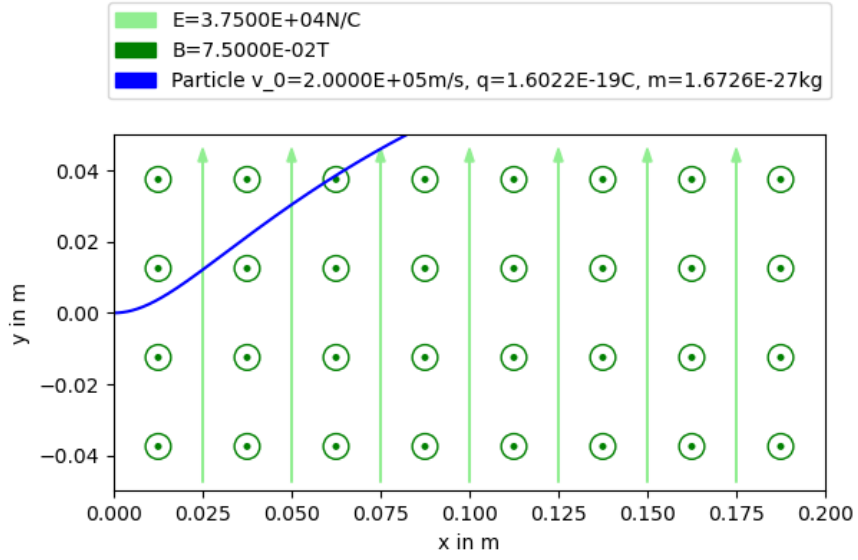


Figure 6: Example with  $E = 0$

