Flugbahn eines Teilchens durch einen Wien Filter

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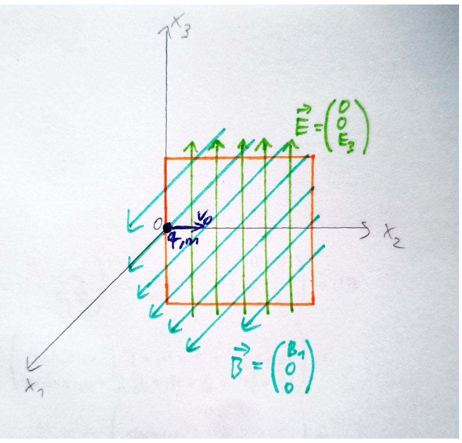
Auf welcher Bahn fliegt ein Teilchen durch einen Wien Filter, wenn seine Anfangsgeschwindigkeit nicht die Durchlassgeschwindigkeit des Filters ist?



gegeben: q; m; $\vec{E} = \begin{pmatrix} 0 \\ 0 \\ E_3 \end{pmatrix}$; $\vec{B} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}$; $\vec{v}(0) = \begin{pmatrix} 0 \\ v_{2;0} \\ 0 \end{pmatrix}$; $\vec{s}(0) = \vec{0}$

$$\vec{F}_{el} = q\vec{E} \tag{1}$$

$$\vec{F}_L = q\vec{v}(t) \times \vec{B} \tag{2}$$

$$\begin{aligned} \vec{F}(t) &= \vec{F}_{el} + \vec{F}_L(t) \\ &= q\vec{E} + q\vec{v}(t) \times \vec{B} \\ &= q(\vec{E} + \vec{v}(t) \times \vec{B}) \end{aligned} \tag{3}$$

$$\vec{a}(t) = \frac{\vec{F}(t)}{m} = \frac{q}{m}(\vec{E} + \vec{v}(t) \times \vec{B}) \tag{4}$$

$$\begin{aligned} \vec{v}(t) &= \int_0^t \vec{a}(x)dx + \vec{v}_0 \\ &= \int_0^t \frac{q}{m}(\vec{E} + \vec{v}(t) \times \vec{B})dx + \vec{v}_0 \end{aligned} \tag{5}$$

$$
\begin{aligned}
\vec{s}(t) &= \int_0^t \vec{v}(x)dx \\
&= \int_0^t \left(\int_0^x \frac{q}{m}(\vec{E} + \vec{v}(z) \times \vec{B})dz + \vec{v}_0\right) dx \\
&= \frac{q}{m}\int_0^t \int_0^x (\vec{E} + \vec{v}(z) \times \vec{B})dzdx + \int_0^t \vec{v}_0 dx \\
&= \frac{q}{m}\int_0^t \int_0^x \vec{E}dzdx + \frac{q}{m} * \int_0^t \int_0^x \vec{v}(z) \times \vec{B}dzdx + \vec{v}_0 * t \\
&= \frac{q}{m} * \vec{E} * \frac{1}{2}t^2 + \frac{q}{m} * \int_0^t \int_0^x \vec{v}(z) \times \vec{B}dzdx + \vec{v}_0 t \\
&= \frac{q\vec{E}}{2m}t^2 + \vec{v}_0 t + \frac{q}{m}\int_0^t \int_0^x \vec{v}(z) \times \vec{B}dzdx
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\int_0^t \int_0^x \vec{v}(z) \times \vec{B}dzdx &= \int_0^t \int_0^x \begin{pmatrix} v_2(z)0 - v_3(z)0 \\ v_3(z)B_1 - v_1(z)0 \\ v_1(z)0 - v_2(z)B_1 \end{pmatrix} dzdx \\
&= \int_0^t \int_0^x \begin{pmatrix} 0 \\ v_3(z)B_1 \\ -v_2(z)B_1 \end{pmatrix} dzdx \\
&= B_1 \int_0^t \int_0^x \begin{pmatrix} 0 \\ v_3(z) \\ -v_2(z) \end{pmatrix} dzdx
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\vec{s}(t) &= \begin{pmatrix} 0 \\ 0 \\ \frac{qE_3}{2m} \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ v_{2;0} \\ 0 \end{pmatrix} t + \frac{q}{m} * B_1 \begin{pmatrix} 0 \\ \int_0^t \int_0^x v_3(z)dzdx \\ \int_0^t \int_0^x -v_2(z)dzdx \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ v_{2;0}t + \frac{q}{m} * B_1 * \int_0^t \int_0^x v_3(z)dzdx \\ \frac{qE_3}{2m}t^2 + \frac{q}{m} * B_1 * \int_0^t \int_0^x -v_2(z)dzdx \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ v_{2;0}t + \frac{qB_1}{m}\int_0^t s_3(x)dx \\ \frac{qE_3}{2m}t^2 + \frac{qB_1}{m}\int_0^t -s_2(x)dx \end{pmatrix}
\end{aligned}
\tag{8}
$$

s_2 in s_3:

$$\begin{aligned}
s_3(t) &= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m}\int_0^t -\left(v_{2;0}x + \frac{qB_1}{m}\int_0^x s_3(z)dz\right)dx \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m}\int_0^t -v_{2;0}x - \frac{qB_1}{m}\int_0^x s_3(z)dzdx \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m}\left(-\int_0^t v_{2;0}xdx - \int_0^t \frac{qB_1}{m}\int_0^x s_3(z)dzdx\right) \\
&= \frac{qE_3}{2m}t^2 + \frac{qB_1}{m}\left(-v_{2;0}\int_0^t xdx - \frac{qB_1}{m}\int_0^t\int_0^x s_3(z)dzdx\right) \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1v_{2;0}}{m}\int_0^t xdx - \frac{q^2B_1^2}{m^2}\int_0^t\int_0^x s_3(z)dzdx \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1v_{2;0}}{m} * \frac{1}{2}t^2 - \frac{q^2B_1^2}{m^2}\int_0^t\int_0^x s_3(z)dzdx \\
&= \frac{qE_3}{2m}t^2 - \frac{qB_1v_{2;0}}{2m}t^2 - \frac{q^2B_1^2}{m^2}\int_0^t\int_0^x s_3(z)dzdx
\end{aligned} \tag{9}$$

subst.: $g(t) = \int_0^t\int_0^x s_3(z)dzdx \Rightarrow \ddot{g}(t) = s_3(t)$

$$\begin{aligned}
\ddot{g}(t) &= \frac{qE_3}{2m}t^2 - \frac{qB_1v_{2;0}}{2m}t^2 - \frac{q^2B_1^2}{m^2}g(t) \\
&= \frac{qE_3 - qB_1v_{2;0}}{2m}t^2 - \frac{q^2B_1^2}{m^2}g(t)
\end{aligned} \tag{10}$$

Das ist eine Differentialgleichung der Form

$$\ddot{f}(x) = a * x^2 - b * f(t) \tag{11}$$

mit $a = \frac{qE_3 - qB_1v_{2;0}}{2m}$ and $b = \frac{q^2B_1^2}{m^2}$
Diese Differentialgleichung hat die Lösung

$$\begin{aligned}
f(t) &= -\frac{2a}{b^2} + \frac{at^2}{b} + c_2\sin\left(\sqrt{b}t\right) + c_1\cos\left(\sqrt{b}t\right) \\
\dot{f}(t) &= \frac{2at}{b} - \sqrt{b}c_1\sin\left(\sqrt{b}t\right) + \sqrt{b}c_2\cos\left(\sqrt{b}t\right) \\
\ddot{f}(t) &= \frac{2a}{b} - bc_2\sin\left(\sqrt{b}t\right) - bc_1\cos\left(\sqrt{b}t\right) \\
\dddot{f}(t) &= b^{\frac{3}{2}}\left(c_1\sin\left(\sqrt{b}t\right) - c_2\cos\left(\sqrt{b}t\right)\right)
\end{aligned} \tag{12}$$

resubst.:

$$\begin{aligned}
s_3(t) = \ddot{g}(t) &= \frac{(qE_3 - qB_1v_{2;0})m}{q^2B_1^2} - \frac{q^2B_1^2}{m^2}c_2\sin\left(\frac{qB_1}{m}t\right) - \frac{q^2B_1^2}{m^2}c_1\cos\left(\frac{qB_1}{m}t\right) \\
&= \frac{E_3m - B_1v_{2;0}m}{qB_1^2} - \frac{q^2B_1^2}{m^2}\left(c_2\sin\left(\frac{qB_1}{m}t\right) + c_1\cos\left(\frac{qB_1}{m}t\right)\right)
\end{aligned} \tag{13}$$

$$
\begin{aligned}
v_3(t) &= \left(\frac{q^2 B_1^2}{m^2}\right)^{\frac{3}{2}} \left(c_1 \sin\left(\frac{qB_1}{m}t\right) - c_2 \cos\left(\frac{qB_1}{m}t\right)\right) \\
&= \frac{q^3 B_1^3}{m^3} \left(c_1 \sin\left(\frac{qB_1}{m}t\right) - c_2 \cos\left(\frac{qB_1}{m}t\right)\right)
\end{aligned} \tag{14}
$$

gegeben: $v_3(0) = 0$:

$$
\begin{aligned}
v_3(0) = \underbrace{\frac{q^3 B_1^3}{m^3}}_{\neq 0}(c_1 \sin(0) - c_2 \cos(0)) &\overset{!}{=} 0 \\
\Rightarrow c_1 \sin(0) - c_2 \cos(0) &= 0 \\
\Leftrightarrow c_1 * 0 - c_2 * 1 &= 0 \\
\Rightarrow c_2 &= 0
\end{aligned} \tag{15}
$$

$c_2 = 0$ in $s_3(t)$ und $v_3(t)$:

$$
s_3(t) = \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} - \frac{q^2 B_1^2}{m^2} c_1 \cos\left(\frac{qB_1}{m}t\right) \tag{16}
$$

$$
v_3(t) = \frac{q^3 B_1^3}{m^3} c_1 \sin\left(\frac{qB_1}{m}t\right) \tag{17}
$$

gegeben: $s_3(0) = 0$:

$$
\begin{aligned}
s_3(t) = \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} - \frac{q^2 B_1^2}{m^2} c_1 \cos(0) &\overset{!}{=} 0 \\
\Leftrightarrow \frac{q^2 B_1^2}{m^2} c_1 &= \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} \\
\Leftrightarrow c_1 &= \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} * \frac{m^2}{q^2 B_1^2}
\end{aligned} \tag{18}
$$

c_1 in $s_3(t)$ and $v_3(t)$:

$$
\begin{aligned}
s_3(t) &= \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} - \frac{E_3 m - B_1 v_{2;0} m}{qB_1^2} \cos\left(\frac{qB_1}{m}t\right) \\
&= m\frac{E_3 - B_1 v_{2;0}}{qB_1^2} \left(1 - \cos\left(\frac{qB_1}{m}t\right)\right) \\
&= \frac{m}{qB_1} \left(\frac{E_3}{B_1} - v_{2;0}\right) \left(1 - \cos\left(\frac{qB_1}{m}t\right)\right) \\
s_3(t) &= -\frac{m}{qB_1} \left(v_{2;0} - \frac{E_3}{B_1}\right) \left(1 - \cos\left(\frac{qB_1}{m}t\right)\right)
\end{aligned} \tag{19}
$$

$$
v_3(t) = -\left(v_{2;0} - \frac{E_3}{B_1}\right) \sin\left(\frac{qB_1}{m}t\right) \tag{20}
$$

s_3 in s_2:

$$\begin{aligned}
s_2(t) &= v_{2;0}t + \frac{qB_1}{m}\int_0^t s_3(x)dx \\
&= v_{2;0}t + \frac{qB_1}{m}\int_0^t m\frac{E_3 - B_1 v_{2;0}}{qB_1^2}\left(1 - \cos\left(\frac{qB_1}{m}x\right)\right)dx \\
&= v_{2;0}t + \frac{E_3 - B_1 v_{2;0}}{B_1}\int_0^t \left(1 - \cos\left(\frac{qB_1}{m}x\right)\right)dx \\
&= v_{2;0}t + \frac{E_3 - B_1 v_{2;0}}{B_1}\left(t - \frac{\sin\left(\frac{qB_1}{m}t\right)m}{qB_1}\right) \\
&= v_{2;0}t - \left(v_{2;0} - \frac{E_3}{B_1}\right)\left(t - \frac{m}{qB_1}\sin\left(\frac{qB_1}{m}t\right)\right) \\
s_2(t) &= \frac{E_3}{B_1}t + \frac{m}{qB_1}\left(v_{2;0} - \frac{E_3}{B_1}\right)\sin\left(\frac{qB_1}{m}t\right)
\end{aligned} \tag{21}$$

$$v_2(t) = \frac{E_3}{B_1} + \left(v_{2;0} - \frac{E_3}{B_1}\right)\cos\left(\frac{qB_1}{m}t\right) \tag{22}$$

Für E = 0:

$$\begin{aligned}
s_2(t) &= v_{2;0}t - v_{2;0}\left(t - \frac{\sin\left(\frac{qB_1}{m}t\right)m}{qB_1}\right) \\
&= \frac{mv_{2;0}}{qB_1}\sin\left(\frac{qB_1}{m}t\right)
\end{aligned} \tag{23}$$

$$\begin{aligned}
s_3(t) &= -m\frac{B_1 v_{2;0}}{qB_1^2}\left(1 - \cos\left(\frac{qB_1}{m}t\right)\right) \\
&= -\frac{mv_{2;0}}{qB_1} + \frac{mv_{2;0}}{qB_1}\cos\left(\frac{qB_1}{m}t\right)
\end{aligned} \tag{24}$$

Das ist, wie erwartet, die Parameterform eines Kreises mit Radius $r = \frac{mv_{2;0}}{qB_1}$.

Figure 1: Beispiel mit $v_0 = \frac{E}{B}$

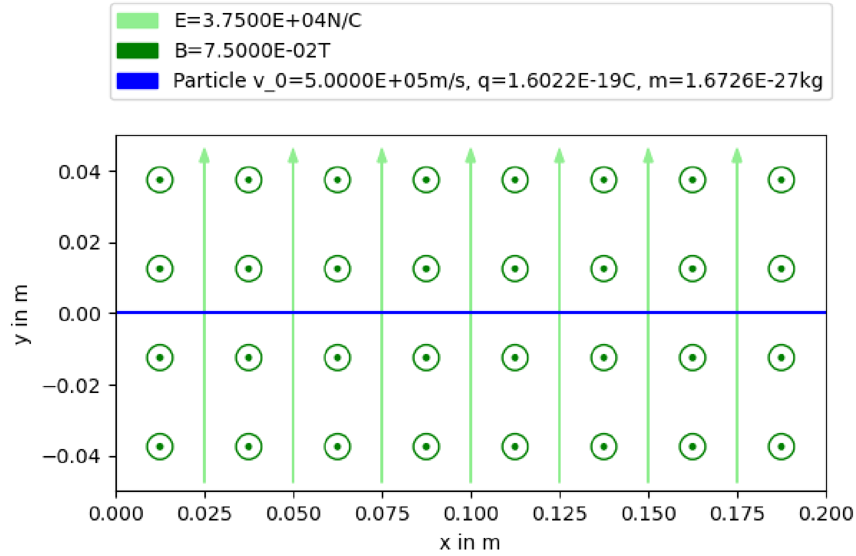


Figure 2: Beispiel mit $v_0 > \frac{E}{B}$

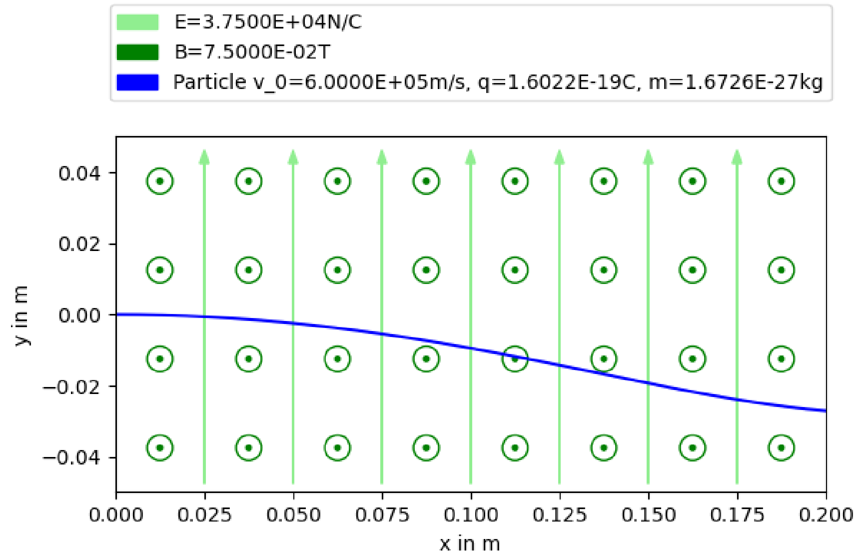


Figure 3: Beispiel mit $v_0 < \frac{E}{B}$

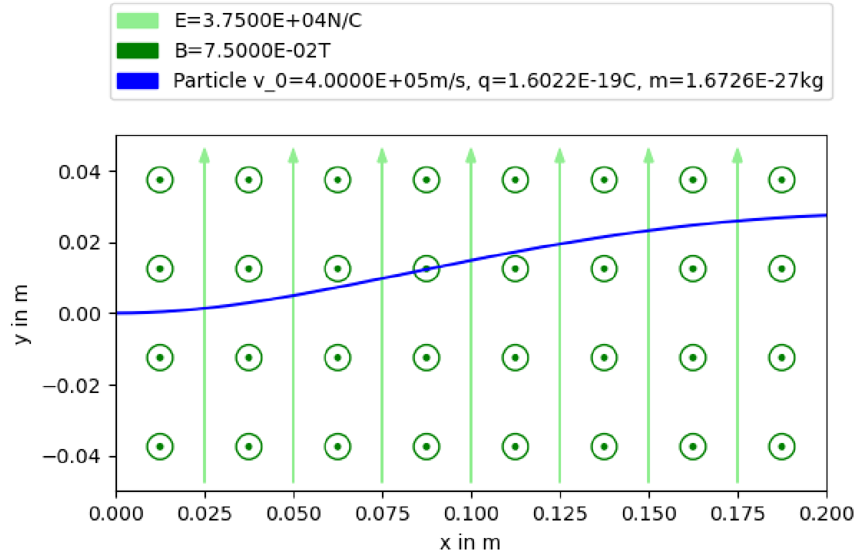


Figure 4: Beispiel mit $v_0 >> \frac{E}{B}$

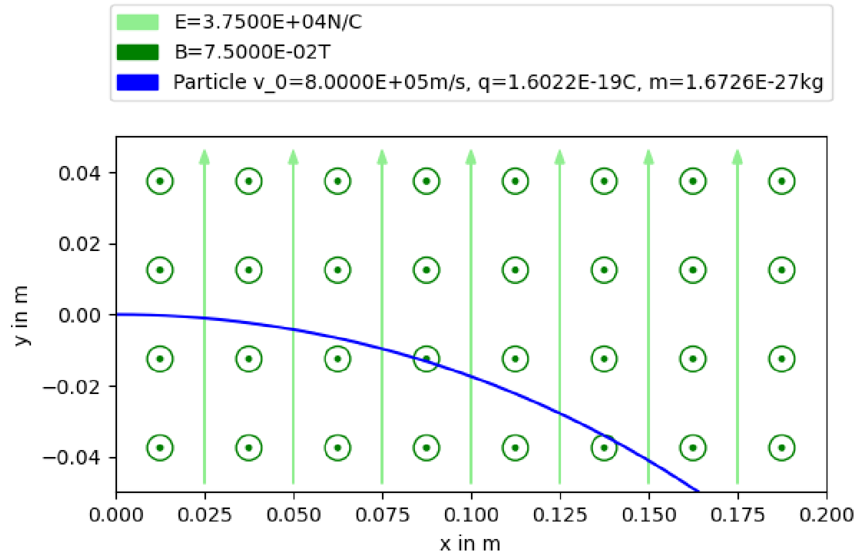


Figure 5: Beispiel mit $v_0 << \frac{E}{B}$

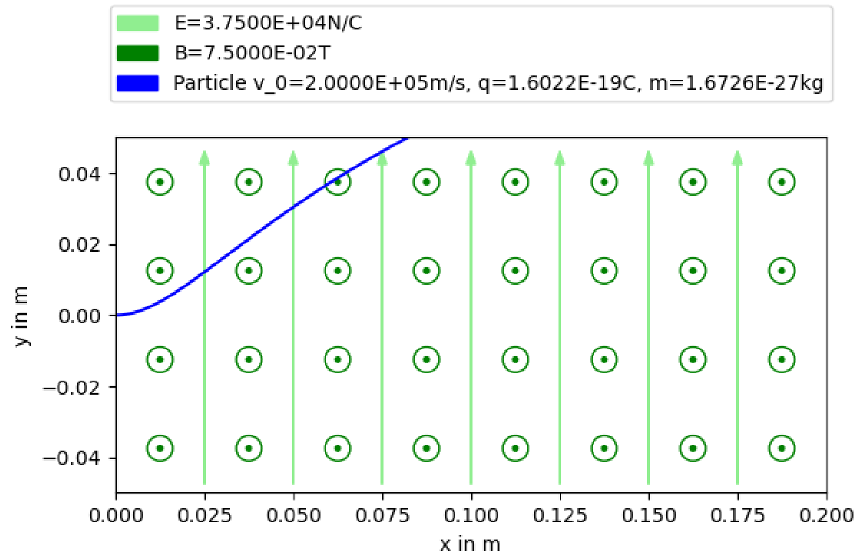


Figure 6: Beispiel mit $E = 0$