Homework 1 Quantitative Risk Management

Group G03
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Question 1

1)

In this particular example with $d = \lambda_1 = 1$, we obtain $V_{1,t} = S_{1,t}$ by applying the formula from example 1. We can also solve for $S_{1,t+1}$ to obtain

$$S_{1,t+1} = S_{1,t}e^{X_{1t}}.$$

Thus,

$$L(t, t+\Delta) = -(V_{1,t+\Delta} - V_{1,t}) = -(S_{1,t+\Delta} - S_{1,t}) = -S_{1,t}(e^{X_{1t}} - 1).$$

This is what we will simulate in the four different cases. The plots can be seen in figure 1-4. We notice that the distributions with higher degrees of freedom look more and more like a normal distribution, but in fact none of these plots are true normal distributions since exponentials of both normal and student distribution is something else.

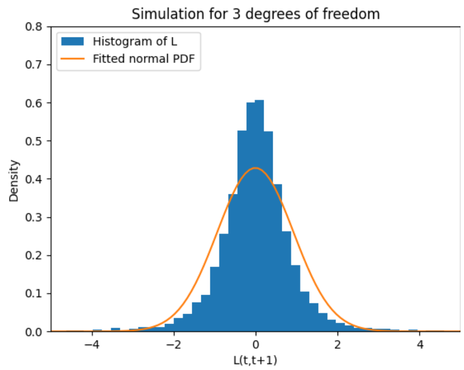

Figure 1: Histogram of 10000 simulations of $L(t, t + \Delta)$ when X_{1t} is a scaled student's t distribution with 3 degrees of freedom. We also fitted a normal distribution based on the mean and variance of the simulations.

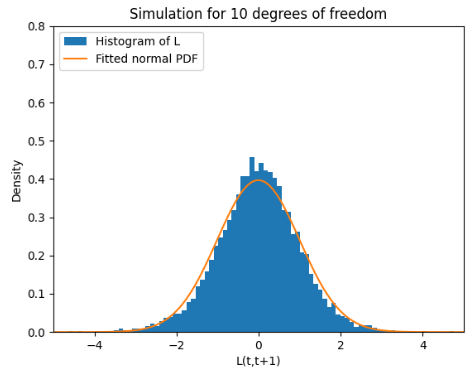

Figure 2: Histogram of 10000 simulations of $L(t, t + \Delta)$ when X_{1t} is a scaled student's t distribution with 10 degrees of freedom.

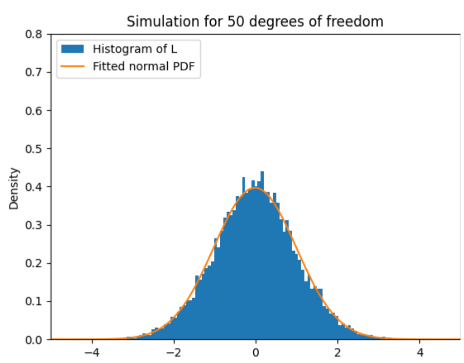

Figure 3: Histogram of 10000 simulations of $L(t, t + \Delta)$ when X_{1t} is a scaled student's t distribution with 50 degrees of freedom.

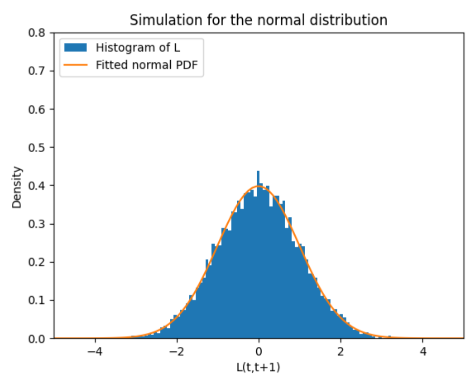

Figure 4: Histogram of 10000 simulations of $L(t, t + \Delta)$ when X_{1t} is a normal distribution with mean 0 and variance $.0^2$.

2)

We have

$$L^{\delta}(t, t+\Delta) = -100X_{1,t}.$$

Since both the student and normal distributions are symmetrical about the y-axis, the above expression has the same distribution as $100X_{1,t}$. For cases a-c, this is a scaled student's t-distribution with the given degrees of freedom with mean

$$\mathbb{E}[L^{\delta}(t, t+\Delta)] = -100 \cdot 0 = 0$$

and standard deviation

$$\sigma = \sqrt{Var(L^{\delta}(t, t+\Delta))} = \sqrt{10000Var(X_{1,t})} = 10000 \cdot .01^2 = 1.$$

In case d, we have the same mean and standard deviation as above but $X_{1,t}$ is normal, meaning $L^{\delta}(t, t+\Delta)$ is the standard normal distribution.

Question 2

1)

We need r and σ to be positive, we can make sure of this by e.g. flipping the sign of negative entries. We have the covariance matrix

$$\text{Cov} = \begin{bmatrix} .01^2 & 0 & -.5 \cdot .01 \cdot 10^{-3} \\ 0 & 10^{-4 \cdot 2} & 0 \\ -.5 \cdot .01 \cdot 10^{-3} & 0 & 10^{-3 \cdot 2} \end{bmatrix}$$

which will be used in our simulation. The result of the simulation can be seen in figure 5.

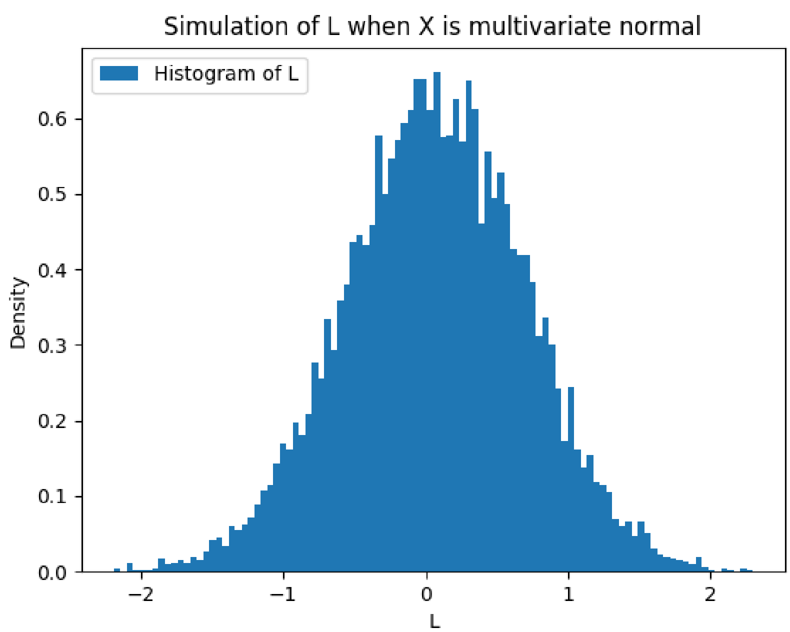

Figure 5: Histogram of 10000 simulations of $L(t, t + \Delta)$ when $\underset{\sim}{X} \sim N((0, 0, 0)^T, \mathrm{Cov})$ with Cov defined above.

2)

We will calculate the greeks theta, delta, rho and vega with respect to the call option price

$$C = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).$$

First of all, we derive the identity

$$\phi(d_2) = \phi(d_1)\frac{S}{K}e^{r\tau}$$

that will be used in several derivations. Here, $\tau = T - t$ is time to maturity, ϕ is the standard normal pdf, Φ is the standard normal cdf, and d_1, d_2 are as defined in the BS framework. We have

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_1-\sigma\sqrt{\tau})^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}e^{d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}} = \phi(d_1)e^{d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}} =$$

$$= \phi(d_1)e^{\log(S/K)+(r+\frac{\sigma^2}{2}\tau)-\frac{\sigma^2\tau}{2}} = \phi(d_1)e^{\log(S/K)+r\tau} = \phi(d_1)\frac{S}{K}e^{r\tau}.$$

Theta

We will make use of the identity derived above. An option's Theta is defined as the its sensitivity to time.

$$\frac{\partial C}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial C}{\partial \tau} = -(\frac{\partial d_1}{\partial \tau}S\phi(d_1) - Kre^{-\tau r}\Phi(d_2) - K\frac{\partial d_2}{\partial \tau}e^{-\tau r}\phi(d_2)) = \{Ke^{-r\tau}\phi(d_2) = S\phi(d_1)\} =$$

$$= S\phi(d_1)(-\frac{\partial d_1}{\partial \tau} + \frac{\partial d_2}{\partial \tau}) - Kre^{-\tau r}\Phi(d_2) = \{-\frac{\partial d_1}{\partial \tau} + \frac{\partial d_2}{\partial \tau} = \frac{\partial}{\partial \tau}(-\sigma\sqrt{\tau}) = -\sigma\frac{1}{2\sqrt{\tau}}\} =$$

$$= -\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}} - Kre^{-\tau r}\Phi(d_2)$$

Delta

An option's Delta is defined as its sensitivity to the underlying asset.

$$\frac{\partial C}{\partial S} = \Phi(d_1) + S\frac{\partial d_1}{\partial S}\phi(d_1) - Ke^{-r\tau}\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial S}\phi(d_2).$$

Now, we make use of the identity again and note that $\frac{\partial d_1}{\partial S} = \frac{\partial(d_1-\sigma\sqrt{\tau})}{\partial S}$. We obtain

$$\Phi(d_1) + S\frac{\partial d_1}{\partial S}\phi(d_1) - Ke^{-r\tau}\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial S}\phi(d_2) = \{Ke^{-r\tau}\phi(d_2) = S\phi(d_1)\} =$$

$$\Phi(d_1) + S\frac{\partial d_1}{\partial S}\phi(d_1) - S\frac{\partial d_1}{\partial S}\phi(d_1) = \Phi(d_1)$$

Rho

An option's Rho is the defined as its sensitivity to the interest rate.

$$\frac{\partial C}{\partial r} = S\frac{\partial d_1}{\partial r}\phi(d_1) + \tau K e^{-r\tau}\Phi(d_2) - K e^{-r\tau}\phi(d_2)\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial r}.$$

Again, we make use of the identity and note that $\frac{\partial d_1}{\partial r} = \frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial r}$. We obtain

$$S\frac{\partial d_1}{\partial r}\phi(d_1) + \tau K e^{-r\tau}\Phi(d_2) - K e^{-r\tau}\phi(d_2)\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial r} = \{K e^{-r\tau}\phi(d_2) = S\phi(d_1)\} =$$

$$= S\frac{\partial d_1}{\partial r}\phi(d_1) + \tau K e^{-r\tau}\Phi(d_2) - S\frac{\partial d_1}{\partial r}\phi(d_1) = \tau K e^{-r\tau}\Phi(d_2)$$

Vega

An option's Vega is defined as its sensitivity to volatility.

$$\frac{\partial C}{\partial \sigma} = S\frac{\partial d_1}{\partial \sigma}\phi(d_1) - K e^{-r\tau}\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial \sigma}\phi(d_2).$$

Once again making use of the same identity,

$$S\frac{\partial d_1}{\partial \sigma}\phi(d_1) - K e^{-r\tau}\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial \sigma}\phi(d_2) = \{K e^{-r\tau}\phi(d_2) = S\phi(d_1)\} =$$

$$= S\frac{\partial d_1}{\partial \sigma}\phi(d_1) - S\frac{\partial(d_1 - \sigma\sqrt{\tau})}{\partial \sigma}\phi(d_1) = \sqrt{\tau}S\phi(d_1)$$

3)

Here, we make use of the greeks derived in 2) and the first order approximation

$$L^{\delta}(t, t+\Delta) = -(\frac{\partial C}{\partial t}\Delta + \frac{\partial C}{\partial S}SX_{1,t} + \frac{\partial C}{\partial r}X_{2,t} + \frac{\partial C}{\partial \sigma}X_{3,t}).$$

The linearized L can be seen in figure 6. To determine which risk factor affected the most, we can calculate the standard deviation of each risk factor's corresdponding term in the linearized L. This can be done analytically, but since we already have 10000 samples, this is easier to do on a computer while still obtaining reliable results.

$$\begin{cases} V(\text{risk1}) = \frac{\partial C}{\partial S}SX_{1,t} = .64^2 \\ V(\text{risk2}) = \frac{\partial C}{\partial r}X_{2,t} = .0053^2 \\ V(\text{risk3}) = \frac{\partial C}{\partial \sigma}X_{3,t} = .037^2 \end{cases}$$

Clearly, the underlying asset had the greatest impact.

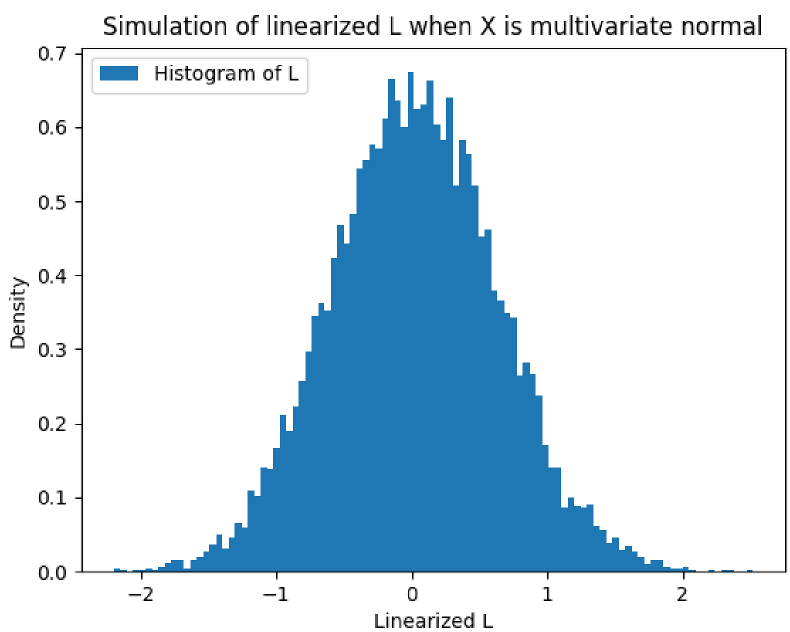

Figure 6: Histogram of 10000 simulations of $L^{\delta}(t, t+\Delta)$ when $\underset{\sim}{X} \sim N((0,0,0)^T, \mathrm{Cov})$ with Cov as defined earlier.

Question 3

Let $X \in N(\mu, \sigma^2)$. Derive the formula:

$$E[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$$

Solution:

$$E[e^X] = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx = \{z = \frac{x-\mu}{\sigma}\} = e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma z} e^{-\frac{1}{2}z^2} dz$$
$$= e^{\mu} e^{\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-\sigma)^2/2} dz = e^{\mu + \frac{1}{2}\sigma^2}$$

Question 4

The first thing the authors of this article presented was that today's risk models do not treat risk as an endogenous process. An example of this is that the actions of market participants cancel each other out in times of calm [page 6]. In times of crisis, on the other hand, this may not be the case since single actions may reinforce the actions of other market participants since, in a crisis, if one individual needs to sell his/her assets, that could indicate that another individual also has the same incentives to sell his/her assets. This will lead to a price fall, and it can no longer be assumed that the market players are heterogeneous. Since VaR relies upon past data, the process will break during crises and will not help estimate risks [page 7].

Another problem with VaR as a measure is that it does not measure the full extent of the distribution beyond the single point that it measures. For example, with a fat tail distribution, the most devastating consequences may be deemed as unlikely with a VaR measure, whereas, for example, an expected shortfall (ES) measure would identify the same distribution as highly risky. The authors argue that when projecting risk, what is really interesting is the distribution of the loss *given* that the unlikely scenario has already occured, even if it's very unlikely. In other words, we would like to know the distribution below some unlikely threshold, say, the bottom 1% quantile [page 9].

Lastly, the VaR regulations might have unexpected consequenses when it comes to liquidity in certain assets. For example, in a market crash, some banks or institutions might have to sell off risky asset to meet regulatory demands. If the market was unregulated, a bank with a higher risk-appetite could purchase these risky assets. But with the VaR regulation, these banks are restricted from doing so. This could lead to a market breakdown, that would otherwise not occur in these markets, were it not for the VaR regulation. [page 7].