Quantitative Risk Management
Assignment 4

Question 1: Suppose $X = \mu + \sqrt{W}Z$ where $Z \sim \mathcal{N}(0,1)$ is independent of W. W is a positive random variable such that $W \in \{k_1, \ldots, k_n\}$ with:

$$\mathbb{P}(W = k_i) = p_i$$

Construct a function of a variable v such that a root of the function v_0 satisfies $v_0 = VaR_\alpha(X)$. Argue that the function you construct has a unique root. (Start by writing down the CDF of X and breaking it up into different terms corresponding to different values of W.)

Question 2: Construct two random variables with zero correlation that are not independent. Prove that they satisfy these requirements.

Question 3: Go back to the daily data for 580 stocks used in Assignment 3 and generate a random portfolio $q \in \mathbb{R}^{580}$. We want to use a rolling window of 1 year to estimate the covariance matrix. To this end, we explore two approaches based on factor models.

1. Choose the factors to be the SPY returns (SP500 ETF), VXX returns (VIX ETF), and GLD returns (Gold ETF). Over each rolling window, compute the betas B matrix and the residuals. Shrink the covariance matrix of the residuals 100% to the diagonal. Then, after computing the covariance matrix as described in the lecture, use it to compute VaR_{95} and VaR_{99}. For both, report the number of breaches you observe and assess the quality of the model.
2. Choose the factors to be the estimated to be the top 5 PCs over the same rolling window and repeat the same analysis as in the previous point.

For the two approaches, plot one year of realised returns, VaR_{95}, VaR_{99} and the breaches you observe. Comment on your results.

Question 4: Copy the following code into the beginning of a Matlab script, or reproduce it in Python:

```
rng(1);
N = 10000;
A = [ 1  0 0 0;
      1  1 0 0;
     -1  2 3 0;
      1 -1 1 1];
x = trnd(5,N,4);
X = (A*x')';
```

This code will generate an array, `X`, which consists of 10,000 rows and 4 columns. Each row represents a single data point observation of a 4-dimensional random vector. In this problem, assume the loss of a portfolio is equal to $L = \sum_{k=1}^{4} X_k$.

1. Based off of the 10,000 observations of $\boldsymbol{X}$, compute $VaR_\alpha(L)$ for $\alpha = 0.95$.
2. What is the eigenvector corresponding to the first principal component of $\boldsymbol{X}$? Can you find a link between the magnitude of some component of this vector and some component in the covariance matrix of $\boldsymbol{X}$? Which of the four components of $\boldsymbol{X}$ you would expect to contribute most to the 1st principle component?

3. Approximate $\boldsymbol{X}$ by using its first two principal components as factors (set the error terms to zero). Write down the steps you take and then recompute $VaR_\alpha(L)$ and compare with the previous result.

The following Matlab functions may be useful for this problem: `cov`, `eig`, `pca`. Be sure to read the documentation on these functions before using them. Matlab may use different conventions for eigenvalue ordering depending on the context.