Quantitative Risk Management
Assignment 10

Question 1: Consider $S_t, \ldots, S_{t+h}$, the daily prices of an asset. As usual, denote as $X_{t+1}, \ldots, X_{t+h}$ the daily log-returns of the same asset.
Provide a formula for the h-period log return $X_{t+h}^{(h)}$ in terms of the daily log-returns and explain why such a return can be considered approximately normal, and under which conditions, even if the daily log-returns are not.

Question 2: Consider the random variable W with Pareto distribution of parameter $\theta > 1$, so that

$$F_W(w) = 1 - w^{-\theta}, \quad w \geq 1.$$

Show that the random vector $X = (X_1, X_2)$ defined by

$$X_1 = \sqrt{W}(Z_1 + Z_2), \quad X_2 = \sqrt{W}(Z_1 - Z_2),$$

with Z_1, Z_2 i.i.d. and $Z_1, Z_2 \sim \mathcal{N}(0, 1)$, is a normal variance mixture and compute its covariance matrix.

Question 3: Consider a loss with distribution function:

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{1+x}, & x \in [1, 3) \\ 1 - \frac{1}{x^2}, & x \geq 3. \end{cases}$$

Compute the value-at-risk for confidence levels 85% and 95% and the expected shortfall at confidence level 85%.

Question 4: Show that the Gaussian copula admits a pdf. In other words, derive the formula in slide 39 of Lecture 7 for a Gaussian copula in dimension 2. Remark: you can derive your expressions in terms of the Gaussian pdf ϕ and the Gaussian cdf Φ.

Question 5: We model two firms' default times as two random variables X, Y defined as follows:

$$X = \min\{\tau_1, \tau\}$$
$$Y = \min\{\tau_2, \tau\},$$

where τ, τ_1 and τ_2 are three independent random times exponentially distributed with parameters $\lambda > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. In other words, we claim that both firms can be subject to the same economic shock which occurs at time τ.

1. What is the survival function of X? What is the survival function of Y?
2. Determine the joint survival probability $\mathbb{P}(X > s, Y > t)$, with $s, t \in \mathbb{R}_+$.
3. Determine the joint CDF of X and Y.
4. Compute the copula $C(u, v)$ of X and Y.
5. Check that it is a valid copula.
6. Derive the density function of the copula.

Question 6: Consider a random variable with CDF

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x^{\theta}}\right)^{\lambda},$$

where $x \geq 0$, $\lambda, \kappa, \theta > 0$. Show that F is in the maximum domain of attraction H_{ξ} and determine ξ as a function of the parameters of the distribution.