HW8 QRM

Group G03
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Q1

A scatter plot of the pseudo copulas can be seen in figure 1. Log-likelihood maximization with the Clayton and Gumbel models resulted in $\theta_{Clayton} = 3.94 \cdot 10^{-9} \approx 0$ and $\theta_{Gumbel} = 1.00$ respectively. Their associated log-likelihoods were virtually zero. The Frank model however, resulted in a log-likelihood of 825.7331 using $\theta_{Frank} = -3.1301$. For this reason, we can conclude that the data was likely generated using a Frank copula.

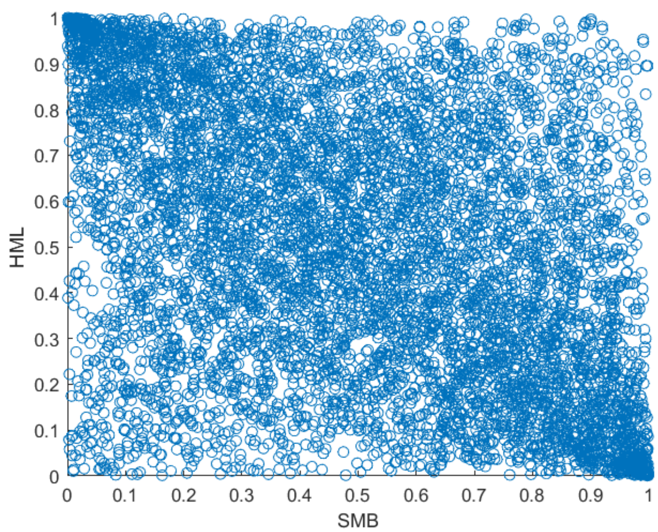

Figure 1: Pseudo copula of the SMB and HML data from the excel file on moodle

Q2

By Sklar's theorem, the copula of X and Y is defined by

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)) = F(x, y)$$

where F is the joint CDF and F_1 and F_2 are the marginal CDF's for X and Y respectively. To make use of this theorem, let us first find the marginals. For F_1 it is simple to see

$$F_1(x) = P(X \leq x) = \Phi(x)$$

where $\Phi(x)$ is the CDF for the standard normal distribution. Thus $F_1^{-1}(u)$ is simply the inverse of this function. Now for F_2, where we will use the fact that for $X \sim \mathcal{N}(0,1)$ we have $\Phi(-x) = 1 - \Phi(x)$. This leads to

$$F_2(y) = P(Y \leq y) = P(ZX \leq y) = P(X \leq y|Z = 1)P(Z = 1) + P(-X \leq y|Z = -1)P(Z = -1)$$

$$= p\Phi(y) + (1-p)(1 - \Phi(-y)) = p\Phi(y) + (1-p)(1 - (1 - \Phi(y))) = p\Phi(y) + (1-p)\Phi(y) = \Phi(y).$$

Thus we have that

$$F_1^{-1}(u) = F_2^{-1}(u) = \Phi(u)^{-1}.$$

Now we need to find the joint distribution of X and Y:

$$F(x, y) = P(X \leq x, Y \leq y) = P(X \leq x, X \leq y|Z = 1)P(Z = 1) + P(X \leq x, -X \leq y|Z = -1)P(Z = -1)$$

$$= P(X \leq x, X \leq y|Z = 1)P(Z = 1) + P(X \leq x, X \geq -y|Z = -1)P(Z = -1)$$

$$pP(X \leq \min\{x, y\}) + (1-p)\max\{\Phi(x) - \Phi(-y), 0\}$$

$$= p\min\{\Phi(x), \Phi(y)\}) + (1-p)\max\{\Phi(x) - (1 - \Phi(y)), 0\}$$

Now by inserting $x = F_1^{-1}(u_1)$ and $y = F_2^{-1}(u_2)$ into this expression we obtain

$$C(u_1, u_2) = p\min\{u_1, u_2\} + (1-p)\max\{u_1 + u_2 - 1, 0\}$$

Which is a parametrization of the comonotonicity copula ($p = 1$) and the countermonotonicity copula ($p = -1$).

Q3

LHS of equation:

$$P(U_1 \leq u_1, ..., U_d \leq u_d) = \int_0^\infty P(U_1 \leq u_1, ..., U_d \leq u_d|V = v)dG(v) =$$

{= u_1 ,..., u_d are conditionally independent given V = }

$$= \int_0^\infty \prod_{i=1}^d F_{U_i|V}(u_i|v)dG(v) = \int_0^\infty e^{-v(\hat{G}^{-1}(u_1)+,...,+\hat{G}^{-1}(u_d)}dG(v) =$$

$$\hat{G}(\hat{G}^{-1}(u_1)+, ... + \hat{G}^{-1}(u_d))$$

Q.E.D

```
clear all
close all

%Read file
table = readtable('Global_3_Factors_Daily.csv','PreserveVariableNames',true);
data1=table2array(table(:,2));
data2=table2array(table(:,3));
len=length(data1);
[sorted_vals1,indices1] = sort(data1,'ascend');
[sorted_vals2,indices2] = sort(data2,'ascend');
r11 = 1:len;
r1(indices1) = r11;
F1=r1/(len+1);
r22 = 1:len;
r2(indices2) = r22;
F2=r2/(len+1);
U=[F1',F2'];
% scatter(F1,F2)
% xlabel('SMB')
% ylabel('HML')

initial_theta=6;
fun=@(theta) gumbel_target_function(U,theta);
[theta,negative_function_value]=fmincon(fun,initial_theta,[],[],[],[],1,Inf) % Change lower bound depending on the family.
function target=target_function(U,theta)
gumbelpdf=copulapdf('Gumbel',U,theta); % Change here based on which copula family is being optimized.
boolean=gumbelpdf>0;                      % To avoid problems in log function.
gumbelpdf=nonzeros(gumbelpdf.*boolean);   % To avoid problems in log function.
target=-sum(log(gumbelpdf));              % Maximizing log-likelihood is the same as minimizing negative log-likelihood.
end
```