Homework 10 Quantitative Risk Management

Group G03
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Question 1

$$X_{t+h}^h = \log\left(\frac{S_{t+h}}{S_t}\right) = \log(S_{t+h}) - \log(S_t) = \sum_{i=t+1}^{t+h} (\log(S_i) - \log(S_{i-1})) =$$
$$= \sum_{i=t+1}^{t+h} \log\left(\frac{S_i}{S_{i-1}}\right) = \sum_{i=t+1}^{t+h} X_i.$$

Assuming that h is large enough, i.e. that we have enough daily log-returns in the sum expression, we can apply the central limit theorem to see that X_{t+h}^h is approximately normal, even if the daily returns X_i are not.

Question 2

A random vector $\mathbf{X}$ follows a normal variance mixture distribution if

$$\mathbf{X} = \mu + \sqrt{W} A \mathbf{Z}$$

where μ is a constant vector, W is a non-negative random variable independent of $\mathbf{Z}$, $\mathbf{\Sigma} = AA^T$ denotes the scale matrix (covariance), and $\mathbf{Z} \sim N(0, I)$ where I is the identity matrix. Clearly, we can write the random vector $(X_1, X_2)^T$ on this form by letting $\mu = (0,0)^T$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since W is non-negative, we conclude that $(X_1, X_2)^T$ is a normal variance mixture.
The independence of W, Z_1 and Z_2, and the fact that

$$\begin{cases} \mathbb{E}[X_1] = \mathbb{E}[X_2] = \mathbb{E}[Z_1] = \mathbb{E}[Z_2] = 0 \\ \mathbb{E}[Z_1^2] = Var(Z_1) = \mathbb{E}[Z_2^2] = Var(Z_2) = 1 \end{cases}$$

gives us

$$\begin{cases} Var(X_1) = \mathbb{E}[(\sqrt{W}(Z_1 + Z_2))^2] = \mathbb{E}[W](\mathbb{E}[Z_1^2] + \mathbb{E}[Z_2^2]) = 2\,\mathbb{E}[W] \\ Var(X_2) = \mathbb{E}[(\sqrt{W}(Z_1 - Z_2))^2] = \mathbb{E}[W](\mathbb{E}[Z_1^2] + \mathbb{E}[Z_2^2]) = 2\,\mathbb{E}[W] \\ Cov(X_1, X_2) = \mathbb{E}[\sqrt{W}(Z_1 + Z_2)\sqrt{W}(Z_1 - Z_2)] = \mathbb{E}[W](\mathbb{E}[Z_1^2] - \mathbb{E}[Z_2^2]) = 0. \end{cases} \tag{1}$$

The pdf of W can be obtained by differentiating the CDF. We obtain

$$f_W(w) = \theta w^{-(\theta+1)},$$

which allows us to find the expectation of W as

$$\mathbb{E}[W] = \int_1^\infty w\theta w^{-(\theta+1)} dw = \int_1^\infty \theta w^{-\theta} dw = \frac{\theta}{\theta - 1}.$$

Now using the expression for $\mathbb{E}[W]$ in (1) finally results in the covariance matix

$$\frac{2\theta}{\theta - 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Question 3

We remember the definition of VaR_α for a given random loss variable X as

$$VaR_\alpha = \inf(x \mid F_X(x) > \alpha).$$

By noting that

$$\lim_{x \to 3^-} 1 - \frac{1}{1+x} = \frac{3}{4} < 0.85 < 0.95$$

and

$$0.85 < 1 - \frac{1}{x^2}\bigg|_{x=3} = \frac{8}{9} < 0.95,$$

we can conclude that $VaR_{0.85} = 3$ by applying the definition. For $VaR_{0.95}$, we solve

$$1 - \frac{1}{(\text{VaR}_{0.95})^2} = 0.95 \implies \text{VaR}_{0.95} = \sqrt{20} \approx 4.47.$$

Now, using the fact that $\text{VaR}_\alpha = 3$ for $0.85 \leq \alpha < \frac{8}{9}$ and $\text{VaR}_\alpha = F^{-1}(\alpha \mid \alpha \geq \frac{8}{9}) = \frac{1}{\sqrt{1-\alpha}}$, we obtain

$$\text{ES}_{0.85} = \frac{1}{1-0.85}\int_{0.85}^{1} \text{VaR}_\alpha d\alpha = \frac{1}{0.15}\left(\int_{0.85}^{\frac{8}{9}} 3dx + \int_{\frac{8}{9}}^{1} \frac{1}{\sqrt{1-\alpha}} d\alpha\right) =$$

$$= \frac{1}{0.15}\left(3(\frac{8}{9} - 0.85) - 2\sqrt{1-\alpha}\bigg|_{\frac{8}{9}}^{1}\right) = \frac{20}{3}\left(\frac{8}{3} - \frac{51}{20} + \frac{2}{3}\right) = \frac{47}{9} \approx 5.22.$$

Question 4

The copula for a 2-dimensional gaussian distribution with mean and covariance matrix

$$\boldsymbol{\mu} = [\mu_1, \mu_2]^\intercal \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}$$

where ρ is the correlation between the random variables, is

$$C^{\text{Gauss}}(u_1, u_2) = \boldsymbol{\Phi}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)).$$

We can obtain the copula density by

$$c(u_1, u_2) = \frac{\boldsymbol{\phi}(\Phi_1^{-1}(u_1), \Phi_2^{-1}(u_2))}{\phi_1(\Phi_1^{-1}(u_1))\phi_2(\Phi_2^{-1}(u_2))}$$

where $\boldsymbol{\phi}$ is the density function of the 2-dimensional gaussian distribution described above, and ϕ_1 and ϕ_2 are marginal distribution functions. Define $v = (\Phi^{-1}(u_1), \Phi^{-1}(u_2))$

$$\frac{\boldsymbol{\phi}(\Phi_1^{-1}(u_1), \Phi_2^{-1}(u_2))}{\phi_1(\Phi_1^{-1}(u_1))\phi_2(\Phi_2^{-1}(u_2))} = \frac{\frac{\exp(-\frac{1}{2}v^\intercal\boldsymbol{\Sigma}^{-1}v)}{\sqrt{(2\pi)^2|\Sigma|}}}{\frac{\exp\left(-\frac{\Phi^{-1}(u_1)^2}{2}\right)}{\sqrt{2\pi}} \frac{\exp\left(-\frac{\Phi^{-1}(u_2)^2}{2}\right)}{\sqrt{2\pi}}} = |\Sigma|^{-\frac{1}{2}} \frac{\exp\left(-\frac{v^\intercal\boldsymbol{\Sigma}^{-1}v}{2}\right)}{\exp\left(-\frac{v^\intercal\boldsymbol{I}_2 v}{2}\right)} = |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{v^\intercal(\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_2)v}{2}\right)$$

where $\boldsymbol{I}_2$ is the 2-dimensional identity matrix.

Question 5

1)

To find the survival functions of X and Y we first need the CDF of the random variables. Let us first consider X:

$$P(X \leq t) = P(\min(\tau_1, \tau) \leq t) = 1 - P(\min(\tau_1, \tau) \geq t) = 1 - P(\tau_1 \geq t \cap \tau \geq t) = 1 - P(\tau_1 \geq t)P(\tau \geq t)$$
$$= 1 - (1 - P(\tau_1 \leq t))(1 - p(\tau \leq t)) = 1 - e^{-(\lambda_1+\lambda)t}$$

where the cumulative distribution of the exponential distributions with parameters λ and λ_1 was inserted in the last step. The derivation for Y is similar. This yields that the survival functions of X and Y respectively are

$$e^{-(\lambda_1+\lambda)t} \quad \text{and} \quad e^{-(\lambda_2+\lambda)t}.$$

2)

By direct computation

$$P(X > s, Y > t) = P(\min(\tau_1, \tau) > s, \min(\tau_2, \tau) > t) = P(\tau_1 > s, \tau > s, \tau_2 > t, \tau > t)$$
$$= P(\tau_1 > s)P(\tau_2 > t)P(\tau > \max(t, s)) = e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda \max(s,t)}$$
$$= e^{-\lambda_1 s - \lambda_2 t - \lambda \max(s,t)}.$$

3)

$$F_{X,Y}(s, t) = P(X \leq s, Y \leq t) = 1 - P(X > s \cup Y > t) = 1 - P(X > s) - P(Y > t) + P(X > s, Y > T)$$

Plug in the results from a and b $\implies$

$$F_{X,Y}(s, t) = 1 - e^{-(\lambda_1+\lambda)s} - e^{-(\lambda_2+\lambda)t} + e^{-\lambda_1 s - \lambda_2 t - \lambda \max(s,t)}$$

4)

Sklar's theorem:

$$C(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))$$

Use our result from 1) $\implies$

$$F_X(t) = 1 - e^{-(\lambda_1+\lambda)t} \implies$$
$$F_X^{-1}(u) = -\frac{1}{\lambda_1 + \lambda} log(1 - u)$$
$$F_Y(t) = 1 - e^{-(\lambda_2+\lambda)t} \implies$$
$$F_Y^{-1}(v) = -\frac{1}{\lambda_2 + \lambda} log(1 - v)$$

Insert our result in the copula $\implies$

$$C(u, v) = F_{X,Y}(-\frac{1}{\lambda_1 + \lambda} log(1 - u), -\frac{1}{\lambda_2 + \lambda} log(1 - v))$$

Use our result form 3 $\implies$

$$C(u, v) =$$
$$= 1 - e^{-(\lambda_1+\lambda)*(-\frac{1}{\lambda_1+\lambda} log(1-u))} - e^{-(\lambda_2+\lambda)*(-\frac{1}{\lambda_2+\lambda} log(1-v))} + e^{-\lambda_1*(-\frac{1}{\lambda_1+\lambda} log(1-u)) - \lambda_2*(-\frac{1}{\lambda_2+\lambda} log(1-v)) - \lambda*max(F_X^{-1}(u), F_Y(v)^{-1})} =$$

$$= 1 - (1 - u) - (1 - v) + e^{-\lambda_1 * F_X^{-1}(u)} e^{-\lambda_2 * F_Y^{-1}(v)} e^{-\lambda*max(F_X^{-1}(u), F_Y(v)^{-1})} =$$
$$= u + v - 1 + e^{-\lambda_1 * F_X^{-1}(u)} e^{-\lambda_2 * F_Y^{-1}(v)} e^{-\lambda*max(F_X^{-1}(u), F_Y(v)^{-1})}$$

Insert $F_X^{-1}(u)$ and $F_Y(v)^{-1}$ $\implies$

$$C(u, v) = u + v - 1 + (1 - u)^{\frac{\lambda_1}{\lambda_1+\lambda}} (1 - v)^{\frac{\lambda_2}{\lambda_2+\lambda}} e^{-\lambda max(-\frac{1}{\lambda_1+\lambda} log(1-u), -\frac{1}{\lambda_2+\lambda} log(1-v))}$$

5)

1) Start by showing that C(u,1) = u:

$$C(u,1) = u + 1 - 1 + (1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}}(1-1)^{\frac{\lambda_2}{\lambda_2+\lambda}} e^{-\lambda max(-\frac{1}{\lambda_1+\lambda}log(1-u), -\frac{1}{\lambda_2+\lambda}log(1-1))} = u$$

2) Show that the copula is strictly increasing:

2.1) Assume that $-\frac{1}{\lambda_1+\lambda}log(1-u) > -\frac{1}{\lambda_2+\lambda}log(1-v)$

$$C(u,v) = u + v - 1 + (1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} e^{-\lambda*(-\frac{1}{\lambda_1+\lambda}log(1-u)))} =$$

$$= u+v-1+(1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} e^{\frac{\lambda}{\lambda_1+\lambda}log(1-u)} = u+v-1+(1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} e^{-\lambda*(-\frac{1}{\lambda_1+\lambda}log(1-u)))} =$$

$$= u+v-1+(1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}}(1-u)^{\frac{\lambda}{\lambda_1+\lambda}} = u+v-1+(1-u)^{\frac{\lambda_1+\lambda}{\lambda_1+\lambda}}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} = u+v-1+(1-u)(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} \implies$$

$$\frac{\partial C(u,v)}{\partial u} = 1 - (1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}} > 0$$

Same thing can be done for the derivative w.r.t. v:

2.2) Assume that $-\frac{1}{\lambda_1+\lambda}log(1-u) < -\frac{1}{\lambda_2+\lambda}log(1-v)$

$$\frac{\partial C(u,v)}{\partial v} = 1 - (1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}} > 0$$

3) Show that $\frac{\partial^2 C(u,v)}{\partial v \partial u} > 0$ We know from 2.2) that

$$\frac{\partial C(u,v)}{\partial v} = 1 - (1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}} \implies$$

$$\frac{\partial^2 C(u,v)}{\partial v \partial u} = \frac{\lambda_1}{\lambda_1+\lambda}(1-u)^{\frac{\lambda_1}{\lambda_1+\lambda}-1} > 0$$

Same could be done with the first assumption from 2.1) and we would have received:

$$\frac{\partial^2 C(u,v)}{\partial v \partial u} = \frac{\lambda_2}{\lambda_2+\lambda}(1-v)^{\frac{\lambda_2}{\lambda_2+\lambda}-1} > 0$$

6)

Since the Copula density is given by: $\frac{\partial^2 C(u,v)}{\partial v \partial u}$ we already solved it in 5) for the different assumptions

Question 6

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x^\theta}\right)^\lambda = 1 - G(x)$$

where $x \geq 0$ and $\lambda, \kappa, \theta > 0$. By the Fisher-Tippett-Gnedenko theorem, we can show that $F \in \text{MDA}(H_\xi)$ for some $\xi > 0$ if and only if

$$G(x) = x^{-1/\xi} L(x)$$

where $L(x)$ is some function slowly varying at infinity. Let us begin by extracting x from the parenthesis of $G(x)$,

$$G(x) = \left(\frac{\kappa}{\kappa + x^\theta}\right)^\lambda = x^{-\theta\lambda}\left(\frac{\kappa}{\frac{\kappa}{x^\theta} + 1}\right)^\lambda = x^{-\theta\lambda} H(x).$$

Now we can see if $H(x)$ is slowly varying at infinity,

$$\lim_{x\to\infty} \frac{H(tx)}{H(x)} = \lim_{x\to\infty} \left(\frac{\frac{\kappa}{\frac{\kappa}{(tx)^\theta}+1}}{\frac{\kappa}{\frac{\kappa}{x^\theta}+1}}\right)^\lambda = \left(\lim_{x\to\infty} \frac{\kappa + x^\theta}{\frac{\kappa}{t} + x^\theta}\right)^\lambda.$$

Since $\theta > 0$ and $\kappa, t > 0$, we have that

$$\lim_{x\to\infty} \frac{\kappa + x^\theta}{\frac{\kappa}{t} + x^\theta} \to \frac{"\infty"}{"\infty"},$$

which means that we can use L'Hôpital's rule to obtain

$$\lim_{x\to\infty} \frac{\kappa + x^\theta}{\frac{\kappa}{t} + x^\theta} = \lim_{x\to\infty} \frac{\theta x^{\theta-1}}{\theta x^{\theta-1}} = 1.$$

Thus, $H(x)$ is slowly varying at infinity, which means that $F \in \text{MDA}(H_\xi)$ where $\xi = \frac{1}{\lambda\theta}$.