Homework 2 Quantitative Risk Management

Group G03
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Question 1

1.1

The estimated values of VaR_α, VaR_α^{mean} and ES_α obtained from the 10,000 simulations of the portefolio are shown respectively in figure 1, along with a graph showing the difference between VaR_α and VaR_α^{mean}.

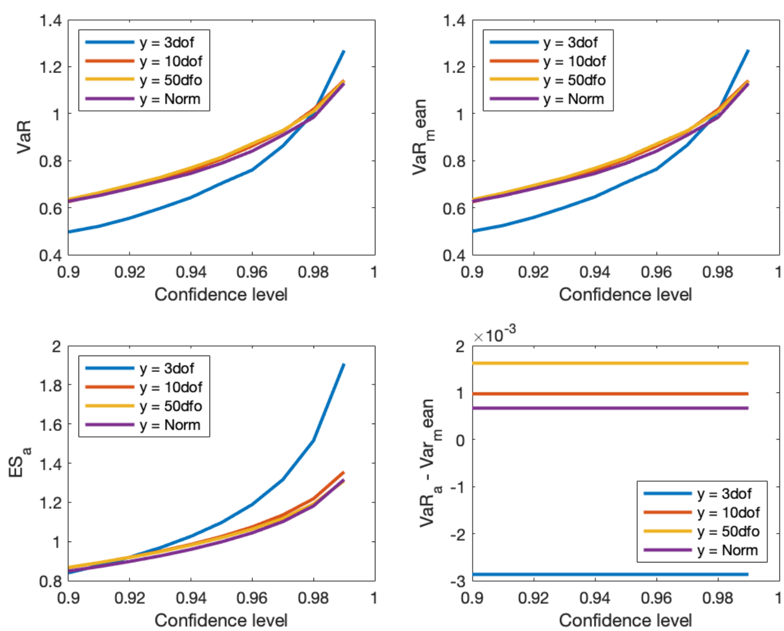

Figure 1: Plots of the results. The bottom right figure coresponds to the second part of this question

1.2a)

Theoretically, $\mu = \mathbb{E}[L]$ should be negative for the following reason

- All four distributions considered are symmetric about the y-axis, this implies that $(e^{X_i} - 1)$ for $i = 1, 2, 3$ has an expected value greater than 1. To understand why, we can calculate the mean when $X_i = -a$ and $X_i = a$. This gives us a mean of

$$\frac{(e^{-a} - 1) + (e^{a} - 1)}{2} = \frac{e^{-a} + e^{a} - 2}{2}.$$

  It is easy to prove that $e^{-a} + e^{a} - 2 > 0$ for all a. Now using the symmetry of the distributions, we have $P(X_i = a) = P(X_i = -a)$ for all a. For this reason, the same argument can be generalised when averaging over all numbers, i.e. taking the expected value.

- Since all X_i are independent and

$$\mu = \mathbb{E}[L] = \mathbb{E}[-(S_1(e^{X_1} - 1) + S_2(e^{X_2} - 1) + S_3(e^{X_3} - 1))],$$

  we can use the result above to see that μ must be negative.

However, due to the small variance of X_i, and relatively low amount of simulations, the bottom right graph for in figure 1 is inconsistent with we would expect. Since, theoretically, $VaR - VaR_{mean} = \mu < 0$, meaning all 4 graphs should be below 0. This would probably not have happened if we ran say $10^6 - 10^7$ simulations.

We can observe from the bottom right graph in figure 1 that the difference between VaR and VaR_{mean} is constant in all levels of alpha. We can also observe that the difference between the two is significantly tiny, with a factor of 10^{-3}. Since the risk factor in our data has a mean of zero combined with low variance, the expected loss is also close to zero, which implies that the difference also should be close to zero, suggesting that our results seem reasonable.

1.2b)

The surprising thing was that three degrees of freedom had the lowest VaR for "small" alpha values and did not surpass the high degrees of freedom until high values of alphas were included. The rationale behind this is probably due to the fat tail behaviour of t-distributions.

Due to the more pronounced fat tail characteristic of lower DoF, we also get a higher ES the lower the degrees of freedom, so the ES graph is not surprising.

Question 2

The estimates of VaR_α, VaR_α^{mean} and ES_α obtained from 10,000 simulations of the portfolio using various methods of estimation are shown in tables 1 and 2 for confidence levels $\alpha = 0.95$ and $\alpha = 0.99$ respectively.

Table 1: The table shows the estimates of VaR_α, VaR_α^{mean} and ES_α for $\alpha = 0.95$. The estimates were obtained using Monte Carlo full revaluation, Monte Carlo on the linearlized loss and the variance-covariance method on 10,000 simulations of the portfolio.

Method	**VaR**	**VaR Mean**	**ES**
Monte Carlo	0.0365	0.00438	0.0376
Linear loss	0.0367	0.00454	0.0378
Variance Covariance	0.0367	0.00453	0.0379

Table 2: The table shows the estimates of VaR_α, VaR_α^{mean} and ES_α for $\alpha = 0.99$. The estimates were obtained using Monte Carlo full revaluation, Monte Carlo on the linearlized loss and the variance-covariance method on 10,000 simulations of the portfolio.

Method	**VaR**	**VaR Mean**	**ES**
Monte Carlo	0.0384	0.0063	0.0392
Linear loss	0.0388	0.0066	0.0396
Variance Covariance	0.0386	0.0064	0.0395

Question 3

The density function of the student's t distribution is given by

$$g_\nu(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}(1+\frac{x^2}{\nu})^{-\frac{\nu+1}{2}}$$

To obtain $ES_\alpha(L)$, we can calculate the expected value of L given that $L > VaR_\alpha$. In other words,

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} x g_\nu(x)dx = \frac{\Gamma(\frac{\nu+1}{2})}{(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\int_{VaR_\alpha}^{\infty} x(1+\frac{x^2}{\nu})^{-\frac{\nu+1}{2}}dx =$$

$$= \{\text{Let } x^2 = y\} = \frac{\Gamma(\frac{\nu+1}{2})}{2(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\int_{VaR_\alpha^2}^{\infty}(1+\frac{y}{\nu})^{-\frac{\nu+1}{2}}dy =$$

$$= \frac{\Gamma(\frac{\nu+1}{2})}{2(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left[-\nu\frac{2}{\nu-1}(1+\frac{y}{\nu})^{-\frac{\nu-1}{2}}\right]_{VaR_\alpha^2}^{\infty}.$$

For $y \to \infty$, the last expression goes to 0 since $\nu > 1$, so what remains is

$$ES_\alpha(L) = \frac{\Gamma(\frac{\nu+1}{2})}{2(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\nu\frac{2}{\nu-1}(1+\frac{VaR_\alpha^2}{\nu})^{-\frac{\nu-1}{2}} =$$

$$= \frac{\nu(1+\frac{VaR_\alpha^2}{\nu})}{\nu-1}(1+\frac{VaR_\alpha^2}{\nu})^{-\frac{\nu+1}{2}}\frac{\Gamma(\frac{\nu+1}{2})}{(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})},$$

where in the last step we factored out $(1+\frac{VaR_\alpha^2}{\nu})$. Now, we use the fact that $VaR_\alpha = t^{-1}(\alpha)$ where t is the cdf of the student's t distribution and identify that

$$(1+\frac{VaR_\alpha^2}{\nu})^{-\frac{\nu+1}{2}}\frac{\Gamma(\frac{\nu+1}{2})}{(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} = g_\nu(VaR_\alpha) = g_\nu(t^{-1}(\alpha)),$$

Thus, we can simplify the expression to

$$ES_\alpha(L) = \frac{\nu(1+\frac{VaR_\alpha^2}{\nu})}{\nu-1}(1+\frac{VaR_\alpha^2}{\nu})^{-\frac{\nu+1}{2}}\frac{\Gamma(\frac{\nu+1}{2})}{(1-\alpha)\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} =$$

$$= \frac{\nu(1+\frac{(t^{-1}(\alpha))^2}{\nu})}{\nu-1}\frac{g_\nu(t^{-1}(\alpha))}{1-\alpha} = \frac{\nu+(t^{-1}(\alpha))^2}{\nu-1}\frac{g_\nu(t^{-1}(\alpha))}{1-\alpha},$$

which is what we wanted to show.

Question 4

a) Gaussian VaR

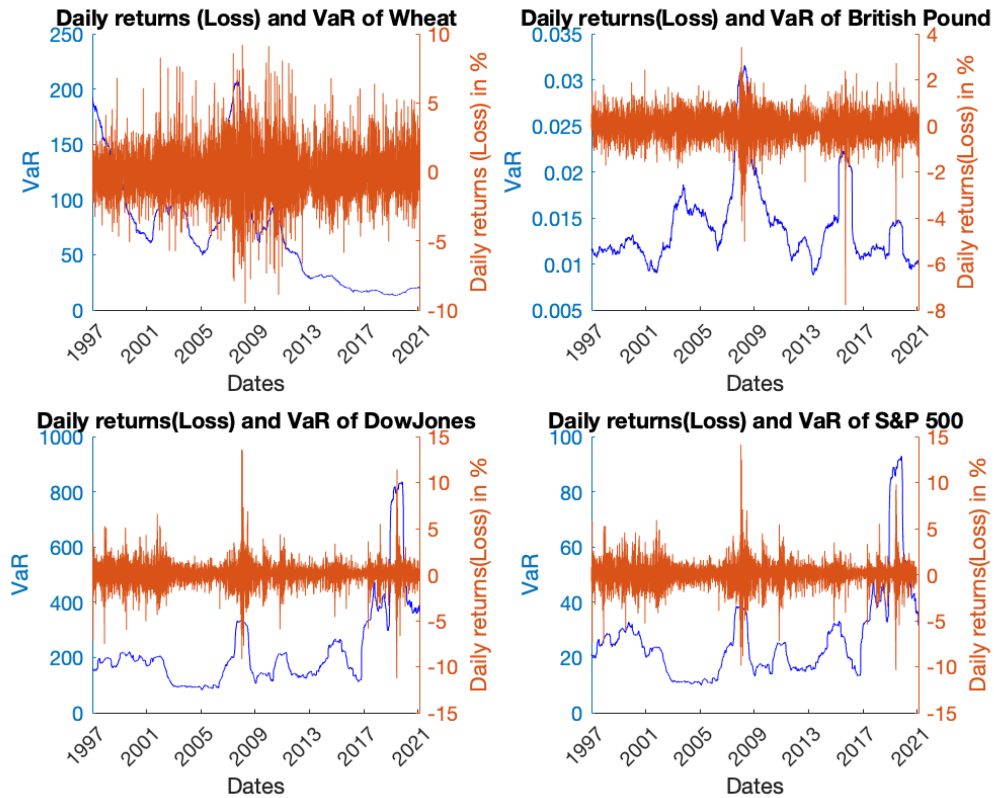

Figure 2:

b) Non-parametric VaR

Comparing the graphs of the two methods, we see that they are similar, as expected. One advantage of the non-parametric method is that we do not make any assumptions on the distribution of the data. For this reason, the non-parametric method may be preferred since returns of commodities, FX-rates and stock indices may not follow a normal distribution. For example, stock market returns are widely regarded as being fat-tail distributed.

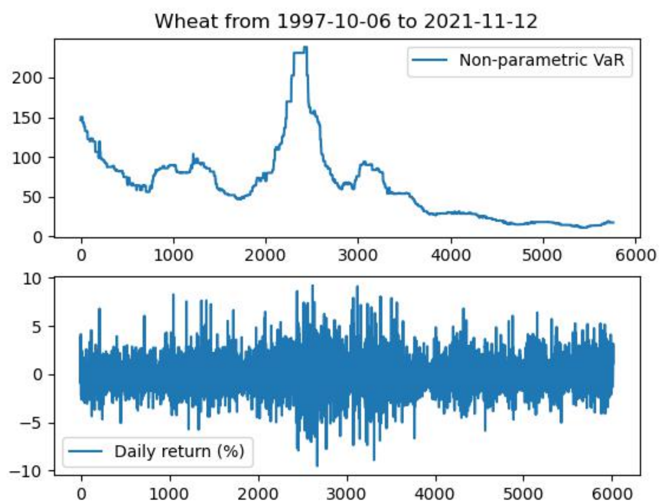

Figure 3:

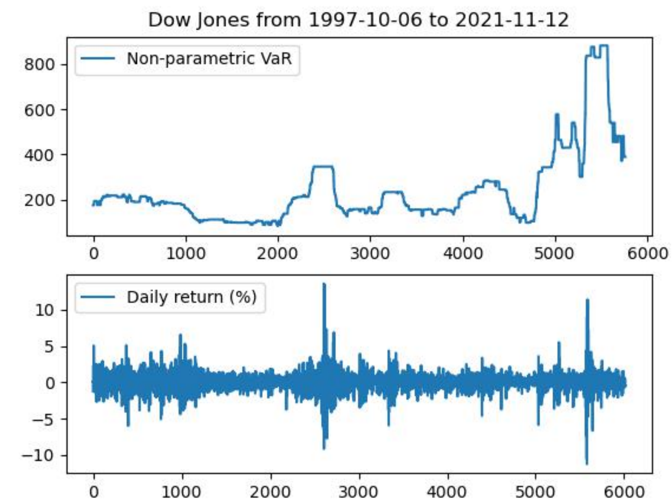

Figure 4:

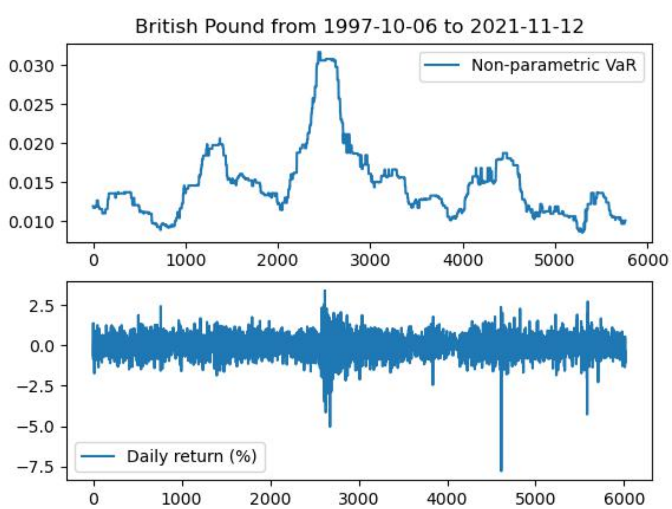

Figure 5:

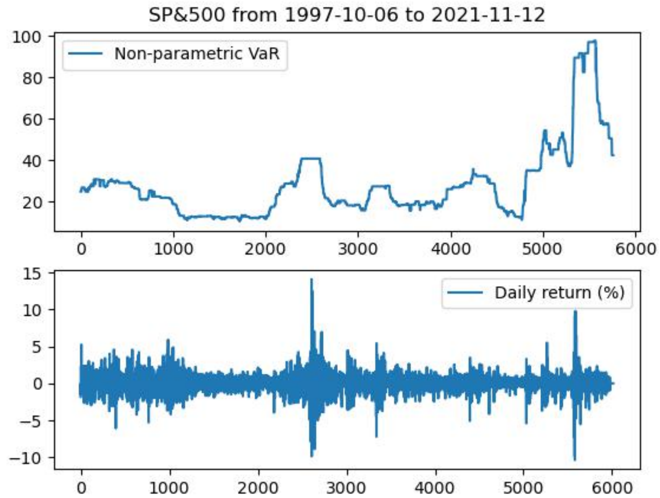
SP&500 from 1997-10-06 to 2021-11-12
Non-parametric VaR
Daily return (%)
100
80
60
40
20
0
1000
2000
3000
4000
5000
6000
15
10
5
0
−5
−10

Figure 6:

1 Matlab code

1.1 Q1

```matlab
S = [100,50,25];
Lambda = [1,3,5];
std = [10^-3,2*10^-3, 3*10^-3];
sim = 10000;
X = [];
L = [];
%risk_factors = log(S);
%Compute VaR, VaR_mean, ES
alpha = [0.90:0.01:0.99];
dof = [3,10,50];
df_1 = zeros(sim,12);
scale = std(1)/ (sqrt(dof(1)/1));
scale_2 = std(2)/(sqrt(dof(2)/8));
scale_3 = std(3)/(sqrt(dof(3)/48));
dof_dif = rdivide(dof,dof-2);
Norm = [];
categories = [];
for i = 1:1:sim
    for x = 1:1:3
        skal =  sqrt(dof_dif)/std(x);
        df_1(i,4*x-3) = trnd(dof(1))/skal(1); % trnd(df) or randn depending
        df_1(i,4*x-2) = trnd(dof(2))/skal(2);
        df_1(i,4*x-1) = trnd(dof(3))/skal(3);
        df_1(i,4*x) = normrnd(0,std(x));
     end
end


x = -(S.*Lambda.*(exp([df_1(:,1),df_1(:,5),df_1(:,9)])-1));
x_2 = -(S.*Lambda.*(exp([df_1(:,2),df_1(:,6),df_1(:,10)])-1));
x_3 = -(S.*Lambda.*(exp([df_1(:,3),df_1(:,7),df_1(:,11)])-1));
x_4 = -(S.*Lambda.*(exp([df_1(:,4),df_1(:,8),df_1(:,12)])-1));

t_3 = sum(x,2);
t_10 = sum(x_2,2);
t_50 = sum(x_3,2);
Norm = sum(x_4,2);
```

```
df_2 = [t_3,t_10,t_50,Norm];


VaR = quantile(df_2,alpha);
subplot(2,2,1)
plot(alpha,VaR,'LineWidth',2.0)
xlabel('Confidence level')
ylabel('VaR')
legend({'y = 3dof','y = 10dof','y = 50dfo','y = Norm'},'Location','northwest')

VaR_mean = VaR - mean(df_2);
subplot(2,2,2)
plot(alpha,VaR_mean,'LineWidth',2.0)
xlabel('Confidence level')
ylabel('VaR_mean')
legend({'y = 3dof','y = 10dof','y = 50dfo','y = Norm'},'Location','northwest')

df_ES = [];
for i = 1:length(alpha)
    for j = 1:4
        var_row_and_col = VaR(i,j);
        df_column = df_2(:,j);
        comparing = df_column((df_column>var_row_and_col));
        df_ES(i,j) = mean(comparing);
    end
end

subplot(2,2,3)
plot(alpha, df_ES,'LineWidth',2.0)
xlabel('Confidence level')
ylabel('ES_a')
legend({'y = 3dof','y = 10dof','y = 50dfo','y = Norm'},'Location','northwest')

diff_VAR_VAR_mean = VaR-VaR_mean;
subplot(2,2,4);
plot(alpha, diff_VAR_VAR_mean,'LineWidth',2.0)
xlabel('Confidence level')
ylabel('VaR_a - Var_mean')
legend({'y = 3dof','y = 10dof','y = 50dfo','y = Norm'},'Location','best')
```

1.2 Q2

```
T = 0.5;
r_t = 0.05;
sig_t = 0.2;
S_t = 100;
K = 100;
Delta = 1/252;
sim = 10000;
Z_1 = log(S_t);
Z_2 = sig_t;
std = [10^-3, 10^-4];
Corr = [-0.5];
mu_vec = [0,0];
a_1 = 0.95;
a_2 = 0.99;
alpha = [a_1,a_2];
COV = [std(1)^2, -0.5*std(1)*std(2); -0.5*std(1)*std(2),
    std(2)^2];
R = mvnrnd(mu_vec, COV, sim);
s_delta = S_t*exp(R(:,1));
r_delta = r_t + R(:,2);

Call_option = call(S_t, r_t, K, sig_t, T);
lambda = -Delta_f(S_t, r_t, K, sig_t, T);
L_f = Call_option - call(s_delta,r_delta,K,sig_t,T-Delta)
     -lambda*(s_delta-S_t) ;



VaR_MC = quantile(L_f,alpha); %VAR MC
VaR_mean_MC = VaR_MC - mean(L_f); %mean VaR MC


df_ES = [];

for j = 1:2
    var_row_and_col = VaR_MC(1,j);
    df_column = L_f(:,1);
    comparing = df_column((df_column>var_row_and_col));
    df_ES(1,j) = mean(comparing);
end

df_ES;%ES_MC


```

```
L_linear = -Theta(S_t, r_t, K, sig_t, T)*Delta - R(:,2)*Vega(S_t, r_t, K, sig_t, T);


VaR_MC_Linear = quantile(L_linear,alpha);%VAR MC linear loss
VaR_mean_MC = VaR_MC_Linear - mean(L_linear); %mean VaR MC loss


df_ES_linear = [];

for j = 1:2
    var_row_and_col = VaR_MC_Linear(1,j);
    df_column = L_f(:,1);
    comparing = df_column((df_column>var_row_and_col));
    df_ES_linear(1,j) = mean(comparing);
end

df_ES_linear;%ES MC Linear


Ex_Linear = mean(L_linear);
Var_Linear = var(L_linear);
STD_Linear = sqrt(Var_Linear);
%how to do inverse of a 1x2??
VaR_VC = Ex_Linear + sqrt(Var_Linear)*norminv(alpha);
VaR_VC_mean = sqrt(Var_Linear)*norminv(alpha);
VaR_VC_ES_1 = Ex_Linear + sqrt(Var_Linear)*normpdf(norminv(alpha(1)))/(1-alpha(1));
VaR_VC_ES_2 = Ex_Linear + sqrt(Var_Linear)*normpdf(norminv(alpha(2)))/(1-alpha(2));

VaR_MC = quantile(L_f,alpha); %VAR MC
VaR_mean_MC = VaR_MC - mean(L_f); %mean VaR MC
df_ES;%ES_MC

VaR_MC_Linear = quantile(L_linear,alpha);%VAR  linear loss
VaR_mean_Linear = VaR_MC_Linear - mean(L_linear); %mean VaR  loss
df_ES_linear;%ES  Linear

VaR_VC = Ex_Linear + sqrt(Var_Linear)*norminv(alpha);
VaR_VC_mean = sqrt(Var_Linear)*norminv(alpha);
```

```
VaR_VC_ES_1 = Ex_Linear + sqrt(Var_Linear)*normpdf(norminv(alpha(1)))/(1-alpha(1));
VaR_VC_ES_2 = Ex_Linear + sqrt(Var_Linear)*normpdf(norminv(alpha(2)))/(1-alpha(2));

%the left number under each column reresent a = 0.95 and the right 0.99
T = table(VaR_MC,VaR_mean_MC,df_ES,VaR_MC_Linear,VaR_mean_Linear,df_ES_linear,VaR_VC,VaR_VC_mean,VaR_VC_ES_1,VaR_VC_ES_2)

function Calloption = call(S_t, r_t, K, sig_t, T)
        d_1 = (log(S_t./K) + (r_t + sig_t.^2 ./2) .* T) ./ (sig_t .* sqrt(T));
        d_2 = d_1 - sig_t.*sqrt(T);
        Calloption = S_t.*normcdf(d_1) - exp(-r_t.*T).*K.*normcdf(d_2);
end

function Delta_f = Delta_f(S, r, K, sig, delta)
    d_1 = (log(S./K) + (r + sig.^2 ./2) .* delta) ./ (sig .* sqrt(delta));
    Delta_f = normcdf(d_1);
end


function Theta = Theta(S_t, r_t, K, sig_t, Delta)
    d_1 = (log(S_t./K) + (r_t + sig_t.^2 ./2) .* Delta) ./ (sig_t .* sqrt(Delta));
    d_2 = d_1 - sig_t.*sqrt(Delta);
    Theta =-S_t*normpdf(d_1)*sig_t/(2*sqrt(Delta)) - r_t*K*exp(-r_t*Delta)*normcdf(d_2);
end

function Vega = Vega(S_t, r_t, K, sig_t, Delta)
    d_1 = (log(S_t./K) + (r_t + sig_t.^2 ./2) .* Delta) ./ (sig_t .* sqrt(Delta));
    d_2 = d_1 - sig_t.*sqrt(Delta);
    Vega = S_t * normpdf(d_1) * sqrt(Delta);
end
```

1.3 Q4a)

```
df = readtable('Data.csv');
dates = df(:,1);
dates_updated = dates(9720:15990,:);


df(:,1)= [];
temp = rmmissing(df);
alpha = 0.95;

%making my table into an array
temp_df = table2array(temp);

%calculating the return
Loss = -diff(temp_df);
returns = 100*(diff(temp_df)./temp_df(1:end-1,:));


Mu= movmean(Loss,256);
STD = movstd(Loss,256);
VaR = Mu + STD*norminv(alpha);


% from here it's just plots...
startlim = '1997-10-07';
endlim = '2021-11-12';
GetRows = isbetween(dates.date, startlim, endlim);
figure
%size(dates.date(GetRows))
%size(VaR(:,1))
subplot(2,2,1)
hold on
yyaxis left
title('Daily returns (Loss) and VaR of Wheat')
xlabel('Dates')
ylabel('VaR')
plot(dates.date(GetRows), VaR(:,1),'b')
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('
    12-Nov-2021'))

yyaxis right
ylabel('Daily returns (Loss) in %')
```

```
plot(dates.date(GetRows), returns(:,1))
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('12-Nov-2021'))
hold off

subplot(2,2,2)
hold on
yyaxis left
title('Daily returns(Loss) and VaR of British Pound')
xlabel('Dates')
ylabel('VaR')
plot(dates.date(GetRows), VaR(:,4),'b')
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('12-Nov-2021'))

yyaxis right
ylabel('Daily returns(Loss) in %')
plot(dates.date(GetRows), returns(:,4))
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('12-Nov-2021'))
hold off

subplot(2,2,3)
hold on
yyaxis left
title('Daily returns(Loss) and VaR of DowJones')
xlabel('Dates')
ylabel('VaR')
plot(dates.date(GetRows), VaR(:,3),'b')
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('12-Nov-2021'))

yyaxis right
ylabel('Daily returns(Loss) in %')
```

```
plot(dates.date(GetRows), returns(:,3))
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('
    12-Nov-2021'))
hold off

subplot(2,2,4)
hold on
yyaxis left
title('Daily returns(Loss) and VaR of S&P 500')
xlabel('Dates')
ylabel('VaR')
plot(dates.date(GetRows), VaR(:,6),'b')
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('
    12-Nov-2021'))

yyaxis right
ylabel('Daily returns(Loss) in %')
plot(dates.date(GetRows), returns(:,6))
grid
xlim([datetime('07-Oct-1997') datetime('12-Nov-2021')])
xtickformat('yyyy')
xticks(datetime('07-Oct-1997') : calyears(4) : datetime('
    12-Nov-2021'))
hold off
```

1.4 Q4b)

Python code:

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
from matplotlib.dates import (YEARLY, DateFormatter,
                              rrulewrapper, RRuleLocator,
                                  drange)
import datetime
df= pd.read_csv('futures.csv')

start_index=9718+256
VaR_nonparametric=[]
price=df.iloc[:,6]
returns=[]
index = 256 * .95+1
def returns_and_VaR(price):
    returns = []
    VaR_nonparametric = []
    diff = [-(x - price[i - 1]) for i, x in enumerate(
        price) if i > 0]
    for i in range(len(diff)-1):
        returns.append(100*(price[i+1]-price[i])/price[i
            ])

    for i in range(len(price)-256-1):
        sorted_L=sorted(diff[i:i+256])
        VaR_nonparametric.append(sorted_L[int(index)])
    return VaR_nonparametric,returns
Wheat_VaR,Wheat_returns=returns_and_VaR(df.iloc[:,1])
DJ_VaR,DJ_returns=returns_and_VaR(df.iloc[:,3])
BP_VaR,BP_returns=returns_and_VaR(df.iloc[:,4])
SP500_VaR,SP500_returns=returns_and_VaR(df.iloc[:,6])


plt.subplot(2,1,1)
plt.title('Wheat from 1997-10-06 to 2021-11-12')
plt.plot(Wheat_VaR[start_index:],label='Non-parametric
    VaR')
plt.legend()
#plt.plot(mean_wheat[start_index+256:]+Zscore*std_wheat[
    start_index+256:],label='Gaussian VaR')
plt.subplot(2,1,2)
```

```
plt.plot(Wheat_returns[start_index:],label='Daily return (%)')
plt.legend()
plt.plot()
plt.show()

plt.subplot(2,1,1)
plt.title('Dow Jones from 1997-10-06 to 2021-11-12')
plt.plot(DJ_VaR[start_index:],label='Non-parametric VaR')
plt.legend()
#plt.plot(mean_wheat[start_index+256:]+Zscore*std_wheat[start_index+256:],label='Gaussian VaR')
plt.subplot(2,1,2)
plt.plot(DJ_returns[start_index:],label='Daily return (%)')
plt.legend()
plt.plot()
plt.show()

plt.subplot(2,1,1)
plt.title('British Pound from 1997-10-06 to 2021-11-12')
plt.plot(BP_VaR[start_index:],label='Non-parametric VaR')
plt.legend()
#plt.plot(mean_wheat[start_index+256:]+Zscore*std_wheat[start_index+256:],label='Gaussian VaR')
plt.subplot(2,1,2)
plt.plot(BP_returns[start_index:],label='Daily return (%)')
plt.legend()
plt.plot()
plt.show()

plt.subplot(2,1,1)
plt.title('SP&500 from 1997-10-06 to 2021-11-12')
plt.plot(SP500_VaR[start_index:],label='Non-parametric VaR')
plt.legend()
#plt.plot(mean_wheat[start_index+256:]+Zscore*std_wheat[start_index+256:],label='Gaussian VaR')
plt.subplot(2,1,2)
plt.plot(SP500_returns[start_index:],label='Daily return (%)')
plt.legend()
plt.plot()
plt.show()
```