# Exercise 1: Vector Analysis 1

**Exercise 1:**

A particle with charge $q$ moves with velocity $\mathbf{v}$ in a uniform electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. The force acting on the particle is then given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

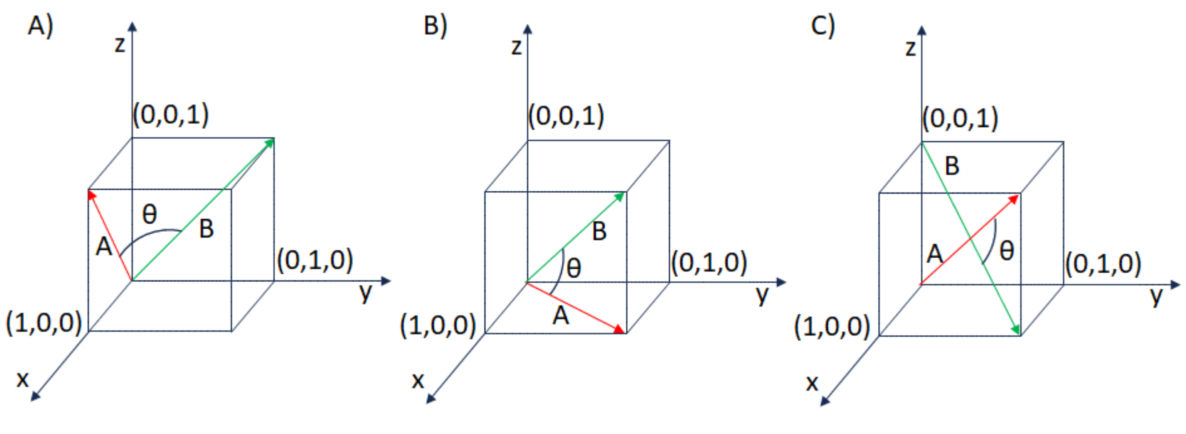
Suppose:

- the particle moves in the positive $x$-direction,
- the magnetic field points in the positive $z$-direction,
- the electric field points in the negative $y$-direction.

**Questions:**

1. What is the direction of the magnetic force $\mathbf{F}_{\text{mag}} = q(\mathbf{v} \times \mathbf{B})$ on the particle if $q > 0$?
2. What is the direction of the magnetic force $\mathbf{F}_{\text{mag}} = q(\mathbf{v} \times \mathbf{B})$ if $q < 0$?
3. If both fields are present, for what sign of charge can the total force be zero?

**Exercise 2:** Calculate the angle $\theta$.




***Hint:*** *Compare the abstract and component form to find $\theta$.*

*Abstract form*: $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$.

*Component form:* $\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}) = A_xB_x+A_yB_y+A_zB_z$

**Exercise 3:**

Consider a scalar field $\phi(x, y)$ and a vector field $\mathbf{F}(x, y)$ defined in the $xy$-plane as:

$$\phi(x, y) = x^2 + y^2$$
$$\mathbf{F}(x, y) = (x - y)\,\hat{\mathbf{x}} + (x + y)\,\hat{\mathbf{y}}$$

**Questions**

1. Gradient (Steepness)
   (a) Find $\nabla\phi$.
   (b) At the point (1,1), determine the direction of $\nabla\phi$.
   (c) Explain the physical meaning of the gradient in (b).
2. Divergence (Spreading)
   (a) Compute $\nabla \cdot \mathbf{F}$.
   (b) Is the field acting as a source, a sink, or neither? What does this indicate physically?
3. Curl (Swirling)
   (a) Compute $\nabla \times \mathbf{F}$.
   (b) What is the direction of the curl vector? What type of motion does this correspond to physically?

**Exercise 4:** Gradient and divergence.

a) Considering the definition of the Nabla ($\nabla$) operator find the gradient of: $v_1(x, z) = x^3 + z^2$

b) Considering the definition of the Nabla ($\nabla$) operator find the divergence of the following vector function: $\mathbf{v}_2 = (2 + x^2)\hat{\mathbf{x}} + (3y + z^2)\hat{\mathbf{z}}$

c) Considering the definition of the Nabla ($\nabla$) operator find the curl of the following vector function: $\mathbf{v}_3 = (2 + yx^2)\hat{\mathbf{x}} + (xy + xz^2)\hat{\mathbf{z}}$

***Hint:***

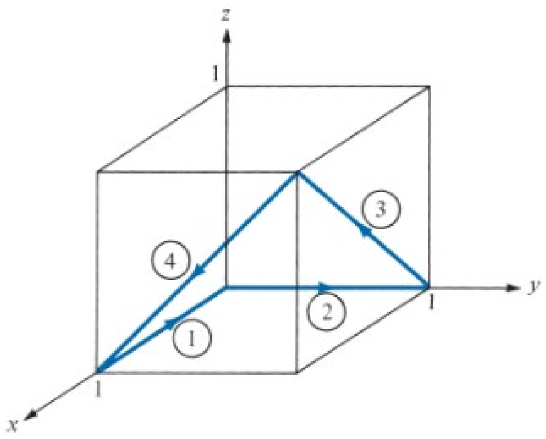
$$\text{Gradient: } \nabla T = \frac{\partial T}{\partial x}\,\hat{x} + \frac{\partial T}{\partial y}\,\hat{y} + \frac{\partial T}{\partial z}\,\hat{z}$$

$$\text{Divergence: } \nabla \cdot T = \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z}$$

$$\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

**Exercise 5:**

Given that $\mathbf{F} = x^2\hat{\mathbf{x}} - xz\hat{\mathbf{y}} - y^2\hat{\mathbf{z}}$, calculate the circulation of $\mathbf{F}$ around the (closed) path shown in the figure.




***Hint:*** *The circulation around the closed path L is found by summing the line integrals along the four segments marked in the figure:*

$$\oint_L \mathbf{F}\cdot d\mathbf{l} = \sum_{i=1}^{4}\int_{L_i} \mathbf{F}\cdot d\mathbf{l}$$

*Make sure to integrate in the correct direction, i.e. in the direction of the arrows in the figure.*

**Exercise 6:**

Find the components of the unit normal vector $\hat{\mathbf{n}}$ in the Cartesian coordinates $(x, y, z)$ for the following surfaces ($i$ and $ii$) in the figure: