ELEMENTS FROM VECTOR ANALYSIS

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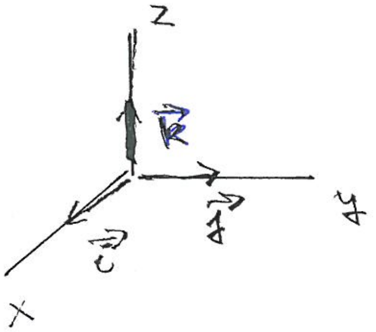
Reference : E. Kreyszig, "Advanced Engineering Mathematics", Chapter 8. (4.ed.)

I. GRADIENT



Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors along the x, y, z-axis. The radius vector $\vec{r}$ is $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, often written as $\vec{r} = (x, y, z)$.

Nabla operator is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

(alternatively written as $\vec{\nabla}$).

If $f(\vec{r})$ is a scalar function, the gradient of f is

$$\text{grad} f \equiv \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

Thus ∇f is a vector, with components $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

Example: Find the directional derivative $\partial f / \partial s$ of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P: (2,1,3)$ in the direction of the vector $\vec{a} = \vec{i} - 2\vec{k}$.

Solution: We calculate $\nabla f = 4x\vec{i} + 6y\vec{j} + 2z\vec{k}$,

thus at P $\nabla f = 8\vec{i} + 6\vec{j} + 6\vec{k}$.

Since the magnitude of $\vec{a}$ is $|\vec{a}| = \sqrt{1+4} = \sqrt{5}$

the unit vector $\hat{a}$ in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}}{\sqrt{5}} - \frac{2\vec{k}}{\sqrt{5}}.$$

Therefore,

$$\underline{\frac{\partial f}{\partial s}} = \hat{a}\cdot\nabla f = \left(\frac{\vec{i}}{\sqrt{5}} - \frac{2\vec{k}}{\sqrt{5}}\right)\cdot\left(8\vec{i}+6\vec{j}+6\vec{k}\right) = \underline{-\frac{4}{\sqrt{5}}}$$

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If f is a constant C, then $\underline{\nabla C = 0}$.

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If ∇ is multiplied with itself, one obtains

$$\nabla^2 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\Rightarrow \quad \underline{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \quad \text{Often called the Laplacian.}$$

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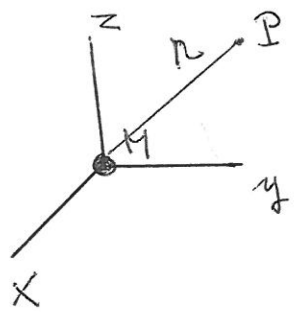
Example:



Given the gravitational potential ϕ at a point P from a mass M situated at the origin:

$$\phi(r) = -\frac{GM}{r}. \qquad r = \sqrt{x^2+y^2+z^2}$$

Here G is Newton's gravitational constant.

Calculate the force $\vec{F} = -\nabla\phi$ on a unit mass at P.

Solution: Calculate $\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{2x}{\sqrt{\;}} = -\frac{x}{r^3}$,

Correspondingly $\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}$, $\frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}$, $\Rightarrow$

$$\nabla\phi = -GM\left(-\frac{x}{r^3}\vec{i} - \frac{y}{r^3}\vec{j} - \frac{z}{r^3}\vec{k}\right) = GM\frac{\vec{r}}{r^3}$$

Thus $\underline{\vec{F} = -\nabla\phi = -GM\frac{\vec{r}}{r^3}}$. Newtonian gravitational force.

II. DIVERGENCE

A vector $\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$ is given.

The divergence of it is

$$\operatorname{div}\vec{V} \equiv \nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad \text{a scalar quantity.}$$

By inserting ∇f as vector, we obtain

$$\operatorname{div}\operatorname{grad} f = \nabla\cdot(\nabla f) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Example: If $\vec{V} = 3xz\vec{i} + 2xy\vec{j} - yz^2\vec{k}$, then

$$\nabla\cdot\vec{V} = 3z + 2x - 2yz$$

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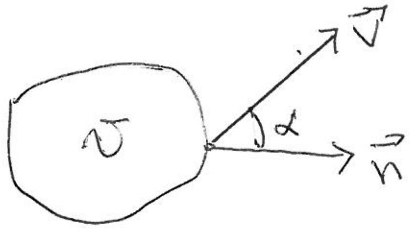
Gauss' theorem:



Let $\vec{V} = (u, v, w)$ be a fluid velocity, which at the boundary of the volume v makes an angle α with the normal vector $\vec{n}$.

Then the volume integral of $\nabla\cdot\vec{V}$ can be converted into a surface integral over the whole surface:

$$\underset{\text{VOLUME}}{\int} \nabla\cdot\vec{V}\, dv = \underset{\text{SURFACE}}{\oint} \vec{V}\cdot\vec{n}\, dA$$

Gauss' theorem.

One may write $V_n = \vec{V}\cdot\vec{n}$, the component of $\vec{V}$ along $\vec{n}$.

Extensions: Green's theorems

Start with $\nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$ where f and g are scalar functions (compare with the rule for ordinary differentiation of a product).

Then

$$\int_{\text{VOLUME}} (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV = \oint_{\text{SURFACE}} f\,\nabla g\cdot\vec{n}\,dA = \oint_{\text{SURFACE}} f\frac{\partial g}{\partial n}\,dA$$

Green's 1. theorem.

Now interchange $f \Leftrightarrow g$:

$$\int_{\text{VOLUME}} (g\nabla^2 f + \nabla g\cdot\nabla f)\,dV = \oint_{\text{SURFACE}} g\frac{\partial f}{\partial n}\,dA$$

Subtract the equations to get

$$\int_{\text{VOLUME}} (f\nabla^2 g - g\nabla^2 f)\,dV = \oint_{\text{SURFACE}} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dA$$

Green's 2. theorem.

III CURL

The curl of a vector $\vec{V} = (u, v, w)$ can in Cartesian coordinates be written as a determinant

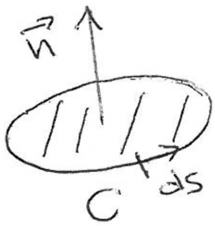
$$\text{curl}\,\vec{V} \equiv \nabla\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} =$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Note that curl grad of a scalar f is always zero:

$$\text{curl grad} f = \nabla\times(\nabla f) = 0.$$

Stokes' theorem:

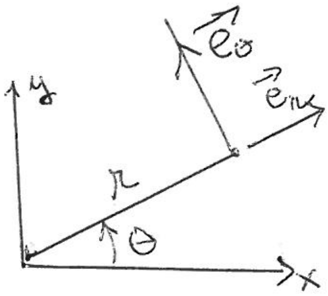


Consider a closed contour C around a surface characterized by a normal vector $\vec{n}$.
Define the line integral $\Gamma = \oint_C \vec{V}\cdot d\vec{s}$

Then the integral of $\nabla\times\vec{V}$ over the surface can be converted to a surface integral,

$$(1) \quad \int_{\text{SURFACE}} (\nabla\times\vec{V})\cdot\vec{n}\,dA = \oint_C \vec{V}\cdot d\vec{s} \qquad \text{Stokes' theorem.}$$

It is here assumed that no singularities are encountered.



Note on polar coordinates r, θ

$\vec{V} = V_r \vec{e}_r + V_\theta \vec{e}_\theta$, where $\vec{e}_r$, $\vec{e}_\theta$ are unit vectors in the r- and θ- directions.

Then

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial \theta}$$

$$\nabla f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta$$

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial (r V_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta}$$

$$(\nabla \times \vec{V})_z = \frac{1}{r} \frac{\partial}{\partial r}(r V_\theta) - \frac{1}{r} \frac{\partial V_r}{\partial \theta}$$

Often useful to note that

$$\frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r, \quad \frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta$$

Example: Check of Stokes' theorem, Eq. ① page 5, in the case of rigid rotation around the z-axis, where $\vec{V} = r\omega \vec{e}_\theta$ (ω constant).

Solution: $(\nabla \times \vec{V})_z = \frac{1}{r} \frac{\partial}{\partial r}(r V_\theta) = \frac{1}{r} \frac{\partial}{\partial r}(r^2 \omega) = 2\omega.$

Left hand side of Eq. ①:

$$\text{LHS} = \int (\nabla \times \vec{V})_z \, dA = 2\omega \int dA = 2\omega \cdot \pi r^2,$$

Right hand side:

$$\text{RHS} = \oint \vec{V} \cdot d\vec{s} = \oint V_\theta \, ds = \int_0^{2\pi} r\omega \cdot r \, d\theta = 2\omega \pi r^2.$$

Thus LHS = RHS, as it should.