Chapter 6.4

Vi bruker at

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}, \qquad n = 0, 1, \dots, \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, \\
\mathcal{L}\{y'\} &= sY - y(0), \\
\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) \\
&= s^2 Y - sy(0) - y'(0), \\
\mathcal{L}\{\delta(t-a)\} &= e^{-as}, \\
\mathcal{L}\{\cos \omega t\} &= \frac{s}{s^2+\omega^2}, \\
\mathcal{L}\{\sin \omega t\} &= \frac{\omega}{s^2+\omega^2}, \\
\mathcal{L}\{e^{at} f(t)\} &= F(s-a), \\
\mathcal{L}\{f(t-a)u(t-a)\} &= e^{-as}F(s).
\end{aligned}$$

6.4:4 Finn, og tegn løsningen til IVP.

$$y'' + 16y = 4\delta(t - 3\pi), \qquad y(0) = 2,\ y'(0) = 0.$$

Løsning:
La $Y(s) = \mathcal{L}\{y\}$. Vi flytter over, transformerer og får

$$\begin{aligned}
0 &= s^2 Y - sy(0) - y'(0) + 16Y - 4e^{3\pi s} \\
&= (s^2 + 16)Y - 2s - 4e^{-3\pi s}.
\end{aligned}$$

Dvs.

$$Y(s) = 2\frac{s}{s^2+4^2} + e^{-3\pi s}\frac{4}{s^2+4^2}$$

som er transformasjonen av

$$\begin{aligned}
y(t) &= 2\cos(4t) + \sin(4(t-3\pi))u(t-3\pi) \\
&= 2\cos(4t) + \sin(4t)u(t-3\pi).
\end{aligned}$$

$\boxed{\text{6.4:10}}$ Finn, og tegn løsningen til IVP.

$$y'' + 5y' + 6y = \delta(t - {}^1\!/\!{}_2\pi) + u(t - \pi)\cos t, \qquad y(0) = 0 = y'(0).$$

Løsning:
Vi har at $\mathcal{L}\left\{u(t-\pi)\cos t\right\} = -e^{-\pi s}\frac{s}{s^2+1}$ fordi $\cos t = -\cos(t-\pi)$.

$$\begin{aligned}
0 &= s^2Y - sy(0) - y'(0) + 5sY - y(0) + 6Y - e^{-1/2\pi s} + e^{-\pi s}\frac{s}{s^2+1} \\
&= (s^2+5s+6)Y - e^{-1/2\pi s} + e^{-\pi s}\frac{s}{s^2+1} \\
&\Rightarrow \\
Y(s) &= e^{-1/2\pi s}\frac{1}{(s+2)(s+3)} - e^{-\pi s}\frac{s}{(s^2+1)(s+2)(s+3)}.
\end{aligned}$$

Delbrøkoppspaltning: Vi viser utregningene for den andre brøken. La A, B, C og D være konstanter. Da er

$$\begin{aligned}
\frac{s}{(s^2+1)(s+2)(s+3)} &= \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3} \\
&\iff \\
s &= As(s+2)(s+3) + B(s+2)(s+3) \\
&\quad + C(s^2+1)(s+3) + D(s^2+1)(s+2) \\
&= (A+C+D)s^3 + (5A+B+3C+2D)s^2 \\
&\quad + (6A+5B+C+D)s + 6B + 3C + 2D.
\end{aligned}$$

Dette gir det lineære ligningssystemet

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 5 & 1 & 3 & 2 \\ 6 & 5 & 1 & 1 \\ 0 & 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

med løsning $A = B = 1/10$, $C = -2/5$ og $D = 3/10$. Vi har nå

$$\begin{aligned}
Y(s) &= e^{-1/2\pi s}\frac{1}{(s+2)(s+3)} - e^{-\pi s}\frac{s}{(s^2+1)(s+2)(s+3)} \\
&= e^{-1/2\pi s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) - \frac{1}{10}e^{-\pi s}\left(\frac{s+1}{s^2+1} - 4\frac{1}{s+2} + 3\frac{1}{s+3}\right)
\end{aligned}$$

som er transformasjonen til

$$\begin{aligned}
y(t) &= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right)u(t-\pi/2) \\
&\quad - \frac{1}{10}\left(\cos(t-\pi) + \sin(t-\pi) - 4e^{-2(t-\pi)} + 3e^{-3(t-\pi)}\right)u(t-\pi) \\
&= \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right)u(t-\pi/2) \\
&\quad + \frac{1}{10}\left(\cos t + \sin t + 4e^{-2(t-\pi)} - 3e^{-3(t-\pi)}\right)u(t-\pi).
\end{aligned}$$

$\boxed{\text{6.4:12}}$ Finn, og tegn løsningen til IVP.

$$y'' + 2y' + 5y = 25t - 100\delta(t-\pi), \qquad y(0) = -2,\, y'(0) = 5.$$

Løsning:

$$\begin{aligned}
0 &= s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + 5Y - 25/s^2 + 100e^{-\pi s} \\
&= (s^2+2s+5)Y + 2s - 5 + 4 - 25/s^2 + 100e^{-\pi s} \\
&\Rightarrow \\
Y(s) &= \frac{1-2s}{s^2+2s+5} + \frac{25}{s^2(s^2+2s+5)} - 100\frac{e^{-\pi s}}{s^2+2s+5} \\
&= \frac{1}{s^2}\frac{-2s^3+s^2+25}{s^2+2s+5} - 100\frac{e^{-\pi s}}{(s+1)^2+4}.
\end{aligned}$$

Ved polynomdivisjon finner vi at $\frac{-2s^3+s^2+25}{s^2+2s+5} = -2s+5$. Dette gir

$$\begin{aligned}
Y(s) &= \frac{-2s+5}{s^2} - 100\frac{e^{-\pi s}}{(s+1)^2+4} \\
&= -\frac{2}{s} + \frac{5}{s^2} - 50e^{-\pi s}\frac{2}{(s+1)^2+2^2}.
\end{aligned}$$

La $W(s) := \frac{2}{(s+1)^2+2^2} = F(s+1)$ der F er transformasjonen til $f(t) := \sin 2t$. Da er

$$w(t) = \mathcal{L}^{-1}\{F(s+1)\} = e^{-t}f(t) = e^{-t}\sin 2t$$

og vi får

$$\begin{aligned}
y(t) &= -2 + 5t - 50\mathcal{L}^{-1}\left\{e^{-\pi s}W(s)\right\} \\
&= -2 + 5t - 50w(t-\pi)u(t-\pi) \\
&= -2 + 5t - 50e^{-(t-\pi)}\sin(2(t-\pi))u(t-\pi) \\
&= -2 + 5t - 50e^{-(t-\pi)}\sin(2t)u(t-\pi).
\end{aligned}$$

Chapter 6.5

Vi bruker at

$$\mathcal{L}\{f\}\mathcal{L}\{g\} = \mathcal{L}\{f * g\}$$

der

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau.$$

$\boxed{\text{6:5.12}}$ Løs integral-ligningen med Laplace-transformasjon.

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,\mathrm{d}\tau = t + e^t.$$

Løsning:
Transformasjonen til

$$y + y * \cosh = t + e^t$$

er

$$Y + Y\frac{s}{s^2-1} = \frac{1}{s^2} + \frac{1}{s-1}.$$

Løser for Y:

$$Y(s) = \frac{1}{s} + \frac{1}{s^2}$$

og løsningen er

$$y(t) = 1 + t.$$

$\boxed{\text{6.5:13}}$ Løs integral-ligningen med Laplace-transformasjon.

$$y(t) + 2e^t \int_0^t y(\tau)e^{-\tau}\,\mathrm{d}\tau = te^t.$$

Løsning:

$$\begin{aligned} te^t &= y(t) + 2e^t \int_0^t y(\tau)e^{-\tau}\,\mathrm{d}\tau \\ &= y(t) + 2\int_0^t y(\tau)e^{t-\tau}\,\mathrm{d}\tau \\ &= y + 2y * \exp \\ &\Rightarrow \\ \frac{1}{(s-1)^2} &= Y + 2Y\frac{1}{s-1} = \frac{s+1}{s-1}Y \\ &\Rightarrow \\ Y(s) &= \frac{1}{(s-1)(s+1)} \\ &= \frac{1}{s^2-1} \end{aligned}$$

Med løsning

$$y(t) = \sinh t.$$

$\boxed{\text{6.5:19}}$ Finn $f(t)$ når

$$\mathcal{L}\{f\} = \frac{2\pi s}{(s^2+\pi^2)^2}.$$

Løsning:
Skriv

$$\mathcal{L}\{f\} = 2\frac{\pi}{s^2+\pi^2}\frac{s}{s^2+\pi^2} = 2\mathcal{L}\{\sin \pi t\}\,\mathcal{L}\{\cos \pi t\}\,.$$

Dette gir

$$\begin{aligned} f(t) &= 2\sin \pi t * \cos \pi t \\ &= 2\int_0^t \sin(\pi\tau)\cos(\pi(t-\tau))\,\mathrm{d}\tau. \end{aligned}$$

Ved de trigonometriske summeformelene og halvvinkel-identitetene finner vi at

$$\begin{aligned} 2\sin\pi\tau\cos(\pi t-\pi\tau) &= 2\sin\pi\tau(\cos\pi t\cos\pi\tau + 2\sin\pi t\sin\pi\tau) \\ &= 2\cos\pi t\sin\pi\tau\cos\pi\tau + 2\sin\pi t\sin^2\pi\tau \\ &= \cos\pi t\sin 2\pi\tau + \sin\pi t(1-\cos 2\pi\tau) \\ &= \sin\pi t + \sin 2\pi\tau\cos\pi t - \cos 2\pi\tau\sin\pi t \\ &= \sin\pi t + \sin(2\pi\tau - \pi t). \end{aligned}$$

Dette gir

$$\begin{aligned} f(t) &= \int_0^t \sin\pi t + \sin(2\pi\tau-\pi t)\,\mathrm{d}\tau \\ &= \sin\pi t\Big|_0^t \tau - \frac{1}{2\pi}\Big|_0^t \cos(2\pi\tau - \pi t) \\ &= t\sin\pi t - \frac{1}{2\pi}(\cos\pi t - \cos(-\pi t)) \\ &= t\sin\pi t. \end{aligned}$$

$\boxed{\text{6.5:22}}$ Finn $f(t)$ når

$$\mathcal{L}\{f\} = \frac{e^{-as}}{s(s-2)}.$$

Løsning:

$$\begin{aligned} F(s) = \mathcal{L}\{f\} &= e^{-as}\mathcal{L}\{1\}\,\mathcal{L}\left\{e^{2t}\right\} \\ &= \mathcal{L}\{u(t-a)\}\,\mathcal{L}\left\{e^{2t}\right\} \\ &= \mathcal{L}\left\{u(t-a) * e^{2t}\right\} \end{aligned}$$

som gir

$$\begin{aligned} f(t) &= u(t-a) * e^{2t} \\ &= \int_0^t u(\tau-a)e^{2(t-\tau)}\,\mathrm{d}\tau \\ &= u(t-a)\int_a^t e^{2(t-\tau)}\,\mathrm{d}\tau \\ &= -\frac{1}{2}u(t-a)e^{2t}\Big|_a^t e^{-2\tau} \\ &= -\frac{1}{2}u(t-a)e^{2t}\left(e^{-2t} - e^{-2a}\right) \\ &= \frac{1}{2}u(t-a)\left(e^{2(t-a)} - 1\right). \end{aligned}$$

Chapter 6.6

Vi bruker
$$\mathcal{L}\{tf(t)\} = -F'(s).$$

$\boxed{\text{6.6:7}}$ Finn $\mathcal{L}\{f\}$ hvis
$$f(t) = t^2 \sinh 2t.$$

Løsning:
Ettersom
$$\begin{aligned}\mathcal{L}\{t \sinh 2t\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{2}{s^2-4} \\ &= \frac{4s}{(s^2-4)^2},\end{aligned}$$
er
$$\begin{aligned}\mathcal{L}\{f\} &= \mathcal{L}\{t \cdot t \sinh 2t\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{4s}{(s^2-4)^2} \\ &= -\frac{4(s^2-4)^2 - 4s \cdot 2(s^2-4)2s}{(s^2-4)^4} \\ &= 4\frac{4+3s^2}{(s^2-4)^3}.\end{aligned}$$

$\boxed{\text{6.6:15}}$ Finn f hvis
$$\mathcal{L}\{f\} = \frac{s}{(s^2-4)^2}.$$

Løsning:
Fra oppgave 6.6:7 ser vi at $f(t) = \frac{1}{4}t \sinh 2t$.

$\boxed{\text{6.6:17}}$ Finn f hvis
$$\mathcal{L}\{f\} = \ln\frac{s}{s-1}.$$

Løsning:

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -F'(s) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\ln\frac{s}{s-1} \\ &= \frac{1}{s-1} - \frac{1}{s}.\end{aligned}$$

Dermed er $tf(t) = e^t - 1$. Dvs.

$$f(t) = \frac{e^t - 1}{t}.$$