Fra Kreyszig (10th), avsnitt 12.6

5 Randbetingelser: $u(0,t) = u(10,t) = 0 \quad \forall t$.
Initalbetingelse: $u(x,0) = f(x) = \sin(0.1\pi x)$.
$f(x)$ er sin egen fourierrekke med $B_1 = 1$ og $B_n = 0 \quad n \in \mathbb{N}\backslash\{1\}$

$$\begin{aligned}\implies u(x,t) &= \sin\left(\frac{1\cdot\pi}{10}x\right)e^{-\left(\frac{c\cdot 1\cdot\pi}{10}\right)^2 t}\\ &= \sin\left(\frac{\pi}{10}x\right)e^{-\frac{1.75\pi^2}{100}t}\end{aligned}$$

der $c^2 = \frac{\kappa}{\rho\sigma}$ med $\kappa = 1.04$, $\rho = 10.6$ og $\sigma = 0.056$.

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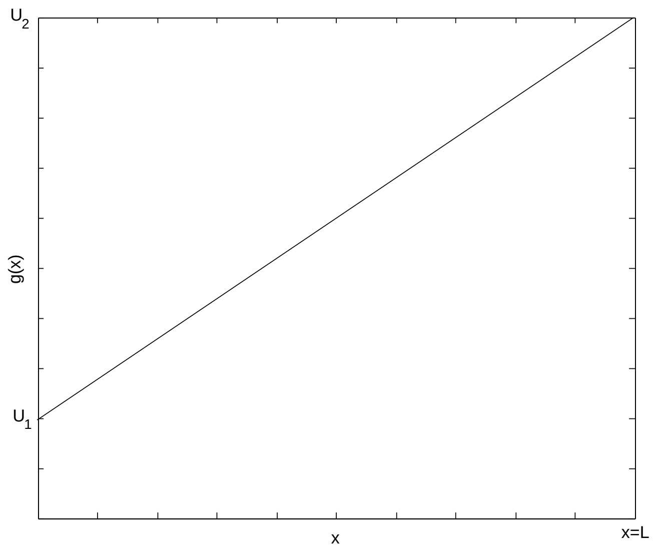
$$\begin{aligned}u_t &= c^2 u_{xx}\\ u(0,t) &= U_1 \quad t \geq 0\\ u(L,t) &= U_2 \quad t \geq 0\\ u(x,0) &= f(x)\end{aligned}$$

Gjetter på formen $g(x) = U_1 + \frac{U_2 - U_1}{L}x$.



La $v(x,t) = u(x,t) - g(x)$

$$\implies v_t = u_t = c^2 u_{xx} = c^2 v_{xx} \quad \text{siden } g^2(x) = 0$$

Rand- og initialbetingelser for v blir

$$\begin{aligned} v(0,t) &= U_1 - g(0) = U_1 - U_1 = 0 \\ v(L,t) &= U_2 - g(L) = U_2 - U_2 = 0 \\ v(x,0) &= f(x) - g(x) \end{aligned}$$

Løsning:

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right) \qquad \text{Se seksjon 12.6, (9)-(10)} \\ B_n &= \frac{2}{L}\int_0^L (f(x) - g(x)) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

$$\implies u(x,t) = g(x) + v(x,t) = g(x) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

Vi har at

$$\begin{aligned} &\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \to 0 \text{ når } t \to \infty \\ &\qquad \implies u(x,t) \to g(x) \text{ når } t \to \infty \end{aligned}$$

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$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \\ u(x,0) &= x \\ L &= \pi \\ c &= 1 \end{aligned}$$

$$\overset{Eks.4}{\implies} u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-n^2 t}$$

$$\begin{aligned} \text{med} \quad A_0 &= \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi x dx = \frac{x^2}{2\pi}\Big|_0^\pi = \frac{\pi}{2} \\ A_n &= \frac{2}{\pi}\int_0^\pi x\cos(nx)dx = \frac{2}{\pi} x \frac{\sin(nx)}{n}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{\sin(nx)}{n} dx \\ &= \frac{2}{\pi}\frac{\cos(nx)}{n^2}\Big|_0^\pi = \frac{2}{\pi n^2}((-1)^n - 1) \end{aligned}$$

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$$u_t = c^2 u_{xx} + H$$
$$L = \pi$$
$$u(0,t) = u(\pi,t) = 0 \qquad t \geq 0$$
$$u(x,0) = f(x)$$

La $u(x,t) = v(x,t) - Hx\frac{x-\pi}{2c^2}$. Dette gir

$$\begin{aligned} u_t(x,t) &= v_t(x,t) \\ u_{xx}(x,t) &= v_{xx}(x,t) - \frac{H}{c^2} \\ \implies v_t(x,t) &= u_t(x,t) = c^2 u_{xx}(x,t) + H = c^2 v_{xx} - H + H = c^2 v_{xx}(x,t) \end{aligned}$$

Vi har dermed

$$\begin{aligned} v_t(x,t) &= c^2 v_{xx}(x,t) \\ v(0,t) &= u(0,t) = 0 \\ v(\pi,t) &= u(\pi,t) = 0 \\ v(x,0) &= u(x,0) + Hx\frac{x-\pi}{2c^2} = f(x) + Hx\frac{x-\pi}{2c^2} \end{aligned}$$

$$\overset{(9)-(10)}{\implies} v(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-\lambda n^2 t}$$

med

$$\lambda_n = cn \qquad \text{og} \qquad B_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) + Hx\frac{x-\pi}{2c^2}\right)\sin(nx)dx$$

Dermed er

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-\lambda n^2 t} - Hx\frac{x-\pi}{2c^2}$$

med

$$\lambda_n = cn \qquad \text{og} \qquad B_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) + Hx\frac{x-\pi}{2c^2}\right)\sin(nx)dx$$

21 Oppgitte betingelser:

$$u(0,y) = u(a,y) = u(x,0) = 0, \qquad \text{og} \qquad u(x,a) = 25, \qquad \text{med } a = 24$$

Steady-state temperatur vil si at temperaturen ikke lenger endrer seg med tiden: $u_t = 0$. Varmeledningsligningen i to dimensjoner blir dermed

$$\nabla^2 u = 0$$
$$u_{xx} + u_{yy} = 0, \qquad (1)$$

som er Laplaces ligning. Bruker seperasjon av variable:

$$u(x,y) = F(x)G(y)$$

Innsatt i (1):

$$\begin{aligned}\frac{\mathrm{d}^2F}{\mathrm{d}x^2}G + F\frac{\mathrm{d}^2G}{\mathrm{d}y^2} &= 0\\ \frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = -\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}y^2} &= k \qquad \text{(en konstant)}\end{aligned}$$

Vi har dermed to ordinære differensialligninger:

$$F'' - kF = 0, \qquad\qquad G'' + kG = 0$$

Med $k = \mu^2 > 0$:

$$F(x) = C_1\mathrm{e}^{\mu x} + C_2\mathrm{e}^{-\mu x}$$

Initialbetingelsene $u(0,y) = u(a,y) = 0$ gir $C_1 = C_2 = 0$.
Med $k = 0$:

$$F(x) = C_3x + C_4$$

Initialbetingelsene $u(0,y) = u(a,y) = 0$ gir $C_3 = C_4 = 0$.
Med $k = -\mu^2 < 0$:

$$F(x) = C_5\cos(\mu x) + C_6\sin(\mu x)$$

Initialbetingelsen $u(0,y) = 0$ gir:

$$\begin{aligned}C_5\cos 0 + C_6\sin 0 &= 0\\ C_5 &= 0\end{aligned}$$

Initialbetingelsen $u(a,y) = 0$ gir:

$$\begin{aligned}C_6\sin(\mu a) &= 0\\ \mu a &= n\pi\\ \mu &= \frac{n\pi}{a} \qquad n = 0, \pm 1, \pm 2, ...\end{aligned}$$

$$F(x) = C_6\sin\left(\frac{n\pi x}{a}\right)$$

Med $k = -\mu^2$ blir ligningen for $G(y)$:

$$G'' - \mu^2G = 0$$

Med løsning

$$G(y) = C_7\mathrm{e}^{\mu y} + C_8\mathrm{e}^{-\mu y}$$

Initialbetingelsen $u(x,0)=0$ gir $C_8=-C_7$:

$$\begin{aligned}G(y)&=C_7\left(\mathrm{e}^{\mu y}-\mathrm{e}^{-\mu y}\right)\\&=2C_7\sinh(\mu y)\\&=2C_7\sinh\left(\frac{n\pi y}{a}\right)\end{aligned}$$

For hver eneste n får vi dermed en løsning:

$$u_n(x,y)=C_9\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

Nå er fortsatt $n=0,\pm1,\pm2$, men siden u_n er odde mtp n (som betyr at u_{-n} er proporsjonal med u_n) holder det å summere $u(x,y)$ for bare positive n. Løsningen for $n=0$ er $u=0$, som ikke er særlig interessant.

Generell løsning:

$$u(x,y)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

Bruker nå siste betingelse: $u(x,a)=25$

$$25=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{a}\right)\sinh(n\pi)$$

$$\begin{aligned}B_n\sinh(n\pi)&=\frac{2}{a}\int_0^a 25\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x\\\frac{B_n\sinh(n\pi)a}{50}&=\left[\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a\\\frac{B_n\sinh(n\pi)}{50}&=\frac{1}{n\pi}(1-(-1)^n)\\B_n&=\frac{50}{n\pi\sinh(n\pi)}(1-(-1)^n)\end{aligned}$$

Som gir løsningen:

$$\underline{u(x,y)=\sum_{n=1}^{\infty}\frac{50}{n\pi\sinh(n\pi)}(1-(-1)^n)\sinh\left(\frac{n\pi y}{24}\right)\sin\left(\frac{n\pi x}{24}\right)}$$

23 Oppgitte betingelser:

$$u(0,y)=0,\quad u(a,y)=f(y),\quad u_y(x,0)=u_y(x,a)=0\qquad\text{med } a=24$$

Samme metode som i oppgave 12.6.23:

$$u_{xx}+u_{yy}=0\qquad\text{(Laplaces ligning)}$$

Seperasjon av variable:

$$u(x,y)=F(x)G(y)$$

$$=>\qquad\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}x^2}=-\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}=k\qquad\text{(en konstant)}$$

$$F''-kF=0,\qquad G''+kG=0$$

Med $k = \mu^2 > 0$:

$$\begin{aligned} G(y) &= C_1 \cos(\mu y) + C_2 \sin(\mu y) \\ G'(y) &= \mu C_2 \cos(\mu y) - \mu C_1 \sin(\mu y) \end{aligned}$$

Initialbetingelsen $u_y(x,a) = 0$ gir:

$$\begin{aligned} -\mu C_1 \sin(\mu a) &= 0 \\ \mu a &= n\pi \\ \mu &= \frac{n\pi}{a}, \qquad n = 0, \pm 1, \pm 2, ... \end{aligned}$$

$$G(y) = C_1 \cos\left(\frac{n\pi y}{a}\right)$$

Med $k = \mu^2 > 0$ blir løsningen for $F(x)$:

$$\begin{aligned} & F'' - \mu^2 F = 0 \\ => \qquad & F(x) = C_3 \mathrm{e}^{\mu x} + C_4 \mathrm{e}^{-\mu x} \end{aligned}$$

Initialbetingelsen $u(0,y) = 0$ gir $C_4 = -C_3$.

$$F(x) = C_3 \left(\mathrm{e}^{\mu x} - \mathrm{e}^{-\mu x}\right) = 2C_3 \sinh(\mu x)$$

$$=> u_n(x,y) = A_n \sinh\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right), \qquad (1), \qquad n = 1, 2, 3, ...$$

(Av samme grunn som i oppgave 12.6.21 får vi ikke noen flere løsninger av å inkludere negative n, mens $n = 0$ bare gir $u = 0$).

Med $k = 0$:

$$G(y) = C_5 y + C_6$$

Initialbetingelsene $u_y(x,0) = u_y(x,a) = 0$ gir begge $C_5 = 0$, slik at $G(y) = C_6$ er en gyldig løsning. Da blir $F(x)$:

$$F(x) = C_7 x + C_8$$

Initialbetingelsen $u(0,y) = 0$ gir $C_8 = 0$. Dermed er en gyldig løsning

$$u(x,y) = F(x)G(y) = A_0 x, \qquad (2) \qquad (A_0 = C_6 C_7)$$

Med $k = -\mu^2$ gir initialbetingelsene at $u(x,y) = 0$.

Den mest generelle løsningen blir dermed en sum av alle løsningene fra (1) pluss løsningen fra (2):

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right)$$

Bruker siste initialbetingelsen $u(a,y) = f(y)$ for å bestemme konstantene:

$$f(y) = A_0 a + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos\left(\frac{n\pi y}{a}\right)$$

$$A_0 a = \frac{1}{a}\int_0^a f(y)\,\mathrm{d}y$$

$$A_n \sinh(n\pi) = \frac{2}{a}\int_0^a f(y)\cos\left(\frac{n\pi y}{a}\right)\,\mathrm{d}y$$

Setter dette inn i uttrykket for $u(x,y)$:

$$\underline{u(x,y) = \left[\frac{1}{24^2}\int_0^{24} f(y)\,\mathrm{d}y\right]x + \sum_{n=1}^{\infty}\left[\frac{1}{12\sinh(n\pi)}\int_0^{24} f(y)\cos\left(\frac{n\pi y}{24}\right)\,\mathrm{d}y\right]\sinh\left(\frac{n\pi x}{24}\right)\cos\left(\frac{n\pi y}{24}\right)}$$

Fra Kreyszig (10th), avsnitt 12.7

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$$f(x) = e^{-|x|}$$

$$u(x,t) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]\,e^{-c^2p^2t}dp$$

$$\implies u(x,0) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]\,dp = f(x)$$

Vi ser dette er et Fourier-integral:

$$\begin{aligned}
A(p) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|v|}\cos(pv)dv \\
&= \frac{1}{\pi}\int_{-\infty}^{0} e^{v}\cos(pv)dv + \frac{1}{\pi}\int_0^\infty e^{-v}\cos(pv)dv \\
&= \frac{1}{\pi}e^v\cos(pv)\Big|_{-\infty}^{0} + \frac{1}{\pi}\int_{-\infty}^{0} e^v\sin(pv)dv - \frac{1}{\pi}e^{-v}cos(pv)\Big|_0^\infty - \frac{1}{\pi}\int_0^\infty e^{-v}\sin(pv)dv \\
&= \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi}e^v\sin(pv)p\Big|_{-\infty}^{0} - \frac{1}{\pi}\int_{-\infty}^{0} e^v\cos(pv)p^2dv + \frac{1}{\pi}e^{-v}\sin(pv)p\Big|_0^\infty \\
&\quad - \frac{1}{\pi}\int_0^\infty e^{-v}\cos(pv)p^2dv \\
&= \frac{2}{\pi} - p^2A(p)
\end{aligned}$$

$$\implies A(p) = \frac{2}{\pi(1+p^2)}$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|v|}\sin(pv)dv = 0 \qquad \text{siden integranden er odde}$$

$$\implies u(x,t) = \int_0^\infty \frac{2}{\pi(1+p^2)}\cos(px)e^{-c^2p^2t}dp$$

Alternativt via Fouriertransformasjon:

$$\hat{u}_t(w,t) = -c^2 w^2 \hat{u}(w,t) \implies \hat{u}(w,t) = C(w)e^{-c^2w^2t} \implies \hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}$$

siden $\hat{u}(w,0) = \hat{f}(w)$.

$$\implies u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) \frac{1}{\sqrt{2c^2t}} e^{-\frac{1}{4c^2t}(x-p)^2}\, dp$$