Fra Kreyszig (10th), avsnitt 6.4

3

$$y'' + 9y = \delta(t - \frac{\pi}{2}), \qquad y(0) = 2, y'(0) = 0$$

La $Y = \mathcal{L}[y] \implies$

$$\begin{aligned}\mathcal{L}[y'' + 9y](s) &= \mathcal{L}[y''](s) + 9\mathcal{L}[y](s) = s^2\mathcal{L}[y](s) - sy(0) - y'(0) + 9\mathcal{L}[y](s)\\ &= (s^2+9)Y(s) - 2s = e^{-\frac{\pi}{2}s},\end{aligned}$$

der

$$\mathcal{L}[\delta(t - \frac{\pi}{2})](s) = e^{-\frac{\pi}{2}s}.$$

$$\begin{aligned}\implies Y(s) = \frac{2s}{s^2+9} + e^{-\frac{\pi}{2}s}\frac{1}{s^2+9} &= 2\mathcal{L}[\cos(3t)](s) + \frac{1}{3}e^{-\frac{\pi}{2}s}\mathcal{L}[\sin(3t)](s)\\ &= 2\mathcal{L}[\cos(3t)](s) + \frac{1}{3}\mathcal{L}[\sin(3(t - \frac{\pi}{2}))u(t - \frac{\pi}{2})](s)\end{aligned}$$

$$\begin{aligned}\implies y(t) &= 2\cos(3t) + \frac{1}{3}\sin(3(t - \frac{\pi}{2}))u(t - \frac{\pi}{2})\\ &= 2\cos(3t) + \frac{1}{3}\cos(3t)u(t - \frac{\pi}{2})\end{aligned}$$

11

$$y'' + 3y' + 2y = u(t-1) + \delta(t-2) \qquad y(0) = 0, y'(0) = 1$$

La $Y = \mathcal{L}[y]$

$$\begin{aligned}\implies \mathcal{L}[y'' + 3y' + 2y](s) &= \mathcal{L}[y''](s) + 3\mathcal{L}[y'](s) + 2\mathcal{L}[y](s)\\ &= s^2\mathcal{L}[y](s) - sy(0) - y'(0) + 3s\mathcal{L}[y](s) - 3y(0) + 2\mathcal{L}[y](s)\\ &= (s^2 + 3s + 2)Y(s) - 1\end{aligned}$$

$$\mathcal{L}[u(t-1) + \delta(t-2)](s) = \frac{e^{-s}}{s} + e^{-2s}$$

$$\begin{aligned}\implies (s^2 + 3s + 2)Y(s) &= 1 + \frac{e^{-s}}{s} + e^{-2s}\\ (s^2 + 3s + 2) &= (s+2)(s+1)\end{aligned}$$

$$\implies Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + e^{-2s}\frac{1}{s+1} - e^{-2s}\frac{1}{s+2} + e^{-s}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)$$

siden

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \implies 1 = A(s^2+3s+2) + B(s^2+2s) + C(s^2+s)$$

$$\begin{aligned}\implies & A+B+C=0\\ & 3A+2B+C=0\\ & 2A=1\\ \implies & A=\frac{1}{2},\ B=-1,\ C=\frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\implies Y(s) &= \mathcal{L}[1](s+1) - \mathcal{L}[1](s+2) + e^{-2s}\left(\mathcal{L}[1](s+1) - \mathcal{L}[1](s+2)\right)\\ &\quad + e^{-s}\left(\mathcal{L}\left[\frac{1}{2}\right](s) - \mathcal{L}[1](s+1)) + \frac{1}{2}\mathcal{L}[1](s+2)\right)\\ &= \mathcal{L}[e^{-t} - e^{-2t}](s) + e^{-2s}\mathcal{L}[e^{-t} - e^{-2t}](s) + e^{-s}\mathcal{L}\left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right](s)\\ &= \mathcal{L}\Big[e^{-t} - e^{-2t} + (e^{-(t-2)} - e^{-2(t-2)})u(t-2)\\ &\quad + \Big(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\Big)u(t-1)\Big](s)\end{aligned}$$

$$\implies y(t) = e^{-t} - e^{-2t} + \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right)u(t-1) + (e^{-(t-2)} - e^{-2(t-2)})u(t-2)$$

12

$$y'' + 2y' + 5y = 25t - 100\delta(t-\pi) \qquad\qquad y(0) = -2, y'(0) = 5$$

La $Y = \mathcal{L}[y] \implies$

$$\begin{aligned}\mathcal{L}[y''+2y'+5y](s) &= s^2\mathcal{L}[y](s) - sy(0) - y'(0) + 2s\mathcal{L}[y](s) - 2y(0) + 5\mathcal{L}[y](s)\\ &= (s^2+2s+5)Y(s) + 2s - 1\end{aligned}$$

$$\mathcal{L}[25t - 100\delta(t-\pi)](s) = 25\frac{1}{s^2} - 100e^{-\pi s}$$

$$\begin{aligned}\implies Y(s) &= \frac{1}{s^2+2s+5}\left(25\frac{1}{s^2} - 100e^{-\pi s} + 1 - 2s)\right)\\ &= \frac{5-2s}{s^2} + \frac{-1+2s}{s^2+2s+5} - 100e^{-\pi s}\frac{1}{s^2+2s+5} + \frac{1}{s^2+2s+5} - 2\frac{s}{s^2+2s+5}\\ &= \frac{5-2s}{s^2} - 100e^{-\pi s}\frac{1}{(s+1)^2+4}\\ &= \mathcal{L}[5t-2](s) - 50e^{-\pi s}\mathcal{L}[\sin(2t)](s+1)\\ &= \mathcal{L}[5t - 2 - 50e^{-(t-\pi)}\sin(2(t-\pi))u(t-\pi)](s)\end{aligned}$$

$$\implies y(t) = 5t - 2 - 50e^{-(t-\pi)}\sin(2(t-\pi))u(t-\pi).$$

Vi brukte

$$\frac{25}{s^2(s^2+2s+5)} = \frac{A+Bs}{s^2} + \frac{C+Ds}{s^2+2s+5} = \frac{(A+Bs)(s^2+2s+5)+(C+Ds)s^2}{s^2(s^2+2s+5)}$$

$$\begin{aligned}\implies 0 &= B+D\\ 0 &= A+2B+C\\ 0 &= 2A+5B\\ 25 &= 5A\\ \implies A &= 5, B=-2, C=-1, D=2\end{aligned}$$

Fra Kreyszig (10th), avsnitt 6.5

10

$$y(t) - \int_0^t y(\tau)\sin(2(t-\tau))d\tau = \sin(2t)$$

La $\mathcal{L}[y] = Y$

$$\begin{aligned}\mathcal{L}[y(t) - \int_0^t y(\tau)\sin(2(t-\tau))d\tau] &= \mathcal{L}[y](s) - \mathcal{L}[y(t)](s)\mathcal{L}[\sin(2t)](s)\\ &= Y(s) - Y(s)\frac{2}{s^2+4}\\ &= \frac{s^2+2}{s^2+4}Y(s)\end{aligned}$$

$$\mathcal{L}[\sin(2t)](s) = \frac{2}{s^2+4}$$

$$\implies Y(s) = \frac{2}{s^2+2} \implies y(t) = \sqrt{2}\sin(\sqrt{2}t)$$

13

$$y(t) + 2e^t\int_0^t y(\tau)e^{-\tau}d\tau = te^t$$

$$y(t) + 2e^t\int_0^t y(\tau)e^{-\tau}d\tau = y(t) + \int_0^t y(\tau)2e^{t-\tau}d\tau$$

La $Y = \mathcal{L}[y] \implies$

$$\begin{aligned}\mathcal{L}[y(t) + \int_0^t y(\tau)2e^{t-\tau}d\tau](s) &= \mathcal{L}[y(t)](s) + 2\mathcal{L}[y(t)](s)\mathcal{L}[e^t](s)\\ &= Y(s) + 2Y(s)\frac{1}{s-1}\end{aligned}$$

$$\mathcal{L}[te^t](s) = \mathcal{L}[t](s-1) = \frac{1}{(s-1)^2} \implies$$

$$\begin{aligned}
&Y(s) = \frac{1}{(s-1)^2}\frac{s-1}{s+1} = \frac{1}{s^2-1} = \mathcal{L}[\sinh(t)](s) \\
\implies\ & y(t) = \sinh(t)
\end{aligned}$$

$\boxed{22}$

$$\begin{aligned}
F(s) = \frac{e^{-as}}{s}\frac{1}{s-2} &= e^{-as}\mathcal{L}[1](s)\mathcal{L}[1](s-2) \\
&= \mathcal{L}[u(t-a)](s)\mathcal{L}[e^{2t}](s) \\
&= \mathcal{L}\left[\int_0^t u(t-a)e^{2(t-\tau)}d\tau\right](s)
\end{aligned}$$

$$f(t) = \int_0^t u(\tau-a)e^{2(t-\tau)}d\tau$$

$\boxed{23}$

$$F(s) = 40.5\frac{1}{s}\frac{1}{s^2-9} = 40.5\mathcal{L}[1](s)\frac{1}{3}\mathcal{L}[\sinh(3t)](s) = \frac{27}{2}\mathcal{L}[1](s)\mathcal{L}[\sinh(3t)](s)$$

$$\implies f(t) = \frac{27}{2}\int_0^t \sinh(3\tau)d\tau = \frac{9}{2}\cosh(3t) - \frac{9}{2}$$

Løsning ved hjelp av delbrøkoppspalting:

$$\frac{1}{s(s^2-9)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+3} = \frac{A(s^2-9) + B(s^2+3s) + C(s^2-3s)}{s(s^2-9)}$$

$$\begin{aligned}
0 &= A + B + C \\
0 &= 3B - 3C \\
1 &= -9A
\end{aligned}$$

$$\implies A = -\frac{1}{9}, B = \frac{1}{18}, C = \frac{1}{18}.$$

$$\begin{aligned}
F(s) = \frac{81}{2}\frac{1}{s(s^2-9)} &= \frac{81}{2}\left(-\frac{1}{9s} + \frac{1}{18(s-3)} + \frac{1}{18(s+3)}\right) \\
&= \frac{9}{2}\left(-\frac{1}{s} + \frac{s}{s^2-9}\right) = \frac{9}{2}\mathcal{L}[-1+\cosh(3t)](s)
\end{aligned}$$

$$\implies f(t) = -\frac{9}{2} + \frac{9}{2}\cosh(3t)$$

$\boxed{25}$

$$F(s) = 3\frac{6}{s^2+6^2}\frac{s}{s^2+6^2} = 3\mathcal{L}[\sin(6t)](s)\mathcal{L}[\cos(6t)](s)$$

$$
\begin{aligned}
\implies f(t) &= 3\int_0^t \sin(6\tau)\cos(6(t-\tau))d\tau = 3\int_0^t \sin(6\tau)\cos(6\tau)\cos(6t)d\tau \\
&+ 3\int_0^t \sin^2(6\tau)\sin(6t)d\tau \\
&= \frac{1}{4}\cos(6t)\sin^2(6t) - \frac{1}{4}\sin^2(6t)\cos(6t) + \frac{3}{2}t\sin(6t) \\
&= \frac{3}{2}t\sin(6t)
\end{aligned}
$$

Der følgende intergraler er brukt:

$$\int_0^t \sin^2(6\tau)d\tau = -\frac{\cos(6\tau)\sin(6\tau)}{6}\Big|_0^t + \int_0^t \cos^2(6\tau)d\tau = -\frac{\cos(6t)\sin(6t)}{6} + t - \int_0^t \sin^2(6\tau)d\tau$$

$$\int_0^t \sin(6\tau)\cos(6\tau)d\tau = \frac{\sin^2(6\tau)}{6}\Big|_0^t - \int_0^t \sin(6\tau)\cos(6\tau)d\tau$$

Fra Kreyszig (10th), avsnitt 6.6

$\boxed{6}$

$$g(t) = \sin(3t) \implies G(s) = \mathcal{L}[g](s) = \frac{3}{s^2+9}$$

$$
\begin{aligned}
&\implies \mathcal{L}[t\sin(3t)](s) = -G'(s) = \frac{3}{(s^2+9)^2}2s = \frac{6s}{(s^2+9)^2} \\
&\implies \mathcal{L}[t^2\sin(3t)](s) = -\frac{6}{(s^2+9)^4}((s^2+9)^2 - s2(s^2+9)2s) \\
&\qquad = -\frac{6}{(s^2+9)^3}(s^2+9-4s^2) = 18\frac{s^2-3}{(s^2+9)^3} \\
&\implies \mathcal{L}[f](s) = 18\frac{s^2-3}{(s^2+9)^3}
\end{aligned}
$$

$\boxed{16}$

$$F(s) = \frac{2s+6}{(s^2+6s+10)^2} = \frac{2s+6}{(s^2+6s+9+1)^2} = \frac{2(s+3)}{((s+3)^2+1)^2}$$

La

$$G(s) = -\frac{1}{(s+3)^2+1} \implies G'(s) = \frac{2(s+3)}{((s+3)^2+1)^2}$$

$$G(s) = \mathcal{L}[-e^{-3t}\sin t](s) \implies G'(s) = \mathcal{L}[te^{-3t}\sin t](s) \implies f(t) = e^{-3t}t\sin t$$

$\boxed{17}$

$$
\begin{aligned}
F(s) &= \ln\left(\frac{s}{s-1}\right) \\
F'(s) &= \frac{s-1}{s}\frac{1}{(s-1)^2}(s-1-s) = -\frac{1}{s(s-1)} = \frac{1}{s} - \frac{1}{s-1} = \mathcal{L}[1-e^t](s)
\end{aligned}
$$

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \ln\left(\frac{s}{s-1}\right) = 0$$

$$\implies F(s) = F(s) - F(\infty) = -\int_s^\infty F'(s)ds = -\mathcal{L}\left[\frac{1-e^t}{t}\right] = \mathcal{L}\left[\frac{e^t-1}{t}\right]$$

$$\implies f(t) = \frac{e^t-1}{t}$$

Fra Kreyszig (10th), avsnitt 6.7

$\boxed{6}$

$$\begin{aligned} y_1' &= 5y_1 + y_2 & \qquad y_1(0) &= 0 \\ y_2' &= y_1 + 5y_2 & \qquad y_2(0) &= -3 \end{aligned}$$

La $Y_1 = \mathcal{L}[y_1]$ og $Y_2 = \mathcal{L}[y_2]$.

$$\begin{aligned} sY_1 - y_1(0) &= 5Y_1 + Y_2 \\ sY_2 - y_2(0) &= Y_1 + 5Y_2 \end{aligned}$$

$$\implies \begin{aligned} &(5-s)Y_1 + Y_2 = -1 \\ &Y_1 + (5-s)Y_2 = 3 \end{aligned}$$

$$\implies \begin{aligned} &(5-s)Y_1 + Y_2 = -1 \\ &(5-s)Y_1 + (5-s)^2Y_2 = 3(5-s) \end{aligned}$$

$$\implies ((s-5)^2-1)Y_2 = 3(5-s) + 1$$

$$\implies Y_2(s) = 3\frac{5-s}{(s-5)^2-1} + \frac{1}{(s-5)^2-1} = -3\mathcal{L}[e^{5t}\cosh t](s) + \mathcal{L}[e^{5t}\sinh t)](s)$$

$$Y_1(s) = 3 - 3\frac{(5-s)^2}{(s-5)^2-1} - \frac{(5-s)}{(s-5)^2-1} = -3\frac{1}{(s-5)^2-1} + \frac{(s-5)}{(s-5)^2-1}$$

$$= \mathcal{L}[-3e^{5t}\sinh t + e^{5t}\cosh t](s)$$

$$\implies \begin{aligned} y_1(t) &= -3e^{5t}\sinh t + e^{5t}\cosh t \\ y_2(t) &= -3e^{5t}\cosh t + e^{5t}\sinh t \end{aligned}$$