Vi har at

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$$

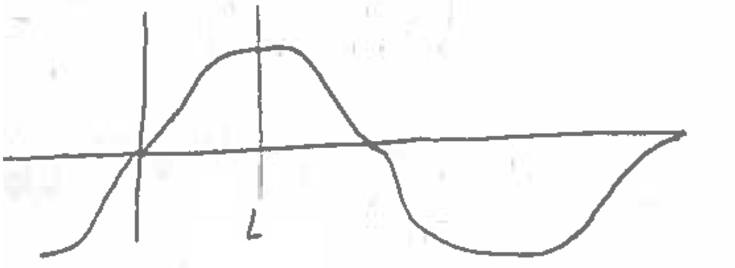
Randbetingelsene

$$\begin{aligned} u(0,t) &= 0 \qquad \forall t \\ u_x(L,t) &= 0 \qquad \forall t \end{aligned}$$

gir

$$\begin{aligned} 0 = u(0,t) &= \frac{1}{2}\left(f(ct) + f(-ct)\right) \implies f(ct) = f(-ct) \implies \text{odde funksjon} \\ 0 = u_x(L,t) &= \frac{1}{2}(f'(L+ct) + f'(L-ct) \\ &\implies f'(L+ct) = -f'(L-ct) \implies f'(L+x) = -f'(L-x)) \end{aligned}$$

$\implies f$ skal utvides til en odde funksjon med periode $4L$.



$$\begin{aligned} &\implies f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2L}x\right) \quad \text{med} \quad b_n = \frac{2}{2L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{2L}x\right)dx \\ &\implies b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{2L}x\right)dx \\ &\qquad = \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx + \int_L^{2L} f(x)\sin\left(\frac{n\pi}{2L}x\right)dx\right) \\ &\qquad = \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx + \int_0^L f(x+L)\sin\left(\frac{n\pi}{2L}(x+L)\right)dx\right) \end{aligned}$$

$n = 2k$ gir

$$\begin{aligned} b_{2k} &= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx + \int_0^L f(x+L)\sin\left(\frac{k\pi}{L}(x+L)\right)dx\right) \\ &= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx + \int_0^L f(L-x)\sin\left(\frac{k\pi}{L}(x-L+2L)\right)dx\right) \\ &= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx - \int_0^L f(L-x)\sin\left(\frac{k\pi}{L}(L-x)\right)dx\right) \\ &= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx + \int_L^0 f(x)\sin\left(\frac{k\pi}{L}x\right)dx\right) = 0 \end{aligned}$$

$n = 2k+1$ gir

$$\begin{aligned}
b_{2k+1} &= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{(2k+1)\pi}{2L}x\right)dx + \int_0^L f(x+L)\sin\left(\frac{(2k+1)\pi}{2L}(x+L)\right)dx\right)\\
&= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{(2k+1)\pi}{2L}x\right)dx + \int_0^L f(L-x)\sin\left(\frac{(2k+1)\pi}{2L}(x-L+2L)\right)dx\right)\\
&= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{(2k+1)\pi}{2L}x\right)dx + \int_L^0 f(z)\sin\left(\frac{(2k+1)\pi}{2L}z-(2k+1)\pi\right)dz\right)\\
&= \frac{1}{L}\left(\int_0^L f(x)\sin\left(\frac{(2k+1)\pi}{2L}x\right)dx - \int_0^L f(z)\sin\left(\frac{(2k+1)\pi}{2L}z\right)(-1)^{2k+1}dz\right)\\
&= \frac{2}{L}\left(\int_0^L f(x)\sin\left(\frac{(2k+1)\pi}{2L}x\right)dx\right)
\end{aligned}$$

$$\begin{aligned}
\implies f(x) &= \sum_{n=0}^{\infty} b_{2n+1}\sin\left(\frac{(2n+1)\pi}{2L}\pi x\right)\\
\implies u(x) &= \frac{1}{2}\sum_{n=0}^{\infty} b_{2n+1}\left(\sin\left(\frac{(2n+1)\pi}{2L}\pi(x+ct)\right)+\sin\left(\frac{(2n+1)\pi}{2L}\pi(x-ct)\right)\right)\\
&= \frac{1}{2}\sum_{n=0}^{\infty} b_{2n+1}\sin\left(\frac{(2n+1)\pi}{2L}x\right)\cos\left(\frac{(2n+1)\pi}{2L}ct\right)
\end{aligned}$$