Fra Kreyszig (10th), avsnitt 11.4

4 Oppgave 11.1.14(øving 3)

$$\implies f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx)$$
$$\implies F(x) = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4}{n^2}(-1)^n \cos(nx) \qquad \text{for } N = 1, 2, ...$$

Minimum square error:

$$E^* = \int_{-\pi}^{\pi} f(x)^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]$$

Her: $f(x) = x^2$, $a_0 = \frac{\pi^2}{3}$, $a_1 = -4$, $a_2 = 1$, $a_3 = \frac{-4}{9}$, $a_4 = \frac{1}{4}$, $a_5 = -\frac{4}{25}$, $b_n = 0$
$n = 1, 2, 3, 4, 5.$

$$\implies N = 1 \quad E^* = \frac{2\pi^5}{5} - \pi\left[\frac{2\pi^4}{9} + 16\right] \approx 4.14$$
$$N = 2 \quad E^* = \frac{2\pi^5}{5} - \pi\left[\frac{2\pi^4}{9} + 16 + 1\right] \approx 1$$
$$N = 3 \quad E^* = \frac{2\pi^5}{5} - \pi\left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81}\right] \approx 0.38$$
$$N = 4 \quad E^* = \frac{2\pi^5}{5} - \pi\left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16}\right] \approx 0.18$$
$$N = 5 \quad E^* = \frac{2\pi^5}{5} - \pi\left[\frac{2\pi^4}{9} + 16 + 1 + \frac{16}{81} + \frac{1}{16} + \frac{16}{625}\right] \approx 0.1$$

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$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Det trigonometriske polynomet med minst square error er rett og slett Fourier-rekka til $f(x)$. Legger merke til at $f(x)$ er en odde funksjon, som betyr at

$$a_0 = 0 \quad \text{og} \quad a_n = 0$$

Regner ut b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\,\mathrm{d}x \\
&= \frac{2}{n\pi}(1-(-1)^n)
\end{aligned}$$

Dermed blir Fourier-rekka (eller rettere sagt den N'te partialsummen til Fourier-rekka):

$$\underline{F(x) = \sum_{n=1}^{N} \frac{2}{n\pi}(1-(-1)^n)\sin(nx)}$$

Plugger inn b_n i formelen for square error:

$$\begin{aligned}
E_N^* &= \int_{-\pi}^{\pi} f^2\,\mathrm{d}x - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2+b_n^2)\right] \\
&= \int_{-\pi}^{\pi} 1\,\mathrm{d}x - \pi\sum_{n=1}^{N}\left(\frac{2}{n\pi}(1-(-1)^n)\right)^2 \\
&= 2\pi - \pi\sum_{n=1}^{N}\left(\frac{4}{n^2\pi^2}(1-2(-1)^n+(-1)^{2n})\right) \\
&= 2\pi - \frac{8}{\pi}\sum_{n=1}^{N}\frac{1-(-1)^n}{n^2}
\end{aligned}$$

Som gir

$$\begin{aligned}
E_1^* = E_2^* &= 2\pi - \frac{16}{\pi} \approx \underline{1.1902} \\
E_3^* = E_4^* &= 2\pi - \frac{16}{\pi}\left(1+\frac{1}{9}\right) \approx \underline{0.6243} \\
E_5^* &= 2\pi - \frac{16}{\pi}\left(1+\frac{1}{9}+\frac{1}{25}\right) \approx \underline{0.4206}
\end{aligned}$$

12 Oppgave 11.1.14 (øving 3):

$$\begin{aligned}
f(x) = x^2 &\implies f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx) \\
&\implies a_0 = \frac{\pi^2}{3}, a_n^2 = \frac{16}{n^4}
\end{aligned}$$

Parsevals identitet:

$$
\begin{aligned}
2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx \\
\implies \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4} &= \frac{1}{\pi}\frac{2\pi^5}{5} = \frac{2\pi^4}{5} \\
\implies 16\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{8\pi^4}{45} \\
\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{90}
\end{aligned}
$$

Fra Kreyszig (9th), avsnitt 11.4

$\boxed{11}$ Vi skal finne den komplekse Fourierrekka til $f(x) = x^2, \ -\pi < x < \pi$.

Formel (6) på side 497 i Kreyszig gir

$$
c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx
$$

Vi må skille mellom $n = 0$ og $n \neq 0$ og får (ved to delvis integrasjoner når $n \neq 0$):

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}, \qquad (n=0) \\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\, dx \\
&= \frac{1}{2\pi}\left[\frac{x^2}{-in}e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{i\pi n}\int_{-\pi}^{\pi} xe^{-inx}\, dx \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{in}e^{in\pi} - \frac{\pi^2}{in}e^{-in\pi}\right) + \frac{1}{i\pi n}\left[\frac{x}{-in}e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{\pi n^2}\int_{-\pi}^{\pi} e^{-inx}\, dx \\
&= \frac{1}{\pi n}\left(\frac{\pi}{n}e^{-in\pi} + \frac{\pi}{n}e^{in\pi}\right) - \frac{1}{\pi n^2}\frac{1}{in}\left[e^{-inx}\right]_{-\pi}^{\pi} \\
&= 2\frac{(-1)^n}{n^2} - \frac{1}{\pi i n^3}\left(e^{-in\pi} - e^{in\pi}\right) \\
&= 2\frac{(-1)^n}{n^2}, \qquad (n \neq 0)
\end{aligned}
$$

der vi har brukt at $e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi = \cos n\pi = (-1)^n$.

Ergo har $f(x)$ kompleks Fourierrekke

$$
f(x) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2(-1)^n}{n^2}e^{inx}
$$

Vi merker oss her at $c_n = c_{-n}$, så vi har at $a_n = 2c_n$ og $b_n = 0$. Altså er

$$
f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx
$$

Fra Kreyszig (10th), avsnitt 11.7

$\boxed{1}$

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{\pi}{2} & \text{for } x = 0 \\ \pi e^{-x} & \text{for } x > 0 \end{cases}$$

$$\implies f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]\,dw$$

med

$$\begin{aligned} A(w) &= \frac{1}{\pi}\int_{\mathbb{R}} f(v)\cos(vw)dv \\ B(w) &= \frac{1}{\pi}\int_{\mathbb{R}} f(v)\sin(vw)dv \end{aligned}$$

$$\begin{aligned} \implies \pi A(w) &= \int_0^\infty \pi e^{-v}\cos(vw)dv = -\pi e^{-v}\cos(vw)\Big|_0^\infty - \int_0^\infty \pi e^{-v}\sin(vw)w dv \\ &= \pi + \pi e^{-v}\sin(vw)w\Big|_0^\infty - \int_0^\infty \pi w^2 e^{-v}\cos(vw)dv \\ &= \pi - \pi w^2 A(w) \end{aligned}$$

$$\implies A(w) = \frac{1}{1+w^2} \quad \text{og} \quad \pi A(w) = \pi - \pi w B(w) \implies B(w) = \frac{1}{w}\left(1 - \frac{1}{1+w^2}\right) = \frac{w}{1+w^2}$$

$$\implies f(x) = \int_0^\infty \frac{\cos(wx) + w\sin(wx)}{1+w^2}dw$$

Fra Kreyszig (10th), avsnitt 11.9

$\boxed{5}$

$$f(x) = \begin{cases} e^x & \text{for } -a < x < a \\ 0 & \text{ellers} \end{cases}$$

Fouriertransformasjon:

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-iwx}dx \\ \implies \hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^x e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{(1-iw)x}dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{1-iw}e^{(1-iw)x}\Big|_{-a}^{a} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{1-iw}(e^{(1-iw)a} - e^{-(1-iw)a}) \end{aligned}$$

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$$f(x) = \mathrm{e}^{-|x|} \qquad (-\infty < x < \infty)$$

Bruker definisjonen på Fourier-transform:

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\mathrm{e}^{-i\omega x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \mathrm{e}^{x}\mathrm{e}^{-i\omega x}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x}\mathrm{e}^{-i\omega x}\,\mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \mathrm{e}^{(1-i\omega)x}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-(1+i\omega)x}\,\mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} \left[\mathrm{e}^{(1-i\omega)x}\right]_{-\infty}^{0} - \frac{1}{1+i\omega} \left[\mathrm{e}^{-(1+i\omega)x}\right]_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right)
\end{aligned}$$

Har her brukt at

$$\lim_{T\to-\infty} \mathrm{e}^{(1-i\omega)T} = 0 \qquad \text{og} \qquad \lim_{T\to\infty} \mathrm{e}^{-(1+i\omega)T} = 0$$

Her er det realdelen i eksponenten som gjør at funksjonene går mot null. Fortegnet og størrelsen på imaginærdelen har ingen innvirkning på denne grenseverdien.

Vi ender opp med svaret

$$\underline{\hat{f}(\omega) = \frac{\sqrt{2}}{\sqrt{\pi}(1+\omega^2)}}$$

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$$f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{ellers} \end{cases}$$

Bruker definisjonen på Fourier-transform:

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{-x} e^{-i\omega x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{(-1-i\omega)x} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{1}{-1-i\omega} x e^{(-1-i\omega)x} \right]_{-1}^{0} - \frac{1}{-1-i\omega} \int_{-1}^{0} e^{(-1-i\omega)x} \mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{1+i\omega} e^{(1+i\omega)} - \frac{1}{(1+i\omega)^2} \left(1 - e^{(1+i\omega)} \right) \right) \\
&= \underline{-\frac{1}{\sqrt{2\pi}} \frac{1}{1+iw} e^{1+iw} - \frac{1}{\sqrt{2\pi}} \frac{1}{(1+iw)^2} + \frac{1}{\sqrt{2\pi}} \frac{1}{(1+iw)^2} e^{1+iw}}
\end{aligned}
$$

$\boxed{9}$

$$
f(x) = \begin{cases} |x| & \text{for } -1 < x < 1 \\ 0 & \text{ellers} \end{cases}
$$

$$
\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -x e^{-iwx} dx + \int_{0}^{1} x e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} x e^{-iwx} \Big|_{-1}^{0} - \int_{-1}^{0} \frac{1}{iw} e^{-iwx} dx - \frac{1}{iw} x e^{-iwx} \Big|_{0}^{1} + \int_{0}^{1} \frac{1}{iw} e^{-iwx} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} + \frac{1}{(iw)^2} e^{-iwx} \Big|_{-1}^{0} - \frac{1}{iw} e^{-iw} - \frac{1}{(iw)^2} e^{-iwx} \Big|_{0}^{1} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iw} - \frac{1}{w^2} + \frac{1}{w^2} e^{iw} - \frac{1}{iw} e^{-iw} + \frac{1}{w^2} e^{-iw} - \frac{1}{w^2} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{2}{w^2} + \frac{2}{w^2} w \sin w + \frac{2}{w^2} \cos w \right) \\
&= \frac{\sqrt{2}}{\sqrt{\pi} w^2} (\cos w + w \sin w - 1)
\end{aligned}
$$

Brukte at $e^{iw} = \cos w + i \sin w$