Fra Kreyszig (10th), avsnitt 15.3

$\boxed{5}$

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^n$$

·) Cauchy-Hadamard:

$$R = \lim_{n\to\infty} \frac{n(n-1)}{4^n} \frac{4^{n+1}}{(n+1)n} = \lim_{n\to\infty} \frac{4(n-1)}{n+1} = 4$$

·)

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^n = (z-2i)^2 \sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}(z-2i)^{n-2}$$

Denne summen svarer til to ganger leddvis derivasjon av

$$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{4^n} \to \left|\frac{z-2i}{4}\right| < 1.$$

Dette gir $R = 4$ siden den deriverte har samme konvergensradius.

$\boxed{8}$

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$$

·) Cauchy-Hadamard:

$$R = \lim_{n\to\infty} \frac{3^n}{n(n+1)} \frac{(n+1)(n+2)}{3^{n+1}} = \lim_{n\to\infty} \frac{n+2}{n}\frac{1}{3} = \frac{1}{3}$$

·) Alternativt:

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n = \frac{3}{z}\sum_{n=1}^{\infty} \frac{3^{n-1}}{n(n+1)} z^{n+1} = \frac{3}{z}\sum_{n=0}^{\infty} \frac{3^n}{(n+1)(n+2)} z^{n+2}$$

To ganger leddvis derivasjon av summen gir

$$\sum_{n=0}^{\infty} 3^n z^n \to |3z| < 1 \implies R = \frac{1}{3}$$

$\boxed{16}$ $f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad f(-z) = \sum_{n=0}^{\infty} a_n(-z)^n \quad \forall |z| < R$

$$f(-z) = f(z) \quad \forall z \in \mathbb{R} \implies \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n \quad \forall |z| < R$$

$$\overset{15.3.2}{\implies} a_n = a_n(-1)^n \quad \forall n \in \mathbb{N}$$

$$\implies a_{2n+1} = -a_{2n+1} \quad \forall n \in \mathbb{N}$$

$$\implies a_{2n+1} = 0 \quad \forall n \in \mathbb{N}$$

Fra Kreyszig (10th), avsnitt 15.4

$\boxed{3}$ Bruker rekken til $\sin z$ som konvergerer for alle z:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\Rightarrow \quad \sin\left(\frac{z^2}{2}\right) &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{z^2}{2}\right)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{2^{2n+1}(2n+1)!} \\
&= \underline{\frac{1}{2}\left(z^2 - \frac{z^6}{2^2 3!} + \frac{z^{10}}{2^4 5!} - ...\right)}
\end{aligned}$$

Rekken konvergerer for alle z: $\quad \underline{R = \infty}$.

$\boxed{5}$ Bruker rekken

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

som konvergerer for $|z| < 1$.

$$\begin{aligned}
\frac{1}{8+z^4} &= \frac{1}{8} \cdot \frac{1}{1 - \left(\frac{-z^4}{8}\right)} \\
&= \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{-z^4}{8}\right)^n \\
&= \underline{\sum_{n=0}^{\infty} (-1)^n \frac{z^{4n}}{8^{n+1}} = \frac{1}{8}\left(1 - \frac{z^4}{8} + \frac{z^8}{64} - ...\right)}
\end{aligned}$$

Rekken konvergerer for

$$\begin{aligned}
\left|\frac{-z^4}{8}\right| &< 1 \\
|z|^4 &< 8 \\
|z| &< 2^{3/4} \qquad \Rightarrow \quad \underline{R = 2^{3/4}}
\end{aligned}$$

$\boxed{7}$ $2\sin^2\left(\frac{z}{2}\right)$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) \implies 2\sin^2(x) = 1 - \cos(2x)$$

$$\text{Vet at: } \cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} \quad R = \infty$$

$$\implies 2\sin^2\left(\frac{z}{2}\right) = 1 - \cos(z) = 1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = -\sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} \quad R = \infty$$

$\boxed{23}$

$$\frac{1}{(z-i)^2}, \quad z_0 = -i$$

$$\text{Vet at } \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z| < 1$$

$$\cdot) \quad \frac{1}{i-z} = \frac{1}{2i - z - i} = \frac{1}{2i - (z+i)} = \frac{1}{2i}\frac{1}{1 - \frac{z+i}{2i}} = \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2i)^{n+1}}(z+i)^n \qquad |z+i| < 2$$

$$\cdot) \left(\frac{1}{i-z}\right)' = -\frac{1}{(i-z)^2}(-1) = \frac{1}{(i-z)^2} = \frac{1}{(z-i)^2} = \sum_{n=1}^{\infty} \frac{1}{(2i)^{n+1}} n(z+i)^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2i)^{n+2}}(n+1)(z+i)^n \qquad |z+i| < 2 \implies R = 2$$

$\boxed{24}$

$$\begin{aligned} \mathrm{e}^{z(z-2)} &= \mathrm{e}^{z^2-2z} \\ &= \mathrm{e}^{(z-1)^2-1} \end{aligned}$$

Kan nå bruke rekken for e^z, som konvergerer for alle z:

$$\begin{aligned} \mathrm{e}^{(z-1)^2-1} &= \mathrm{e}^{-1}\sum_{n=0}^{\infty}\frac{\left((z-1)^2\right)^n}{n!} \\ &= \mathrm{e}^{-1}\sum_{n=0}^{\infty}\frac{(z-1)^{2n}}{n!} \\ &= \underline{\frac{1}{\mathrm{e}}\left(1 + (z-1)^2 + \frac{(z-1)^4}{2} + \frac{(z-1)^6}{6} + \ldots\right)} \end{aligned}$$

med konvergensradius $\underline{R = \infty}$.

Fra Kreyszig (10th), avsnitt 16.1

$\boxed{2}$

$$\frac{e^{-\frac{1}{z^2}}}{z^2}$$

$$\text{Vet at } e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad R = \infty$$

$$\cdot) \quad e^{-\frac{1}{z^2}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! z^{2n}}$$

$$\cdot) \quad \frac{e^{-\frac{1}{z^2}}}{z^2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! z^{2n+2}}$$

$$\cdot) \quad \left| \frac{z^{2n+2} n!}{(n+1)! z^{2n+4}} \right| = \left| \frac{1}{(n+1) z^2} \right| \to 0 < 1$$

$$\implies \text{konvergent for } \left| \frac{1}{z} \right| < \infty \implies \text{konvergent for} |z| > 0$$

$\boxed{5}$

$$\begin{aligned} \frac{\mathrm{e}^z}{(z-1)^2} &= \frac{\mathrm{e}^{(z-1)+1}}{(z-1)^2} \\ &= \frac{\mathrm{e}}{(z-1)^2} \mathrm{e}^{(z-1)} \end{aligned}$$

Bruker nå rekken til eksponentialfunksjonen:

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \text{konvergerer for alle z}$$

$$\begin{aligned} \Rightarrow \quad \frac{\mathrm{e}}{(z-1)^2} \mathrm{e}^{(z-1)} &= \frac{\mathrm{e}}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \\ &= \underline{\sum_{n=0}^{\infty} \frac{\mathrm{e}(z-1)^{n-2}}{n!} = \mathrm{e} \left(\frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2} + \frac{(z-1)}{6} + ... \right)} \end{aligned}$$

Rekken til $\mathrm{e}^{(z-1)}$ konvergerer for alle z, så konvergensområdet blir

$$\underline{|z-1| > 0}$$

$\boxed{6}$

$$\frac{1}{z^2(z-i)}, \qquad z_0 = i.$$

Vi har (se oppgave 16.1.12)

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)(z-i)^n}{(-1)^{n+1}} \quad \text{for } |z-i| < 1$$

$$\implies \frac{1}{z^2(z-i)} = \frac{1}{z-i} \sum_{n=0}^{\infty} \frac{(n+1)(z-i)^n}{(-i)^{n+1}} = \sum_{n=0}^{\infty} \frac{(n+1)(z-i)^{n-1}}{(-i)^{n+1}}$$

$$= \sum_{n=-1}^{\infty} \frac{(n+2)(z-i)^n}{(-i)^{n+2}}$$

$\implies$ Konvergent for $0 < |z-i| < 1$

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$$\frac{1}{z^2}, \quad z_0 = i. \qquad \text{Vet at } \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

1)

$$\cdot) \quad -\frac{1}{z} = \frac{1}{-i-(z-i)} = \frac{1}{-i} \frac{1}{1-\left(\frac{z-i}{-i}\right)} = \frac{1}{-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{-i}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(-i)^{n+1}} (z-i)^n$$

$\implies$ Konvergent for $\left|\frac{z-i}{-i}\right| < 1 \implies$ Konvergent for $|z-i| < 1$

$$\cdot) \quad \left(-\frac{1}{z}\right)' = \frac{1}{z^2} = \sum_{n=1}^{\infty} \frac{n(z-i)^{n-1}}{(-i)^{n+1}} = \sum_{n=0}^{\infty} \frac{(n+1)(z-i)^n}{(-i)^{n+2}}$$

$\implies$ Konvergent for $|z-i| < 1$

2)

$$\cdot) -\frac{1}{z} = \frac{1}{-i-(z-i)} = \frac{1}{(z-i)} \frac{1}{\frac{-i}{z-i} - 1} = \frac{-1}{(z-i)} \frac{1}{1 - \frac{-i}{z-i}} =$$

$$-\frac{1}{z-i} \sum_{n=0}^{\infty} \left(\frac{-i}{z-i}\right)^n = -\sum_{n=0}^{\infty} \frac{(-i)^n}{(z-i)^{n+1}}$$

$\implies$ Konvergent for $\left|\frac{1}{z-i}\right| < 1 \implies$ Konvergent for $|z-i| > 1$

$$\cdot) \left(-\frac{1}{z}\right)' = \frac{1}{z^2} = \sum_{n=0}^{\infty} (n+1) \frac{(-1)^n}{(z-i)^{n+2}} \implies \text{Konvergent for } |z-i| > 1$$

Obs: $z = 0$ eneste pol $\implies$ Vet at vi får forskjellige rekker for $|z-i| < 1$ og $|z-i| > 1$ siden $|i| = 1$