Vector Calculus And Continuum Conservation Equations In Curvilinear Orthogonal Coordinates

Robert Marskar: November 25, 2008

In order to rewrite the conservation equations(continuity, momentum, energy) to some curvilinear ortogonal coordinate system(polar, spherical, elliptical etc.) it is a useful strategy to first review vector analysis.

A displacement vector $\mathbf{r}$ in some three-dimensional ortogonal metric coordinate system (q_1, q_2, q_3) can be written by its base vectors by

$$d\mathbf{r} = dq_1\boldsymbol{\varepsilon}_1 + dq_2\boldsymbol{\varepsilon}_2 + dq_3\boldsymbol{\varepsilon}_3 \tag{1}$$

where $\boldsymbol{\varepsilon}$ corresponds to a *base vector*. The base vectors do not necessarily have unit magnitude and dimension length such as is the case in cartesian coordinates. One can expand this in terms of unit vectors instead by letting $\boldsymbol{\varepsilon}_i = h_i\mathbf{e}_i$ where h_i is a scaling factor for the base vector, i.e. its length. The $\mathbf{e}_i$'s are mutually orthogonal and have unit magnitude. This gives for the differential displacement

$$d\mathbf{r} = dq_1h_1\mathbf{e}_1 + dq_2h_2\mathbf{e}_2 + dq_3h_3\mathbf{e}_3$$

An example of this is the displacement vector in cylindrical polar coordinates $(q_1, q_2, q_3) = (\rho, \varphi, \zeta)$, it can be written as

$$d\mathbf{r} = d\rho\hat{\boldsymbol{\rho}} + d\varphi\rho\hat{\boldsymbol{\varphi}} + d\zeta\hat{\boldsymbol{\zeta}}$$

where h_1, h_2, h_3 are easily recognized on inspection. It transpires that these "scaling factors" does not necessarily have unit magnitude and in fact could carry different dimensions. In contrast, a displacement vector in cartesian coordinates is written

$$d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$

where all the h's have dimension and length of unity. The net differential displacement still carries the dimension length.

Consider now a differential displacement in the space (q_1, q_2, q_3), its magnitude is

$$d\mathbf{r}\cdot d\mathbf{r} = (ds)^2 = (h_1dq_1)^2 + (h_2dq_2)^2 + (h_3dq_3)^2 \tag{2}$$

since the unit vectors are mutually orthogonal.

Gradient

The component i of the gradient of a scalar ψ is

$$\mathbf{e_i}\cdot\nabla\psi = \frac{\partial\psi}{\partial s_i} = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}$$

A generalization gives the the gradient

$$\nabla\psi = \sum_i \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\mathbf{e}_i \tag{3}$$

For alternative derivations, see for example Arfken&Weber[1].

Divergence

The *divergence* of a vector $\mathbf{v}$ is defined[1]

$$\text{div }\mathbf{v} = \lim_{\int d\tau\to 0}\frac{\oint \mathbf{v}\cdot d\boldsymbol{\sigma}}{\int d\tau} \tag{4}$$

For convenience, a right-hand system has been chosen. A differential area in the chosen space can be written

$$d\boldsymbol{\sigma} = ds_2ds_3\mathbf{e}_1 + ds_3ds_1\mathbf{e}_2 + ds_1ds_2\mathbf{e}_3$$

where now, should it not be clear, $ds_i = h_idq_i$. In cartesian coordinates for instance, this would simply reduce to $d\boldsymbol{\sigma} = (dydz)\hat{\mathbf{x}} + (dzdx)\hat{\mathbf{y}} + (dxdy)\hat{\mathbf{z}}$ which is a slanted square face. The above equation is a generalization to curved faces.

This lets us evaluate the surface integral, the vector product $\mathbf{v}\cdot d\boldsymbol{\sigma}$ is integrated termwise. Doing the $v_1d\sigma_1$ term first one finds that the integral for the two faces in the $\mathbf{e}_1$-direction is

$$\int v_1d\sigma_1 = -v_1ds_2ds_3 + \left[v_1ds_2ds_3 + \frac{\partial\left(v_1ds_2ds_3\right)}{\partial q_1}dq_1\right] \tag{5}$$

where the previously mentioned limit has been taken.

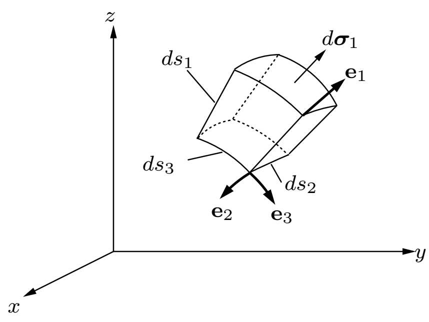

FIG. 1: Differential volume and surfaces.

We will now explain this, see FIG. 1 for reference. The first term is simply the surface integral over the face opposite the label $d\boldsymbol{\sigma}_1$ in FIG. 1. The second term comes from a Taylor expansion of the vector component v_1 and the differential area. This just means that the v_1 component changes by

$$v_1(q_1+dq_1) = v_1(q_1) + \frac{\partial v_1}{\partial q_1}dq_1 + \mathcal{O}(dq_1^2)$$

Similarly the surface area has changed by

$$d\sigma_1(q_1+dq_1) = d\sigma_1(q_1) + \frac{\partial d\sigma_1}{\partial q_1}dq_1 + \mathcal{O}(dq_1^2)$$

For notational reasons we now denote $v_1(q_1)$ by v_1 and a similar formalism for $d\sigma_1$. Then the surface integral over the second face is found by multiplication of $v_1(q_1+dq_1)$ and $d\sigma_1(q_1+dq_1)$

$$\begin{aligned} &= v_1d\sigma_1 + v_1\frac{\partial d\sigma_1}{\partial q_1}dq_1 + d\sigma_1\frac{\partial v_1}{\partial q_1}dq_1 + \mathcal{O}(dq_1^2) \\ &= v_1d\sigma_1 + \frac{\partial\left(v_1d\sigma_1\right)}{\partial q_1}dq_1 + \mathcal{O}(dq_1^2) \end{aligned}$$

which follows from the product rule. Imposing the limit $\int d\tau \to 0$ leaves only the first order terms and EQ. 5 is fulfilled. EQ. 5 now reduces to

$$\int v_1d\sigma_1 = \frac{\partial(v_1ds_2ds_3)}{\partial q_1}dq_1 = \frac{\partial(v_1h_2h_3)}{\partial q_1}dq_1dq_2dq_3$$

We have now integrated over two of the faces. The result is of course the same for the other four faces, with a cyclic permutation of 123. In other words

$$\int v_2d\sigma_2 = \frac{\partial(v_2ds_3ds_1)}{\partial q_2}dq_2 = \frac{\partial(v_2h_3h_1)}{\partial q_2}dq_1dq_2dq_3$$

$$\int v_3d\sigma_3 = \frac{\partial(v_3ds_1ds_2)}{\partial q_1}dq_3 = \frac{\partial(v_3h_1h_2)}{\partial q_2}dq_1dq_2dq_3$$

Addition of these three integrals gives the closed surface integral, as in EQ. 4. The differential volume is simply $ds_1ds_2ds_3 =$

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$h_1h_2h_3dq_1dq_2dq_3$. By collecting terms and imposing the limit we conclude that

$$\text{div } \mathbf{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(v_1h_2h_3)}{\partial q_1} + \frac{\partial(v_2h_3h_1)}{\partial q_2} + \frac{\partial(v_3h_1h_2)}{\partial q_3}\right] \tag{6}$$

As previously mentioned, in cartesian coordinates the h_i's are equal to one. This allows us to write the divergence as an inner product of ∇ and $\mathbf{v}$.

It is however common to adopt the notation $\nabla\cdot\mathbf{v} = \text{div } \mathbf{v}$ even in curvilinear coordinates. This is bad notation since it suggests that the divergence can be written as an inner product which is not the case.

Laplacian

The Laplacian of a scalar $\psi(q_1, q_2, q_3)$ can be developed by taking the gradient of the divergence. That is, we choose the components of $\mathbf{v}$ as the vector components of EQ. 3. This gives without dwelling on details

$$\begin{aligned}\text{div } \mathbf{v} &= \text{div grad } \psi \\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right)\right. \\ &\quad \left. + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]\end{aligned} \tag{7}$$

This is by definition the Laplacian and is sometimes written $\nabla^2\psi$.

Some people seem to like confusing the Laplacian of a scalar with the Laplacian of a vector. A component of the Laplacian of a vector is not the same as the Laplacian of the vector component, as will be shown later.

Curl

The curl can be found using Stokes' theorem

$$\int_S (\text{curl } \mathbf{v})\cdot d\boldsymbol{\sigma} = \oint \mathbf{v}\cdot d\mathbf{s} \tag{8}$$

One will merit from doing this component-wise on an infinitesimal area. Thus let the surface be made up of three surfaces, each with q_1, q_3 or q_3 kept constant.

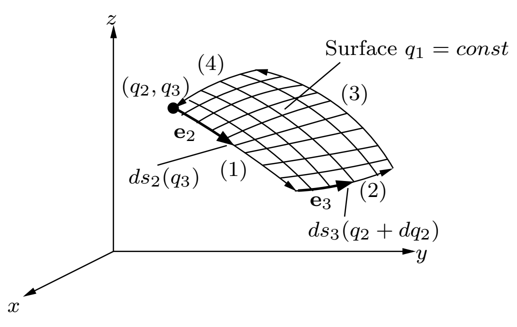

FIG. 2: Termwise integration of EQ. 8

Consider first the surface where q_1 is constant. Following the loop in the figure counterclockwise from the leftmost corner, the line integral is

$$\begin{aligned}\oint \mathbf{v}\cdot d\mathbf{s} =& v_2ds_2 + \left[v_3ds_3 + \frac{\partial(v_3ds_3)}{\partial q_2}dq_2\right] \\ &- \left[v_2ds_2 + \frac{\partial(v_2ds_2)}{\partial q_3}dq_3\right] - v_3ds_3 \\ =& \left[\frac{\partial}{\partial q_2}(h_3v_3) - \frac{\partial}{\partial q_3}(h_2v_2)\right]dq_2dq_3\end{aligned} \tag{9}$$

This also follows by Taylor expanding the vector components. The first integral along the line (1) is simply

$$= v_2ds_2$$

For the second line integral (2), the infinitesimal length and the vector component has changed according to a Taylor series

$$v_3(q_2 + dq_2) = v_3(q_2) + \frac{\partial v_3(q_2)}{\partial q_2}dq_2 + \mathcal{O}(dq_2^2)$$
$$ds_3(q_2 + dq_2) = ds_3(q_2) + \frac{\partial ds_3(q_2)}{\partial q_2}dq_2 + \mathcal{O}(dq_2^2)$$

Using the same formalism as earlier, $v_3(q_2)$ is denoted v_3 etc., combined with the product rule gives that the line integral (2) is

$$= v_3ds_3 + \frac{\partial(v_3ds_3)}{\partial q_2}dq_2 + \mathcal{O}(dq_2^2)$$

The remaining terms in EQ. 9 is found by identical argumentation on the two remaining line integrals. For an infinitesimal area, the left hand side of EQ. 8 can be written

$$\int_S (\text{curl } \mathbf{v})\cdot d\boldsymbol{\sigma} = \mathbf{e}_1\cdot(\text{curl } \mathbf{v})ds_2ds_3$$

since the first surface is oriented in the q_1 direction. Combining these results with the limit of an infinitesimal area gives

$$[\text{curl } \mathbf{v}]_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial q_2}(h_3v_3) - \frac{\partial}{\partial q_3}(h_2v_2)\right]$$

since the higher-order terms vanish in the limit. The remaining two components are found in a similar way, or by cyclic permutation of 123. Thus

$$[\text{curl } \mathbf{v}]_2 = \frac{1}{h_3h_1}\left[\frac{\partial}{\partial q_3}(h_1v_1) - \frac{\partial}{\partial q_1}(h_3v_3)\right]$$
$$[\text{curl } \mathbf{v}]_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(h_2v_2) - \frac{\partial}{\partial q_2}(h_1v_1)\right]$$

Or in index notation

$$[\text{curl } \mathbf{v}]_i = \frac{1}{h_jh_k}\varepsilon_{ijk}\partial_j(h_kv_k) \tag{10}$$

where ε_{ijk} denotes the Levi-Civita pseudo-tensor.

Occasionally, but rarely, it is useful to write this as a determinant. It can then be written

$$\text{curl } \mathbf{v} = \frac{1}{h_1h_2h_3}\begin{vmatrix}\mathbf{e}_1h_1 & \mathbf{e}_2h_2 & \mathbf{e}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1v_1 & h_2v_2 & h_3v_3\end{vmatrix} \tag{11}$$

Next we will develop the divergence and curl in cylindrical and spherical polar coordinates.

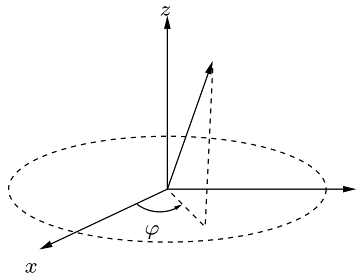

FIG. 3: Cylindrical polar coordinates.

Cylindrical Polar Coordinates

In cylindrical polar coordinates $(q_1, q_2, q_3) = (\rho, \varphi, \zeta)$ the relation between the cylindrical coordinates and the cartesian xyz is

$$x = \rho\cos\varphi; \quad y = \rho\sin\varphi; \quad z = \zeta$$

A differential displacement can be written

$$d\mathbf{r} = d\rho\hat{\boldsymbol{\rho}} + d\varphi\rho\hat{\boldsymbol{\varphi}} + d\zeta\hat{\boldsymbol{\zeta}}$$

which again gives

$$d\mathbf{r}\cdot d\mathbf{r} = (ds)^2 = (d\rho)^2 + (\rho d\varphi)^2 + (d\zeta)^2$$

One should now recognize

$$h_1 = h_\rho = 1; \quad h_2 = h_\varphi = \rho; \quad h_3 = h_\zeta = 1$$

Going to EQ. 6 and EQ. 11, or EQ. 10, the divergence and curl is found by direct insertion to be

$$\begin{aligned}
\text{div }\mathbf{v} &= \frac{1}{\rho}\frac{\partial(\rho v_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_\zeta}{\partial\zeta}\\
[\text{curl }\mathbf{v}]_\rho &= \frac{1}{\rho}\frac{\partial v_\zeta}{\partial\varphi} - \frac{\partial v_\varphi}{\partial\zeta}\\
[\text{curl }\mathbf{v}]_\varphi &= \frac{\partial v_\rho}{\partial\zeta} - \frac{\partial v_\zeta}{\partial\rho}\\
[\text{curl }\mathbf{v}]_\zeta &= \frac{1}{\rho}\frac{\partial(\rho v_\varphi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial v_\rho}{\partial\varphi}
\end{aligned}$$

If one absolutely insists this can be written in matrix form

$$\text{curl }\mathbf{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{\zeta}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial\zeta} \\ v_\rho & \rho v_\varphi & v_\zeta \end{vmatrix}$$

Spherical Polar Coordinates

In spherical cylindrical coordinates $(q_1, q_2, q_3) = (\rho, \theta, \varphi)$ the relation to the cartesian xyz is

$$x = \rho\sin\theta\cos\varphi; \quad y = \rho\sin\theta\sin\varphi; \quad z = \rho\cos\theta$$

A differential displacement can be written

$$d\mathbf{r} = d\rho\hat{\boldsymbol{\rho}} + d\theta\rho\hat{\boldsymbol{\theta}} + d\varphi\rho\sin\theta\hat{\boldsymbol{\varphi}}$$

One should now recognize

$$h_1 = h_\rho = 1; \quad h_2 = h_\theta = \rho; \quad h_3 = h_\varphi = \rho\sin\theta$$

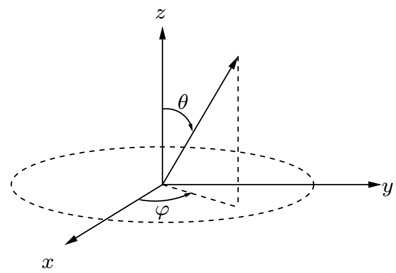

FIG. 4: Spherical polar coordinates.

Going to EQ. 6 gives the divergence of a vector $\mathbf{v}$

$$\text{div }\mathbf{v} = \frac{1}{\rho^2}\frac{\partial(\rho^2 v_\rho)}{\partial\rho} + \frac{1}{\rho\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{\rho\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

From EQ. 10 the curl components can be found

$$\begin{aligned}
[\text{curl }\mathbf{v}]_\rho &= \frac{1}{\rho\sin\theta}\left[\frac{\partial(v_\varphi\sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right]\\
[\text{curl }\mathbf{v}]_\theta &= \frac{1}{\rho\sin\theta}\left[\frac{\partial v_\rho}{\partial\varphi} - \sin\theta\frac{\partial(\rho v_\varphi)}{\partial\rho}\right]\\
[\text{curl }\mathbf{v}]_\varphi &= \frac{1}{\rho}\frac{\partial(\rho v_\theta)}{\partial\rho} - \frac{1}{\rho}\frac{v_\rho}{\partial\theta}
\end{aligned}$$

This can be written in matrix form following EQ. 11

$$\text{curl }\mathbf{v} = \frac{1}{\rho^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\theta}} & \rho\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ v_\rho & \rho v_\theta & \rho\sin\theta v_\varphi \end{vmatrix}$$

This is rarely seen in practice.

The Conservation Equations

In this text we neglect the rewriting of the energy equation and focus on the continuity equation and the incompressible Navier-Stokes equation. From here on we adopt the notations curl $\mathbf{v} = \nabla\times\mathbf{v}$ and div $\mathbf{v} = \nabla\cdot\mathbf{v}$. The fully compressible equations are in their vector form given by[3]

$$\frac{\partial\rho}{\partial t} + \text{div }(\rho\mathbf{v}) = 0 \tag{12}$$

and

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right]\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}_{ij} \tag{13}$$

where ρ denotes the density of the fluid and $\boldsymbol{\tau}_{ij}$ denotes the full viscous stress tensor including the second coefficient of viscosity. Including compressibility into the Navier-Stokes equations makes for a too cumbersome calculation.

In the case of constant density the final term on the right hand of EQ. 13 side reduces to $\nu\nabla^2\mathbf{v}$. We will start by rewriting the Navier-Stokes equations[2]. Two vector identities are needed

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\mathbf{v}\times(\nabla\times\mathbf{v}) + \frac{1}{2}\nabla|\mathbf{v}|^2 \tag{14a}$$

$$\nabla^2\mathbf{v} = \nabla\,(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) \tag{14b}$$

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This can be found in any elementary book on calculus, and is in fact quite easy to show using the Levi-Civita tensor. This puts the Navier-Stokes equations on the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v}\times\boldsymbol{\omega} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu\left[\nabla\left(\nabla\cdot\mathbf{v}\right) - \nabla\times\boldsymbol{\omega}\right] \quad (15)$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. The fluid velocity vector is denoted by $\mathbf{v}$ and the components by v_1, v_2, v_3.

The problem with the vector Laplacian is that we have

$$\nabla^2\left(\hat{\boldsymbol{\rho}}v_\rho + \hat{\boldsymbol{\varphi}}v_\varphi + \hat{\boldsymbol{z}}v_z\right) = \nabla^2\left(\hat{\boldsymbol{\rho}}v_\rho + \hat{\boldsymbol{\varphi}}v_\varphi\right) + \hat{\boldsymbol{z}}\nabla^2 v_z \quad (16)$$

Writing out the first paranthesis on the right hand side would require several pages of equations. Conversely, the Laplacian in cartesian coordinates is

$$\nabla^2\left(\hat{\boldsymbol{x}}v_x + \hat{\boldsymbol{y}}v_y + \hat{\boldsymbol{z}}v_z\right) = \hat{\boldsymbol{x}}\nabla^2 v_x + \hat{\boldsymbol{y}}\nabla^2 v_y + \hat{\boldsymbol{z}}\nabla^2 v_z$$

This may seem confusing at first, but remember that we are not looking for the Laplacian of a component v_i, but looking for component i of the Laplacian of $\mathbf{v}$. Confusing these is a common mistake!

Cylindrical Polar Coordinates

In cylindrical polar coordinates, the gradient is given by EQ. 3, $\boldsymbol{\omega}$ and $\nabla\times\boldsymbol{\omega}$ are given by EQ. 10 or EQ. 11. In order not to confuse the density with the radial coordinate, we denote the latter by r. Starting with the term $\nabla(\nabla\cdot\mathbf{v})$, these equations give after some extensive rewriting

$$[\nabla(\nabla\cdot\mathbf{v})]_r = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r}\frac{\partial^2 v_\varphi}{\partial\varphi\partial r} - \frac{1}{r^2}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial^2 v_\zeta}{\partial r\partial\zeta}$$

$$[\nabla(\nabla\cdot\mathbf{v})]_\varphi = \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial v_r}{\partial\varphi} + \frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\varphi} + \frac{1}{r}\frac{\partial^2 v_\zeta}{\partial\varphi\partial\zeta}$$

The terms $\nabla\times\boldsymbol{\omega}$ are(after some rewriting)

$$[\nabla\times\boldsymbol{\omega}]_r = -\left[\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\zeta^2}\right]v_r + \frac{1}{r}\frac{\partial^2 v_\varphi}{\partial r\partial\varphi} + \frac{1}{r^2}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial^2}{\partial r\partial\zeta}$$

$$[\nabla\times\boldsymbol{\omega}]_\varphi = -\left[\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]v_\varphi + \frac{v_\varphi}{r^2} + \frac{1}{r}\frac{\partial^2 v_r}{\partial\varphi\partial r} - \frac{1}{r^2}\frac{\partial v_r}{\partial\varphi} + \frac{1}{r}\frac{\partial^2 v_\zeta}{\partial\zeta\partial\varphi}$$

Notice that it is not necessary to calculate $[\nabla(\nabla\cdot\mathbf{v}) - \nabla\times\boldsymbol{\omega}]_\zeta$ because by EQ. 16 one finds that the vector Laplacian $[\nabla^2\mathbf{v}]_\zeta = \nabla^2 v_\zeta$. This is not true for the other two components which is why they need to be explicitly calculated. In addition, the term $\mathbf{v}\times\boldsymbol{\omega}$ is

$$[\mathbf{v}\times\boldsymbol{\omega}]_r = \frac{v_\varphi}{r}\left[\frac{\partial(rv_\varphi)}{\partial r} - \frac{\partial v_r}{\partial\varphi}\right] - v_\zeta\left[\frac{\partial v_r}{\partial\zeta} - \frac{\partial v_\zeta}{\partial r}\right]$$

$$[\mathbf{v}\times\boldsymbol{\omega}]_\varphi = v_\zeta\left[\frac{1}{r}\frac{\partial v_\zeta}{\partial\varphi} - \frac{\partial v_\varphi}{\partial\zeta}\right] - v_r\left[\frac{1}{r}\frac{\partial(rv_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\varphi}\right]$$

$$[\mathbf{v}\times\boldsymbol{\omega}]_\zeta = v_r\left[\frac{\partial v_r}{\partial\zeta} - \frac{\partial v_\zeta}{\partial r}\right] - v_\varphi\left[\frac{1}{r}\frac{\partial v_\zeta}{\partial\varphi} - \frac{\partial v_\varphi}{\partial\zeta}\right]$$

There is now only one more vector term which needs treatment, and that is the gradient of $\frac{1}{2}|\mathbf{v}|^2$. By EQ. 3 this is

$$\left[\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_r = v_r\frac{\partial v_r}{\partial r} + v_\varphi\frac{\partial v_\varphi}{\partial r} + v_\zeta\frac{\partial v_\zeta}{\partial r}$$

$$\left[\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_\varphi = \frac{v_r}{r}\frac{\partial v_r}{\partial\varphi} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\zeta}{r}\frac{\partial v_\zeta}{\partial\varphi}$$

$$\left[\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_\zeta = v_r\frac{\partial v_r}{\partial\zeta} + v_\varphi\frac{\partial v_\varphi}{\partial\zeta} + v_\zeta\frac{\partial v_\zeta}{\partial\zeta}$$

We are now in position to write down the momentum equations by use of EQ. 15. Some viscous terms can be put into the Laplacian by EQ. 7, these terms are put first on the right hand side of the equations above. For the radial component $\hat{\boldsymbol{r}}$ one finds

$$\frac{\partial v_r}{\partial t} + (\mathbf{v}\cdot\nabla)\,v_r - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}\right)$$

Similarly the azimuthal component is

$$\frac{\partial v_\varphi}{\partial t} + (\mathbf{v}\cdot\nabla)\,v_\varphi - \frac{v_r v_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi} + g_\varphi + \nu\left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}\right)$$

Finally, the vertical component is

$$\frac{\partial v_\zeta}{\partial t} + (\mathbf{v}\cdot\nabla)\,v_\zeta = -\frac{1}{\rho}\frac{\partial p}{\partial\zeta} + g_\zeta + \nu\nabla^2 v_\zeta$$

where the convective time-derivative is

$$\mathbf{v}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\varphi}{r}\frac{\partial}{\partial\varphi} + v_\zeta\frac{\partial}{\partial\zeta}$$

It is easy to be led astray at this point since one might intuively think that since $[(\mathbf{v}\cdot\nabla)\mathbf{v}]_x = (\mathbf{v}\cdot\nabla)v_x$ this also applies in curvilinear coordinates. This, as in the case of the vector Laplacian, is not true.

One should now have learned the difference between the Laplacian of a vector component and the component of a vector Laplacian. To summarize

$$[\nabla^2\mathbf{v}]_r = \nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}$$

$$[\nabla^2\mathbf{v}]_\varphi = \nabla^2 v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}$$

$$[\nabla^2\mathbf{v}]_\zeta = \nabla^2 v_\zeta$$

The equation of continuity can be written from EQ. 4

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\varphi)}{\partial\varphi} + \frac{\partial(\rho v_\zeta)}{\partial\zeta}$$

The energy equation is left out of this text.

Spherical Polar Coordinates

The same terms are needed, $\mathbf{v}\times\boldsymbol{\omega}$, $\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)$, $\nabla(\nabla\cdot\mathbf{v})$ and $\nabla\times\boldsymbol{\omega}$. We will however not go into the explicit details on the calculation but cite the result. By use of EQ. 3, 7, 10 and EQ. 14b one finds after extensive use of the product rule and rewriting that the vector Laplacian(the final term in EQ. 15) is

$$[\nabla^2\mathbf{v}]_r = \nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_\theta}{r^2\tan\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

$$[\nabla^2\mathbf{v}]_\theta = \nabla^2 v_\theta - \frac{v_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

$$[\nabla^2\mathbf{v}]_\varphi = \nabla^2 v_\varphi - \frac{v_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}$$

For the left hand side of the Navier-Stokes equations one finds

$$\left[-\mathbf{v}\times\boldsymbol{\omega} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_r = (\mathbf{v}\cdot\nabla)v_r - \frac{v_\theta^2 + v_\varphi^2}{r}$$

$$\left[-\mathbf{v}\times\boldsymbol{\omega} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_\theta = (\mathbf{v}\cdot\nabla)v_\theta - \frac{v_r v_\theta - v_\varphi^2\cot\theta}{r}$$

$$\left[-\mathbf{v}\times\boldsymbol{\omega} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]_\varphi = (\mathbf{v}\cdot\nabla)v_\varphi - \frac{v_r v_\varphi + v_\theta v_\varphi\cot\theta}{r}$$

where the convective term is

$$\mathbf{v}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

To summarize, the incompressible Navier-Stokes equations are (starting with the radial component)

$$\frac{\partial v_r}{\partial t}+(\mathbf{v}\cdot\nabla)v_r - \frac{v_\theta^2+v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{2v_r}{r^2}\right.$$
$$\left. - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_\theta}{r^2\tan\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}\right)$$

The $\hat{\boldsymbol{\theta}}$ component:

$$\frac{\partial v_\theta}{\partial t}+(\mathbf{v}\cdot\nabla)v_\theta - \frac{v_r v_\theta - v_\varphi\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + g_\theta + \nu\left(\nabla^2 v_\theta\right.$$
$$\left. - \frac{v_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi}\right)$$

and finally the azimuthal component

$$\frac{\partial v_\varphi}{\partial t}+(\mathbf{v}\cdot\nabla)v_\varphi - \frac{v_r v_\varphi + v_\theta v_\varphi\cot\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + g_\varphi$$
$$+\nu\left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}\right)$$

The equation of continuity can be written from EQ. 4 and is

$$\frac{\partial\rho}{\partial t} + \left[\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\varphi)}{\partial\varphi}\right] = 0$$

For constant density, ρ vanishes. The Navier-Stokes equations are quite messy but are sometimes useful. The energy equation is best transformed by transforming each element in the viscous stress tensor.

*

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