Method - Open Loop Planning

Open Loop Planning

We are considering for a real-life system, which we model using an ordinary differential equation

$$\dot{x} = f(x, u)$$

with state $x \in \mathbb{R}^{n_x}$, and control $u \in \mathbb{R}^{n_u}$. We want to find a find a control strategy $u(t)$ such that the trajectory $x(t)$ that the system (hopefully) follows in the future, is optimal in the sense of some cost function.

Discrete Dynamics

For simplicity, we assume in the following that we approximate the continuous state trajectory $x(t)$ on grid points $t_0, t_1, \dots, t_k, t_{k+1}, \dots$ as $x(t_k) \approx x_k$. Also, for simplicity, we assume that over each interval the control is constant: $u(t) = u_k, \forall t \in [t_k, t_{k+1}]$, and that the intervals have the same constant duration, $h = t_{k+1} - t_k$.
Then we can find discrete dynamics:

$$x_{k+1} = F(x_k, u_k)$$

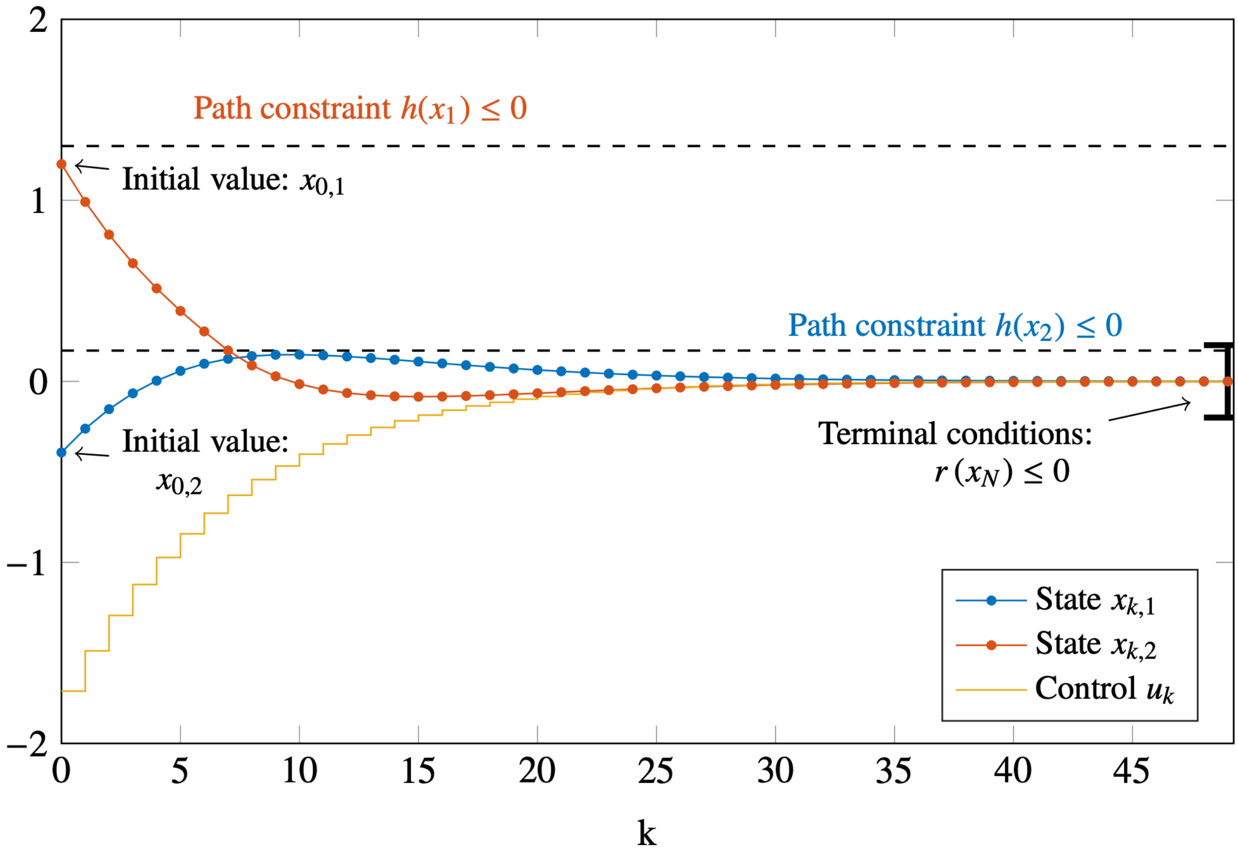
which can be obtained from a continuous-time ODE using a single (or multiple) steps of a one-step integration method. A commonly used integration method is the Runge-Kutta method of order 4:

$$x_{k+1} = \dots$$

We already implemented such a function in the code, a single step:

```
F = model.discreteDynamics(h)
x_1 = F(x_0, u_0)
```

Discrete Optimal Control Problem



Given the system model and constraints, a quite generic discrete time optimal control problem can be formulated as the following constrained NLP:

$$
\begin{aligned}
\min_{x_0, u_0, x_1, u_1} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) + E(x_N) \\
\text{s.t.} \quad & 0 = x_0 - \bar{x}_0 \\
& 0 = x_{k+1} - F(x_k, u_k), \qquad k = 0, \ldots, N-1 \\
& 0 \leq h(x_k, u_k), \qquad k = 0, \ldots, N-1
\end{aligned}
$$

- The decision variables of the problem contain the *discrete* state and control trajectories on the time grid. We have $N+1$ variables

$$x_0, x_1, \ldots, x_N$$

for the state trajectory each of which is of a vector of n_x variables, and N variables for the control trajectory:

$$u_0, u_1, \ldots, u_{N-1}$$

each of which is a vector of size n_u.

- The trajectory should satisfy some constraints, for example simple bounds, this is expressed in the inequality for each state and control pair

$$h(x_k, u_k)$$

- Most importantly, the trajectory that we plan, should satisfy the discrete dynamics of the system, and should start at some initial point $\bar{x}_0 \in \mathbb{R}^{n_x}$, given by the equality constraints:

$$\begin{aligned} 0 &= x_0 - \bar{x}_0 \\ 0 &= x_{k+1} - F(x_k, u_k), \qquad k = 0, \ldots, N-1 \end{aligned}$$

- The cost function is divided into a *stage cost* $l(x_k, u_k)$ for each interval and a terminal cost $E(x_N)$ for the terminal node. A very common example is a *tracking cost*

$$\sum (x_k - \bar{x}_k)^\top Q (x_k - \bar{x}_k) + (u_k - \bar{u}_k)^\top R (u_k - \bar{u}_k)$$

when we want to find a control which makes the system follow a given reference of states $\bar{x}_0, \bar{x}_1, \ldots,$ and controls $\bar{u}_0, \bar{u}_1, \ldots,$. Here Q and R are (typically diagonal) *weighting matrices*, to emphasise the importance of either control or state tracking.

Practical Solution of the Nonlinear Programm

The nonlinear program above is of the general form

$$
\begin{aligned}
\min_{w} \quad & f(w) \\
\text{s.t.} \quad & 0 = g(w) \\
& 0 \leq h(w)
\end{aligned}
$$

with variables w, objective function f, equality constraints g and inequality constraints h. Such an NLP can be formulated and solved using a number of tools:

- The `Python/Matlab` framework `CasAdi` which, when installed for Python using `pip install casadi` comes natively with the very robust solver `IPOPT` which is called using the `nlpsol` function.
- The Python library `scipy`, provides the function `scipy.optimize.minimize`.
- Matlabs `optim` toolbox provides the function `fmincon`.

Extended Reading:

- **Moritz Diehl and Sébastien Gros**, *Numerical Optimal Control*. Available online: http://www.syscop.de/numericaloptimalcontrol.