Tether drag in the awebox

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1 Introduction

Our goal is to model the tether drag on a segment of the tether - the straight, inelastic segment between an upper node and a lower node. If we want to include this tether drag into the Lagrangian dynamics of the nodes in our system, we need to split our modelled drag between the two nodes according to some rules that have physical meaning.

In this document we:

- (Section 2.1) specify some naming conventions and reference frames that will be used to determine the forces applied to the upper and lower segment nodes;
- (Section 2.2) model the drag force (and its corresponding moment) on an element of the segment, such that the total segment drag and moment are the sum of the element drag and moment for all elements; and
- (Section 2.3) divide the total segment drag into two forces that can be applied at the upper and lower nodes, such that the segment moment is equivalent.

Then, in Section 3, we explain the options available to an awebox user with respect to the tether drag model.

2 Tether drag modelling in the awebox

In the following subsections, we describe the specifics of how tether drag is modelled in the awebox.

2.1 Naming conventions and reference frames

The naming conventions and reference frames are all shown in Figure 1.

The upper node (with tree-index n_u) is located at position $\boldsymbol{q}_\mathrm{u}$ and has a velocity $\dot{\boldsymbol{q}}_\mathrm{u}$. The lower node (with tree-index $n_\mathrm{l} = P\left(n_\mathrm{u}\right)$) is located at position $\boldsymbol{q}_\mathrm{l}$ and has a velocity $\dot{\boldsymbol{q}}_\mathrm{l}$.

In addition to the usual earth-fixed reference frame of $\hat{\boldsymbol{x}}$ along the dominant wind, $\hat{\boldsymbol{y}}$ to the horizontal side, and $\hat{\boldsymbol{z}}$ against gravity, we also introduce a tether segment reference frame. In this new reference frame, $\hat{e}_{z,\mathrm{T},n_\mathrm{u}}$ points from the upper node to the lower node, $\hat{e}_{x,\mathrm{T},n_\mathrm{u}}$ points perpendicular to the tether segment (and $\hat{\boldsymbol{y}}$), and $\hat{e}_{y,\mathrm{T},n_\mathrm{u}}$ completes the right-handed frame. Then, $\underline{\boldsymbol{R}}_{\mathrm{T},n_\mathrm{u}} = \left[\hat{e}_{x,\mathrm{T},n_\mathrm{u}}, \hat{e}_{y,\mathrm{T},n_\mathrm{u}}, \hat{e}_{z,\mathrm{T},n_\mathrm{u}}\right]$.

We can split the segment into N_E elements of equal length. These elements are labelled with indices $e \in [0, N_\mathrm{E}-1]$ sequentially, such that the element with the index $e = 0$ has its lower side coincident with node n_l. Then, the element with index $e = (N_\mathrm{E}-1)$ has its upper side coincident with node n_u. For a particular element e, its upper side is located at position $\boldsymbol{q}_{e,\mathrm{u}}$ and has a velocity $\dot{\boldsymbol{q}}_{e,\mathrm{u}}$. The lower side of element e is located at position $\boldsymbol{q}_{e,\mathrm{l}}$ and has a velocity $\dot{\boldsymbol{q}}_{e,\mathrm{l}}$. For a straight segment with equal-sized elements:

$$\boldsymbol{q}_{e,\mathrm{l}} = \boldsymbol{q}_\mathrm{l} + \left(\boldsymbol{q}_\mathrm{u} - \boldsymbol{q}_\mathrm{l}\right)\phi_{e,\mathrm{l}} \quad (1)$$

$$\boldsymbol{q}_{e,\mathrm{u}} = \boldsymbol{q}_\mathrm{l} + \left(\boldsymbol{q}_\mathrm{u} - \boldsymbol{q}_\mathrm{l}\right)\phi_{e,\mathrm{u}} \quad (2)$$

$$\dot{\boldsymbol{q}}_{e,\mathrm{l}} = \dot{\boldsymbol{q}}_\mathrm{l} + \left(\dot{\boldsymbol{q}}_\mathrm{u} - \dot{\boldsymbol{q}}_\mathrm{l}\right)\phi_{e,\mathrm{l}} \quad (3)$$

$$\dot{\boldsymbol{q}}_{e,\mathrm{u}} = \dot{\boldsymbol{q}}_\mathrm{l} + \left(\dot{\boldsymbol{q}}_\mathrm{u} - \dot{\boldsymbol{q}}_\mathrm{l}\right)\phi_{e,\mathrm{u}} \quad (4)$$

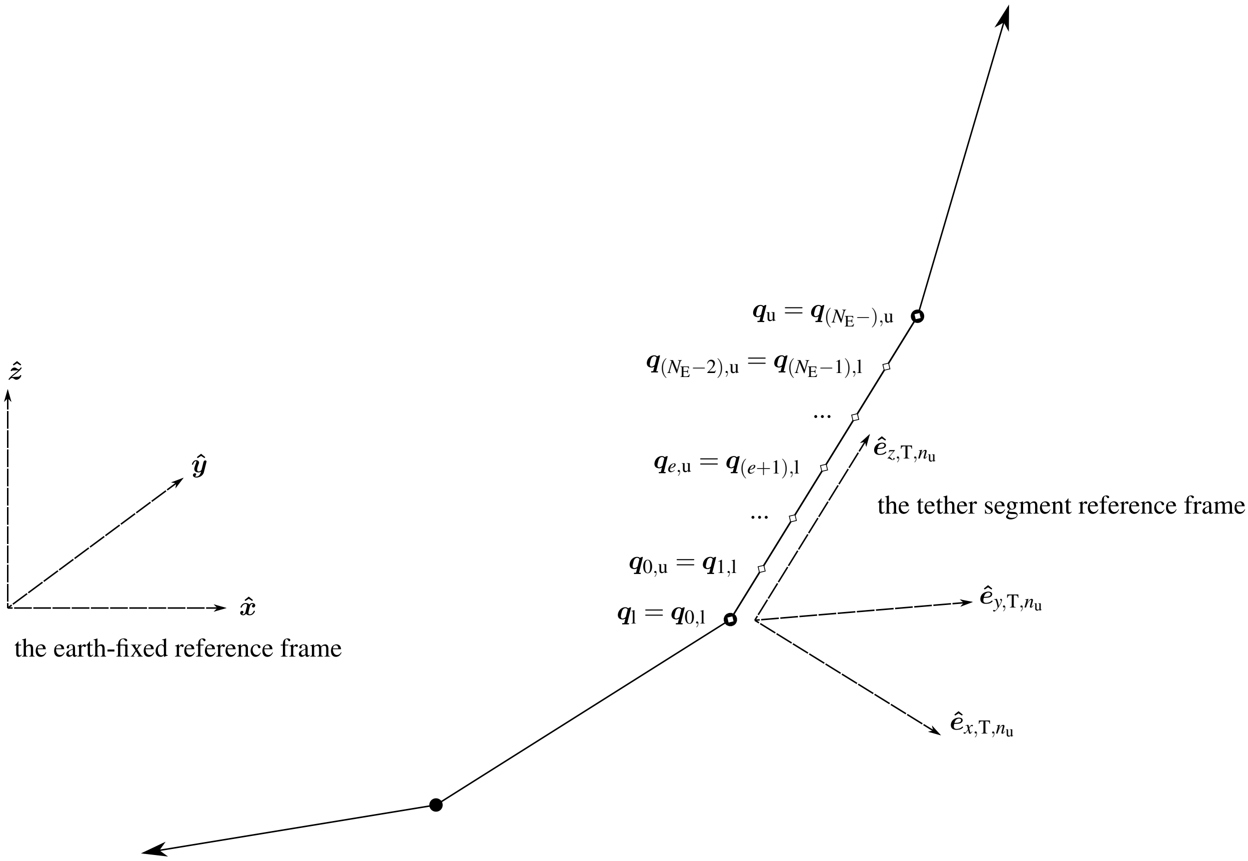

Figure 1: reference frames and naming conventions

where the linear parameters $\phi_{e,\mathrm{l}} = e/N_{\mathrm{E}}$ and $\phi_{e,\mathrm{u}} = (e+1)/N_{\mathrm{E}}$.

There is an average position for the segment $\overline{\boldsymbol{q}} = \left(\boldsymbol{q}_{\mathrm{l}} + \boldsymbol{q}_{\mathrm{u}}\right)/2$; as well as, an average position for the element $\overline{\boldsymbol{q}}_e = \left(\boldsymbol{q}_{e,\mathrm{l}} + \boldsymbol{q}_{e,\mathrm{u}}\right)/2$. Similarly, there is the same sort of average velocity of the segment $\overline{\dot{\boldsymbol{q}}} = \left(\dot{\boldsymbol{q}}_{\mathrm{l}} + \dot{\boldsymbol{q}}_{\mathrm{u}}\right)/2$; and for the element $\overline{\dot{\boldsymbol{q}}}_e = \left(\dot{\boldsymbol{q}}_{e,\mathrm{l}} + \dot{\boldsymbol{q}}_{e,\mathrm{u}}\right)/2$.

2.2 Drag force and moment acting on an element

We're going to approximate the environmental conditions of the element, based on the average vertical position of the element $\overline{z}_e = \overline{\boldsymbol{q}}_e^\top \hat{\boldsymbol{z}}$. That is, the average air density $\overline{\rho}_e$ is the air density distribution evaluated at the average vertical position:

$$\overline{\rho}_e = \rho\left(\overline{z}_e\right), \tag{5}$$

as is the average wind velocity:

$$\overline{\boldsymbol{u}}_{\infty,e} = \boldsymbol{u}_\infty\left(\overline{z}_e\right). \tag{6}$$

Based on the average element velocity and the average wind velocity, we can approximate the apparent velocity of the element:

$$\boldsymbol{u}_{\mathrm{a},e} = \overline{\boldsymbol{u}}_{\infty,e} - \overline{\dot{\boldsymbol{q}}}_e. \tag{7}$$

In order to lead to functions that are differentiable, we're going to use the smooth norm $\|\boldsymbol{v}\|_\varepsilon = \left(\boldsymbol{v}^\top \boldsymbol{v} + \varepsilon^2\right)^{\frac{1}{2}}$ for any vector $\boldsymbol{v}$. Using this smooth norm, we'll find the element's apparent speed $u_{\mathrm{a},e}$, as well as the direction of the element's apparent velocity $\hat{\boldsymbol{e}}_{u_{\mathrm{a}},e}$:

$$u_{\mathrm{a},e} = \left\|\boldsymbol{u}_{\mathrm{a},e}\right\|_\varepsilon, \qquad \hat{\boldsymbol{e}}_{u_{\mathrm{a}},e} = \boldsymbol{u}_{\mathrm{a},e}/u_{\mathrm{a},e}. \tag{8}$$

We can use (a smoothed version of) the Pythagorean theorem to find the length of the element that is perpendicular to the average apparent velocity $\ell_{\perp,e}$:

$$\ell_{\perp,e} \quad = \quad \left(\boldsymbol{t}_e^\top \boldsymbol{t}_e - \left(\boldsymbol{t}_e^\top \hat{\boldsymbol{e}}_{u_{\mathrm{a}},e} \right)^2 + \varepsilon^2 \right)^{\frac{1}{2}}, \tag{9}$$

where $\boldsymbol{t}_e = \boldsymbol{q}_{e,\mathrm{u}} - \boldsymbol{q}_{e,\mathrm{l}}$ is the vector from the upper side of the element to the lower side of the element.

If desired, you can determine the element's drag coefficient based on the element's Reynolds number. That will be described later. For the moment, we'll assume that the element's drag coefficient $C_{\mathrm{D,T},e}$ is known.

Finally, we use the known diameter d_e of the element, to approximate the drag force $\boldsymbol{D}_e$ on the element as:

$$\boldsymbol{D}_e = C_{\mathrm{D,T},e} \left(\frac{1}{2} \overline{\rho}_e u_{\mathrm{a},e} \right) \left(d_e \ell_{\perp,e} \right) \boldsymbol{u}_{\mathrm{a},e} \tag{10}$$

This drag force creates a moment on the center of the segment, of:

$$\boldsymbol{\tau}_e = \left(\overline{\boldsymbol{q}} - \overline{\boldsymbol{q}}_e \right) \times \boldsymbol{D}_e \tag{11}$$

2.3 Conversion from total segment force and moment to two equivalent forces

Let's assume - for the moment - that we know the "true" drag force $\boldsymbol{D}_{\mathrm{true}}$ and moment $\boldsymbol{\tau}_{\mathrm{true}}$ on the segment. We'd like to use these known drag force and moments to determine two equivalent forces.

That is, given:

- the "true" drag $\boldsymbol{D}_{\mathrm{true}} = D_x \hat{\boldsymbol{e}}_{x,\mathrm{T},n_{\mathrm{u}}} + D_y \hat{\boldsymbol{e}}_{y,\mathrm{T},n_{\mathrm{u}}} + D_z \hat{\boldsymbol{e}}_{z,\mathrm{T},n_{\mathrm{u}}}$, and
- the "true" moment about the segment center $\boldsymbol{\tau}_{\mathrm{true}} = \tau_x \hat{\boldsymbol{e}}_{x,\mathrm{T},n_{\mathrm{u}}} + \tau_y \hat{\boldsymbol{e}}_{y,\mathrm{T},n_{\mathrm{u}}} + \tau_z \hat{\boldsymbol{e}}_{z,\mathrm{T},n_{\mathrm{u}}}$

we want to find:

- the equivalent force on the upper node $\boldsymbol{F}_{\mathrm{u}} = F_{x,\mathrm{u}} \hat{\boldsymbol{e}}_{x,\mathrm{T},n_{\mathrm{u}}} + F_{y,\mathrm{u}} \hat{\boldsymbol{e}}_{y,\mathrm{T},n_{\mathrm{u}}} + F_{z,\mathrm{u}} \hat{\boldsymbol{e}}_{z,\mathrm{T},n_{\mathrm{u}}}$, and
- the equivalent force on the lower node $\boldsymbol{F}_{\mathrm{l}} = F_{x,\mathrm{l}} \hat{\boldsymbol{e}}_{x,\mathrm{T},n_{\mathrm{u}}} + F_{y,\mathrm{l}} \hat{\boldsymbol{e}}_{y,\mathrm{T},n_{\mathrm{u}}} + F_{z,\mathrm{l}} \hat{\boldsymbol{e}}_{z,\mathrm{T},n_{\mathrm{u}}}$.

We'd like our equivalent forces to have certain characteristics. First, we want the sum of both equivalent forces to equal the true total drag force - in all three directions. Second, we'd like the moments from the equivalent forces (except about the central axis of the tether segment) to equal the true total moment due to drag. Third, we want the equivalent forces to have an equal share of the force-along-the-tether-direction.

Expressed mathematically, these characteristics read as:

$$\underline{\boldsymbol{A}} \begin{pmatrix} F_{x,\mathrm{u}} \\ F_{y,\mathrm{u}} \\ F_{z,\mathrm{u}} \\ F_{x,\mathrm{l}} \\ F_{y,\mathrm{l}} \\ F_{z,\mathrm{l}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \frac{\ell}{2} & 0 & 0 & -\frac{\ell}{2} & 0 & 0 \\ 0 & \frac{\ell}{2} & 0 & 0 & -\frac{\ell}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} F_{x,\mathrm{u}} \\ F_{y,\mathrm{u}} \\ F_{z,\mathrm{u}} \\ F_{x,\mathrm{l}} \\ F_{y,\mathrm{l}} \\ F_{z,\mathrm{l}} \end{pmatrix} = \begin{pmatrix} D_x \\ D_y \\ D_z \\ \tau_x \\ \tau_y \\ 0 \end{pmatrix}, \tag{12}$$

where $\ell = \left\| \boldsymbol{q}_{\mathrm{u}} - \boldsymbol{q}_{\mathrm{l}} \right\|_2$ is the length of the tether.

We can see - then - that we can find our equivalent forces as:

$$\begin{pmatrix} F_{x,\mathrm{u}} & F_{y,\mathrm{u}} & F_{z,\mathrm{u}} & F_{x,\mathrm{l}} & F_{y,\mathrm{l}} & F_{z,\mathrm{l}} \end{pmatrix}^\top = \underline{\boldsymbol{A}}^{-1} \begin{pmatrix} D_x & D_y & D_z & \tau_x & \tau_y & 0 \end{pmatrix}^\top. \tag{13}$$

Notice, here, that the above choice of characteristics has (luckily) given us an invertible $\underline{\boldsymbol{A}}$:

$$\underline{\boldsymbol{A}}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{\ell} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{\ell} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{\ell} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{\ell} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{14}$$

We can now approximate the true segment drag force and moment with the sum of those drag force and moments found over all of the elements, as transformed into the tether segment reference frame:

$$\begin{pmatrix} F_{x,\mathrm{u}} & F_{y,\mathrm{u}} & F_{z,\mathrm{u}} & F_{x,\mathrm{l}} & F_{y,\mathrm{l}} & F_{z,\mathrm{l}} \end{pmatrix}^\top \approx \underline{\boldsymbol{A}}^{-1} \begin{pmatrix} \widetilde{D}_x & \widetilde{D}_y & \widetilde{D}_z & \widetilde{\tau}_x & \widetilde{\tau}_y & 0 \end{pmatrix}^\top, \tag{15}$$

where the elements labeled with a "tilde" are the components of the approximate tether segment drag force and moment, in the tether segment reference frame:

$$\widetilde{\boldsymbol{D}} = \widetilde{D}_x \hat{\boldsymbol{e}}_{x,\mathrm{T},n_\mathrm{u}} + \widetilde{D}_y \hat{\boldsymbol{e}}_{y,\mathrm{T},n_\mathrm{u}} + \widetilde{D}_z \hat{\boldsymbol{e}}_{z,\mathrm{T},n_\mathrm{u}} \tag{16}$$

$$= \underline{\boldsymbol{R}}_{\mathrm{T},n_\mathrm{u}}^{-1} \left(\sum_{e=0}^{N_\mathrm{E}-1} \boldsymbol{D}_e \right) \tag{17}$$

$$\widetilde{\boldsymbol{\tau}} = \widetilde{\tau}_x \hat{\boldsymbol{e}}_{x,\mathrm{T},n_\mathrm{u}} + \widetilde{\tau}_y \hat{\boldsymbol{e}}_{y,\mathrm{T},n_\mathrm{u}} + \widetilde{\tau}_z \hat{\boldsymbol{e}}_{z,\mathrm{T},n_\mathrm{u}} \tag{18}$$

$$= \underline{\boldsymbol{R}}_{\mathrm{T},n_\mathrm{u}}^{-1} \left(\sum_{e=0}^{N_\mathrm{E}-1} \boldsymbol{\tau}_e \right) \tag{19}$$

Naturally, when we find our equivalent forces $\boldsymbol{F}_\mathrm{u}$ and $\boldsymbol{F}_\mathrm{l}$, they are also in the tether segment reference frame. We have to convert them into the earth-fixed reference frame in order to apply them in the Lagrangian dynamics construction. This can be done, again, using the rotation matrix of the tether segment reference frame:

$$\boldsymbol{F}_{\mathrm{EF,u}} = \underline{\boldsymbol{R}}_{\mathrm{T},n_\mathrm{u}} \boldsymbol{F}_\mathrm{u}, \qquad \boldsymbol{F}_{\mathrm{EF,l}} = \underline{\boldsymbol{R}}_{\mathrm{T},n_\mathrm{u}} \boldsymbol{F}_\mathrm{l}. \tag{20}$$

Finally, we have generalized forces that can be applied to the upper and lower nodes of the segment, to represent the effect of drag on that segment.

3 Code options

Within the awebox, the user has a choice of tether drag model:

- *split* where only one large element is considered over the whole segment ($N_\mathrm{E} = 1$), and the total drag is simply applied "half-half" to the upper and lower nodes of the segment. This model results in quickly-solvable problems, but the use of only one large element is known to underestimate the impact of drag on the segment - especially when there is a large change in apparent wind speed along the tether length, as on a secondary tether. To explain this effect we only need to know that drag is proportional to apparent speed squared, and make a simple thought experiment.

  Consider a segment, where u_a varies linearly between 0 m/s at the lower node and 8 m/s at the upper node. The average speed is then 4 m/s. With one element, our total drag will be proportional to $(4\mathrm{m/s})^2 = 16$ $\mathrm{m}^2/\mathrm{s}^2$. With two elements, our total drag will be proportional to $\frac{1}{2}((2\mathrm{m/s})^2 + (6\mathrm{m/s})^2) = 20$ $\mathrm{m}^2/\mathrm{s}^2$; with four, 21 $\mathrm{m}^2/\mathrm{s}^2$. In fact, the "true" value would be proportional to the integral $\int_0^1 u_\mathrm{a}(l)^2 \mathrm{d}l$, or 21.33 $\mathrm{m}^2/\mathrm{s}^2$.

- *single* where only one large element is considered over the whole segment ($N_\mathrm{E} = 1$), but the equivalent forces are calculated as given in Section (2.3). Notice, that this is physically equivalent to the model "split", because the moment arm has zero length when there is only one element, and $\widetilde{\boldsymbol{\tau}} = \mathbf{0}$. This model exists within the awebox as a testing option, and is not recommended for standard use.

- *multi* where the user selects some number of elements N_E, and the full procedure above is applied. Clearly, the more elements the user selects, the better the approximation towards the "true" drag force and moment will be. However, increasing the number of elements leads to more complex and slower-to-solve problems. Exactly how much slower this problem will be, as N_E grows, depends on the problem tuning and other specifics.

The user should chose the model that best fits the purposes of their problem, by considering the trade-offs between the model complexity and accuracy.

4 Conclusion

In this document, we have described how the tether drag can be computed, as well as how the options an awebox user has with respect to the tether drag model.

We have seen that the simple ”split” and ”single” models are equivalent to an $N_{\mathrm{E}} = 1$ case of the ”multi” model. And, we have considered the relative advantages in terms of accuracy and disadvantages in terms of problem complexity resulting from model selection.

From this, we recommend that users consider their intended purposes carefully before making a tether drag model selection.

References