

# PLACEHOLDER-TITLE: Functional Linear Regression in a Scalar-on-Function Setting with Applications to SOMETHING

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# 1 Introduction

- Describe the idea of regressing a scalar on functional data
- Describing the difference to multiple linear regression intuitively
- Giving an intuitive example

Functional Data Analysis (FDA) is a relatively new field (roots in the 1940s Grenander and Karhunen) which is gaining more attention as researchers from different fields collect data that is functional in nature. This data can often be processed by classical statistical methods, but only FDA allows extracting information given by the smoothness of the underlying process (cf. Levitin et al. 2007). As Kokoszka and Reimherr 2017 describe, FDA should be considered when one can view variables or units of a given data set as smooth curves or functions and the interest is in analyzing samples of curves (cf. Kokoszka and Reimherr 2017, S. 17).

To motivate scalar-on-function regression, consider the case of a data set containing a scalar response and observations of an underlying continuous process. In economics, one application could be the regression of stock market correlations on the Global Crisis Index (GCI), where the regression allows to assess the relationship between the correlation and the GCI at every point within a window (cf. Das et al. 2019).

The focus of this paper is to introduce Functional Linear Regression (FLR) in a scalar-on-function setting. We will be using the standard FLR framework, which relates functional predictors to a scalar response as follows: (I don't set up any interval for  $s$  here we might do later...)

$$Y_i = \beta_0 + \int X_i(s)\beta(s)ds + \epsilon_i, \quad i = 1, \dots, n \quad (1)$$

where the  $X_i$  are observations of a random function  $X$ ,  $Y_i$  are realizations of the response variable and  $\beta(s)$  is the coefficient function. The distinct feature of this framework is that the regressor is a function, which necessitates a different approach to estimation. As in the well-known framework of scalar linear regression, this is motivated by an interest in  $\beta(s)$  for prediction. For instance, fluctuation in  $X$  at a point  $s_0$  will not have any effect on  $Y$  if  $\beta(s_0) = 0$ .

Estimation of  $\beta(s)$  is inherently an infinite dimensional problem. In Section 2, after introducing the necessary theoretical concepts, we describe three methods of estimating a scalar coefficient function using a concept called truncated basis expansion. The results of the Monte-Carlo simulation regarding these three different methods are reported in Section 3. Finally, in Section 4, we test the prediction of FLR in a real world setting. (We may put some simple descriptions of results about each of MC and Application)

## 2 Theory

### 2.1 Detailed Draft

- Motivate random functions from introduction and the general concept of random variables
- Formalize random function in this context as random variables realizing in a Hilbert space
- Introduce  $L^2[0, 1]$  as the Hilbert space of square integrable functions on  $[0, 1]$

- Specialize to Hilbert space being  $\mathbf{L}^2[0, 1]$  for this context
- Define mean and covariance function of a random function realizing in  $\mathbf{L}^2[0, 1]$
- Introduce the concept of a basis of a Hilbert space and specialize to  $\mathbf{L}^2[0, 1]$
- Introduce b-spline and Fourier bases
- Introduce eigenfunctions and FPCA on the basis of covariance function (Karhunen-Loève expansion)
- explain similarities to Eigenvalues and Eigenvectors of matrix + PCA (fraction of explained variance etc...)
- Introduce functional observations in this context as realizations of a random variable realizing in  $\mathbf{L}^2[0, 1]$
- Explain the concept of iid data in a functional setting
- Define point-wise mean (sample), point-wise standard deviation (sample) and sample covariance function
- Explain approximations of functional observations using truncated basis representations
- Introduce linear operator  $L_1$  and sufficient condition associated with it
- Motivate Scalar-on-function regression from multivariate linear regression with a scalar response variable

## 2.2 Kokoszka Reimherr (2017) p51-53

There are several important aspects of functional regression in this functional setting that separate it from usual multiple regression according to Kokoszka and Reimherr 2017: In functional linear regression, the aim is not only to obtain an estimate of the function  $\beta(s)$  because this function also needs to have a useful interpretation. Without it, there might be prediction of the scalar responses from new explanatory functions, but the increase in understanding of the underlying question will be minimal. Thereby, intervals with larger values of  $|\beta(s)|$  will contribute more to the response than small values of  $|\beta(s)|$ . The sign of  $\beta(s)$  indicates either negative or positive influence of  $X(s)$  for a specific value of  $s$ . To get a useful interpretation, the estimate  $\beta$  should not jump in a seemingly random fashion, because a useful interpretation of this erratic behavior will often be impossible and predictions from these model tend to have larger variances and to center around the mean of the responses. (see Kokoszka and Reimherr 2017)

- Explain problem of naively extending multivariate linear regression to infinite dimensions

Assuming a data generating process of the form:

$$Y = \int \beta(s)X(s) dt + \epsilon \quad (2)$$

it becomes clear that we cannot compute the estimate of  $\beta(t)$  as we would do in a classical multivariate setup because of the infinite dimensionality of the underlying objects. Define therefore

$$c_X(t, s) = E[X(t)X(s)], \quad c_{XY}(t) = E[X(t)Y], \quad (3)$$

Under the assumption that  $X$  is independent from  $\epsilon$  we obtain

$$c_{XY}(t) = E[X(t) \int \beta(s)X(s) ds] + \epsilon \quad (4)$$

$$c_{XY}(t) = \int \beta(s) ds E[X(s)X(t)|X] + E[\epsilon|X] \quad (5)$$

$$c_{XY}(t) = \int c_X(t, s)\beta(s) ds \quad (6)$$

In practice, this results in a large and often difficult to solve system of equations. Even if solved, the result is often a noisy function  $\beta(\cdot)$  that is not useful for interpretation since it does not utilize the intuition of smooth functions. Another reason why estimation is not feasible using this approach is colinearity.

If we approximate the scalar-on-functional regression as

$$Y_i = \sum_{j=1}^n \beta(t_j)X_i(t_j) + \epsilon_i \quad (7)$$

, it becomes clear that if  $t_i$  is close to  $t_{i'}$ ,  $X_i(t_i)$  is close to  $X_{i'}(t_{i'})$  there will be vectors  $X_i$  that are highly correlated and thus lead to large variances.

- Solution: estimation using truncated basis expansion to approximate data (theoretical description)
- Problem: truncation error  $\delta$  and how to deal with it?
- Explain how to address truncation error in standard errors
- Motivate three estimation procedures
  1. truncated b-spline basis expansion without addressing truncation error
  2. truncated b-spline basis expansion WITH addressing truncation error
  3. truncated Eigenbasis expansion (advantages: low number of basis functions get low approximation error)

## 2.3 Draft-Overview

- Motivate Karhunen-Loeve-Expansion and Eigenbasis from PCA
- Explain Scalar-on-Function Regression
- Estimation through basis-expansion (incl. Eigenbasis) [and estimation with roughness penalty]
- Address approximation error due to basis-truncation

## 2.4 Literature

- Kokoszka and Reimherr 2017
- Hsing and Eubank 2015
- Ramsay and Silverman 2005
- Horváth and Kokoszka 2012
- Cai and Hall 2006
- Levitin et al. 2007

## 3 Simulation

### 3.1 Draft-Overview

- Motivate Simulation for some data generating process from application
- Describe Simulation Setting from technical standpoint (DGP, set-up for replication, ...)
- Compare estimation with
  1. b-spline basis without addressing approximation error
  2. ... including proper treatment of approximation error
  3. Eigenbasis constructed from observations
- Prediction not Inference (Alternative: Focused on a testing procedure motivated by the application)
- Present Results
- Explain relevance for application

### 3.2 Literature

- Shonkwiler and Mendivil 2009
- R-packages: fda, refund, mgcv

## 4 Application

### 4.1 Draft-Overview

- Prediction not Inference (Alternative: Focused on a testing procedure motivated by the data set)
- IID data set (no dependence between the curves, don't want to do functional time series)
- Not necessarily data from economics (like biology, sports, whatever)
- Smooth curves or random walk (both fine)
- <https://functionaldata.wordpress.ncsu.edu/resources/>

### 4.2 Literature

- Carey et al. 2002

## 5 Outlook

### 5.1 Literature

- James, Wang, and Zhu 2009 (shape-restrictions)

## 6 Appendix



## 7 Bibliography

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