Bugni and Horowitz (2021) Permutation Tests for the Equality of Distributions of Functional Data

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1 Introduction

In modern economics, it is becoming more and more common to use data measured at a very high frequency. As the frequency of observing a variable increases, it often becomes more natural to view the data not as a sequence of distinct observation points but as a smooth curve that describes the variable over time.
This idea, to think of observations as measurements of a continuous process, is the motivating thought behind functional data analysis. Functional data analysis is a branch of statistics that has its beginnings in the 1940s and 1950s in the works of Ulf Grenander and Kari Karhunen. It gained traction during the following decades and focused more on possible applications during the 1990s. In Economics, functional data analysis is still a relatively exotic field, but it is beginning to become more established, which can be seen in the works of, for example, hier Autoren einfuegen.
A typical question in economics is whether observations from two or more data sets, e.g., data generated by treatment and control groups, are systematically different across groups. In statistical terms, this can be formulated as whether observations in both data sets can be seen as if the same stochastic process generated them.

This question can also occur in functional data analysis, where each observation in a data set is itself a smooth curve. Bugni and Horowitz 2021 develop a permutation test that tries to answer this question by combining two distinct test statistics. To explore their approach, it is first necessary to introduce some theoretical concepts. Section 2 introduces the necessary concepts from functional data analysis. Section 3 explores the theory around Cramér-von Mises tests. Section 4 introduces the Bonferroni Correction for multiple testing problems and Section 5 finally introduces the necessary background in Permutation Testing.
After explaining these concepts, section 6 focuses on the test developed in Bugni and Horowitz 2021 for the case of a two sample test. Section 7 replicates the results from the simulation study in the paper and section 8 explores their usefulness in an application to Thema der Anwendung.

- Introduce general idea and possible hypothesis to test
- Maybe focus on two sample setting

2 Functional Data Analysis

The overarching concept of functional data analysis is to incorporate observations that are functional in nature. In this context, a functional observation can often be understood as a smooth curve. A classical example of this is shown in Figure 1. It presents data provided

in the R package *fda*[1] and shows growth curves of 93 humans up to the age of 18.

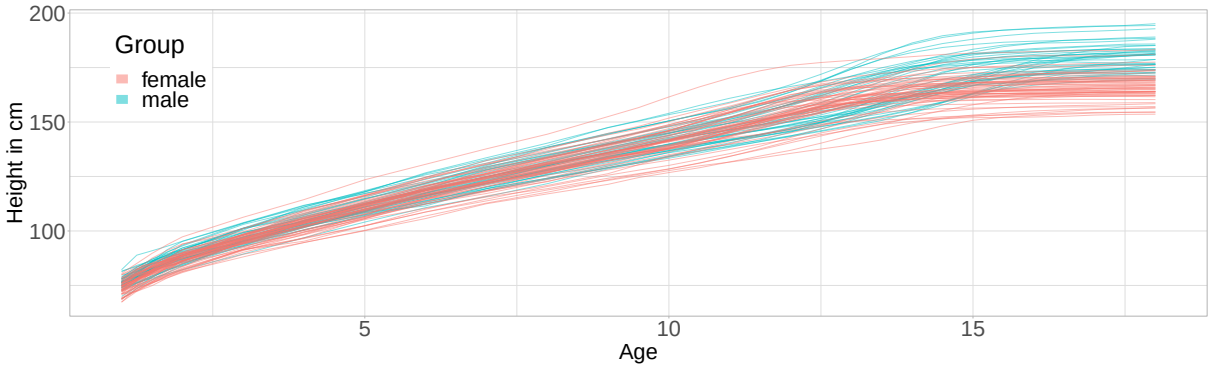

Figure 1: Human Growth Curves up to the Age of 18

In many cases functional data analysis restricts its scope to subsets of the functions $f : \mathbb{R} \to \mathbb{R}$. As these are inherently infinite-dimensional, it is necessary to introduce additional theory to appropriately deal with their unique properties.

- Ramsay and Silverman 2005
- Kokoszka and Reimherr 2021
- Hsing and Eubank 2015

2.1 Hilbert Space of Square Integrable Functions

Definition 1 (Inner Product)
A function $\langle \cdot, \cdot \rangle : \mathbb{V}^2 \to \mathbb{R}$ on a vector space $\mathbb{V}$ is called an inner product if the following four conditions hold for all $v, v_1, v_2 \in \mathbb{V}$ and $a_1, a_2 \in \mathbb{R}$.

1. $\langle v, v \rangle \geq 0$
2. $\langle v, v \rangle = 0$ if $v = 0$
3. $\langle a_1 v_1 + a_2 v_2, v \rangle = a_1 \langle v_1, v \rangle + a_2 \langle v_2, v \rangle$
4. $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$

Hsing and Eubank 2015

Definition 2 (Inner Product Space)
A vector space with an associated inner product is called an inner product space.

Hsing and Eubank 2015

Definition 3 (Hilbert Space)
A complete inner product space is called a Hilbert space.

Definition 4 (Basis of a Hilbert Space)
content...

[1] Ramsay, Graves, and Hooker 2021.

Definition 5 (Separable Hilbert Space)
content...

Definition 6 (Hilbert Space of Square Integrable Functions)
The space of square integrable functions on a closed interval A together with the norm $\langle f, g\rangle = \int_A f(t)g(t)\mathrm{d}t$ is a Hilbert space. A function $f : A \to \mathbb{R}$ is called square integrable if the following condition holds.

$$\int_A [f(t)]^2 \,\mathrm{d}t < \infty \tag{1}$$

The Hilbert space of all square integrable functions on A is denoted by $\mathbb{L}_2(A)$.

2.2 Bases of $\mathbb{L}_2$

- Orthogonality
- Orthonormality
- Fourier Basis

2.3 Random Functions

2.4 Probability Measures on $\mathbb{L}_2$

- Kolmogorov Extension Theorem
- Gihman and Skorokhod 2004

2.5 Functional Integration on $\mathbb{L}_2$

- Skorohod 1974
- Perturbation theory

Functional Integral:

$$\int_{\mathbb{L}_2(\mathcal{I})} G\,[f]\,[Df] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G\,[f] \prod_x \mathrm{d}f(x) \tag{2}$$

If a representation in terms of an orthogonal functional basis is possible:

$$\int_{\mathbb{L}_2(\mathcal{I})} G\,[f]\,[Df] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G\,(f_1, f_2, \dots) \prod_n \mathrm{d}f_n \tag{3}$$

3 Cramér-von Mises Tests

- Darling 1957
- Anderson and Darling 1952
- Büning and Trenkler 2013

3.1 Empirical Distribution Functions

Gibbons and Chakraborti 2021

Definition 7 (Order Statistic)
Let $\{x_i \mid i = 1, \ldots, n\}$ be a random sample from a population with continuous cumulative distribution function F_X. Then there almost surely exists a unique ordered arrangement within the sample.

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

$X_{(r)} \quad r \in \{1, \ldots, n\}$ is called the rth-order statistic.

Definition 8 (Empirical Distribution Function)

$$F_n(x) = \begin{cases} 0 & \text{if} \quad x < x_{(1)} \\ \frac{r}{n} & \text{if} \quad x_{(r)} \leq x < x_{(r+1)} \\ 1 & \text{if} \quad x \geq x_{(n)} \end{cases} \tag{4}$$

3.2 Nullhypothesis

3.3 Cramér-von Mises Statistic

Büning and Trenkler 2013

$$C_{m,n} = \left(\frac{nm}{n+m}\right) \int_{-\infty}^{\infty} \left(F_m(x) - G_n(x)\right)^2 \mathrm{d}\left(\frac{mF_m(x) + nG_n(x)}{m+n}\right) \tag{5}$$

3.4 Asymptotic Distributions

4 Multiple Testing

When testing statistical hypotheses, it is often helpful or even necessary to test multiple hypotheses independently of each other. One setting where this could be useful is when we want to combine the desirable properties of two tests, as is done by Bugni and Horowitz

2021. If the tests do not perfectly depend on each other[2], this creates a problem relating to the size of the combined test.

- Dunn 1961

4.1 Bonferroni Correction

Bonferroni Inequality / Boole's Inequality

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}\left[A_i\right] \tag{6}$$

for a countable set of events $A_1, A_2, \ldots$.

5 Permutation Tests

In layman's terms, the idea of a permutation test is the following: if two samples show distinctly different properties, that will lead to differences in an appropriately chosen summary statistic. If we were to permute the elements of the groups randomly, we would expect these differences to disappear. Permutation tests formalize this intuition.

- Lehmann and Romano 2005
- Vaart and Wellner 1996

5.1 Functional Principle of Permutation Tests

Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ be two data sets.
Number of Permutations: $(n+m)!$
Number of Combinations: $\binom{m+n}{m}$

For my implementation, I chose the latter variant.

5.2 Size and Power

6 Test by Bugni and Horowitz (2021)

- Bugni and Horowitz 2021
- Bugni, Hall, et al. 2009

Distribution Functions

$$\begin{aligned} F_X(z) &= \mathbb{P}\left[X(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \quad z \in \mathbb{L}_2(\mathcal{I}) \\ F_Y(z) &= \mathbb{P}\left[Y(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \quad z \in \mathbb{L}_2(\mathcal{I}) \end{aligned} \tag{7}$$

[2] If the tests perfectly depend on each other, one of the tests is superfluous.

6.1 Nullhypothesis

$$\begin{aligned} H_0: &\quad F_X(z) = F_Y(z) \quad \forall z \in \mathbb{L}_2(\mathcal{I}) \\ H_1: &\quad \mathbb{P}_\mu \left[F_X(Z) \neq F_Y(Z) \right] > 0 \end{aligned} \tag{8}$$

Here, μ is a probability measure on $\mathbb{L}_2(\mathcal{I})$ and Z is a random function with probability distribution μ. Doesn't this leave out the case where the Probability functions only differ on a set of μ-measure zero?

6.2 Assumptions

Assumption 1
Contains two assumptions

1. $X(t)$ and $Y(t)$ are separable, μ-measurable stochastic processes.
2. $\{X_i(t) \mid i = 1, \dots, n\}$ is an independent random sample of the process $X(t)$. $\{Y_i(t) \mid i = 1, \dots, m\}$ is an independent random sample of $Y(t)$ and is independent of $\{X_i(t) \mid i = 1, \dots, n\}$.

Assumption 2
$\mathbb{E}X(t)$ and $\mathbb{E}Y(t)$ exist and are finite for all $t \in [0, T]$.

Assumption 3
$X_i(t)$ and $Y_i(t)$ are observed for all $t \in \mathcal{I}$.

6.3 Cramér-von Mises type Test

Empirical Distribution Functions

$$\hat{F}_X(z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1} \left[X_i(t) \leq z(t) \ \forall t \in \mathcal{I} \right] \qquad \hat{F}_Y(z) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1} \left[Y_i(t) \leq z(t) \ \forall t \in \mathcal{I} \right] \tag{9}$$

Test statistic

$$\tau = \int_{\mathbb{L}_2(\mathcal{I})} \left[F_X(z) - F_Y(z) \right]^2 \mathrm{d}\mu(z) \tag{10}$$

Sample analog:

$$\tau_{n,m} = (n+m) \int_{\mathbb{L}_2(\mathcal{I})} \left[\hat{F}_X(z) - \hat{F}_Y(z) \right]^2 \mathrm{d}\mu(z) \tag{11}$$

Critical values for Permutation Test Statistic

$$t^*_{n,m}(1-\alpha) = \inf \left\{ t \in \mathbb{R} \quad | \quad \frac{1}{Q} \sum_{i=1}^{Q} \mathbb{1} \left[\tau_{n,m,q} \leq t \right] \geq 1 - \alpha \right\} \tag{12}$$

6.4 Mean focused Test

Test statistic

$$\nu = \int_{\mathcal{I}} \left[\mathbb{E}X(t) - \mathbb{E}Y(t)\right]^2 \mathrm{d}t \tag{13}$$

Mean Estimators

$$\hat{\mathbb{E}}X(t) = \frac{1}{n}\sum_{i=1}^{n} X_i(t) \qquad\qquad \hat{\mathbb{E}}Y(t) = \frac{1}{m}\sum_{i=1}^{m} Y_i(t) \tag{14}$$

Sample Analog

$$\nu_{n,m} = (n+m)\int_{\mathcal{I}} \left[\hat{\mathbb{E}}X(t) - \hat{\mathbb{E}}Y(t)\right]^2 \mathrm{d}t \tag{15}$$

Critical values for Permutation Test Statistic

$$t^*_{n,m}(1-\alpha) = \inf\left\{ t \in \mathbb{R} \quad | \quad \frac{1}{Q}\sum_{i=1}^{Q} \mathbb{1}\left[\nu_{n,m,q} \leq t\right] \geq 1-\alpha \right\} \tag{16}$$

6.5 Combined Permutation Test

Define for the two underlying tests the following objects.

$$\phi_{n,m} = \begin{cases} 1 & \text{if } \tau_{n,m} > t^*_{n,m}(1-\alpha_\tau) \\ a_\tau & \text{if } \tau_{n,m} = t^*_{n,m}(1-\alpha_\tau) \\ 0 & \text{if } \tau_{n,m} < t^*_{n,m}(1-\alpha_\tau) \end{cases} \qquad \tilde{\phi}_{n,m} = \begin{cases} 1 & \text{if } \nu_{n,m} > t^*_{n,m}(1-\alpha_\nu) \\ a_\nu & \text{if } \nu_{n,m} = t^*_{n,m}(1-\alpha_\nu) \\ 0 & \text{if } \nu_{n,m} < t^*_{n,m}(1-\alpha_\nu) \end{cases} \tag{17}$$

a_τ and a_ν are given by the following equations to ensure that the expected values of ϕ and $\tilde{\phi}$ have the desired values.

- $a_\tau = \frac{Q\alpha_\tau - Q^+_\tau}{Q^0_\tau}$
- $Q^+_\tau = \sum_{q=1}^{Q} \mathbb{1}\left[\tau_{n,m,q} > t^*_{n,m}(1-\alpha_\tau)\right]$
- $Q^0_\tau = \sum_{q=1}^{Q} \mathbb{1}\left[\tau_{n,m,q} = t^*_{n,m}(1-\alpha_\tau)\right]$
- $a_\nu = \frac{Q\alpha_\nu - Q^+_\nu}{Q^0_\nu}$
- $Q^+_\nu = \sum_{q=1}^{Q} \mathbb{1}\left[\nu n, m, q > t^*_{n,m}(1-\alpha_\nu)\right]$
- $Q^0_\nu = \sum_{q=1}^{Q} \mathbb{1}\left[\nu n, m, q = t^*_{n,m}(1-\alpha_\nu)\right]$

Bonferroni inequality under H_0 leads to

$$\max(\alpha_\tau, \alpha_\nu) \leq \mathbb{P}\left[(\phi_{n,m} > 0) \cup (\tilde{\phi}_{n,m} > 0)\right] \leq \alpha_\tau + \alpha_\nu \tag{18}$$

6.6 Properties

7 Simulation Study

7.1 Implementation as an R package

All analyses in this thesis have been conducted with R[3]. I implemented the two-sample variant of the test presented taken from Bugni and Horowitz 2021 in an R package called *PermFDATest*. The R package and all code that has been used to produce the following results are publicly available as part of a GitHub repository[4] that complements this thesis.

- Ramsay, Graves, and Hooker 2021
- Wickham et al. 2019
- Goldsmith et al. 2021

7.2 Use of High-Performance Computing

The simulations presented as part of this thesis have been conducted on bonna[5]. bonna is the high performance computing cluster provided by the University of Bonn. The implementation is heavily parallelized and makes use of a SLURM scheduling system. However, slight modifications of the provided code suffice to run it on personal computers.

7.3 Simulation Setup

7.4 Results

8 Application

9 Outlook

[3] R Core Team 2022.
[4] https://github.com/JakobJuergens/Masters_Thesis
[5] https://www.dice.uni-bonn.de/de/hpc/hpc-a-bonn/infrastruktur

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11 Appendix

Versicherung an Eides statt

Ich versichere hiermit, dass ich die vorstehende Masterarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass die vorgelegte Arbeit noch an keiner anderen Hochschule zur Prüfung vorgelegt wurde und dass sie weder ganz noch in Teilen bereits veröffentlicht wurde. Wörtliche Zitate und Stellen, die anderen Werken dem Sinn nach entnommen sind, habe ich in jedem einzelnen Fall kenntlich gemacht.

Bonn, XX.XX.2021 _____

Jakob R. Juergens