

# Bugni and Horowitz (2021) Permutation Tests for the Equality of Distributions of Functional Data

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# **1 Introduction**

- Introduce general idea and possible hypothesis to test
- Maybe focus on two sample setting

# **2 Functional Data Analysis**

- Ramsay and Silverman 2005
- Kokoszka and Reimherr 2021
- Hsing and Eubank 2015

## **2.1 Hilbert Space of Square Integrable Functions**

## **2.2 Random Functions**

## **2.3 Probability Measures on $\mathbb{L}^2$**

# **3 Cramér-von Mises Tests**

- Darling 1957
- Anderson and Darling 1952

## **3.1 Empirical Distribution Functions**

## **3.2 Cramér-von Mises Statistics**

## **3.3 Asymptotic Distributions**

# **4 Permutation Tests**

- Lehmann and Romano 2005
- Vaart and Wellner 1996

## **4.1 Functional Principle of Permutation Tests**

## **4.2 Size and Power**

## **4.3 Permutation Test for Equality of scalar-valued Distributions**

# **5 Test by Bugni and Horowitz (2021)**

- Bugni and Horowitz 2021
- Bugni, Hall, et al. 2009

### **5.1 Nullhypothesis**

### **5.2 Cramér-von Mises type Test**

### **5.3 Mean focused Test**

### **5.4 Combined Permutation Test**

### **5.5 Properties**

## **6 Simulation Study**

### **6.1 Use of High-Performance Computing**

- bonna - HPC/A-Cluster der Universität Bonn
- <https://www.dice.uni-bonn.de/de/hpc/hpc-a-bonn/infrastruktur>

### **6.2 Simulation Setup**

### **6.3 Results**

## **7 Application**

## 8 Bibliography

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## 9 Appendix

## 10 Versicherung an Eides statt

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