# Bugni and Horowitz (2021) Permutation Tests for the Equality of Distributions of Functional Data

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# Contents

# 1 Introduction

- Introduce general idea and possible hypothesis to test
- Maybe focus on two sample setting

# 2 Functional Data Analysis

- Ramsay and Silverman 2005
- Kokoszka and Reimherr 2021
- Hsing and Eubank 2015

## 2.1 Hilbert Space of Square Integrable Functions

**Definition 1** (Inner Product)
A function $\langle \cdot, \cdot \rangle : \mathbb{V}^2 \to \mathbb{R}$ on a vector space $\mathbb{V}$ is called an inner product if the following four conditions hold for all $v, v_1, v_2 \in \mathbb{V}$ and $a_1, a_2 \in \mathbb{R}$.

1. $\langle v, v \rangle \geq 0$
2. $\langle v, v \rangle = 0$ if $v = 0$
3. $\langle a_1 v_1 + a_2 v_2, v \rangle = a_1 \langle v_1, v \rangle + a_2 \langle v_2, v \rangle$
4. $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$

Hsing and Eubank 2015

**Definition 2** (Inner Product Space)
A vector space with an associated inner product is called an inner product space.

Hsing and Eubank 2015

**Definition 3** (Hilbert Space)
A complete inner product space is called a Hilbert space.

**Definition 4** (Basis of a Hilbert Space)
content...

**Definition 5** (Separable Hilbert Space)
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**Definition 6** (Hilbert Space of Square Integrable Functions)
The space of square integrable functions on a closed interval $A$ together with the norm $\langle f, g \rangle = \int_A f(t) g(t) \mathrm{d}t$ is a Hilbert space. A function $f : A \to \mathbb{R}$ is called square integrable if the following condition holds.

$$\int_A [f(t)]^2 \, \mathrm{d}t < \infty \tag{1}$$

The Hilbert space of all square integrable functions on $A$ is denoted by $\mathbb{L}_2(A)$.

### 2.2 Bases of $\mathbb{L}_2$

- Orthogonality
- Orthonormality
- Fourier Basis

### 2.3 Random Functions

### 2.4 Probability Measures on $\mathbb{L}_2$

- Kolmogorov Extension Theorem
- Gihman and Skorokhod 2004

### 2.5 Functional Integration on $\mathbb{L}_2$

- Skorohod 1974
- Perturbation theory

Functional Integral:

$$\int_{\mathbb{L}_2(\mathcal{I})} G[f]\,[Df] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G[f] \prod_{x} \mathrm{d}f(x) \tag{2}$$

If a representation in terms of an orthogonal functional basis is possible:

$$\int_{\mathbb{L}_2(\mathcal{I})} G[f]\,[Df] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(f_1, f_2, \dots) \prod_{n} \mathrm{d}f_n \tag{3}$$

## 3 Cramér-von Mises Tests

- Darling 1957
- Anderson and Darling 1952
- Büning and Trenkler 2013

### 3.1 Empirical Distribution Functions

Gibbons and Chakraborti 2021

**Definition 7**
Order Statistic
Let $\{x_i \mid i = 1, \dots, n\}$ be a random sample from a population with continuous cumulative distribution function $F_X$. Then there almost surely exists a unique ordered arrangement within the sample.

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

$X_{(r)} \quad r \in \{1, \dots, n\}$ is called the $r$th-order statistic.

**Definition 8**

Empirical Distribution Function

$$F_n(x) = \begin{cases} 0 & \text{if} \quad x < x_{(1)} \\ \frac{r}{n} & \text{if} \quad x_{(r)} \leq x < x_{(r+1)} \\ 1 & \text{if} \quad x \geq x_{(n)} \end{cases} \tag{4}$$

### 3.2 Nullhypothesis

### 3.3 Cramér-von Mises Statistic

Büning and Trenkler 2013

$$C_{m,n} = \left(\frac{nm}{n+m}\right) \int_{-\infty}^{\infty} (F_m(x) - G_n(x))^2 \, \mathrm{d}\left(\frac{mF_m(x) + nG_n(x)}{m+n}\right) \tag{5}$$

### 3.4 Asymptotic Distributions

## 4 Multiple Testing

- Dunn 1961

### 4.1 Bonferroni Correction

Bonferroni Inequality / Boole's Inequality

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}\left[A_i\right] \tag{6}$$

for a countable set of events $A_1, A_2, \ldots$.

## 5 Permutation Tests

- Lehmann and Romano 2005
- Vaart and Wellner 1996

### 5.1 Functional Principle of Permutation Tests

### 5.2 Size and Power

## 6 Test by Bugni and Horowitz (2021) - Two Samples

- Bugni and Horowitz 2021
- Bugni, Hall, et al. 2009

Distribution Functions

$$\begin{aligned} F_X(z) &= \mathbb{P}\left[X(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \quad z \in \mathbb{L}_2(\mathcal{I}) \\ F_Y(z) &= \mathbb{P}\left[Y(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \quad z \in \mathbb{L}_2(\mathcal{I}) \end{aligned} \tag{7}$$

## 6.1 Nullhypothesis

$$\begin{aligned} H_0: &\quad F_X(z) = F_Y(z) \quad \forall z \in \mathbb{L}_2(\mathcal{I}) \\ H_1: &\quad \mathbb{P}_\mu\left[F_X(Z) \neq F_Y(Z)\right] > 0 \end{aligned} \tag{8}$$

Here, $\mu$ is a probability measure on $\mathbb{L}_2(\mathcal{I})$ and $Z$ is a random function with probability distribution $\mu$. Doesn't this leave out the case where the Probability functions only differ on a set of $\mu$-measure zero?

## 6.2 Assumptions

**Assumption 1**

Contains two assumptions

1. $X(t)$ and $Y(t)$ are separable, $\mu$-measurable stochastic processes.
2. $\{X_i(t) \mid i = 1, \ldots, n\}$ is an independent random sample of the process $X(t)$. $\{Y_i(t) \mid i = 1, \ldots, m\}$ is an independent random sample of $Y(t)$ and is independent of $\{X_i(t) \mid i = 1, \ldots, n\}$.

**Assumption 2**

$\mathbb{E}X(t)$ and $\mathbb{E}Y(t)$ exist and are finite for all $t \in [0, T]$.

**Assumption 3**

$X_i(t)$ and $Y_i(t)$ are observed for all $t \in \mathcal{I}$.

## 6.3 Cramér-von Mises type Test

Empirical Distribution Functions

$$\begin{aligned} \hat{F}_X(z) &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\left[X_i(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \\ \hat{F}_Y(z) &= \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}\left[Y_i(t) \leq z(t) \quad \forall t \in \mathcal{I}\right] \end{aligned} \tag{9}$$

Test statistic

$$\tau = \int_{\mathbb{L}_2(\mathcal{I})} \left[F_X(z) - F_Y(z)\right]^2 \mathrm{d}\mu(z) \tag{10}$$

Sample analog:

$$\tau_{n,m} = (n+m)\int_{\mathbb{L}_2(\mathcal{I})} \left[\hat{F}_X(z) - \hat{F}_Y(z)\right]^2 \mathrm{d}\mu(z) \tag{11}$$

Critical values for Permutation Test Statistic

$$t^*_{n,m}(1-\alpha) = \inf\left\{\frac{1}{Q}\sum_{i=1}^{Q} \mathbb{1}\left[\tau_{n,m,q} \leq t\right] \geq 1-\alpha \quad \Big| \quad t \in \mathbb{R}\right\} \tag{12}$$

## 6.4 Mean focused Test

Test statistic

$$\nu = \int_{\mathcal{I}} \left[\mathbb{E}X(t) - \mathbb{E}Y(t)\right]^2 \mathrm{d}t \tag{13}$$

Mean Estimators

$$\hat{\mathbb{E}}X(t) = \frac{1}{n}\sum_{i=1}^{n} X_i(t) \tag{14}$$

$$\hat{\mathbb{E}}Y(t) = \frac{1}{m}\sum_{i=1}^{m} Y_i(t) \tag{15}$$

Sample Analog

$$\nu_{n,m} = (n+m)\int_{\mathcal{I}} \left[\hat{\mathbb{E}}X(t) - \hat{\mathbb{E}}Y(t)\right]^2 \mathrm{d}t \tag{16}$$

Critical values for Permutation Test Statistic

$$t^*_{n,m}(1-\alpha) = \inf\left\{\frac{1}{Q}\sum_{i=1}^{Q} \mathbb{1}\left[\nu_{n,m,q} \leq t\right] \geq 1-\alpha \quad | \quad t \in \mathbb{R}\right\} \tag{17}$$

### 6.5 Combined Permutation Test

Bonferroni inequality under $H_0$ leads to

$$\max(\alpha_\tau, \alpha_\nu) \leq \mathbb{P}\left[(\phi_{n,m} > 0) \cup (\tilde{\phi}_{n,m} > 0)\right] \leq \alpha_\tau + \alpha_\nu \tag{18}$$

### 6.6 Properties

## 7 Simulation Study

### 7.1 Use of High-Performance Computing

- bonna - HPC/A-Cluster der Universität Bonn
- `https://www.dice.uni-bonn.de/de/hpc/hpc-a-bonn/infrastruktur`

### 7.2 Simulation Setup

### 7.3 Results

## 8 Application

## 9 Outlook

# 10 Bibliography


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## 11 Appendix

# 12 Versicherung an Eides statt

Ich versichere hiermit, dass ich die vorstehende Masterarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass die vorgelegte Arbeit noch an keiner anderen Hochschule zur Prüfung vorgelegt wurde und dass sie weder ganz noch in Teilen bereits veröffentlicht wurde. Wörtliche Zitate und Stellen, die anderen Werken dem Sinn nach entnommen sind, habe ich in jedem einzelnen Fall kenntlich gemacht.

Bonn, XX.XX.2021 ________________________________________

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