Fourier Transformation
Signal and Image Processing

February 16, 2023

You can work on this assignment and submit your solution (report and code) as a GROUP. This assignment counts towards your grade and have to be submitted in order to pass the course. You must follow the report guidelines found in `guidelines.pdf`. The page limit for this assignment is **10 pages** including everything, i.e. illustrations and code snippets.

1. **Fourier Transform – Theory**: The following investigate theoretical properties of Fourier series and transform.

   1.1. (1 point) What is the difference between a Fourier series and the Fourier Transform?

   1.2. (1 point) Prove that the continuous Fourier transform of a real and even function is real and even.

   1.3. (1 point) Derive the continuous Fourier transform of $\delta(x-d)+\delta(x+d)$ for $x \in \mathbb{R}$ and some constant $d \in \mathbb{R}$.

   1.4. Consider the box function

   $$b_a(x) = \begin{cases} 1/a & \text{if } |x| \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

   i. (1 point) Show that $\int_{-\infty}^{\infty} b_a(x)\,dx = 1$

   ii. (1 point) Show that the continuous Fourier transform of b_a, using the definition of the Fourier transform given in Bracewell Chapter 2 (system 1), is $B_a(k) = \frac{1}{ak\pi}\sin(ak\pi)$. Rewrite $B_a(k)$ using the $\text{sinc}(x) = \frac{\sin x}{x}$.

   iii. (1 point) Show that $\lim_{a\to 0} B_a(k) = 1$ (Hint: $\lim_{x\to 0}\frac{\sin x}{x} = 1$).

   iv. (1 point) When a is near zero, then b_a is narrow in the x space domain. In this case, would you consider its Fourier transform narrow or wide in the frequency domain, k? What is the relation, when a is large? Explain your answer.

   Deliverables: Include crucial steps in derivations, and a short comment to each answer.

2. **Fourier Transform – Practice:**

   2.1. (1 point) Write two programs: 1) that implements convolution as a nested for loop of the spatial representation of the kernel and image, and 2) that implements the same convolution using Fast Fourier Transformation (`scipy.fft.fft2` and its inverse `scipy.fft.ifft2`). Compare the two implementations, both in terms of the result and the computation time for a number of kernel sizes and image sizes.

   2.2. (1 point) Write a program that adds the function $a_0 cos(v_0 x + w_0 y)$ to `cameraman.tif`. Compute and describe the power spectrum of the result. Design a filter, which removes any such planar waves given v_0 and w_0. You don't need to automate this process, i.e., this design step is allowed to involve you hand-crafting a filter for a particular combination of v_0, w_0.

   Deliverables: Each of theses answers should include examples of the input and output, possibly crucial Python code snippets, and definitely a description of which problems were solved, how, and an evaluation of the results.