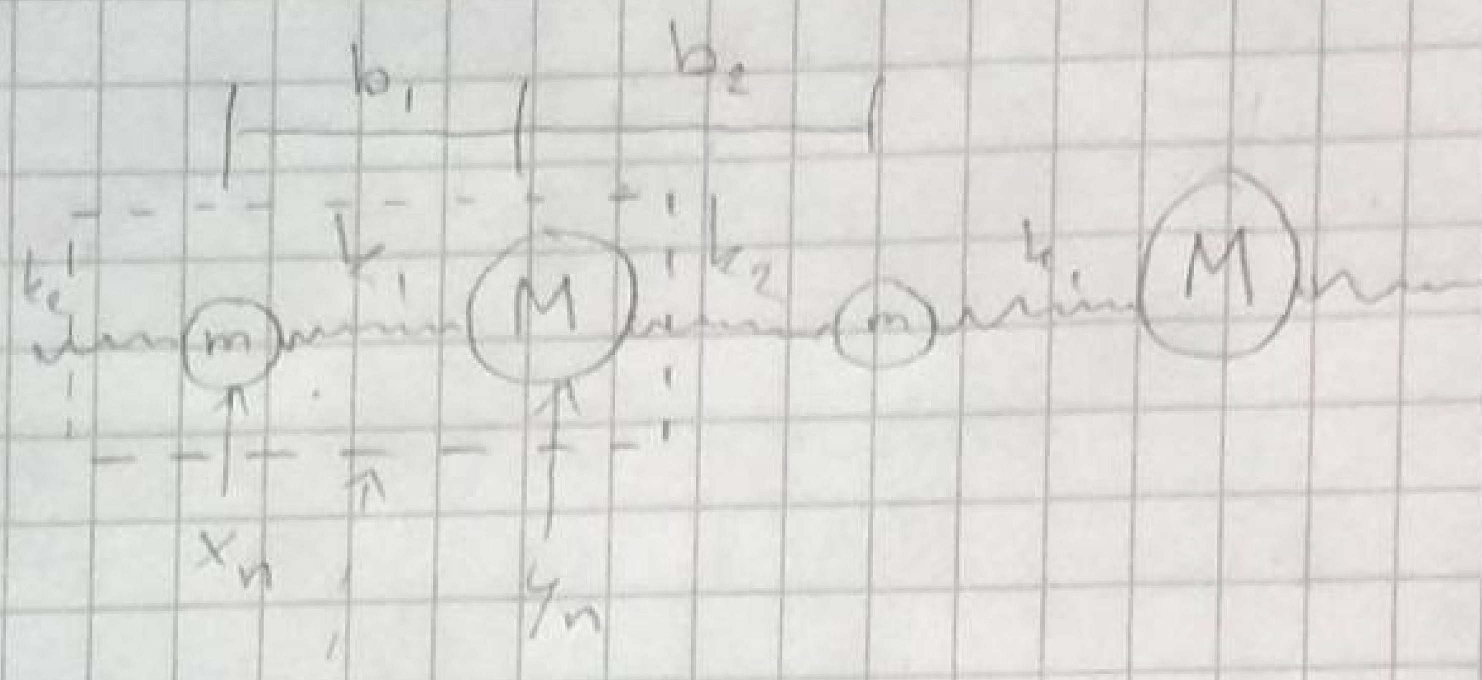
CMP PS2 PWN274

ⓐ




$a = b_1 + b_2$

ⓑ I know the potential energy of any single spring is $V = \frac{1}{2}kx^2$ where $x$ is the distance from equilibrium. This gives:

$$V_n^x = \frac{1}{2}k_1(y_n - x_n - b_1)^2 + \frac{1}{2}k_2(x_n - y_{n-1} - b_2)^2$$

$$V_n^y = \frac{1}{2}k_1(y_n - x_n - b_1)^2 + \frac{1}{2}k_2(x_{n+1} - y_n - b_2)^2$$

The total is therefore:

$$V = \sum_n (V_n^x + V_n^y)$$

ⓒ The forces are found for m and M:

$$F_n^x = -\frac{\partial V}{\partial x_n} = k_1(y_n - x_n - b_1) - k_2(x_n - y_{n-1} - b_2)$$

$$F_n^y = -\frac{\partial V}{\partial y_n} = -k_1(y_n - x_n - b_1) + k_2(x_{n+1} - y_n - b_2)$$

As the terms in parentheses simply is the differerence in offset from equilibrium, this simplifies to:

$$F_n^x = k_1(\partial y_n - \partial x_n) - k_2(\partial x_n - \partial y_{n-1}) \quad \Big| \quad F_n^y = k_2(\partial x_{n+1} - \partial y_n) - k_1(\partial y_n - \partial x_n)$$

Scanned by TapScanner

I now use Newton and look at:

$F_n^x = m\ddot{x}_n$ and $F_n^y = M\ddot{y}_n$

I have the two ans$\frac{d^2a}{dt^2}$tze:

$\partial x_n = A_x e^{i(\omega t - qna)}$ and $\partial y_n = A_y e^{i(\omega t - qna)}$

where $q$ is the wave-number.

I insert using $\phi \equiv i(\omega t - qna)$

for x: $-m\omega^2 A_x e^{\phi} = k_2 A_y e^{\phi} - k_2 A_x e^{\phi} + k_1 A_y e^{\phi + iqa} - k_1 A_x e^{\phi}$

for y: $-M\omega^2 A_y e^{\phi} = k_1 A_x e^{\phi - iqa} - k_1 A_y e^{\phi} + k_2 A_x e^{\phi} - k_2 A_y e^{\phi}$

Dividing through with $e^{\phi}$, this equates the following matrix equation:

$$\omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} (k_1+k_2)/m & -(k_1 e^{iqa} + k_2)/m \\ -(k_1 e^{iqa} + k_2)/M & (k_1+k_2)/M \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

d) Setting $q=0$, it is easy to see $\omega=0$ is a solution:

$$\omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} (k_1+k_2)/m & -(k_1+k_2)/m \\ -(k_1+k_2)/M & (k_1+k_2)/M \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Here $\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigen vector with eigenvalue $0$ ie. $\omega = 0$

This shows us, that for both masses nothing happens, since they experience a wave with $0$ frequency / infinite period

Finding the frequency, I try to solve the characteristic polynomial:

$$\left| \begin{pmatrix} (k_1+k_2)/m - \omega^2 & -(k_1+k_2)/m \\ -(k_1+k_2)/M & (k_1+k_2)/M - \omega^2 \end{pmatrix} \right| = 0$$

$$0 = \left[(k_1+k_2)/m - \omega^2\right]\left[(k_1+k_2)/M - \omega^2\right] - \frac{(k_1+k_2)^2}{mM}$$

$$= \frac{(k_1+k_2)^2}{mM} - \frac{(k_1+k_2)}{m}\omega^2 - \frac{(k_1+k_2)}{M}\omega^2 + \omega^4 - \frac{(k_1+k_2)^2}{mM}$$

$$= \omega^2 - \frac{(k_1+k_2)}{m} - \frac{(k_1+k_2)}{M} \iff \omega^2 = (k_1+k_2)\left(\frac{1}{m} + \frac{1}{M}\right)$$

Now using $m/M \to 0$ I get:

$\omega = \sqrt{\frac{(k_1+k_2)}{m}}$ where we choose the positive

This eigenvalue corresponds to the eigenvector

This is very much expected, as we are looking at the case of M being so big it basically is not moving, which can be seen in the amplitude $A_y$ being 0.

(f) Here I do the same as in the previous part, but using $q = \pi/a$.

I am also using $k_1 = k_2 \equiv \frac{1}{2}k$ i.e. $k_1 + k_2 = k$

$$\left|\begin{pmatrix} k/m - \omega^2 & -(\frac{1}{2}ke^{i\pi} + \frac{1}{2}k)/m \\ -(\frac{1}{2}ke^{i\pi} + \frac{1}{2}k)/M & k/M - \omega^2 \end{pmatrix}\right| = \left|\begin{pmatrix} k/m - \omega^2 & 0 \\ 0 & k/M - \omega^2 \end{pmatrix}\right|$$

$$\Rightarrow 0 = (k/m - \omega^2)(k/M - \omega^2)$$

As this is alredy factored, I see the two eigenvalues are $\omega^2 = k/M$ and $\omega^2 = k/m$

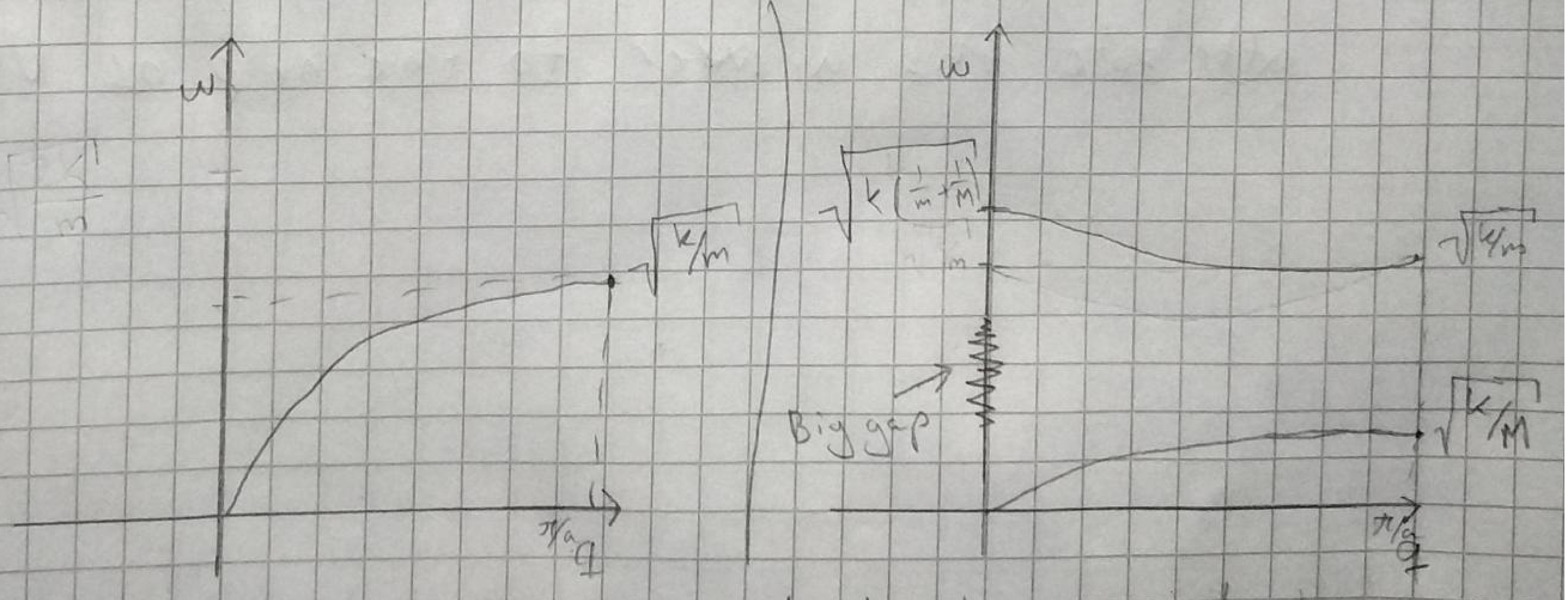
This means the splitting at the edge is

$$\Delta\omega = \sqrt{k}\left(\frac{1}{\sqrt{M}} - \frac{1}{\sqrt{m}}\right) \quad \text{for } k_1 = k_2$$

(g)

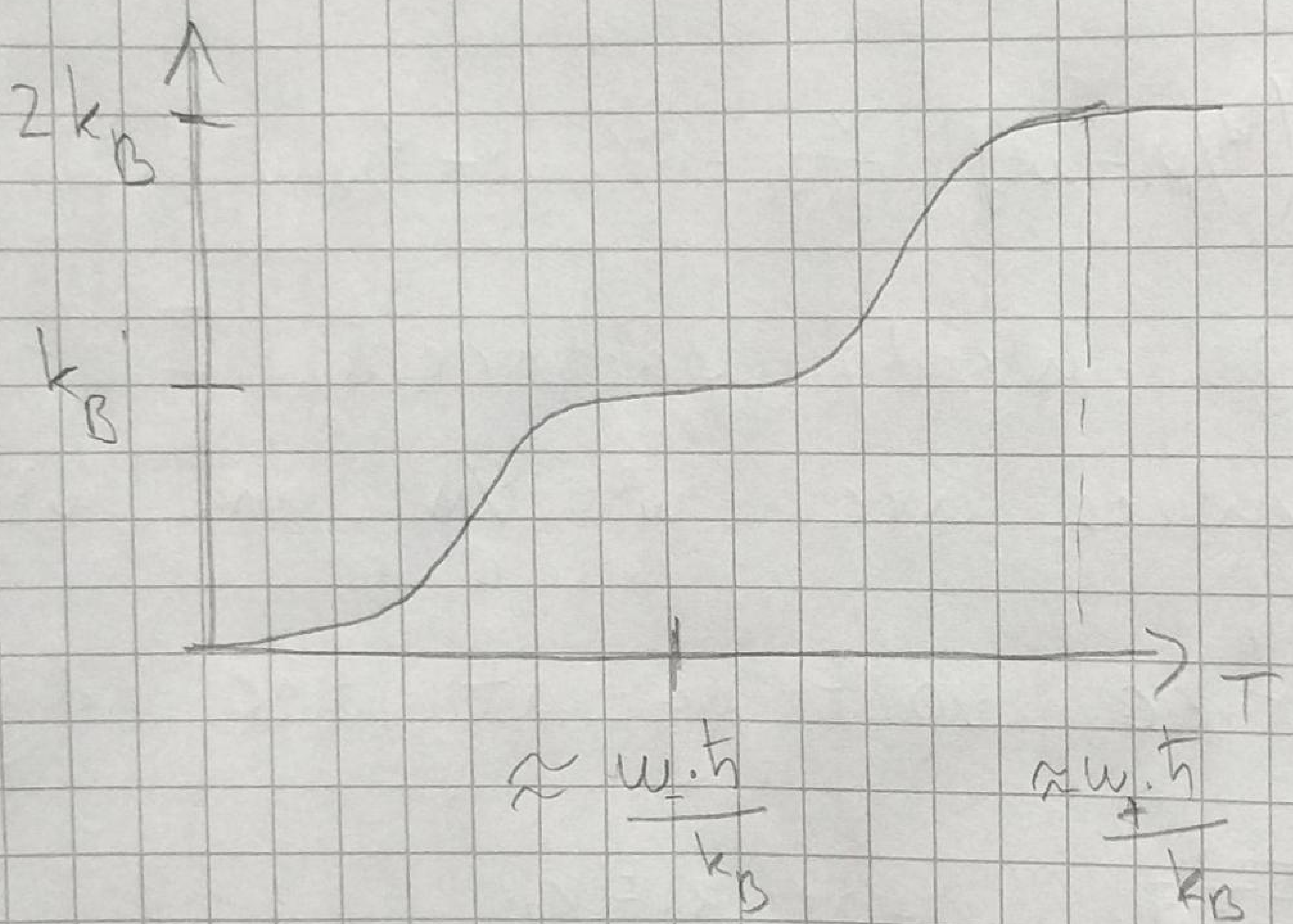
$\boxed{k = k_1 + k_2}$

$m = M$ $M \gg m$




As we for $m = M$ and $k_1 = k_2$ simply have the monoatomic chain, there is no optical

To solve this, I will simply use the equipartition theorem.
When only the acoustic modes are active we have 2 degrees of freedom.
When the optic gets going we have 4.
This gives:




I was having a lot of trouble finding the x-axis without calculations, so I attribute this answer to the help of Ulrik!

Scanned by TapScanner